0.1 Introduce Substitutions

0.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{1}$$

0.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{2}$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{4}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6}$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{7}$$
$$x\,[\sigma, x := v] = v \tag{8}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{9}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{10}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{11}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{12}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{13}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{14}$$
$$\tag{15}$$

0.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma : \Gamma$$

by:

- (Nil) $\dfrac{\Gamma'\mathtt{Ok}}{\Gamma' \vdash \diamond : \diamond}$

- (Extend) $\dfrac{\Gamma' \vdash \sigma : \Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Gamma' \vdash v : A}{\Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma : \Gamma$ then: **TODO: Number these**

Property 1: Γ**Ok and** Γ'**Ok** Since Γ'Ok holds by the Nil-axiom. ΓOk holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ **implies** $\Gamma'' \vdash \sigma : \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v : A$ holds if $\Gamma' \vdash v : A$ holds.

Property 3: $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma''))$ **implies** $(\Gamma', x : A) \vdash (\sigma, x := x) : (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma : \Gamma$$

In addition, $\Gamma', x : A \vdash x : A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) : (\Gamma, x : A) \tag{16}$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g : \tau \wedge \Gamma' \vdash \sigma : \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{17}$$

Assuming $\Gamma' \vdash \sigma : \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t : \tau \rightarrow \Gamma' \vdash t : \tau$

0.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x : A \tag{18}$$

So by inversion, since $\Gamma' \vdash \sigma : \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v : A \tag{19}$$

By the defintion of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Gamma' \vdash x\,[\sigma] : A \tag{20}$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken}) \frac{()\frac{\Delta}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x : A} \tag{21}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma' : \Gamma'' \tag{22}$$

So by induction,

$$\Gamma' \vdash x\,[\sigma'] : A \tag{23}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Gamma' \vdash x\,[\sigma] : A \tag{24}$$

0.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn})\dfrac{()\dfrac{\Delta}{\Gamma,x:A\vdash C:\overline{\mathrm{M}}_\epsilon B}}{\Gamma\vdash \lambda x:A.C:A\to \mathrm{M}_\epsilon B} \tag{25}$$

Using alpha equivalence, we pick $x\notin(\mathrm{dom}(\Gamma)\cup\mathrm{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma',x:A)\vdash(\sigma,x:=x):\Gamma,x:A \tag{26}$$

So by induction using $\sigma,x:=x$, we have Δ' such that:

$$(\text{Fn})\dfrac{()\dfrac{\Delta'}{\Gamma',x:A\vdash C[\sigma,x:=v]:\overline{\mathrm{M}}_\epsilon B}}{\Gamma\vdash \lambda x:A.C\,[\sigma,x:=x]:A\to \mathrm{M}_\epsilon B} \tag{27}$$

Since $\lambda x:A.(C\,[\sigma,x:=x])=\lambda x:A.(C\,[\sigma])=(\lambda x:A.C)\,[\sigma]$, we have a typing derivation for $\Gamma'\vdash(\lambda x:A.C)\,[\sigma]:A\to \mathrm{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $(),\texttt{true},\texttt{false},\mathtt{C}^A$:
$\Gamma\vdash\sigma:\Gamma\Rightarrow\Gamma'\texttt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma]=\mathtt{C}^A \tag{28}$$

So

$$(\text{Const})\dfrac{\Gamma'\texttt{Ok}}{\Gamma'\vdash \mathtt{C}^A:A} \tag{29}$$

0.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return})\dfrac{()\dfrac{\Delta_1}{\Gamma\vdash v:A}}{\Gamma\vdash \texttt{return}\,v:\mathrm{M}_1 A} \tag{30}$$

By induction, we have Δ_1' such that

$$(\text{Return})\dfrac{()\dfrac{\Delta_1'}{\Gamma'\vdash v[\sigma]:A}}{\Gamma'\vdash \texttt{return}(v\,[\sigma]):\mathrm{M}_1 A} \tag{31}$$

Since $(\texttt{return}\,v)\,[\sigma]=\texttt{return}(v\,[\sigma])$, the type derivation above holds for $\Gamma'\vdash(\texttt{return}\,v)\,[\sigma]:\mathrm{M}_1 A$.

Case Apply By inversion, we have Δ_1,Δ_2 such that:

$$(\text{Apply})\dfrac{()\dfrac{\Delta_1}{\Gamma\vdash v_1:A\to\overline{\mathrm{M}}_\epsilon B}\quad ()\dfrac{\Delta_2}{\Gamma\vdash v_2:A}}{\Gamma\vdash v_1\ v_2:\mathrm{M}_\epsilon B} \tag{32}$$

By induction on Δ_1,Δ_2, we have Δ_1',Δ_2' such that

$$(\text{Apply})\dfrac{()\dfrac{\Delta_1'}{\Gamma'\vdash v_1[\sigma]:A\to\overline{\mathrm{M}}_\epsilon B}\quad ()\dfrac{\Delta_2'}{\Gamma'\vdash v_2[\sigma]:A}}{\Gamma'\vdash (v_1\,[\sigma])\ (v_2\,[\sigma]):\mathrm{M}_\epsilon B} \tag{33}$$

Since $(v_1\ v_2)\,[\sigma]=(v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Gamma'\vdash(v_1\ v_2)\,[\sigma]:\mathrm{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)}\dfrac{()\frac{\Delta_1}{\Gamma \vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma \vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma \vdash C_2:\texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A} \; v \; \texttt{then} \; C_1 \; \texttt{else} \; C_2:\texttt{M}_\epsilon A} \tag{34}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$\text{(If)}\dfrac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma]:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Gamma' \vdash C_1[\sigma]:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3'}{\Gamma' \vdash C_2[\sigma]:\texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A} \; (v\,[\sigma]) \; \texttt{then} \; (C_1\,[\sigma]) \; \texttt{else} \; (C_2\,[\sigma]):\texttt{M}_\epsilon A} \tag{35}$$

Since $(\texttt{if}_{\epsilon,A} \; v \; \texttt{then} \; C_1 \; \texttt{else} \; C_2)\,[\sigma] = \texttt{if}_{\epsilon,A} \; (v\,[\sigma]) \; \texttt{then} \; (C_1\,[\sigma]) \; \texttt{else} \; (C_2\,[\sigma])$ The derivation above holds for $\Gamma' \vdash (\texttt{if}_{\epsilon,A} \; v \; \texttt{then} \; C_1 \; \texttt{else} \; C_2)\,[\sigma]:\texttt{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)}\dfrac{()\frac{\Delta_1}{\Gamma \vdash C_1:\texttt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma,x:A \vdash C_2:\texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do} \; x \leftarrow C_1 \; \texttt{in} \; C_2:\texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{36}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x : A) \vdash (\sigma, x := x):(\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\dfrac{()\frac{\Delta_1'}{\Gamma' \vdash C_1[\sigma]:\texttt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma',x:A \vdash C_2[\sigma,x:=x]:\texttt{M}_{\epsilon_2} B}}{\Gamma' \vdash \texttt{do} \; x \leftarrow (C_1\,[\sigma]) \; \texttt{in} \; (C_2\,[\sigma, x := x]):\texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{37}$$

Since $(\texttt{do} \; x \leftarrow C_1 \; \texttt{in} \; C_2)\,[\sigma] = \texttt{do} \; x \leftarrow (C_1\,[\sigma]) \; \texttt{in} \; (C_2\,[\sigma]) = \texttt{do} \; x \leftarrow (C_1\,[\sigma]) \; \texttt{in} \; (C_2\,[\sigma, x := x])$, the above derivation holds for $\Gamma' \vdash (\texttt{do} \; x \leftarrow C_1 \; \texttt{in} \; C_2)\,[\sigma]:\texttt{M}_{\epsilon_1 \cdot \epsilon_2} B$

0.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$\text{(sub-type)}\dfrac{()\frac{\Delta}{\Gamma \vdash v:A} \quad A \leq: B}{\Gamma \vdash v:B} \tag{38}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\dfrac{()\frac{\Delta'}{\Gamma' \vdash v[\sigma]:A} \quad A \leq: B}{\Gamma \vdash v\,[\sigma] : B} \tag{39}$$

Case Sub-effect By inversion, there exists Δ such that

$$\text{(sub-effect)}\dfrac{()\frac{\Delta}{\Gamma \vdash C:\texttt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma \vdash C:\texttt{M}_{\epsilon_2} B} \tag{40}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-effect)}\dfrac{()\frac{\Delta'}{\Gamma' \vdash C[\sigma]:\texttt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma' \vdash C\,[\sigma] : \texttt{M}_{\epsilon_2} B} \tag{41}$$

0.3   Semantics of Substitution

0.3.1   Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$\llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M : \Gamma' \to \Gamma \tag{42}$$

- (Nil) $\dfrac{\Gamma' \texttt{Ok}}{\llbracket \Gamma' \vdash \diamond{:}\diamond \rrbracket_M = \langle \rangle_{\Gamma'}}$

- (Extend) $\dfrac{f = \llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \quad g = \llbracket \Gamma' \vdash v{:}A \rrbracket_M}{\llbracket \Gamma' \vdash (\sigma, x := v){:}(\Gamma, x{:}A) \rrbracket_M = \langle f, g \rangle : \Gamma' \to (\Gamma \times A)}$

0.3.2   Extension Lemma

If $\Gamma' \vdash \sigma{:}\Gamma$ and $x \notin (\texttt{dom}(\Gamma') \cup \texttt{dom}(\Gamma))$ then the substitution in property 3 has denotation:

$$\llbracket (\Gamma', x : A) \vdash (\sigma, x := x){:}(\Gamma, x : A) \rrbracket_M = (\llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \times \texttt{Id}_A) \tag{43}$$

This holds since

$$\llbracket \Gamma', x : A \vdash x{:}A \rrbracket_M = \pi_2 \tag{44}$$

And $\iota\pi : (\Gamma', x : A) \rhd \Gamma'$

$$\llbracket \iota\pi : (\Gamma', x : A) \rhd \Gamma' \rrbracket_M = \pi_1 \tag{45}$$

So for each denotation $\llbracket \Gamma' \vdash v{:}B \rrbracket_M$ of each $y := v$ in σ, we can prepend the denotation with the weakening denotation to yield:

$$\llbracket \Gamma', x : A \vdash v{:}B \rrbracket_M = \llbracket \Gamma' \vdash v{:}B \rrbracket_M \circ \pi_1 \tag{46}$$

Since π_1 appears in every branch of $\llbracket \Gamma', x : A \vdash \sigma{:}\Gamma \rrbracket_M$, it can be factored out to yield:

$$\llbracket \Gamma', x : A \vdash \sigma{:}\Gamma \rrbracket_M = \llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \circ \pi_1 \tag{47}$$

Hence,

$$\llbracket (\Gamma', x : A) \vdash (\sigma, x := x){:}\Gamma, x : A \rrbracket_M = \langle \llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \circ \pi_1, \pi_2 \rangle = (\llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \times \texttt{Id}_A) \tag{48}$$

0.3.3   Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma \vdash t{:}\tau$ and $\Gamma' \vdash \sigma{:}\Gamma$ then the derivation Δ' deriving $\Gamma' \vdash t\,[\sigma]{:}\tau$ satisfies:

$$\Delta' = \Delta \circ \llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M \tag{49}$$

This is proved by induction over the derivation of $\Gamma \vdash t{:}\tau$. We shall use σ to denote $\llbracket \Gamma' \vdash \sigma{:}\Gamma \rrbracket_M$ where it is clear from the context.

0.3.4   Proof For Value Terms

Case Var   By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var}) \dfrac{\Gamma \texttt{Ok}}{\Gamma'', x : A \vdash x{:}A} \tag{50}$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v{:}A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{51}$$

$$\Delta = [\![\Gamma'', x : A \vdash x \colon A]\!]_M = \pi_2 \tag{52}$$

$$\tag{53}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{54}$$

$$= \Delta' \quad \text{By product property} \tag{55}$$

Case Weaken   By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken}) \frac{() \frac{\Delta_1}{\Gamma'' \vdash x \colon A}}{\Gamma'', y : B \vdash x \colon A} \tag{56}$$

Also by inversion of the well-formedness of $\Gamma' \vdash \sigma \colon \Gamma$, we have $\Gamma' \vdash \sigma' \colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M = \langle [\![\Gamma' \vdash \sigma \colon \Gamma'']\!]_M, [\![\Gamma' \vdash v \colon B]\!]_M \rangle \tag{57}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$() \frac{\Delta_1'}{\Gamma' \vdash x\,[\sigma] \colon A} \tag{58}$$

Hence

$$\Delta' = \Delta_1' \quad \text{By definition} \tag{59}$$

$$= \Delta_1 \circ \sigma' \quad \text{By induction} \tag{60}$$

$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v \colon B]\!]_M \rangle \quad \text{By product property} \tag{61}$$

$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.} \tag{62}$$

Case Constants   The logic for all constant terms $(\texttt{true}, \texttt{false}, ()\texttt{C}^A)$ is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{63}$$

$$\Delta' = c \circ \langle \rangle_{\Gamma'} \quad \text{By Definition} \tag{64}$$

$$= c \circ \langle \rangle_G \circ \sigma \quad \text{Terminal property} \tag{65}$$

$$= \Delta \circ \sigma \quad \text{By definition} \tag{66}$$

Case Lambda   By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn}) \frac{() \frac{\Delta_1}{\Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C \colon A \to \texttt{M}_\epsilon B} \tag{67}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn}) \frac{() \frac{\Delta_1'}{\Gamma', x : A \vdash (C[\sigma]) \colon \texttt{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A.C)\,[\sigma] \colon A \to \texttt{M}_\epsilon B} \tag{68}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{69}$$

6

Hence:

$$\Delta' = \mathtt{cur}(\Delta'_1) \quad \text{By definition} \tag{70}$$
$$= \mathtt{cur}(\Delta_1 \circ (\sigma \times \mathtt{Id}_A)) \quad \text{By induction and extension lemma.} \tag{71}$$
$$= \mathtt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} \tag{72}$$
$$= \Delta \circ \sigma \quad \text{By Definition} \tag{73}$$
$$\tag{74}$$

Case Sub-type By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Gamma \vdash v:A} \quad A \leq: B}{\Gamma \vdash v:B} \tag{75}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v[\sigma]:A} \quad A \leq: B}{\Gamma' \vdash v\,[\sigma]:B} \tag{76}$$

Hence,

$$\Delta' = [\![A \leq: B]\!]_M \circ \Delta'_1 \quad \text{By definition} \tag{77}$$
$$= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \sigma \quad \text{By induction} \tag{78}$$
$$= \Delta \circ \sigma \quad \text{By definition} \tag{79}$$
$$\tag{80}$$

0.3.5 Proof For Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v:A}}{\Gamma \vdash \mathtt{return}v:\mathtt{M_1}A} \tag{81}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v[\sigma]:A}}{\Gamma' \vdash (\mathtt{return}v)\,[\sigma]:\mathtt{M_1}A} \tag{82}$$

Hence,

$$\Delta' = \eta_A \circ \Delta'_1 \quad \text{By Definition} \tag{83}$$
$$= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} \tag{84}$$
$$= \Delta \circ \sigma \quad \text{By Definition} \tag{85}$$
$$\tag{86}$$

Case Apply By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma \vdash v_1:A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma \vdash v_2:A}}{\Gamma \vdash v_1\ v_2:\mathtt{M}_\epsilon B} \tag{87}$$

7

By induction we find Δ_1', Δ_2' such that

$$\Delta_1' = \Delta_1 \circ \sigma \tag{88}$$
$$\Delta_2' = \Delta_2 \circ \sigma \tag{89}$$
$$\tag{90}$$

And

$$\Delta' = (\text{Apply}) \frac{() \frac{\Delta_1'}{\Gamma' \vdash v_1[\sigma]:A \to \mathtt{M}_\epsilon B} \quad () \frac{\Delta_2'}{\Gamma' \vdash v_2[\sigma]:A}}{\Gamma' \vdash (v_1\ v_2)\,[\sigma]:\mathtt{M}_\epsilon B} \tag{91}$$

Hence

$$\Delta' = \mathtt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Definition} \tag{92}$$
$$= \mathtt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} \tag{93}$$
$$= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} \tag{94}$$
$$= \Delta \circ \sigma \quad \text{By Definition} \tag{95}$$
$$\tag{96}$$

Case If By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If}) \frac{() \frac{\Delta_1}{\Gamma \vdash v:\mathtt{Bool}} \quad () \frac{\Delta_2}{\Gamma \vdash C_1:\mathtt{M}_\epsilon A} \quad () \frac{\Delta_3}{\Gamma \vdash C_2:\mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A} \tag{97}$$

By induction we find $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta_1' = \Delta_1 \circ \sigma \tag{98}$$
$$\Delta_2' = \Delta_2 \circ \sigma \tag{99}$$
$$\Delta_3' = \Delta_3 \circ \sigma \tag{100}$$
$$\tag{101}$$

And

$$\Delta' = (\text{If}) \frac{() \frac{\Delta_1'}{\Gamma' \vdash v[\sigma]:\mathtt{Bool}} \quad () \frac{\Delta_2'}{\Gamma' \vdash C_1[\sigma]:\mathtt{M}_\epsilon A} \quad () \frac{\Delta_3'}{\Gamma' \vdash C_2[\sigma]:\mathtt{M}_\epsilon A}}{\Gamma' \vdash (\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma]:\mathtt{M}_\epsilon A} \tag{102}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\sigma = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{103}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \mathtt{cur}(f) \tag{104}$$

And so:

---

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

$$\Delta' = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} \tag{105}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} \tag{106}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2 \circ (\mathtt{Id_1} \times \sigma)), \mathtt{cur}(\Delta_3 \circ \pi_2 \circ (\mathtt{Id_1} \times \sigma))] \circ \Delta_1 \circ \sigma) \times \mathtt{Id}_{\Gamma'} \quad \text{By product property} \tag{107}$$

$$= \mathtt{app} \circ (([(T_\epsilon A)^\sigma \circ \mathtt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} \tag{108}$$

$$= \mathtt{app} \circ (((T_\epsilon A)^\sigma \circ [\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} \tag{109}$$

$$= \mathtt{app} \circ ((T_\epsilon A)^\sigma \times \mathtt{Id}_{\Gamma'}) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\sigma \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} \tag{110}$$

$$= \mathtt{app} \circ (\mathtt{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\sigma \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \mathtt{app}, (T_\epsilon A)^\sigma \tag{111}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} \tag{112}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \circ \sigma \quad \text{By Definition of the diagonal morphism.} \tag{113}$$

$$= \Delta \circ \sigma \tag{114}$$

Case Bind By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\mathrm{Bind}) \dfrac{() \frac{\Delta_1}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \quad () \frac{\Delta_2}{\Gamma, x : A \vdash C_1 : \mathtt{M}_\epsilon B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{115}$$

By property 3,

$$(\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A) \tag{116}$$

With denotation (extension lemma)

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A)]\!]_M = \sigma \times \mathtt{Id}_A \tag{117}$$

By induction, we derive Δ_1', Δ_2' such that:

$$\Delta_1' = \Delta_1 \circ \sigma \tag{118}$$

$$\Delta_2' = \Delta_2 \circ (\sigma \times \mathtt{Id}_A) \quad \text{By Extension Lemma} \tag{119}$$

And:

$$\Delta' = (\mathrm{Bind}) \dfrac{() \frac{\Delta_1'}{\Gamma' \vdash C_1[\sigma] : \mathtt{M}_\epsilon A} \quad () \frac{\Delta_2'}{\Gamma', x : A \vdash C_1[\sigma] : \mathtt{M}_\epsilon B}}{\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\,[\sigma] : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{120}$$

Hence:

$$\Delta' = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta'_2 \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} \tag{121}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} \tag{122}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} \tag{123}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} \tag{124}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} \tag{125}$$

$$= \Delta \circ \sigma \quad \text{By Defintion} \tag{126}$$

$$\tag{127}$$

Case Subeffect By inversion, there exists derivation Δ_1 such that:

$$\Delta = \text{(Sub-effect)} \frac{() \dfrac{\Delta_1}{\Gamma \vdash C : \mathtt{M}_{\epsilon_1}A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \mathtt{M}_{\epsilon_2}B} \tag{128}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = \text{(Sub-effect)} \frac{() \dfrac{\Delta'_1}{\Gamma' \vdash C[\sigma] : \mathtt{M}_{\epsilon_1}A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C\,[\sigma]\,:\mathtt{M}_{\epsilon_2}B} \tag{129}$$

Hence, Let

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{130}$$

$$g = [\![A \leq: B]\!]_M \tag{131}$$

$$\Delta' = h_B \circ T_{\epsilon_1}g \circ \Delta'_1 \quad \text{By definition} \tag{132}$$

$$= h_B \circ T_{\epsilon_1}g \circ \Delta_1 \circ \sigma \quad \text{By induction} \tag{133}$$

$$= \Delta \circ \sigma \quad \text{By definition} \tag{134}$$

$$\tag{135}$$

0.4 The Identity Substitution

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$

- $I_{(\Gamma,x:A} = (I_\Gamma, x := x)$

0.4.1 Properties of the Identity Substitution

Property 1 If $\Gamma \mathtt{Ok}$ then $\Gamma \vdash I_\Gamma : \Gamma$, proved trivially by induction over the well formedness relation.

Property 2 $[\![\Gamma \vdash I_\Gamma : \Gamma]\!]_M = \mathtt{Id}_\Gamma$, proved trivially by induction over the definition of I_Γ

0.5 Single Substitution

If $\Gamma \vdash v \colon A$, let the single substitution $\Gamma \vdash [v/x] \colon \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{136}$$

Then by properties 1, 2 of the identity substitution, we have:

$$[\![\Gamma \vdash [v/x] \colon \Gamma, x : A]\!]_M = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle : \Gamma \to (\Gamma \times A) \tag{137}$$

0.5.1 The Semantics of Single Substitution

The following diagram commutes:

$$[\![\Gamma \vdash t\,[v/x] \colon \tau]\!]_M = [\![\Gamma, x : A \vdash t \colon \tau]\!]_M \circ \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle \tag{138}$$

TODO: Again, there is code here to draw a Commutative diagram, but for some reason pdflatex hangs when compiling it Since $[\![\Gamma \vdash (I_\Gamma, x := v) \colon (\Gamma, x : A)]\!]_M = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle$ And $\mathtt{true}\,[v/x] = \mathtt{true}\,[I_\Gamma, x := v]$