Contents

Chapter 1

Language Definition

1.1 Terms

1.1.1 Value Terms

$$
\begin{aligned}
v ::= & x \\
& \mid \lambda x : A.C \\
& \mid \mathtt{C}^{A} \\
& \mid \mathtt{()} \\
& \mid \mathtt{true} \mid \mathtt{false}
\end{aligned}
\tag{1.1}
$$

1.1.2 Computation Terms

$$
\begin{aligned}
C ::= & \mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2 \\
& \mid v_1 \quad v_2 \\
& \mid \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 \\
& \mid \mathtt{return} v
\end{aligned}
\tag{1.2}
$$

1.2 Type System

1.2.1 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Value Types

$$A, B, C ::= \gamma \mid A \to \mathtt{M}_{\epsilon} B$$

Computation Types Computation types are of the form $\mathtt{M}_{\epsilon} A$

1.2.2 Sub-typing

There exists a sub-typing pre-order relation $\leq:_{\gamma}$ over ground types that is:

- (Reflexive)$\frac{}{A \leq:_{\gamma} A}$
- (Transitive)$\frac{A \leq:_{\gamma} B B \leq:_{\gamma} C}{A \leq:_{\gamma} C}$

We extend this relation with the function sub-typing rule to yield the full sub-typing relation $\leq:$

- (ground) $\frac{A \leq:_\gamma B}{A \leq: B}$
- (Fn) $\frac{A \leq: A' B' \leq: B \epsilon \leq \epsilon'}{A' \to \mathtt{M}_{\epsilon'} B' \leq: A \to \mathtt{M}_\epsilon B}$

1.2.3 Type Environments

An environment, $G ::= \diamond \mid \Gamma, x : A$

Domain Function

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

`Ok` Predicate

- (Atom) $\frac{}{\diamond \mathtt{Ok}}$
- (Var) $\frac{\Gamma \mathtt{Ok} x \notin \mathtt{dom}(\Gamma)}{\Gamma, x:A \mathtt{Ok}}$

1.2.4 Type Rules

Value Typing Rules

- (Const) $\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A}$
- (Unit) $\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{()} : \mathtt{Unit}}$
- (True) $\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{true} : \mathtt{Bool}}$
- (False) $\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{false} : \mathtt{Bool}}$
- (Var) $\frac{\Gamma, x:A \mathtt{Ok}}{\Gamma, x:A \vdash X : A}$
- (Weaken) $\frac{\Gamma \vdash x:A}{\Gamma, y:B \vdash X:A}$(if $x \neq y$)
- (Fn) $\frac{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B}{\Gamma \vdash \lambda x:A.C : A \to \mathtt{M}_\epsilon B}$
- (Sub) $\frac{\Gamma \vdash v:A A \leq: B}{\Gamma \vdash v:B}$

Computation typing rules

- (Return) $\frac{\Gamma \vdash v:A}{\Gamma \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}} A}$
- (Apply) $\frac{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B \Gamma \vdash v_2 : A}{\Gamma \vdash v_1 v_2 : \mathtt{M}_\epsilon B}$
- (if) $\frac{\Gamma \vdash v:\mathtt{Bool} \Gamma \vdash C_1 : \mathtt{M}_\epsilon A \Gamma \vdash C_2 : \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon, A} V \mathtt{then} C_1 \mathtt{else} C_2 : \mathtt{M}_\epsilon A}$
- (Do) $\frac{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A \Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do} x \leftarrow C_1 \mathtt{in} C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (Subeffect) $\frac{\Gamma \vdash C : \mathtt{M}_{\epsilon_1} A A \leq: B \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \mathtt{M}_{\epsilon_2} B}$

1.2.5 Ok Lemma

If $\Gamma \vdash t\colon \tau$ then Γ`Ok`.

Proof If $\Gamma, x : A$`Ok` then by inversion Γ`Ok` Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require Γ`Ok`. And all non-axiom derivations preserve the `Ok` property.

Chapter 2

Category Requirements

2.1 CCC

The section should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

2.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endofunctor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot\, \mathtt{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

2.2.1 Left Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon, \mathtt{1}, A}"] \\
& T_\epsilon A
\end{tikzcd}$$

2.2.2 Right Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1}, \epsilon, A}"] \\
& T_\epsilon A
\end{tikzcd}$$

2.2.3 Associativity

$$\begin{array}{ccc} T_{\epsilon_1}T_{\epsilon_2}T_{\epsilon_3}A & \xrightarrow{\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3}A}} & T_{\epsilon_1\cdot\epsilon_2}T_{\epsilon_3}A \\ \downarrow{\scriptstyle T_{\epsilon_1}\mu_{\epsilon_2,\epsilon_3,A}} & & \downarrow{\scriptstyle \mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,A}} \\ T_{\epsilon_1}T_{\epsilon_2\cdot\epsilon_3}A & \xrightarrow{\mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,A}} & T_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}A \end{array}$$

2.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$$

Satisfying the following rules:

2.3.1 Left Naturality

$$\begin{array}{ccc} A \times T_\epsilon B & \xrightarrow{\mathtt{Id}_A \times T_\epsilon f} & A \times T_\epsilon B' \\ \downarrow{\scriptstyle \mathtt{t}_{\epsilon,A,B}} & & \downarrow{\scriptstyle \mathtt{t}_{\epsilon,A,B'}} \\ T_\epsilon(A \times B) & \xrightarrow{T_\epsilon(\mathtt{Id}_A \times f)} & T_\epsilon(A \times B') \end{array}$$

2.3.2 Right Naturality

$$\begin{array}{ccc} A \times T_\epsilon B & \xrightarrow{f \times \mathtt{Id}_{T_\epsilon B}} & A' \times T_\epsilon B \\ \downarrow{\scriptstyle \mathtt{t}_{\epsilon,A,B}} & & \downarrow{\scriptstyle \mathtt{t}_{\epsilon,A',B}} \\ T_\epsilon(A \times B) & \xrightarrow{T_\epsilon(f \times \mathtt{Id}_B)} & T_\epsilon(A' \times B) \end{array}$$

2.3.3 Unitor Law

$$\begin{tikzcd} \mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"'] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\ & T_\epsilon A \end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutivity of π_2 with $\mathtt{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B)$$

And $\pi_{2,\mathtt{1}}$ is the left unitor, so by tensorial strength:

$$\begin{aligned} T_\epsilon\pi_2 \circ \mathtt{t}_{\epsilon,A,B} &= T_\epsilon\pi_{2,\mathtt{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon,A,B} \\ &= T_\epsilon\pi_{2,1,B} \circ \mathtt{t}_{\epsilon,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\ &= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\ &= \pi_2 \end{aligned} \tag{2.1}$$

So the following commutes:

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{t}_{\epsilon,A,B}"] \arrow[dr, "\pi_2"'] & T_\epsilon(A \times B) \arrow[d, "T_\epsilon \pi_2"] \\
& T_\epsilon B
\end{tikzcd}$$

2.3.4 Commutativity with Join

$$\begin{tikzcd}
A \times T_{\epsilon_1} T_{\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1} T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A\times B}"] \\
& A \times T_{\epsilon_1 \cdot \epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1 \cdot \epsilon_2,A,B}"] & T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
\end{tikzcd}$$

2.4 Commutivity with Unit

$$\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"'] & A \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] \\
& T_\epsilon(A \times B)
\end{tikzcd}$$

2.5 Commutivity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times T_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,T_\epsilon C}"] & & T_\epsilon((A \times B) \times C) \arrow[d, "T_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times T_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times T_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & T_\epsilon(A \times (B \times C))
\end{tikzcd}$$

TODO: Needed?

2.6 Subeffecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

2.6.1 Subeffecting and Tensor Strength

$$\begin{tikzcd}
A \times T_{\epsilon_1} B \arrow[r, "\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2} B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "[\![\epsilon_1 \leq \epsilon_2]\!]_{A\times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}$$

2.7 Subtyping

The denotation of ground types $[\![\_]\!]_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function subtyping to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)}\frac{f = [\![A' \leq: A]\!]_M \qquad g = [\![B \leq: B']\!]_M \qquad h = [\![\epsilon_1 \leq \epsilon_2]\!]}{rhs = [\![A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B']\!]_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned} rhs =&(h_{B'} \circ T_{\epsilon_1} g)^A \circ (T_{\epsilon_1} B)^f \\ =&\texttt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \texttt{app}) \circ \texttt{cur}(\texttt{app} \circ (\texttt{Id} \times f)) \end{aligned} \tag{2.2}$$

2.8 If natural transformation

There exists a natural transformation $\texttt{If}_A : (\texttt{Bool} \times (A \times A)) \to A$ Satisfying the following:

- $\texttt{If}_A \circ \langle [\![\texttt{true}]\!]_M \circ \langle\rangle_\Gamma, \langle t, f \rangle\rangle = t$
- $\texttt{If}_A \circ \langle [\![\texttt{false}]\!]_M \circ \langle\rangle_\Gamma, \langle t, f \rangle\rangle = f$

Chapter 3

Denotations

3.1 Denotations of Types

3.1.1 Denotation of Type Environments

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an $\texttt{Ok}$ type environment Γ.

$$[\![\diamond]\!]_M = \texttt{1}$$

$$[\![\Gamma, x : A]\!]_M = ([\![\Gamma]\!]_M \times [\![A]\!]_M)$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x : A]\!]_M = \Gamma \times A$$

3.1.2 Denotation of Computation Type

Given a function $[\![\_]\!]_M$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\texttt{M}_\epsilon A$ as so:

$$[\![\texttt{M}_\epsilon A]\!]_M = T_\epsilon [\![A]\!]_M$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\texttt{M}_\epsilon A]\!]_M = T_\epsilon A$$

3.1.3 Denotation of Function Types

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to \texttt{M}_\epsilon B$ as so:

$$[\![A \to \texttt{M}_\epsilon B]\!]_M = (T_\epsilon [\![B]\!]_M)^{[\![A]\!]_M}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$[\![A \to \texttt{M}_\epsilon B]\!]_M = (T_\epsilon B)^A$$

3.2 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$\llbracket \Gamma \vdash t \colon \tau \rrbracket_M : \Gamma \to \llbracket \tau \rrbracket_M$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t \colon \tau$ are equal, we need not refer to the derivation that yielded each denotation.

3.2.1 Denotation of Value Terms

- (Unit) $\frac{\Gamma \mathtt{Ok}}{\llbracket \Gamma \vdash \mathtt{()} : \mathtt{Unit} \rrbracket_M = \llbracket \mathtt{()} \rrbracket_M \circ \langle \rangle_\Gamma : \Gamma \to \llbracket \mathtt{Unit} \rrbracket_M}$

- (Const) $\frac{\Gamma \mathtt{Ok}}{\llbracket \Gamma \vdash \mathtt{C}^A : A \rrbracket_M = \llbracket \mathtt{C}^A \rrbracket_M \circ \langle \rangle_\Gamma : \Gamma \to \llbracket A \rrbracket_M}$

- (True) $\frac{\Gamma \mathtt{Ok}}{\llbracket \Gamma \vdash \mathtt{true} : \mathtt{Bool} \rrbracket_M = \llbracket \mathtt{true} \rrbracket_M \circ \langle \rangle_\Gamma : \Gamma \to \llbracket \mathtt{Bool} \rrbracket_M}$

- (False) $\frac{\Gamma \mathtt{Ok}}{\llbracket \Gamma \vdash \mathtt{false} : \mathtt{Bool} \rrbracket_M = \llbracket \mathtt{false} \rrbracket_M \circ \langle \rangle_\Gamma : \Gamma \to \llbracket \mathtt{Bool} \rrbracket_M}$

- (Var) $\frac{\Gamma \mathtt{Ok}}{\llbracket \Gamma , x : A \vdash x : A \rrbracket_M = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\frac{f = \llbracket \Gamma \vdash x : A \rrbracket_M : \Gamma \to A}{\llbracket \Gamma , y : B \vdash x : A \rrbracket_M = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\frac{f = \llbracket \Gamma , x : A \rrbracket_M C \mathtt{M}_\epsilon B : \Gamma \times A \to T_\epsilon B}{\llbracket \Gamma \vdash \lambda x : A . C : A \to \mathtt{M}_\epsilon B \rrbracket_M = \mathtt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\frac{f = \llbracket \Gamma \vdash v : A \rrbracket_M : \Gamma \to A g = \llbracket A \leq : B \rrbracket_M}{\llbracket \Gamma \vdash v : B \rrbracket_M = g \circ f : \Gamma \to B}$

3.2.2 Denotation of Computation Terms

- (Return) $\frac{f = \llbracket \Gamma \vdash v : A \rrbracket_M}{\llbracket \Gamma \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}} A \rrbracket_M = \eta_A \circ f}$

- (If) $\frac{f = \llbracket \Gamma \vdash v : \mathtt{Bool} \rrbracket_M g = \llbracket \Gamma \vdash C_1 : \mathtt{M}_\epsilon A \rrbracket_M h = \llbracket \Gamma \vdash C_2 : \mathtt{M}_\epsilon A \rrbracket_M}{\llbracket \Gamma \vdash \mathtt{if}_{\epsilon, A} v \mathtt{then} C_1 \mathtt{else} C_2 : \mathtt{M}_\epsilon A \rrbracket_M = \mathtt{If}_{\mathtt{M}_\epsilon B} \circ \langle f , \langle g , h \rangle \rangle : \Gamma \to T_\epsilon A}$

- (Bind) $\frac{f = \llbracket \Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A : \Gamma \to T_{\epsilon_1} A g = \llbracket \Gamma , x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B \rrbracket_M \rrbracket_M : \Gamma \times A \to T_{\epsilon_2} B}{\llbracket \Gamma \vdash \mathtt{do} x \leftarrow C_1 \mathtt{in} C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} \rrbracket_M = \mu_{\epsilon_1 , \epsilon_2 , B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma , A , \epsilon_1} \circ \left\langle \mathtt{Id}_\Gamma , f \right\rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$

- (Subeffect) $\frac{f = \llbracket \Gamma \vdash c : \mathtt{M}_{\epsilon_1} A \rrbracket_M : \Gamma \to T_{\epsilon_1} A g = \llbracket A \leq : B \rrbracket_M h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket}{\llbracket \Gamma \vdash C : \mathtt{M}_{\epsilon_2} B \rrbracket_M = h_B \circ T_{\epsilon_1} g \circ f}$

- (Apply) $\frac{f = \llbracket \Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B \rrbracket_M : \Gamma \to (T_\epsilon B)^A g = \llbracket \Gamma \vdash v_2 : A \rrbracket_M : \Gamma \to A}{\llbracket \Gamma \vdash v_1 v_2 : \mathtt{M}_\epsilon B \rrbracket_M = \mathtt{app} \circ \langle f , g \rangle : \Gamma \to T_\epsilon B}$

Chapter 4

Unique Denotations

4.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, **lambda** or **apply** rule.

4.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t \colon \tau$,there exists at most one reduced derivation of $\Gamma \vdash t \colon \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

4.2.1 Constants

For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A} \qquad A \leq: B}{\Gamma \vdash \mathtt{C}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

4.2.2 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C \colon A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.B \colon A \to \mathtt{M}_\epsilon B} \qquad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C \colon A' \to \mathtt{M}_{\epsilon'} B'}$$

Where Δ is the reduced derivation of $\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B$ if it exists.

Case Subtype TODO: Do we need to write anything here? (Probably needs an explanation)

4.2.3 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\frac{()\frac{\Delta}{\Gamma \vdash v \colon A}}{\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_{\mathtt{1}} A} \qquad A \leq: A' \leq: B \qquad \mathtt{1} \leq \epsilon}{\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_\epsilon B}$$

Where

$$(\text{Subtype})\frac{()\frac{\Delta}{\Gamma\vdash v:A} \qquad A \leq: A'}{\Gamma \vdash v\colon A'}$$

is the reduced derivation of $\Gamma \vdash v\colon A'$

Case Apply If

$$(\text{Subtype})\frac{()\frac{\Delta}{\Gamma\vdash v_1:A\rightarrow \mathtt{M}_\epsilon BB} \qquad A \rightarrow \mathtt{M}_\epsilon BB \leq: A' \rightarrow \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1\colon A' \rightarrow \mathtt{M}_{\epsilon'} B'}$$

and

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash v_2:A''} \qquad A'' \leq: A'}{\Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1\colon A' \rightarrow \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2\colon A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1 \quad v_2\colon \mathtt{M}_{\epsilon'} B'$ as

$$(\text{Subeffect})\frac{(\text{Apply})\frac{()\frac{\Delta}{\Gamma\vdash v_1:A\rightarrow \mathtt{M}_\epsilon B} \qquad (\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash v:A''} \qquad A''\leq:A}{\Gamma\vdash v:A}}{\Gamma\vdash v_1 \quad v_2:\mathtt{M}_\epsilon B} \qquad B \leq: B' \qquad \epsilon \leq \epsilon'}{\Gamma \vdash v_1 \quad v_2\colon \mathtt{M}_{\epsilon'} B'}$$

Case If

Case Bind

Case Subeffect

4.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t\colon \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed. **TODO: Fill in these cases with actual maths**

4.3.1 Constants

TODO: *reduce* just appends the identity subtype rule to the derivation, trivially preserves denotation

4.3.2 Value Types

Lambda TODO: Recursively call reduce on C then push subtyping through using currying

Subtype TODO: Recursively call reduce then merge subtypes

4.3.3 Computation Types

Return TODO: Recursively call reduce then use naturality to push subtyping into subeffect

Apply TODO: Recursively call reduce, then construct the reduced apply as in the proof of uniqueness

If TODO: Recursively call reduce, then leave tree otherwise unchanged.

Bind TODO: Recursively call reduce then push subtyping rules through the bind

Subeffect TODO: Recursively call reduce, then merge subeffecting rules

4.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t\colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce \Delta]\!]_M = [\![reduce \Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t\colon \tau]\!]_M$ is unique.

Chapter 5

Weakening

5.1 Weakening Definition

5.1.1 Relation

We define the ternary weaking relation $w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Gamma \mathtt{Ok}}{\iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\omega : \Gamma' \triangleright \Gamma x \notin \mathtt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\omega : \Gamma' \triangleright \Gamma \times \notin \mathtt{dom}(\Gamma') A \leq : B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B}$

5.1.2 `Ok` definition

5.1.3 `Dom` definition

5.1.4 Weakening Denotations

5.2 Weakening Theorems

5.2.1 Theorem 1

If $\omega : \Gamma' \triangleright \Gamma$ and $\Gamma \mathtt{Ok}$ then $\Gamma' \mathtt{Ok}$

Proof **TODO: this**

5.2.2 Theorem 2

If $\Gamma \vdash t\colon \tau$ and $\omega : \Gamma' \triangleright \Gamma$ then $\Gamma' \vdash t\colon \tau$

Proof Proved in parallel with theorem 3 below

5.2.3 Theorem 3

If $\omega : \Gamma' \triangleright \Gamma$ and $\Delta = [\![\Gamma \vdash t\colon \tau]\!]_M$ and $\Delta' = [\![\Gamma' \vdash t\colon \tau]\!]_M$ then

$$\Delta \circ [\![\omega]\!]_M = \Delta' : \Gamma' \to [\![\tau]\!]_M$$

Proof **TODO: this, induct over typing relation/definition of Denotations**

Chapter 6

Substitution

6.1 Introduce Substitutions

6.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{6.1}$$

6.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{6.2}$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{6.3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{6.4}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{6.5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6.6}$$

6.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{6.7}$$
$$x\,[\sigma, x := v] = v \tag{6.8}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{6.9}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.10}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.11}$$
$$(\mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A} \quad v\,[\sigma] \quad \mathtt{then} \quad C_1\,[\sigma] \quad \mathtt{else} \quad C_2\,[\sigma] \tag{6.12}$$
$$(v_1 \quad v_2)\,[\sigma] = (v_1\,[\sigma]) \quad v_2\,[\sigma] \tag{6.13}$$
$$(\mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2) = \mathtt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \mathtt{in} \quad (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.14}$$
$$\tag{6.15}$$

6.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil) $\frac{\Gamma' \texttt{Ok}}{\Gamma' \vdash \diamond \colon \diamond}$
- (Extend) $\frac{\Gamma' \vdash \sigma \colon \Gamma x \notin \texttt{dom}(\Gamma) \Gamma' \vdash v \colon A}{\Gamma' \vdash (\sigma, x := v) \colon (\Gamma, x \colon A)}$

6.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma \colon \Gamma$ then: **TODO: Number these**

$\Gamma\texttt{Ok}$ and $\Gamma'\texttt{Ok}$ Since $\Gamma'\texttt{Ok}$ holds by the Nil-axiom. $\Gamma\texttt{Ok}$ holds by induction on the well-formed-ness relation.

$\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v \colon A$ holds if $\Gamma' \vdash v \colon A$ holds.

$x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{6.16}$$

6.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g \colon \tau \wedge \Gamma' \vdash \sigma \colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{6.17}$$

TODO: Proof by induction over type relation Assuming $\Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t \colon \tau \rightarrow \Gamma' \vdash t \colon \tau$

6.2.1 Variables

Case Var **TODO: The more difficult case. case split on the structure of σ**

Case Weaken **TODO:**

6.2.2 Other Value Terms

Case Lambda **TODO:**

Case Constants **TODO:**

Case Unit **TODO:**

Case True **TODO:**

False **TODO:**

6.2.3 Computation Terms

Case Return **TODO: Induct using preconditions, then construct new tree**

Case Apply **TODO:**

Case If **TODO:**

Case Bind **TODO:**

6.2.4 Sub-typing and Sub-effecting

Case Sub-type **TODO:**

Case Sub-effect **TODO:**

6.3 Semantics of Substitution

6.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma' \vdash \sigma : \Gamma]\!]_M : \Gamma' \to \Gamma \tag{6.18}$$

- (Nil) $\frac{\Gamma' \texttt{Ok}}{[\![\Gamma' \vdash \diamond : \diamond]\!]_M = \langle\rangle_{\Gamma'}}$
- (Extend) $\frac{f = [\![\Gamma' \vdash \sigma : \Gamma]\!]_M \, g = [\![\Gamma' \vdash v : A]\!]_M}{[\![\Gamma' \vdash (\sigma, x := v : (\Gamma, x : A)]\!]_M = \langle f, g \rangle : \Gamma' \to (\Gamma \times A)}$

6.3.2 Lemma

TODO: Fill in from p98

6.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If $\Gamma \vdash t : \tau$ and $\Gamma' \vdash \sigma : \Gamma$ then

6.4 Single Substitution

Chapter 7

Beta Eta Equivalence (Soundness)

7.1 Beta and Eta Equivalence

7.1.1 Beta conversions

- (Lambda) $\frac{\Gamma, x:A \vdash C:\mathtt{M}_\epsilon B \quad \Gamma \vdash v:A}{\Gamma \vdash (\lambda x:A.C)v =_{\beta\eta} C[x/v]:\mathtt{M}_\epsilon B}$

- (Left Unit) $\frac{\Gamma \vdash v:A \quad \Gamma, x:A \vdash C:\mathtt{M}_\epsilon B}{\Gamma \vdash \mathtt{do}\, x \leftarrow \mathtt{return}\, v\, \mathtt{in}\, C =_{\beta\eta} C[V/x]:\mathtt{M}_\epsilon B}$

- (Right Unit) $\frac{\Gamma \vdash C:\mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{do}\, x \leftarrow C\, \mathtt{in}\, \mathtt{return}\, x =_{\beta\eta} C:\mathtt{M}_\epsilon A}$

- (Associativity) $\frac{\Gamma \vdash C_1:\mathtt{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2:\mathtt{M}_{\epsilon_2} B \quad \Gamma, y:B \vdash C_3:\mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do}\, x \leftarrow C_1\, \mathtt{in}(\mathtt{do}\, y \leftarrow C_2\, \mathtt{in}\, C_3) =_{\beta\eta} \mathtt{do}\, y \leftarrow (\mathtt{do}\, x \leftarrow C_1\, \mathtt{in}\, C_2)\mathtt{in}\, C_3:\mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$

- (Eta) $\frac{\Gamma \vdash v:A \to \mathtt{M}_\epsilon B}{\Gamma \vdash \lambda x:A.(vx) =_{\beta\eta} v:A \to \mathtt{M}_\epsilon B}$

- (if-true) $\frac{\Gamma \vdash C_1:\mathtt{M}_\epsilon A \quad \Gamma \vdash C_2:\mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\, \mathtt{true}\, \mathtt{then}\, C_1\, \mathtt{else}\, C_2 =_{\beta\eta} C_1:\mathtt{M}_\epsilon A}$

- (if-false) $\frac{\Gamma \vdash C_2:\mathtt{M}_\epsilon A \quad \Gamma \vdash C_1:\mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\, \mathtt{false}\, \mathtt{then}\, C_1\, \mathtt{else}\, C_2 =_{\beta\eta} C_2:\mathtt{M}_\epsilon A}$

7.1.2 Equivalence Relation

- (Reflexive) $\frac{\Gamma \vdash t:\tau}{\Gamma \vdash t =_{\beta\eta} t:\tau}$

- (Symmetric) $\frac{\Gamma \vdash t_1 =_{\beta\eta} t_2:\tau}{\Gamma \vdash t_2 =_{\beta\eta} t_1:\tau}$

- (Transitive) $\frac{\Gamma \vdash t_1 =_{\beta\eta} t_2:\tau \quad \Gamma \vdash t_2 =_{\beta\eta} t_3:\tau}{\Gamma \vdash t_1 =_{\beta\eta} t_3:\tau}$

7.1.3 Congruences

- (Lambda) $\frac{\Gamma, x:A \vdash C_1 =_{\beta\eta} C_2:\mathtt{M}_\epsilon B}{\Gamma \vdash \lambda x:A.C_1 =_{\beta\eta} \lambda x:A.C_2:A \to \mathtt{M}_\epsilon B}$

- (Return) $\frac{\Gamma \vdash v_1 =_{\beta\eta} v_2:A}{\Gamma \vdash \mathtt{return}\, v_1 =_{\beta\eta} \mathtt{return}\, v_2:\mathtt{M}_{\mathtt{1}} A}$

- (Apply) $\frac{\Gamma \vdash v_1 =_{\beta\eta} v_1':A \to \mathtt{M}_\epsilon B \quad \Gamma \vdash v_2 =_{\beta\eta} v_2':A}{\Gamma \vdash v_1 v_2 =_{\beta\eta} v_1' v_2':\mathtt{M}_\epsilon B}$

- (Bind) $\frac{\Gamma \vdash C_1 =_{\beta\eta} C_1':\mathtt{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2 =_{\beta\eta} C_2':\mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do}\, x \leftarrow C_1\, \mathtt{in}\, C_2 =_{\beta\eta} \mathtt{do}\, c \leftarrow C_1'\, \mathtt{in}\, C_2':\mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

- (If) $\frac{\Gamma\vdash v =_{\beta\eta} v' \colon \texttt{Bool}\Gamma\vdash C_1 =_{\beta\eta} C_1' \colon \texttt{M}_\epsilon A\Gamma\vdash C_2 =_{\beta\eta} C_2' \colon \texttt{M}_\epsilon A}{\Gamma\vdash \texttt{if}_{\epsilon,A} v \texttt{then} C_1 \texttt{else} C_2 =_{\beta\eta} \texttt{if}_{\epsilon,A} v \texttt{then} C_1' \texttt{else} C_2' \colon \texttt{M}_\epsilon A}$
- (Subtype) $\frac{\Gamma\vdash v =_{\beta\eta} v' \colon AA \leq \colon B}{\Gamma\vdash v =_{\beta\eta} v' \colon B}$
- (Subeffect) $\frac{\Gamma\vdash C =_{\beta\eta} C' \colon \texttt{M}_{\epsilon_1} AA \leq \colon B\epsilon_1 \leq \epsilon_2}{\Gamma\vdash C =_{\beta\eta} C' \colon \texttt{M}_{\epsilon_2} B}$

7.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

Each derivation of $\Gamma \vdash t =_{\beta\eta} t' \colon \tau$ can be converted to a derivation of $\Gamma \vdash t \colon \tau$ and $\Gamma \vdash t' \colon \tau$ by induction over the beta-eta equivalence relation derivation.

7.2.1 Equivalence Relations

Case Reflexive By inversion we have a derivation of $\Gamma \vdash t \colon \tau$.

Case Symmetric By inversion $\Gamma \vdash t' =_{\beta\eta} t \colon \tau$. Hence by induction, derivations of $\Gamma \vdash t' \colon \tau$ and $\Gamma \vdash t \colon \tau$ are given.

Case Transitive By inversion, there exists t_2 such that $\Gamma \vdash t_1 =_{\beta\eta} t_2 \colon \tau$ and $\Gamma \vdash t_2 =_{\beta\eta} t_3 \colon \tau$. Hence by induction, we have derivations of $\Gamma \vdash t_1 \colon \tau$ and $\Gamma \vdash t_3 \colon \tau$

7.2.2 Beta conversions

Case Lambda By inversion, we have $\Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B$ and $\Gamma \vdash v \colon A$. Hence by the typing rules, we have:

$$\text{(Apply)}\frac{\text{(Lambda)}\frac{\Gamma, x:A \vdash C \colon \texttt{M}_\epsilon B}{\Gamma \vdash \lambda x:A.C \colon A \to \texttt{M}_\epsilon B} \qquad \Gamma \vdash v \colon A}{\Gamma \vdash (\lambda x : A.C) \quad v \colon \texttt{M}_\epsilon A}$$

By the substitution rule **TODO: which?**, we have

$$\text{(Substitution)}\frac{\Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B \qquad \Gamma \vdash v \colon A}{\Gamma \vdash C\,[v/x] : \texttt{M}_\epsilon B}$$

Case Left Unit By inversion, we have $\Gamma \vdash v \colon A$ and $\Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B$

Hence we have:

$$\text{(Bind)}\frac{\text{(Return)}\frac{\Gamma \vdash v \colon A}{\Gamma \vdash \texttt{return} v \colon \texttt{M}_{\texttt{1}} A} \qquad \Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B}{\Gamma \vdash \texttt{do} \quad x \leftarrow \texttt{return} v \quad \texttt{in} \quad C \colon \texttt{M}_{\texttt{1}\cdot\epsilon} B = \texttt{M}_\epsilon B} \tag{7.1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Gamma \vdash C\,[v/x] : \texttt{M}_\epsilon B \tag{7.2}$$

Case Right Unit By inversion, we have $\Gamma \vdash C \colon \texttt{M}_\epsilon A$.

Hence we have:

$$\text{(Bind)}\frac{\Gamma \vdash C \colon \texttt{M}_\epsilon A \qquad \text{(Return)}\frac{\text{(var)}\frac{}{\Gamma, x:A \vdash x:A}}{\Gamma, x:A \vdash \texttt{return} v \colon \texttt{M}_{\texttt{1}} A}}{\Gamma \vdash \texttt{do} \quad x \leftarrow C \quad \texttt{in} \quad \texttt{return} x \colon \texttt{M}_{\epsilon\cdot\texttt{1}} A = \texttt{M}_\epsilon A} \tag{7.3}$$

Case Associativity By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$(\iota\pi\times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Gamma, x : A, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \text{(Bind)}\frac{\Gamma, x{:}A \vdash C_2 {:} \mathtt{M}_{\epsilon_2} B \qquad \Gamma, x{:}A, y{:}B \vdash C_3 {:} \mathtt{M}_{\epsilon_3} C}{\Gamma, x{:}A \vdash x C_2 C_3 {:} \mathtt{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad (\mathtt{do} \quad y \leftarrow C_2 \quad \mathtt{in} \quad C_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\frac{\Gamma \vdash C_1 {:} \mathtt{M}_{\epsilon_1} A \qquad \Gamma, x{:}A \vdash C_2 {:} \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 {:} \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \qquad \Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do} \quad y \leftarrow (\mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2) \quad \mathtt{in} \quad C_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.5}$$

Case Eta By inversion, we have $\Gamma \vdash v \colon A \to \mathtt{M}_\epsilon B$

By weakening, we have $\iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\frac{\text{(App)}\frac{(\Gamma, x{:}A) \vdash x{:}A \qquad \text{(weakening)}\frac{\Gamma \vdash v{:}A \to \mathtt{M}_\epsilon B \qquad \iota\pi{:}\Gamma, x{:}A \triangleright \Gamma}{\Gamma, x{:}A \vdash v{:}A \to \mathtt{M}_\epsilon B}}{\Gamma, x{:}A \vdash v \quad x{:}\mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.(v \quad x) \colon A \to \mathtt{M}_\epsilon B} \tag{7.6}$$

Case If True By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A$. Hence by the typing lemma **TODO: Which?**, we have $\Gamma \mathtt{Ok}$ so $\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \qquad \Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A \qquad \Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon, A} \quad \mathtt{true} \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2 \colon \mathtt{M}_\epsilon A} \tag{7.7}$$

Case If False As above,

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \qquad \Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A \qquad \Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon, A} \quad \mathtt{false} \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2 \colon \mathtt{M}_\epsilon A} \tag{7.8}$$

7.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda By inversion, $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_\epsilon B$. Hence by induction $\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B$, and $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B$.

So

$$\Gamma \vdash \lambda x : A.C_1 \colon A \to \mathtt{M}_\epsilon B \tag{7.9}$$

and

$$\Gamma \vdash \lambda x : A.C_2 \colon A \to \mathtt{M}_\epsilon B \tag{7.10}$$

Hold.

Case Return By inversion, $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, so by induction

$$\Gamma \vdash v_1 \colon A$$

and

$$\Gamma \vdash v_2 \colon A$$

Hence we have

$$\Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M_1} A$$

and

$$\Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M_1} A$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \mathtt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$. Hence we have by induction $\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B$, $\Gamma \vdash v_2 \colon A$, $\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B$, and $\Gamma \vdash v_2' \colon A$.

So we have:

$$\Gamma \vdash v_1 \quad v_2 \colon \mathtt{M}_\epsilon B \tag{7.11}$$

and

$$\Gamma \vdash v_1' \quad v_2' \colon \mathtt{M}_\epsilon B \tag{7.12}$$

Case Bind By inversion, we have: $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.13}$$

$$\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1' \quad \mathtt{in} \quad C_2' \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.14}$$

Case If By inversion, we have: $\Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$, and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon A$.

Hence by induction, we have:
$\Gamma \vdash v \colon \mathtt{Bool}$, $\Gamma \vdash v' \colon \mathtt{Bool}$,
$\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A$,
$\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A$, and $\Gamma \vdash C_2' \colon \mathtt{M}_\epsilon A$.
So

$$\Gamma \vdash \mathtt{if}_{\epsilon, A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2 \colon \mathtt{M}_\epsilon A \tag{7.15}$$

and

$$\Gamma \vdash \mathtt{if}_{\epsilon, A} \quad v \quad \mathtt{then} \quad C_1' \quad \mathtt{else} \quad C_2' \colon \mathtt{M}_\epsilon A \tag{7.16}$$

Hold.

Case Subtype By inversion, we have $A \leq: B$ and $\Gamma \vdash v =_{\beta\eta} v' \colon A$. By induction, we therefore have $\Gamma \vdash v \colon A$ and $\Gamma \vdash v' \colon A$.

Hence we have

$$\Gamma \vdash v \colon B \tag{7.17}$$

$$\Gamma \vdash v' \colon B \tag{7.18}$$

Case subeffect By inversion we have: $A \leq: B$, $\epsilon_1 \leq \epsilon_2$, and $\Gamma \vdash C =_{\beta\eta} C' \colon \mathtt{M}_{\epsilon_1} A$.
Hence by inductive hypothesis, we have $\Gamma \vdash C \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma \vdash C' \colon \mathtt{M}_{\epsilon_1} A$.
Hence,

$$\Gamma \vdash C \colon \mathtt{M}_{\epsilon_2} B \tag{7.19}$$

and

$$\Gamma \vdash C' \colon \mathtt{M}_{\epsilon_2} B \tag{7.20}$$

hold.

7.3 Beta-Eta equivalent terms have equal denotations

If $t \vdash t' =_{\beta\eta} \tau$: then $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$
By induction over Beta-eta equivalence relation.

7.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

Case Reflexive Equality is reflexive, so if $\Gamma \vdash t \colon \tau$ then $[\![\Gamma \vdash t \colon \tau]\!]_M$ is equal to itself.

Case Symmetric By inversion, if $\Gamma \vdash t =_{\beta\eta} t' \colon \tau$ then $\Gamma \vdash t' =_{\beta\eta} t \colon \tau$, so by induction $[\![\Gamma \vdash t' \colon \tau]\!]_M = [\![\Gamma \vdash t \colon \tau]\!]_M$ and hence $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$

Case Transitive There must exist t_2 such that $\Gamma \vdash t_1 =_{\beta\eta} t_2 \colon \tau$ and $\Gamma \vdash t_2 =_{\beta\eta} t_3 \colon \tau$, so by induction, $[\![\Gamma \vdash t_1 \colon \tau]\!]_M = [\![\Gamma \vdash t_2 \colon \tau]\!]_M$ and $[\![\Gamma \vdash t_2 \colon \tau]\!]_M = [\![\Gamma \vdash t_3 \colon \tau]\!]_M$. Hence by transitivity of equality, $[\![\Gamma \vdash t_1 \colon \tau]\!]_M = [\![\Gamma \vdash t_3 \colon \tau]\!]_M$

7.3.2 Beta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

Case Lambda Let $f = [\![\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B]\!]_M : (\Gamma \times A) \to T_\epsilon B$
Let $g = [\![\Gamma \vdash v \colon A]\!]_M : \Gamma \to A$
By the substitution denotation,

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned} [\![\Gamma \vdash (\lambda x : A.C) \quad v \colon \mathtt{M}_\epsilon B]\!]_M &= \mathtt{app} \circ \langle \mathtt{cur}(f), g \rangle \\ &= \mathtt{app} \circ (\mathtt{cur}(f) \times \mathtt{Id}_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\ &= f \circ \langle \mathtt{Id}_\Gamma, g \rangle \\ &= [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M \end{aligned} \tag{7.21}$$

Case Left Unit Let $f = \llbracket \Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B \rrbracket_M$
Let $g = \llbracket \Gamma \vdash v \colon A \rrbracket_M : \Gamma \to A$
By the substitution denotation,

$$\llbracket \Gamma \vdash [v/x] : \Gamma, x : A \rrbracket_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B \rrbracket_M = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned}
\llbracket \Gamma \vdash \mathtt{do} \quad x \leftarrow \mathtt{return} v \quad \mathtt{in} \quad C \colon \mathtt{M}_\epsilon B \rrbracket_M =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \eta_A \circ g \rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times \eta_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \eta_{(\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\
=& \mu_{\mathbf{1},\epsilon,B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\
=& f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\
=& \llbracket \Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B \rrbracket_M
\end{aligned} \tag{7.22}$$

Case Right Unit Let $f = \llbracket \Gamma \vdash C \colon \mathtt{M}_\epsilon A \rrbracket_M$

$$\begin{aligned}
\llbracket \Gamma \vdash \mathtt{do} \quad x \leftarrow C \quad \mathtt{in} \quad \mathtt{return} x \colon \mathtt{M}_\epsilon A \rrbracket_M &= \mu_{\epsilon,\mathbf{1},A} \circ T_\epsilon(\eta_A \circ \pi_2) \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= \pi_2 \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= f
\end{aligned} \tag{7.23}$$

Case Associative Let

$$f = \llbracket \Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A \rrbracket_M \tag{7.24}$$
$$g = \llbracket \Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B \rrbracket_M \tag{7.25}$$
$$h = \llbracket \Gamma, y : B \vdash C_3 \colon \mathtt{M}_\epsilon C \rrbracket_M \tag{7.26}$$

We also have the weakening:

$$\iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \tag{7.27}$$

With denotation:

$$\llbracket \iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \rrbracket_M = (\pi_1 \times \mathtt{Id}_B) \tag{7.28}$$

We need to prove that the following are equal

$$lhs = \llbracket \Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad (\mathtt{do} \quad y \leftarrow C_2 \quad \mathtt{in} \quad C_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2} \rrbracket_M \tag{7.29}$$
$$= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g \right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \tag{7.30}$$
$$rhs = \llbracket \Gamma \vdash \mathtt{do} \quad y \leftarrow (\mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2) \quad \mathtt{in} \quad C_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2} \rrbracket_M \tag{7.31}$$
$$= \mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_1\cdot\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_1\cdot\epsilon_2,\Gamma,B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \tag{7.32}$$
$$\tag{7.33}$$

Let's look at fragment F of rhs.

$$F = \mathtt{t}_{\epsilon_1\cdot\epsilon_2,\Gamma,B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \tag{7.34}$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \tag{7.35}$$

$$\begin{aligned}
F &= \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\
&= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathtt{t}_{\epsilon_2, \Gamma, B} \circ \mathtt{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\mathtt{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{By } \textbf{TODO: ref: mu+tstrength} \\
&= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{By naturality of t-strength}
\end{aligned} \tag{7.36}$$

Since $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F$,

$$\begin{aligned}
rhs =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\
=& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\
=& \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle
\end{aligned} \tag{7.37}$$

Let's now look at the fragment G of rhs

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \tag{7.38}$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B}) \circ G \tag{7.39}$$

By folding out the $\langle ..., ... \rangle$, we have

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ (\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1, \Gamma, A}) \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f \rangle \rangle \tag{7.40}$$

From the rule **TODO: Ref** showing the commutivity of tensor strength with α, the following commutes

$$\begin{tikzcd}
\Gamma \arrow[r, "\langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f \rangle \rangle"] & \Gamma \times (\Gamma \times T_{\epsilon_1} A) \arrow[d, "\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1, \Gamma, A}"] & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \arrow[l, "\alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}"'] \arrow[d, "\mathtt{t}_{\epsilon_1, (\Gamma \times \Gamma), A}"] \\
 & \Gamma \times T_{\epsilon_1}(\Gamma \times A) \arrow[d, "\mathtt{t}_{\epsilon_1, \Gamma, \Gamma \times A}"] & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \arrow[dl, "T_{\epsilon_1} \alpha_{\Gamma, \Gamma, A}"] \\
 & T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) &
\end{tikzcd}$$

Where $\alpha : ((\_ \times \_) \times \_) \to (\_ \times (\_ \times \_))$ is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle \tag{7.41}$$

$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle \tag{7.42}$$

So:

$$\begin{aligned}
G =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathtt{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \alpha^{-1}_{\Gamma, \Gamma, (T_{\epsilon_1} A)} \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathtt{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \times \mathtt{Id}_{T_{\epsilon_1} A}) \circ \langle \mathtt{Id}_\Gamma, f \rangle \quad \text{By definition of } \alpha \text{ and products} \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \quad \text{By tensor strength's left-naturality} \\
=& T_{\epsilon_1}((\pi_1 \times \mathtt{Id}_{T_{\epsilon_2} B}) \circ \left\langle \mathtt{Id}_{(\Gamma \times A)}, g \right\rangle) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle
\end{aligned} \tag{7.43}$$

Since

$$rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{7.44}$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\ &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \mathtt{Id}_B)) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By Left-Tensor Stre} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \tag{7.45}$$

Case Eta Let

$$f = [\![\Gamma \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \tag{7.46}$$

By weakening, we have

$$[\![\Gamma, x : A \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M = f \circ \pi_1 : \Gamma \times A \to (T_\epsilon B)^A \tag{7.47}$$

$$[\![\Gamma, x : A \vdash v \quad x\colon \mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle \tag{7.48}$$

$$\tag{7.49}$$

Hence, we have

$$\begin{aligned} [\![\Gamma \vdash \lambda x : A.(v \quad x)\colon A \to \mathtt{M}_\epsilon B]\!]_M &= \mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) \\ \mathtt{app} \circ ([\![\Gamma \vdash \lambda x : A.(v \quad x)\colon A \to \mathtt{M}_\epsilon B]\!]_M \times \mathtt{Id}_A) &= \mathtt{app} \circ (\mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) \times \mathtt{Id}_A) \\ &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle \\ &= \mathtt{app} \circ (f \times \mathtt{Id}_A) \end{aligned} \tag{7.50}$$

Hence, by the fact that $\mathtt{cur}(f)$ is unique in a cartesian closed category,

$$[\![\Gamma \vdash \lambda x : A.(v \quad x)\colon A \to \mathtt{M}_\epsilon B]\!]_M = f = [\![\Gamma \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.51}$$

Case If-True Let

$$f =[\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.52}$$

$$g =[\![\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.53}$$

$$\tag{7.54}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{true},A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2\colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle [\![\mathtt{true}]\!]_M \circ \langle\rangle_\Gamma, \langle f, g\rangle\rangle \\ &= f \\ &= [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{7.55}$$

Case If-False Let

$$f =[\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.56}$$

$$g =[\![\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.57}$$

$$\tag{7.58}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{false},A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2\colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle [\![\mathtt{false}]\!]_M \circ \langle\rangle_\Gamma, \langle f, g\rangle\rangle \\ &= g \\ &= [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{7.59}$$

7.3.3 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of subexpressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

Case Lambda By inversion, we have $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M \tag{7.60}$$

And so

$$[\![\Gamma \vdash \lambda x : A.C_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) = [\![\Gamma \vdash \lambda x : A.C_2 \colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.61}$$

Case Return By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M \tag{7.62}$$

And so

$$[\![\Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M_1} A]\!]_M = \eta_A \circ f = [\![\Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M_1} A]\!]_M \tag{7.63}$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \mathtt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B]\!]_M$ and $[\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.64}$$

$$g = [\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M \tag{7.65}$$

And so

$$[\![\Gamma \vdash v_1 \quad v_2 \colon \mathtt{M}_\epsilon A]\!]_M = \mathtt{app} \circ \langle f, g \rangle = [\![\Gamma \vdash v_1' \quad v_2' \colon \mathtt{M}_\epsilon A]\!]_M \tag{7.66}$$

Case Bind By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{7.67}$$

$$g = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{7.68}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= [\![\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{7.69}$$

Case If By inversion, we have $\Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$ and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon A$ By induction, we therefore have $[\![\Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M$, $[\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M \tag{7.70}$$

$$g = [\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{7.71}$$

$$h = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{7.72}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\epsilon, A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2 \colon]\!]_{\mathbf{M}} &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle f, \langle g, h \rangle \rangle \\ &= [\![\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{7.73}$$

Case Subtype By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, and $A \leq: B$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{7.74}$$

$$g = [\![A \leq: B]\!]_M \tag{7.75}$$

And so

$$[\![\Gamma \vdash v_1 \colon B]\!]_M = g \circ f = [\![\Gamma \vdash v_1 \colon B]\!]_M \tag{7.76}$$

Case subeffect By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_{\epsilon_1} A$, and $A \leq: B$ and $\epsilon_1 \leq \epsilon_2$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon_1} A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{7.77}$$

$$g = [\![A \leq: B]\!]_M \tag{7.78}$$

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{7.79}$$

And so

$$[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = h_B \circ T_{\epsilon_1} g \circ f = [\![\Gamma \vdash C_2 \mathtt{M}_{\epsilon_2} B \colon]\!]_{\mathbf{M}} \tag{7.80}$$

□