0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t\colon \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t\colon \tau$,there exists at most one reduced derivation of $\Gamma \vdash t\colon \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Variables

To find the unique derivation of $\Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is, if $A' \leq: A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\frac{\Gamma', x:A' \,\mathtt{Ok}}{\Gamma, x:A' \vdash x:A'} \quad A' \leq: A}{\Gamma', x : A' \vdash x\colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Gamma \vdash x\colon A$ holds, then so must $\Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma' \vdash x:A'} \quad A' \leq: A}{\Gamma' \vdash x\colon A} \tag{2}$$

Be the unique reduced derivation of $\Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\frac{()\frac{\Delta}{\Gamma, x:A' \vdash x:A'}}{\Gamma \vdash x:A'} \quad A' \leq: A}{\Gamma \vdash x\colon A} \tag{3}$$

0.2.2 Constants

For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A} \quad A \leq: B}{\Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

0.2.3 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Gamma, x:A \vdash C:\mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x:A.B:A \to \mathtt{M}_\epsilon B} \quad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'}$$

Where

$$\text{(Sub-Effect)}\frac{()\frac{\Delta}{\Gamma, x{:}A \vdash C{:}\,\mathtt{M}_{\epsilon} B} \quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma, x : A \vdash C{:}\,\mathtt{M}_{\epsilon'} B'} \tag{4}$$

is the reduced derivation of $\Gamma, x : A \vdash C{:}\,\mathtt{M}_{\epsilon} B$ if it exists.

Case Subtype **TODO: Do we need to write anything here? (Probably needs an explanation)**

0.2.4 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return} v{:}\,\mathtt{M}_{\epsilon} B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\frac{()\frac{\Delta}{\Gamma \vdash v{:}A}}{\Gamma \vdash \mathtt{return} v{:}\mathtt{M}_{\mathtt{1}} A} \quad A \leq: B \quad \mathtt{1} \leq \epsilon}{\Gamma \vdash \mathtt{return} v{:}\,\mathtt{M}_{\epsilon} B}$$

Where

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma \vdash v{:}A} \quad A \leq: B}{\Gamma \vdash v{:}\, B}$$

is the reduced derivation of $\Gamma \vdash v{:}\, B$

Case Apply If

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma \vdash v_1{:}A \to \mathtt{M}_{\epsilon} B} \quad A \to \mathtt{M}_{\epsilon} B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1{:}\, A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Gamma \vdash v_2{:}A''} \quad A'' \leq: A'}{\Gamma \vdash v_2{:}\, A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1{:}\, A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2{:}\, A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1\ v_2{:}\,\mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subeffect)}\frac{\text{(Apply)}\frac{()\frac{\Delta}{\Gamma \vdash v_1{:}A \to \mathtt{M}_{\epsilon} B} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Gamma \vdash v{:}A''} \quad A'' \leq: A}{\Gamma \vdash v{:}A}}{\Gamma \vdash v_1\ v_2{:}\mathtt{M}_{\epsilon} B} \quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma \vdash v_1\ v_2{:}\,\mathtt{M}_{\epsilon'} B'}$$

Case If Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma \vdash v{:}B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v{:}\,\mathtt{Bool}} \tag{5}$$

$$\text{(Subeffect)}\frac{()\frac{\Delta'}{\Gamma \vdash C_1{:}\mathtt{M}_{\epsilon'} A'} \quad A' \leq: A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1{:}\,\mathtt{M}_{\epsilon} A} \tag{6}$$

$$\text{(Subeffect)}\frac{()\frac{\Delta''}{\Gamma \vdash C_2{:}\mathtt{M}_{\epsilon''} A''} \quad A'' \leq: A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2{:}\,\mathtt{M}_{\epsilon} A} \tag{7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v{:}\,\mathtt{Bool}$, $\Gamma \vdash C_1{:}\,\mathtt{M}_{\epsilon} A$, $\Gamma \vdash C_2{:}\,\mathtt{M}_{\epsilon} A$.
Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{:}\,\mathtt{M}_{\epsilon} A$ is:
TODO: Scale this properly

$$\text{(Subtype)}\frac{\text{(If)}\frac{\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma \vdash v{:}B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v{:}\mathtt{Bool}} \quad \text{(Subeffect)}\frac{()\frac{\Delta'}{\Gamma \vdash C_1{:}\mathtt{M}_{\epsilon'} A'} \quad A' \leq: A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1{:}\mathtt{M}_{\epsilon} A} \quad \text{(Subeffect)}\frac{()\frac{\Delta''}{\Gamma \vdash C_2{:}\mathtt{M}_{\epsilon''} A''} \quad A'' \leq: A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2{:}\mathtt{M}_{\epsilon} A}}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{:}\mathtt{M}_{\epsilon} A} \quad \epsilon \leq \epsilon \quad A \leq: A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{:}\,\mathtt{M}_{\epsilon} A} \tag{8}$$

Case Bind Let

$$(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1 : \mathtt{M}_{\epsilon_1} A} \quad A \leq: A' \quad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma, x{:}A\vdash C_2 : \mathtt{M}_{\epsilon_2} B} \quad B \leq: B' \quad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms]

By weakening, $\iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2 : \mathtt{M}_\epsilon B$, there's also one of $\Gamma, x : A \vdash C_2 : \mathtt{M}_\epsilon B$.

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$(\text{Subeffect})\frac{(\text{Bind})\frac{(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1} A} \quad A\leq: A' \quad \epsilon_1\leq\epsilon_1'}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1'} A'} \quad (\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma,x{:}A\vdash C_2:\mathtt{M}_{\epsilon_2} B} \quad B\leq: B' \quad \epsilon_2\leq\epsilon_2'}{\Gamma,x{:}A\vdash C_2:\mathtt{M}_{\epsilon_2'} B'}}{\Gamma\vdash \mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \quad B \leq: B' \quad \epsilon_1\cdot\epsilon_2 \leq \epsilon_1'\cdot\epsilon_2'}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B'} \tag{11}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

0.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$reduce((\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A : A}) = (\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A : A}$

0.3.2 Value Types

Var

$$reduce((\text{Var})\frac{\Gamma\mathtt{Ok}}{\Gamma, x : A \vdash x : A}) = (\text{Var})\frac{\Gamma\mathtt{Ok}}{\Gamma, x : A \vdash x : A} \tag{12}$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{()\frac{\Delta}{\Gamma\vdash x:A}}{\Gamma, y : B \vdash x : A}) \tag{13}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash x:A} \quad A' \leq: A}{\Gamma \vdash x : A} = reduce(\Delta) \tag{14}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma\vdash x:A'}}{\Gamma, y{:}B\vdash x:A'} \quad A' \leq: A}{\Gamma, y : B \vdash x : A} \tag{15}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$[\![(\text{Weaken})\frac{()\frac{\Delta}{\Gamma\vdash x\colon A}}{\Gamma, y : B \vdash x\colon A}]\!]_M = \Delta \circ \pi_1 \tag{16}$$

Similarly, the denotation of the reduced denotation is:

$$[\![(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma\vdash x:A'}}{\Gamma,y:B\vdash x:A'} \quad A' \leq: A}{\Gamma, y : B \vdash x\colon A}]\!]_M = [\![A' \leq: A]\!]_M \circ \Delta' \circ \pi_1 \tag{17}$$

By induction on *reduce* preserving denotations and the reduction of Δ (14), we have:

$$\Delta = [\![A' \leq: A]\!]_M \circ \Delta' \tag{18}$$

So the denotations of the un-reduced and reduced derivations are equal.

Lambda

reduce **definition** To find:

$$\mathit{reduce}((\text{Fn})\frac{()\frac{\Delta}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_2}B}}{\Gamma \vdash \lambda x : A.C\colon A \to \mathtt{M}_{\epsilon_2}B}) \tag{19}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta'}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_1}B'} \quad \epsilon_1 \leq \epsilon_2 \quad B' \leq: B}{\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon_2}B} = \mathit{reduce}(\Delta) \tag{20}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{\Delta'}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_1}B'} \quad A \to \mathtt{M}_{\epsilon_1}B' \leq: A \to \mathtt{M}_{\epsilon_2}B}{\Gamma \vdash \lambda x : A.C\colon A \to \mathtt{M}_{\epsilon_2}B} \tag{21}$$

Preserves Denotation Let

$$f = [\![\mathtt{M}_{\epsilon_1}B' \leq: \mathtt{M}_{\epsilon_2}B]\!]_M = [\![\epsilon_1 \leq \epsilon_2]\!]_{M,B} \circ T_{\epsilon_1}([\![B' \leq: B]\!]_M) \tag{22}$$

$$[\![A \to \mathtt{M}_{\epsilon_1}B' \leq: A \to \mathtt{M}_{\epsilon_2}B]\!]_M = f^A = \mathtt{cur}(f \circ \mathtt{app}) \tag{23}$$

Then

$$\begin{aligned}
\mathit{before} &= \mathtt{cur}(\Delta) \quad \text{By definition} && (24)\\
&= \mathtt{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta && (25)\\
&= f^A \circ \mathtt{cur}(\Delta') \quad \text{By the property of } f^X \circ \mathtt{cur}(g) = \mathtt{cur}(f \circ g) && (26)\\
&= \mathit{after} \quad \text{By definition} && (27)\\
& && (28)
\end{aligned}$$

Subtype

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Gamma\vdash v:A} \quad A \leq: B}{\Gamma \vdash v: B}) \tag{29}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash x:A} \quad A' \leq: A}{\Gamma \vdash x: A} = reduce(\Delta) \tag{30}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash v:A'} \quad A' \leq: A \leq: B}{\Gamma \vdash v: B} \tag{31}$$

Preserves Denotation

$$\begin{aligned} before &= [\![A \leq: B]\!]_M \circ \Delta && (32)\\ &= [\![A \leq: B]\!]_M \circ ([\![A' \leq: A]\!]_M \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. && (33)\\ &= [\![A' \leq: B]\!]_M \circ \Delta' \quad \text{Subtyping relations are unique} && (34)\\ &= after && (35)\\ & && (36) \end{aligned}$$

0.3.3 Computation Types

Return

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Gamma\vdash v:A}}{\Gamma \vdash \mathtt{return}v: \mathtt{M}_{\mathtt{1}} A}) \tag{37}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Gamma\vdash v:A'} \quad A' \leq: A}{\Gamma \vdash v: A} = reduce(\Delta) \tag{38}$$

In

$$(\text{Sub-effect})\frac{(\text{Return})\frac{\Delta'}{\Gamma\vdash v:A} \quad \mathtt{1} \leq \mathtt{1} \quad A' \leq: A}{\Gamma \vdash \mathtt{return}v: \mathtt{M}_{\mathtt{1}} A} \tag{39}$$

Then

$$\begin{aligned} before &= \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} && (40)\\ &= \eta_A \circ [\![A' \leq: A]\!]_M \circ \Delta' \quad \text{BY reduction of } \Delta && (41)\\ &= T_{\mathtt{1}}[\![A' \leq: A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta && (42)\\ &= [\![\mathtt{1} \leq \mathtt{1}]\!]_{M,A} \circ T_{\mathtt{1}}[\![A' \leq: A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{Since } [\![\mathtt{1} \leq \mathtt{1}]\!]_M \text{ is the identity Nat-Trans} && (43)\\ &= after \quad \text{By definition} && (44)\\ & && (45) \end{aligned}$$

Apply

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B}) \tag{46}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Gamma \vdash v_1 : A' \to \mathtt{M}_{\epsilon'} B'} \quad A' \to \mathtt{M}_{\epsilon'} B' \leq: A \to \mathtt{M}_\epsilon B}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} = reduce(\Delta_1) \tag{47}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Gamma \vdash v : A'} \quad A' \leq: A}{\Gamma \vdash v_1 : A} = reduce(\Delta_2) \tag{48}$$

In

$$(\text{Sub-effect})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma \vdash v_1 : A' \to \mathtt{M}_{\epsilon'} B'} \quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Gamma \vdash v_2 : A''} \quad A'' \leq: A \leq: A'}{\Gamma \vdash v_2 : A'}}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'} \quad \epsilon' \leq \epsilon \quad B' \leq: B}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B} \tag{49}$$

Preserves Denotation Let

$$f = [\![A \leq: A']\!]_M : A \to A' \tag{50}$$
$$f' = [\![A'' \leq: A]\!]_M : A'' \to A \tag{51}$$
$$g = [\![B' \leq: B]\!]_M : B' \to B \tag{52}$$
$$h = [\![\epsilon' \leq \epsilon]\!]_M : T_{\epsilon'} \to T_\epsilon \tag{53}$$

Hence

$$[\![A' \to \mathtt{M}_{\epsilon'} B' \leq: A \to \mathtt{M}_\epsilon B]\!]_M = (h_B \circ T_{\epsilon'} g)^A \circ (T_{\epsilon'} B')^f \tag{54}$$
$$= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f)) \tag{55}$$
$$= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \tag{56}$$

Then

$$before = \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} \tag{57}$$
$$= \mathtt{app} \circ \langle \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \circ \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 \tag{58}$$
$$= \mathtt{app} \circ (\mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \times \mathtt{Id}_A) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{Factoring out} \tag{59}$$
$$= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By the exponential property} \tag{60}$$
$$= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ \langle \Delta_1', f \circ f' \circ \Delta_2' \rangle \tag{61}$$
$$= after \quad \text{By defintion} \tag{62}$$

If

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Gamma \vdash v : \mathtt{Bool}} \quad ()\frac{\Delta_2}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma \vdash C_2 : \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Gamma \vdash v : \mathtt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \quad ()\frac{reduce(\Delta_3)}{\Gamma \vdash C_2 : \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A} \tag{63}$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Bind

reduce **definition** To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1}A}\quad ()\frac{\Delta_2}{\Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2}B}}{\Gamma\vdash \mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{64}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta'_1}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon'_1}A'}\quad \epsilon'_1\leq:\epsilon_1\quad A'\leq: A}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1}A} = reduce(\Delta_1) \tag{65}$$

Since $i,\times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq: A$, and by Δ_2, $(\Gamma, x : A) \vdash C_2{:}\,\mathtt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $(\Gamma, x : A') \vdash C_2{:}\,\mathtt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta'_3}{\Gamma,x:A'\vdash C_2:\mathtt{M}_{\epsilon'_2}B'}\quad \epsilon'_2\leq:\epsilon_2\quad B'\leq: B}{\Gamma, x : A'\vdash C_2:\mathtt{M}_{\epsilon_2}B} = reduce(\Delta_3) \tag{66}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon'_1$ and $\epsilon_2 \leq \epsilon'_2$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon'_1 \cdot \epsilon'_2$
Then the result of reduction of the whole bind expression is:

$$(\text{Sub-effect})\frac{(\text{Bind})\frac{()\frac{\Delta'_1}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon'_1}A'}\quad ()\frac{\Delta'_3}{\Gamma,x:A'\vdash C_2:\mathtt{M}_{\epsilon'_2}B'}}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon'_1\cdot\epsilon'_2}B}\quad B'\leq: B\quad \epsilon'_1\cdot\epsilon'_2\leq\epsilon_1\cdot\epsilon_2}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{67}$$

Preserves Denotation Let

$$f = [\![A' \leq: A]\!]_M : A' \to A \tag{68}$$
$$g = [\![B' \leq: B]\!]_M : B' \to B \tag{69}$$
$$h_1 = [\![\epsilon'_1 \leq \epsilon_1]\!]_M : T_{\epsilon'_1} \to T_{\epsilon_1} \tag{70}$$
$$h_2 = [\![\epsilon'_2 \leq \epsilon_2]\!]_M : T_{\epsilon'_2} \to T_{\epsilon_2} \tag{71}$$
$$h = [\![\epsilon'_1 \cdot \epsilon'_2 \leq \epsilon_1 \cdot \epsilon_2]\!]_M : T_{\epsilon'_1\cdot\epsilon'_2} \to T_{\epsilon_1\cdot\epsilon_2} \tag{72}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3 = \Delta_2 \circ (\mathtt{Id}_\Gamma \times f) \tag{73}$$

And due to the reduction of Δ_3, we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3 \tag{74}$$

So:

$$\begin{aligned}
before &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1\rangle \quad \text{By definition.} && (75)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon_1'} f \circ \Delta_1'\right\rangle \quad \text{By reduction of } \Delta_1. && (76)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1'\right\rangle \quad \text{Factor out } h_1 && (77)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma\times A)} \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1'\right\rangle \quad \text{Tensor strength and sub-effecting } h_1 && (78)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1'\right\rangle \quad \text{Naturality of } h_1 && (79)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1'} f) \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{Factor out pairing again} && (80)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{Tensorstrength} && (81)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_3) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{By the definition of } \Delta_3 && (82)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3') \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{By the reduction of } \Delta_3 && (83)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'} h_{2,B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{Factor out the functor} && (84)\\
&= h_B \circ \mu_{\epsilon_1',\epsilon_2',B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{By the } \mu \text{ and Sub-effect rule} && (85)\\
&= h_B \circ T_{\epsilon_1'\cdot\epsilon_2'} g \circ \mu_{\epsilon_1',\epsilon_2',B'} \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1'\rangle \quad \text{By naturality of } \mu_{,,} && (86)\\
&= after \quad \text{By definition} && (87)
\end{aligned}$$

Subeffect

reduce **definition** To find:

$$reduce((\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C:\mathtt{M}_{\epsilon'}B'} \quad \epsilon' \leq \epsilon \quad B' \leq: B}{\Gamma \vdash C : \mathtt{M}_\epsilon B}) \tag{88}$$

Let

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \quad \epsilon'' \leq \epsilon' \quad \mathtt{Bool}'' \leq: B}{\Gamma \vdash C : \mathtt{M}_{\epsilon'} B} = reduce(\Delta) \tag{89}$$

in

$$(\text{subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \quad \epsilon'' \leq \epsilon \quad B'' \leq: B}{\Gamma \vdash C : \mathtt{M}_\epsilon B} \tag{90}$$

Preserves Denotation Let

$$\begin{aligned}
f &= [\![B' \leq: B]\!]_M && (91)\\
g &= [\![B'' \leq: B']\!]_M && (92)\\
h_1 &= [\![\epsilon' \leq \epsilon]\!]_M && (93)\\
h_2 &= [\![\epsilon' \leq \epsilon']\!]_M && (94)\\
f \circ g &= [\![B'' \leq: B]\!]_M && (95)\\
h_1 \circ h_2 &= [\![\epsilon'' \leq \epsilon']\!]_M && (96)\\
& && (97)
\end{aligned}$$

Hence we can find the denotation of the derivation before reduction.

$$
\begin{aligned}
before &= h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} && (98)\\
&= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta && (99)\\
&= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \qquad = after \quad \text{By definition.} && (100)
\end{aligned}
$$

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t \colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce \Delta]\!]_M = [\![reduce \Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t \colon \tau]\!]_M$ is unique.