0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, **lambda** or **apply** rule.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t \colon \tau$,there exists at most one reduced derivation of $\Gamma \vdash t \colon \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Constants

For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{}{\Gamma\vdash \mathtt{C}^A : A} \qquad A \leq: B}{\Gamma \vdash \mathtt{C}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

0.2.2 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C \colon A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x:A.B : A \to \mathtt{M}_\epsilon B} \qquad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C \colon A' \to \mathtt{M}_{\epsilon'} B'}$$

Where Δ is the reduced derivation of $\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B$ if it exists.

Case Subtype TODO: Do we need to write anything here? (Probably needs an explanation)

0.2.3 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\frac{()\frac{\Delta}{\Gamma\vdash v:A}}{\Gamma \vdash \mathtt{return} v : \mathtt{M}_1 A} \qquad A \leq: A' \leq: B \qquad 1 \leq \epsilon}{\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_\epsilon B}$$

Where

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma\vdash v:A} \qquad A \leq: A'}{\Gamma \vdash v \colon A'}$$

is the reduced derivation of $\Gamma \vdash v \colon A'$

Case Apply If

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma\vdash v_1 : A \to \mathtt{M}_\epsilon BB} \qquad A \to \mathtt{M}_\epsilon BB \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1 \colon A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Gamma\vdash v_2 : A''} \qquad A'' \leq: A'}{\Gamma \vdash v_2 \colon A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1 \colon A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2 \colon A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1 \quad v_2 \colon \mathtt{M}_{\epsilon'} B'$ as

$$
(\text{Subeffect})\frac{(\text{Apply})\dfrac{()\dfrac{\Delta}{\Gamma \vdash v_1 : A \to \mathtt{M}_{\epsilon} B} \qquad (\text{Subtype})\dfrac{()\dfrac{\Delta'}{\Gamma \vdash v : A''} \qquad A'' \leq: A}{\Gamma \vdash v : A}}{\Gamma \vdash v_1 \quad v_2 : \mathtt{M}_{\epsilon} B} \qquad B \leq: B' \qquad \epsilon \leq \epsilon'}{\Gamma \vdash v_1 \quad v_2 : \mathtt{M}_{\epsilon'} B'}
$$

Case If

Case Bind

Case Subeffect

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed. **TODO: Fill in these cases with actual maths**

0.3.1 Constants

TODO: ***reduce*** **just appends the identity subtype rule to the derivation, trivially preserves denotation**

0.3.2 Value Types

Lambda **TODO: Recursively call reduce on C then push subtyping through using currying**

Subtype **TODO: Recursively call reduce then merge subtypes**

0.3.3 Computation Types

Return **TODO: Recursively call reduce then use naturality to push subtyping into subeffect**

Apply **TODO: Recursively call reduce, then construct the reduced apply as in the proof of uniqueness**

If **TODO: Recursively call reduce, then leave tree otherwise unchanged.**

Bind **TODO: Recursively call reduce then push subtyping rules through the bind**

Subeffect **TODO: Recursively call reduce, then merge subeffecting rules**

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t : \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce \Delta]\!]_M = [\![reduce \Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t : \tau]\!]_M$ is unique.