Given a set based S-Category $\mathbb{C}$ which is a model of the non-polymorphic effect calculus, we generate an indexed category capable of modelling the polymorphic effect calculus.

0.1 The Non-Polymorphic Model

Since $\mathbb{C}$ is a model of the non-polymorphic calculus,

- $\mathbb{C}$ is cartesian closed.
- $\mathbb{C}$ has a strong graded monad: $\mathtt{T}^0 : (E, \cdot, \leq_0, \mathtt{1}) \to [\mathbb{C}, \mathbb{C}]$
- $\mathbb{C}$ has a co-product on the terminal object $\mathtt{1}$.

In addition, we require that

- $\mathbb{C}$ should be complete (e.g a sub-category of `Set`)
- E should be small.
- The Non-polymorphic effect calculus has a least effect $\perp \in E$ such that $\forall \epsilon \in E. \perp \leq_0 \epsilon$.

0.2 Base Category

We construct the base category, `Eff` as follows:

- $U = E$, the set of ground effects in the non-polymorphic language.
- $\mathtt{1}$ is a singleton set.
- $U^n = E^n$, set of n-wide tuples of effects, $\vec{\epsilon}$

Hence when we treat effects that are well formed in Φ as morphisms, $E^n \to E$ in `Eff`, we should treat them as functions $f : E^n \to E$. Ground effects become point functions: $e : \mathtt{1} \to E$, so the denotation of a ground effect is the constant value function: $[\![\Phi \vdash e\colon \mathtt{Effect}]\!]_M = \vec{\epsilon} \mapsto e$

We extend the multiplication of ground effects to multiplication on effect functions, giving us our `Mul` operation

$$\mathtt{Mul}(f, g) = f \cdot g \tag{1}$$
$$(f \cdot g)(\vec{\epsilon}) = (f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \tag{2}$$
$$\tag{3}$$

This satisfies naturality of `Mul`.

$$((f \cdot g) \circ \theta)\vec{\epsilon} = (f(\theta\vec{\epsilon})) \cdot (g(\theta\vec{\epsilon})) = ((f \circ \theta) \cdot (g \circ \theta))\vec{\epsilon} \tag{4}$$

0.3 S-Categories

The semantic category, $\mathbb{C}_0$ of the effect-environment $\diamond$ is $\mathbb{C}$.

Since each effect-environment is alpha equivalent to a natural number, the semantic category for Φ shall be represented as $\mathbb{C}_\Phi = \mathbb{C}_n = [E^n, \mathbb{C}]$, the category of functions $E^n \to \mathbb{C}$.

Objects in $[E^n, \mathbb{C}]$ are functions and we describe them by their actions on a generic vector of ground effects, $\vec{\epsilon}$.

Morphisms in $[E^n, \mathbb{C}]$ are natural transformations between the functions. So:

$$\begin{aligned} m : &\quad A \to B \quad \text{In } [E^n, \mathbb{C}] && (5)\\ m\vec{\epsilon} : &\quad A\vec{\epsilon} \to B\vec{\epsilon} \quad \text{In } \mathbb{C} && (6)\\ (f \circ g)\vec{\epsilon} = &\quad (f\vec{\epsilon}) \circ (g\vec{\epsilon}) && (7)\\ \mathtt{1}(\vec{\epsilon}) = &\quad \mathtt{1} && (8) \end{aligned}$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in $\mathbb{C}$.

0.3.1 Each S-Category is a CCC

Since $\mathbb{C}$ is complete and a CCC, and E^n is small, since E is small, $[E^n, \mathbb{C}]$ is a CCC.

$$\begin{aligned} (A \times B)\vec{\epsilon} &= (A\vec{\epsilon}) \times (B\vec{\epsilon}) && (9)\\ \mathtt{1}\vec{\epsilon} &= \mathtt{1} && (10)\\ (B^A)\vec{\epsilon} &= (B\vec{\epsilon})^{(A\vec{\epsilon})} && (11)\\ \pi_1\vec{\epsilon} &= \pi_1 && (12)\\ \pi_2\vec{\epsilon} &= \pi_2 && (13)\\ \mathtt{app}\vec{\epsilon} &= \mathtt{app} && (14)\\ \mathtt{cur}(f)\vec{\epsilon} &= \mathtt{cur}(f\vec{\epsilon}) && (15)\\ \langle f, g\rangle\,\vec{\epsilon} &= \langle f\vec{\epsilon}, g\vec{\epsilon}\rangle && (16)\\ & && (17) \end{aligned}$$

0.3.2 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation $[\![\gamma]\!]_M \in \mathtt{obj}\ \mathbb{C}$. The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$\begin{aligned} [\![\gamma]\!]_M : &\quad E^n \to \mathtt{obj}\ \mathbb{C} && (18)\\ \vec{\epsilon} \mapsto &\quad [\![\gamma]\!]_M && (19)\\ & && (20) \end{aligned}$$

Each constant term $\mathtt{C}^A$ in the non-polymorphic calculus has a fixed denotation $[\![\mathtt{C}^A]\!]_M \in \mathbb{C}(\mathtt{1}, A)$.

So the morphism $[\![\mathtt{C}^A]\!]_M$ in $[E^n, \mathbb{C}]$ is the corresponding constant dependently typed morphism.

$$\begin{aligned}
[\![\mathtt{C}^A]\!]_M : &\quad [E^n, \mathbb{C}](\mathtt{1}, A) && (21)\\
\vec{\epsilon} \mapsto &\quad [\![\mathtt{C}^A]\!]_M && (22)
\end{aligned}$$

0.3.3 Graded Monad

Given the strong graded monad $(\mathtt{T}^0, \eta^0, \mu^0_{,,}, \mathtt{t}^0_{,,})$ on $\mathbb{C}$ we can construct an appropriate graded monad on $[E^n, \mathbb{C}]$.

$$\begin{aligned}
\mathtt{T}^n : &\quad (E^n, \cdot, \leq_n, \mathtt{1}_n) \to [[E^n, \mathbb{C}], [E^n, \mathbb{C}]] && (23)\\
(\mathtt{T}^n_f A)\vec{\epsilon} = &\quad \mathtt{T}^0_{(f\vec{\epsilon})} A\vec{\epsilon} && (24)\\
(\eta^n_A)\vec{\epsilon} = &\quad \eta^0_{A\vec{\epsilon}} && (25)\\
(\mu^n_{f,g,A})\vec{\epsilon} = &\quad \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} && (26)\\
(\mathtt{t}^n_{f,A,B})\vec{\epsilon} = &\quad \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} && (27)
\end{aligned}$$

Through some mechanical proof and the naturality of the $\mathbb{C}$ strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in $[E^n, \mathbb{C}]$

Naturality

$$\begin{tikzcd}
A\vec{\epsilon} \arrow[r, "\eta^0_{(A\vec{\epsilon})}"] \arrow[d, "f\vec{\epsilon}"] & \mathtt{T}^0_{\mathtt{1}}(A\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{\mathtt{1}}(f\vec{\epsilon})"] \\
B\vec{\epsilon} \arrow[r, "\eta^0_{(B\vec{\epsilon})}"] & \mathtt{T}^0_{\mathtt{1}}(B\vec{\epsilon})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(A\vec{\epsilon}) \arrow[r, "\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}"] \arrow[d, "\mathtt{T}^0_{f\vec{\epsilon}}\mathtt{T}^0_{g\vec{\epsilon}}m\vec{\epsilon}"] & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}m\vec{\epsilon}"] \\
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(B\vec{\epsilon}) \arrow[r, "\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}"] & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon})
\end{tikzcd}$$

$$\begin{tikzcd}
A\vec{\epsilon} \times \mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}"] \arrow[d, "(m\vec{\epsilon} \times \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}B})"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})}(m\vec{\epsilon} \times \mathtt{Id}_{B\vec{\epsilon}})"] \\
A'\vec{\epsilon} \times \mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A'\vec{\epsilon}),(B\vec{\epsilon})}"] & \mathtt{T}^0_{f\vec{\epsilon}}(A'\vec{\epsilon} \times B\vec{\epsilon})
\end{tikzcd}$$

$$\begin{tikzcd}
A\vec{\epsilon} \times \mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}"] \arrow[d, "(\mathtt{Id}_{A\vec{\epsilon}} \times \mathtt{T}^0_{f\vec{\epsilon}}(m\vec{\epsilon}))"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})}(\mathtt{Id}_{A\vec{\epsilon}} \times m\vec{\epsilon})"] \\
A\vec{\epsilon} \times \mathtt{T}^0_{f\vec{\epsilon}}(B'\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B'\vec{\epsilon})}"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B'\vec{\epsilon})
\end{tikzcd}$$

Monad Laws

Left Unit

$$
\begin{aligned}
(\mu^n_{f,\mathtt{1},A} \circ \mathtt{T}^n_f \eta^n_A)\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),\mathtt{1},(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\eta^0_{A\vec{\epsilon}}) && (28)\\
&= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} && (29)\\
&= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon} && (30)
\end{aligned}
$$

Right Unit

$$
\begin{aligned}
(\mu^n_{\mathtt{1},g,A} \circ \eta^n_{\mathtt{T}^n_f A})\vec{\epsilon} &= \mu^0_{\mathtt{1},(f\vec{\epsilon}),(A\vec{\epsilon})} \circ (\eta^0_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}}) && (31)\\
&= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} && (32)\\
&= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon} && (33)
\end{aligned}
$$

Monad Associativity

$$
\begin{aligned}
((\mu^n_{f,(g\cdot h),A}) \circ \mathtt{T}^n_f(\mu^n_{g,h,A}))\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),((g\vec{\epsilon})\cdot(h\vec{\epsilon})),(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}\mu^0_{(h\vec{\epsilon}),(g\vec{\epsilon}),A\vec{\epsilon}} && (34)\\
&= \mu^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(h\vec{\epsilon}),(A\vec{\epsilon})} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(\mathtt{T}^0_{h\vec{\epsilon}}(A\vec{\epsilon}))} && (35)\\
&= (\mu^n_{f\cdot g,h,A} \circ \mu^n_{f,g,\mathtt{T}^0_h A})\vec{\epsilon} && (36)
\end{aligned}
$$

Tensorial Strength

Unitor Law

$$
\begin{aligned}
(\mathtt{T}^n_f \pi_2)\vec{\epsilon} &= \mathtt{T}^0_{(f\vec{\epsilon})}(\pi_2\vec{\epsilon}) && (37)\\
&= \mathtt{T}^0_{(f\vec{\epsilon})}(\pi_2) && (38)\\
&= \pi_2 && (39)\\
&= \pi_2\vec{\epsilon} && (40)
\end{aligned}
$$

Bind Law

$$
\begin{tikzcd}
A \times \mathtt{T}^n_f\mathtt{T}^n_g B \arrow[r, "\mathtt{t}_{f,A,\mathtt{T}^n_g B}"] \arrow[dr, "\mathtt{Id}_A \times \mu^n_{f,g,B}"] & \mathtt{T}^n_f(A \times \mathtt{T}^n_g B) \arrow[r, "\mathtt{T}^n_f \mathtt{t}_{g,A,B}"] & \mathtt{T}^n_f\mathtt{T}^n_g(A \times B) \arrow[d, "\mu^n_{f,g,A\times B}"] \\
& A \times \mathtt{T}^n_{f\cdot g} B \arrow[r, "\mathtt{t}_{f\cdot g,A,B}"] & \mathtt{T}^n_{f\cdot g}(A \times B)
\end{tikzcd}
$$

$$
\begin{aligned}
(\mathtt{t}^n_{(f\cdot g),A,B} \circ (\mathtt{Id}_A \times \mu^n_{f,g,B}))\vec{\epsilon} &= (\mathtt{t}^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mu^n_{(f\vec{\epsilon}),(g\vec{\epsilon}),(B\vec{\epsilon})})) && (41)\\
&= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\times B)\vec{\epsilon}} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}) \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),\mathtt{T}^0_{g\vec{\epsilon}}(B\vec{\epsilon})} && (42)\\
&= (\mu^n_{f,g,(A\times B)} \circ \mathtt{T}^n_f(\mathtt{t}^n_{g,A,B}) \circ \mathtt{t}^n_{f,A,\mathtt{T}^n_g(B)})\vec{\epsilon} && (43)
\end{aligned}
$$

Commutativity with Unit

$$
\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_{\mathtt{1}} B \arrow[d, "\mathtt{t}_{\mathtt{1},A,B}"] \\
& \mathtt{T}^n_{\mathtt{1}}(A \times B)
\end{tikzcd}
$$

$$(\mathtt{t}^n_{1,A,B} \circ (\mathtt{Id}_A \times \eta^n_A))\vec{\epsilon} = \mathtt{t}^0_{1,(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \eta^0_{A\vec{\epsilon}}) \tag{44}$$
$$= \eta^0_{A\vec{\epsilon}\times B\vec{\epsilon}} \tag{45}$$
$$= (\eta^n_{A\times B})\vec{\epsilon} \tag{46}$$

Commutativity with α Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times \mathtt{T}^n_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,\mathtt{T}^n_\epsilon C}"] & & \mathtt{T}^n_\epsilon((A \times B) \times C) \arrow[d, "\mathtt{T}^n_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times \mathtt{T}^n_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times \mathtt{T}^n_\epsilon (B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & \mathtt{T}^n_\epsilon(A \times (B \times C))
\end{tikzcd}$$

$$(\mathtt{T}^n_f \alpha_{A,B,C} \circ \mathtt{t}^n_{f,A\times B,C})\vec{\epsilon} = \mathtt{T}^0_{f\vec{\epsilon}} \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\times B)\vec{\epsilon},(C\vec{\epsilon})} \tag{47}$$
$$= \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon}\times C\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mathtt{t}^0_{(f\vec{\epsilon}),(B\vec{\epsilon}),(C\vec{\epsilon})}) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \tag{48}$$
$$= (\mathtt{t}^n_{f,A,(B\times C)} \circ (\mathtt{Id}_A \times \mathtt{t}^n_{f,B,C}) \circ \alpha_{A,B,C})\vec{\epsilon} \tag{49}$$
$$\tag{50}$$

0.3.4 Sub-Effecting

Given a collection of sub-effecting natural transformation in $\mathbb{C}$,

$$[\![\epsilon_1 \leq_0 \epsilon_2]\!]_M : \quad \mathtt{T}^0_{\epsilon_1} \to \mathtt{T}^0_{\epsilon_2} \tag{51}$$

We can form sub-effect natural transformations in $[E^n, \mathbb{C}]$:

$$[\![f \leq_n g]\!]_M : \quad \mathtt{T}^n_f \to \mathtt{T}^n_g \tag{52}$$
$$[\![f \leq_n g]\!]_M A\vec{\epsilon} : \quad \mathtt{T}^n_{f\vec{\epsilon}}(A\vec{\epsilon}) \to \mathtt{T}^n_{g\vec{\epsilon}}(B\vec{\epsilon}) \tag{53}$$
$$= \quad [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_M A\vec{\epsilon} \tag{54}$$

Naturality

$$\begin{tikzcd}
\mathtt{T}^0_{f\vec{\epsilon}} A\vec{\epsilon} \arrow[r, "{[\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_M A\vec{\epsilon}}"] \arrow[d, "\mathtt{T}^0_{f\vec{\epsilon}} m\vec{\epsilon}"] & \mathtt{T}^0_{g\vec{\epsilon}} A\vec{\epsilon} \arrow[d, "\mathtt{T}^0_{g\vec{\epsilon}} m\vec{\epsilon}"] \\
\mathtt{T}^0_{f\vec{\epsilon}} B\vec{\epsilon} \arrow[r, "{[\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_M B\vec{\epsilon}}"] & \mathtt{T}^0_{g\vec{\epsilon}} B\vec{\epsilon}
\end{tikzcd}$$

Commutes With Tensor Strength

$$\begin{tikzcd}
A \times \mathtt{T}^n_f B \arrow[r, "{\mathtt{Id}_A \times [\![f \leq_n g]\!]_B}"] \arrow[d, "\mathtt{t}^n_{f,A,B}"] & A \times \mathtt{T}^n_g B \arrow[d, "\mathtt{t}^n_{g,A,B}"] \\
\mathtt{T}^n_f (A \times B) \arrow[r, "{[\![f \leq_n g]\!]_{A\times B}}"] & \mathtt{T}^n_g (A \times B)
\end{tikzcd}$$

$$
\begin{aligned}
(\mathtt{t}^n_{g,A,B} \circ (\mathtt{Id}_A \times [\![f \leq_n g]\!]_B))\vec{\epsilon} &= \mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{B\vec{\epsilon}}) && (55)\\
&= [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{(A\times B)\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} && (56)\\
&= ([\![f \leq_n g]\!]_{(A\times B)} \circ \mathtt{t}^n_{f,A,B})\vec{\epsilon} && (57)\\
& && (58)
\end{aligned}
$$

Commutes with Join

$$
\begin{tikzcd}
\mathtt{T}^n_f\mathtt{T}^n_g \arrow[r, "\mathtt{T}^n_f[\![g\leq_n g']\!]_M"] \arrow[d, "\mu^n_{f,g,}"] & \mathtt{T}^n_f\mathtt{T}^n_{g'} \arrow[r, "{[\![f\leq_n f']\!]_{M,\mathtt{T}^n_{g'}}}"] & \mathtt{T}^n_{f'}\mathtt{T}^n_{g'} \arrow[d, "\mu^n_{f',g',}"] \\
\mathtt{T}^n_{f\cdot g} \arrow[rr, "{[\![f\cdot g\leq_n f'\cdot g']\!]_M}"] & & \mathtt{T}^n_{f'\cdot g'}
\end{tikzcd}
$$

$$
\begin{aligned}
([\![f\cdot g \leq_n f'\cdot g']\!]_A \circ \mu^n_{f,g,A})\vec{\epsilon} &= [\![(f\vec{\epsilon})\cdot(g\vec{\epsilon}) \leq_0 (f'\vec{\epsilon})\cdot(g\vec{\epsilon})]\!]_{A\vec{\epsilon}} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} && (59)\\
&= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \circ [\![f\vec{\epsilon} \leq_0 f'\vec{\epsilon}]\!]_{\mathtt{T}^0_{g'\vec{\epsilon}}(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}[\![g\vec{\epsilon} \leq_0 g'\vec{\epsilon}]\!]_{(A\vec{\epsilon})} && (60)\\
&= \mu^n_{f,g,A} \circ [\![f \leq_n f']\!]_{\mathtt{T}^n_{g'}A} \circ \mathtt{T}^n_f[\![g \leq_n g']\!]_A && (61)
\end{aligned}
$$

0.3.5 Sub-Typing

Sub-typing in $[E^n,\mathbb{C}]$ holds via sub-typing in $\mathbb{C}$

$$
\begin{aligned}
[\![A \leq:_n B]\!]_M : &\quad A \to B && (62)\\
[\![A \leq:_n B]\!]_M\vec{\epsilon} = &\quad [\![A\vec{\epsilon} \leq:_0 B\vec{\epsilon}]\!]_M && (63)
\end{aligned}
$$

So the subtyping relation $A \leq: B$ forms a morphism in $[E^n,\mathbb{C}]$

0.4 Functors Between S-Categories

For a function $\theta : E^m \to E^n$, the re-indexing functor θ^* is defined as follows:

$$
\begin{aligned}
\theta^* : &\quad [E^n,\mathbb{C}] \to [E^m,\mathbb{C}] && (64)\\
\theta^*(A)\vec{\epsilon_m} = &\quad A(\theta(\vec{\epsilon_m})) && (65)\\
f : &\quad A \to B \in [E^n,\mathbb{C}] && (66)\\
\theta^*(f)\vec{\epsilon_m} = &\quad f(\theta(\vec{\epsilon_m})) : A(\theta(\vec{\epsilon_m}) \to B(\theta(\vec{\epsilon_m}))) && (67)
\end{aligned}
$$

0.4.1 Quantification

We need to define $\forall_{E^n} : [E^{n+1},\mathbb{C}] \to [E^n,\mathbb{C}]$

So

$$
\begin{aligned}
(\forall_{E^n} A)\vec{\epsilon_n} = \quad & \Pi_{\epsilon \in E} A(\vec{\epsilon_n}, \epsilon) && (68)\\
m: \quad & A \to B && (69)\\
(\forall_{E^n} m): \quad & \forall_{E^n} A \to \forall_{E^n} B && (70)\\
(\forall_{E^n} m)\vec{\epsilon_n} = & \Pi_{\epsilon \in E} m(\vec{\epsilon_n}, \epsilon) && (71)\\
& && (72)
\end{aligned}
$$

0.4.2 Adjunction

It is the case that:

$$\pi_1^* \dashv \forall_{E^n}$$

With unit:

$$
\begin{aligned}
\boldsymbol{\eta}_A: \quad & A \to \forall_{E^n} \pi_1^* A && (73)\\
\boldsymbol{\eta}_A(\vec{\epsilon_n}) = \quad & \left\langle \mathtt{Id}_{A(\vec{\epsilon_n}, e)} \right\rangle_{\epsilon \in E} && (74)
\end{aligned}
$$

And co-unit

$$
\begin{aligned}
\boldsymbol{\epsilon}_B: \quad & \pi_1^* \forall_{E^n} B \to B && (75)\\
\boldsymbol{\epsilon}_B(\vec{\epsilon_n}, \epsilon) = \quad & \pi_\epsilon : \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) \to \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) && (76)
\end{aligned}
$$

We then define the natural bijection as so:

$$\overline{(-)} : [E^{n_1}, \mathbb{C}](\pi_1^* A, B) \leftrightarrow [E^n, \mathbb{C}](A, \forall_{E^n} B) : \widehat{(-)} \tag{77}$$

$$
\begin{aligned}
m: \quad & \pi_1^* A \to B && (78)\\
\overline{m}: \quad & A \to \forall_{E^n} B && (79)\\
\overline{m}(\vec{\epsilon_n}) = \quad & \langle m(\vec{\epsilon_n}, \epsilon) \rangle_{e \in E} && (80)
\end{aligned}
$$

$$
\begin{aligned}
n: \quad & A \to \forall_{E^n} B && (81)\\
\widehat{n}: \quad & \pi_1^* A \to B && (82)\\
\widehat{n}(\vec{\epsilon_n}, \epsilon_{n+1}) = \quad & \pi_\epsilon \circ g(\vec{\epsilon_n}) && (83)
\end{aligned}
$$

This is an Adjunction

For any $g : \pi_1^* A \to B$,

$$
\begin{aligned}
(\boldsymbol{\epsilon}_B \circ \pi_1^*(\overline{g}))(\vec{\epsilon_n}, \epsilon_{n+1}) &= \pi_{\epsilon_{n+1}} \circ \langle g(\vec{\epsilon_n}, \epsilon') \rangle_{\epsilon' \in E} && (84)\\
&= g(\vec{\epsilon_n}, \epsilon_{n+1}) && (85)
\end{aligned}
$$

0.4.3 Beck-Chevalley Condition

For $\theta : E^m \to E^n$:

$$\begin{aligned}
((\theta^* \circ \forall_{E^n})A)\vec{\epsilon_n} &= \theta^*(\forall_{E^n} A)\vec{\epsilon_n} && (86)\\
&= (\forall_{E^n} A)(\theta(\vec{\epsilon_n})) && (87)\\
&= \Pi_{\epsilon\in E}(A(\theta(\vec{\epsilon_n}), \epsilon)) && (88)\\
&= \Pi_{\epsilon\in E}((((\theta \times \mathtt{Id}_U)^* A)(\vec{\epsilon_n}, \epsilon)) && (89)\\
&= \forall_{E^m}((\theta \times \mathtt{Id}_E)^* A)\vec{\epsilon_n} && (90)\\
&= ((\forall_{E^m} \circ (\theta \times \mathtt{Id}_E)^*)A)\vec{\epsilon_n} && (91)
\end{aligned}$$

And $\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}} = \mathtt{Id}_{\theta^* \circ \forall_I}$.

$$\begin{aligned}
\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A}\vec{\epsilon} &= \langle (\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A(\vec{\epsilon}, \epsilon) \rangle_{\epsilon\in E} && (92)\\
&= \langle \boldsymbol{\epsilon}_A(\theta\vec{\epsilon}, \epsilon) \rangle_{e\in E} && (93)\\
&= \langle \pi_\epsilon \rangle_{\epsilon\in E} : \Pi_{\epsilon\in E} A(\theta\vec{\epsilon}, \epsilon) \to \Pi_{\epsilon\in E} A(\theta\vec{\epsilon}, \epsilon) && (94)\\
&= \mathtt{Id}_{\Pi_{\epsilon\in E} A(\theta\vec{\epsilon}, \epsilon)} && (95)\\
&= \mathtt{Id}_{\forall_{I'} \circ (\theta \times \mathtt{Id}_U)^* A}\vec{\epsilon} && (96)\\
&= \mathtt{Id}_{\theta^* \circ \forall_I} && (97)
\end{aligned}$$