Contents

Chapter 1

Preliminaries

1.1 Base Category Requirements

There are 3 distinct objects in the base category, $\mathbb{C}$:

- U - The kind of `Effect`
- W - The kind of `Type`
- $\mathtt{1}$ - A terminal object

And we have finite products on U.

- $U^0 = \mathtt{1}$
- $U^{n+1} = U^n \times U$

We also have the following natural operations on morphisms in $\mathbb{C}$.
Let $I = U^n$.

- $\diamond : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates exponential types.
- $\square : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates products of types.
- $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I, W)$ - generates quantified types.
- $\mathtt{Eff} : \mathbb{C}(I, U) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - generates monad types.
- $\mathtt{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \to \mathbb{C}(I, U)$ - Generates multiplication of effects.

With naturality conditions which mean, for $\theta : \mathtt{Unit}^m \to \mathtt{Unit}^n (I' \to I)$,

- $\diamond(\phi, \psi) \circ \theta = \diamond(\phi \circ \theta, \psi \circ \theta)$
- $\square(\phi, \psi) \circ \theta = \square(\phi \circ \theta, \psi \circ \theta)$
- $\forall_I(\phi) \circ \theta = \forall_{I'}(\phi \circ (\theta \times \mathtt{Id}_U))$
- $\mathtt{Eff}(\phi, \psi) \circ \theta = \mathtt{Eff}(\phi \circ \theta, \psi \circ \theta)$
- $\mathtt{Mul}(\phi, \psi) \circ \theta = \mathtt{Mul}(\phi \circ \theta, \psi \circ \theta)$

1.2 Well-Formed-ness

Each instance of the well-formed-ness relation on effects, $\Phi \vdash \epsilon$ has a denotation in $\mathbb{C}$:

$$[\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]_M : I \to U \tag{1.1}$$

Each instance of the well-formed-ness relation on types, $\Phi \vdash A$ has a denotation in $\mathbb{C}$:

$$[\![P \vdash A\colon \texttt{Type}]\!]_M : I \to W \tag{1.2}$$

It should also be the case that

$$\texttt{Mul}([\![\Phi \vdash \epsilon_1\colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2\colon \texttt{Effect}]\!]_M) = [\![\Phi \vdash \epsilon_1 \cdot \epsilon_2\colon \texttt{Effect}]\!]_M \in \mathbb{C}(I, U) \tag{1.3}$$

That is, Mul should be have identity $[\![\Phi \vdash \texttt{1}\colon \texttt{Effect}]\!]_M$ and be associative.

1.3 Substitution and Weakening of the Effect Environment

For each instance of the well-formed-ness relation on substitution of effects $\Phi' \vdash \sigma\colon \Phi$, there exists a denotation in $\mathbb{C}$:

$$[\![\Phi' \vdash \sigma\colon \Phi]\!]_M : I' \to I \tag{1.4}$$

For each instance of the well-formed weakening relation on effect-environments, $\omega : \Phi' \triangleright \Phi$ there exists a denotation in $\mathbb{C}$:

$$[\![\omega : \Phi' \triangleright \Phi]\!]_M : I' \to I \tag{1.5}$$

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1.4 Fibre Categories

Each set of morphisms $\mathbb{C}(I, W)$ forms the objects of a semantic-closed (S-closed) category. That is, a category satisfying all the properties needed for the non-polymorphic language:

- Cartesian Closed
- Co-product of the terminal object with itself ($\texttt{1} + \texttt{1}$)
- Ground morphisms for each ground constant ($\texttt{C}^A : \texttt{1} \to A$)
- Partial order morphisms on ground types ($[\![A \leq:_\gamma]\!]_M B$)
- A strong, monad, graded by the po-monoid $(E_\Phi, \cdot_\Phi, \leq_\Phi, \texttt{1})$.

1.5 Re-indexing Functors

For each morphism $f : I' \to I$ in $\mathbb{C}$, there should be a co-variant, re-indexing functor $f^* : \mathbb{C}(I, W) \to \mathbb{C}(I', W)$, which is S-closed. That is, it preserves the S-closed properties of $\mathbb{C}(I, W)$. (See below).

$(-)^*$ should be a contra-variant functor in its $\mathbb{C}$ argument and co-variant in its right argument.

- $(g \circ f)^*(a) = f^*(\gamma^*(a))$
- $\mathtt{Id}_I^*(a) = a$
- $f^*(\mathtt{Id}_A) = \mathtt{Id}_{f^*(A)}$
- $f^*(a \circ b) = f^*(a) \circ f^*(b)$

1.5.1 f^* Preserves Products

If $\langle g, h \rangle : \mathbb{C}(I, W)(Z, X \times Y)$ Then

$$
\begin{aligned}
f^*(X \times Y) &= f^*(X) \times f^*(Y) && && (1.6)\\
f^*(\langle g, h \rangle) &= \langle f^*(g), f^*h \rangle && : \mathbb{C}(I', W)(f^*Z, f^*(X) \times f^*(Y)) && (1.7)\\
f^*(\pi_1) &= \pi_1 && : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(X)) && (1.8)\\
f^*(\pi_2) &= \pi_2 && : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(Y)) && (1.9)
\end{aligned}
$$

1.5.2 f^* Preserves Terminal Object

If $\mathtt{Id}_A : \mathbb{C}(I, W)(A, \mathtt{1})$ Then

$$
\begin{aligned}
f^*(\mathtt{1}) &= \mathtt{1} && && (1.10)\\
f^*(\langle\rangle_A) &= \langle\rangle_{f^*(A)} && : \mathbb{C}(I', W)(f^*A, \mathtt{1}) && (1.11)\\
& && && (1.12)
\end{aligned}
$$

1.5.3 f^* Preserves Exponentials

$$
\begin{aligned}
f^*(Z^X) &= (f^*(Z))^{(f^*(X))} && && (1.13)\\
f^*(\mathtt{app}) &= \mathtt{app} && : \mathbb{C}(I', W)(f^*(Z^X) \times f^*(X), f^*(Z)) && (1.14)\\
f^*(\mathtt{cur}(g)) &= \mathtt{cur}(f^*(g)) && : \mathbb{C}(I', W)(f^*(Y) \times f^*(X), f^*(Z)^{f^*(X)}) && (1.15)
\end{aligned}
$$

1.5.4 f^* Preserves Co-product on Terminal

$$
\begin{aligned}
f^*(\mathtt{1} + \mathtt{1}) &= \mathtt{1} + \mathtt{1} && && (1.16)\\
f^*(\mathtt{inl}) &= \mathtt{inl} && : \mathbb{C}(I', W)(\mathtt{1}, \mathtt{1} + \mathtt{1}) && (1.17)\\
f^*(\mathtt{inr}) &= \mathtt{inr} && : \mathbb{C}(I', W)(\mathtt{1}, \mathtt{1} + \mathtt{1}) && (1.18)\\
f^*([g, h]) &= [f^*(g), f^*(h)] && : \mathbb{C}(I', W)(\mathtt{1} + \mathtt{1}, f^*(Z)) && (1.19)
\end{aligned}
$$

1.5.5 f^* Preserves Graded Monad

$$
\begin{aligned}
f^*(T_\epsilon A) &= T_{f^*(\epsilon)} f^*(A) && : \mathbb{C}(I', W) && (1.20)\\
f^*(\mathtt{1}) &= \mathtt{1} \quad \text{The unit effect} && && (1.21)\\
f^*(\eta_A) &= \eta_{f^*(A)} && : \mathbb{C}(I', W)(f^*(A), f^*(T_{\mathtt{1}} A)) && (1.22)\\
f^*(\mu_{\epsilon_1, \epsilon_2, A}) &= \mu_{f^*(\epsilon_1), f^*(\epsilon_2), f^*(A)} && : \mathbb{C}(I', W)(f^*(T_{\epsilon_1} T_{\epsilon_2} A), f^*(T_{f^*(\epsilon_1) \cdot f^*(\epsilon_2)} f^*(A))) && (1.23)\\
f^*(\epsilon_1 \cdot \epsilon_2) &= f^*(\epsilon_1) \cdot f^*(\epsilon_2) && && (1.24)\\
& && && (1.25)
\end{aligned}
$$

1.5.6 f^* Preserves Tensor Strength

$$f^*(\mathtt{t}_{\epsilon,A,B}) = \mathtt{t}_{f^*(\epsilon),f^*(A),f^*(B)} \qquad : \mathbb{C}(I',W)(f^*(A \times T_\epsilon B), f^*(T_\epsilon(A \times B))) \tag{1.26}$$

1.5.7 f^* Preserves Ground Constants

For each ground constant $[\![\mathtt{C}^A]\!]_M$ in $\mathbb{C}(I,W)$,

$$f^*([\![\mathtt{C}^A]\!]_M) = \mathtt{C}^{f^*(A)} : \mathbb{C}(I',W)(\mathtt{1}, f^*(A)) \tag{1.27}$$

1.5.8 f^* Preserves Ground Sub-effecting

For ground effects e_1, e_2 such that $e_1 \leq e_2$

$$f^*(e) = e : \mathbb{C}(I',U) \tag{1.28}$$
$$f^*[\![\epsilon_1 \leq e_2]\!]_A = [\![e_1 \leq e_2]\!]_{f^*(A)} : \mathbb{C}(I',W)f^*(T_{e_1}A), f^*(T_{e_2}A) \tag{1.29}$$
$$\tag{1.30}$$

1.5.9 f^* Preserves Ground Sub-typing

For ground types γ_1, γ_2 such that $\gamma_1 \leq:_\gamma \gamma_2$

$$f^*\gamma = \gamma : \mathbb{C}(I',W)(\mathtt{1},\gamma) \tag{1.31}$$
$$f^*([\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M) = [\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M \qquad : \mathbb{C}(I',W)(\gamma_1,\gamma_2) \tag{1.32}$$
$$\tag{1.33}$$

1.5.10 Action on Objects

It follows that the action of f^* on an object A in $\mathbb{C}(I,W)$ (i.e. a morphism $I \to U$ in $\mathbb{C}$) is:

$$f^*(A) = A \circ f : I' \to I \to W \tag{1.34}$$

1.6 Naturality Properties

1.7 The $\forall_I$ functor

We expand $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I,W)$ to be a functor which is right adjoint to the re-indexing functor π_1^*.

$$\overline{(\_)} : \mathbb{C}(I \times U, W)(\pi_1^* A, B) \leftrightarrow \mathbb{C}(I,W)(A, \forall_I B) : \widehat{(\_)} \tag{1.35}$$

For $A : \mathbb{C}(I,W)$, $B : \mathbb{C}(I \times U, W)$.

Hence the action of $\forall_I$ on a morphism $l : A \to A'$ is as follows:

$$\forall_I(l) = \overline{l \circ \epsilon_A} \tag{1.36}$$

Where $\epsilon_A : \mathbb{C}(I \times U, W)(\pi_1^* \forall_I A \to A)$ is the co-unit of the adjunction.

1.8 Naturality Corollaries

Here are some simple corollaries of the adjunction between π_1^* and $\forall_I$.

1.8.1 Naturality

By the definition of an adjunction:

$$\overline{f \circ \pi_1^*(n)} = \overline{f} \circ n \tag{1.37}$$

1.8.2 $\overline{(-)}$ and Re-indexing Functors

TODO: Why does this occur? it comes from page 222 of Crole?

$$\begin{aligned}
\theta^*(\overline{f}) &= (\pi_1 \circ (\theta \times \mathtt{Id}_U))^*(\overline{f}) && (1.38)\\
&= (\theta \times \mathtt{Id}_U)^*(\pi_1^*(\overline{f})) && (1.39)\\
& && (1.40)\\
& && (1.41)\\
&= \overline{(\theta \times \mathtt{Id}_U)^* f} && (1.42)\\
& && (1.43)\\
& && (1.44)
\end{aligned}$$

1.8.3 $(\hat{-})$ and Re-Indexing Functors

$$\begin{aligned}
\theta^*(\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m})) &= (\langle \mathtt{Id}_I, \rho\rangle \circ \theta)^*(\widehat{m}) && (1.45)\\
&= ((\theta \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_I, \rho\rangle)^*(\widehat{m}) && (1.46)\\
&= \langle \mathtt{Id}_I, \rho \circ \theta\rangle^* (\theta \times \mathtt{Id}_U)^*(\widehat{m}) && (1.47)\\
&= \langle \mathtt{Id}_I, \theta^* \rho\rangle^* (\theta^*(\widehat{m})) && (1.48)
\end{aligned}$$

1.8.4 Pushing Morphisms into f^*

$$\begin{aligned}
\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ n &= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ \langle \mathtt{Id}_I, \rho\rangle^* \pi_1^*(n) && (1.49)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m} \circ \pi_1^*(n)) && (1.50)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m \circ n}) && (1.51)
\end{aligned}$$

Chapter 2

Denotations

2.1 Effects

For each instance of the well-formed-ness relation on effects, we define a morphism $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M : \mathbb{C}(I, U)$

- $[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M = [\![\epsilon]\!]_M \circ \langle\rangle_I : \to U$
- $[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M = \pi_2 : I \times U \to U$
- $[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 : I \times U \to U$
- $[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Effect}]\!]_M = \texttt{Mul}([\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M) : I \to U$

2.2 Types

For each instance of the well-formed-ness relation on types, we define a morphism $[\![\Phi \vdash A \colon \texttt{Type}]\!]_M : \mathbb{C}(I, W)$.

$[\![\texttt{Unit}]\!]_M$ is the morphism generating the terminal object of $\mathbb{C}(I, W)$. `Bool` is the morphism generating the co-product of this terminal object, $\mathtt{1} + \mathtt{1}$.

- $[\![\Phi \vdash \texttt{Unit} \colon \texttt{Type}]\!]_M = [\![\texttt{Unit}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \texttt{Bool} \colon \texttt{Type}]\!]_M = [\![\texttt{Bool}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \gamma \colon \texttt{Type}]\!]_M = [\![\gamma]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash A \to B \colon \texttt{Type}]\!]_M = \diamond([\![\Phi \vdash A \colon \texttt{Type}]\!]_M, [\![\Phi \vdash B \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \texttt{M}_\epsilon A \colon \texttt{Type}]\!]_M = \texttt{Eff}([\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash A \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \forall \alpha . A \colon \texttt{Type}]\!]_M = \forall_I([\![\Phi, \alpha \vdash A \colon \texttt{Type}]\!]_M) : I \to W$

2.3 Effect Substitution

For each effect-substitution well-formed-ness-relation, define a denotation morphism, $[\![\Phi' \vdash \sigma \colon \Phi]\!]_M : \mathbb{C}(I', I)$

- $[\![\Phi' \vdash \diamond\colon \diamond]\!]_M = \langle\rangle_I : \mathbb{C}(I', \mathtt{1})$
- $[\![\Phi' \vdash (\sigma, \alpha := \epsilon)\colon \Phi, \alpha]\!]_M = \langle [\![\Phi' \vdash \sigma\colon \Phi]\!]_M, [\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M \rangle : \mathbb{C}(I', I \times U)$

2.4 Effect Weakening

For each instance of the effect-environment weakening relation, define a denotation morphism: $[\![\omega : \Phi' \triangleright P]\!]_M : \mathbb{C}(I', I)$

- $[\![\iota : \Phi \triangleright \Phi]\!]_M = \mathtt{Id}_I : I \to I$
- $[\![w\pi : \Phi', \alpha \triangleright \Phi]\!]_M = [\![\omega : \Phi' \triangleright \Phi]\!]_M \circ \pi_1 : I' \times U \to I$
- $[\![w\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = ([\![\omega : \Phi' \triangleright \Phi]\!]_M \times \mathtt{Id}_U) : I' \times U \to I \times U$

2.5 Sub-Typing

For each instance of the sub-typing relation with respect to an effect environment, there exists a denotation, $[\![A \leq :_\Phi B]\!]_M : \mathbb{C}(I, W)(A, B)$.

- $[\![\gamma_1 \leq :_\Phi \gamma_2]\!]_M = [\![\gamma_1 \leq :_\gamma \gamma_2]\!]_M : \mathbb{C}(I, W)(\gamma_1, \gamma_2)$
- $[\![A \to B \leq :_\Phi A' \to B']\!]_M = [\![B \leq :_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq :_\Phi A]\!]_M}$
- $[\![\mathtt{M}_{\epsilon_1} A \leq :_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M \circ T_{\epsilon_1} [\![A \leq :_\Phi B]\!]_M$
- $[\![\forall \alpha.A \leq :_\Phi \forall \alpha.B]\!]_M = \forall_I [\![A \leq :_{\Phi,\alpha} B]\!]_M$

2.6 Type-Environments

For each instance of the well-formed relation on type environments, define an object in $[\![I \vdash W \mathtt{Ok}]\!]_M \in \mathbb{C}(I, W)$.

- $[\![\Phi \vdash \diamond \mathtt{Ok}]\!]_M = \mathtt{1} : \mathbb{C}(I, W)$
- $[\![\Phi \vdash \Gamma, x : A \mathtt{Ok}]\!]_M = \Box([\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M)$

2.7 Terms

For each instance of the typing relation, define a denotation morphism: $[\![\Phi \mid \Gamma \vdash v\colon A]\!]_M : \mathbb{C}(I, W)(\Gamma_I, A_I)$. Writing Γ_I and A_I for $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$ and $[\![\Phi \vdash A\colon \mathtt{Type}]\!]_M$.

For each ground constant, $\mathtt{C}^A$, there exists $c : \mathtt{1} \to A_I$ in $\mathbb{C}(I, W)$.

- (Unit) $\dfrac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{()} : \mathtt{Unit}]\!]_M = \langle\rangle_\Gamma : \Gamma_I \to \mathtt{1}}$
- (Const) $\dfrac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{C}^A : A]\!]_M = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]_M}$
- (True) $\dfrac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{true} : \mathtt{Bool}]\!]_M = \mathtt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = \mathtt{1} + \mathtt{1}}$

- (False) $\dfrac{\Phi \vdash \Gamma\, \mathtt{Ok}}{[\![\Phi|\Gamma \vdash \mathtt{false} : \mathtt{Bool}]\!]_M = \mathtt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = \mathtt{1}+\mathtt{1}}$

- (Var) $\dfrac{\Phi \vdash \Gamma\, \mathtt{Ok}}{[\![\Phi|\Gamma, x : A \vdash x : A]\!]_M = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Phi|\Gamma \vdash x : A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma, y : B \vdash x : A]\!]_M = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\dfrac{f = [\![\Phi|\Gamma, x : A \vdash C : \mathtt{M}_\epsilon B]\!]_M : \Gamma \times A \to T_\epsilon B}{[\![\Phi|\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\dfrac{f = [\![\Phi|\Gamma \vdash v : A]\!]_M : \Gamma \to A \quad g = [\![A \leq : B]\!]_M}{[\![\Phi|\Gamma \vdash v : B]\!]_M = g \circ f : \Gamma \to B}$

- (Return) $\dfrac{f = [\![\Phi|\Gamma \vdash v : A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{return}\, v : \mathtt{M}_{\mathtt{1}} A]\!]_M = \eta_A \circ f}$

- (If) $\dfrac{f = [\![\Phi|\Gamma \vdash v : \mathtt{Bool}]\!]_M : \Gamma \to \mathtt{1}+\mathtt{1} \quad g = [\![\Phi|\Gamma \vdash C_1 : \mathtt{M}_\epsilon A]\!]_M \quad h = [\![\Phi|\Gamma \vdash C_2 : \mathtt{M}_\epsilon A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A]\!]_M = \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \to T_\epsilon A}$

- (Bind) $\dfrac{f = [\![\Phi|\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A : \Gamma \to T_{\epsilon_1} A \quad g = [\![\Phi|\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B]\!]_M]\!]_M : \Gamma \times A \to T_{\epsilon_2} B}{[\![\Phi|\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2}]\!]_M = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma, A, \epsilon_1} \circ \left\langle \mathtt{Id}_\Gamma, f \right\rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$

- (Apply) $\dfrac{f = [\![\Phi|\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \quad g = [\![\Phi|\Gamma \vdash v_2 : A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f, g \rangle : \Gamma \to T_\epsilon B}$

- (Effect-Lambda) $\dfrac{f = [\![\Phi, \alpha|\Gamma \vdash v : A]\!]_M : \mathbb{C}(I \times U, W)(\Gamma, A)}{[\![\Phi|\Gamma \vdash \Lambda \alpha.A : \forall \epsilon.A]\!]_M = \overline{f} : \mathbb{C}(I, W)(\Gamma, \forall_I(A))}$

- (Effect-App) $\dfrac{g = [\![\Phi|\Gamma \vdash v : \forall \alpha.A]\!]_M : \mathbb{C}(I, W)(\Gamma, \forall_I(A)) \quad h = [\![\Phi \vdash \epsilon : \mathtt{Effect}]\!]_M : \mathbb{C}(I, U)}{[\![\Phi|\Gamma \vdash v\ \epsilon : A[\epsilon/\alpha]]\!]_M = \left\langle \mathtt{Id}_I, h \right\rangle^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ g : \mathbb{C}(I, W)(\Gamma, A[\epsilon/\alpha])}$

Chapter 3

Effect Substitution Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-variable substitution upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the substituted relation, $\Delta' = \sigma^*(\Delta)$.

3.1 Effects

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash \sigma(\epsilon) \colon \texttt{Effect}]\!]_M = \sigma^*[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M$

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M \circ \sigma &= [\![e]\!]_M \circ \langle\rangle_I \circ \sigma && (3.1)\\
&= [\![e]\!]_M \circ \langle\rangle_{I'} && (3.2)\\
&= [\![\Phi' \vdash e \colon \texttt{Type}]\!]_M && (3.3)\\
& && (3.4)
\end{aligned}$$

Case Var:

$$\begin{aligned}
[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \sigma' &= \pi_2 \circ \langle \sigma, [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle \quad \text{By inversion } \sigma' = (\sigma, \alpha := \epsilon) && (3.5)\\
&= [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M && (3.6)\\
&= [\![\Phi' \vdash \sigma'(\alpha) \colon \texttt{Effect}]\!]_M && (3.7)\\
& && (3.8)
\end{aligned}$$

Case Weaken:

$$
\begin{aligned}
[\![\Phi, \beta \vdash \alpha \colon \texttt{Type}]\!]_M \circ \sigma' &= [\![\Phi \vdash \alpha \colon \texttt{Type}]\!]_M \circ \pi_1 \circ \langle \sigma, [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle \quad \text{By inversion, } \sigma' = (\sigma, \beta := \epsilon) && (3.9)\\
&= [\![\Phi \vdash \alpha \colon \texttt{Type}]\!]_M \circ \sigma && (3.10)\\
&= [\![\Phi' \vdash \sigma(\alpha) \colon \texttt{Type}]\!]_M && (3.11)\\
&= [\![\Phi' \vdash \sigma'(\alpha) \colon \texttt{Type}]\!]_M && (3.12)\\
& && (3.13)
\end{aligned}
$$

Case Multiply:

$$
\begin{aligned}
[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Type}]\!]_M \circ \sigma &= \texttt{Mul}([\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M) \circ \sigma && (3.14)\\
&= \texttt{Mul}([\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M \circ \sigma) \quad \text{By Naturality} && (3.15)\\
&= \texttt{Mul}([\![\Phi' \vdash \sigma(\epsilon_1) \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \sigma(\epsilon_2) \colon \texttt{Effect}]\!]_M) && (3.16)\\
&= [\![\Phi' \vdash \sigma(\epsilon_1) \cdot \sigma(\epsilon_2) \colon \texttt{Effect}]\!]_M && (3.17)\\
&= [\![\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2) \colon \texttt{Effect}]\!]_M && (3.18)\\
& && (3.19)
\end{aligned}
$$

3.2 Types

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash A\,[\sigma] : \texttt{Type}]\!]_M = \sigma^* [\![\Phi \vdash A \colon \texttt{Type}]\!]_M = [\![\Phi \vdash A \colon \texttt{Type}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash A \colon \texttt{Type}]\!]_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$
\begin{aligned}
[\![\Phi \vdash \gamma \colon \texttt{Type}]\!]_M \circ \sigma &= [\![\gamma]\!]_M \circ \langle\rangle_I \circ \sigma && (3.20)\\
&= [\![\gamma]\!]_M \circ \langle\rangle_{I'} && (3.21)\\
&= [\![\Phi' \vdash \gamma \colon \texttt{Type}]\!]_M && (3.22)\\
&= [\![\Phi' \vdash \gamma\,[\sigma] : \texttt{Type}]\!]_M && (3.23)
\end{aligned}
$$

Case Monad:

$$
\begin{aligned}
[\![\Phi \vdash \texttt{M}_\epsilon A \colon \texttt{Type}]\!]_M \circ \sigma &= \texttt{Eff}([\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash A \colon \texttt{Type}]\!]_M) \circ \sigma && (3.24)\\
&= \texttt{Eff}([\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash A \colon \texttt{Type}]\!]_M \circ \sigma) \quad \text{By naturality} && (3.25)\\
&= \texttt{Eff}([\![\Phi' \vdash \sigma(\epsilon) \colon \texttt{Effect}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \texttt{Type}]\!]_M) && (3.26)\\
&= [\![\Phi' \vdash \texttt{M}_{\sigma(\epsilon)} A\,[\sigma] \colon \texttt{Type}]\!]_M && (3.27)\\
&= [\![\Phi' \vdash (\texttt{M}_\epsilon A)\,[\sigma] : \texttt{Type}]\!]_M && (3.28)
\end{aligned}
$$

Case Quantification:

$$
\begin{aligned}
[\![\Phi \vdash \forall\alpha.A\colon \mathtt{Type}]\!]_M \circ \sigma &= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M) \circ \sigma && (3.29)\\
&= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M \circ (\sigma \times \mathtt{Id}_U)) && (3.30)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma, \alpha := \epsilon] : \mathtt{Type}]\!]_M) && (3.31)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.32)\\
&= [\![\Phi' \vdash \forall\alpha.A\,[\sigma] : \mathtt{Type}]\!]_M && (3.33)\\
&= [\![\Phi' \vdash (\forall\alpha.A)\,[\sigma] : \mathtt{Type}]\!]_M && (3.34)\\
& && (3.35)
\end{aligned}
$$

Case Function:

$$
\begin{aligned}
[\![\Phi \vdash A \to B\colon \mathtt{Type}]\!]_M \circ \sigma &= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M) \circ \sigma && && (3.36)\\
&= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \sigma, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M \circ \sigma) && \text{By Naturality} && (3.37)\\
&= \diamond([\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M, [\![\Phi' \vdash B\,[\sigma] : \mathtt{Type}]\!]_M) && && (3.38)\\
&= [\![\Phi' \vdash (A\,[\sigma]) \to (B\,[\sigma])\colon \mathtt{Type}]\!]_M && && (3.39)\\
&= [\![\Phi' \vdash (A \to B)\,[\sigma] : \mathtt{Type}]\!]_M && && (3.40)\\
& && && (3.41)
\end{aligned}
$$

3.3 Sub-typing

If $\sigma = [\![\Phi' \vdash \sigma\colon \Phi]\!]_M$ then $[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M = \sigma^*[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $[\![A \leq:_\Phi B]\!]_M$. Using S-closure of σ^*

Case Ground:

$$\sigma^*(\gamma_1 \leq:_\gamma \gamma_2) = (\gamma_1 \leq:_\gamma \gamma_2) \tag{3.42}$$

Since σ^* is s-closed.

Case Monad:

$$
\begin{aligned}
\sigma^*[\![\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M &= \sigma^*([\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M) \circ \sigma^*(T_{\epsilon_1}([\![A \leq:_\Phi B]\!]_M)) && && (3.43)\\
&= [\![\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)]\!]_M \circ T_{\sigma(\epsilon_1)}[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M && \text{By S-Closure} && (3.44)\\
&= [\![\mathtt{M}_{\sigma(\epsilon_1)} A\,[\sigma] \leq:_{\Phi'} \mathtt{M}_{\sigma(\epsilon_2)} B\,[\sigma]]\!]_M && && (3.45)\\
&= [\![(\mathtt{M}_{\epsilon_1} A)\,[\sigma] \leq:_{\Phi'} \mathtt{M}_{\epsilon_2} B\,[\sigma]\,]\!]_M && && (3.46)\\
& && && (3.47)
\end{aligned}
$$

Case For All:

$$
\begin{aligned}
\sigma^*[\![\forall\alpha.A \leq:_\Phi \forall\alpha.B]\!]_M &= \sigma^*(\forall_I([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.48)\\
&= \forall_{I'}((\sigma \times \mathtt{Id}_U)^*([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.49)\\
&= \forall_{I'}([\![A\,[\sigma, \alpha := \alpha] \leq:_{\Phi',\alpha} B\,[\sigma, \alpha := \alpha]\,]\!]_M) && (3.50)\\
&= [\![(\forall\alpha.A)\,[\sigma] \leq:_{\Phi'} (\forall\alpha.B)\,[\sigma]\,]\!]_M && (3.51)\\
& && (3.52)
\end{aligned}
$$

Case Fn:

$$
\begin{aligned}
\sigma^* [\![(A \to B) \leq:_\Phi A' \to B']\!]_M &= \sigma^*([\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}) && (3.53)\\
&= \sigma^*(\mathtt{cur}([\![B \leq:_\Phi B']\!]_M \circ \mathtt{app})) \circ \sigma^*(\mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq:_\Phi A]\!]_M))) && (3.54)\\
&= \mathtt{cur}(\sigma^*([\![B \leq:_\Phi B']\!]_M) \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times \sigma^*([\![A' \leq:_\Phi A]\!]_M))) && (3.55)\\
&= \mathtt{cur}([\![B\,[\sigma] \leq:_{\Phi'} B'\,[\sigma]\,]\!]_M \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{B[\sigma]} \times [\![A'\,[\sigma] \leq:_{\Phi'} A\,[\sigma]\,]\!]_M)) && (3.56)\\
&= [\![(A\,[\sigma]) \to (B\,[\sigma]) \leq:_{\Phi'} (A'\,[\sigma]) \to (B'\,[\sigma])]\!]_M && (3.57)\\
&= [\![(A \to B)\,[\sigma] \leq:_{\Phi'} (A' \to B')\,[\sigma]\,]\!]_M && (3.58)
\end{aligned}
$$

3.4 Type Environments

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}]\!]_M = \sigma^* [\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M = [\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M \circ \sigma : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$
\begin{aligned}
\sigma^* [\![\Phi \vdash \diamond \mathtt{Ok}]\!]_M &= \langle\rangle_I \circ \sigma && (3.59)\\
&= \langle\rangle_{I'} && (3.60)\\
&= [\![\Phi' \vdash \diamond \mathtt{Ok}]\!]_M && (3.61)\\
[\![\Phi' \vdash \diamond\,[\sigma]\,\mathtt{Ok}]\!]_M & && (3.62)\\
& && (3.63)
\end{aligned}
$$

Case Var:

$$
\begin{aligned}
\sigma^* [\![\Phi \vdash \Gamma, x : A \mathtt{Ok}]\!]_M &= \sigma^*(\square([\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M)) && (3.64)\\
&= \square([\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M) \circ \sigma && (3.65)\\
&= \square([\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M \circ \sigma, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M \circ \sigma) && (3.66)\\
&= \square([\![\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.67)\\
&= [\![\Phi' \vdash \Gamma\,[\sigma]\,, x : A\,[\sigma]\,\mathtt{Ok}]\!]_M && (3.68)\\
&= [\![\Phi' \vdash (\Gamma, x : A)\,[\sigma]\,\mathtt{Ok}]\!]_M && (3.69)\\
& && (3.70)
\end{aligned}
$$

3.5 Terms

If

$$
\begin{aligned}
\sigma &= [\![\Phi' \vdash \sigma \colon \Phi]\!]_M && (3.71)\\
\Delta &= [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M && (3.72)\\
\Delta' &= [\![\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]\,]\!]_M && (3.73)\\
& && (3.74)
\end{aligned}
$$

Then

$$\Delta' = \sigma^*(\Delta) \tag{3.75}$$

Proof: By induction over the derivation of Δ. Using the S-Closure of σ^*. We use Γ_I to indicate $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$, an A_I to indicate $[\![\Phi \vdash A\colon \mathtt{Effect}]\!]_M$

Case Unit:

$$\Delta = \langle\rangle_{\Gamma_I} \tag{3.76}$$

So

$$\sigma^*(\Delta) = \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.77}$$

Case True, False: Giving the case for true as false is the same but using `inr`

$$\Delta = \mathtt{inl} \circ \langle\rangle_{\Gamma_I} \tag{3.78}$$

So

$$\sigma^*(\Delta) = \mathtt{inl} \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.79}$$

Since σ^* is S-closed.

Case Constant:

$$\Delta = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I} \tag{3.80}$$

So

$$\sigma^*(\Delta) = \sigma^*[\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = [\![\mathtt{C}^{A[\sigma]}]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.81}$$

Since σ^* is S-closed.

Case Subtype: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M \tag{3.82}$$

Then

$$\Delta = [\![A \leq:_{\Phi} B]\!]_M \circ \Delta_1 \tag{3.83}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*[\![A \leq:_{\Phi} B]\!]_M \circ \sigma^*\Delta_1 && (3.84)\\
&= [\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M \circ \Delta_1' \quad \text{By induction} && (3.85)\\
&= D' && (3.86)
\end{aligned}$$

Case Lambda: Let

$$\Delta_1 = [\![\Phi \mid \Gamma, x : A \vdash v \colon B]\!]_M \tag{3.87}$$

Then

$$\Delta = \mathtt{cur}(()\Delta_1) \tag{3.88}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mathtt{cur}(\Delta_1)) && (3.89)\\
&= \mathtt{cur}(\sigma^*(\Delta_1)) \quad \text{By S-closure} && (3.90)\\
&= \mathtt{cur}(\Delta_1') \quad \text{By induction} && (3.91)\\
&= \Delta' && (3.92)
\end{aligned}$$

Case Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v_1 \colon A \to B]\!]_M && (3.93)\\
\Delta_2 &= [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M && (3.94)
\end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \tag{3.95}$$

So

$$\begin{aligned}
\sigma^*\Delta &= \sigma^*(\mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle) && (3.96)\\
&= \mathtt{app} \circ \langle \sigma^*(\Delta_1), \sigma^*(\Delta_2) \rangle \quad \text{By S-closure} && (3.97)\\
&= \mathtt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Induction} && (3.98)\\
&= \Delta' && (3.99)
\end{aligned}$$

Case Return: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M \tag{3.100}$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \tag{3.101}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\eta_{A_I} \circ \Delta_1) && (3.102)\\
&= \eta_{A_{I'}} \circ \sigma^*(\Delta_1) \quad \text{By S-closure} && (3.103)\\
&= \eta_{A_{I'}} \circ \Delta_1' && (3.104)\\
&= \Delta' && (3.105)
\end{aligned}$$

Case Bind: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{3.106}$$
$$\Delta_2 = [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{3.107}$$

Then

$$\Delta = \mathtt{M}_{\epsilon_1}\epsilon_2 A_I \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma_I,A_I} \circ \langle \mathtt{Id}_{\Gamma_I}, \Delta_1 \rangle \tag{3.108}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle) && (3.109)\\
&= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A}) \circ \sigma^*(T_{\epsilon_1}\Delta_2) \circ \sigma^*(\mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (3.110)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)}\sigma^*(\Delta_2) \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (3.111)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)}\Delta_2' \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \Delta_1' \right\rangle \quad \text{By Induction} && (3.112)\\
&= \Delta' && (3.113)\\
& && (3.114)
\end{aligned}$$

Case If: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}]\!]_M \tag{3.115}$$
$$\Delta_2 = [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M \tag{3.116}$$
$$\Delta_3 = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M \tag{3.117}$$
$$\tag{3.118}$$

Then

$$\Delta = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{3.119}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma) && (3.120)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\sigma^*(\Delta_2) \circ \pi_2), \mathtt{cur}(\sigma^*(\Delta_3) \circ \pi_2)] \circ \sigma^*(\Delta_1)) \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By S-Closure} && (3.121)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By Induction} && (3.122)\\
&= \Delta' && (3.123)\\
& && (3.124)
\end{aligned}$$

Case Effect-Lambda: Let

$$\Delta_1 = [\![\Phi, \alpha \mid \Gamma \vdash v \colon A]\!]_M \tag{3.125}$$

Then

$$\Delta = \widehat{\Delta_1} \tag{3.126}$$

And also

$$\sigma \times \mathtt{Id} = [\![(\Phi', \alpha) \vdash (\sigma, \alpha := \epsilon) \colon (\Phi, \alpha)]\!]_M \tag{3.127}$$

So

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\widehat{\Delta_1}) && (3.128)\\
&= \widehat{(\sigma \times \mathtt{Id}_U)^* \Delta_1} \quad \text{By naturality} && (3.129)\\
&= \widehat{\Delta_1'} \quad \text{By induction} && (3.130)\\
&= \Delta' && (3.131)
\end{aligned}$$

Case Effect-Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v \colon \forall \alpha. A]\!]_M && (3.132)\\
h &= [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M && (3.133)\\
& && (3.134)
\end{aligned}$$

Then

$$\Delta = \langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1 \tag{3.135}$$

So Due to the substitution theorem on effects

$$h \circ \sigma = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \sigma = [\![\Phi' \vdash \sigma(\epsilon) \colon \mathtt{Effect}]\!]_M = h' \tag{3.136}$$

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1) && (3.137)\\
&= (\langle \mathtt{Id}_\Gamma, h \rangle \circ \sigma)^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \sigma^*(\Delta_1) && (3.138)\\
&= ((\sigma \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_\Gamma, h \circ \sigma \rangle)^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1)' && (3.139)\\
&= (\langle \mathtt{Id}_\Gamma, h' \rangle)^* ((\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' && (3.140)\\
& && (3.141)
\end{aligned}$$

Looking at the inner part of the functor application: Let

$$\begin{aligned}
A &= [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M && (3.142)\\
& && (3.143)
\end{aligned}$$

$$\begin{aligned}
(\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M} &= (\sigma \times \mathtt{Id}_U)^* \epsilon_A && (3.144)\\
&= (\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}}) && (3.145)\\
&= \widehat{\overline{(\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}})}} \quad \text{By bijection} && (3.146)\\
&= \widehat{\sigma^* (\overline{\widehat{\mathtt{Id}_{\forall_I(A)}}})} \quad \text{By naturality} && (3.147)\\
&= \widehat{\sigma^* (\mathtt{Id}_{\forall_I(A)})} \quad \text{By bijection} && (3.148)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A \circ (\sigma \times \mathtt{Id}_U))}} \quad \text{By S-Closure, naturality} && (3.149)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A[\sigma, \alpha := \alpha])}} \quad \text{By Substitution theorem} && (3.150)\\
&= \epsilon_{A[\sigma]} && (3.151)
\end{aligned}$$

Going back to the original expression:

$$\sigma^*\Delta = (\langle \mathtt{Id}_\Gamma, h' \rangle)^*(\epsilon_{A[\sigma]}) \circ \Delta_1)' \tag{3.152}$$

$$= \Delta' \tag{3.153}$$

$$\tag{3.154}$$

Chapter 4

Effect Weakening Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-weakening upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the weakened relation, $\Delta' = \omega^*(\Delta)$.

4.1 Effects

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $\Phi' \vdash \epsilon\colon \mathtt{Effect} = \omega^*[\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M = [\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M \circ \omega$

Proof: By induction on the derivation on $[\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M$

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash e\colon \mathtt{Effect}]\!]_M \circ \omega &= [\![e]\!]_M \circ \langle\rangle_I \circ \omega && (4.1)\\
&= [\![e]\!]_M \circ \langle\rangle_{I'} && (4.2)\\
&= [\![\Phi' \vdash e\colon \mathtt{Type}]\!]_M && (4.3)\\
& && (4.4)
\end{aligned}$$

Case Var: Case split on ω.

Case: $\omega = \iota$ Then $\Phi' = \Phi$ and $\omega = \mathtt{Id}_I$. So the theorem holds trivially.

Case: $\omega = \omega'\times$ Then

$$\begin{aligned}
[\![\Phi, \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega &= \pi_2 \circ (\omega' \times \mathtt{Id}_U) && (4.5)\\
&= \pi_2 && (4.6)\\
&= [\![\Phi', \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M && (4.7)
\end{aligned}$$

Case: $\omega = \omega'\pi_1$ Then

$$[\![\Phi, \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M = \pi_2 \circ \omega' \circ \pi_1 \qquad (4.8)$$

Where $\Phi' = \Phi, \beta$ and $\omega' : \Phi'' \triangleright \Phi$.

So

$$\pi_2 \circ \omega' = [\![\Phi'' \vdash \alpha\colon \mathtt{Effect}]\!]_M \tag{4.9}$$
$$\pi_2 \circ \omega' \circ \pi_1 = [\![\Phi'', \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \qquad = [\![\Phi' \vdash \alpha\colon \mathtt{Effect}]\!]_M \tag{4.10}$$

Case Weaken:

$$[\![\Phi, \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega = [\![\Phi \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \pi_1 \circ \omega \tag{4.11}$$

Case split of structure of w

Case: $\omega = \iota$ Then $\Phi' = \Phi, \beta$ so $\omega = \mathtt{Id}_I$ So $[\![\Phi, \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega = [\![\Phi' \vdash \alpha\colon \mathtt{Effect}]\!]_M$

Case: $\omega = \omega'\pi_1$ Then $\Phi' = \Phi'', \gamma$ and $\omega = \omega' \circ \pi_1$ Where $\omega' : \Phi'' \triangleright \Phi, \beta$. So

$$\begin{aligned}
[\![\Phi, \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega &= [\![\Phi, \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega' \circ \pi_1 && (4.12)\\
&= \Phi'' \vdash \alpha\colon \mathtt{Effect} \circ \pi_1 && (4.13)\\
&= \Phi'', \gamma \vdash \alpha\colon \mathtt{Effect} && (4.14)\\
&= \Phi' \vdash \alpha\colon \mathtt{Effect} && (4.15)\\
& && (4.16)
\end{aligned}$$

Case: $\omega = \omega' \times$ Then $\Phi' = \Phi'', \beta$ and $\omega' : \Phi' \triangleright \Phi$

So

$$\begin{aligned}
[\![\Phi, \beta \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega &= [\![\Phi \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \pi_1 \circ (\omega' \times \mathtt{Id}_U) && (4.17)\\
&= [\![\Phi \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega' \circ \pi_1 && (4.18)\\
&= [\![\Phi'' \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \pi_1 && (4.19)\\
&= [\![\Phi' \vdash \alpha\colon \mathtt{Effect}]\!]_M && (4.20)\\
& && (4.21)
\end{aligned}$$

Case Multiply:

$$\begin{aligned}
[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2\colon \mathtt{Type}]\!]_M \circ \omega &= \mathtt{Mul}([\![\Phi \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M) \circ \omega && (4.22)\\
&= \mathtt{Mul}([\![\Phi \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M \circ \omega, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M \circ \omega) \quad \text{By Naturality} && (4.23)\\
&= \mathtt{Mul}([\![\Phi' \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M) && (4.24)\\
&= [\![\Phi \vdash \epsilon_1 \cdot \epsilon_2\colon \mathtt{Effect}]\!]_M && (4.25)
\end{aligned}$$

4.2 Types

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![\Phi' \vdash A\colon \mathtt{Type}]\!]_M = \omega^*[\![\Phi \vdash A\colon \mathtt{Type}]\!]_M = [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \omega$.

Proof: By induction on the derivation on $[\![\Phi \vdash A\colon \mathtt{Type}]\!]_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$
\begin{aligned}
[\![\Phi \vdash \gamma\colon \texttt{Type}]\!]_M \circ \omega &= [\![\gamma]\!]_M \circ \langle\rangle_I \circ \omega && (4.26)\\
&= [\![\gamma]\!]_M \circ \langle\rangle_{I'} && (4.27)\\
&= [\![\Phi' \vdash \gamma\colon \texttt{Type}]\!]_M && (4.28)\\
&= [\![\Phi' \vdash \gamma\colon \texttt{Type}]\!]_M && (4.29)
\end{aligned}
$$

Case Monad:

$$
\begin{aligned}
[\![\Phi \vdash \texttt{M}_\epsilon A\colon \texttt{Type}]\!]_M \circ \omega &= \texttt{Eff}([\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]_M, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M) \circ \omega && (4.30)\\
&= \texttt{Eff}([\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]_M \circ \omega, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M \circ \omega) \quad \text{By naturality} && (4.31)\\
&= \texttt{Eff}([\![\Phi' \vdash \epsilon\colon \texttt{Effect}]\!]_M, [\![\Phi' \vdash A\colon \texttt{Type}]\!]_M) && (4.32)\\
&= [\![\Phi' \vdash (\texttt{M}_\epsilon A)\colon \texttt{Type}]\!]_M && (4.33)
\end{aligned}
$$

Case Quantification: Note $[\![\omega\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = \omega \times \texttt{Id}_U$

$$
\begin{aligned}
[\![\Phi \vdash \forall\alpha.A\colon \texttt{Type}]\!]_M \circ \omega &= \forall_I([\![\Phi, \alpha \vdash A\colon \texttt{Type}]\!]_M) \circ \omega && (4.34)\\
&= \forall_I([\![\Phi, \alpha \vdash A\colon \texttt{Type}]\!]_M \circ (\omega \times \texttt{Id}_U)) \quad \text{By naturality} && (4.35)\\
&= \forall_I([\![\Phi', \alpha \vdash A\colon \texttt{Type}]\!]_M) \quad \text{By induction} && (4.36)\\
&= [\![\Phi' \vdash \forall\alpha.A\colon \texttt{Type}]\!]_M && (4.37)\\
&= [\![\Phi' \vdash (\forall\alpha.A)\colon \texttt{Type}]\!]_M && (4.38)\\
& && (4.39)
\end{aligned}
$$

Case Function:

$$
\begin{aligned}
[\![\Phi \vdash A \to B\colon \texttt{Type}]\!]_M \circ \omega &= \diamond([\![\Phi \vdash A\colon \texttt{Type}]\!]_M, [\![\Phi \vdash B\colon \texttt{Type}]\!]_M) \circ \omega && (4.40)\\
&= \diamond([\![\Phi \vdash A\colon \texttt{Type}]\!]_M \circ \omega, [\![\Phi \vdash B\colon \texttt{Type}]\!]_M \circ \omega) \quad \text{By Naturality} && (4.41)\\
&= \diamond([\![\Phi' \vdash A\colon \texttt{Type}]\!]_M, [\![\Phi' \vdash B\colon \texttt{Type}]\!]_M) && (4.42)\\
&= [\![\Phi' \vdash (A \to B)\colon \texttt{Type}]\!]_M && (4.43)\\
& && (4.44)
\end{aligned}
$$

4.3 Sub-typing

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![A \leq:_{\Phi'} B]\!]_M = \omega^*[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $[\![A \leq:_\Phi B]\!]_M$. Using S-closure of ω^*

Case Ground:

$$\omega^*(\gamma_1 \leq:_\gamma \gamma_2) = (\gamma_1 \leq:_\gamma \gamma_2) \tag{4.45}$$

Since ω^* is s-closed.

Case Monad:

$$
\begin{aligned}
\omega^*[\![\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M &= \omega^*([\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M) \circ \omega^*(T_{\epsilon_1}([\![A \leq:_\Phi B]\!]_M)) && (4.46)\\
&= [\![\epsilon_1 \leq_{\Phi'} \epsilon_2]\!]_M \circ T_{\epsilon_1}[\![A \leq:_{\Phi'} B]\!]_M \quad \text{By S-Closure} && (4.47)\\
&= [\![\mathtt{M}_{\epsilon_1} A \leq:_{\Phi'} \mathtt{M}_{\epsilon_2} B]\!]_M && (4.48)\\
&= [\![(\mathtt{M}_{\epsilon_1} A) \leq:_{\Phi'} \mathtt{M}_{\epsilon_2} B]\!]_M && (4.49)\\
& && (4.50)
\end{aligned}
$$

Case For All: Note $[\![\omega\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = (\omega \times \mathtt{Id}_U)$

$$
\begin{aligned}
\omega^*[\![\forall\alpha.A \leq:_\Phi \forall\alpha.B]\!]_M &= \omega^*(\forall_I([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (4.51)\\
&= \forall_{I'}((\omega \times \mathtt{Id}_U)^*([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (4.52)\\
&= \forall_{I'}([\![A \leq:_{\Phi',\alpha} B]\!]_M) && (4.53)\\
&= [\![(\forall\alpha.A) \leq:_{\Phi'} (\forall\alpha.B)]\!]_M && (4.54)\\
& && (4.55)
\end{aligned}
$$

Case Fn:

$$
\begin{aligned}
\omega^*[\![(A \to B) \leq:_\Phi A' \to B']\!]_M &= \omega^*([\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}) && (4.56)\\
&= \omega^*(\mathtt{cur}([\![B \leq:_\Phi B']\!]_M \circ \mathtt{app})) \circ \omega^*(\mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq:_\Phi A]\!]_M))) && (4.57)\\
&= \mathtt{cur}(\omega^*([\![B \leq:_\Phi B']\!]_M) \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times \omega^*([\![A' \leq:_\Phi A]\!]_M))) && (4.58)\\
&= \mathtt{cur}([\![B \leq:_{\Phi'} B']\!]_M \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq:_{\Phi'} A]\!]_M)) && (4.59)\\
&= [\![(A \to B) \leq:_{\Phi'} (A' \to B')]\!]_M && (4.60)
\end{aligned}
$$

4.4 Type Environments

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma\mathtt{Ok}]\!]_M = \omega^*[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M = [\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M \circ \omega : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$
\begin{aligned}
\omega^*[\![\Phi \vdash \diamond\mathtt{Ok}]\!]_M &= \langle\rangle_I \circ \omega && (4.61)\\
&= \langle\rangle_{I'} && (4.62)\\
&= [\![\Phi' \vdash \diamond\mathtt{Ok}]\!]_M && (4.63)\\
& && (4.64)
\end{aligned}
$$

Case Var:

$$\begin{aligned}
\omega^*[\![\Phi \vdash \Gamma, x : A\mathtt{Ok}]\!]_M &= \omega^*(\Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M)) && (4.65)\\
&= \Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M) \circ \omega && (4.66)\\
&= \Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M \circ \omega, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \omega) && (4.67)\\
&= \Box([\![\Phi' \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi' \vdash A\colon \mathtt{Type}]\!]_M) && (4.68)\\
&= [\![\Phi' \vdash (\Gamma, x : A)\mathtt{Ok}]\!]_M && (4.69)\\
& && (4.70)
\end{aligned}$$

4.5 Terms

4.6 Terms

If

$$\begin{aligned}
\omega &= [\![\omega : \Phi' \triangleright \Phi]\!]_M && (4.71)\\
\Delta &= [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M && (4.72)\\
\Delta' &= [\![\Phi' \mid \Gamma \vdash v\colon A]\!]_M && (4.73)\\
& && (4.74)
\end{aligned}$$

Then

$$\Delta' = \omega^*(\Delta) \tag{4.75}$$

Proof: By induction over the derivation of Δ. Using the S-Closure of ω^*. We use Γ_I to indicate $[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M$, an A_I to indicate $[\![\Phi \vdash A\colon \mathtt{Effect}]\!]_M$

Case Unit:

$$\Delta = \langle\rangle_{\Gamma_I} \tag{4.76}$$

So

$$\omega^*(\Delta) = \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.77}$$

Case True, False: Giving the case for true as false is the same but using `inr`

$$\Delta = \mathtt{inl} \circ \langle\rangle_{\Gamma_I} \tag{4.78}$$

So

$$\omega^*(\Delta) = \mathtt{inl} \circ \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.79}$$

Since ω^* is S-closed.

Case Constant:
$$\Delta = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I} \tag{4.80}$$

So

$$\omega^*(\Delta) = \omega^*[\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_{I'}} = [\![\mathtt{C}^{A_{I'}}]\!]_M \circ \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.81}$$

Since ω^* is S-closed.

Case Subtype: Let
$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M \tag{4.82}$$

Then

$$\Delta = [\![A \leq:_\Phi B]\!]_M \circ \Delta_1 \tag{4.83}$$

So
$$\begin{aligned}
\omega^*(\Delta) &= \omega^*[\![A \leq:_\Phi B]\!]_M \circ \omega^*\Delta_1 && (4.84)\\
&= [\![A_{I'} \leq:_{\Phi'} B_{I'}]\!]_M \circ \Delta_1' \quad \text{By induction} && (4.85)\\
&= D' && (4.86)
\end{aligned}$$

Case Lambda: Let
$$\Delta_1 = [\![\Phi \mid \Gamma, x : A \vdash v\colon B]\!]_M \tag{4.87}$$

Then

$$\Delta = \mathtt{cur}(()\Delta_1) \tag{4.88}$$

So
$$\begin{aligned}
\omega^*(\Delta) &= \omega^*(\mathtt{cur}(\Delta_1)) && (4.89)\\
&= \mathtt{cur}(\omega^*(\Delta_1)) \quad \text{By S-closure} && (4.90)\\
&= \mathtt{cur}(\Delta_1') \quad \text{By induction} && (4.91)\\
&= \Delta' && (4.92)
\end{aligned}$$

Case Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v_1\colon A \to B]\!]_M && (4.93)\\
\Delta_2 &= [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]_M && (4.94)
\end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \tag{4.95}$$

So

$$\begin{aligned}
\omega^*\Delta &= \omega^*(\mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle) && (4.96)\\
&= \mathtt{app} \circ \langle \omega^*(\Delta_1), \omega^*(\Delta_2) \rangle \quad \text{By S-closure} && (4.97)\\
&= \mathtt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Induction} && (4.98)\\
&= \Delta' && (4.99)
\end{aligned}$$

Case Return: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M \tag{4.100}$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \tag{4.101}$$

So

$$\begin{aligned}
\omega^*(\Delta) &= \omega^*(\eta_{A_I} \circ \Delta_1) && (4.102)\\
&= \eta_{A_{I'}} \circ \omega^*(\Delta_1) \quad \text{By S-closure} && (4.103)\\
&= \eta_{A_{I'}} \circ \Delta_1' && (4.104)\\
&= \Delta' && (4.105)
\end{aligned}$$

Case Bind: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M && (4.106)\\
\Delta_2 &= [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M && (4.107)
\end{aligned}$$

Then

$$\Delta = \mathtt{M}_{\epsilon_1}\epsilon_2 A_I \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma_I,A_I} \circ \langle \mathtt{Id}_{\Gamma_I}, \Delta_1 \rangle \tag{4.108}$$

So

$$\begin{aligned}
\omega^*(\Delta) &= \omega^*(\mu_{\epsilon_1,\epsilon_2,A} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle) && (4.109)\\
&= \omega^*(\mu_{\epsilon_1,\epsilon_2,A}) \circ \omega^*(T_{\epsilon_1}\Delta_2) \circ \omega^*(\mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \omega^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (4.110)\\
&= \mu_{\epsilon_1,\epsilon_2,A_{I'}} \circ T_{\epsilon_1}\omega^*(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma_{I'},A_{I'}} \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \omega^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (4.111)\\
&= \mu_{\epsilon_1,\epsilon_2,A_{I'}} \circ T_{\epsilon_1}\Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma_{I'},A_{I'}} \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \Delta_1' \right\rangle \quad \text{By Induction} && (4.112)\\
&= \Delta' && (4.113)\\
& && (4.114)
\end{aligned}$$

Case If: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}]\!]_M && (4.115)\\
\Delta_2 &= [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M && (4.116)\\
\Delta_3 &= [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M && (4.117)\\
& && (4.118)
\end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{4.119}$$

So

$$\omega^*(\Delta) = \omega^*(\mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma) \tag{4.120}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(\omega^*(\Delta_2) \circ \pi_2), \mathtt{cur}(\omega^*(\Delta_3) \circ \pi_2)] \circ \omega^*(\Delta_1)) \times \mathtt{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By S-Closure} \tag{4.121}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By Induction} \tag{4.122}$$
$$= \Delta' \tag{4.123}$$
$$\tag{4.124}$$

Case Effect-Lambda: Let

$$\Delta_1 = [\![\Phi, \alpha \mid \Gamma \vdash v \colon A]\!]_M \tag{4.125}$$

Then

$$\Delta = \widehat{\Delta_1} \tag{4.126}$$

And also

$$\omega \times \mathtt{Id} = [\![\omega\times : (\Phi', \alpha) \triangleright (\Phi, \alpha)]\!]_M \tag{4.127}$$

So

$$\omega^*\Delta = \omega^*(\widehat{\Delta_1}) \tag{4.128}$$
$$= \widehat{(\omega \times \mathtt{Id}_U)^*\Delta_1} \quad \text{By naturality} \tag{4.129}$$
$$= \widehat{\Delta_1'} \quad \text{By induction} \tag{4.130}$$
$$= \Delta' \tag{4.131}$$

Case Effect-Application: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon \forall \alpha. A]\!]_M \tag{4.132}$$
$$h = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \tag{4.133}$$
$$\tag{4.134}$$

Then

$$\Delta = \langle \mathtt{Id}_\Gamma, h\rangle^* \, (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1 \tag{4.135}$$

So due to the substitution theorem on effects

$$h \circ \omega = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \omega = [\![\Phi' \vdash \epsilon \colon \mathtt{Effect}]\!]_M = h' \tag{4.136}$$

Also note $(\omega \times \mathtt{Id}_U) = [\![\omega\times : \Phi', \alpha \triangleright \Phi\alpha]\!]_M$

$$\omega^*\Delta = \omega^*(\langle \mathtt{Id}_\Gamma, h\rangle^* \, (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1) \tag{4.137}$$
$$= (\langle \mathtt{Id}_\Gamma, h\rangle \circ \omega)^*(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \omega^*(\Delta_1) \tag{4.138}$$
$$= ((\omega \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_\Gamma, h \circ \omega\rangle)^*(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' \tag{4.139}$$
$$= (\langle \mathtt{Id}_\Gamma, h'\rangle)^*((\omega \times \mathtt{Id}_U)^*\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' \tag{4.140}$$
$$\tag{4.141}$$

Looking at the inner part of the functor application: Let

$$
\begin{align}
A &= [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \texttt{Type}]\!]_M \tag{4.142} \\
& \tag{4.143}
\end{align}
$$

$$
\begin{align}
(\omega \times \texttt{Id}_U)^* \epsilon_{[\![\Phi,\beta\vdash A[\beta/\alpha]:\texttt{Type}]\!]_M} &= (\omega \times \texttt{Id}_U)^* \epsilon_A \tag{4.144} \\
&= (\omega \times \texttt{Id}_U)^* (\widehat{\texttt{Id}_{\forall_I(A)}}) \tag{4.145} \\
&= \widehat{\overline{(\omega \times \texttt{Id}_U)^* (\widehat{\texttt{Id}_{\forall_I(A)}})}} \quad \text{By bijection} \tag{4.146} \\
&= \widehat{\omega^* (\overline{\widehat{\texttt{Id}_{\forall_I(A)}}})} \quad \text{By naturality} \tag{4.147} \\
&= \widehat{\omega^* (\texttt{Id}_{\forall_I(A)})} \quad \text{By bijection} \tag{4.148} \\
&= \widehat{\texttt{Id}_{\forall_{I'}(A \circ (\omega \times \texttt{Id}_U))}} \quad \text{By S-Closure, naturality} \tag{4.149} \\
&= \widehat{\texttt{Id}_{\forall_{I'}(A)}} \quad \text{By Substitution theorem} \tag{4.150} \\
&= \epsilon_{A_{I'}} \tag{4.151}
\end{align}
$$

Going back to the original expression:

$$
\begin{align}
\omega^* \Delta &= (\langle \texttt{Id}_\Gamma, h' \rangle)^* (\epsilon_{A_{I'}}) \circ \Delta_1)' \tag{4.152} \\
&= \Delta' \tag{4.153} \\
& \tag{4.154}
\end{align}
$$

4.7 Term-Substitution

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$, then $[\![\Phi' \mid \Gamma' \vdash \sigma \colon \Gamma]\!]_M = \omega^* [\![\Phi \mid \Gamma' \vdash \sigma \colon \Gamma]\!]_M$.

Proof: By induction on the structure of σ, making use of the weakening of term denotations above.

Case Nil: Then $\sigma = \langle\rangle_{\Gamma'_I}$, so $\omega^*(\sigma) = \langle\rangle_{\Gamma'_{I'}} = [\![\Phi' \mid \Gamma' \vdash \sigma \colon \Gamma]\!]_M$

Case Var: Then $\sigma = (\sigma', x := v)$

$$
\begin{align}
\omega^* \sigma &= \omega * \langle \sigma', [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M \rangle \tag{4.155} \\
&= \langle \omega^* \sigma', \omega^* [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M \rangle \tag{4.156} \\
&= \langle [\![\Phi' \mid \Gamma' \vdash \sigma' \colon \Gamma]\!]_M, [\![\Gamma' \mid \Phi' \vdash v \colon A]\!]_M \rangle \tag{4.157} \\
&= [\![\Phi' \mid \Gamma' \vdash \sigma \colon \Gamma, x : A]\!]_M \tag{4.158}
\end{align}
$$

4.8 Term-Weakening

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$, then $[\![\Phi' \vdash \omega_1 : \Gamma' \triangleright \Gamma]\!]_M = \omega^* [\![\Phi \vdash \omega_1 : \Gamma' \triangleright \Gamma]\!]_M$.

Proof: By induction on the structure of ω_1.

Case Id: Then $\omega_1 = \iota$, so its denotation is $\omega_1 = \mathtt{Id}_{\Gamma_I}$

So

$$\omega^*(\mathtt{Id}_{\Gamma_I}) = \mathtt{Id}_{\Gamma_{I'}} = [\![\Phi' \vdash \iota : \Gamma \triangleright \Gamma]\!]_M \tag{4.159}$$

Case Project: Then $\omega_1 = \omega_1'\pi$

$$(\text{Project})\frac{\Phi \vdash \omega_1' : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1\pi : \Gamma', x : A \triangleright \Gamma} \tag{4.160}$$

So $\omega_1 = \omega_1' \circ \pi_1$

Hence

$$\begin{aligned}
\omega^*(\omega_1) &= \omega^*(\omega_1') \circ \omega^*(\pi_1) && (4.161)\\
&= [\![\Phi' \vdash \omega_1' : \Gamma' \triangleright \Gamma]\!]_M \circ \pi_1 && (4.162)\\
&= [\![\Phi' \vdash \omega_1'\pi : \Gamma', x : A \triangleright \Gamma]\!]_M && (4.163)\\
&= [\![\Phi' \vdash \omega_1 : \Gamma', x : A \triangleright \Gamma]\!]_M && (4.164)
\end{aligned}$$

Case Extend: Then $\omega_1 = \omega_1'\times$

$$(\text{Extend})\frac{\Phi \vdash \omega_1' : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1\times : \Gamma', x : A \triangleright \Gamma, x : A} \tag{4.165}$$

So $\omega_1 = \omega_1' \times \mathtt{Id}_{A_I}$

Hence

$$\begin{aligned}
\omega^*(\omega_1) &= (\omega^*(\omega_1') \times \omega^*(\mathtt{Id}_{A_I}) && (4.166)\\
&= ([\![\Phi' \vdash \omega_1' : \Gamma' \triangleright \Gamma]\!]_M \times \mathtt{Id}_{A_I}) && (4.167)\\
&= [\![\Phi' : \omega_1 \triangleright \Gamma', x : A\Gamma, x : A]\!]_M && (4.168)
\end{aligned}$$

Chapter 5

Value Substitution Theorem

If Δ derives $\Phi \mid \Gamma \vdash v\colon A$ and $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$ then the derivation Δ' deriving $\Phi \mid \Gamma' \vdash v\,[\sigma] : A$ satisfies:

$$\Delta' = \Delta \circ [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M \tag{5.1}$$

This is proved by induction over the derivation of $\Phi \mid \Gamma \vdash v\colon A$. We shall use σ to denote $[\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M$ where it is clear from the context.

Case Var: By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var})\frac{\Phi \vdash \Gamma\,\mathtt{Ok}}{\Phi \mid \Gamma'', x : A \vdash x\colon A} \tag{5.2}$$

By inversion, $\sigma = \sigma', x := v$ and $\Phi \mid \Gamma' \vdash v\colon A$.

Let

$$\sigma = [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{5.3}$$

$$\Delta = [\![\Phi \mid \Gamma'', x : A \vdash x\colon A]\!]_M = \pi_2 \tag{5.4}$$

$$\tag{5.5}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{5.6}$$

$$= \Delta' \quad \text{By product property} \tag{5.7}$$

Case Weaken: By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x:A}}{\Phi \mid \Gamma'', y : B \vdash x\colon A} \tag{5.8}$$

Also by inversion of the well-formed-ness of $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$, we have $\Phi \mid \Gamma' \vdash \sigma'\colon \Gamma''$ and

$$[\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma'']\!]_M, [\![\Phi \mid \Gamma' \vdash v\colon B]\!]_M \rangle \tag{5.9}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$()\frac{\Delta_1'}{\Phi \mid \Gamma' \vdash x\,[\sigma] : A} \tag{5.10}$$

Hence

$$\begin{aligned}
\Delta' &= \Delta_1' \quad \text{By definition} && (5.11)\\
&= \Delta_1 \circ \sigma' \quad \text{By induction} && (5.12)\\
&= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Phi \mid \Gamma' \vdash v : B]\!]_M \rangle \quad \text{By product property} && (5.13)\\
&= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma && (5.14)\\
&= \Delta \circ \sigma \quad \text{By defintion.} && (5.15)
\end{aligned}$$

Case Constants: The logic for all constant terms ($\texttt{true}, \texttt{false}, \texttt{()}, \texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{5.16}$$

$$\begin{aligned}
\Delta' &= c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} && (5.17)\\
&= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} && (5.18)\\
&= \Delta \circ \sigma \quad \text{By definition} && (5.19)
\end{aligned}$$

Case Lambda: By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to B} \tag{5.20}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma', x : A \vdash (v[\sigma]) : B}}{\Phi \mid \Gamma \vdash (\lambda x : A.v)\,[\sigma] : A \to B} \tag{5.21}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{5.22}$$

Hence:

$$\begin{aligned}
\Delta' &= \texttt{cur}(\Delta_1') \quad \text{By definition} && (5.23)\\
&= \texttt{cur}(\Delta_1 \circ (\sigma \times \texttt{Id}_A)) \quad \text{By induction and extension lemma.} && (5.24)\\
&= \texttt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (5.25)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.26)\\
& && (5.27)
\end{aligned}$$

Case Sub-type: By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : A} \quad A \leq : B}{\Phi \mid \Gamma \vdash v : B} \tag{5.28}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:A} \quad A \leq: B}{\Phi \mid \Gamma' \vdash v\,[\sigma] : B} \tag{5.29}$$

Hence,

$$\begin{aligned}
\Delta' &= [\![A \leq: B]\!]_M \circ \Delta'_1 \quad \text{By definition} && (5.30)\\
&= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \sigma \quad \text{By induction} && (5.31)\\
&= \Delta \circ \sigma \quad \text{By definition} && (5.32)\\
& && (5.33)
\end{aligned}$$

Case Return: By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return} v : \mathtt{M_1} A} \tag{5.34}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash (\mathtt{return} v)\,[\sigma] : \mathtt{M_1} A} \tag{5.35}$$

Hence,

$$\begin{aligned}
\Delta' &= \eta_A \circ \Delta'_1 \quad \text{By Definition} && (5.36)\\
&= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} && (5.37)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.38)\\
& && (5.39)
\end{aligned}$$

Case Apply: By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2 : B} \tag{5.40}$$

By induction we find Δ'_1, Δ'_2 such that

$$\begin{aligned}
\Delta'_1 &= \Delta_1 \circ \sigma && (5.41)\\
\Delta'_2 &= \Delta_2 \circ \sigma && (5.42)\\
& && (5.43)
\end{aligned}$$

And

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v_1[\sigma]:A\to B} \quad ()\frac{\Delta'_2}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (v_1\ v_2)\,[\sigma] : B} \tag{5.44}$$

Hence

$$\begin{aligned}
\Delta' &= \mathtt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} && (5.45)\\
&= \mathtt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} && (5.46)\\
&= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} && (5.47)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.48)\\
& && (5.49)
\end{aligned}$$

Case If: By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{5.50}$$

By induction we find $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\begin{aligned}
\Delta'_1 &= \Delta_1 \circ \sigma && (5.51)\\
\Delta'_2 &= \Delta_2 \circ \sigma && (5.52)\\
\Delta'_3 &= \Delta_3 \circ \sigma && (5.53)\\
& && (5.54)
\end{aligned}$$

And

$$\Delta' = (\text{If})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:\mathtt{Bool}} \quad ()\frac{\Delta'_2}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta'_3}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)\,[\sigma] : A} \tag{5.55}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\sigma = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{5.56}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \mathtt{cur}(f) \tag{5.57}$$

And so:

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

$$
\begin{aligned}
\Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta_2' \circ \pi_2), \texttt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (5.58)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \texttt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (5.59)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_{\texttt{1}} \times \sigma)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_{\texttt{1}} \times \sigma))] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (5.60)\\
&= \texttt{app} \circ (([(T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} && (5.61)\\
&= \texttt{app} \circ (((T_\epsilon A)^\sigma \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (5.62)\\
&= \texttt{app} \circ ((T_\epsilon A)^\sigma \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (5.63)\\
&= \texttt{app} \circ (\texttt{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (T_\epsilon A)^\sigma && (5.64)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (5.65)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \sigma \quad \text{By Definition of the diagonal morphism.} && (5.66)\\
&= \Delta \circ \sigma && (5.67)
\end{aligned}
$$

Case Bind: By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_1:B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.68}$$

By property 3,

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x \colon (\Gamma, x : A) \tag{5.69}$$

With denotation (extension lemma)

$$[\![\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x \colon (\Gamma, x : A)]\!]_M = \sigma \times \texttt{Id}_A \tag{5.70}$$

By induction, we derive Δ_1', Δ_2' such that:

$$
\begin{aligned}
\Delta_1' &= \Delta_1 \circ \sigma && (5.71)\\
\Delta_2' &= \Delta_2 \circ (\sigma \times \texttt{Id}_A) \quad \text{By Extension Lemma} && (5.72)
\end{aligned}
$$

And:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta_2'}{\Phi|\Gamma',x:A\vdash v_1[\sigma]:B}}{\Phi \mid \Gamma' \vdash (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.73}$$

Hence:

$$
\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta'_2 \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} && (5.74)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} && (5.75)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} && (5.76)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} && (5.77)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} && (5.78)\\
&= \Delta \circ \sigma \quad \text{By Defintion} && (5.79)\\
& && (5.80)
\end{aligned}
$$

Case Effect-Lambda: By inversion, we have Δ_1 such that

$$\Delta = (\text{Effect-Fn}) \frac{()\frac{\Delta_1}{\Phi,\alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\epsilon.A} \tag{5.81}$$

By induction, we derive Δ'_1 such that

$$\Delta' = (\text{Effect-Fn}) \frac{()\frac{\Delta'_1}{\Phi,\alpha \mid \Gamma' \vdash v[\sigma] : A}}{\Phi \mid \Gamma' \vdash (\Lambda\alpha.v)\,[\sigma] : \forall\epsilon.A} \tag{5.82}$$

Where

$$
\begin{aligned}
\Delta'_1 &= \Delta_1 \circ [\![\Phi, \alpha \mid \Gamma' \vdash \sigma : \Gamma]\!]_M && (5.83)\\
&= \Delta_1 \circ [\![\iota\pi : \Phi, a \triangleright \Phi]\!]^*_M(\sigma && (5.84)\\
&= \Delta_1 \circ \pi^*_1(\sigma) && (5.85)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Delta \circ \sigma &= \overline{\Delta_1} \circ \sigma && (5.86)\\
&= \overline{\Delta_1 \circ \pi^*_1(\sigma)} && (5.87)\\
&= \overline{\Delta'_1} && (5.88)\\
&= \Delta' && (5.89)
\end{aligned}
$$

Case Effect-Application: By inversion, we derive Δ_1 such that

$$\Delta = (\text{Effect-App}) \frac{()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon : A\,[\epsilon/\alpha]} \tag{5.90}$$

By induction, we derive Δ'_1 such that

$$\Delta' = (\text{Effect-App}) \frac{()\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v : \forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash v\ \epsilon : A\,[\epsilon/\alpha]} \tag{5.91}$$

Where

$$\Delta'_1 = \Delta \circ \sigma \tag{5.92}$$

Hence, if $h = [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M$

$$\Delta \circ \sigma = \langle \texttt{Id}_I, h \rangle^* \left(\epsilon [\![\Phi, \beta \vdash A\,[\alpha/\beta] : \texttt{Effect}]\!]_M\right) \circ \Delta_1 \circ \sigma \tag{5.93}$$

$$= \langle \texttt{Id}_I, h \rangle^* \left(\epsilon [\![\Phi, \beta \vdash A\,[\alpha/\beta] : \texttt{Effect}]\!]_M\right) \circ \Delta_1' \tag{5.94}$$

$$= \Delta' \tag{5.95}$$

Chapter 6

Type-Environment Weakening Theorem

Chapter 7

Unique Denotation Theorem

Chapter 8

Beta-Eta-Equivalence Theorem (Soundness)