0.1 Introduce Substitutions

0.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{1}$$

0.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{2}$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{4}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6}$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{7}$$
$$x\,[\sigma, x := v] = v \tag{8}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{9}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{10}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{11}$$
$$(\mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A} \quad v\,[\sigma] \quad \mathtt{then} \quad C_1\,[\sigma] \quad \mathtt{else} \quad C_2\,[\sigma] \tag{12}$$
$$(v_1 \quad v_2)\,[\sigma] = (v_1\,[\sigma]) \quad v_2\,[\sigma] \tag{13}$$
$$(\mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C_2) = \mathtt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \mathtt{in} \quad (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{14}$$
$$\tag{15}$$

0.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma : \Gamma$$

by:

- (Nil) $\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Gamma' \vdash \sigma : \Gamma x \notin \mathtt{dom}(\Gamma) \Gamma' \vdash v : A}{\Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma : \Gamma$ then: **TODO: Number these**

Property 1: $\Gamma \texttt{Ok}$ and $\Gamma' \texttt{Ok}$ Since $\Gamma' \texttt{Ok}$ holds by the Nil-axiom. $\Gamma \texttt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v \colon A$ holds if $\Gamma' \vdash v \colon A$ holds.

Property 3: $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{16}$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g \colon \tau \wedge \Gamma' \vdash \sigma \colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{17}$$

TODO: Proof by induction over type relation Assuming $\Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t \colon \tau \rightarrow \Gamma' \vdash t \colon \tau$

0.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x \colon A \tag{18}$$

So by inversion, since $\Gamma' \vdash \sigma \colon \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v \colon A \tag{19}$$

By the defintion of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Gamma' \vdash x\,[\sigma] : A \tag{20}$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{()\frac{\Delta}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x \colon A} \tag{21}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma' \colon \Gamma'' \tag{22}$$

So by induction,

$$\Gamma' \vdash x\,[\sigma'] : A \tag{23}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Gamma' \vdash x\,[\sigma] : A \tag{24}$$

0.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn})\frac{()\frac{\Delta}{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_\epsilon B} \tag{25}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma', x : A) \vdash (\sigma, x := x) : \Gamma, x : A \tag{26}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{()\frac{\Delta'}{\Gamma', x:A \vdash C[\sigma, x := v] : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\,[\sigma, x := x] : A \to \mathtt{M}_\epsilon B} \tag{27}$$

Since $\lambda x : A.(C\,[\sigma, x := x]) = \lambda x : A.(C\,[\sigma]) = (\lambda x : A.C)\,[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x : A.C)\,[\sigma] : A \to \mathtt{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:
$\Gamma \vdash \sigma : \Gamma \Rightarrow \Gamma' \mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{28}$$

So

$$(\text{Const})\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A : A} \tag{29}$$

0.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}} A} \tag{30}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma] : A}}{\Gamma' \vdash \mathtt{return}(v\,[\sigma]) : \mathtt{M}_{\mathtt{1}} A} \tag{31}$$

Since $(\mathtt{return} v)\,[\sigma] = \mathtt{return}(v\,[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\mathtt{return} v)\,[\sigma] : \mathtt{M}_{\mathtt{1}} A$.

Case Apply By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{()\frac{\Delta_1}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \qquad ()\frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1 \qquad v_2 : \mathtt{M}_\epsilon B} \tag{32}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v_1[\sigma] : A \to \mathtt{M}_\epsilon B} \qquad ()\frac{\Delta_2'}{\Gamma' \vdash v_2[\sigma] : A}}{\Gamma' \vdash (v_1\,[\sigma]) \qquad (v_2\,[\sigma]) : \mathtt{M}_\epsilon B} \tag{33}$$

Since $(v_1 \qquad v_2)\,[\sigma] = (v_1\,[\sigma]) \qquad (v_2\,[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1 \qquad v_2)\,[\sigma] : \mathtt{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \qquad ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \qquad ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma\vdash \texttt{if}_{\epsilon,A} \quad v \quad \texttt{then} \quad C_1 \quad \texttt{else} \quad C_2 : \texttt{M}_\epsilon A} \tag{34}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1'}{\Gamma'\vdash v[\sigma]:\texttt{Bool}} \qquad ()\frac{\Delta_2'}{\Gamma'\vdash C_1[\sigma]:\texttt{M}_\epsilon A} \qquad ()\frac{\Delta_3'}{\Gamma'\vdash C_2[\sigma]:\texttt{M}_\epsilon A}}{\Gamma'\vdash \texttt{if}_{\epsilon,A} \quad (v\,[\sigma]) \quad \texttt{then} \quad (C_1\,[\sigma]) \quad \texttt{else} \quad (C_2\,[\sigma]) : \texttt{M}_\epsilon A} \tag{35}$$

Since $(\texttt{if}_{\epsilon,A} \quad v \quad \texttt{then} \quad C_1 \quad \texttt{else} \quad C_2)\,[\sigma] = \texttt{if}_{\epsilon,A} \quad (v\,[\sigma]) \quad \texttt{then} \quad (C_1\,[\sigma]) \quad \texttt{else} \quad (C_2\,[\sigma])$
The derivation above holds for $\Gamma' \vdash (\texttt{if}_{\epsilon,A} \quad v \quad \texttt{then} \quad C_1 \quad \texttt{else} \quad C_2)\,[\sigma] : \texttt{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1} A} \qquad ()\frac{\Delta_2}{\Gamma, x:A\vdash C_2:\texttt{M}_{\epsilon_2} B}}{\Gamma\vdash \texttt{do} \quad x \leftarrow C_1 \quad \texttt{in} \quad C_2 : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{36}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x : A) \vdash (\sigma, x := x) : (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1'}{\Gamma'\vdash C_1[\sigma]:\texttt{M}_{\epsilon_1} A} \qquad ()\frac{\Delta_2}{\Gamma', x:A\vdash C_2[\sigma, x:=x]:\texttt{M}_{\epsilon_2} B}}{\Gamma'\vdash \texttt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \texttt{in} \quad (C_2\,[\sigma, x := x]) : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{37}$$

Since $(\texttt{do} \quad x \leftarrow C_1 \quad \texttt{in} \quad C_2)\,[\sigma] = \texttt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \texttt{in} \quad (C_2\,[\sigma]) = \texttt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \texttt{in} \quad (C_2\,[\sigma, x :=$
the above derivation holds for $\Gamma' \vdash (\texttt{do} \quad x \leftarrow C_1 \quad \texttt{in} \quad C_2)\,[\sigma] : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B$

0.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$\text{(sub-type)}\frac{()\frac{\Delta}{\Gamma\vdash v:A} \qquad A \leq: B}{\Gamma\vdash v : B} \tag{38}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\frac{()\frac{\Delta'}{\Gamma'\vdash v[\sigma]:A} \qquad A \leq: B}{\Gamma\vdash v\,[\sigma] : B} \tag{39}$$

Case Sub-effect By inversion, there exists Δ such that

$$\text{(sub-effect)}\frac{()\frac{\Delta}{\Gamma\vdash C:\texttt{M}_{\epsilon_1} A} \qquad A \leq: B \qquad \epsilon_1 \leq: \epsilon_2}{\Gamma\vdash C : \texttt{M}_{\epsilon_2} B} \tag{40}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-effect)}\frac{()\frac{\Delta'}{\Gamma'\vdash C[\sigma]:\texttt{M}_{\epsilon_1} A} \qquad A \leq: B \qquad \epsilon_1 \leq: \epsilon_2}{\Gamma'\vdash C\,[\sigma] : \texttt{M}_{\epsilon_2} B} \tag{41}$$

0.3 Semantics of Substitution

0.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma' \vdash \sigma : \Gamma]\!]_M : \Gamma' \to \Gamma \tag{42}$$

- (Nil) $\frac{\Gamma' \mathtt{Ok}}{[\![\Gamma' \vdash \diamond : \diamond]\!]_M = \langle\rangle_{\Gamma'}}$
- (Extend) $\frac{f = [\![\Gamma' \vdash \sigma : \Gamma]\!]_M \, g = [\![\Gamma' \vdash v : A]\!]_M}{[\![\Gamma' \vdash (\sigma, x := v : (\Gamma, x : A)]\!]_M = \langle f, g\rangle : \Gamma' \to (\Gamma \times A)}$

0.3.2 Lemma

TODO: Fill in from p98

0.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If $\Gamma \vdash t : \tau$ and $\Gamma' \vdash \sigma : \Gamma$ then

0.4 Single Substitution