TODO: Add in contributions slide

A Denotational Semantics for Polymorphic Effect Systems

Part III Project

Alexander Taylor, at736

University of Cambridge

May 13, 2019

Motivating Polymorphic Effect Analysis

```
def logAction(
    action: Unit => String
): Unit {
    log.info(action())
}

logAction(() => FireMissiles(); "Launched Missiles)

logAction(() => throwError("My Error"))

logAction(() => readEnvironmentVariables)
```

What is denotational Semantics?

A mapping $[\![-]\!]$: Language Structure $\to$ Mathematical Structure

In particular want to define $[\![\Gamma \vdash t \colon A]\!]$

Needs to be *compositional*, *sound*

And *adequate* for our needs

Denotational Semantics using Category Theory

Interested in: Objects, Morphisms, and Functors

$[\![A]\!], [\![\Gamma]\!] \in \texttt{obj}\ \mathbb{C}$

$[\![\Gamma \vdash t \colon A]\!] : [\![\Gamma]\!] \to [\![A]\!]$

Language features (1) - Lambda Calculus

A cartesian closed category (CCC) consists of:

Products $A \times B$ - models tuples

A terminal object 1 - models the `Unit` type

Exponential objects B^A - models functions as first-class objects

Language features (2.A) - Monads

A (strong) monad consists of:

- A functor $T : \mathbb{C} \to \mathbb{C}$
- Join and Unit natural transformations
  - $\mu_A : TTA \to TA$
  - $\eta_A : A \to TA$
- Tensor strength natural transformation $\mathtt{t}_{A,B} : A \times TB \to T(A \times B)$

Language features (2.B) - Graded Monads

A (strong) graded monad consists of:

An indexed functor $T_\epsilon : \mathbb{C} \to \mathbb{C}$

Indexed Join and Unit natural transformations

$\mu_{\epsilon_1,\epsilon_2,A} : T_{\epsilon_1} T_{\epsilon_2} A \to T_{\epsilon_1 \cdot \epsilon_2} A$
$\eta_A : A \to T_1 A$

Tensor strength natural transformation $\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$

Language Features (3.A) - If-Expressions

- If expression example - Co-product diagram

$\texttt{if}_{A}\ b\ \texttt{then}\ t\ \texttt{else}\ f$

If expressions modelled by co-products

$$
\begin{tikzcd}
 & A & \\
\Gamma \arrow[ur, "t"] \arrow[r, "\texttt{inl}"] & (\Gamma + \Gamma) \arrow[u, "{[t,f]}"] & \Gamma \arrow[ul, "f"'] \arrow[l, "\texttt{inr}"']
\end{tikzcd}
$$

Language Features 3.B - An Issue

Consider this:

```
if (UserConfirms) then Save() else pass;
```

Branches have different effects

So have different types!

This doesn't type correctly

Language Features 3.C - Subtyping

$$(\text{Subtype})\frac{\Gamma \vdash t\colon A \qquad A \leq: B}{\Gamma \vdash t\colon B}$$

This needs a denotation

So introduce $[\![A \leq: B]\!]$

$[\![\Gamma \vdash t\colon B]\!] = [\![A \leq: B]\!] \circ [\![\Gamma \vdash t\colon A]\!]$

An Effectful Language

$$v ::= \mathtt{k}^{A} \mid x \mid \mathtt{true} \mid \mathtt{false} \mid () \mid \lambda x \colon A.v \mid v_1\ v_2 \mid \mathtt{return}\ v$$
$$\mid \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \mid \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2$$

$$A, B, C ::= \gamma \mid A \rightarrow B \mid \mathtt{M}_\epsilon A$$

$$(\text{Return})\frac{\Gamma \vdash v \colon A}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_1 A} \quad (\text{Apply})\frac{\Gamma \vdash v_1 \colon A \rightarrow B \qquad \Gamma \vdash v_2 \colon A}{\Gamma \vdash v_1\ v_2 \colon B}$$

Semantics of EC

Can build a model of EC when we have

- CCC
- Co-product
- Strong Graded Monad
- Subtyping morphisms

We'll call this an S-category

$$\text{(Return)}\frac{f = [\![\Gamma \vdash v \colon A]\!]}{[\![\Gamma \vdash \texttt{return}\ v : \mathtt{M}_1 A]\!] = \eta_A \circ f} \qquad \text{(Fn)}\frac{f = [\![\Gamma, x \colon A \vdash v \colon B]\!] : \Gamma \times A \to B}{[\![\Gamma \vdash \lambda x \colon A.v : A \to B]\!] = \mathtt{cur}(f) : \Gamma \to B^A}$$

An Ugly Example

let *twiceIO* $= \lambda$ *action*: $\mathtt{M}_{\mathtt{IO}}\mathtt{Unit}$. (
    do _ <- *action* in *action*
)

let *twiceState* $= \lambda$ *action*: $\mathtt{M}_{\mathtt{State}}\mathtt{Unit}$. (
    do _ <- *action* in *action*
)

do _ <- *twiceState*(increment) in *twiceIO*(writeLog)

Let's Add Polymorphism

$$v ::= .. \mid \Lambda\alpha.v \mid v\ \epsilon$$

$$A, B, C ::= ... \mid \forall\alpha.A$$

$$\epsilon ::= e \mid \alpha \mid \epsilon \cdot \epsilon$$

$$(\text{Effect-Gen})\frac{\Phi, \alpha \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\alpha.A} \quad (\text{Effect-Spec})\frac{\Phi \mid \Gamma \vdash v : \forall\alpha.A \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon : A[\epsilon/\alpha]}$$

An Ugly Example - With a Makeover

```
let twice = Λ eff.(
    λ action: M_eff Unit. (
        do _ <- action in action
    )
)

do _ <- (twice State increment) in (twice IO writeLog)
```

How do we Model the Semantics of a Polymorphic Language?

For a fixed effect variable environment Φ and terms with no polymorphic sub-terms, we have EC

Effect-variable environments of length n are isomorphic by α-equivalence

How do we Model the Semantics of a Polymorphic Language?

- Stack of S-categories and their morphisms
- type rule for generalisation - "Need functors"

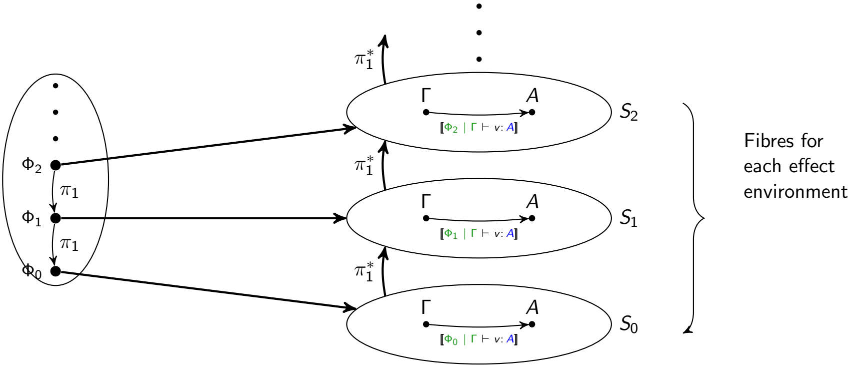
Base Category

- We need a way of reasoning about effect-variable environment categorically

- We can model effects and environments in new category.

- Objects: $\mathtt{1}$, U, U^n (write I for U^n) - Morphisms: $[\![e]\!] : \mathtt{1} \to U$ - Monoidal operator $\mathtt{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \to \mathbb{C}(I, U)$ - Can represent each effect environment as an object I, and common transformations between environments, such as weakening and substitutions, are morphisms between effect environments.

Indexed Category



Quantification

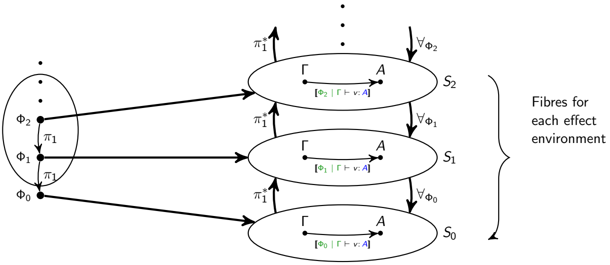
What about effect-generalisation?

$$(\text{Effect-Gen})\frac{\Phi, \alpha \mid \Gamma \vdash v\colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A}$$

Need to map $[\![\Phi, \alpha \mid \Gamma \vdash v\colon A]\!]$ to $[\![\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A]\!]$

For specialisation to work, needs: $\pi_1^* \dashv \forall_I$

Instantiating a Model (1)



Can we actually instantiate a category with the required structure?

Starting point a model of EC in $\mathtt{Set}$

Instantiating a Model (2) - Base Category

Use `Eff` - category of monotone functions of tuples of ground effects to ground effects

$[\![\diamond, \alpha, \beta \vdash \beta \cdot (\alpha \cdot \texttt{IO})\colon \texttt{Effect}]\!] = (e_1, e_2) \mapsto e_2 \cdot (e_1 \cdot \texttt{IO})$

$\texttt{Mul}(f, g)\vec{\epsilon} = (f\vec{\epsilon}) \cdot (g\vec{\epsilon})$

Instantiating a Model (3) - Fibres

The fibre $\mathbb{C}(n)$ is the category of functors $[E^n, \texttt{Set}]$

I.E. objects are functions that take a vector of ground effects and return a set.

Morphisms are functions that return functions in $\texttt{Set}$

S-Category features

Instantiating a Model (4) - Functors and Adjunctions

Re-indexing functors act by pre-composition

$$\begin{aligned} A \in & \quad [E^n, \texttt{Set}] \\ \theta^*(A)\vec{\epsilon_m} = & \quad A(\theta(\vec{\epsilon_m})) \\ \theta^*(f)\vec{\epsilon_m} = & \quad f(\theta(\vec{\epsilon_m})) : \theta^*(A) \to \theta^*(B) \end{aligned}$$

The quantification functor takes a product over all ground effects

$$\forall_{E^n}(A)\vec{\epsilon_n} = \Pi_{\epsilon \in E} A(\vec{\epsilon_n}, \epsilon)$$

The End

- Dissertation and github links