Analysis of Variance
Single Factor
Part 2

Dr. Supaporn Erjongmanee

Department of Computer Engineering
Kasetsart University
fengspe@ku.ac.th

Outline

- Single-Factor ANOVA:
  - Review: Equal sample size
    - F-test
    - Multiple comparison
  - Unequal Sample Size
    - F-test
    - Multiple comparison
  - Random Effects Model

Single-Factor ANOVA : Equal Sample Size

- Compare two or more populations on one factor
- Let
  - μ_1 = mean of treatment (population) 1
  - μ_2 = mean of treatment (population) 2
  - …
  - μ_I = mean of treatment (population) I

Two population -> Can perform hypothesis test on two data sets

I = number of compared treatments

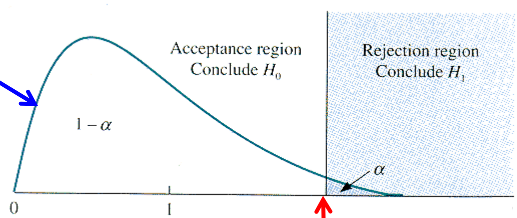
- Hypothesis
  - H_0: $\mu_1 = \mu_2 = \ldots = \mu_I$
  - H_a: Not all μ_i's are equal (at least two of the μ_i's are different)

F-Test

- Test statistic:

$$F_{I-1,I(J-1)} = \frac{\frac{SSTr}{\sigma^2}/(I-1)}{\frac{SSE}{\sigma^2}/(I(J-1))} = \frac{SSTr/(I-1)}{SSE/(I(J-1))} = \frac{MSTr}{MSE}$$

F distribution for v_1 and v_2



$F_{\alpha,\, v1,\, v2}$

If test statistic > $F_{\alpha,\, v1,\, v2}$, reject H_0

Image source: http://www.unc.edu/~nielsen/soci708/m16/m2009.gif

Summary : ANOVA

- Test statistic:

$$F_{I-1,I(J-1)} = \frac{\frac{SSTr}{\sigma^2}/(I-1)}{\frac{SSE}{\sigma^2}/(I(J-1))} = \frac{SSTr/(I-1)}{SSE/(I(J-1))} = \frac{MSTr}{MSE}$$

- When f is large, we are about to reject H_0

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	I-1	SSTr	SSTr/(I-1)	MSTr / MSE
Error	I(J-1)	SSE	SSE/(I(J-1))	
Total	IJ-1	SST		

Summary : ANOVA (cont.)

$\sum_i \sum_j (x_{ij} - \bar{x})^2 = \sum_i \sum_j (x_{ij} - \bar{x}_i + \bar{x}_i - \bar{x})^2$

- Test statistic:

$$F_{I-1,I(J-1)} = \frac{\frac{SSTr}{\sigma^2}/(I-1)}{\frac{SSE}{\sigma^2}/(I(J-1))} = \frac{SSTr/(I-1)}{SSE/(I(J-1))} = \frac{MSTr}{MSE}$$

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	I-1	$J \sum_i (\bar{x}_i - \bar{x})^2$	SSTr/(I-1)	MSTr / MSE
Error	I(J-1)	$\sum_i \sum_j (x_{ij} - \bar{x}_i)^2$ = SST - SSTr	SSE/(I(J-1))	
Total	IJ-1	$\sum_i \sum_j (x_{ij} - \bar{x})^2$		

Summary : ANOVA (cont.)

- Test statistic:

$$F_{I-1,I(J-1)} = \frac{\frac{SSTr}{\sigma^2}/(I-1)}{\frac{SSE}{\sigma^2}/(I(J-1))} = \frac{SSTr/(I-1)}{SSE/(I(J-1))} = \frac{MSTr}{MSE}$$

Another option: Sample-based computation

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	I-1	$\boxed{\frac{1}{J}\sum_{i=1}^{I}(\sum_{j=1}^{J} x_{ij})^2 - \frac{1}{IJ}(\sum_{i=1}^{I}\sum_{j=1}^{J} x_{ij})^2}$	SSTr/(I-1)	MSTr / MSE
Error	I(J-1)	SST - SSTr	SSE/(I(J-1))	
Total	IJ-1	$\boxed{\sum_{i=1}^{I}\sum_{j=1}^{J} x_{ij}^2 - \frac{1}{IJ}(\sum_{i=1}^{I}\sum_{j=1}^{J} x_{ij})^2}$		

Example 1

- Experiment degree of soiling on 3 mixtures of fabric and polymer
- Prove whether 3 mixture means are the same at $\alpha = 0.01$

Mixture	Degree of soiling				
1:	0.56	1.12	0.90	1.07	0.94
2:	0.72	0.69	0.87	0.78	0.91
3:	0.62	1.08	1.07	0.99	0.93

- Let
  - μ_1 = mean of mixture 1
  - μ_2 = mean of mixture 2
  - μ_3 = mean of mixture 3
  - I = 3, J = 5
- Hypothesis
  - H_0: $\mu_1 = \mu_2 = \mu_3$
  - H_a: Not all μ_i's are equal (at least two of the μ_i's are different)

Example 1 (cont.)

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	2	0.0608	0.0304	0.99
Error	12	0.3701 = 0.4309 – 0.0608	0.0308	
Total	14	0.4309		

- Rejection region: given $\alpha = 0.01$
  - $F_{0.01,\ 2,\ 12} = 6.93$
- H_0 is not rejected.
- All mixture means are equals.

Supaporn Erjongmanee
fengspe@ku.ac.th

Department of Computer Engineering
Kasetsart University

Example 2

- Experiment consistency lab measurements from 7 labs

Lab 1	Lab 2	Lab 3	Lab 4	Lab 5	Lab 6	Lab 7
4.13	3.86	4.00	3.88	4.02	4.02	4.00
4.07	3.85	4.02	3.88	3.95	3.86	4.02
4.04	4.08	4.01	3.91	4.02	3.96	4.03
4.07	4.11	4.01	3.95	3.89	3.97	4.04
4.05	4.08	4.04	3.92	3.91	4.00	4.10
4.04	4.01	3.99	3.97	4.01	3.82	3.81
4.02	4.02	4.03	3.92	3.89	3.98	3.91
4.06	4.04	3.97	3.90	3.89	3.99	3.96
4.10	3.97	3.98	3.97	3.99	4.02	4.05
4.04	3.95	3.98	3.90	4.00	3.93	4.06

Supaporn Erjongmanee
fengspe@ku.ac.th

Department of Computer Engineering
Kasetsart University

Example 2

- Experiment consistency lab measurements from 7 labs
- Let
  - μ_1 = mean of measurements from lab 1
  - μ_2 = mean of measurements from lab 2
  - …
  - μ_7 = mean of measurements from lab 7
  - I = 7, J = 10
- Hypothesis
  - H_0: $\mu_1 = \mu_2 = \ldots = \mu_7$
  - H_a: Not all μ_i's are equal (at least two of the μ_i's are different)

Example 2

- Experiment consistency lab measurements from 7 labs

Lab 1	Lab 2	Lab 3	Lab 4	Lab 5	Lab 6	Lab 7
4.13	3.86	4.00	3.88	4.02	4.02	4.00
4.07	3.85	4.02	3.88	3.95	3.86	4.02
4.04	4.08	4.01	3.91	4.02	3.96	4.03
4.07	4.11	4.01	3.95	3.89	3.97	4.04
4.05	4.08	4.04	3.92	3.91	4.00	4.10
4.04	4.01	3.99	3.97	4.01	3.82	3.81
4.02	4.02	4.03	3.92	3.89	3.98	3.91
4.06	4.04	3.97	3.90	3.89	3.99	3.96
4.10	3.97	3.98	3.97	3.99	4.02	4.05
4.04	3.95	3.98	3.90	4.00	3.93	4.06

$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{x}_5$	$\bar{x}_6$	$\bar{x}_7$
4.062	3.997	4.003	3.92	3.957	3.955	3.998

$\bar{x}$=3.985

Example 2 (cont.)

7 labs. Each lab collects 10 samples.

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	**6**	**0.125**	**0.0208**	**5.66**
Error	**63**	**0.356-0.125 = 0.231**	**0.0037**	
Total	**69**	**0.356**		

- There is no $F_{\alpha, 6, 63}$ in the f-table
- Examine closest F critical values: 60 vs. 70
  - $F_{0.1, 6, 60} = 1.8747$, $F_{0.05, 6, 60} = 2.2541$, $F_{0.01, 6, 60} = 3.1187$
  - $F_{0.1, 6, 70} = 1.8600$, $F_{0.05, 6, 70} = 2.2312$, $F_{0.01, 6, 70} = 3.0712$
  - For common values of α, $F_{\alpha, 6, 63} < f$
- H_0 is rejected.
- There are differences in 7 means.

```
> qf(0.99, df1=6, df2=63)
[1] 3.102767
> qf(0.95, df1=6, df2=63)
[1] 2.246408
> qf(0.90, df1=6, df2=63)
[1] 1.869819
>
```


Example 2 (cont.)

7 labs. Each lab collects 10 samples.

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	**6**	**0.125**	**0.0208**	**5.66**
Error	**63**	**0.356-**		
Total	**69**	**0.356**		

- Reject H_0 for large test statistic -> choose small α
- Do not reject H_0 for small test statistic -> choose large α

- There is no $F_{\alpha, 6, 63}$ in the f-table
- Examine closest F critical values: 60 vs. 70
  - $F_{0.1, 6, 60} = 1.8747$, $F_{0.05, 6, 60} = 2.2541$, $F_{0.01, 6, 60} = 3.1187$
  - $F_{0.1, 6, 70} = 1.8600$, $F_{0.05, 6, 70} = 2.2312$, $F_{0.01, 6, 70} = 3.0712$
  - For common values of α, $F_{\alpha, 6, 63} < f$
- H_0 is rejected.
- There are differences in 7 means.

How to choose α for rejecting null hypothesis?
Small or Large?

Outline

- Single-Factor ANOVA:
  - Review: Equal sample size
    - F-test
    - Multiple comparison
  - Unequal Sample Size
    - F-test
    - Multiple comparison
  - Random Effects Model

Multiple Comparisons

- Question:
  - When H_0 for ANOVA is rejected, how many means are different from each other?
- Procedures:
  - Find confidence interval of pairwise difference $\mu_i - \mu_j$
  - If confidence interval for any pairwise difference $\mu_i - \mu_j$ does not include zero, we determine that μ_i, μ_j are significantly different from each other

Studentized Range Distribution (cont.)

$$1-\alpha = P\left(\frac{|\bar{X}_i - \bar{X}_j - (\mu_i - \mu_j)|}{\sqrt{MSE/J}} \le Q_{\alpha,I,I(J-1)} \; for \; all \; i,j\right)$$

$$= P\left(-Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}} \le \bar{X}_i - \bar{X}_j - (\mu_i - \mu_j) \le Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}} \; for \; all \; i,j\right)$$

$$= P\left(\underbrace{\boxed{\bar{X}_i - \bar{X}_j - Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}} \le \mu_i - \mu_j \le \boxed{\bar{X}_i - \bar{X}_j + Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}}}_{} \; for \; all \; i,j\right)$$

Confidence intervals between one pair of true mean difference $\mu_i - \mu_j$

- There are $\binom{I}{2} = \frac{I(I-1)}{2}$ confidence intervals of $\mu_i - \mu_j$

Studentized Range Distribution (cont.)

$$1-\alpha = P\left(\boxed{\bar{X}_i - \bar{X}_j - Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}} \le \mu_i - \mu_j \le \boxed{\bar{X}_i - \bar{X}_j + Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}} \; for \; all \; i,j\right)$$

Expect difference between sample mean difference $(\bar{X}_i - \bar{X}_j)$ and true mean difference $(\mu_i - \mu_j)$ is not more than this value $Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$

- The value w = $Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$ is called
  Tukey's honestly significantly difference (HSD)

Example 1

- Experiment degree of soiling on 3 mixtures
- Group mixture means at α = 0.01

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	2	0.0608	0.0304	0.99
Error	12	0.3701 = 0.4309 – 0.0608	0.0308	
Total	14	0.4309		

Mixture	Degree of soiling					$\bar{x}_i$
1:	0.56	1.12	0.90	1.07	0.94	0.918
2:	0.72	0.69	0.87	0.78	0.91	0.794
3:	0.62	1.08	1.07	0.99	0.93	0.938

$$HSD_\alpha = q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}} = q_{0.01,3,12}\sqrt{\frac{0.0308}{5}} = 5.05\sqrt{\frac{0.0308}{5}} = 0.396$$

```
> qtukey(0.01, nmeans=anovaResult$rank,
+        df=anovaResult$df.residual, lower.tail=F)
[1] 5.045934
```

- Sort sample means: 0.794, 0.918, 0.938 ($\bar{x}_2$, $\bar{x}_1$, $\bar{x}_3$)

  0.124 0.020

- One group of mixture means

Note that H_0 is not rejected.

Supaporn Erjongmanee
fengspe@ku.ac.th

Statistics in Computer Engineering Applications
Slide 19

Department of Computer Engineering
Kasetsart University

Example 1

```
> tk <- TukeyHSD(anovaResult, conf.level = 0.99)
> plot(tk)
```

- Experiment degree of soiling on 3 mixtures
- Group mixture means at α = 0.01

Mixture	Degree of soiling					$\bar{x}_i$
1:	0.56	1.12	0.90	1.07	0.	
2:	0.72	0.69	0.87	0.78	0.	
3:	0.62	1.08	1.07	0.99	0.	

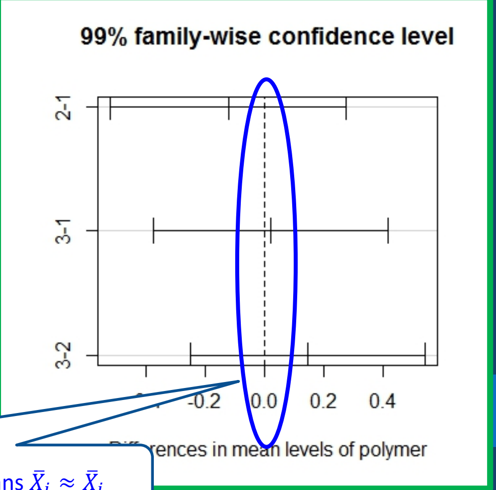
- Sort sample means:

  $\bar{x}_2$ $\bar{x}_1$ $\bar{x}_3$

  0.794, 0.918, 0.938

  0.124 0.020

- One group of mixture means



Confidence interval of difference $(\bar{X}_i - \bar{X}_j)$ includes zero. This means $\bar{X}_i \approx \bar{X}_j$

Supap
fengspe
etsart University

Example 3

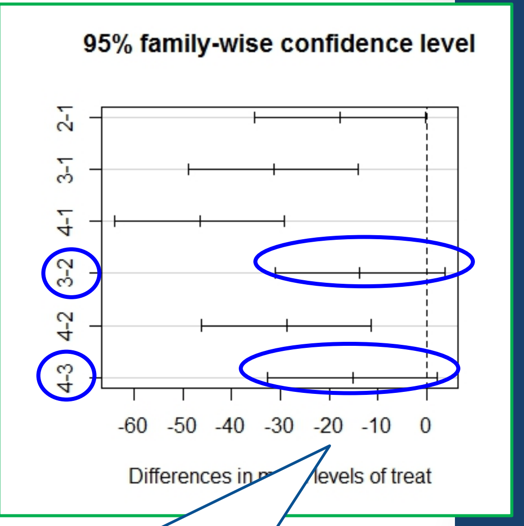
- For another data set:
  - $\bar{x}_1$= 79.28, $\bar{x}_2$= 61.54, $\bar{x}_3$= 47.92, $\bar{x}_4$= 32.76

$$HSD_\alpha = 4.05\sqrt{\frac{932.9625}{5}} = 17.47$$

- Sort sample means:

$\bar{x}_4$ $\bar{x}_3$ $\bar{x}_2$ $\bar{x}_1$

32.76, 47.92, 61.54, 79.28



- 2 groups of means
  - $\bar{x}_4, \bar{x}_3, \bar{x}_2$ are not significantly different from each other
  - $\bar{x}_1$ is significantly different from each other

Confidence intervals of $(\bar{X}_2 - \bar{X}_3)$ and $(\bar{X}_3 - \bar{X}_4)$ include zero. This means $\bar{X}_2 \approx \bar{X}_3 \approx \bar{X}_4$

More on Multiple Comparison

- Is it possible that H_0 is not rejected but result to multiple group of means?
- Is it also possible that H_0 is rejected but have one group of means?

Measured by MSE

ANOVA tests on <u>ALL means</u> whether they are identical

Multiple comparison tests on <u>PAIRWISE means</u>

Not measured by MSE

ANOVA detects variability among all means

ANOVA test is more sensitive than multiple comparison

More on Multiple Comparison (cont.)

ANOVA tests on <u>ALL means</u> whether they are identical

Multiple comparison tests on <u>PAIRWISE means</u>

ANOVA detects lower variability among all means

ANOVA test is more sensitive than multiple comparison

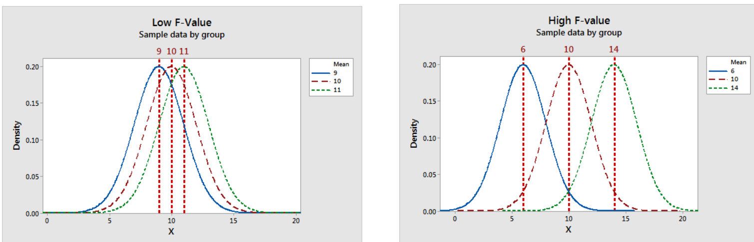


Image source: http://blog.minitab.com/blog/adventures-in-statistics-2/understanding-analysis-of-variance-anova-and-the-f-test

Outline

- Single-Factor ANOVA:
  - Review: Equal sample size
    - F-test
    - Multiple comparison
  - Unequal Sample Size
    - F-test
    - Multiple comparison
  - Random Effects Model

Example 1

- Measure strength of Mg-based alloys from 3 processes
- Show that alloys from 3 processes are the same are the same at α = 0.001

Process	Observations							
Permanent Molding (1)	45.5	45.3	45.4	44.4	44.6	43.9	44.6	44.0
Die casting (2)	44.2	43.9	44.7	44.2	44.0	43.8	44.6	43.1
Plaster molding (3)	46.0	45.9	44.8	46.2	45.1	45.5		

Single ANOVA with Unequal Size

- ANOVA table:
  - N = Number of all samples = $\sum_{i=1}^{I} J_i$
  - $\boldsymbol{I}$ = Number of treatments
  - J_i = Number of samples for treatment i, i = 1, 2, …, $\boldsymbol{I}$

$$SST_r = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(\bar{x}_{ij} - \bar{x})^2 = \sum_{i=1}^{I}\frac{1}{J_i}(\sum_{j=1}^{J_i} X_{ij})^2 - \frac{1}{n}(\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij})^2 \quad \boxed{\text{df = I-1}}$$

$$SST = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(x_{ij} - \bar{x})^2 = \sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij}^2 - \frac{1}{n}(\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij})^2 \quad \boxed{\text{df = N-1}}$$

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(x_{ij} - \bar{x}_i)^2 = SST - SSTr \quad \boxed{\text{df = N-I}}$$

Single ANOVA with Unequal Size

- ANOVA table:
  - N = Number of all samples = $\sum_{i=1}^{I} J_i$
  - $\boldsymbol{I}$ = Number of treatments
  - J_i = Number of samples for treatment i, i = 1, 2, ..., $\boldsymbol{I}$

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	$\boldsymbol{I}$ - 1	$\sum_{i=1}^{I}\sum_{j=1}^{J_i}(\bar{x}_{ij} - \bar{x})^2$	SSTr/($\boldsymbol{I}$-1)	MSTr / MSE
Error	N - $\boldsymbol{I}$	SSE = $\sum_{i=1}^{I}\sum_{j=1}^{J_i}(x_{ij} - \bar{x}_i)^2$ = SST - SSTr	SSE/(N - $\boldsymbol{I}$)	
Total	N - 1	$\sum_{i=1}^{I}\sum_{j=1}^{J_i}(x_{ij} - \bar{x})^2$		

Single ANOVA with Unequal Size

- ANOVA table:
  - N = Number of all samples = $\sum_{i=1}^{I} J_i$
  - $\boldsymbol{I}$ = Number of treatments
  - J_i = Number of samples for treatment i, , 2, ..., $\boldsymbol{I}$

Another option: Sample-based computation

	df	Sum of Squares (SS)	Mean Square (MS)	f
Treatment	$\boldsymbol{I}$ - 1	$\sum_{i=1}^{I}\frac{1}{J_i}(\sum_{j=1}^{J_i} X_{ij})^2 - \frac{1}{n}(\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij})^2$	SSTr/($\boldsymbol{I}$-1)	MSTr / MSE
Error	N - $\boldsymbol{I}$	SSE = $\sum_{i=1}^{I}\sum_{j=1}^{J_i}(X_{ij} - \bar{X}_i)^2$ = SST - SSTr	SSE/(N - $\boldsymbol{I}$)	
Total	N - 1	$\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij}^2 - \frac{1}{n}(\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij})^2$		

Example 1

- Measure strength of Mg-based alloys from 3 processes
- Show that alloys from 3 processes are the same are the same at $\alpha = 0.001$

Process	Observations							
Permanent Molding (1)	45.5	45.3	45.4	44.4	44.6	43.9	44.6	44.0
Die casting (2)	44.2	43.9	44.7	44.2	44.0	43.8	44.6	43.1
Plaster molding (3)	46.0	45.9	44.8	46.2	45.1	45.5		

- Let
  - μ_1 = mean of alloy 1, μ_2 = mean of alloy 2, μ_3 = mean of alloy 3
  - $I = 3, J_1 = J_2 = 8, J_3 = 6, N = 22$
- Hypothesis
  - $H_0: \mu_1 = \mu_2 = \mu_3$
  - H_a: Not all μ_i's are equal (at least two of the μ_i's are different)

Example 1 (cont.)

Process	Observations								$\sum_{j=1}^{J_i} x_{ij}$
Permanent Molding (1)	45.5	45.3	45.4	44.4	44.6	43.9	44.6	44.0	357.7
Die casting (2)	44.2	43.9	44.7	44.2	44.0	43.8	44.6	43.1	352.5
Plaster molding (3)	46.0	45.9	44.8	46.2	45.1	45.5			273.5

- Fill ANOVA table

	df	Sum of Squares (SS)	Mean Square(MS)	f
Treatment	**2**	**7.93**	**3.97**	**12.56**
Error	**19**	**13.93 – 7.93 = 6.00**	**0.32**	
Total	**21**	**13.93**		

Example 1 (cont.)

	df	Sum of Squares (SS)	Mean Square(MS)	f
Treatment	2	7.93	3.97	12.56
Error	19	13.93 – 7.93 = 6.00	0.32	
Total	21	13.93		

- Rejection region: given α = 0.001
  - $F_{0.001, 2, 19}$

Not exist in f-table. Use R.

```
> qf (0.999, df1=2, df2=19)
[1] 10.15681
>
```

- H_0 is rejected.
- Alloys from 3 processes are not the same.

Outline

- Single-Factor ANOVA:
  - Review: Equal sample size
    - F-test
    - Multiple comparison
  - Unequal Sample Size
    - F-test
    - Multiple comparison
  - Random Effects Model

Multiple Comparisons : Unequal Sample Size

- When some means are not all equal, how to specify which mean is different from others
- Procedure
  1. Find Tukey's Honestly Significant Difference (HSD)

$$HSD_\alpha = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}(\frac{1}{J_i}+\frac{1}{J_j})}$$

     - $q_{\alpha,\,I,\,N-I}$ = q-value from studentized range distribution with 2 degrees of freedom *I, N-I*
  2. Sort sample means in increasing order
  3. Underline pairs that differ less than HSD_α
  4. Any pair without underline are considered as significantly different.

Example 1 : Mg-based Alloy (cont.)

$$HSD_{\alpha,ij} = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}(\frac{1}{J_i}+\frac{1}{J_j})}$$

- $\bar{x}_1$= 44.71, $\bar{x}_2$= 44.06, $\bar{x}_3$= 45.58

$$w_{12} = HSD_\alpha = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}(\frac{1}{J_i}+\frac{1}{J_j})} = q_{0.05,3,19}\sqrt{\frac{0.32}{2}(\frac{1}{8}+\frac{1}{8})} = 3.59\sqrt{\frac{0.32}{2}(\frac{1}{8}+\frac{1}{8})} = 0.718$$

$$w_{13} = w_{23} = HSD_\alpha = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}(\frac{1}{J_i}+\frac{1}{J_j})} = q_{0.05,3,19}\sqrt{\frac{0.32}{2}(\frac{1}{8}+\frac{1}{6})} = 3.59\sqrt{\frac{0.32}{2}(\frac{1}{8}+\frac{1}{6})} = 0.775$$

- Sort sample means: 44.06 ($\bar{x}_2$), 44.71 ($\bar{x}_1$), 45.58 ($\bar{x}_3$)

  0.65 **0.87**

- 2 groups of means

Example 1 : Mg-based Alloy (cont.)

- $\bar{x}_1 = 44.71$, $\bar{x}_2 = 44.06$, $\bar{x}_3 = 45.58$

$$w_{12} = HSD_{\alpha} = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}\left(\frac{1}{J_i}+\frac{1}{J_j}\right)} = q_{0.05,3,19}\sqrt{\frac{0.32}{2}\left(\frac{1}{8}+\frac{1}{8}\right)} = 3.5$$

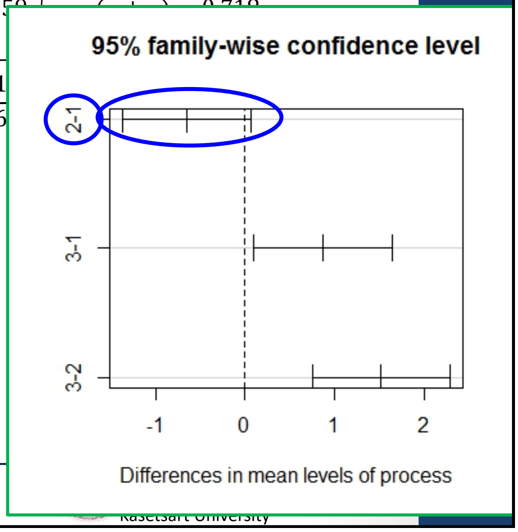
$$w_{13} = w_{23} = HSD_{\alpha} = q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}\left(\frac{1}{J_i}+\frac{1}{J_j}\right)} = q_{0.05,3,19}\sqrt{\frac{0.32}{2}\left(\frac{1}{8}+\frac{1}{6}\right)}$$

- Sort sample means:

$\bar{x}_2$ $\bar{x}_1$ $\bar{x}_3$

44.06, 44.71, 45.58

- 2 groups of means



Outline

- Single-Factor ANOVA:
  - Review: Equal sample size
    - F-test
    - Multiple comparison
  - Unequal Sample Size
    - F-test
    - Multiple comparison
  - Random Effects Model

Random Effects Model

- Single factor can be considered as fixed-effects ANOVA model
- The single-factor fixed-effects model is defined as

$$\boxed{X_{ij} = \mu \quad + \alpha_i + \varepsilon_{ij}} \qquad \boxed{\sum \alpha_i = 0}$$

- X_{ij} = random sample j of treatment i
- μ = overall mean of all treatment i's
- α_i = effect of treatment i
- ε_{ij}= random error in sample j of treatment i
  - Assumed to be independent and normally distributed with mean = 0, variance = σ^2

Random Effects Model (cont.)

$$\boxed{X_{ij} = \mu \quad + \alpha_i + \varepsilon_{ij}} \qquad \boxed{\sum \alpha_i = 0}$$

- Explanations of the single-factor fixed-effects model:
  - Sample is corrupted by random errors
  - Error in one sample is independent from error of other samples
  - Expected response of treatment i

$$\boxed{E(X_{ij}) = \mu \quad + \ \alpha_i}$$

  - If $\alpha_i = 0$, then all treatment i's have the same response

$$\boxed{E(X_{ij}) = \mu}$$

That's what we try to prove in single-factor ANOVA

Null Hypothesis: Single-Factor ANOVA

$E(X_{ij}) = \mu + \alpha_i$

- Besides the prior null hypothesis in single-factor ANOVA
  - H_0: $\mu_1 = \mu_2 = \ldots = \mu_I$
- Sometimes, the following null hypothesis is also used instead:
  - H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$

References

1. J.L. Devore and K.N.Berk, Modern Mathematical Statistics with Applications, Springer, 2012.
2. S. Few, Now You See It: Simple Visualization Techniques for Quantitative Analysis, Analytics Press, 2009