0.1 Introduce Substitutions

0.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{1}$$

0.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{2}$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{4}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6}$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{7}$$
$$x\,[\sigma, x := v] = v \tag{8}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{9}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{10}$$
$$(\lambda x : A.C\ )\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{11}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\ )\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{12}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{13}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2\ ) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{14}$$
$$\tag{15}$$

0.1.4 Well Formed-ness

Define the relation

$$\Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil) $\dfrac{\Gamma'\mathtt{Ok}}{\Gamma' \vdash \diamond \colon \diamond}$
- (Extend) $\dfrac{\Gamma' \vdash \sigma \colon \Gamma \qquad x \notin \mathtt{dom}(\Gamma) \qquad \Gamma' \vdash v \colon A}{\Gamma' \vdash (\sigma, x := v) \colon (\Gamma, x : A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma\colon \Gamma$ then: **TODO: Number these**

Property 1: $\Gamma\,\mathtt{Ok}$ and $\Gamma'\,\mathtt{Ok}$ Since $\Gamma'\,\mathtt{Ok}$ holds by the Nil-axiom. $\Gamma\,\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma\colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v\colon A$ holds if $\Gamma' \vdash v\colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x)\colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma\colon \Gamma$$

In addition, $\Gamma', x : A \vdash x\colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v)\colon (\Gamma, x : A) \tag{16}$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g\colon \tau \wedge \Gamma' \vdash \sigma\colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{17}$$

Assuming $\Gamma' \vdash \sigma\colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t\colon \tau \to \Gamma' \vdash t\colon \tau$

0.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x\colon A \tag{18}$$

So by inversion, since $\Gamma' \vdash \sigma\colon \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v\colon A \tag{19}$$

By the definition of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Gamma' \vdash x\,[\sigma] : A \tag{20}$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{\dfrac{\Delta}{\Gamma'' \vdash x\colon A}}{\Gamma'', y : B \vdash x\colon A} \tag{21}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma' \colon \Gamma'' \tag{22}$$

So by induction,

$$\Gamma' \vdash x\,[\sigma'] \colon A \tag{23}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Gamma' \vdash x\,[\sigma] \colon A \tag{24}$$

0.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn})\frac{\dfrac{\Delta}{\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_\epsilon B} \tag{25}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma', x : A) \vdash (\sigma, x := x) \colon \Gamma, x : A \tag{26}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{\dfrac{\Delta'}{\Gamma', x : A \vdash C\,[\sigma, x := v] \colon \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\,[\sigma, x := x]\ : A \to \mathtt{M}_\epsilon B} \tag{27}$$

Since $\lambda x : A.(C\,[\sigma, x := x]) = \lambda x : A.(\ C\,[\sigma]) = (\lambda x : A.C\ )\,[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x : A.C\ )\,[\sigma] \colon A \to \mathtt{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:

$\Gamma \vdash \sigma \colon \Gamma \Rightarrow \Gamma' \mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{28}$$

So

$$(\text{Const})\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A \colon A} \tag{29}$$

0.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{\dfrac{\Delta_1}{\Gamma \vdash v \colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_{\mathtt{1}} A} \tag{30}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\,[\sigma] : A}}{\Gamma' \vdash \texttt{return}\ (v\,[\sigma]) : \texttt{M}_{\texttt{1}} A} \tag{31}$$

Since $(\texttt{return}\ v\ )\,[\sigma] = \texttt{return}\ (v\,[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\texttt{return}\ v\ )\,[\sigma] : \texttt{M}_{\texttt{1}} A$.

Case Apply By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1 : A \to \texttt{M}_\epsilon B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 : \texttt{M}_\epsilon B} \tag{32}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v_1\,[\sigma] : A \to \texttt{M}_\epsilon B} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash v_2\,[\sigma] : A}}{\Gamma' \vdash (v_1\,[\sigma])\ (v_2\,[\sigma]) : \texttt{M}_\epsilon B} \tag{33}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \texttt{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$(\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v : \texttt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash C_1 : \texttt{M}_\epsilon A} \qquad \dfrac{\Delta_3}{\Gamma \vdash C_2 : \texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 : \texttt{M}_\epsilon A} \tag{34}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$(\text{If})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\,[\sigma] : \texttt{Bool}} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash C_1\,[\sigma] : \texttt{M}_\epsilon A} \qquad \dfrac{\Delta_3'}{\Gamma' \vdash C_2\,[\sigma] : \texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A}\ (v\,[\sigma])\ \texttt{then}\ (C_1\,[\sigma])\ \texttt{else}\ (C_2\,[\sigma]) : \texttt{M}_\epsilon A} \tag{35}$$

Since $(\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ )\,[\sigma] = \texttt{if}_{\epsilon,A}\ (v\,[\sigma])\ \texttt{then}\ (C_1\,[\sigma])\ \texttt{else}\ (C_2\,[\sigma])$ The derivation above holds for $\Gamma' \vdash (\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ )\,[\sigma] : \texttt{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$(\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash C_1 : \texttt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash C_2 : \texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{36}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x : A) \vdash (\sigma, x := x) : (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\frac{\dfrac{\Delta_1'}{\Gamma' \vdash C_1\,[\sigma] : \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma', x : A \vdash C_2\,[\sigma, x := x] : \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma, x := x]) : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{37}$$

Since $(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2\ )\,[\sigma] = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma])\ = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma, x := x])$, the above derivation holds for $\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2\ )\,[\sigma] : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B$

0.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$\text{(sub-type)}\frac{\dfrac{\Delta}{\Gamma \vdash v \colon A} \qquad A \leq: B}{\Gamma \vdash v \colon B} \tag{38}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\frac{\dfrac{\Delta'}{\Gamma' \vdash v\,[\sigma] : A} \qquad A \leq: B}{\Gamma \vdash v\,[\sigma] : B} \tag{39}$$

Case Sub-effect By inversion, there exists Δ such that

$$\text{(sub-effect)}\frac{\dfrac{\Delta}{\Gamma \vdash C \colon \mathtt{M}_{\epsilon_1} A} \qquad \text{(Computation)}\dfrac{\epsilon_1 \leq_\Phi \epsilon_2 \qquad A \leq:_\Phi B}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash C \colon \mathtt{M}_{\epsilon_2} B} \tag{40}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-effect)}\frac{\dfrac{\Delta'}{\Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_1} A} \qquad \text{(Computation)}\dfrac{\epsilon_1 \leq_\Phi \epsilon_2 \qquad A \leq:_\Phi B}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_2} B} \tag{41}$$

0.3 Semantics of Substitution

0.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma' \vdash \sigma \colon \Gamma]\!] : \Gamma' \to \Gamma \tag{42}$$

- $\text{(Nil)}\dfrac{\Gamma' \mathtt{Ok}}{[\![\Gamma' \vdash \diamond \colon \diamond]\!] = \langle\rangle_{\Gamma'}}$
- $\text{(Extend)}\dfrac{f = [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \qquad g = [\![\Gamma' \vdash v \colon A]\!]}{[\![\Gamma' \vdash (\sigma, x := v \colon (\Gamma, x : A)]\!] = \langle f, g\rangle : \Gamma' \to (\Gamma \times A)}$

0.3.2 Extension Lemma

If $\Gamma' \vdash \sigma \colon \Gamma$ and $x \notin (\mathtt{dom}(\Gamma') \cup \mathtt{dom}(\Gamma))$ then the substitution in property 3 has denotation:

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)]\!] = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!] \times \mathtt{Id}_A) \tag{43}$$

This holds since

$$[\![\Gamma', x : A \vdash x \colon A]\!] = \pi_2 \tag{44}$$

And $\iota\pi : (\Gamma', x : A) \triangleright \Gamma'$

$$[\![\iota\pi : (\Gamma', x : A) \triangleright \Gamma']\!] = \pi_1 \tag{45}$$

So for each denotation $[\![\Gamma' \vdash v \colon B]\!]$ of each $y := v$ in σ, we can pre-pend the denotation with the weakening denotation to yield:

$$[\![\Gamma', x : A \vdash v \colon B]\!] = [\![\Gamma' \vdash v \colon B]\!] \circ \pi_1 \tag{46}$$

Since π_1 appears in every branch of $[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!]$, it can be factored out to yield:

$$[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!] = [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \circ \pi_1 \tag{47}$$

Hence,

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x) \colon \Gamma, x : A]\!] = \langle [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \circ \pi_1, \pi_2 \rangle = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!] \times \mathtt{Id}_A) \tag{48}$$

0.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma \vdash t \colon \tau$ and $\Gamma' \vdash \sigma \colon \Gamma$ then the derivation Δ' deriving $\Gamma' \vdash t\,[\sigma] : \tau$ satisfies:

$$\Delta' = \Delta \circ [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \tag{49}$$

This is proved by induction over the derivation of $\Gamma \vdash t \colon \tau$. We shall use σ to denote $[\![\Gamma' \vdash \sigma \colon \Gamma]\!]$ where it is clear from the context.

0.3.4 Proof For Value Terms

Case Var By inversion $\Gamma = \Gamma'', x : A$

$$(\mathrm{Var})\frac{\Gamma\mathtt{Ok}}{\Gamma'', x : A \vdash x \colon A} \tag{50}$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v \colon A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma \colon \Gamma]\!] = \langle \sigma', \Delta' \rangle \tag{51}$$
$$\Delta = [\![\Gamma'', x : A \vdash x \colon A]\!] = \pi_2 \tag{52}$$
$$\tag{53}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{54}$$
$$= \Delta' \quad \text{By product property} \tag{55}$$

Case Weaken By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{\dfrac{\Delta_1}{\Gamma'' \vdash x\colon A}}{\Gamma'', y : B \vdash x\colon A} \tag{56}$$

Also by inversion of the well-formed-ness of $\Gamma' \vdash \sigma\colon \Gamma$, we have $\Gamma' \vdash \sigma'\colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma\colon \Gamma]\!] = \langle [\![\Gamma' \vdash \sigma\colon \Gamma'']\!], [\![\Gamma' \vdash v\colon B]\!]\rangle \tag{57}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$\frac{\Delta_1'}{\Gamma' \vdash x\,[\sigma] : A} \tag{58}$$

Hence

$$\begin{aligned}
\Delta' &= \Delta_1' \quad \text{By definition} && (59)\\
&= \Delta_1 \circ \sigma' \quad \text{By induction} && (60)\\
&= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v\colon B]\!]\rangle \quad \text{By product property} && (61)\\
&= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.} && (62)
\end{aligned}$$

Case Constants The logic for all constant terms ($\texttt{true}, \texttt{false}$, $\texttt{()}\texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!] \tag{63}$$

$$\begin{aligned}
\Delta' &= c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} && (64)\\
&= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} && (65)\\
&= \Delta \circ \sigma \quad \text{By definition} && (66)
\end{aligned}$$

Case Lambda By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{\dfrac{\Delta_1}{\Gamma, x : A \vdash C\colon \texttt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \texttt{M}_\epsilon B} \tag{67}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{\dfrac{\Delta_1'}{\Gamma', x : A \vdash (C\,[\sigma])\colon \texttt{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A.C\ )\,[\sigma] : A \to \texttt{M}_\epsilon B} \tag{68}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{69}$$

Hence:

$$\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By definition} && (70)\\
&= \mathtt{cur}(\Delta_1 \circ (\sigma \times \mathtt{Id}_A)) \quad \text{By induction and extension lemma.} && (71)\\
&= \mathtt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (72)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (73)\\
& && (74)
\end{aligned}$$

Case Sub-type By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{\dfrac{\Delta_1}{\Gamma \vdash v\colon A} \qquad A \leq: B}{\Gamma \vdash v\colon B} \tag{75}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\,[\sigma] : A} \qquad A \leq: B}{\Gamma' \vdash v\,[\sigma] : B} \tag{76}$$

Hence,

$$\begin{aligned}
\Delta' &= [\![A \leq: B]\!] \circ \Delta_1' \quad \text{By definition} && (77)\\
&= [\![A \leq: B]\!] \circ \Delta_1 \circ \sigma \quad \text{By induction} && (78)\\
&= \Delta \circ \sigma \quad \text{By definition} && (79)\\
& && (80)
\end{aligned}$$

0.3.5 Proof For Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{\dfrac{\Delta_1}{\Gamma \vdash v\colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M_1} A} \tag{81}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\,[\sigma] : A}}{\Gamma' \vdash (\mathtt{return}\ v\ )\,[\sigma] : \mathtt{M_1} A} \tag{82}$$

Hence,

$$\begin{aligned}
\Delta' &= \eta_A \circ \Delta_1' \quad \text{By Definition} && (83)\\
&= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} && (84)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (85)\\
& && (86)
\end{aligned}$$

Case Apply By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B} \tag{87}$$

By induction we find Δ_1', Δ_2' such that

$$\Delta_1' = \Delta_1 \circ \sigma \tag{88}$$
$$\Delta_2' = \Delta_2 \circ \sigma \tag{89}$$
$$\tag{90}$$

And

$$\Delta' = (\text{Apply})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v_1\,[\sigma] : A \to \mathtt{M}_\epsilon B} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash v_2\,[\sigma] : A}}{\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \mathtt{M}_\epsilon B} \tag{91}$$

Hence

$$\Delta' = \mathtt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Definition} \tag{92}$$
$$= \mathtt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} \tag{93}$$
$$= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} \tag{94}$$
$$= \Delta \circ \sigma \quad \text{By Definition} \tag{95}$$
$$\tag{96}$$

Case If By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v : \mathtt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \qquad \dfrac{\Delta_3}{\Gamma \vdash C_2 : \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A} \tag{97}$$

By induction we find $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta_1' = \Delta_1 \circ \sigma \tag{98}$$
$$\Delta_2' = \Delta_2 \circ \sigma \tag{99}$$
$$\Delta_3' = \Delta_3 \circ \sigma \tag{100}$$
$$\tag{101}$$

And

$$\Delta' = (\text{If})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\,[\sigma] : \mathtt{Bool}} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash C_1\,[\sigma] : \mathtt{M}_\epsilon A} \qquad \dfrac{\Delta_3'}{\Gamma' \vdash C_2\,[\sigma] : \mathtt{M}_\epsilon A}}{\Gamma' \vdash (\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\ )\,[\sigma] : \mathtt{M}_\epsilon A} \tag{102}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\sigma = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{103}$$

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

. And hence, we have:

$$\texttt{cur}(f \circ (\texttt{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \texttt{cur}(f) \tag{104}$$

And so:

$$\begin{aligned}
\Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta_2' \circ \pi_2), \texttt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (105)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \texttt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (106)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma))] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (107)\\
&= \texttt{app} \circ (([(T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} && (108)\\
&= \texttt{app} \circ (((T_\epsilon A)^\sigma \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (109)\\
&= \texttt{app} \circ ((T_\epsilon A)^\sigma \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (110)\\
&= \texttt{app} \circ (\texttt{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (T_\epsilon A)^\sigma && (111)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (112)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \sigma \quad \text{By Definition of the diagonal morphism.} && (113)\\
&= \Delta \circ \sigma && (114)
\end{aligned}$$

Case Bind By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash C_1 \colon \texttt{M}_\epsilon A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash C_1 \colon \texttt{M}_\epsilon B}}{\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{115}$$

By property 3,

$$(\Gamma', x : A) \vdash (\sigma, x := x \colon (\Gamma, x : A) \tag{116}$$

With denotation (extension lemma)

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x \colon (\Gamma, x : A)]\!] = \sigma \times \texttt{Id}_A \tag{117}$$

By induction, we derive Δ_1', Δ_2' such that:

$$\begin{aligned}
\Delta_1' &= \Delta_1 \circ \sigma && (118)\\
\Delta_2' &= \Delta_2 \circ (\sigma \times \texttt{Id}_A) \quad \text{By Extension Lemma} && (119)
\end{aligned}$$

And:

$$\Delta' = (\text{Bind})\frac{\dfrac{\Delta'_1}{\Gamma' \vdash C_1\,[\sigma] : \mathtt{M}_\epsilon A} \qquad \dfrac{\Delta'_2}{\Gamma', x : A \vdash C_1\,[\sigma] : \mathtt{M}_\epsilon B}}{\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2\ )\,[\sigma] : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{120}$$

Hence:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta'_2 \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} && (121)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} && (122)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} && (123)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} && (124)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} && (125)\\
&= \Delta \circ \sigma \quad \text{By Defintion} && (126)\\
& && (127)
\end{aligned}$$

Case Subeffect By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-effect})\frac{\dfrac{\Delta_1}{\Gamma \vdash C : \mathtt{M}_{\epsilon_1} A} \qquad (\text{Computation})\dfrac{\epsilon_1 \leq_\Phi \epsilon_2 \qquad A \leq:_\Phi B}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash C : \mathtt{M}_{\epsilon_2} B} \tag{128}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-effect})\frac{\dfrac{\Delta'_1}{\Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_1} A} \qquad A \leq: B}{\epsilon_1 \leq \epsilon_2} \Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_2} B \tag{129}$$

Hence, Let

$$h = [\![\epsilon_1 \leq \epsilon_2]\!] \tag{130}$$
$$g = [\![A \leq: B]\!] \tag{131}$$

$$\begin{aligned}
\Delta' &= h_B \circ T_{\epsilon_1} g \circ \Delta'_1 \quad \text{By definition} && (132)\\
&= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \sigma \quad \text{By induction} && (133)\\
&= \Delta \circ \sigma \quad \text{By definition} && (134)\\
& && (135)
\end{aligned}$$

0.4 The Identity Substitution

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma, x:A} = (I_\Gamma, x := x)$

0.4.1 Properties of the Identity Substitution

Property 1 If $\Gamma \mathtt{Ok}$ then $\Gamma \vdash I_\Gamma \colon \Gamma$, proved trivially by induction over the well formed-ness relation.

Property 2 $[\![\Gamma \vdash I_\Gamma \colon \Gamma]\!] = \mathtt{Id}_\Gamma$, proved trivially by induction over the definition of I_Γ

0.5 Single Substitution

If $\Gamma \vdash v \colon A$, let the single substitution $\Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{136}$$

Then by properties 1, 2 of the identity substitution, we have:

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!] = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!] \rangle : \Gamma \to (\Gamma \times A) \tag{137}$$

0.5.1 The Semantics of Single Substitution

The following diagram commutes:

$$[\![\Gamma \vdash t\,[v/x] : \tau]\!] = [\![\Gamma, x : A \vdash t \colon \tau]\!] \circ \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!] \rangle \tag{138}$$

TODO: Again, there is code here to draw a Commutative diagram, but for some reason pdflatex hangs when compiling it Since $[\![\Gamma \vdash (I_\Gamma, x := v) \colon (\Gamma, x : A)]\!] = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!] \rangle$ And $\mathtt{true}\,[v/x] = \mathtt{true}\,[I_\Gamma, x := v]$