0.1 Terms

Making the language no-longer differentiate between values and computations.

0.1.1 Value Terms

$$
\begin{aligned}
v ::= & x \\
& \mid \lambda x : A.v \\
& \mid \mathtt{C}^A \\
& \mid \mathtt{()} \\
& \mid \mathtt{true} \mid \mathtt{false} \\
& \mid \Lambda\alpha.v \\
& \mid v\ \epsilon \\
& \mid \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \\
& \mid v_1\ v_2 \\
& \mid \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \\
& \mid \mathtt{return}\ v
\end{aligned}
\tag{1}
$$

0.2 Type System

0.2.1 Ground Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, \mathtt{1}, \leq)$ with ground elements e.

0.2.2 Effect Po-Monoid Under an Effect Environment

Derive a new Po-Monoid for each Φ:

$$(E_\Phi, \cdot_\Phi, \mathtt{1}, \leq_\Phi) \tag{2}$$

Where meta-variables, ϵ, range over E_Φ Where

$$E_\Phi = E \cup \{\alpha \mid \alpha \in \Phi\} \tag{3}$$

And

$$\frac{\epsilon_3 = \epsilon_1 \cdot \epsilon_2}{\epsilon_3 = \epsilon_1 \cdot_\Phi \epsilon_2} \tag{4}$$

Otherwise, $\cdot_\Phi$ is symbolic in nature.

$$\epsilon_1 \leq_\Phi \epsilon_2 \Leftrightarrow \forall \sigma \downarrow . \epsilon_1 [\sigma \downarrow] \leq \epsilon_2 [\sigma \downarrow] \tag{5}$$

Where $\sigma \downarrow$ denotes any ground-substitution of Φ. That is any substitution of all effect-variables in Φ to ground effects. Where it is obvious from the context, I shall use $\leq$ instead of $\leq_\Phi$.

0.2.3 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Term Types

$$A, B, C ::= \gamma \mid A \to B \mid \mathtt{M}_\epsilon A \mid \forall \alpha . A$$

0.2.4 Type and Effect Environments

A type environment is a snoc-list of term-variable, type pairs, $G ::= \diamond \mid \Gamma, x : A$. An effect environment is a snoc-list of effect-variables.

$\Phi ::= \diamond \mid \Phi, \alpha$

Domain Function on Type Environments

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

Membership of Effect Environments Informally, $\alpha \in \Phi$ if α appears in the list represented by Φ.

$\mathtt{Ok}$ Predicate On Effect Environments

- (Atom) $\dfrac{}{\diamond \mathtt{Ok}}$
- (A) $\dfrac{\Phi \mathtt{Ok}}{\Phi, \alpha \mathtt{Ok}}$ (if $\alpha \notin \Phi$)

Well-Formed-ness of effects We define a relation $\Phi \vdash \epsilon$.

- (Ground) $\dfrac{\Phi \mathtt{Ok}}{\Phi \vdash e}$
- (Var) $\dfrac{\Phi, \alpha \mathtt{Ok}}{\Phi, \alpha \vdash \alpha}$
- (Weaken) $\dfrac{\Phi \vdash \alpha}{\Phi, \beta \vdash \alpha}$ (if $\alpha \neq \beta$)
- (Monoid Op) $\dfrac{\Phi \vdash \epsilon_1 \qquad \Phi \vdash \epsilon_2}{\Phi \vdash \epsilon_1 \cdot \epsilon_2}$

Well-Formed-ness of Types We define a relation $\Phi \vdash \tau$ on types.

- (Ground) $\dfrac{}{\Phi \vdash \gamma}$
- (Lambda) $\dfrac{\Phi \vdash A \qquad \Phi \vdash B}{\Phi \vdash A \to B}$
- (Computation) $\dfrac{\Phi \vdash A \qquad \Phi \vdash \epsilon}{\Phi \vdash \mathtt{M}_\epsilon A}$
- (For-All) $\dfrac{\Phi, \alpha \vdash A}{\Phi \vdash \forall \alpha . A}$

Ok Predicate on Type Environments We now define a predicate on type environments and effect environments: $\Phi \vdash \Gamma\mathtt{Ok}$

- (Nil)$\dfrac{}{\Phi \vdash \diamond\mathtt{Ok}}$
- (Var)$\dfrac{\Phi \vdash \Gamma\mathtt{Ok} \qquad x \notin \mathtt{dom}(\Gamma)}{\Phi \vdash A}\Phi \vdash \Gamma, x : A\mathtt{Ok}$

0.2.5 Sub-typing

There exists a sub-typing pre-order relation $\leq:_\gamma$ over ground types that is:

- (Reflexive)$\dfrac{}{A \leq:_\gamma A}$
- (Transitive)$\dfrac{A \leq:_\gamma B \qquad B \leq:_\gamma C}{A \leq:_\gamma C}$

We extend this relation with the function and effect-lambda sub-typing rules to yield the full sub-typing relation under an effect environment, Φ, $\leq:_\Phi$

- (ground)$\dfrac{A \leq:_\gamma B}{A \leq:_\Phi B}$
- (Fn)$\dfrac{A \leq:_\Phi A' \qquad B' \leq:_\Phi B}{A' \to B' \leq:_\Phi A \to B}$
- (All)$\dfrac{A \leq:_\Phi A'}{\forall \alpha.A \leq:_\Phi \forall a.A'}$
- (Effect)$\dfrac{A \leq:_\Phi B \qquad \epsilon_1 \leq_\Phi \epsilon_2}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B}$

0.2.6 Type Rules

- (Const)$\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \vdash A}\Phi \mid \Gamma \vdash \mathtt{C}^A \colon A$
- (Unit)$\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{()} \colon \mathtt{Unit}}$
- (True)$\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}}$
- (False)$\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool}}$
- (Var)$\dfrac{\Phi \vdash \Gamma, x : A\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A}$
- (Weaken)$\dfrac{\Phi \mid \Gamma \vdash x \colon A}{\Phi \vdash B}$(if $\Phi \mid \Gamma, y : B \vdash x \colon A$)$x \neq y$

- (Fn) $\dfrac{\Phi \mid \Gamma, x : A \vdash v \colon \beta}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to B}$

- (Sub) $\dfrac{\Phi \mid \Gamma \vdash v \colon A \qquad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v \colon B}$

- (Effect-Abs) $\dfrac{\Phi, \alpha \mid \Gamma \vdash v \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A}$

- (Effect-apply) $\dfrac{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]}$

- (Return) $\dfrac{\Phi \mid \Gamma \vdash v \colon A}{\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_{\mathtt{1}} A}$

- (Apply) $\dfrac{\Phi \mid \Gamma \vdash v_1 \colon A \to \mathtt{M}_{\epsilon} B \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon} B}$

- (If) $\dfrac{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool} \qquad \Phi \mid \Gamma \vdash v_1 \colon A \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ V\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A}$

- (Do) $\dfrac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

0.2.7 Ok Lemma

If $\Phi \mid \Gamma \vdash t \colon \tau$ then $\Phi \vdash \Gamma\mathtt{Ok}$.

Proof If $\Gamma, x : A\mathtt{Ok}$ then by inversion $\Gamma\mathtt{Ok}$ Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Phi \vdash \Gamma\mathtt{Ok}$. And all non-axiom derivations preserve the `Ok` property.