We need to define substutions of effects on effects, effects on types, effects on terms, terms on terms.

0.1 Effect Substitutions

Define a substitution, σ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \tag{1}$$

0.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \tag{2}$$

$$\sigma(e) = e \tag{3}$$
$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \tag{4}$$
$$\diamond(\alpha) = \alpha \tag{5}$$
$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \tag{6}$$
$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \tag{7}$$

0.1.2 Action of Effect Substitution on Types

Define the effect of applying an effect substitution, σ to a type τ as:

$$\tau\,[\sigma]$$

Defined as so **TODO: Define #**

$$\gamma\,[\sigma] = \gamma \tag{8}$$
$$(A \to \mathtt{M}_\epsilon B)\,[\sigma] = (A\,[\sigma]) \to \mathtt{M}_{\sigma(\epsilon)}(B\,[\sigma]) \tag{9}$$
$$(\mathtt{M}_\epsilon A)\,[\sigma] = \mathtt{M}_{\sigma(\epsilon)}(A\,[\sigma]) \tag{10}$$
$$(\forall \alpha.A)\,[\sigma] = \forall \alpha.(A\,[\sigma]) \quad \text{If } \alpha \# \sigma \tag{11}$$

0.1.3 Action of Effect Substitution on Terms

Define the effect of effect-substitution on terms:

$$x\,[\sigma] = x \tag{12}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^{(A[\sigma])} \tag{13}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : (A\,[\sigma]).(C\,[\sigma]) \tag{14}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\sigma(\epsilon),(A[\sigma])}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{15}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{16}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \tag{17}$$
$$(\Lambda \alpha.v)\,[\sigma] = \Lambda \alpha.(v\,[\sigma]) \quad \text{If } \alpha \# \sigma \tag{18}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \sigma(\epsilon) \tag{19}$$
$$\tag{20}$$

0.1.4 Well-Formed-ness

0.2 Term-Term Substitutions

0.2.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{21}$$

0.2.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{22}$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{23}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{24}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{25}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{26}$$

0.2.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{27}$$
$$x\,[\sigma, x := v] = v \tag{28}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{29}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{30}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{31}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{32}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{33}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{34}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \tag{35}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \epsilon \tag{36}$$
$$\tag{37}$$

0.2.4 Well Formedness

0.2.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma : \Gamma$ then: **TODO: Number these**

Property 1: $\Gamma\mathtt{Ok}$ and $\Gamma'\mathtt{Ok}$ Since $\Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ **implies** $\Gamma'' \vdash \sigma\colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v\colon A$ holds if $\Gamma' \vdash v\colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ **implies** $(\Gamma', x : A) \vdash (\sigma, x := x)\colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma\colon \Gamma$$

In addition, $\Gamma', x : A \vdash x\colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v)\colon (\Gamma, x : A) \tag{38}$$

0.3 Substitution Preserves Typing

0.3.1 Variables

Case Var

Case Weaken

0.3.2 Other Value Terms

Case Lambda

Case Constants

0.3.3 Computation Terms

Case Return

Case Apply

Case If

Case Bind

0.3.4 Sub-typing and Sub-effecting

Case Sub-type

Case Sub-effect

0.4 Semantics of Substitution

0.4.1 Denotation of Substitutions

0.4.2 Extension Lemma

0.4.3 Substitution Theorem

0.4.4 Proof For Value Terms

Case Var

Case Weaken

Case Constants

Case Lambda

Case Sub-type

0.4.5 Proof For Computation Terms

Case Return

Case Apply

Case If

Case Bind

Case Subeffect

0.5 The Identity Substitution

0.5.1 Properties of the Identity Substitution

Property 1

Property 2