0.1 CCC

The category at each index should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

Further more, it should have a co-product of the terminal object $\mathtt{1}$. This is required for the beta-eta equivalence of `if-then-else` terms.

$\mathtt{1} \xrightarrow{inl} A \xleftarrow{inr} \mathtt{1}$

For each:

$\mathtt{1} \xrightarrow{f} A \xleftarrow{g} \mathtt{1}$

There exists unique $[f, g] : \mathtt{1} + \mathtt{1} \to A$ such that:

$$\begin{tikzcd} & A & \\ \mathtt{1} \arrow[ur, "f"] \arrow[r, "\mathtt{inl}"'] & \mathtt{1}+\mathtt{1} \arrow[u, "{[f,g]}"] & \mathtt{1} \arrow[ul, "g"'] \arrow[l, "\mathtt{inr}"] \end{tikzcd}$$

0.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endo-functor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot\, \mathtt{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

0.2.1 Left Unit

$$\begin{tikzcd} T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon, \mathtt{1}, A}"] \\ & T_\epsilon A \end{tikzcd}$$

0.2.2 Right Unit

$$\begin{tikzcd} T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1}, \epsilon, A}"] \\ & T_\epsilon A \end{tikzcd}$$

0.2.3 Associativity

$$\begin{tikzcd}
T_{\epsilon_1}T_{\epsilon_2}T_{\epsilon_3}A \arrow[r, "\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3}A}"] \arrow[d, "T_{\epsilon_1}\mu_{\epsilon_2,\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2}T_{\epsilon_3}A \arrow[d, "\mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,A}"] \\
T_{\epsilon_1}T_{\epsilon_2\cdot\epsilon_3}A \arrow[r, "\mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}A
\end{tikzcd}$$

0.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$$

Satisfying the following rules:

0.3.1 Left Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{Id}_A \times T_\epsilon f"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A \times T_\epsilon B' \arrow[d, "\mathtt{t}_{\epsilon,A,B'}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(\mathtt{Id}_A \times f)"] & T_\epsilon(A \times B')
\end{tikzcd}$$

0.3.2 Right Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "f \times \mathtt{Id}_{T_\epsilon B}"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A' \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A',B}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(f \times \mathtt{Id}_B)"] & T_\epsilon(A' \times B)
\end{tikzcd}$$

0.3.3 Unitor Law

$$\begin{tikzcd}
\mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"'] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\
& T_\epsilon A
\end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutativity of π_2 with $\mathtt{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B)$$

And $\pi_{2,\mathtt{1}}$ is the left unitor, so by tensorial strength:

$$\begin{aligned}
T_\epsilon\pi_2 \circ \mathtt{t}_{\epsilon,A,B} &= T_\epsilon\pi_{2,\mathtt{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon,A,B} \\
&= T_\epsilon\pi_{2,1,B} \circ \mathtt{t}_{\epsilon,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_2
\end{aligned} \tag{1}$$

So the following commutes:

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{t}_{\epsilon,A,B}"] \arrow[dr, "\pi_2"] & T_\epsilon(A \times B) \arrow[d, "T_\epsilon \pi_2"] \\
& T_\epsilon B
\end{tikzcd}$$

0.3.4 Commutativity with Join

$$\begin{tikzcd}
A \times T_{\epsilon_1} T_{\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1} T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A\times B}"] \\
& A \times T_{\epsilon_1\cdot\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1\cdot\epsilon_2,A,B}"] & T_{\epsilon_1\cdot\epsilon_2}(A \times B)
\end{tikzcd}$$

0.4 Commutativity with Unit

$$\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_{\mathbf{1}} B \arrow[d, "\mathtt{t}_{\mathbf{1},A,B}"] \\
& T_{\mathbf{1}}(A \times B)
\end{tikzcd}$$

0.5 Commutativity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times T_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,T_\epsilon C}"] & & T_\epsilon((A \times B) \times C) \arrow[d, "T_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times T_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times T_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & T_\epsilon(A \times (B \times C))
\end{tikzcd}$$

TODO: Needed?

0.6 Sub-effecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

0.6.1 Sub-effecting and Tensor Strength

$$\begin{tikzcd}
A \times T_{\epsilon_1} B \arrow[r, "\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2} B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "[\![\epsilon_1 \leq \epsilon_2]\!]_{A\times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}$$

0.6.2 Sub-effecting and Monadic Join

Since the monoid operation on effects is monotone, we can introduce the following diagram.

$$
\begin{tikzcd}
T_{\epsilon_1}T_{\epsilon_2} \arrow[r, "T_{\epsilon_1}\llbracket \epsilon_2 \leq \epsilon_2' \rrbracket_M"] \arrow[d, "\mu_{\epsilon_1,\epsilon_2,}"] & T_{\epsilon_1}T_{\epsilon_2'} \arrow[r, "\llbracket \epsilon_1 \leq \epsilon_1' \rrbracket_{M,T_{\epsilon_2'}}"] & T_{\epsilon_1'}T_{\epsilon_2'} \arrow[d, "\mu_{\epsilon_1',\epsilon_2',}"] \\
T_{\epsilon_1 \cdot \epsilon_2} \arrow[rr, "\llbracket \epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \epsilon_2' \rrbracket_M"] & & T_{\epsilon_1' \cdot \epsilon_2'}
\end{tikzcd}
$$

0.7 Sub-typing

The denotation of ground types $\llbracket \_ \rrbracket_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function sub-typing to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$
\text{(Function Subtyping)}\frac{f = \llbracket A' \leq: A \rrbracket_M \quad g = \llbracket B \leq: B' \rrbracket_M}{rhs = \llbracket A \to B \leq: A' \to B' \rrbracket_M : (B)^A \to (B')^{A'}} \tag{2}
$$

$$
\begin{aligned}
rhs =& (g)^{A'} \circ (B)^f && (3) \\
=& \mathtt{cur}(g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{B^{A'}} \times f)) && (4) \\
& && (5)
\end{aligned}
$$