Given a set based S-Category which is a model of the non-polymorphic effect calculus, we generate an indexed category capable of modelling the polymorphic effect calculus.

0.1 The Non-Polymorphic Model

Since `Set` is a model of the non-polymorphic calculus,

- `Set` is cartesian closed.

- `Set` has a strong graded monad: $T^0 : (E, \cdot, \leq_0, 1) \to [\mathtt{Set}, \mathtt{Set}]$

- `Set` has a co-product on the terminal object `1`.

In addition, we require that

- E should be small.

0.2 Base Category

We construct the base category, `Eff` as follows:

- $U = E$, the set of ground effects in the non-polymorphic language.

- `1` is a singleton set.

- $U^n = E^n$, set of n-wide tuples of effects, $\vec{\epsilon}$

Hence when we treat effects that are well formed in Φ as morphisms, $E^n \to E$ in `Eff`, we should treat them as functions $f : E^n \to E$. Ground effects become point functions: $e : 1 \to E$, so the denotation of a ground effect is the constant value function: $[\![\Phi \vdash e : \mathtt{Effect}]\!] = \vec{\epsilon} \mapsto e$

We extend the multiplication of ground effects to multiplication on effect functions, giving us our `Mul` operation

$$\mathtt{Mul}(f, g) = f \cdot g \tag{1}$$
$$(f \cdot g)(\vec{\epsilon}) = (f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \tag{2}$$
$$\tag{3}$$

This satisfies naturality of `Mul`.

$$((f \cdot g) \circ \theta)\vec{\epsilon} = (f(\theta\vec{\epsilon})) \cdot (g(\theta\vec{\epsilon})) = ((f \circ \theta) \cdot (g \circ \theta))\vec{\epsilon} \tag{4}$$

0.3 S-Categories

The semantic category, $[E^0, \mathtt{Set}]$ of the effect-environment $\diamond$ is isomorphic to `Set`. Since each effect-environment is alpha equivalent to a natural number, the semantic category for Φ shall be represented as $\mathbb{C}(\Phi) = \mathbb{C}(n) = [E^n, \mathtt{Set}]$, the category of functions $E^n \to \mathtt{Set}$. Objects in $[E^n, \mathtt{Set}]$ are functions and we describe them by their actions on a generic vector of ground effects, $\vec{\epsilon}$.

Morphisms in $[E^n, \mathtt{Set}]$ are natural transformations between the functions. So:

$$m: \quad A \to B \quad \text{In } [E^n, \mathtt{Set}] \tag{5}$$

$$m\vec{\epsilon}: \quad A\vec{\epsilon} \to B\vec{\epsilon} \quad \text{In } \mathtt{Set} \tag{6}$$

$$(f \circ g)\vec{\epsilon} = \quad (f\vec{\epsilon}) \circ (g\vec{\epsilon}) \tag{7}$$

$$\mathbf{1}(\vec{\epsilon}) = \quad \mathbf{1} \tag{8}$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in $\mathtt{Set}$.

0.3.1 Each S-Category is a CCC

Since $\mathtt{Set}$ is complete and a CCC, and E^n is small, since E is small, $[E^n, \mathtt{Set}]$ is a CCC.

$$(A \times B)\vec{\epsilon} = (A\vec{\epsilon}) \times (B\vec{\epsilon}) \tag{9}$$

$$\mathbf{1}\vec{\epsilon} = \mathbf{1} \tag{10}$$

$$(B^A)\vec{\epsilon} = (B\vec{\epsilon})^{(A\vec{\epsilon})} \tag{11}$$

$$\pi_1\vec{\epsilon} = \pi_1 \tag{12}$$

$$\pi_2\vec{\epsilon} = \pi_2 \tag{13}$$

$$\mathtt{app}\vec{\epsilon} = \mathtt{app} \tag{14}$$

$$\mathtt{cur}(f)\vec{\epsilon} = \mathtt{cur}(f\vec{\epsilon}) \tag{15}$$

$$\langle f, g \rangle\vec{\epsilon} = \langle f\vec{\epsilon}, g\vec{\epsilon} \rangle \tag{16}$$

$$\tag{17}$$

0.3.2 The Terminal Co-Product

We can define the co-product point-wise.

$$(1 + 1)\vec{\epsilon} = (1\vec{\epsilon} + 1\vec{\epsilon}) \tag{18}$$

$$= (1 + 1) \tag{19}$$

$$\mathtt{inl}\vec{\epsilon} = \mathtt{inl} \tag{20}$$

$$\mathtt{inr}\vec{\epsilon} = \mathtt{inr} \tag{21}$$

$$[f, g]\vec{\epsilon} = [f\vec{\epsilon}, g\vec{\epsilon}] \tag{22}$$

$$\tag{23}$$

This preserves the co-product diagram.

$$([f, g] \circ \mathtt{inl})\vec{\epsilon} = [f\vec{\epsilon}, g\vec{\epsilon}] \circ \mathtt{inl} \tag{24}$$

$$= f\vec{\epsilon} \tag{25}$$

$$\square \tag{26}$$

$$([f, g] \circ \mathtt{inr})\vec{\epsilon} = [f\vec{\epsilon}, g\vec{\epsilon}] \circ \mathtt{inr} \tag{27}$$

$$= f\vec{\epsilon} \tag{28}$$

$$\square \tag{29}$$

$$\tag{30}$$

$[f, g]$ is also unique in $[E^n, \mathtt{Set}]$. Suppose $l \circ \mathtt{inl} = f$ and $l \circ \mathtt{inr} = g$ in $[E^n, \mathtt{Set}]$. Then $l\vec{\epsilon} \circ \mathtt{inl} = f\vec{\epsilon}$ and $l\vec{\epsilon} \circ \mathtt{inr} = g\vec{\epsilon}$. Hence by the co-product in $\mathtt{Set}$, $l = [f\vec{\epsilon}, g\vec{\epsilon}]$ so $l = [f, g]$.

0.3.3 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation $[\![\gamma]\!] \in \mathtt{obj}\ \mathtt{Set}$. The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$[\![\gamma]\!]: \quad E^n \to \mathtt{obj}\ \mathtt{Set} \tag{31}$$
$$\vec{\epsilon} \mapsto \quad [\![\gamma]\!] \tag{32}$$
$$\tag{33}$$

Each constant term $\mathtt{C}^A$ in the non-polymorphic calculus has a fixed denotation $[\![\mathtt{C}^A]\!] \in \mathtt{Set}(1, A)$. So the morphism $[\![\mathtt{C}^A]\!]$ in $[E^n, \mathtt{Set}]$ is the corresponding constant dependently typed morphism returning the $[\![\mathtt{C}^A]\!]$ function in $\mathtt{Set}$.

$$[\![\mathtt{C}^A]\!]: \quad [E^n, \mathtt{Set}](1, A) \tag{34}$$
$$\vec{\epsilon} \mapsto \quad [\![\mathtt{C}^A]\!] \tag{35}$$

0.3.4 Graded Monad

Given the strong graded monad $(\mathtt{T}^0, \eta^0, \mu^0, \mathtt{t}^0)$ on $\mathtt{Set}$, we can construct an appropriate graded monad $(\mathtt{T}^n, \eta^n, \mu^n, \mathtt{t}^n)$ on $[E^n, \mathtt{Set}]$. Through some mechanical proof and the naturality of the $\mathtt{Set}$ strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in $[E^n, \mathtt{Set}]$.

$$\mathtt{T}^n: \quad (E^n, \cdot, \leq_n, 1_n) \to [[E^n, \mathtt{Set}], [E^n, \mathtt{Set}]] \tag{36}$$
$$(\mathtt{T}^n_f A)\vec{\epsilon} = \quad \mathtt{T}^0_{(f\vec{\epsilon})} A\vec{\epsilon} \tag{37}$$
$$(\eta^n_A)\vec{\epsilon} = \quad \eta^0_{A\vec{\epsilon}} \tag{38}$$
$$(\mu^n_{f,g,A})\vec{\epsilon} = \quad \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \tag{39}$$
$$(\mathtt{t}^n_{f,A,B})\vec{\epsilon} = \quad \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \tag{40}$$

Naturality

$$
\begin{array}{ccc}
A\vec{\epsilon} & \xrightarrow{\eta^0_{(A\vec{\epsilon})}} & \mathtt{T}^0_1(A\vec{\epsilon}) \\
\downarrow{\scriptstyle f\vec{\epsilon}} & & \downarrow{\scriptstyle \mathtt{T}^0_1(f\vec{\epsilon})} \\
B\vec{\epsilon} & \xrightarrow{\eta^0_{(B\vec{\epsilon})}} & \mathtt{T}^0_1(B\vec{\epsilon})
\end{array}
$$

$$
\begin{array}{ccc}
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(A\vec{\epsilon}) & \xrightarrow{\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}} & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon}) \\
\downarrow{\scriptstyle \mathtt{T}^0_{f\vec{\epsilon}}\mathtt{T}^0_{g\vec{\epsilon}}m\vec{\epsilon}} & & \downarrow{\scriptstyle \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}m\vec{\epsilon}} \\
\mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(B\vec{\epsilon}) & \xrightarrow{\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}} & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon})
\end{array}
$$

$$A\vec{\epsilon} \times \mathrm{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \xrightarrow{\mathbf{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}} \mathrm{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B\vec{\epsilon})$$

$$\downarrow^{(m\vec{\epsilon} \times \mathrm{Id}_{\mathrm{T}^0_{f\vec{\epsilon}}B})} \qquad\qquad \downarrow^{\mathrm{T}^0_{(f\vec{\epsilon})}(m\vec{\epsilon} \times \mathrm{Id}_{B\vec{\epsilon}})}$$

$$A'\vec{\epsilon} \times \mathrm{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \xrightarrow{\mathbf{t}^0_{f\vec{\epsilon},(A'\vec{\epsilon}),(B\vec{\epsilon})}} \mathrm{T}^0_{f\vec{\epsilon}}(A'\vec{\epsilon} \times B\vec{\epsilon})$$

$$A\vec{\epsilon} \times \mathrm{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) \xrightarrow{\mathbf{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}} \mathrm{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B\vec{\epsilon})$$

$$\downarrow^{(\mathrm{Id}_{A\vec{\epsilon}} \times \mathrm{T}^0_{f\vec{\epsilon}}(m\vec{\epsilon}))} \qquad\qquad \downarrow^{\mathrm{T}^0_{(f\vec{\epsilon})}(\mathrm{Id}_{A\vec{\epsilon}} \times m\vec{\epsilon})}$$

$$A\vec{\epsilon} \times \mathrm{T}^0_{f\vec{\epsilon}}(B'\vec{\epsilon}) \xrightarrow{\mathbf{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B'\vec{\epsilon})}} \mathrm{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon} \times B'\vec{\epsilon})$$

Monad Laws

Left Unit

$$(\mu^n_{f,\mathbf{1},A} \circ \mathrm{T}^n_f \eta^n_A)\vec{\epsilon} = \mu^0_{(f\vec{\epsilon}),\mathbf{1},(A\vec{\epsilon})} \circ \mathrm{T}^0_{f\vec{\epsilon}}(\eta^0_{A\vec{\epsilon}}) \tag{41}$$

$$= \mathrm{Id}_{\mathrm{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} \tag{42}$$

$$= (\mathrm{Id}_{\mathrm{T}^n_f A})\vec{\epsilon} \tag{43}$$

Right Unit

$$(\mu^n_{\mathbf{1},g,A} \circ \eta^n_{\mathrm{T}^n_f A})\vec{\epsilon} = \mu^0_{\mathbf{1},(f\vec{\epsilon}),(A\vec{\epsilon})} \circ (\eta^0_{\mathrm{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}}) \tag{44}$$

$$= \mathrm{Id}_{\mathrm{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} \tag{45}$$

$$= (\mathrm{Id}_{\mathrm{T}^n_f A})\vec{\epsilon} \tag{46}$$

Monad Associativity

$$((\mu^n_{f,(g\cdot h),A}) \circ \mathrm{T}^n_f(\mu^n_{g,h,A}))\vec{\epsilon} = \mu^0_{(f\vec{\epsilon}),((g\vec{\epsilon})\cdot(h\vec{\epsilon})),(A\vec{\epsilon})} \circ \mathrm{T}^0_{f\vec{\epsilon}}\mu^0_{(h\vec{\epsilon}),(g\vec{\epsilon}),A\vec{\epsilon}} \tag{47}$$

$$= \mu^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(h\vec{\epsilon}),(A\vec{\epsilon})} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(\mathrm{T}^0_{h\vec{\epsilon}}(A\vec{\epsilon}))} \tag{48}$$

$$= (\mu^n_{f\cdot g,h,A} \circ \mu^n_{f,g,\mathrm{T}^0_h A})\vec{\epsilon} \tag{49}$$

Tensorial Strength

Unitor Law

$$(\mathrm{T}^n_f \pi_2)\vec{\epsilon} = \mathrm{T}^0_{(f\vec{\epsilon})}(\pi_2\vec{\epsilon}) \tag{50}$$

$$= \mathrm{T}^0_{(f\vec{\epsilon})}(\pi_2) \tag{51}$$

$$= \pi_2 \tag{52}$$

$$= \pi_2\vec{\epsilon} \tag{53}$$

Bind Law

$$A \times \mathrm{T}^n_f \mathrm{T}^n_g B \xrightarrow{\mathbf{t}_{f,A,\mathrm{T}^n_g B}} \mathrm{T}^n_f(A \times \mathrm{T}^n_g B) \xrightarrow{\mathrm{T}^n_f \mathbf{t}_{g,A,B}} \mathrm{T}^n_f \mathrm{T}^n_g(A \times B)$$

$$\searrow^{\mathrm{Id}_A \times \mu^n_{f,g,B}} \qquad\qquad \downarrow^{\mu^n_{f,g,A\times B}}$$

$$A \times \mathrm{T}^n_{f\cdot g} B \xrightarrow{\mathbf{t}_{f\cdot g,A,B}} \mathrm{T}^n_{f\cdot g}(A \times B)$$

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$$(\mathtt{t}^n_{(f\cdot g),A,B} \circ (\mathtt{Id}_A \times \mu^n_{f,g,B}))\vec{\epsilon} = (\mathtt{t}^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mu^n_{(f\vec{\epsilon}),(g\vec{\epsilon}),(B\vec{\epsilon})})) \tag{54}$$

$$= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\times B)\vec{\epsilon}} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}) \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),\mathtt{T}^0_{g\vec{\epsilon}}(B\vec{\epsilon})} \tag{55}$$

$$= (\mu^n_{f,g,(A\times B)} \circ \mathtt{T}^n_f(\mathtt{t}^n_{g,A,B}) \circ \mathtt{t}^n_{f,A,\mathtt{T}^n_g(B)})\vec{\epsilon} \tag{56}$$

Commutativity with Unit

$$A \times B \xrightarrow{\mathtt{Id}_A \times \eta_B} A \times \mathtt{T}_1 B$$
$$\searrow \eta_{A\times B} \quad \downarrow \mathtt{t}_{1,A,B}$$
$$\mathtt{T}^n_1(A \times B)$$

$$(\mathtt{t}^n_{1,A,B} \circ (\mathtt{Id}_A \times \eta^n_A))\vec{\epsilon} = \mathtt{t}^0_{1,(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \eta^0_{A\vec{\epsilon}}) \tag{57}$$

$$= \eta^0_{A\vec{\epsilon}\times B\vec{\epsilon}} \tag{58}$$

$$= (\eta^n_{A\times B})\vec{\epsilon} \tag{59}$$

Commutativity with α Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$(A \times B) \times \mathtt{T}^n_\epsilon C \xrightarrow{\mathtt{t}_{\epsilon,(A\times B),C}} \mathtt{T}^n_\epsilon((A \times B) \times C)$$
$$\downarrow \alpha_{A,B,\mathtt{T}^n_\epsilon C} \qquad\qquad\qquad \downarrow \mathtt{T}^n_\epsilon \alpha_{A,B,C}$$
$$A \times (B \times \mathtt{T}^n_\epsilon C) \xrightarrow{\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}} A \times \mathtt{T}^n_\epsilon(B \times C) \xrightarrow{\mathtt{t}_{\epsilon,A,(B\times C)}} \mathtt{T}^n_\epsilon(A \times (B \times C))$$

$$(\mathtt{T}^n_f \alpha_{A,B,C} \circ \mathtt{t}^n_{f,A\times B,C})\vec{\epsilon} = \mathtt{T}^0_{f\vec{\epsilon}} \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\times B)\vec{\epsilon},(C\vec{\epsilon})} \tag{60}$$

$$= \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon}\times C\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mathtt{t}^0_{(f\vec{\epsilon}),(B\vec{\epsilon}),(C\vec{\epsilon})}) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \tag{61}$$

$$= (\mathtt{t}^n_{f,A,(B\times C)} \circ (\mathtt{Id}_A \times \mathtt{t}^n_{f,B,C}) \circ \alpha_{A,B,C})\vec{\epsilon} \tag{62}$$

$$\tag{63}$$

0.3.5   Sub-Effecting

Given a collection of sub-effecting natural transformation in $\mathtt{Set}$,

$$[\![\epsilon_1 \leq_0 \epsilon_2]\!] : \quad \mathtt{T}^0_{\epsilon_1} \to \mathtt{T}^0_{\epsilon_2} \tag{64}$$

We can form sub-effect natural transformations in $[E^n, \mathtt{Set}]$:

$$[\![f \leq_n g]\!] : \quad \mathtt{T}^n_f \to \mathtt{T}^n_g \tag{65}$$

$$[\![f \leq_n g]\!] A\vec{\epsilon} : \quad \mathtt{T}^n_{f\vec{\epsilon}}(A\vec{\epsilon}) \to \mathtt{T}^n_{g\vec{\epsilon}}(B\vec{\epsilon}) \tag{66}$$

$$= \quad [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!] A\vec{\epsilon} \tag{67}$$

Naturality

$$\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon} \xrightarrow{[\![f\vec{\epsilon}\leq_0 g\vec{\epsilon}]\!]A\vec{\epsilon}} \mathtt{T}^0_{g\vec{\epsilon}}A\vec{\epsilon}$$
$$\downarrow{}_{\mathtt{T}^0_{f\vec{\epsilon}}m\vec{\epsilon}} \qquad \downarrow{}_{\mathtt{T}^0_{g\vec{\epsilon}}m\vec{\epsilon}}$$
$$\mathtt{T}^0_{f\vec{\epsilon}}B\vec{\epsilon} \xrightarrow{[\![f\vec{\epsilon}\leq_0 g\vec{\epsilon}]\!]B\vec{\epsilon}} \mathtt{T}^0_{g\vec{\epsilon}}B\vec{\epsilon}$$

Commutes With Tensor Strength

$$A \times \mathtt{T}^n_f B \xrightarrow{\mathtt{Id}_A \times [\![f\leq_n g]\!]_B} A \times \mathtt{T}^n_g B$$
$$\downarrow{}_{\mathtt{t}^n_{f,A,B}} \qquad \downarrow{}_{\mathtt{t}^n_{g,A,B}}$$
$$\mathtt{T}^n_f(A \times B) \xrightarrow{[\![f\leq_n g]\!]_{A\times B}} \mathtt{T}^n_g(A \times B)$$

$$(\mathtt{t}^n_{g,A,B} \circ (\mathtt{Id}_A \times [\![f \leq_n g]\!]_B))\vec{\epsilon} = \mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{B\vec{\epsilon}}) \tag{68}$$
$$= [\![f\vec{\epsilon} \leq_0 g\vec{\epsilon}]\!]_{(A\times B)\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \tag{69}$$
$$= ([\![f \leq_n g]\!]_{(A\times B)} \circ \mathtt{t}^n_{f,A,B})\vec{\epsilon} \tag{70}$$
$$\tag{71}$$

Commutes with Join

$$\mathtt{T}^n_f\mathtt{T}^n_g \xrightarrow{\mathtt{T}^n_f[\![g\leq_n g']\!]} \mathtt{T}^n_f\mathtt{T}^n_{g'} \xrightarrow{[\![f\leq_n f']\!]_{M,\mathtt{T}^n_{g'}}} \mathtt{T}^n_{f'}\mathtt{T}^n_{g'}$$
$$\downarrow{}_{\mu^n_{f,g,}} \qquad \downarrow{}_{\mu^n_{f',g',}}$$
$$\mathtt{T}^n_{f\cdot g} \xrightarrow{[\![f\cdot g\leq_n f'\cdot g']\!]} \mathtt{T}^n_{f'\cdot g'}$$

$$([\![f \cdot g \leq_n f' \cdot g']\!]_A \circ \mu^n_{f,g,A})\vec{\epsilon} = [\![(f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \leq_0 (f'\vec{\epsilon}) \cdot (g\vec{\epsilon})]\!]_{A\vec{\epsilon}} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \tag{72}$$
$$= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \circ [\![f\vec{\epsilon} \leq_0 f'\vec{\epsilon}]\!]_{\mathtt{T}^0_{g'\vec{\epsilon}}(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}}[\![g\vec{\epsilon} \leq_0 g'\vec{\epsilon}]\!]_{(A\vec{\epsilon})} \tag{73}$$
$$= \mu^n_{f,g,A} \circ [\![f \leq_n f']\!]_{\mathtt{T}^n_{g'}A} \circ \mathtt{T}^n_f[\![g \leq_n g']\!]_A \tag{74}$$

0.3.6 Sub-Typing

Sub-typing in $[E^n, \mathtt{Set}]$ holds via sub-typing in $\mathtt{Set}$

$$[\![A \leq:_n B]\!]: \quad A \to B \tag{75}$$
$$[\![A \leq:_n B]\!]\vec{\epsilon} = \quad [\![A\vec{\epsilon} \leq:_0 B\vec{\epsilon}]\!] \tag{76}$$

So the subtyping relation $A \leq: B$ forms a morphism in $[E^n, \mathtt{Set}]$

0.4 Functors Between S-Categories

For a function $\theta: E^m \to E^n$, the re-indexing functor θ^* is defined as follows:

$$\theta^* : \quad [E^n, \mathtt{Set}] \to [E^m, \mathtt{Set}] \tag{77}$$

$$\theta^*(A)\vec{\epsilon_m} = \quad A(\theta(\vec{\epsilon_m})) \tag{78}$$

$$f : \quad A \to B \in [E^n, \mathtt{Set}] \tag{79}$$

$$\theta^*(f)\vec{\epsilon_m} = \quad f(\theta(\vec{\epsilon_m})) : A(\theta(\vec{\epsilon_m})) \to B(\theta(\vec{\epsilon_m}))) \tag{80}$$

0.4.1 θ^* is S-closed.

CCC

$$(\theta^*(A \times B))\vec{\epsilon} = (A \times B)(\theta\vec{\epsilon}) \tag{81}$$

$$= (A(\theta\vec{\epsilon}) \times B(\theta\vec{\epsilon})) \tag{82}$$

$$= (\theta^* A \times \theta^* B)\vec{\epsilon} \tag{83}$$

$$\tag{84}$$

$$(\theta^* \pi_1)\vec{\epsilon} = \pi_1(\theta\vec{\epsilon}) \tag{85}$$

$$= \pi_1 \quad \text{Constant function} \tag{86}$$

$$= \pi_1 \vec{\epsilon} \tag{87}$$

$$(\theta^* \pi_2)\vec{\epsilon} = \pi_2(\theta\vec{\epsilon}) \tag{88}$$

$$= \pi_2 \quad \text{Constant function} \qquad\qquad = \pi_2\vec{\epsilon} \tag{89}$$

$$(\theta^* \langle f, g \rangle)\vec{\epsilon} = (\langle f, g \rangle)(\theta\vec{\epsilon}) \tag{90}$$

$$= \langle f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon}) \rangle \tag{91}$$

$$= \langle \theta^* f, \theta^* g \rangle \vec{\epsilon} \tag{92}$$

$$(\theta^*(A^B))\vec{\epsilon} = (A^B)(\theta\vec{\epsilon}) \tag{93}$$

$$= (A(\theta\vec{\epsilon}))^{(B(\theta\vec{\epsilon}))} \tag{94}$$

$$= (\theta^* A)^{(\theta^* B)} \vec{\epsilon} \tag{95}$$

$$\tag{96}$$

$$(\theta^* \mathtt{app})\vec{\epsilon} = \mathtt{app}(\theta\vec{\epsilon}) \tag{97}$$

$$= \mathtt{app} \quad \text{Constant fn} \tag{98}$$

$$= \mathtt{app}\vec{\epsilon} \tag{99}$$

$$(\theta^* \mathtt{cur}(f))\vec{\epsilon} = \mathtt{cur}(f)(\theta\vec{\epsilon}) \tag{100}$$

$$= \mathtt{cur}(f(\theta\vec{\epsilon})) \tag{101}$$

$$= \mathtt{cur}(\theta^* f) \tag{102}$$

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$$(\theta^* 1)\vec{\epsilon} = 1(\theta\vec{\epsilon}) \tag{103}$$
$$= 1 \tag{104}$$
$$= 1\vec{\epsilon} \tag{105}$$
$$\tag{106}$$

$$(\theta^* \langle\rangle_A)\vec{\epsilon} = \langle\rangle_A (\theta\vec{\epsilon}) \tag{107}$$
$$= \langle\rangle_{A(\theta\vec{\epsilon})} \tag{108}$$
$$= \langle\rangle_{\theta^* A} \vec{\epsilon} \tag{109}$$

Co-Product

$$(\theta^*(1+1))\vec{\epsilon} = (1+1)(\theta\vec{\epsilon}) \tag{110}$$
$$= (1+1) \quad \text{Constant function} \tag{111}$$
$$= (1+1)\vec{\epsilon} \tag{112}$$

$$(\theta^*\texttt{inl})\vec{\epsilon} = \texttt{inl}(\theta\vec{\epsilon}) \tag{113}$$
$$= \texttt{inl} \quad \text{Constant Fn} \tag{114}$$
$$= \texttt{inl}\vec{\epsilon} \tag{115}$$

$$(\theta^*\texttt{inr})\vec{\epsilon} = \texttt{inr}(\theta\vec{\epsilon}) \tag{116}$$
$$= \texttt{inr} \quad \text{Constant Fn} \tag{117}$$
$$= \texttt{inr}\vec{\epsilon} \tag{118}$$

$$(\theta^*[f,g])\vec{\epsilon} = [f,g](\theta\vec{\epsilon}) \tag{119}$$
$$= [f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})] \tag{120}$$
$$= [\theta^* f, \theta^* g]\vec{\epsilon} \tag{121}$$
$$\tag{122}$$

Strong Graded Monad

$$(\theta^*\mathrm{T}_f^n A)\vec{\epsilon} = \mathrm{T}_f^n A(\theta\vec{\epsilon}) \tag{123}$$
$$= \mathrm{T}_{(f(\theta\vec{\epsilon}))}^0 (A(\theta\vec{\epsilon})) \tag{124}$$
$$= (\mathrm{T}_{(f\circ\theta)}^m \theta^* A)\vec{\epsilon} \tag{125}$$
$$\tag{126}$$

$$(\theta^*\eta_A^n)\vec{\epsilon} = \eta_A^n(\theta\vec{\epsilon}) \tag{127}$$
$$= \eta_{A(\theta\vec{\epsilon})}^0 \tag{128}$$
$$= \eta_{\theta^*A}^m\vec{\epsilon} \tag{129}$$
$$\tag{130}$$

$$(\theta^*\mathbf{t}_{f,A,B}^n)\vec{\epsilon} = \mathbf{t}_{f,A,B}^n(\theta\vec{\epsilon}) \tag{131}$$
$$= \mathbf{t}_{(f(\theta\vec{\epsilon})),(A(\theta\vec{\epsilon})),(B(\theta\vec{\epsilon}))}^0 \tag{132}$$
$$= \mathbf{t}_{f\circ\theta,\theta^*A,\theta^*B}\vec{\epsilon} \tag{133}$$
$$\tag{134}$$

Sub-Effecting

$$(\theta^*([\![f \leq_n g]\!]A))\vec{\epsilon} = ([\![f \leq_n g]\!]A)(\theta\vec{\epsilon}) \tag{135}$$
$$= ([\![f(\theta\vec{\epsilon}) \leq_n g(\theta\vec{\epsilon})]\!](A(\theta\vec{\epsilon}))) \tag{136}$$
$$= ([\![\theta^*f \leq_m \theta^*g]\!](\theta^*A))\vec{\epsilon} \tag{137}$$
$$\tag{138}$$

Ground Sub-Typing

$$\theta^*([\![A \leq:_\gamma B]\!])\vec{\epsilon} = [\![A \leq:_\gamma B]\!](\theta\vec{\epsilon}) \tag{139}$$
$$= [\![A \leq: B]\!] \quad \text{Constant Function} \tag{140}$$
$$= [\![A \leq: B]\!]\vec{\epsilon} \tag{141}$$
$$\tag{142}$$

0.4.2 Quantification

We need to define $\forall_{E^n} : [E^{n+1}, \mathtt{Set}] \to [E^n, \mathtt{Set}]$

So

$$(\forall_{E^n}A)\vec{\epsilon_n} = \ \Pi_{\epsilon\in E}A(\vec{\epsilon_n},\epsilon) \tag{143}$$
$$m: \ A \to B \tag{144}$$
$$(\forall_{E^n}m): \ \forall_{E^n}A \to \forall_{E^n}B \tag{145}$$
$$(\forall_{E^n}m)\vec{\epsilon_n} = \Pi_{\epsilon\in E}m(\vec{\epsilon_n},\epsilon) \tag{146}$$
$$\tag{147}$$

0.4.3 Adjunction

It is the case that:
$$\pi_1^* \dashv \forall_{E^n}$$

With unit:

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$$\boldsymbol{\eta}_A : \quad A \to \forall_{E^n} \pi_1^* A \tag{148}$$

$$\boldsymbol{\eta}_A(\vec{\epsilon_n}) = \quad \langle \mathtt{Id}_{A(\vec{\epsilon_n},e)} \rangle_{\epsilon \in E} \tag{149}$$

And co-unit

$$\boldsymbol{\epsilon}_B : \quad \pi_1^* \forall_{E^n} B \to B \tag{150}$$

$$\boldsymbol{\epsilon}_B(\vec{\epsilon_n}, \epsilon) = \quad \pi_\epsilon : \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) \to \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) \tag{151}$$

We then define the natural bi-jection as so:

$$\overline{(-)} : [E^{n_1}, \mathtt{Set}](\pi_1^* A, B) \leftrightarrow [E^n, \mathtt{Set}](A, \forall_{E^n} B) : \widehat{(-)} \tag{152}$$

$$m : \quad \pi_1^* A \to B \tag{153}$$

$$\overline{m} : \quad A \to \forall_{E^n} B \tag{154}$$

$$\overline{m}(\vec{\epsilon_n}) = \quad \langle m(\vec{\epsilon_n}, \epsilon) \rangle_{e \in E} \tag{155}$$

$$n : \quad A \to \forall_{E^n} B \tag{156}$$

$$\widehat{n} : \quad \pi_1^* A \to B \tag{157}$$

$$\widehat{n}(\vec{\epsilon_n}, \epsilon_{n+1}) = \quad \pi_\epsilon \circ g(\vec{\epsilon_n}) \tag{158}$$

This is an Adjunction

For any $g : \pi_1^* A \to B$,

$$(\boldsymbol{\epsilon}_B \circ \pi_1^*(\overline{g}))(\vec{\epsilon_n}, \epsilon_{n+1}) = \pi_{\epsilon_{n+1}} \circ \langle g(\vec{\epsilon_n}, \epsilon') \rangle_{\epsilon' \in E} \tag{159}$$

$$= g(\vec{\epsilon_n}, \epsilon_{n+1}) \tag{160}$$

0.4.4 Beck-Chevalley Condition

For $\theta : E^m \to E^n$:

$$((\theta^* \circ \forall_{E^n}) A)\vec{\epsilon_n} = \theta^* (\forall_{E^n} A)\vec{\epsilon_n} \tag{161}$$

$$= (\forall_{E^n} A)(\theta(\vec{\epsilon_n})) \tag{162}$$

$$= \Pi_{\epsilon \in E}(A(\theta(\vec{\epsilon_n}), \epsilon)) \tag{163}$$

$$= \Pi_{\epsilon \in E}(((\theta \times \mathtt{Id}_U)^* A)(\vec{\epsilon_n}, \epsilon)) \tag{164}$$

$$= \forall_{E^m}((\theta \times \mathtt{Id}_E)^* A)\vec{\epsilon_n} \tag{165}$$

$$= ((\forall_{E^m} \circ (\theta \times \mathtt{Id}_E)^*) A)\vec{\epsilon_n} \tag{166}$$

And $\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}} = \mathtt{Id}_{\theta^* \circ \forall_I}$.

$$\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A} \vec{\epsilon} = \langle (\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A(\vec{\epsilon}, \epsilon) \rangle_{\epsilon \in E} \tag{167}$$

$$= \langle \boldsymbol{\epsilon}_A(\theta \vec{\epsilon}, \epsilon) \rangle_{e \in E} \tag{168}$$

$$= \langle \pi_\epsilon \rangle_{\epsilon \in E} : \Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon) \to \Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon) \tag{169}$$

$$= \mathtt{Id}_{\Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon)} \tag{170}$$

$$= \mathtt{Id}_{\forall_{I'} \circ (\theta \times \mathtt{Id}_U)^* A} \vec{\epsilon} \tag{171}$$

$$= \mathtt{Id}_{\theta^* \circ \forall_I} \tag{172}$$