0.1 Effect Weakening Definition

Introduce a relation $\omega : \Phi' \triangleright \Phi$ relating effect-environments.

0.1.1 Relation

- (Id) $\frac{\Phi \texttt{Ok}}{\iota : \Phi \triangleright \Phi}$
- (Project) $\frac{\omega : \Phi' \triangleright \Phi}{\omega\pi : (\Phi', \alpha) \triangleright \Phi}$
- (Extend) $\frac{\omega : \Phi' \triangleright \Phi}{\omega\times : (\Phi', \alpha) \triangleright (\Phi, \alpha)}$

0.1.2 Weakening Properties

0.1.3 Effect Weakening Preserves Ok

$$\omega : \Phi' \triangleright \Phi \wedge \Phi \texttt{Ok} \Leftarrow \Phi' \texttt{Ok} \tag{1}$$

Proof

Case: ι

$$\Phi \texttt{Ok} \wedge \iota : \Phi \triangleright \Phi \Leftarrow \Phi \texttt{Ok}$$

Case: $\omega\pi$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{2}$$

So, by induction, $\Phi' \texttt{Ok}$ and hence $(\Phi', \alpha)\texttt{Ok}$

Case: $\omega\times$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{3}$$

So

$$\begin{aligned} (\Phi, \alpha)\texttt{Ok} &\Rightarrow \Phi \texttt{Ok} && (4)\\ &\Rightarrow \Phi' \texttt{Ok} && (5)\\ &\Rightarrow (\Phi', \alpha)\texttt{Ok} && (6)\\ & && (7) \end{aligned}$$

0.1.4 Domain Lemma

$$\omega : \Phi' \triangleright \Phi \Rightarrow (\alpha \notin \Phi \Rightarrow \alpha \notin \Phi')$$

Proof By trivial Induction.

0.1.5 Weakening Preserves Effect Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ then $\Phi \vdash \epsilon \implies \Phi' \vdash \epsilon$

Proof By induction over the well-formed-ness of effects

Case Ground By inversion, $\Phi \texttt{Ok} \wedge \epsilon \in E$. Hence by the ok-property, $\Phi' \texttt{Ok}$ So $\Phi' \vdash \epsilon$

Case Var $\Phi = \Phi'', \alpha$
So either:

Case: $\Phi' = \Phi''', \alpha$ So $\omega = \omega' \times$ So $\omega' : \Phi''' \triangleright \Phi''$, and hence:

$$(\text{Var})\frac{\Phi''', \alpha \texttt{Ok}}{\Phi''', \alpha \vdash \alpha} \tag{8}$$

Case: $\Phi' = \Phi''', \beta$ and $\beta \neq \alpha$
So $\omega = \omega' \pi$
By induction, $\omega' : \Phi''' \triangleright \Phi$ so

$$(\text{Weaken})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{9}$$

Case Weaken By inversion, $\Phi = \Phi'', \beta$.
So $\omega = \omega' \times$
And, $\Phi' = \Phi''', \beta$ So By inversion$\omega' : \Phi''' \triangleright \pi_1''$
So by induction

$$(\text{weak})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{10}$$

Case Monoid By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$. So by induction, $\Phi' \vdash \epsilon_1$ and $\Phi' \vdash \epsilon_2$, and so:

$$\Phi' \vdash \epsilon_1 \cdot \epsilon_2 \tag{11}$$

0.1.6 Weakening Preserves Type-Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ and $\Phi \vdash A$ then $\Phi' \vdash A$.

Proof:

Case Ground: By inversion, $\Phi\texttt{Ok}$, hence by property 1 of weakening, $\Phi'\texttt{Ok}$. Hence $\Phi' \vdash \gamma$.

Case Function: By inversion, $\Phi \vdash A$, $\Phi \vdash B$. So by induction $\Phi' \vdash A$, $\Phi' \vdash B$, hence,

$$\Phi' \vdash A \to B$$

Case Computation: By inversion$\Phi \vdash A$, and $\Phi \vdash \epsilon$.
So by induction and the effect-well-formed-ness theorem,
$\Phi' \vdash A$ and $\Phi' \vdash \epsilon$
So

$$\Phi' \vdash \texttt{M}_\epsilon A$$

Case For All: By inversion, $\Phi, \alpha \vdash A$ Picking $\alpha \notin \Phi'$ using α-conversion.
So $\omega \times : (\Phi', \alpha) \triangleright (\Phi, \alpha)$
So $(\Phi', \alpha) \vdash A$
So $\Phi \vdash \forall \alpha.A$

0.1.7 Corollary

$$\omega : \Phi' \triangleright \Phi \wedge \Phi \vdash \Gamma\texttt{Ok} \implies \Phi' \vdash \Gamma\texttt{Ok}$$

Case Nil: By inversion$\Phi \mathtt{Ok}$ so $\Phi \vdash \diamond \mathtt{Ok}$

Case Var: By inversion$\Phi \vdash \Gamma \mathtt{Ok}$, $x \in \mathtt{dom}(\Gamma)$, $\Phi \vdash A$
So by induction $\Phi' \vdash \Gamma \mathtt{Ok}$, and $\pi_1' \vdash \Gamma \mathtt{Ok}$
So $\Phi' \vdash (\Gamma, x : A)\mathtt{Ok}$

0.1.8 Effect Weakening preserves Type Relations

$$\Phi \mid \Gamma \vdash v\colon A \wedge \omega : \Phi' \triangleright \Phi \implies \Phi' \mid \Gamma \vdash v\colon A \tag{12}$$

Proof:

Case Constants: If $\Phi \vdash \Gamma \mathtt{Ok}$ then $\Phi' \vdash \Gamma \mathtt{Ok}$ so:

$$(\text{Const})\frac{\Phi' \vdash \Gamma \mathtt{Ok}}{\Phi' \mid \Gamma \vdash \mathtt{C}^A\colon A} \tag{13}$$

Case Variables: If $\Phi \vdash \Gamma \mathtt{Ok}$ then $\Phi' \vdash \Gamma \mathtt{Ok}$ so: So, $\Phi' \mid G \vdash x\colon A$, if $\Phi \mid G \vdash x\colon A$

Case Lambda: By inversion, $\Phi \mid \Gamma, x : A \vdash v\colon B$, so by induction $\Phi' \mid \Gamma, x : A \vdash v\colon B$.
So,

$$\Phi' \mid \Gamma \vdash \lambda x : A.v\colon A \to B \tag{14}$$

Case Apply: By inversion$\Phi \mid \Gamma \vdash v_1\colon A \to B$ and $\Phi \mid \Gamma \vdash v_2\colon A$.
Hence by induction, $\Phi' \mid \Gamma \vdash v_1\colon A \to B$ and $\Phi' \mid \Gamma \vdash v_2\colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{app} v_1 v_2\colon B$$

Case Return: By inversion$\Phi \mid \Gamma \vdash v\colon A$
So by induction $\Phi' \mid \Gamma \vdash v\colon A$
Hence $\Phi' \mid \Gamma \vdash \mathtt{return} v\colon \mathtt{M}_{\mathbf{1}} A$

Case Bind: By inversion$\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash \epsilon_2\colon \mathtt{M}_{\epsilon_2} A$.
Hence by induction $\Phi' \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A$ and $\Phi' \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B \tag{15}$$

Case If: By inversion$\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1\colon A$, and $\Phi \mid \Gamma \vdash v_2\colon A$.
Hence by induction $\Phi' \mid \Gamma \vdash v\colon \mathtt{Bool}$, $\Phi' \mid \Gamma \vdash v_1\colon A$, and $\Phi' \mid \Gamma \vdash v_2\colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{if}_{A,}\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A \tag{16}$$

Case Subtype: By inversion$\Phi \mid \Gamma \vdash v\colon A$, and $A \leq: B$.
So by induction: $\Phi' \mid \Gamma \vdash v\colon A$, and $A \leq: B$.
So

$$\Phi' \mid \Gamma \vdash v\colon B \tag{17}$$

Case Effect-Lambda: By inversion$\Phi, \alpha \mid \Gamma \vdash v \colon A$

By picking $\alpha \notin \Phi'$ using α-conversion.

$$\omega\times : \Phi', \alpha \rhd \Phi, \alpha \tag{18}$$

So by induction, $\Phi', \alpha \mid \Gamma \vdash v \colon A$

Hence,

$$\Phi' \mid \Gamma \vdash \Lambda\alpha.v \colon \forall a.A \tag{19}$$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v \colon \forall\alpha.A$, and $\Phi \vdash \epsilon$.

So by induction, $\Phi' \mid \Gamma \vdash v \colon \forall\alpha.A$

And by the well-formed-ness-theorem $\Phi' \vdash \epsilon$

Hence,

$$\Phi' \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha] \tag{20}$$

0.2 Type Environment Weakening

0.2.1 Relation

We define the ternary weakening relation $\Phi \vdash w : \Gamma' \rhd \Gamma$ using the following rules.

- (Id)$\frac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \vdash \iota : \Gamma \rhd \Gamma}$
- (Project)$\frac{\Phi \vdash \omega : \Gamma' \rhd \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \rhd \Gamma}$
- (Extend)$\frac{\Phi \vdash \omega : \Gamma' \rhd \Gamma \quad x \notin \mathtt{dom}(\Gamma') \quad A \leq : B}{\Phi \vdash w\times : \Gamma', x : A \rhd \Gamma, x : B}$

0.2.2 Domain Lemma

If $\Phi \vdash \omega : \Gamma' \rhd \Gamma$, then $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$.

Proof:

Case Id: Then $\Gamma' = \Gamma$ and so $\mathtt{dom}(\Gamma') = \mathtt{dom}(\Gamma)$.

Case Project: By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma') \subseteq \mathtt{dom}(\Gamma' \cup \{x\})$

Case Extend: By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$ so

$$\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\} \subseteq \mathtt{dom}(\Gamma') \cup \{x\} = \mathtt{dom}(\Gamma', x : A)$$

0.2.3 Theorem 1

If $\Phi \vdash \omega : \Gamma' \rhd \Gamma$ and $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi \vdash \Gamma'\mathtt{Ok}$

Proof:

Case Id:

$$\text{(Id)}\frac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \vdash \iota : \Gamma \rhd \Gamma}$$

By inversion, $\Phi \vdash \Gamma\mathtt{Ok}$.

Case Project:

$$(\text{Project})\frac{\Phi \vdash \omega : \Gamma' \rhd \Gamma \quad x \notin \texttt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \rhd \Gamma}$$

By inversion, $\Phi \vdash \omega : \Gamma' \rhd \Gamma$ and $x \notin \texttt{dom}(\Gamma')$.

Hence by induction $\Phi \vdash \Gamma'\texttt{Ok}$, $\Phi \vdash \Gamma\texttt{Ok}$. Since $x \notin \texttt{dom}(\Gamma')$, we have $\Phi \vdash \Gamma', x : A\texttt{Ok}$.

Case Extend: $(\text{Extend})\frac{\Phi\vdash\omega:\Gamma'\rhd\Gamma \quad x\notin\texttt{dom}(\Gamma') \quad A\leq:B}{\Phi\vdash w\times:\Gamma',x:A\rhd\Gamma,x:B}$,

By inversion, we have

$\Phi \vdash \omega : \Gamma' \rhd \Gamma$, $x \notin \texttt{dom}(\Gamma')$.

Hence we have $\Phi \vdash \Gamma\texttt{Ok}$, $\Phi \vdash \Gamma'\texttt{Ok}$, and by the domain Lemma, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$, hence $x \notin \texttt{dom}(\Gamma)$. Hence, we have $\Phi \vdash \Gamma, x : A\texttt{Ok}$ and $\Phi \vdash \Gamma', x : A\texttt{Ok}$

0.2.4 Theorem 2

If $\Phi \mid \Gamma \vdash t\colon\tau$ and $\Phi \vdash \omega : \Gamma' \rhd \Gamma$ then there is a derivation of $\Phi \mid \Gamma' \vdash t\colon\tau$

Proof: We induct over the structure of typing derivations of $\Phi \mid \Gamma \vdash t\colon\tau$, assuming $\Phi \vdash \omega : \Gamma' \rhd \Gamma$ holds.

Case Var and Weaken: We case split on the weakening ω.

Case: $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Phi \mid \Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

Case: $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\Phi \vdash \omega' : \Gamma'' \rhd \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Phi \mid \Gamma'' \vdash x\colon A$, such that:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma'', x' : A' \vdash x\colon A} \tag{21}$$

Case: $\omega = \omega'\times$ Then

$$\Gamma' = \Gamma''', x' : B \tag{22}$$
$$\Gamma = \Gamma'', x' : A' \tag{23}$$
$$B \leq: A \tag{24}$$

Case: $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$(\text{Sub-type})\frac{(\text{var})\frac{}{\Phi|\Gamma''',x:B\vdash x:B} \quad B \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{25}$$

Case: $x \neq x'$ Then

$$\Delta = (\text{Weaken})\frac{()\frac{\Delta_1}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma \vdash x\colon A} \tag{26}$$

By induction with $\Phi \vdash \omega : \Gamma''' \rhd \Gamma''$, we have a derivation Δ_1 of $\Phi \mid \Gamma''' \vdash x\colon A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'''\vdash x:A}}{\Phi \mid \Gamma' \vdash x\colon A} \tag{27}$$

Case Constant: The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$\text{(Const)}\frac{\Gamma\,\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A} \tag{28}$$

By inversion, we have $\Phi \vdash \Gamma\mathtt{Ok}$, so we have $\Phi \vdash \Gamma'\mathtt{Ok}$.
Hence

$$\text{(Const)}\frac{\Phi \vdash \Gamma'\mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A \colon A} \tag{29}$$

Holds.

Case Lambda: By inversion, we have a derivation Δ_1 giving

$$\Delta = \text{(Fn)}\frac{()\frac{\Delta_1}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A \to B} \tag{30}$$

Since $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we have:

$$\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{31}$$

Hence, by induction, using $\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = \text{(Fn)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma',x:A\vdash v:B}}{\Phi \mid \Gamma', x : A \vdash \lambda x : A.v \colon A \to B} \tag{32}$$

Case Sub-typing:

$$\text{(Sub-type)}\frac{\Phi \mid \Gamma \vdash v \colon A \quad A \leq: B}{\Phi \mid \Gamma \vdash v \colon B} \tag{33}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v \colon A} \tag{34}$$

So by induction, we have a derivation Δ_1' such that:

$$\text{(Sub-type)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:a} \quad A \leq: B}{\Phi \mid \Gamma' \vdash v \colon B} \tag{35}$$

Case Return: We have the sub-derivation Δ_1 such that

$$\Delta = \text{(Return)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return}v \colon \mathtt{M}_1 A} \tag{36}$$

Hence, by induction, with $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ_1' such that:

$$\Delta' = \text{(Return)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \mathtt{return}v \colon \mathtt{M}_1 A} \tag{37}$$

Case Apply: By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1 : A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B} \tag{38}$$

By induction, this gives us the respective derivations: Δ_1', Δ_2' such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1 : A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2 : A}}{\Phi \mid \Gamma' \vdash v_1\ v_2 \colon B} \tag{39}$$

Case If: By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v : \texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1 : A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2 : A}}{\Phi \mid \Gamma \vdash \texttt{if}_{A,}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{40}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v : \texttt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1 : A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2 : A}}{\Phi \mid \Gamma' \vdash \texttt{if}_{A,}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{41}$$

Case Bind: By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1 : \texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma, x : A\vdash v_2 : \texttt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{42}$$

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then $\Phi \vdash \omega\times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ_1', Δ_2' such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1 : \texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2'}{\Phi|\Gamma', x : A\vdash v_2 : \texttt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{43}$$

Case Effect-Abstraction: By inversion,we have derivation Δ_1 deriving

$$(\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{44}$$

By α-conversion, we have $\iota\pi : \Phi, \alpha \triangleright \Phi$, So we have $\Phi, \alpha \vdash \omega : \Gamma' \triangleright \Gamma$ so by induction, there exists Δ_1 deriving:

$$\Delta' = (\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma'\vdash v : A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{45}$$

Case Effect-Application: By inversionwe have derivation Δ_1 deriving

$$(\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v : \forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{46}$$

So by induction, we have Δ_1' deriving

$$(\text{Effect-App})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v : \forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{47}$$