Naturality

$$(\texttt{Mul}_m(f,g)\circ\theta)\vec{\epsilon} = (f(\theta\vec{\epsilon}))\cdot(g(\theta\vec{\epsilon})) = (\texttt{Mul}_n(f\circ\theta),(g\circ\theta))\vec{\epsilon}$$

Left and Right Identity

$$\begin{aligned}\texttt{Mul}_n([\![\Phi\vdash \texttt{1}\colon \texttt{Effect}]\!], f)(\vec{\epsilon}) &= (\texttt{1}\vec{\epsilon})\cdot(f\vec{\epsilon})\\ &= (\texttt{1})\cdot(f\vec{\epsilon})\\ &= f\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}\texttt{Mul}_n(f, [\![\Phi\vdash \texttt{1}\colon \texttt{Effect}]\!])(\vec{\epsilon}) &= (f\vec{\epsilon})\cdot(\texttt{1}\vec{\epsilon})\\ &= (f\vec{\epsilon})\cdot(\texttt{1})\\ &= f\vec{\epsilon}\end{aligned}$$

Monoid Associativity

$$\begin{aligned}\texttt{Mul}_n(f,\texttt{Mul}_n(g,h))(\vec{\epsilon}) &= (f\vec{\epsilon})\cdot((g\vec{\epsilon})\cdot(h\vec{\epsilon}))\\ &= ((f\vec{\epsilon})\cdot(g\vec{\epsilon}))\cdot(h\vec{\epsilon})\\ &= \texttt{Mul}_n(\texttt{Mul}_n(f,g),h)(\vec{\epsilon})\end{aligned}$$

Figure 1: Our pick for `Mul` is natural and is a monoid at each n.

0.1 Base Category is Monoidal

We construct the base category, `Eff`, as follows:

- $U = E$, the set of ground effects in the non-polymorphic language.
- $\texttt{1}$ is a singleton set.
- $U^n = E^n$, set of n-wide tuples of effects, $\vec{\epsilon}$

Hence when we treat effects that are well formed in Φ as morphisms, $E^n \to E$ in `Eff`, we should treat them as functions $f : E^n \to E$. Ground effects become point functions: $e : \texttt{1} \to E$, so the denotation of a ground effect is the constant value function: $[\![\Phi \vdash e\colon \texttt{Effect}]\!] = \vec{\epsilon} \mapsto e$. We extend the multiplication of ground effects to multiplication on effect functions, giving us our $\texttt{Mul}_n$ operation $\texttt{Mul}_n(f,g)(\vec{\epsilon}) = (f\vec{\epsilon})\cdot(g\vec{\epsilon})$. This $\texttt{Mul}_n$ satisfies the naturality and monoidal requirements, as seen in figure 1. It is also trivially the case that $\texttt{Mul}_n([\![\Phi\vdash e_1\colon\texttt{Effect}]\!],[\![\Phi\vdash e_2\colon\texttt{Effect}]\!]) = [\![\Phi\vdash e_1\cdot e_2\colon\texttt{Effect}]\!]$.

0.2 S-Categories

The semantic category, $[E^0, \texttt{Set}]$ of the effect-environment $\diamond$ is isomorphic to $\texttt{Set}$. Since each effect-environment is alpha equivalent to a natural number, the semantic category for Φ shall be represented as $\mathbb{C}(\Phi) = \mathbb{C}(n) = [E^n, \texttt{Set}]$, the category of functions $E^n \to \texttt{Set}$. Objects in $[E^n, \texttt{Set}]$ are functions and we describe them by their actions on a generic vector of ground effects, $\vec{\epsilon}$. Morphisms in $[E^n, \texttt{Set}]$ are natural transformations between the functions. So:

$$
\begin{aligned}
m : \quad & A \to B \quad \text{In } [E^n, \texttt{Set}] \\
m\vec{\epsilon} : \quad & A\vec{\epsilon} \to B\vec{\epsilon} \quad \text{In } \texttt{Set} \\
(f \circ g)\vec{\epsilon} = \quad & (f\vec{\epsilon}) \circ (g\vec{\epsilon}) \\
\texttt{Id}_A(\vec{\epsilon}) = \quad & \texttt{Id}_{A\vec{\epsilon}}
\end{aligned}
$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in $\texttt{Set}$.

0.2.1 Each S-Category is a CCC

Since $\texttt{Set}$ is complete and a CCC, and E^n is small, since E is small, $[E^n, \texttt{Set}]$ is a CCC.

$$
\begin{aligned}
(A \times B)\vec{\epsilon} &= (A\vec{\epsilon}) \times (B\vec{\epsilon}) \\
\texttt{1}\vec{\epsilon} &= \texttt{1} \\
(B^A)\vec{\epsilon} &= (B\vec{\epsilon})^{(A\vec{\epsilon})} \\
\pi_1\vec{\epsilon} &= \pi_1 \\
\pi_2\vec{\epsilon} &= \pi_2 \\
\texttt{app}\vec{\epsilon} &= \texttt{app} \\
\texttt{cur}(f)\vec{\epsilon} &= \texttt{cur}(f\vec{\epsilon}) \\
\langle f, g \rangle\vec{\epsilon} &= \langle f\vec{\epsilon}, g\vec{\epsilon} \rangle
\end{aligned}
$$

0.2.2 The Terminal Co-Product

We can define the co-product point-wise.

$$
\begin{aligned}
(\texttt{1} + \texttt{1})\vec{\epsilon} &= (\texttt{1}\vec{\epsilon} + \texttt{1}\vec{\epsilon}) \\
&= (\texttt{1} + \texttt{1}) \\
\texttt{inl}\vec{\epsilon} &= \texttt{inl} \\
\texttt{inr}\vec{\epsilon} &= \texttt{inr} \\
[f, g]\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}]
\end{aligned}
$$

This preserves the co-product diagram.

$$
\begin{aligned}
([f,g] \circ \mathtt{inl})\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}] \circ \mathtt{inl} \\
&= f\vec{\epsilon} \\
&\Box
\end{aligned}
$$

$$
\begin{aligned}
([f,g] \circ \mathtt{inr})\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}] \circ \mathtt{inr} \\
&= f\vec{\epsilon} \\
&\Box
\end{aligned}
$$

$[f,g]$ is also unique in $[E^n, \mathtt{Set}]$. Suppose $l \circ \mathtt{inl} = f$ and $l \circ \mathtt{inr} = g$ in $[E^n, \mathtt{Set}]$. Then $l\vec{\epsilon} \circ \mathtt{inl} = f\vec{\epsilon}$ and $l\vec{\epsilon} \circ \mathtt{inr} = g\vec{\epsilon}$. Hence by the co-product in Set, $l = [f\vec{\epsilon}, g\vec{\epsilon}]$ so $l = [f,g]$.

0.2.3 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation $[\![\gamma]\!] \in \mathtt{obj\ Set}$. The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$
\begin{aligned}
[\![\gamma]\!] :&\quad E^n \to \mathtt{obj\ Set} \\
\vec{\epsilon} \mapsto&\quad [\![\gamma]\!]
\end{aligned}
$$

Each constant term $\mathtt{C}^A$ in the non-polymorphic calculus has a fixed denotation $[\![\mathtt{C}^A]\!] \in \mathtt{Set}(\mathtt{1}, A)$. So the morphism $[\![\mathtt{C}^A]\!]$ in $[E^n, \mathtt{Set}]$ is the corresponding constant dependently typed morphism returning the $[\![\mathtt{C}^A]\!]$ function in Set.

$$
\begin{aligned}
[\![\mathtt{C}^A]\!] :&\quad [E^n, \mathtt{Set}](\mathtt{1}, A) \\
\vec{\epsilon} \mapsto&\quad [\![\mathtt{C}^A]\!]
\end{aligned}
$$

0.2.4 Graded Monad

Given the strong graded monad $(\mathtt{T}^0, \eta^0, \mu^0, \mathtt{t}^0)$ on Set, we can construct an appropriate graded monad $(\mathtt{T}^n, \eta^n, \mu^n, \mathtt{t}^n)$ on $[E^n, \mathtt{Set}]$. Through some mechanical proof and the naturality of the Set strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in $[E^n, \mathtt{Set}]$.

$$
\begin{aligned}
\mathtt{T}^n :&\quad (E^n, \cdot, \leq_n, \mathtt{1}_n) \to [[E^n, \mathtt{Set}], [E^n, \mathtt{Set}]] \\
(\mathtt{T}^n_f A)\vec{\epsilon} =&\quad \mathtt{T}^0_{(f\vec{\epsilon})} A\vec{\epsilon} \\
(\eta^n_A)\vec{\epsilon} =&\quad \eta^0_{A\vec{\epsilon}} \\
(\mu^n_{f,g,A})\vec{\epsilon} =&\quad \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \\
(\mathtt{t}^n_{f,A,B})\vec{\epsilon} =&\quad \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}
\end{aligned}
$$

Naturality of the Graded Monad Transformations

$$\begin{tikzcd}
A\vec{\epsilon} \arrow[r, "\eta^0_{(A\vec{\epsilon})}"] \arrow[d, "f\vec{\epsilon}"] & \mathtt{T}^0_{\mathtt{1}}(A\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{\mathtt{1}}(f\vec{\epsilon})"] \\
B\vec{\epsilon} \arrow[r, "\eta^0_{(B\vec{\epsilon})}"] & \mathtt{T}^0_{\mathtt{1}}(B\vec{\epsilon})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(A\vec{\epsilon}) \arrow[r, "\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}"] \arrow[d, "\mathtt{T}^0_{f\vec{\epsilon}}\mathtt{T}^0_{g\vec{\epsilon}}m\vec{\epsilon}"] & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}m\vec{\epsilon}"] \\
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(B\vec{\epsilon}) \arrow[r, "\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}"] & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon})
\end{tikzcd}$$

$$\begin{tikzcd}
A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}"] \arrow[d, "(m\vec{\epsilon}\times\mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}B})"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})}(m\vec{\epsilon}\times\mathtt{Id}_{B\vec{\epsilon}})"] \\
A'\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A'\vec{\epsilon}),(B\vec{\epsilon})}"] & \mathtt{T}^0_{f\vec{\epsilon}}(A'\vec{\epsilon}\times B\vec{\epsilon})
\end{tikzcd}
\qquad
\begin{tikzcd}
A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}"] \arrow[d, "(\mathtt{Id}_{A\vec{\epsilon}}\times\mathtt{T}^0_{f\vec{\epsilon}}(m\vec{\epsilon}))"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B\vec{\epsilon}) \arrow[d, "\mathtt{T}^0_{(f\vec{\epsilon})}(\mathtt{Id}_{A\vec{\epsilon}}\times m\vec{\epsilon})"] \\
A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B'\vec{\epsilon}) \arrow[r, "\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B'\vec{\epsilon})}"] & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B'\vec{\epsilon})
\end{tikzcd}$$

Figure 2: Naturality Squares for the graded monad natural transformations. The naturality of each transformation α in $[E^n, \mathtt{Set}]$ derives from the naturality of each of its component transformations $\alpha\vec{\epsilon}$ in $\mathtt{Set}$

Monad Laws

Left Unit

$$\begin{aligned}
(\mu^n_{\mathtt{1},g,A} \circ \eta^n_{\mathtt{T}^n_f A})\vec{\epsilon} &= \mu^0_{\mathtt{1},(f\vec{\epsilon}),(A\vec{\epsilon})} \circ (\eta^0_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}}) \\
&= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} \\
&= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon}
\end{aligned}$$

Right Unit

$$\begin{aligned}
(\mu^n_{f,\mathtt{1},A} \circ \mathtt{T}^n_f \eta^n_A)\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),\mathtt{1},(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\eta^0_{A\vec{\epsilon}}) \\
&= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} \\
&= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon}
\end{aligned}$$

Monad Associativity

$$\begin{aligned}
((\mu^n_{f,(g\cdot h),A}) \circ \mathtt{T}^n_f(\mu^n_{g,h,A}))\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),((g\vec{\epsilon})\cdot(h\vec{\epsilon})),(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}\mu^0_{(h\vec{\epsilon}),(g\vec{\epsilon}),A\vec{\epsilon}} \\
&= \mu^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(h\vec{\epsilon}),(A\vec{\epsilon})} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(\mathtt{T}^0_{h\vec{\epsilon}}(A\vec{\epsilon}))} \\
&= (\mu^n_{f\cdot g,h,A} \circ \mu^n_{f,g,\mathtt{T}^0_h A})\vec{\epsilon}
\end{aligned}$$

Figure 3: The monad laws for the graded monad $(\mathtt{T}^n, \eta^n, \mu^n, \mathtt{t}^n)$ can be proved component-wise from the graded monad $(\mathtt{T}^0)$ on $\mathtt{Set}$.

Tensor Strength Laws

Bind Law

$$\begin{tikzcd}
A \times \mathtt{T}^n_f \mathtt{T}^n_g B \arrow[r, "\mathtt{t}_{f,A,\mathtt{T}^n_g B}"] \arrow[dr, "\mathtt{Id}_A \times \mu^n_{f,g,B}"] & \mathtt{T}^n_f(A \times \mathtt{T}^n_g B) \arrow[r, "\mathtt{T}^n_f \mathtt{t}_{g,A,B}"] & \mathtt{T}^n_f \mathtt{T}^n_g (A \times B) \arrow[d, "\mu^n_{f,g,A\times B}"] \\
& A \times \mathtt{T}^n_{f\cdot g} B \arrow[r, "\mathtt{t}_{f\cdot g,A,B}"] & \mathtt{T}^n_{f\cdot g}(A \times B)
\end{tikzcd}$$

$$\begin{aligned}
&(\mathtt{t}^n_{(f\cdot g),A,B} \circ (\mathtt{Id}_A \times \mu^n_{f,g,B}))\vec{\epsilon} \\
&= (\mathtt{t}^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mu^n_{(f\vec{\epsilon}),(g\vec{\epsilon}),(B\vec{\epsilon})})) \\
&= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\times B)\vec{\epsilon}} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}) \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),\mathtt{T}^0_{g\vec{\epsilon}}(B\vec{\epsilon})} \\
&= (\mu^n_{f,g,(A\times B)} \circ \mathtt{T}^n_f(\mathtt{t}^n_{g,A,B}) \circ \mathtt{t}^n_{f,A,\mathtt{T}^n_g(B)})\vec{\epsilon}
\end{aligned}$$

Commutativity with Unit

$$\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_{\mathtt{1}} B \arrow[d, "\mathtt{t}_{\mathtt{1},A,B}"] \\
& \mathtt{T}^n_{\mathtt{1}}(A \times B)
\end{tikzcd}$$

$$\begin{aligned}
(\mathtt{t}^n_{\mathtt{1},A,B} \circ (\mathtt{Id}_A \times \eta^n_A))\vec{\epsilon} &= \mathtt{t}^0_{\mathtt{1},(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \eta^0_{A\vec{\epsilon}}) \\
&= \eta^0_{A\vec{\epsilon}\times B\vec{\epsilon}} \\
&= (\eta^n_{A\times B})\vec{\epsilon}
\end{aligned}$$

Commutativity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times \mathtt{T}^n_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,\mathtt{T}^n_\epsilon C}"] & & \mathtt{T}^n_\epsilon((A \times B) \times C) \arrow[d, "\mathtt{T}^n_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times \mathtt{T}^n_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times \mathtt{T}^n_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & \mathtt{T}^n_\epsilon(A \times (B \times C))
\end{tikzcd}$$

$$\begin{aligned}
&(\mathtt{T}^n_f \alpha_{A,B,C} \circ \mathtt{t}^n_{f,A\times B,C})\vec{\epsilon} \\
&= \mathtt{T}^0_{f\vec{\epsilon}} \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\times B)\vec{\epsilon},(C\vec{\epsilon})} \\
&= \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon}\times C\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mathtt{t}^0_{(f\vec{\epsilon}),(B\vec{\epsilon}),(C\vec{\epsilon})}) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \\
&= (\mathtt{t}^n_{f,A,(B\times C)} \circ (\mathtt{Id}_A \times \mathtt{t}^n_{f,B,C}) \circ \alpha_{A,B,C})\vec{\epsilon}
\end{aligned}$$

Figure 4: The tensor strength laws can be proved component wise from the strength of the monad $\mathtt{T}^0$

Subeffect Natural Transformation

$$
\begin{aligned}
[\![f \le_n g]\!] : \quad & \mathtt{T}^n_f \to \mathtt{T}^n_g \\
[\![f \le_n g]\!] A\vec{\epsilon} : \quad & \mathtt{T}^n_{f\vec{\epsilon}}(A\vec{\epsilon}) \to \mathtt{T}^n_{g\vec{\epsilon}}(B\vec{\epsilon}) \\
= \quad & [\![f\vec{\epsilon} \le_0 g\vec{\epsilon}]\!] A\vec{\epsilon}
\end{aligned}
$$

Figure 5: Definition of subeffecting natural transformations

0.2.5 Subeffecting

Given a collection of subeffecting natural transformation in $\mathtt{Set}$, $[\![\epsilon_1 \le_0 \epsilon_2]\!] : \quad \mathtt{T}^0_{\epsilon_1} \to \mathtt{T}^0_{\epsilon_2}$ we can form subeffect natural transformations in $[E^n, \mathtt{Set}]$ as seen in figure 5. This natural transformation has all the required properties. These are proved in figure 6.

0.3 Re-indexing Functors

For a function $\theta : E^m \to E^n$, the re-indexing functor θ^* is defined as follows:

$$
\begin{aligned}
\theta^* : \quad & [E^n, \mathtt{Set}] \to [E^m, \mathtt{Set}] \\
\theta^*(A)\vec{\epsilon_m} = \quad & A(\theta(\vec{\epsilon_m})) \\
f : \quad & A \to B \in [E^n, \mathtt{Set}] \\
\theta^*(f)\vec{\epsilon_m} = \quad & f(\theta(\vec{\epsilon_m})) : A(\theta(\vec{\epsilon_m}) \to B(\theta(\vec{\epsilon_m})))
\end{aligned}
$$

This functor preserves all the S-category properties. These can be seen in figures 7-10

0.3.1 Quantification

We need to define $\forall_{E^n} : [E^{n+1}, \mathtt{Set}] \to [E^n, \mathtt{Set}]$

So

$$
\begin{aligned}
(\forall_{E^n} A)\vec{\epsilon_n} = \quad & \Pi_{\epsilon \in E} A(\vec{\epsilon_n}, \epsilon) \\
m : \quad & A \to B \\
(\forall_{E^n} m) : \quad & \forall_{E^n} A \to \forall_{E^n} B \\
(\forall_{E^n} m)\vec{\epsilon_n} = & \Pi_{\epsilon \in E} m(\vec{\epsilon_n}, \epsilon)
\end{aligned}
$$

Subeffect Natural Transformation Properties

Naturality

$$\begin{tikzcd} \mathtt{T}^0_{f\vec{\epsilon}} A\vec{\epsilon} \arrow[r, "{[\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!] A\vec{\epsilon}}"] \arrow[d, "\mathtt{T}^0_{f\vec{\epsilon}} m\vec{\epsilon}"] & \mathtt{T}^0_{g\vec{\epsilon}} A\vec{\epsilon} \arrow[d, "\mathtt{T}^0_{g\vec{\epsilon}} m\vec{\epsilon}"] \\ \mathtt{T}^0_{f\vec{\epsilon}} B\vec{\epsilon} \arrow[r, "{[\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!] B\vec{\epsilon}}"] & \mathtt{T}^0_{g\vec{\epsilon}} B\vec{\epsilon} \end{tikzcd}$$

Commutes With Tensor Strength

$$\begin{tikzcd} A \times \mathtt{T}^n_f B \arrow[r, "{\mathtt{Id}_A \times [\![f \leq_n g]\!]_B}"] \arrow[d, "\mathtt{t}^n_{f,A,B}"] & A \times \mathtt{T}^n_g B \arrow[d, "\mathtt{t}^n_{g,A,B}"] \\ \mathtt{T}^n_f (A \times B) \arrow[r, "{[\![f \leq_n g]\!]_{A \times B}}"] & \mathtt{T}^n_g (A \times B) \end{tikzcd}$$

$$\begin{aligned} &(\mathtt{t}^n_{g,A,B} \circ (\mathtt{Id}_A \times [\![f \leq_n g]\!]_B))\vec{\epsilon} \\ &= \mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{B\vec{\epsilon}}) \\ &= [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{(A\times B)\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \\ &= ([\![f \leq_n g]\!]_{(A\times B)} \circ \mathtt{t}^n_{f,A,B})\vec{\epsilon} \end{aligned}$$

Commutes with Join

$$\begin{tikzcd} \mathtt{T}^n_f \mathtt{T}^n_g \arrow[r, "{\mathtt{T}^n_f [\![g \leq_n g']\!]}"] \arrow[d, "{\mu^n_{f,g,}}"] & \mathtt{T}^n_f \mathtt{T}^n_{g'} \arrow[r, "{[\![f \leq_n f']\!]_{M,\mathtt{T}^n_{g'}}}"] & \mathtt{T}^n_{f'} \mathtt{T}^n_{g'} \arrow[d, "{\mu^n_{f',g',}}"] \\ \mathtt{T}^n_{f\cdot g} \arrow[rr, "{[\![f\cdot g \leq_n f'\cdot g']\!]}"] & & \mathtt{T}^n_{f'\cdot g'} \end{tikzcd}$$

$$\begin{aligned} &([\![f \cdot g \leq_n f' \cdot g']\!]_A \circ \mu^n_{f,g,A})\vec{\epsilon} \\ &= [\![(f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \leq_0 (f'\vec{\epsilon}) \cdot (g\vec{\epsilon})]\!]_{A\vec{\epsilon}} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \\ &= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \circ [\![f\vec{\epsilon} \leq_0 f'\vec{\epsilon}]\!]_{\mathtt{T}^0_{g'\vec{\epsilon}}(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}} [\![g\vec{\epsilon} \leq_0 g'\vec{\epsilon}]\!]_{(A\vec{\epsilon})} \\ &= \mu^n_{f,g,A} \circ [\![f \leq_n f']\!]_{\mathtt{T}^n_{g'},A} \circ \mathtt{T}^n_f [\![g \leq_n g']\!]_A \end{aligned}$$

Figure 6: The required properties of the subeffect natural transformations can be proved component-wise from the appropriate of the subeffect natural transformation on `Set`.

θ^* is Cartesian Closed

$$\begin{aligned}(\theta^*(A \times B))\vec{\epsilon} &= (A \times B)(\theta\vec{\epsilon})\\ &= (A(\theta\vec{\epsilon}) \times B(\theta\vec{\epsilon}))\\ &= (\theta^* A \times \theta^* B)\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\pi_1)\vec{\epsilon} &= \pi_1(\theta\vec{\epsilon})\\ &= \pi_1 \quad \text{Constant function}\\ &= \pi_1\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\pi_2)\vec{\epsilon} &= \pi_2(\theta\vec{\epsilon})\\ &= \pi_2 \quad \text{Constant function}\\ &= \pi_2\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\langle f, g\rangle)\vec{\epsilon} &= (\langle f, g\rangle)(\theta\vec{\epsilon})\\ &= \langle f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})\rangle\\ &= \langle \theta^* f, \theta^* g\rangle\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*(A^B))\vec{\epsilon} &= (A^B)(\theta\vec{\epsilon})\\ &= (A(\theta\vec{\epsilon}))^{(B(\theta\vec{\epsilon}))}\\ &= (\theta^* A)^{(\theta^* B)}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{app})\vec{\epsilon} &= \mathtt{app}(\theta\vec{\epsilon})\\ &= \mathtt{app} \quad \text{Constant fn}\\ &= \mathtt{app}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{cur}(f))\vec{\epsilon} &= \mathtt{cur}(f)(\theta\vec{\epsilon})\\ &= \mathtt{cur}(f(\theta\vec{\epsilon}))\\ &= \mathtt{cur}(\theta^* f)\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{1})\vec{\epsilon} &= \mathtt{1}(\theta\vec{\epsilon})\\ &= \mathtt{1}\\ &= \mathtt{1}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^* \langle\rangle_A)\vec{\epsilon} &= \langle\rangle_A (\theta\vec{\epsilon})\\ &= \langle\rangle_{A(\theta\vec{\epsilon})}\\ &= \langle\rangle_{\theta^* A}\,\vec{\epsilon}\end{aligned}$$

Figure 7: Proof of the CCC-preserving property of re-indexing functors.

θ^* Preserves Co-products

$$\begin{aligned}(\theta^*(\mathtt{1}+\mathtt{1}))\vec{\epsilon} &= (\mathtt{1}+\mathtt{1})(\theta\vec{\epsilon})\\ &= (\mathtt{1}+\mathtt{1}) \quad \text{Constant function}\\ &= (\mathtt{1}+\mathtt{1})\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{inl})\vec{\epsilon} &= \mathtt{inl}(\theta\vec{\epsilon})\\ &= \mathtt{inl} \quad \text{Constant Fn}\\ &= \mathtt{inl}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{inr})\vec{\epsilon} &= \mathtt{inr}(\theta\vec{\epsilon})\\ &= \mathtt{inr} \quad \text{Constant Fn}\\ &= \mathtt{inr}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*[f, g])\vec{\epsilon} &= [f, g](\theta\vec{\epsilon})\\ &= [f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})]\\ &= [\theta^* f, \theta^* g]\vec{\epsilon}\end{aligned}$$

Figure 8: Proof that re-indexing functors preserve the $\mathtt{1}+\mathtt{1}$ co-product.

θ^* Preserves the Graded Monad

$$\begin{aligned}(\theta^*\mathtt{T}^n_f A)\vec{\epsilon} &= \mathtt{T}^n_f A(\theta\vec{\epsilon})\\ &= \mathtt{T}^0_{(f(\theta\vec{\epsilon}))}(A(\theta\vec{\epsilon}))\\ &= (\mathtt{T}^m_{(f\circ\theta)}\theta^* A)\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\eta^n_A)\vec{\epsilon} &= \eta^n_A(\theta\vec{\epsilon})\\ &= \eta^0_{A(\theta\vec{\epsilon})}\\ &= \eta^m_{\theta^* A}\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*\mu^n_{f,g,A})\vec{\epsilon} &= \mu^n_{f,g,A}(\theta\vec{\epsilon})\\ &= \mu^0_{f(\theta\vec{\epsilon}),g(\theta\vec{\epsilon}),A(\theta\vec{\epsilon})}\\ &= \mu^m_{f\circ\theta,g\circ\theta,\theta^*(A)}(\vec{\epsilon})\end{aligned}$$

$$\begin{aligned}(\theta^*\mathtt{t}^n_{f,A,B})\vec{\epsilon} &= \mathtt{t}^n_{f,A,B}(\theta\vec{\epsilon})\\ &= \mathtt{t}^0_{(f(\theta\vec{\epsilon})),(A(\theta\vec{\epsilon})),(B(\theta\vec{\epsilon}))}\\ &= \mathtt{t}^m_{f\circ\theta,\theta^* A,\theta^* B}\vec{\epsilon}\end{aligned}$$

Figure 9: Re-indexing functors preserve the graded monad structure

θ^* preserves Ground Subtyping and Subeffecting

$$\begin{aligned}\theta^*(\llbracket A \leq:_\gamma B\rrbracket)\vec{\epsilon} &= \llbracket A \leq:_\gamma B\rrbracket(\theta\vec{\epsilon})\\ &= \llbracket A \leq: B\rrbracket \quad \text{Constant Function}\\ &= \llbracket A \leq: B\rrbracket\vec{\epsilon}\end{aligned}$$

$$\begin{aligned}(\theta^*(\llbracket f \leq_n g\rrbracket A))\vec{\epsilon} &= (\llbracket f \leq_n g\rrbracket A)(\theta\vec{\epsilon})\\ &= (\llbracket f(\theta\vec{\epsilon}) \leq_n g(\theta\vec{\epsilon})\rrbracket(A(\theta\vec{\epsilon})))\\ &= (\llbracket \theta^* f \leq_m \theta^* g\rrbracket(\theta^* A))\vec{\epsilon}\end{aligned}$$

Figure 10: Re-indexing functors preserve the ground subtyping and subeffecting morphisms.