0.1 CCC

The section should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

0.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endofunctor indexed by the pre-order on effects: $T : (\mathbb{E}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1,\epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

0.2.1 Left Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"'] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon,\mathtt{1},A}"] \\
 & T_\epsilon A
\end{tikzcd}$$

0.2.2 Right Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"'] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1},\epsilon,A}"] \\
 & T_\epsilon A
\end{tikzcd}$$

0.2.3 Associativity

$$\begin{tikzcd}
T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3}A}"] \arrow[d, "T_{\epsilon_1}\mu_{\epsilon_2,\epsilon_3,A}"] & T_{\epsilon_1 \cdot \epsilon_2} T_{\epsilon_3} A \arrow[d, "\mu_{\epsilon_1 \cdot \epsilon_2,\epsilon_3,A}"] \\
T_{\epsilon_1} T_{\epsilon_2 \cdot \epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2 \cdot \epsilon_3,A}"] & T_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} A
\end{tikzcd}$$

0.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon (A \times B)$$

Satisfying the following rules:

0.3.1 Left Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{Id}_A \times T_\epsilon f"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A \times T_\epsilon B' \arrow[d, "\mathtt{t}_{\epsilon,A,B'}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(\mathtt{Id}_A \times f)"] & T_\epsilon(A \times B')
\end{tikzcd}$$

0.3.2 Right Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "f \times \mathtt{Id}_{T_\epsilon B}"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A' \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A',B}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(f \times \mathtt{Id}_B)"] & T_\epsilon(A' \times B)
\end{tikzcd}$$

0.3.3 Unitor Law

$$\begin{tikzcd}
\mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\
& T_\epsilon A
\end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

0.3.4 Commutativity with Join

$$\begin{tikzcd}
A \times T_{\epsilon_1} T_{\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1} T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A \times B}"] \\
& A \times T_{\epsilon_1 \cdot \epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1 \cdot \epsilon_2,A,B}"] & T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
\end{tikzcd}$$

0.4 Subeffecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

0.4.1 Subeffecting and Tensor Strength

$$\begin{tikzcd}
A \times T_{\epsilon_1} B \arrow[r, "\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2} B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "[\![\epsilon_1 \leq \epsilon_2]\!]_{A \times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}$$

0.5 Subtyping

The denotation of ground types $[\![\_]\!]_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function subtyping to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)}\frac{f = [\![A' \leq: A]\!]_M \qquad g = [\![B \leq: B']\!]_M \qquad h = [\![\epsilon_1 \leq \epsilon_2]\!]}{rhs = [\![A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B']\!]_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned} rhs =& (h_{B'} \circ T_{\epsilon_1} g)^A \circ (T_{\epsilon_1} B)^f \\ =& \mathtt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f)) \end{aligned} \tag{1}$$

0.6 If natural transformation

There exists a natural transformation $\mathtt{If}_A : (\mathtt{Bool} \times (A \times A)) \to A$ Satisfying the following:

- $\mathtt{If}_A \circ \langle [\![\mathtt{true}]\!]_M \circ \langle\rangle_\Gamma , \langle t, f \rangle \rangle = t$
- $\mathtt{If}_A \circ \langle [\![\mathtt{false}]\!]_M \circ \langle\rangle_\Gamma , \langle t, f \rangle \rangle = f$