

0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule. **TODO: No-lambda?**

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t : \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t : \tau$, there exists at most one reduced derivation of $\Gamma \vdash t : \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Variables

To find the unique derivation of $\Gamma \vdash x : A$, we case split on the type-environment, Γ .

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x : A$ is, if $A' \leq A$, as below:

$$\text{(Subtype)} \frac{(\text{Var}) \frac{\Gamma', x : A' \text{Ok}}{\Gamma, x : A' \vdash x : A'} \quad A' \leq A}{\Gamma', x : A' \vdash x : A} \quad (1)$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Gamma \vdash x : A$ holds, then so must $\Gamma' \vdash x : A$.

Let

$$\text{(Subtype)} \frac{() \frac{\Delta}{\Gamma' \vdash x : A'} \quad A' \leq A}{\Gamma' \vdash x : A} \quad (2)$$

Be the unique reduced derivation of $\Gamma' \vdash x : A$.

Then the unique reduced derivation of $\Gamma \vdash x : A$ is:

$$\text{(Subtype)} \frac{(\text{Weaken}) \frac{() \frac{\Delta}{\Gamma, x : A' \vdash x : A'} \quad A' \leq A}{\Gamma \vdash x : A'}}{\Gamma \vdash x : A} \quad (3)$$

0.2.2 Constants

For each of the constants, (\mathbb{C}^A , **true**, **false**, $()$), there is exactly one possible derivation for $\Gamma \vdash c : A$ for a given A. I shall give examples using the case \mathbb{C}^A

$$\text{(Subtype)} \frac{(\text{Const}) \frac{\Gamma \text{Ok}}{\Gamma \vdash \mathbb{C}^A : A} \quad A \leq B}{\Gamma \vdash \mathbb{C}^A : B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq A$).

0.2.3 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C : A' \rightarrow \mathbb{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)} \frac{(\text{Lambda}) \frac{() \frac{\Delta}{\Gamma, x : A \vdash C : \mathbb{M}_{\epsilon} B}}{\Gamma \vdash \lambda x : A.B : A \rightarrow \mathbb{M}_{\epsilon} B} \quad A \rightarrow \mathbb{M}_{\epsilon} B \leq A' \rightarrow \mathbb{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C : A' \rightarrow \mathbb{M}_{\epsilon'} B'}$$

Where

$$\text{(Sub-Effect)} \frac{() \frac{\Delta}{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B} \quad B \leq B' \quad \epsilon \leq \epsilon'}{\Gamma, x : A \vdash C : \mathbb{M}_{\epsilon'} B'} \quad (4)$$

is the reduced derivation of $\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B$ if it exists.

Case Subtype **TODO:** Do we need to write anything here? (Probably needs an explanation)

0.2.4 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \text{return } v : \mathbb{M}_\epsilon B$ is

$$\text{(Subtype)} \frac{(\text{Return}) \frac{() \frac{\Delta}{\Gamma \vdash v : A} \quad A \leq B \quad 1 \leq \epsilon}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A}}{\Gamma \vdash \text{return } v : \mathbb{M}_\epsilon B}$$

Where

$$\text{(Subtype)} \frac{() \frac{\Delta}{\Gamma \vdash v : A} \quad A \leq B}{\Gamma \vdash v : B}$$

is the reduced derivation of $\Gamma \vdash v : B$

Case Apply If

$$\text{(Subtype)} \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B} \quad A \rightarrow \mathbb{M}_\epsilon B \leq A' \rightarrow \mathbb{M}_{\epsilon'} B'}{\Gamma \vdash v_1 : A' \rightarrow \mathbb{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)} \frac{() \frac{\Delta'}{\Gamma \vdash v_2 : A''} \quad A'' \leq A'}{\Gamma \vdash v_2 : A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1 : A' \rightarrow \mathbb{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2 : A'$

Then we can construct the reduced derivation of $\Gamma \vdash v_1 \quad v_2 : \mathbb{M}_{\epsilon'} B'$ as

$$\text{(Subeffect)} \frac{(\text{Apply}) \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B} \quad (\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash v_2 : A''} \quad A'' \leq A'}{\Gamma \vdash v_2 : A} \quad B \leq B' \quad \epsilon \leq \epsilon'}{\Gamma \vdash v_1 \quad v_2 : \mathbb{M}_{\epsilon'} B'}}$$

Case If Let

$$\text{(Subtype)} \frac{() \frac{\Delta}{\Gamma \vdash v : B} \quad B \leq \text{Bool}}{\Gamma \vdash v : \text{Bool}} \quad (5)$$

$$\text{(Subeffect)} \frac{() \frac{\Delta'}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'} A'} \quad A' \leq A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 : \mathbb{M}_\epsilon A} \quad (6)$$

$$\text{(Subeffect)} \frac{() \frac{\Delta''}{\Gamma \vdash C_2 : \mathbb{M}_{\epsilon''} A''} \quad A'' \leq A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 : \mathbb{M}_\epsilon A} \quad (7)$$

Be the unique reduced reduced derivations of $\Gamma \vdash v : \text{Bool}$, $\Gamma \vdash C_1 : \mathbb{M}_\epsilon A$, $\Gamma \vdash C_2 : \mathbb{M}_\epsilon A$.

Then the only reduced derivation of $\Gamma \vdash \text{if}_{\epsilon, A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A$ is:

TODO: Scale this properly

$$\text{(Subtype)} \frac{(\text{If}) \frac{(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v : B} \quad B \leq \text{Bool}}{\Gamma \vdash v : \text{Bool}} \quad (\text{Subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'} A'} \quad A' \leq A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 : \mathbb{M}_\epsilon A} \quad (\text{Subeffect}) \frac{() \frac{\Delta''}{\Gamma \vdash C_2 : \mathbb{M}_{\epsilon''} A''} \quad A'' \leq A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 : \mathbb{M}_\epsilon A} \quad \epsilon \leq \epsilon \quad A \leq A}{\Gamma \vdash \text{if}_{\epsilon, A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A}}{\Gamma \vdash \text{if}_{\epsilon, A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A} \quad (8)$$

Case Bind Let

$$\text{(Subeffect)} \frac{() \frac{\Delta}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A} \quad A \leq : A' \quad \epsilon_1 \leq \epsilon'_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'_1} A'} \quad (9)$$

$$\text{(Subeffect)} \frac{() \frac{\Delta'}{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B} \quad B \leq : B' \quad \epsilon_2 \leq \epsilon'_2}{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon'_2} B'} \quad (10)$$

Be the respective unique reduced type derivations of the subterms]

By weakening, $\iota \times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2 : \mathbb{M}_{\epsilon} B$, there's also one of $\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon} B$.

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon'_1$ and $\epsilon_2 \leq \epsilon'_2$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon'_1 \cdot \epsilon'_2$

Hence the reduced type derivation of $\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C - 2 : \mathbb{M}_{\epsilon'_1 \cdot \epsilon'_2} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Subeffect)} \frac{\text{(Bind)} \frac{\text{(Subeffect)} \frac{() \frac{\Delta}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A} \quad A \leq : A' \quad \epsilon_1 \leq \epsilon'_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'_1} A'} \quad \text{(Subeffect)} \frac{() \frac{\Delta'}{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B} \quad B \leq : B' \quad \epsilon_2 \leq \epsilon'_2}{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon'_2} B'}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C - 2 : \mathbb{M}_{\epsilon'_1 \cdot \epsilon'_2} B'} \quad (11)$$

Case Subeffect **TODO: Do I want to talk about this?**

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed. **TODO: Fill in these cases with actual maths**

0.3.1 Constants

For the constants **true**, **false**, \mathcal{C}^A , etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$$\text{reduce}((\text{Const}) \frac{\Gamma 0k}{\Gamma \vdash \mathcal{C}^A : A}) = (\text{Const}) \frac{\Gamma 0k}{\Gamma \vdash \mathcal{C}^A : A}$$

0.3.2 Value Types

Var

$$\text{reduce}((\text{Var}) \frac{\Gamma 0k}{\Gamma, x : A \vdash x : A}) = (\text{Var}) \frac{\Gamma 0k}{\Gamma, x : A \vdash x : A} \quad (12)$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$\text{reduce}((\text{Weaken}) \frac{() \frac{\Delta}{\Gamma \vdash x : A}}{\Gamma, y : B \vdash x : A}) \quad (13)$$

Let

$$\text{(Subtype)} \frac{() \frac{\Delta'}{\Gamma \vdash x : A} \quad A' \leq : A}{\Gamma \vdash x : A} = \text{reduce}(\Delta) \quad (14)$$

In

$$\text{(Subtype)} \frac{\text{(Weaken)} \frac{() \frac{\Delta'}{\Gamma \vdash x:A'}}{\Gamma, y:B \vdash x:A'} \quad A' \leq A}{\Gamma, y:B \vdash x:A} \quad (15)$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$\llbracket \text{(Weaken)} \frac{() \frac{\Delta}{\Gamma \vdash x:A}}{\Gamma, y:B \vdash x:A} \rrbracket_M = \Delta \circ \pi_1 \quad (16)$$

Similarly, the denotation of the reduced denotation is:

$$\llbracket \text{(Subtype)} \frac{\text{(Weaken)} \frac{() \frac{\Delta'}{\Gamma \vdash x:A'}}{\Gamma, y:B \vdash x:A'} \quad A' \leq A}{\Gamma, y:B \vdash x:A} \rrbracket_M = \llbracket A' \leq A \rrbracket_M \circ \Delta' \circ \pi_1 \quad (17)$$

By induction on *reduce* preserving denotations and the reduction of Δ (14), we have:

$$\Delta = \llbracket A' \leq A \rrbracket_M \circ \Delta' \quad (18)$$

So the denotations of the un-reduced and reduced derivations are equal.

Lambda

reduce **definition** To find:

$$\text{reduce}((\text{Fn}) \frac{() \frac{\Delta}{\Gamma, x:A \vdash C: \mathbb{M}_{\epsilon_2} B}}{\Gamma \vdash \lambda x : A.C : A \rightarrow \mathbb{M}_{\epsilon_2} B}) \quad (19)$$

Let

$$\text{(Sub-effect)} \frac{() \frac{\Delta'}{\Gamma, x:A \vdash C: \mathbb{M}_{\epsilon_1} B'} \quad \epsilon_1 \leq \epsilon_2 \quad B' \leq B}{\Gamma, x : A \vdash C: \mathbb{M}_{\epsilon_2} B} = \text{reduce}(\Delta) \quad (20)$$

In

$$\text{(Sub-type)} \frac{\text{(Fn)} \frac{\Delta'}{\Gamma, x:A \vdash C: \mathbb{M}_{\epsilon_1} B'} \quad A \rightarrow \mathbb{M}_{\epsilon_1} B' \leq A \rightarrow \mathbb{M}_{\epsilon_2} B}{\Gamma \vdash \lambda x : A.C : A \rightarrow \mathbb{M}_{\epsilon_2} B} \quad (21)$$

Preserves Denotation Let

$$f = \llbracket \mathbb{M}_{\epsilon_1} B' \leq \mathbb{M}_{\epsilon_2} B \rrbracket_M = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_{M,B} \circ T_{\epsilon_1}(\llbracket B' \leq B \rrbracket_M) \quad (22)$$

$$\llbracket A \rightarrow \mathbb{M}_{\epsilon_1} B' \leq A \rightarrow \mathbb{M}_{\epsilon_2} B \rrbracket_M = f^A = \text{cur}(f \circ \text{app}) \quad (23)$$

Then

$$\text{before} = \text{cur}(\Delta) \quad \text{By definition} \quad (24)$$

$$= \text{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta \quad (25)$$

$$= f^A \circ \text{cur}(\Delta') \quad \text{By the property of } f^X \circ \text{cur}(g) = \text{cur}(f \circ g) \quad (26)$$

$$= \text{after} \quad \text{By definition} \quad (27)$$

$$(28)$$

Subtype

reduce **definition** To find:

$$reduce((\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v:A} \quad A \leq B}{\Gamma \vdash v:B}) \quad (29)$$

Let

$$(\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash x:A} \quad A' \leq A}{\Gamma \vdash x:A} = reduce(\Delta) \quad (30)$$

In

$$(\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash v:A'} \quad A' \leq A \leq B}{\Gamma \vdash v:B} \quad (31)$$

Preserves Denotation

$$before = \llbracket A \leq B \rrbracket_M \circ \Delta \quad (32)$$

$$= \llbracket A \leq B \rrbracket_M \circ (\llbracket A' \leq A \rrbracket_M \circ \Delta') \quad \text{by Denotation of reduction of } \Delta. \quad (33)$$

$$= \llbracket A' \leq B \rrbracket_M \circ \Delta' \quad \text{Subtyping relations are unique} \quad (34)$$

$$= after \quad (35)$$

$$(36)$$

0.3.3 Computation Types

Return

reduce **definition** To find:

$$reduce((\text{Return}) \frac{() \frac{\Delta}{\Gamma \vdash v:A}}{\Gamma \vdash \text{return } v: \mathbf{M}_1 A}) \quad (37)$$

Let

$$(\text{Sub-type}) \frac{() \frac{\Delta'}{\Gamma \vdash v:A'} \quad A' \leq A}{\Gamma \vdash v:A} = reduce(\Delta) \quad (38)$$

In

$$(\text{Sub-effect}) \frac{(\text{Return}) \frac{\Delta'}{\Gamma \vdash v:A} \quad 1 \leq 1 \quad A' \leq A}{\Gamma \vdash \text{return } v: \mathbf{M}_1 A} \quad (39)$$

Then

$$before = \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} \quad (40)$$

$$= \eta_A \circ \llbracket A' \leq A \rrbracket_M \circ \Delta' \quad \text{BY reduction of } \Delta \quad (41)$$

$$= T_1 \llbracket A' \leq A \rrbracket_M \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta \quad (42)$$

$$= \llbracket 1 \leq 1 \rrbracket_{M,A} \circ T_1 \llbracket A' \leq A \rrbracket_M \circ \eta_{A'} \circ \Delta' \quad \text{Since } \llbracket 1 \leq 1 \rrbracket_M \text{ is the identity Nat-Trans} \quad (43)$$

$$= after \quad \text{By definition} \quad (44)$$

$$(45)$$

Apply

reduce **definition** To find:

$$reduce((Apply) \frac{() \frac{\Delta_1}{\Gamma \vdash v_1 : A \rightarrow M_\epsilon B}}{\Gamma \vdash v_1} \quad \frac{() \frac{\Delta_2}{\Gamma \vdash v_2 : A}}{v_2 : M_\epsilon B}) \quad (46)$$

Let

$$(Subtype) \frac{() \frac{\Delta'_1}{\Gamma \vdash v_1 : A' \rightarrow M_{\epsilon'} B'}}{\Gamma \vdash v_1 : A \rightarrow M_\epsilon B} \quad A' \rightarrow M_{\epsilon'} B' \leq A \rightarrow M_\epsilon B = reduce(\Delta_1) \quad (47)$$

$$(Subtype) \frac{() \frac{\Delta'_2}{\Gamma \vdash v : A'}}{\Gamma \vdash v_1 : A} \quad A' \leq A = reduce(\Delta_2) \quad (48)$$

In

$$(Sub-effect) \frac{(Apply) \frac{() \frac{\Delta'_1}{\Gamma \vdash v_1 : A' \rightarrow M_{\epsilon'} B'}}{\Gamma \vdash v_1} \quad (Sub-type) \frac{() \frac{\Delta'_2}{\Gamma \vdash v_2 : A''} \quad A'' \leq A \leq A'}{\Gamma \vdash v_2 : A'}}{\Gamma \vdash v_1 \quad v_2 : M_{\epsilon'} B'} \quad \epsilon' \leq \epsilon \quad B' \leq B}{\Gamma \vdash v_1 \quad v_2 : M_\epsilon B} \quad (49)$$

Preserves Denotation Let

$$f = \llbracket A \leq A' \rrbracket_M : A \rightarrow A' \quad (50)$$

$$f' = \llbracket A'' \leq A \rrbracket_M : A'' \rightarrow A \quad (51)$$

$$g = \llbracket B' \leq B \rrbracket_M : B' \rightarrow B \quad (52)$$

$$h = \llbracket \epsilon' \leq \epsilon \rrbracket_M : T_{\epsilon'} \rightarrow T_\epsilon \quad (53)$$

Hence

$$\llbracket A' \rightarrow M_{\epsilon'} B' \leq A \rightarrow M_\epsilon B \rrbracket_M = (h_B \circ T_{\epsilon'} g)^A \circ (T_{\epsilon'} B')^f \quad (54)$$

$$= \text{cur}(h_B \circ T_{\epsilon'} g \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id} \times f)) \quad (55)$$

$$= \text{cur}(h_B \circ T_{\epsilon'} g \circ \text{app} \circ (\text{Id} \times f)) \quad (56)$$

Then

$$before = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} \quad (57)$$

$$= \text{app} \circ \langle \text{cur}(h_B \circ T_{\epsilon'} g \circ \text{app} \circ (\text{Id} \times f)) \circ \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 \quad (58)$$

$$= \text{app} \circ (\text{cur}(h_B \circ T_{\epsilon'} g \circ \text{app} \circ (\text{Id} \times f)) \times \text{Id}_A) \circ \langle \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{Factoring out} \quad (59)$$

$$= h_B \circ T_{\epsilon'} g \circ \text{app} \circ (\text{Id} \times f) \circ \langle \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{By the exponential property} \quad (60)$$

$$= h_B \circ T_{\epsilon'} g \circ \text{app} \circ \langle \Delta'_1, f \circ f' \circ \Delta'_2 \rangle \quad (61)$$

$$= after \quad \text{By definition} \quad (62)$$

If

reduce **definition**

$$reduce((If) \frac{() \frac{\Delta_1}{\Gamma \vdash v : \text{Bool}}}{\Gamma \vdash \text{if}_{\epsilon, A} v} \quad \frac{() \frac{\Delta_2}{\Gamma \vdash C_1 : M_\epsilon A}}{C_1} \quad \text{then} \quad \frac{() \frac{\Delta_3}{\Gamma \vdash C_2 : M_\epsilon A}}{C_2 : M_\epsilon A} \quad \text{else} \quad C_2 : M_\epsilon A) = (If) \frac{() \frac{reduce(\Delta_1)}{\Gamma \vdash v : \text{Bool}}}{\Gamma \vdash \text{if}_{\epsilon, A} v} \quad \frac{() \frac{reduce(\Delta_2)}{\Gamma \vdash C_1 : M_\epsilon A}}{C_1} \quad \text{then} \quad \frac{() \frac{reduce(\Delta_3)}{\Gamma \vdash C_2 : M_\epsilon A}}{C_2 : M_\epsilon A} \quad \text{else} \quad C_2 : M_\epsilon A \quad (63)$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Bind

reduce definition To find

$$reduce((Bind) \frac{() \frac{\Delta_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A}}{\Gamma \vdash \text{do } x \leftarrow C_1} \frac{() \frac{\Delta_2}{\Gamma, x:A \vdash C_2 : \mathbb{M}_{\epsilon_2} B}}{\text{in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B}) \quad (64)$$

Let

$$(Sub-effect) \frac{() \frac{\Delta'_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'_1} A'} \quad \epsilon'_1 \leq \epsilon_1 \quad A' \leq A}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A} = reduce(\Delta_1) \quad (65)$$

Since $i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq A$, and by Δ_2 , $(\Gamma, x : A) \vdash C_2 : \mathbb{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $(\Gamma, x : A') \vdash C_2 : \mathbb{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(Sub-effect) \frac{() \frac{\Delta'_3}{\Gamma, x:A' \vdash C_2 : \mathbb{M}_{\epsilon'_2} B'} \quad \epsilon'_2 \leq \epsilon_2 \quad B' \leq B}{\Gamma, x : A' \vdash C_2 : \mathbb{M}_{\epsilon_2} B} = reduce(\Delta_3) \quad (66)$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon'_1$ and $\epsilon_2 \leq \epsilon'_2$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon'_1 \cdot \epsilon'_2$. Then the result of reduction of the whole bind expression is:

$$(Sub-effect) \frac{(Bind) \frac{() \frac{\Delta'_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon'_1} A'} \quad \epsilon'_1 \leq \epsilon_1}{\Gamma \vdash \text{do } x \leftarrow C_1} \frac{() \frac{\Delta'_3}{\Gamma, x:A' \vdash C_2 : \mathbb{M}_{\epsilon'_2} B'} \quad \epsilon'_2 \leq \epsilon_2 \quad B' \leq B}{\text{in } C_2 : \mathbb{M}_{\epsilon'_1 \cdot \epsilon'_2} B}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \epsilon'_1 \cdot \epsilon'_2 \leq \epsilon_1 \cdot \epsilon_2 \quad (67)$$

Preserves Denotation **TODO:** Recursively call reduce, then leave tree otherwise unchanged.

Let

$$f = \llbracket A' \leq A \rrbracket_M : A' \rightarrow A \quad (68)$$

$$g = \llbracket B' \leq B \rrbracket_M : B' \rightarrow B \quad (69)$$

$$h_1 = \llbracket \epsilon'_1 \leq \epsilon_1 \rrbracket_M : T_{\epsilon'_1} \rightarrow T_{\epsilon_1} \quad (70)$$

$$h_2 = \llbracket \epsilon'_2 \leq \epsilon_2 \rrbracket_M : T_{\epsilon'_2} \rightarrow T_{\epsilon_2} \quad (71)$$

$$h = \llbracket \epsilon'_1 \cdot \epsilon'_2 \leq \epsilon_1 \cdot \epsilon_2 \rrbracket_M : T_{\epsilon'_1 \cdot \epsilon'_2} \rightarrow T_{\epsilon_1 \cdot \epsilon_2} \quad (72)$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2 , we have

$$\Delta_3 = \Delta_2 \circ (\text{Id}_\Gamma \times f) \quad (73)$$

And due to the reduction of Δ_3 , we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3 \quad (74)$$

So:

$$before = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, \Delta_1 \rangle \quad \text{By definition.} \quad (75)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, h_{1, A} \circ T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{By reduction of } \Delta_1. \quad (76)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ (\text{Id}_\Gamma \times h_{1, A}) \circ \langle \text{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Factor out } h_1 \quad (77)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ h_{1, (\Gamma \times A)} \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Tensor strength and sub-effecting } h_1 \quad (78)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} \Delta_2 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Naturality of } h_1 \quad (79)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} \Delta_2 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ (\text{Id}_\Gamma \times T_{\epsilon'_1} f) \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{Factor out pairing again} \quad (80)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} (\Delta_2 \circ (\text{Id}_\Gamma \times f)) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{Tensorstrength} \quad (81)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} (\Delta_3) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{By the definition of } \Delta_3 \quad (82)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} (h_{2, B} \circ T_{\epsilon'_2} g \circ \Delta'_3) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{By the reduction of } \Delta_3 \quad (83)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1, B} \circ T_{\epsilon'_1} h_{2, B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{Factor out the functor} \quad (84)$$

$$= h_B \circ \mu_{\epsilon'_1, \epsilon'_2, B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{By the } \mu \text{ and Sub-effect rule} \quad (85)$$

$$= h_B \circ T_{\epsilon'_1 \cdot \epsilon'_2} g \circ \mu_{\epsilon'_1, \epsilon'_2, B'} \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_\Gamma, \Delta'_1 \rangle \quad \text{By naturality of } \mu, \quad (86)$$

$$= after \quad \text{By definition} \quad (87)$$

Subeffect

reduce **definition** To find:

$$reduce((\text{Subeffect}) \frac{() \frac{\Delta}{\Gamma \vdash C : \mathbb{M}_{\epsilon'} B'}{\epsilon' \leq \epsilon} \quad B' \leq B}{\Gamma \vdash C : \mathbb{M}_\epsilon B}) \quad (88)$$

Let

$$(\text{Subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C : \mathbb{M}_{\epsilon''} B''} \quad \epsilon'' \leq \epsilon' \quad \text{Bool}'' \leq B}{\Gamma \vdash C : \mathbb{M}_{\epsilon'} B} = reduce(\Delta) \quad (89)$$

in

$$(\text{subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C : \mathbb{M}_{\epsilon''} B''} \quad \epsilon'' \leq \epsilon \quad B'' \leq B}{\Gamma \vdash C : \mathbb{M}_\epsilon B} \quad (90)$$

Preserves Denotation Let

$$f = \llbracket B' \leq B \rrbracket_M \quad (91)$$

$$g = \llbracket B'' \leq B' \rrbracket_M \quad (92)$$

$$h_1 = \llbracket \epsilon' \leq \epsilon \rrbracket_M \quad (93)$$

$$h_2 = \llbracket \epsilon' \leq \epsilon' \rrbracket_M \quad (94)$$

$$f \circ g = \llbracket B'' \leq B \rrbracket_M \quad (95)$$

$$h_1 \circ h_2 = \llbracket \epsilon'' \leq \epsilon' \rrbracket_M \quad (96)$$

$$(97)$$

Hence we can find the denotation of the derivation before reduction.

$$before = h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} \tag{98}$$

$$= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta \tag{99}$$

$$= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \quad = after \quad \text{By definition.} \tag{100}$$

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t : \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $\llbracket \Delta \rrbracket_M = \llbracket reduce \Delta \rrbracket_M = \llbracket reduce \Delta' \rrbracket_M = \llbracket \Delta' \rrbracket_M$, hence the denotation $\llbracket \Gamma \vdash t : \tau \rrbracket_M$ is unique.