Chapter 1

Adequacy of a Model of the Effect Calculus

1.1 Instantiation of the Effect Calculus

Let us instantiate the effect calculus to be a language in which one can write programs which can create an output signal. The effect system of EC is then used to count an upper bound on the number of outputs that a program can make. This language shall be called EC_{put}

1.1.1 Ground Types

We simply use the basic ground types.

$$\gamma ::= \text{Bool} \mid \text{Unit} \tag{1.1}$$

1.1.2 Graded Monad

We grade index the graded monad with a partially ordered monoid derived from the natural numbers.

$$E = (\mathbb{N}, 0, +, \leq) \tag{1.2}$$

This means that the $\text{do } x \leftarrow v \text{ in } v'$ type rule adds together the upper bounds on the two expressions to give an upper bound on the number of outputs of the sequenced expression. The $\text{return } v$ type rule acknowledges that a pure expression does not have any output.

1.1.3 Constants

We extend the set of constant, built in expressions to include a put statement which makes a single output action.

$$\text{C}^A ::= \text{true}^{\text{Bool}} \mid \text{false}^{\text{Bool}} \mid ()^{\text{Unit}} \mid \text{put}^{\text{M}_1 \text{Unit}} \tag{1.3}$$

1.1.4 Subtyping

The ground subtyping relation is the trivial identity relation. This is extended using the subeffect and function subtyping rules given in **TODO: Ref**.

1.2 Instantiation of a Model of the Effect Calculus

Let us now instantiate a model of $\texttt{EC}_{\texttt{put}}$ in $\texttt{Set}$, the category of sets and functions.

1.2.1 Cartesian Closed Category

Is given by the usage of $\texttt{Set}$.

1.2.2 Graded Monad

The strong graded monad is given by tagging values of the underlying type with the number of output operations required to compute that value.

$$T_n A = \{(n', a) \mid n' \leq n \wedge a \in A\} \tag{1.4}$$
$$\mu_{m,n,A} = (m', (n', a)) \mapsto (n' + m', a) \tag{1.5}$$
$$\eta_A = a \mapsto (0, a) \tag{1.6}$$
$$\mathsf{t}_{n,A,B} = (a, (n', b)) \mapsto (n', (a, b)) \tag{1.7}$$

1.2.3 Subeffecting Natural Transformations

These natural transformations are given by inclusion functions (identities), since $(n', a) \in T_n A \implies (n' \leq n \leq m, a \in A) \implies (n', a) \in T_n B$. Other subtyping morphisms are generated using the usual method according to the subtype derivation.

1.2.4 Ground Denotations

We define denotations for ground types as follows:

$$\llbracket \texttt{Unit} \rrbracket = \{*\} \tag{1.8}$$
$$\llbracket \texttt{Bool} \rrbracket = \{\top, \bot\} \tag{1.9}$$

We then define denotations for the constant expressions, including the $\texttt{put}$ operation.

$$\llbracket \texttt{()} \rrbracket = * \mapsto * \tag{1.10}$$
$$\llbracket \texttt{true} \rrbracket = * \mapsto \top \tag{1.11}$$
$$\llbracket \texttt{false} \rrbracket = * \mapsto \bot \tag{1.12}$$
$$\llbracket \texttt{put} \rrbracket = * \mapsto (1, *) \tag{1.13}$$
$$\tag{1.14}$$

1.2.5 Soundness

This category is now an S-category and hence a sound model for $\mathtt{EC_{put}}$.

1.2.6 Denotational Shorthand

In the remaining sections I shall use $[\![\, \vdash v\colon A \,]\!]$ to indicate $[\![\, \diamond \vdash v\colon A \,]\!](*)$.

1.3 Programming With Put

This simple language now has some extra properties which the general EC does not have.

Definition 1.3.1 (Powers of Put as an Equational Equivalence Class). *Define put^n as follows:*

$$\vdash put^0 \approx return\ ()\colon M_0\,Unit \tag{1.15}$$

$$\vdash put^{n+1} \approx do\ \_\leftarrow put^n\ in\ put\colon M_{n+1}\,Unit \tag{1.16}$$

Lemma 1.3.1 (Denotations of Powers of Put). *Powers of put have a simple denotation.* $[\![\, \vdash put^n\colon M_n\,Unit \,]\!] = (n, *)$

Proof: By induction on n.

Case 0:

$$[\![\, \vdash \mathtt{put}^0\colon \mathtt{M_0Unit} \,]\!] = \eta(*) = (0, *) \tag{1.17}$$

Case n+1:

$$[\![\, \vdash \mathtt{put}^{n+1}\colon \mathtt{M_0Unit} \,]\!] = (\mu \circ T_n([\![\, \diamond \vdash \mathtt{put}\colon \mathtt{M_1Unit} \,]\!] \circ \pi_1) \circ \mathtt{t})(*, [\![\, \vdash \mathtt{put}^n\colon \mathtt{M_nUnit} \,]\!]) \tag{1.18}$$

$$= (n + 1, *) \tag{1.19}$$

Theorem 1.3.2 (Put Reordering Theorem). *For all $\vdash v\colon M_n A$, there exist $\vdash v'\colon A$, $n' \leq n$ such that $\vdash v \approx do\ \_\leftarrow put^{n'}\ in\ return\ v'\ \colon M_n A$.*

Proof: By induction over typing rules capable of producing types of the form $\mathtt{M}_n A$.

Case Return:

$$\vdash \mathtt{return}\ v\ \approx \mathtt{do}\ \_\leftarrow \mathtt{return}\ ()\ \mathtt{in\ return}\ v\ \colon \mathtt{M_0} A$$

$v' = v$, $n' = 0$.

Case Subtype:

$$(\text{Subtype})\ \dfrac{\vdash v\colon \mathtt{M}_n A \qquad (\text{Effect})\dfrac{n \leq m \qquad A \leq\colon B}{\mathtt{M}_n A \leq\colon \mathtt{M}_m B}}{\vdash v\colon \mathtt{M}_m B} \tag{1.20}$$

By induction, $\exists n', v'.\vdash v \approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ \colon \mathtt{M}_n A$

$n' \leq n \leq m$, and due to the subtyping equational equivalence rule, $\vdash v \approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ \colon \mathtt{M}_m B$.

Case Bind:

$$(\text{Bind})\dfrac{\vdash v\colon \mathtt{M}_n A \qquad x:A\vdash u\colon \mathtt{M}_m B}{\vdash \mathtt{do}\ x\leftarrow v\ \mathtt{in}\ u\ :\mathtt{M}_{n+m}B}$$

So, by induction, $\vdash v\approx \mathtt{do}\ \_\leftarrow n'\ \mathtt{in}\ \mathtt{return}\ v'\ :\mathtt{M}_n A$. Hence,

$$\vdash \mathtt{do}\ x\leftarrow v\ \mathtt{in}\ u\ \approx \mathtt{do}\ x\leftarrow (\mathtt{do}\ \_\leftarrow n'\ \mathtt{in}\ \mathtt{return}\ v'\ \ \mathtt{in}\ )\ :\mathtt{M}_{n+m}B \tag{1.21}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{do}\ x\leftarrow \mathtt{return}\ v'\ \mathtt{in}\ u \tag{1.22}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ u\,[v'/x] \tag{1.23}$$

By induction $\vdash u\,[v'/x]\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ u'\ :\mathtt{M}_m B$.

So

$$\vdash \mathtt{do}\ x\leftarrow v\ \mathtt{in}\ u\ \approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ u\,[v'/x]\ :\mathtt{M}_{n+m}B \tag{1.24}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ (\mathtt{do}\ \_\leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ u'\ ) \tag{1.25}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n'+m'}\ \mathtt{in}\ \mathtt{return}\ u' \tag{1.26}$$

Where $n'+m'\le n+m$

Case If:

$$(\text{If})\dfrac{\vdash b\colon \mathtt{Bool}\qquad \vdash v_1\colon \mathtt{M}_n A\qquad \vdash v_2\colon \mathtt{M}_n A}{\vdash \mathtt{if}_b\ \mathtt{M}_n A\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ :\mathtt{M}_n A} \tag{1.27}$$

Since we can't construct booleans in any other way, $\vdash b\approx \mathtt{true}\colon \mathtt{Bool}$ or $\vdash b\approx \mathtt{false}\colon \mathtt{Bool}$.
By induction, we have $\vdash v_1\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n_1'}\ \mathtt{in}\ v_1'\ :\mathtt{M}_n A$ and $\vdash v_2\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n_2'}\ \mathtt{in}\ v_2'\ :\mathtt{M}_n A$.
So we can case split on what b reduces to.

Case $\vdash b\approx \mathtt{true}\colon \mathtt{Bool}$:

$$\vdash \mathtt{if}_b\ v_1\ \mathtt{then}\ v_2\ \mathtt{else}\ \approx\ v_1\colon \mathtt{M}_n A \tag{1.28}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n_1'}\ \mathtt{in}\ v_1' \tag{1.29}$$

Case $\vdash b\approx \mathtt{false}\colon \mathtt{Bool}$:

$$\vdash \mathtt{if}_b\ v_1\ \mathtt{then}\ v_2\ \mathtt{else}\ \approx\ v_2\colon \mathtt{M}_n A \tag{1.30}$$

$$\approx \mathtt{do}\ \_\leftarrow \mathtt{put}^{n_2'}\ \mathtt{in}\ v_2' \tag{1.31}$$

Thus in both cases, there exists picks for n' and v'.

Case Apply:

$$(\text{Apply})\dfrac{\vdash v_1\colon A\to \mathtt{M}_n B\qquad \vdash v_2\colon A}{\vdash v_1\ v_2\colon \mathtt{M}_n B} \tag{1.32}$$

So $\vdash v_1\approx \lambda x\colon A'.u : A\to \mathtt{M}_n B$, with $A\le:A'$, since $\mathtt{EC_{put}}$ is strongly normalizing. So using equational equivalence, we have the following:

$$\vdash v_1\, v_2 \approx u\,[v_2/x] : \mathtt{M}_n B \tag{1.33}$$

$$\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ u' \quad \text{By induction} \tag{1.34}$$

Corollary 1.3.2.1 (Denotational Equivalence). *Due to soundness,* $\llbracket \vdash v : \mathtt{M}_n A \rrbracket = (n', \llbracket \vdash v' : A \rrbracket)$

1.4   Logical Relations

$$\lhd_A \in \llbracket A \rrbracket \times \mathtt{EC}_{\mathtt{put}}^A \tag{1.35}$$

1.5   Definition

Definition 1.5.1 (Logical Relation).

$$d \lhd_{\mathit{Unit}} v \Leftrightarrow (d = * \implies \vdash v \approx \mathit{():Unit}) \tag{1.36}$$

$$d \lhd_{\mathit{Bool}} v \Leftrightarrow ((d = \top \implies \vdash v \approx \mathit{true:Bool}) \wedge (d = \bot \implies \vdash v \approx \mathit{false:Bool})) \tag{1.37}$$

$$d \lhd_{A \to B} v \Leftrightarrow (\forall e, u. e \lhd_A u \implies d(e) \lhd_B v\, u) \tag{1.38}$$

$$d \lhd_{\mathtt{M}_n A} v \Leftrightarrow (d = (n', d') \in T_n \llbracket A \rrbracket \implies \exists v'. d' \lhd_A v' \wedge \vdash v' : A \wedge \vdash v \approx \mathit{do}\ \_ \leftarrow \mathit{put}^{n'}\ \mathit{in}\ \mathit{return}\ v' : \mathtt{M}_n A) \tag{1.39}$$

1.6   Subtyping

Theorem 1.6.1 (Logical Relation and Subtyping). *If* $A \leq: B$ *and* $d \lhd_A v$ *then* $d \lhd_B v]$

Proof: By induction on the derivation of $A \leq: B$.

Case Ground: $A \leq: B \implies A = B$, since ground subtyping is the identity relation.

Case Fun: $A \leq: B \implies A = A_1 \to A_2, B = B_1 \to B_2$ where $B_1 \leq: A_1$ and $A_2 \leq: B_2$.
   By the definition of the $\lhd_{A \to B}$ relation, $d \lhd_{A \to B} v \Leftrightarrow (\forall e, u. e \lhd_A u \implies d(e) \lhd_B v\, u)$.
   So

$$\forall e, u. e \lhd_{B_1} u \implies e \lhd_{A_1} u \quad \text{By induction } B_1 \leq: A_1 \tag{1.40}$$

$$\implies d(e) \lhd_{A_2} u\, v \quad \text{By definition} \tag{1.41}$$

$$\implies d(e) \lhd_{B_2} u\, v \quad \text{By induction } A_2 \leq: B_2 \tag{1.42}$$

As required.

Case Effect: $\mathtt{M}_{n_1} A_1 \leq: \mathtt{M}_{n_2} A_2 \implies n_1 \leq n_2, A_1 \leq: A_2$

$$d \lhd_{M_{n_1} A_1} v \implies d = (n'_1, d') \land n'_1 \le n_1 \le n_2 \land \exists v'.(d' \lhd_{A_1} v' \land \vdash v \approx \texttt{do}\ \_\ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ v'\ : M_{n_1} A_1) \tag{1.43}$$

$$\implies\ \vdash v'_1 : A_2 \land d' \lhd_{A_2} v' \land \vdash v \approx \texttt{do}\ \_\ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ v'\ : M_{n_1} A_2 \tag{1.44}$$

$$\implies\ d \lhd_{M_{n_2} A_2} v \tag{1.45}$$

1.7 Fundamental Property

Let $\lhd_\Gamma \in \llbracket \Gamma \rrbracket \times \texttt{EC}^G_{\texttt{put}}$ mean:

$$\rho \lhd_\Gamma \sigma \Leftrightarrow \forall x. \rho(x) \lhd_{\Gamma(x)} \sigma(x) \tag{1.46}$$

Theorem 1.7.1 (Fundamental Theorem)**.** *If $\rho \lhd_\Gamma \sigma$ then $\llbracket \Gamma \vdash v : A \rrbracket \rho \lhd_A v\,[\sigma]$ up to equational equivalence.*

Proof: By induction over the derivation of $\Gamma \vdash v : A$

Case Variables:

$$\llbracket \Gamma \vdash x : \Gamma(x) \rrbracket = \rho(x) \lhd_{\Gamma(x)} \sigma(x) \approx x\,[\sigma] \tag{1.47}$$

Case Constants:

$$\llbracket \Gamma \vdash \texttt{true} : \texttt{Bool} \rrbracket \rho = \top \land \vdash \texttt{true}\,[\sigma] \approx \texttt{true} : \texttt{Bool} \quad \text{So } \top \lhd_{\texttt{Bool}} \texttt{true}\,[\sigma] \tag{1.48}$$

$$\llbracket \Gamma \vdash \texttt{false} : \texttt{Bool} \rrbracket \rho = \bot \land \vdash \texttt{false}\,[\sigma] \approx \texttt{false} : \texttt{Bool} \quad \text{So } \top \lhd_{\texttt{Bool}} \texttt{false}\,[\sigma] \tag{1.49}$$

$$\llbracket \Gamma \vdash () : \texttt{Unit} \rrbracket \rho = * \land \vdash ()\,[\sigma] \approx () : \texttt{Unit} \quad \text{So } * \lhd_{\texttt{Unit}} ()\,[\sigma] \tag{1.50}$$

$$\llbracket \Gamma \vdash \texttt{put} : M_1 \texttt{Unit} \rrbracket \rho = (1, *) \land \vdash \texttt{put} \approx \texttt{do}\ \_\ \leftarrow \texttt{put}^1\ \texttt{in}\ \texttt{return}\ ()\ : M_1 \texttt{Unit} \tag{1.51}$$

$$\tag{1.52}$$

Case Subtype:

$$\llbracket \Gamma \vdash v : B \rrbracket \rho = \llbracket \Gamma \vdash v : A \rrbracket \lhd_A v\,[\sigma] \tag{1.53}$$

Since $A \le: B \land d \lhd_A v \implies d \lhd_B v$, $\llbracket \Gamma \vdash v : B \rrbracket \lhd_B v\,[\sigma]$.

Case Fn: For all $d \lhd_A u$,

$$(\llbracket \Gamma \vdash \lambda x : A.v : A \to B \rrbracket \rho)d = (\texttt{cur}(\llbracket \Gamma, x : A \vdash v : B \rrbracket)\rho)d \tag{1.54}$$

$$= \llbracket \Gamma, x : A \vdash v : B \rrbracket(\rho[x \mapsto d]) \tag{1.55}$$

$$\tag{1.56}$$

Since $d \lhd_A u$, $(\rho[x \mapsto d]) \lhd_{\Gamma, x : A} (\sigma, x := u)$, so by induction

$$(\llbracket \Gamma \vdash \lambda x : A.v : A \to B \rrbracket \rho)d = \llbracket \Gamma, x : A \vdash v : B \rrbracket(\rho[x \mapsto d]) \lhd_B v\,[\sigma, x := u] \tag{1.57}$$

$$\lhd_B v\,[\sigma]\,[u/x] \tag{1.58}$$

$$\approx (\lambda x : A.(v\,[\sigma]))\,u \tag{1.59}$$

Case Apply:
$$\llbracket \Gamma \vdash v\, u \colon B \rrbracket \rho = (\llbracket \Gamma \vdash v \colon A \to B \rrbracket \rho)(\llbracket \Gamma \vdash u \colon A \rrbracket \rho) \tag{1.60}$$

By induction $\llbracket \Gamma \vdash v \colon A \to B \rrbracket \rho \lhd_{A \to B} v\, [\sigma]$ and $\llbracket \Gamma \vdash u \colon A \rrbracket \rho \lhd_A u\, [\sigma]$. So by the definition of $\lhd_{A \to B}$,

$$\llbracket \Gamma \vdash v\, u \colon B \rrbracket \rho = (\llbracket \Gamma \vdash v \colon A \to B \rrbracket \rho)(\llbracket \Gamma \vdash u \colon A \rrbracket \rho) \lhd_B (v\, [\sigma])\, (u\, [\sigma]) \approx (v\, u)\, [\sigma] \tag{1.61}$$

Case Return:
$$\llbracket \Gamma \vdash v \colon \mathtt{M}_0 A \rrbracket \rho = (0, \llbracket \Gamma \vdash v \colon A \rrbracket) \tag{1.62}$$

By induction, $\llbracket \Gamma \vdash v \colon A \rrbracket \lhd_A v\, [\sigma]$, so by picking $v' = v\, [\sigma]$

$$\vdash (\mathtt{return}\ v\,)\, [\sigma] \approx \mathtt{return}\, (v\, [\sigma]) \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^0\ \mathtt{in}\ \mathtt{return}\ v'\ \colon \mathtt{M}_0 A \tag{1.63}$$

So $\llbracket \Gamma \vdash \mathtt{return}\ v \colon \mathtt{M}_0 A \rrbracket \lhd_{\mathtt{M}_0 A} (\mathtt{return}\ v\,)\, [\sigma]$

Case Bind: By inversion, $\llbracket \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u \colon \mathtt{M}_{m+n} B \rrbracket \rho = (m' + n', d_u)$, where $(n', d_u) = \llbracket \Gamma, x \colon A \vdash u \colon \mathtt{M}_n B \rrbracket (\rho[x \mapsto d_v])$, and $(n', d_v) = \llbracket \Gamma \vdash v \colon \mathtt{M}_m A \rrbracket \rho$.

By induction, $(m', t_v) \lhd_{\mathtt{M}_m A} v[[\sigma]]$. So $\exists v'$ such that $\vdash v\, [\sigma] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v'\ \colon \mathtt{M}_m A$. So $(\rho[x \mapsto d_v]) \lhd_{\Gamma, x \colon A} ([\sigma], x \colon= v')$.

So by induction $\llbracket \Gamma, x \colon A \vdash u \colon \mathtt{M}_n B \rrbracket (\rho[x \mapsto d_v]) \lhd_{\mathtt{M}_n B} u\, [\sigma, x \colon= v']$.

Hence, $\exists u'$ such that $\vdash u\, [\sigma, x \colon= v'] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'} \mathtt{return}\ u'\ \mathtt{in} \colon \mathtt{M}_{m+n} B$ and $t_u \lhd_{\mathtt{M}_n B} u'$.

Hence,

$$\vdash \mathtt{do}\ x \leftarrow v\, [\sigma]\ \mathtt{in}\ u\, [\sigma]\ \approx \mathtt{do}\ x \leftarrow (\mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v'\,)\ \mathtt{in}\ (u\, [\sigma]) \colon \mathtt{M}_{m+n} B \tag{1.64}$$
$$\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ u\, [\sigma, x \colon= v'] \tag{1.65}$$
$$\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'+n'}\ \mathtt{in}\ \mathtt{return}\ u' \tag{1.66}$$

So $\llbracket \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u \colon \mathtt{M}_{m+n} B \rrbracket \rho \lhd_{\mathtt{M}_{m+n} B} (\mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u\,)\, [\sigma]$.

Case If: By inversion, $\llbracket \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \rrbracket \rho = \begin{cases} \llbracket \Gamma \vdash v_1 \colon A \rrbracket \rho & \text{If } \llbracket \Gamma \vdash b \colon \mathtt{Bool} \rrbracket \rho = \top \\ \llbracket \Gamma \vdash v_2 \colon A \rrbracket \rho & \text{If } \llbracket \Gamma \vdash b \colon \mathtt{Bool} \rrbracket \rho = \bot \end{cases}$.

By induction,

$$\llbracket \Gamma \vdash b \colon \mathtt{Bool} \rrbracket \rho \lhd_{\mathtt{Bool}} b\, [\sigma] \tag{1.67}$$
$$\llbracket \Gamma \vdash v_1 \colon A \rrbracket \rho \lhd_A v_1\, [\sigma] \tag{1.68}$$
$$\llbracket \Gamma \vdash v_2 \colon A \rrbracket \rho \lhd_A v_2\, [\sigma] \tag{1.69}$$

Case: $\llbracket \Gamma \vdash b \colon \mathtt{Bool} \rrbracket \rho = \top$

$$\llbracket \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \rrbracket \rho = \llbracket \Gamma \vdash v_1 \colon A \rrbracket \rho \lhd_A v_1\, [\sigma] \approx (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\,)\, [\sigma] \tag{1.70}$$

Case: $\llbracket \Gamma \vdash b \colon \mathtt{Bool} \rrbracket \rho = \bot$

$$\llbracket \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \rrbracket \rho = \llbracket \Gamma \vdash v_2 \colon A \rrbracket \rho \lhd_A v_2\, [\sigma] \approx (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\,)\, [\sigma] \tag{1.71}$$

1.8 Adequacy

Theorem 1.8.1 (Adequacy)**.** *For G defined as:*

$$G ::= \textit{Bool} \mid \textit{Unit} \mid M_n G \tag{1.72}$$

Equality of denotations implies equational equality.

$$[\![\vdash v \colon G]\!] = [\![\vdash u \colon G]\!] \implies \; \vdash v \approx u \colon G \tag{1.73}$$

Proof: By induction on the structure of G, making use of the fundamental property 1.7.

Case Boolean: Let $d = [\![\vdash v \colon \texttt{Bool}]\!] = [\![\vdash v \colon \texttt{Bool}]\!] \in \{\top, \bot\}$. By the fundamental property, $d \lhd_{\texttt{Bool}} v$ and $d \lhd_{\texttt{Bool}} v$.

    Case: $d = \top$ Then $\vdash v \approx \texttt{true} \approx u \colon \texttt{Bool}$

    Case: $d = \bot$ Then $\vdash v \approx \texttt{false} \approx u \colon \texttt{Bool}$

Case Unit: Let $* = [\![\vdash v \colon \texttt{Unit}]\!] = [\![\vdash v \colon \texttt{Unit}]\!] \in \{*\}$. By the fundamental property, $d \lhd_{\texttt{Unit}} v$ and $d \lhd_{\texttt{Unit}} v$. Hence $\vdash v \approx () \approx u \colon \texttt{Unit}$.

Case Effect: Let $(n', d) = [\![\vdash v \colon M_n G]\!] = [\![\vdash u \colon M_n G]\!]$. By the fundamental property, $(n', d) \lhd_{M_n G} v$ and $(n', d) \lhd_{M_n G} u$. So there exists u', v' such that $d' \lhd_G u'$ and $d' \lhd_G u'$ and:

$$\vdash v \approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ v'\ : M_n G \tag{1.74}$$

$$\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ u' \tag{1.75}$$

$$\approx u \tag{1.76}$$

$\square$