

0.1 Introduce Substitutions

0.1.1 Substitutions as SNOc lists

$$\sigma ::= \diamond \mid \sigma, x := v \quad (1)$$

0.1.2 Trivial Properties of substitutions

$\text{fv}(\sigma)$

$$\text{fv}(\diamond) = \emptyset \quad (2)$$

$$\text{fv}(\sigma, x := v) = \text{fv}(\sigma) \cup \text{fv}(v) \quad (3)$$

$\text{dom}(\sigma)$

$$\text{dom}(\diamond) = \emptyset \quad (4)$$

$$\text{dom}(\sigma, x := v) = \text{dom}(\sigma) \cup \{x\} \quad (5)$$

$x \# \sigma$

$$x \# \sigma \Leftrightarrow x \notin (\text{fv}(\sigma) \cup \text{dom}(\sigma')) \quad (6)$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t[\sigma]$$

$$x[\diamond] = x \quad (7)$$

$$x[\sigma, x := v] = v \quad (8)$$

$$x[\sigma, x' := v'] = x[\sigma] \quad \text{If } x \neq x' \quad (9)$$

$$\mathbf{C}^A[\sigma] = \mathbf{C}^A \quad (10)$$

$$(\lambda x : A. C)[\sigma] = \lambda x : A. (C[\sigma]) \quad \text{If } x \# \sigma \quad (11)$$

$$(\text{if}_{\epsilon, A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2)[\sigma] = \text{if}_{\epsilon, A} \quad v[\sigma] \quad \text{then} \quad C_1[\sigma] \quad \text{else} \quad C_2[\sigma] \quad (12)$$

$$(v_1 \quad v_2)[\sigma] = (v_1[\sigma]) \quad v_2[\sigma] \quad (13)$$

$$(\text{do} \quad x \leftarrow C_1 \quad \text{in} \quad C_2) = \text{do} \quad x \leftarrow (C_1[\sigma]) \quad \text{in} \quad (C_2[\sigma]) \quad \text{If } x \# \sigma \quad (14)$$

$$(15)$$

0.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma : \Gamma$$

by:

- (Nil) $\frac{\Gamma' \mathbf{Ok}}{\Gamma' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Gamma' \vdash \sigma : \Gamma \quad x \notin \text{dom}(\Gamma) \quad \Gamma' \vdash v : A}{\Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma : \Gamma$ then: **TODO: Number these**

Property 1: ΓOk and $\Gamma' \text{Ok}$ Since $\Gamma' \text{Ok}$ holds by the Nil-axiom. ΓOk holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ **implies** $\Gamma'' \vdash \sigma : \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ , $\Gamma'' \vdash v : A$ holds if $\Gamma' \vdash v : A$ holds.

Property 3: $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma''))$ **implies** $(\Gamma', x : A) \vdash (\sigma, x := x) : (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma : \Gamma$$

In addition, $\Gamma', x : A \vdash x : A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) : (\Gamma, x : A) \quad (16)$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g : \tau \wedge \Gamma' \vdash \sigma : \Gamma \Rightarrow \Gamma' \vdash t[\sigma] : \tau \quad (17)$$

TODO: Proof by induction over type relation Assuming $\Gamma' \vdash \sigma : \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t : \tau \rightarrow \Gamma' \vdash t : \tau$

0.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x : A \quad (18)$$

So by inversion, since $\Gamma' \vdash \sigma : \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v : A \quad (19)$$

By the definition of the effect of substitutions, $x[\sigma] = v$, So

$$\Gamma' \vdash x[\sigma] : A \quad (20)$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken}) \frac{() \frac{\Delta}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x : A} \quad (21)$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma' : \Gamma'' \quad (22)$$

So by induction,

$$\Gamma' \vdash x[\sigma'] : A \quad (23)$$

And so by definition of the effect of σ , $x[\sigma] = x[\sigma']$

$$\Gamma' \vdash x[\sigma] : A \quad (24)$$

0.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn}) \frac{() \frac{\Delta}{\Gamma, x:A \vdash C : \mathbb{M}_\epsilon B}}{\Gamma \vdash \lambda x : A. C : A \rightarrow \mathbb{M}_\epsilon B} \quad (25)$$

Using alpha equivalence, we pick $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma', x : A) \vdash (\sigma, x := x) : \Gamma, x : A \quad (26)$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn}) \frac{() \frac{\Delta'}{\Gamma', x:A \vdash C[\sigma, x := x] : \mathbb{M}_\epsilon B}}{\Gamma \vdash \lambda x : A. C[\sigma, x := x] : A \rightarrow \mathbb{M}_\epsilon B} \quad (27)$$

Since $\lambda x : A. (C[\sigma, x := x]) = \lambda x : A. (C[\sigma]) = (\lambda x : A. C)[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x : A. C)[\sigma] : A \rightarrow \mathbb{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $()$, **true**, **false**, \mathbb{C}^A :

$\Gamma \vdash \sigma : \Gamma \Rightarrow \Gamma' 0k$ and:

$$\mathbb{C}^A[\sigma] = \mathbb{C}^A \quad (28)$$

So

$$(\text{Const}) \frac{\Gamma' 0k}{\Gamma' \vdash \mathbb{C}^A : A} \quad (29)$$

0.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return}) \frac{() \frac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A} \quad (30)$$

By induction, we have Δ'_1 such that

$$(\text{Return}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v[\sigma] : A}}{\Gamma' \vdash \text{return}(v[\sigma]) : \mathbb{M}_1 A} \quad (31)$$

Since $(\text{return } v)[\sigma] = \text{return}(v[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\text{return } v)[\sigma] : \mathbb{M}_1 A$.

Case Apply By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply}) \frac{() \frac{\Delta_1}{\Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B} \quad () \frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1 \ v_2 : \mathbb{M}_\epsilon B} \quad (32)$$

By induction on Δ_1, Δ_2 , we have Δ'_1, Δ'_2 such that

$$(\text{Apply}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v_1[\sigma] : A \rightarrow \mathbb{M}_\epsilon B} \quad () \frac{\Delta'_2}{\Gamma' \vdash v_2[\sigma] : A}}{\Gamma' \vdash (v_1 \ v_2)[\sigma] : \mathbb{M}_\epsilon B} \quad (33)$$

Since $(v_1 \ v_2)[\sigma] = (v_1[\sigma] \ v_2[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1 \ v_2)[\sigma] : \mathbb{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$(If) \frac{() \frac{\Delta_1}{\Gamma \vdash v: \mathbf{Bool}} \quad () \frac{\Delta_2}{\Gamma \vdash C_1: \mathbb{M}_\epsilon A} \quad () \frac{\Delta_3}{\Gamma \vdash C_2: \mathbb{M}_\epsilon A}}{\Gamma \vdash \mathbf{if}_{\epsilon, A} \quad v \quad \mathbf{then} \quad C_1 \quad \mathbf{else} \quad C_2: \mathbb{M}_\epsilon A} \quad (34)$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta'_1, \Delta'_2, \Delta'_3$ such that:

$$(If) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v[\sigma]: \mathbf{Bool}} \quad () \frac{\Delta'_2}{\Gamma' \vdash C_1[\sigma]: \mathbb{M}_\epsilon A} \quad () \frac{\Delta'_3}{\Gamma' \vdash C_2[\sigma]: \mathbb{M}_\epsilon A}}{\Gamma' \vdash \mathbf{if}_{\epsilon, A} \quad (v[\sigma]) \quad \mathbf{then} \quad (C_1[\sigma]) \quad \mathbf{else} \quad (C_2[\sigma]): \mathbb{M}_\epsilon A} \quad (35)$$

Since $(\mathbf{if}_{\epsilon, A} \quad v \quad \mathbf{then} \quad C_1 \quad \mathbf{else} \quad C_2)[\sigma] = \mathbf{if}_{\epsilon, A} \quad (v[\sigma]) \quad \mathbf{then} \quad (C_1[\sigma]) \quad \mathbf{else} \quad (C_2[\sigma])$
The derivation above holds for $\Gamma' \vdash (\mathbf{if}_{\epsilon, A} \quad v \quad \mathbf{then} \quad C_1 \quad \mathbf{else} \quad C_2)[\sigma]: \mathbb{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$(Bind) \frac{() \frac{\Delta_1}{\Gamma \vdash C_1: \mathbb{M}_{\epsilon_1} A} \quad () \frac{\Delta_2}{\Gamma, x: A \vdash C_2: \mathbb{M}_{\epsilon_2} B}}{\Gamma \vdash \mathbf{do} \quad x \leftarrow C_1 \quad \mathbf{in} \quad C_2: \mathbb{M}_{\epsilon_1, \epsilon_2} B} \quad (36)$$

Using alpha-equivalence, we pick $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x: A) \vdash (\sigma, x := x): (\Gamma, x: A)$$

By induction on Δ_1, Δ_2 , we have Δ'_1, Δ'_2 such that:

$$(Bind) \frac{() \frac{\Delta'_1}{\Gamma' \vdash C_1[\sigma]: \mathbb{M}_{\epsilon_1} A} \quad () \frac{\Delta'_2}{\Gamma', x: A \vdash C_2[\sigma, x := x]: \mathbb{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathbf{do} \quad x \leftarrow (C_1[\sigma]) \quad \mathbf{in} \quad (C_2[\sigma, x := x]): \mathbb{M}_{\epsilon_1, \epsilon_2} B} \quad (37)$$

Since $(\mathbf{do} \quad x \leftarrow C_1 \quad \mathbf{in} \quad C_2)[\sigma] = \mathbf{do} \quad x \leftarrow (C_1[\sigma]) \quad \mathbf{in} \quad (C_2[\sigma]) = \mathbf{do} \quad x \leftarrow (C_1[\sigma]) \quad \mathbf{in} \quad (C_2[\sigma, x := x])$
the above derivation holds for $\Gamma' \vdash (\mathbf{do} \quad x \leftarrow C_1 \quad \mathbf{in} \quad C_2)[\sigma]: \mathbb{M}_{\epsilon_1, \epsilon_2} B$

0.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$(\text{sub-type}) \frac{() \frac{\Delta}{\Gamma \vdash v: A} \quad A \leq: B}{\Gamma \vdash v: B} \quad (38)$$

By induction on Δ we derive Δ' such that:

$$(\text{sub-type}) \frac{() \frac{\Delta'}{\Gamma' \vdash v[\sigma]: A} \quad A \leq: B}{\Gamma \vdash v[\sigma]: B} \quad (39)$$

Case Sub-effect By inversion, there exists Δ such that

$$(\text{sub-effect}) \frac{() \frac{\Delta}{\Gamma \vdash C: \mathbb{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma \vdash C: \mathbb{M}_{\epsilon_2} B} \quad (40)$$

By induction on Δ we derive Δ' such that:

$$(\text{sub-effect}) \frac{() \frac{\Delta'}{\Gamma' \vdash C[\sigma]: \mathbb{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma' \vdash C[\sigma]: \mathbb{M}_{\epsilon_2} B} \quad (41)$$

0.3 Semantics of Substitution

0.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M : \Gamma' \rightarrow \Gamma \quad (42)$$

- (Nil) $\frac{\Gamma' 0k}{\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M = \langle \rangle_{\Gamma'}}$
- (Extend) $\frac{f = \llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \quad g = \llbracket \Gamma' \vdash v : A \rrbracket_M}{\llbracket \Gamma' \vdash (\sigma, x := v : (\Gamma, x : A)) \rrbracket_M = \langle f, g \rangle : \Gamma' \rightarrow (\Gamma \times A)}$

0.3.2 Extension Lemma

If $\Gamma' \vdash \sigma : \Gamma$ and $x \notin (\text{dom}(\Gamma') \cup \text{dom}(\Gamma))$ then the substitution in property 3 has denotation:

$$\llbracket (\Gamma', x : A) \vdash (\sigma, x := x) : (\Gamma, x : A) \rrbracket_M = (\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \times \text{Id}_A) \quad (43)$$

This holds since

$$\llbracket \Gamma', x : A \vdash x : A \rrbracket_M = \pi_2 \quad (44)$$

And $\iota\pi : (\Gamma', x : A) \triangleright \Gamma'$

$$\llbracket \iota\pi : (\Gamma', x : A) \triangleright \Gamma' \rrbracket_M = \pi_1 \quad (45)$$

So for each denotation $\llbracket \Gamma' \vdash v : B \rrbracket_M$ of each $y := v$ in σ , we can prepend the denotation with the weakening denotation to yield:

$$\llbracket \Gamma', x : A \vdash v : B \rrbracket_M = \llbracket \Gamma' \vdash v : B \rrbracket_M \circ \pi_1 \quad (46)$$

Since π_1 appears in every branch of $\llbracket \Gamma', x : A \vdash \sigma : \Gamma \rrbracket_M$, it can be factored out to yield:

$$\llbracket \Gamma', x : A \vdash \sigma : \Gamma \rrbracket_M = \llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \circ \pi_1 \quad (47)$$

Hence,

$$\llbracket (\Gamma', x : A) \vdash (\sigma, x := x) : \Gamma, x : A \rrbracket_M = \langle \llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \circ \pi_1, \pi_2 \rangle = (\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \times \text{Id}_A) \quad (48)$$

0.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma \vdash t : \tau$ and $\Gamma' \vdash \sigma : \Gamma$ then the derivation Δ' deriving $\Gamma' \vdash t[\sigma] : \tau$ satisfies:

$$\Delta' = \Delta \circ \llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M \quad (49)$$

This is proved by induction over the derivation of $\Gamma \vdash t : \tau$. We shall use σ to denote $\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M$ where it is clear from the context.

0.3.4 Proof For Value Terms

Case Var By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var}) \frac{\Gamma 0k}{\Gamma'', x : A \vdash x : A} \quad (50)$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v : A$.

Let

$$\sigma = \llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M = \langle \sigma', \Delta' \rangle \quad (51)$$

$$\Delta = \llbracket \Gamma'', x : A \vdash x : A \rrbracket_M = \pi_2 \quad (52)$$

$$(53)$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \quad (54)$$

$$= \Delta' \quad \text{By product property} \quad (55)$$

Case Weaken By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken}) \frac{() \frac{\Delta_1}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x : A} \quad (56)$$

Also by inversion of the well-formedness of $\Gamma' \vdash \sigma : \Gamma$, we have $\Gamma' \vdash \sigma' : \Gamma''$ and

$$\llbracket \Gamma' \vdash \sigma : \Gamma \rrbracket_M = \langle \llbracket \Gamma' \vdash \sigma' : \Gamma'' \rrbracket_M, \llbracket \Gamma' \vdash v : B \rrbracket_M \rangle \quad (57)$$

Hence by induction on Δ_1 we have Δ'_1 such that

$$() \frac{\Delta'_1}{\Gamma' \vdash x [\sigma] : A} \quad (58)$$

Hence

$$\Delta' = \Delta'_1 \quad \text{By definition} \quad (59)$$

$$= \Delta_1 \circ \sigma' \quad \text{By induction} \quad (60)$$

$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', \llbracket \Gamma' \vdash v : B \rrbracket_M \rangle \quad \text{By product property} \quad (61)$$

$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By definition of the denotation of } \sigma \quad = \Delta \circ \sigma \quad \text{By definition.} \quad (62)$$

Case Constants The logic for all constant terms (`true`, `false`, $()\mathbb{C}^A$) is the same. Let

$$c = \llbracket \mathbb{C}^A \rrbracket_M \quad (63)$$

$$\Delta' = c \circ \langle \rangle_{\Gamma'} \quad \text{By Definition} \quad (64)$$

$$= c \circ \langle \rangle_G \circ \sigma \quad \text{Terminal property} \quad (65)$$

$$= \Delta \circ \sigma \quad \text{By definition} \quad (66)$$

Case Lambda By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn}) \frac{() \frac{\Delta_1}{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B}}{\Gamma \vdash \lambda x : A. C : A \rightarrow \mathbb{M}_\epsilon B} \quad (67)$$

By induction of Δ_1 we have Δ'_1 such that

$$\Delta' = (\text{Fn}) \frac{() \frac{\Delta'_1}{\Gamma', x : A \vdash (C[\sigma]) : \mathbb{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A. C) [\sigma] : A \rightarrow \mathbb{M}_\epsilon B} \quad (68)$$

By induction and the extension lemma, we have:

$$\Delta'_1 = \Delta_1 \circ (\sigma \times \text{Id}_A) \quad (69)$$

Hence:

$$\Delta' = \text{cur}(\Delta'_1) \quad \text{By definition} \tag{70}$$

$$= \text{cur}(\Delta_1 \circ (\sigma \times \text{Id}_A)) \quad \text{By induction and extension lemma.} \tag{71}$$

$$= \text{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} \tag{72}$$

$$= \Delta \circ \sigma \quad \text{By Definition} \tag{73}$$

$$\tag{74}$$

Case Subtype By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type}) \frac{() \frac{\Delta_1}{\Gamma \vdash v : A} \quad A \leq B}{\Gamma \vdash v : B} \tag{75}$$

By induction on Δ_1 , we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v[\sigma] : A} \quad A \leq B}{\Gamma' \vdash v[\sigma] : B} \tag{76}$$

0.3.5 Proof For Computation Terms

Case Return

Case Apply

Case If

Case Bind

Case Subeffect

0.4 Single Substitution