

Contents

1	Preliminaries	3
1.1	Base Category Requirements	3
1.2	Well-Formed-ness	4
1.3	Substitution and Weakening of the Effect Environment	4
1.4	Fibre Categories	4
1.5	Re-indexing Functors	4
1.5.1	f^* Preserves Products	5
1.5.2	f^* Preserves Terminal Object	5
1.5.3	f^* Preserves Exponentials	5
1.5.4	f^* Preserves Co-product on Terminal	5
1.5.5	f^* Preserves Graded Monad	5
1.5.6	f^* Preserves Tensor Strength	6
1.5.7	f^* Preserves Ground Constants	6
1.5.8	f^* Preserves Ground Sub-effecting	6
1.5.9	f^* Preserves Ground Sub-typing	6
1.5.10	Action on Objects	6
1.6	Naturality Properties	6
1.7	The \forall_I functor	6
1.8	Naturality Corollaries	7
1.8.1	Naturality	7
1.8.2	$\overline{(-)}$ and Re-indexing Functors	7
1.8.3	$\hat{(-)}$ and Re-Indexing Functors	7
1.8.4	Pushing Morphisms into f^*	7
2	Denotations	8
2.1	Effects	8
2.2	Types	8
2.3	Effect Substitution	8
2.4	Effect Weakening	9
2.5	Sub-Typing	9

2.6	Type-Environments	9
2.7	Terms	9
3	Effect Substitution Theorem	11
3.1	Effects	11
3.2	Types	12
3.3	Sub-typing	13
3.4	Type Environments	14
3.5	Terms	14
4	Effect Weakening Theorem	20
4.1	Effects	20
4.2	Types	21
4.3	Sub-typing	22
4.4	Type Environments	23
4.5	Terms	24
4.6	Terms	24
4.7	Term-Substitution	28
4.8	Term-Weakening	28
5	Value Substitution Theorem	30
6	Type-Environment Weakening Theorem	31
7	Unique Denotation Theorem	32
8	Beta-Eta-Equivalence Theorem (Soundness)	33

Chapter 1

Preliminaries

1.1 Base Category Requirements

There are 3 distinct objects in the base category, \mathbb{C} :

- U - The kind of **Effect**
- W - The kind of **Type**
- 1 - A terminal object

And we have finite products on U .

- $U^0 = 1$
- $U^{n+1} = U^n \times U$

We also have the following natural operations on morphisms in \mathbb{C} .

Let $I = U^n$.

- $\diamond : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$ - Generates exponential types.
- $\square : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$ - Generates products of types.
- $\forall_I : \mathbb{C}(I \times U, W) \rightarrow \mathbb{C}(I, W)$ - generates quantified types.
- $\text{Eff} : \mathbb{C}(I, U) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$ - generates monad types.
- $\text{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \rightarrow \mathbb{C}(I, U)$ - Generates multiplication of effects.

With naturality conditions which mean, for $\theta : \text{Unit}^m \rightarrow \text{Unit}^n(I' \rightarrow I)$,

- $\diamond(\phi, \psi) \circ \theta = \diamond(\phi \circ \theta, \psi \circ \theta)$
- $\square(\phi, \psi) \circ \theta = \square(\phi \circ \theta, \psi \circ \theta)$
- $\forall_I(\phi) \circ \theta = \forall_{I'}(\phi \circ (\theta \times \text{Id}_U))$
- $\text{Eff}(\phi, \psi) \circ \theta = \text{Eff}(\phi \circ \theta, \psi \circ \theta)$
- $\text{Mul}(\phi, \psi) \circ \theta = \text{Mul}(\phi \circ \theta, \psi \circ \theta)$

1.2 Well-Formed-ness

Each instance of the well-formed-ness relation on effects, $\Phi \vdash \epsilon$ has a denotation in \mathbb{C} :

$$\llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M : I \rightarrow U \quad (1.1)$$

Each instance of the well-formed-ness relation on types, $\Phi \vdash A$ has a denotation in \mathbb{C} :

$$\llbracket P \vdash A : \mathbf{Type} \rrbracket_M : I \rightarrow W \quad (1.2)$$

It should also be the case that

$$\mathbf{Mul}(\llbracket \Phi \vdash \epsilon_1 : \mathbf{Effect} \rrbracket_M, \llbracket \Phi \vdash \epsilon_2 : \mathbf{Effect} \rrbracket_M) = \llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \mathbf{Effect} \rrbracket_M \in \mathbb{C}(I, U) \quad (1.3)$$

That is, \mathbf{Mul} should be have identity $\llbracket \Phi \vdash 1 : \mathbf{Effect} \rrbracket_M$ and be associative.

1.3 Substitution and Weakening of the Effect Environment

For each instance of the well-formed-ness relation on substitution of effects $\Phi' \vdash \sigma : \Phi$, there exists a denotation in \mathbb{C} :

$$\llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M : I' \rightarrow I \quad (1.4)$$

For each instance of the well-formed weakening relation on effect-environments, $\omega : \Phi' \triangleright \Phi$ there exists a denotation in \mathbb{C} :

$$\llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M : I' \rightarrow I \quad (1.5)$$

1.4 Fibre Categories

Each set of morphisms $\mathbb{C}(I, W)$ forms the objects of a semantic-closed (S-closed) category. That is, a category satisfying all the properties needed for the non-polymorphic language:

- Cartesian Closed
- Co-product of the terminal object with itself ($1 + 1$)
- Ground morphisms for each ground constant ($\mathbb{C}^A : 1 \rightarrow A$)
- Partial order morphisms on ground types ($\llbracket A \leq_\gamma B \rrbracket_M$)
- A strong, monad, graded by the po-monoid $(E_\Phi, \cdot_\Phi, \leq_\Phi, 1)$.

1.5 Re-indexing Functors

For each morphism $f : I' \rightarrow I$ in \mathbb{C} , there should be a co-variant, re-indexing functor $f^* : \mathbb{C}(I, W) \rightarrow \mathbb{C}(I', W)$, which is S-closed. That is, it preserves the S-closed properties of $\mathbb{C}(I, W)$. (See below).

$(-)^*$ should be a contra-variant functor in its \mathbb{C} argument and co-variant in its right argument.

- $(g \circ f)^*(a) = f^*(\gamma^*(a))$
- $\text{Id}_I^*(a) = a$
- $f^*(\text{Id}_A) = \text{Id}_{f^*(A)}$
- $f^*(a \circ b) = f^*(a) \circ f^*(b)$

1.5.1 f^* Preserves Products

If $\langle g, h \rangle : \mathbb{C}(I, W)(Z, X \times Y)$ Then

$$f^*(X \times Y) = f^*(X) \times f^*(Y) \quad (1.6)$$

$$f^*(\langle g, h \rangle) = \langle f^*(g), f^*(h) \rangle \quad : \mathbb{C}(I', W)(f^*(Z), f^*(X) \times f^*(Y)) \quad (1.7)$$

$$f^*(\pi_1) = \pi_1 \quad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(X)) \quad (1.8)$$

$$f^*(\pi_2) = \pi_2 \quad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(Y)) \quad (1.9)$$

1.5.2 f^* Preserves Terminal Object

If $\text{Id}_A : \mathbb{C}(I, W)(A, 1)$ Then

$$f^*(1) = 1 \quad (1.10)$$

$$f^*(\langle \rangle_A) = \langle \rangle_{f^*(A)} \quad : \mathbb{C}(I', W)(f^*(A), 1) \quad (1.11)$$

$$(1.12)$$

1.5.3 f^* Preserves Exponentials

$$f^*(Z^X) = (f^*(Z))^{f^*(X)} \quad (1.13)$$

$$f^*(\text{app}) = \text{app} \quad : \mathbb{C}(I', W)(f^*(Z^X) \times f^*(X), f^*(Z)) \quad (1.14)$$

$$f^*(\text{cur}(g)) = \text{cur}(f^*(g)) \quad : \mathbb{C}(I', W)(f^*(Y) \times f^*(X), f^*(Z)^{f^*(X)}) \quad (1.15)$$

1.5.4 f^* Preserves Co-product on Terminal

$$f^*(1 + 1) = 1 + 1 \quad (1.16)$$

$$f^*(\text{inl}) = \text{inl} \quad : \mathbb{C}(I', W)(1, 1 + 1) \quad (1.17)$$

$$f^*(\text{inr}) = \text{inr} \quad : \mathbb{C}(I', W)(1, 1 + 1) \quad (1.18)$$

$$f^*([g, h]) = [f^*(g), f^*(h)] \quad : \mathbb{C}(I', W)(1 + 1, f^*(Z)) \quad (1.19)$$

1.5.5 f^* Preserves Graded Monad

$$f^*(T_\epsilon A) = T_{f^*(\epsilon)} f^*(A) \quad : \mathbb{C}(I', W) \quad (1.20)$$

$$f^*(1) = 1 \quad \text{The unit effect} \quad (1.21)$$

$$f^*(\eta_A) = \eta_{f^*(A)} \quad : \mathbb{C}(I', W)(f^*(A), f^*(T_1 A)) \quad (1.22)$$

$$f^*(\mu_{\epsilon_1, \epsilon_2, A}) = \mu_{f^*(\epsilon_1), f^*(\epsilon_2), f^*(A)} \quad : \mathbb{C}(I', W)(f^*(T_{\epsilon_1} T_{\epsilon_2} A), f^*(T_{f^*(\epsilon_1) \cdot f^*(\epsilon_2)} f^*(A))) \quad (1.23)$$

$$f^*(\epsilon_1 \cdot \epsilon_2) = f^*(\epsilon_1) \cdot f^*(\epsilon_2) \quad (1.24)$$

$$(1.25)$$

1.5.6 f^* Preserves Tensor Strength

$$f^*(\mathfrak{t}_{\epsilon, A, B}) = \mathfrak{t}_{f^*(\epsilon), f^*(A), f^*(B)} : \mathbb{C}(I', W)(f^*(A \times T_\epsilon B), f^*(T_\epsilon(A \times B))) \quad (1.26)$$

1.5.7 f^* Preserves Ground Constants

For each ground constant $\llbracket \mathfrak{C}^A \rrbracket_M$ in $\mathbb{C}(I, W)$,

$$f^*(\llbracket \mathfrak{C}^A \rrbracket_M) = \mathfrak{C}^{f^*(A)} : \mathbb{C}(I', W)(1, f^*(A)) \quad (1.27)$$

1.5.8 f^* Preserves Ground Sub-effecting

For ground effects e_1, e_2 such that $e_1 \leq e_2$

$$f^*(e) = e : \mathbb{C}(I', U) \quad (1.28)$$

$$f^*(\llbracket e_1 \leq e_2 \rrbracket_A) = \llbracket e_1 \leq e_2 \rrbracket_{f^*(A)} : \mathbb{C}(I', W)(f^*(T_{e_1} A), f^*(T_{e_2} A)) \quad (1.29)$$

$$(1.30)$$

1.5.9 f^* Preserves Ground Sub-typing

For ground types γ_1, γ_2 such that $\gamma_1 \leq_{\gamma} \gamma_2$

$$f^*\gamma = \gamma : \mathbb{C}(I', W)(1, \gamma) \quad (1.31)$$

$$f^*(\llbracket \gamma_1 \leq_{\gamma} \gamma_2 \rrbracket_M) = \llbracket \gamma_1 \leq_{\gamma} \gamma_2 \rrbracket_M : \mathbb{C}(I', W)(\gamma_1, \gamma_2) \quad (1.32)$$

$$(1.33)$$

1.5.10 Action on Objects

It follows that the action of f^* on an object A in $\mathbb{C}(I, W)$ (i.e. a morphism $I \rightarrow U$ in \mathbb{C}) is:

$$f^*(A) = A \circ f : I' \rightarrow I \rightarrow W \quad (1.34)$$

1.6 Naturality Properties

1.7 The \forall_I functor

We expand $\forall_I : \mathbb{C}(I \times U, W) \rightarrow \mathbb{C}(I, W)$ to be a functor which is right adjoint to the re-indexing functor π_1^* .

$$\overline{(-)} : \mathbb{C}(I \times U, W)(\pi_1^* A, B) \leftrightarrow \mathbb{C}(I, W)(A, \forall_I B) : \widehat{(-)} \quad (1.35)$$

For $A : \mathbb{C}(I, W)$, $B : \mathbb{C}(I \times U, W)$.

Hence the action of \forall_I on a morphism $l : A \rightarrow A'$ is as follows:

$$\forall_I(l) = \overline{l \circ \epsilon_A} \quad (1.36)$$

Where $\epsilon_A : \mathbb{C}(I \times U, W)(\pi_1^* \forall_I A \rightarrow A)$ is the co-unit of the adjunction.

1.8 Naturality Corollaries

Here are some simple corollaries of the adjunction between π_1^* and \forall_I .

1.8.1 Naturality

By the definition of an adjunction:

$$\overline{f \circ \pi_1^*(n)} = \overline{f} \circ n \quad (1.37)$$

1.8.2 $\overline{(-)}$ and Re-indexing Functors

TODO: Why does this occur? it comes from page 222 of Crole?

$$\theta^*(\overline{f}) = (\pi_1 \circ (\theta \times \text{Id}_U))^*(\overline{f}) \quad (1.38)$$

$$= (\theta \times \text{Id}_U)^*(\pi_1^*(\overline{f})) \quad (1.39)$$

$$(1.40)$$

$$(1.41)$$

$$= \overline{(\theta \times \text{Id}_U)^* f} \quad (1.42)$$

$$(1.43)$$

$$(1.44)$$

1.8.3 $\widehat{(-)}$ and Re-Indexing Functors

$$\theta^*(\langle \text{Id}_I, \rho \rangle^* (\widehat{m})) = (\langle \text{Id}_I, \rho \rangle \circ \theta)^* (\widehat{m}) \quad (1.45)$$

$$= ((\theta \times \text{Id}_U) \circ \langle \text{Id}_I, \rho \rangle)^* (\widehat{m}) \quad (1.46)$$

$$= \langle \text{Id}_I, \rho \circ \theta \rangle^* (\theta \times \text{Id}_U)^* (\widehat{m}) \quad (1.47)$$

$$= \langle \text{Id}_I, \theta^* \rho \rangle^* (\theta^* (\widehat{m})) \quad (1.48)$$

1.8.4 Pushing Morphisms into f^*

$$\langle \text{Id}_I, \rho \rangle^* (\widehat{m}) \circ n = \langle \text{Id}_I, \rho \rangle^* (\widehat{m}) \circ \langle \text{Id}_I, \rho \rangle^* \pi_1^*(n) \quad (1.49)$$

$$= \langle \text{Id}_I, \rho \rangle^* (\widehat{m} \circ \pi_1^*(n)) \quad (1.50)$$

$$= \langle \text{Id}_I, \rho \rangle^* (\widehat{m \circ n}) \quad (1.51)$$

Chapter 2

Denotations

2.1 Effects

For each instance of the well-formed-ness relation on effects, we define a morphism $\llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M : \mathbb{C}(I, U)$

- $\llbracket \Phi \vdash e : \mathbf{Effect} \rrbracket_M = \llbracket \epsilon \rrbracket_M \circ \langle \rangle_I : I \rightarrow U$
- $\llbracket \Phi, \alpha \vdash \alpha : \mathbf{Effect} \rrbracket_M = \pi_2 : I \times U \rightarrow U$
- $\llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M = \llbracket \Phi \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \pi_1 : I \times U \rightarrow U$
- $\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \mathbf{Effect} \rrbracket_M = \mathbf{Mul}(\llbracket \Phi \vdash \epsilon_2 : \mathbf{Effect} \rrbracket_M, \llbracket \Phi \vdash \epsilon_1 : \mathbf{Effect} \rrbracket_M) : I \rightarrow U$

2.2 Types

For each instance of the well-formed-ness relation on types, we define a morphism $\llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M : \mathbb{C}(I, W)$.

$\llbracket \mathbf{Unit} \rrbracket_M$ is the morphism generating the terminal object of $\mathbb{C}(I, W)$. \mathbf{Bool} is the morphism generating the co-product of this terminal object, $1 + 1$.

- $\llbracket \Phi \vdash \mathbf{Unit} : \mathbf{Type} \rrbracket_M = \llbracket \mathbf{Unit} \rrbracket_M \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash \mathbf{Bool} : \mathbf{Type} \rrbracket_M = \llbracket \mathbf{Bool} \rrbracket_M \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash \gamma : \mathbf{Type} \rrbracket_M = \llbracket \gamma \rrbracket_M \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash A \rightarrow B : \mathbf{Type} \rrbracket_M = \diamond(\llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M, \llbracket \Phi \vdash B : \mathbf{Type} \rrbracket_M) : I \rightarrow W$
- $\llbracket \Phi \vdash \mathbf{M}_\epsilon A : \mathbf{Type} \rrbracket_M = \mathbf{Eff}(\llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M, \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M) : I \rightarrow W$
- $\llbracket \Phi \vdash \forall \alpha. A : \mathbf{Type} \rrbracket_M = \forall_I(\llbracket \Phi, \alpha \vdash A : \mathbf{Type} \rrbracket_M) : I \rightarrow W$

2.3 Effect Substitution

For each effect-substitution well-formed-ness-relation, define a denotation morphism, $\llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M : \mathbb{C}(I', I)$

- $\llbracket \Phi' \vdash \diamond : \diamond \rrbracket_M = \langle \rangle_I : \mathbb{C}(I', 1)$
- $\llbracket \Phi' \vdash (\sigma, \alpha := \epsilon) : \Phi, \alpha \rrbracket_M = \langle \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M, \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M \rangle : \mathbb{C}(I', I \times U)$

2.4 Effect Weakening

For each instance of the effect-environment weakening relation, define a denotation morphism: $\llbracket \omega : \Phi' \triangleright P \rrbracket_M : \mathbb{C}(I', I)$

- $\llbracket \iota : \Phi \triangleright \Phi \rrbracket_M = \text{Id}_I : I \rightarrow I$
- $\llbracket w\pi : \Phi', \alpha \triangleright \Phi \rrbracket_M = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M \circ \pi_1 : I' \times U \rightarrow I$
- $\llbracket w\times : \Phi', \alpha \triangleright \Phi, \alpha \rrbracket_M = (\llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M \times \text{Id}_U) : I' \times U \rightarrow I \times U$

2.5 Sub-Typing

For each instance of the sub-typing relation with respect to an effect environment, there exists a denotation, $\llbracket A \leq_{:\Phi} B \rrbracket_M : \mathbb{C}(I, W)(A, B)$.

- $\llbracket \gamma_1 \leq_{:\Phi} \gamma_2 \rrbracket_M = \llbracket \gamma_1 \leq_{:\gamma} \gamma_2 \rrbracket_M : \mathbb{C}(I, W)(\gamma_1, \gamma_2)$
- $\llbracket A \rightarrow B \leq_{:\Phi} A' \rightarrow B' \rrbracket_M = \llbracket B \leq_{:\Phi} B' \rrbracket_M^{A'} \circ B[A' \leq_{:\Phi} A]_M$
- $\llbracket \mathbf{M}_{\epsilon_1} A \leq_{:\Phi} \mathbf{M}_{\epsilon_2} B \rrbracket_M = \llbracket \epsilon_1 \leq_{\Phi} \epsilon_2 \rrbracket_M \circ T_{\epsilon_1} \llbracket A \leq_{:\Phi} B \rrbracket_M$
- $\llbracket \forall \alpha. A \leq_{:\Phi} \forall \alpha. B \rrbracket_M = \forall_I \llbracket A \leq_{:\Phi, \alpha} B \rrbracket_M$

2.6 Type-Environments

For each instance of the well-formed relation on type environments, define an object in $\llbracket I \vdash W\mathbf{Ok} \rrbracket_M \in \mathbb{C}(I, W)$.

- $\llbracket \Phi \vdash \diamond \mathbf{Ok} \rrbracket_M = 1 : \mathbb{C}(I, W)$
- $\llbracket \Phi \vdash \Gamma, x : A\mathbf{Ok} \rrbracket_M = \square(\llbracket \Phi \vdash \Gamma\mathbf{Ok} \rrbracket_M, \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M)$

2.7 Terms

For each instance of the typing relation, define a denotation morphism: $\llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M : \mathbb{C}(I, W)(\Gamma_I, A_I)$. Writing Γ_I and A_I for $\llbracket \Phi \vdash \Gamma\mathbf{Ok} \rrbracket_M$ and $\llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M$.

For each ground constant, \mathbf{C}^A , there exists $c : 1 \rightarrow A_I$ in $\mathbb{C}(I, W)$.

- (Unit) $\frac{\Phi \vdash \Gamma\mathbf{Ok}}{\llbracket \Phi \mid \Gamma \vdash () : \mathbf{Unit} \rrbracket_M = \langle \rangle_{\Gamma : \Gamma_I \rightarrow 1}}$
- (Const) $\frac{\Phi \vdash \Gamma\mathbf{Ok}}{\llbracket \Phi \mid \Gamma \vdash \mathbf{C}^A : A \rrbracket_M = \llbracket \mathbf{C}^A \rrbracket_M \circ \langle \rangle_{\Gamma : \Gamma_I \rightarrow \llbracket A \rrbracket_M}}$
- (True) $\frac{\Phi \vdash \Gamma\mathbf{Ok}}{\llbracket \Phi \mid \Gamma \vdash \mathbf{true} : \mathbf{Bool} \rrbracket_M = \mathbf{inl} \circ \langle \rangle_{\Gamma : \Gamma_I \rightarrow \llbracket \mathbf{Bool} \rrbracket_M = 1 + 1}}$

- (False) $\frac{\Phi \vdash \Gamma \mathbf{Ok}}{\llbracket \Phi | \Gamma \vdash \mathbf{false} : \mathbf{Bool} \rrbracket_M = \mathbf{inr} \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket \mathbf{Bool} \rrbracket_M = \mathbf{1} + \mathbf{1}}$
- (Var) $\frac{\Phi \vdash \Gamma \mathbf{Ok}}{\llbracket \Phi | \Gamma, x : A \vdash x : A \rrbracket_M = \pi_2 : \Gamma \times A \rightarrow A}$
- (Weaken) $\frac{f = \llbracket \Phi | \Gamma \vdash x : A \rrbracket_M : \Gamma \rightarrow A}{\llbracket \Phi | \Gamma, y : B \vdash x : A \rrbracket_M = f \circ \pi_1 : \Gamma \times B \rightarrow A}$
- (Lambda) $\frac{f = \llbracket \Phi | \Gamma, x : A \vdash C : \mathbf{M}_{\epsilon} B \rrbracket_M : \Gamma \times A \rightarrow T_{\epsilon} B}{\llbracket \Phi | \Gamma \vdash \lambda x : A. C : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M = \mathbf{cur}(f) : \Gamma \rightarrow (T_{\epsilon} B)^A}$
- (Subtype) $\frac{f = \llbracket \Phi | \Gamma \vdash v : A \rrbracket_M : \Gamma \rightarrow A \quad g = \llbracket A \leq B \rrbracket_M}{\llbracket \Phi | \Gamma \vdash v : B \rrbracket_M = g \circ f : \Gamma \rightarrow B}$
- (Return) $\frac{f = \llbracket \Phi | \Gamma \vdash v : A \rrbracket_M}{\llbracket \Phi | \Gamma \vdash \mathbf{return} v : \mathbf{M}_1 A \rrbracket_M = \eta_A \circ f}$
- (If) $\frac{f = \llbracket \Phi | \Gamma \vdash v : \mathbf{Bool} \rrbracket_M : \Gamma \rightarrow \mathbf{1} + \mathbf{1} \quad g = \llbracket \Phi | \Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A \rrbracket_M \quad h = \llbracket \Phi | \Gamma \vdash C_2 : \mathbf{M}_{\epsilon} A \rrbracket_M}{\llbracket \Phi | \Gamma \vdash \mathbf{if}_{\epsilon, A} v \mathbf{then} C_1 \mathbf{else} C_2 : \mathbf{M}_{\epsilon} A \rrbracket_M = \mathbf{app} \circ ((\llbracket \mathbf{cur}(g \circ \pi_2), \mathbf{cur}(h \circ \pi_2) \rrbracket \circ f) \times \mathbf{Id}_{\Gamma}) \circ \delta_{\Gamma} : \Gamma \rightarrow T_{\epsilon} A}$
- (Bind) $\frac{f = \llbracket \Phi | \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A : \Gamma \rightarrow T_{\epsilon_1} A \quad g = \llbracket \Phi | \Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M : \Gamma \times A \rightarrow T_{\epsilon_2} B}{\llbracket \Phi | \Gamma \vdash \mathbf{do} x \leftarrow C_1 \mathbf{in} C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B \rrbracket_M = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \tau_{\Gamma, A, \epsilon_1} \circ \langle \mathbf{Id}_{\Gamma}, f \rangle : \Gamma \rightarrow T_{\epsilon_1 \cdot \epsilon_2} B}$
- (Apply) $\frac{f = \llbracket \Phi | \Gamma \vdash v_1 : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M : \Gamma \rightarrow (T_{\epsilon} B)^A \quad g = \llbracket \Phi | \Gamma \vdash v_2 : A \rrbracket_M : \Gamma \rightarrow A}{\llbracket \Phi | \Gamma \vdash v_1 v_2 : \mathbf{M}_{\epsilon} B \rrbracket_M = \mathbf{app} \circ (f, g) : \Gamma \rightarrow T_{\epsilon} B}$
- (Effect-Lambda) $\frac{f = \llbracket \Phi, \alpha | \Gamma \vdash v : A \rrbracket_M : \mathbb{C}(I \times U, W)(\Gamma, A)}{\llbracket \Phi | \Gamma \vdash \Lambda \alpha. A : \forall \epsilon. A \rrbracket_M = \bar{f} : \mathbb{C}(I, W)(\Gamma, \forall_I(A))}$
- (Effect-App) $\frac{g = \llbracket \Phi | \Gamma \vdash v : \forall \alpha. A \rrbracket_M : \mathbb{C}(I, W)(\Gamma, \forall_I(A)) \quad h = \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M : \mathbb{C}(I, U)}{\llbracket \Phi | \Gamma \vdash v : \epsilon : A[\epsilon/\alpha] \rrbracket_M = \langle \mathbf{Id}_I, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \mathbf{Type} \rrbracket_M}) \circ g : \mathbb{C}(I, W)(\Gamma, A[\epsilon/\alpha])}$

Chapter 3

Effect Substitution Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-variable substitution upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the substituted relation, $\Delta' = \sigma^*(\Delta)$.

3.1 Effects

If $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M$ then $\llbracket \Phi' \vdash \sigma(\epsilon) : \mathbf{Effect} \rrbracket_M = \sigma^* \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M = \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M \circ \sigma$.

Proof: By induction on the derivation on $\llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M$

Case Ground:

$$\llbracket \Phi \vdash e : \mathbf{Effect} \rrbracket_M \circ \sigma = \llbracket e \rrbracket_M \circ \langle \rangle_I \circ \sigma \quad (3.1)$$

$$= \llbracket e \rrbracket_M \circ \langle \rangle_{I'} \quad (3.2)$$

$$= \llbracket \Phi' \vdash e : \mathbf{Type} \rrbracket_M \quad (3.3)$$

$$(3.4)$$

Case Var:

$$\llbracket \Phi, \alpha \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \sigma' = \pi_2 \circ \langle \sigma, \llbracket \Phi' \vdash \epsilon : \mathbf{Effect} \rrbracket_M \rangle \quad \text{By inversion } \sigma' = (\sigma, \alpha := \epsilon) \quad (3.5)$$

$$= \llbracket \Phi' \vdash \epsilon : \mathbf{Effect} \rrbracket_M \quad (3.6)$$

$$= \llbracket \Phi' \vdash \sigma'(\alpha) : \mathbf{Effect} \rrbracket_M \quad (3.7)$$

$$(3.8)$$

Case Weaken:

$$\llbracket \Phi, \beta \vdash \alpha : \text{Type} \rrbracket_M \circ \sigma' = \llbracket \Phi \vdash \alpha : \text{Type} \rrbracket_M \circ \pi_1 \circ \langle \sigma, \llbracket \Phi' \vdash \epsilon : \text{Effect} \rrbracket_M \rangle \quad \text{By inversion, } \sigma' = (\sigma, \beta := \epsilon) \quad (3.9)$$

$$= \llbracket \Phi \vdash \alpha : \text{Type} \rrbracket_M \circ \sigma \quad (3.10)$$

$$= \llbracket \Phi' \vdash \sigma(\alpha) : \text{Type} \rrbracket_M \quad (3.11)$$

$$= \llbracket \Phi' \vdash \sigma'(\alpha) : \text{Type} \rrbracket_M \quad (3.12)$$

$$(3.13)$$

Case Multiply:

$$\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Type} \rrbracket_M \circ \sigma = \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket_M, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket_M) \circ \sigma \quad (3.14)$$

$$= \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket_M \circ \sigma, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket_M \circ \sigma) \quad \text{By Naturality} \quad (3.15)$$

$$= \text{Mul}(\llbracket \Phi' \vdash \sigma(\epsilon_1) : \text{Effect} \rrbracket_M, \llbracket \Phi \vdash \sigma(\epsilon_2) : \text{Effect} \rrbracket_M) \quad (3.16)$$

$$= \llbracket \Phi' \vdash \sigma(\epsilon_1) \cdot \sigma(\epsilon_2) : \text{Effect} \rrbracket_M \quad (3.17)$$

$$= \llbracket \Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2) : \text{Effect} \rrbracket_M \quad (3.18)$$

$$(3.19)$$

3.2 Types

If $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M$ then $\llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket_M = \sigma^* \llbracket \Phi \vdash A : \text{Type} \rrbracket_M = \llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \sigma$.

Proof: By induction on the derivation on $\llbracket \Phi \vdash A : \text{Type} \rrbracket_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$\llbracket \Phi \vdash \gamma : \text{Type} \rrbracket_M \circ \sigma = \llbracket \gamma \rrbracket_M \circ \langle \rangle_I \circ \sigma \quad (3.20)$$

$$= \llbracket \gamma \rrbracket_M \circ \langle \rangle_{I'} \quad (3.21)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket_M \quad (3.22)$$

$$= \llbracket \Phi' \vdash \gamma[\sigma] : \text{Type} \rrbracket_M \quad (3.23)$$

Case Monad:

$$\llbracket \Phi \vdash \mathbb{M}_\epsilon A : \text{Type} \rrbracket_M \circ \sigma = \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M) \circ \sigma \quad (3.24)$$

$$= \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \circ \sigma, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \sigma) \quad \text{By naturality} \quad (3.25)$$

$$= \text{Eff}(\llbracket \Phi' \vdash \sigma(\epsilon) : \text{Effect} \rrbracket_M, \llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket_M) \quad (3.26)$$

$$= \llbracket \Phi' \vdash \mathbb{M}_{\sigma(\epsilon)} A[\sigma] : \text{Type} \rrbracket_M \quad (3.27)$$

$$= \llbracket \Phi' \vdash (\mathbb{M}_\epsilon A)[\sigma] : \text{Type} \rrbracket_M \quad (3.28)$$

Case Quantification:

$$\llbracket \Phi \vdash \forall \alpha. A : \text{Type} \rrbracket_M \circ \sigma = \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket_M) \circ \sigma \quad (3.29)$$

$$= \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket_M \circ (\sigma \times \text{Id}_U)) \quad (3.30)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A[\sigma, \alpha := \epsilon] : \text{Type} \rrbracket_M) \quad (3.31)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A[\sigma] : \text{Type} \rrbracket_M) \quad (3.32)$$

$$= \llbracket \Phi' \vdash \forall \alpha. A[\sigma] : \text{Type} \rrbracket_M \quad (3.33)$$

$$= \llbracket \Phi' \vdash (\forall \alpha. A) [\sigma] : \text{Type} \rrbracket_M \quad (3.34)$$

$$(3.35)$$

Case Function:

$$\llbracket \Phi \vdash A \rightarrow B : \text{Type} \rrbracket_M \circ \sigma = \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket_M, \llbracket \Phi \vdash B : \text{Type} \rrbracket_M) \circ \sigma \quad (3.36)$$

$$= \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \sigma, \llbracket \Phi \vdash B : \text{Type} \rrbracket_M \circ \sigma) \quad \text{By Naturality} \quad (3.37)$$

$$= \diamond(\llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket_M, \llbracket \Phi' \vdash B[\sigma] : \text{Type} \rrbracket_M) \quad (3.38)$$

$$= \llbracket \Phi' \vdash (A[\sigma]) \rightarrow (B[\sigma]) : \text{Type} \rrbracket_M \quad (3.39)$$

$$= \llbracket \Phi' \vdash (A \rightarrow B) [\sigma] : \text{Type} \rrbracket_M \quad (3.40)$$

$$(3.41)$$

3.3 Sub-typing

If $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M$ then $\llbracket A[\sigma] \leq_{:\Phi'} B[\sigma] \rrbracket_M = \sigma^* \llbracket A \leq_{:\Phi} B \rrbracket_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $\llbracket A \leq_{:\Phi} B \rrbracket_M$. Using S-closure of σ^*

Case Ground:

$$\sigma^*(\gamma_1 \leq_{:\gamma} \gamma_2) = (\gamma_1 \leq_{:\gamma} \gamma_2) \quad (3.42)$$

Since σ^* is s-closed.

Case Monad:

$$\sigma^* \llbracket \mathbb{M}_{\epsilon_1} A \leq_{:\Phi} \mathbb{M}_{\epsilon_2} B \rrbracket_M = \sigma^*(\llbracket \epsilon_1 \leq_{\Phi} \epsilon_2 \rrbracket_M) \circ \sigma^*(T_{\epsilon_1}(\llbracket A \leq_{:\Phi} B \rrbracket_M)) \quad (3.43)$$

$$= \llbracket \sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2) \rrbracket_M \circ T_{\sigma(\epsilon_1)} \llbracket A[\sigma] \leq_{:\Phi'} B[\sigma] \rrbracket_M \quad \text{By S-Closure} \quad (3.44)$$

$$= \llbracket \mathbb{M}_{\sigma(\epsilon_1)} A[\sigma] \leq_{:\Phi'} \mathbb{M}_{\sigma(\epsilon_2)} B[\sigma] \rrbracket_M \quad (3.45)$$

$$= \llbracket (\mathbb{M}_{\epsilon_1} A) [\sigma] \leq_{:\Phi'} \mathbb{M}_{\epsilon_2} B [\sigma] \rrbracket_M \quad (3.46)$$

$$(3.47)$$

Case For All:

$$\sigma^* \llbracket \forall \alpha. A \leq_{:\Phi} \forall \alpha. B \rrbracket_M = \sigma^*(\forall_I(\llbracket A \leq_{:\Phi, \alpha} B \rrbracket_M)) \quad (3.48)$$

$$= \forall_{I'}((\sigma \times \text{Id}_U)^*(\llbracket A \leq_{:\Phi, \alpha} B \rrbracket_M)) \quad (3.49)$$

$$= \forall_{I'}(\llbracket A[\sigma, \alpha := \alpha] \leq_{:\Phi', \alpha} B[\sigma, \alpha := \alpha] \rrbracket_M) \quad (3.50)$$

$$= \llbracket (\forall \alpha. A) [\sigma] \leq_{:\Phi'} (\forall \alpha. B) [\sigma] \rrbracket_M \quad (3.51)$$

$$(3.52)$$

Case Fn:

$$\sigma^* \llbracket (A \rightarrow B) \leq_{:\Phi} A' \rightarrow B' \rrbracket_M = \sigma^* (\llbracket B \leq_{:\Phi} B' \rrbracket_M^{A'} \circ B \llbracket A' \leq_{:\Phi} A \rrbracket_M) \quad (3.53)$$

$$= \sigma^* (\text{cur}(\llbracket B \leq_{:\Phi} B' \rrbracket_M \circ \text{app}) \circ \sigma^* (\text{cur}(\text{app} \circ (\text{Id}_B \times \llbracket A' \leq_{:\Phi} A \rrbracket_M)))) \quad (3.54)$$

$$= \text{cur}(\sigma^* (\llbracket B \leq_{:\Phi} B' \rrbracket_M) \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_B \times \sigma^* (\llbracket A' \leq_{:\Phi} A \rrbracket_M))) \quad (3.55)$$

$$= \text{cur}(\llbracket B[\sigma] \leq_{:\Phi'} B'[\sigma] \rrbracket_M \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_{B[\sigma]} \times \llbracket A'[\sigma] \leq_{:\Phi'} A[\sigma] \rrbracket_M)) \quad (3.56)$$

$$= \llbracket (A[\sigma]) \rightarrow (B[\sigma]) \leq_{:\Phi'} (A'[\sigma]) \rightarrow (B'[\sigma]) \rrbracket_M \quad (3.57)$$

$$= \llbracket (A \rightarrow B)[\sigma] \leq_{:\Phi'} (A' \rightarrow B')[\sigma] \rrbracket_M \quad (3.58)$$

3.4 Type Environments

If $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M$ then $\llbracket \Phi' \vdash \Gamma[\sigma] \mathbf{0k} \rrbracket_M = \sigma^* \llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M = \llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M \circ \sigma : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $\llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M$. Using Naturality.

Case Nil:

$$\sigma^* \llbracket \Phi \vdash \diamond \mathbf{0k} \rrbracket_M = \langle \rangle_I \circ \sigma \quad (3.59)$$

$$= \langle \rangle_{I'} \quad (3.60)$$

$$= \llbracket \Phi' \vdash \diamond \mathbf{0k} \rrbracket_M \quad (3.61)$$

$$\llbracket \Phi' \vdash \diamond [\sigma] \mathbf{0k} \rrbracket_M \quad (3.62)$$

$$(3.63)$$

Case Var:

$$\sigma^* \llbracket \Phi \vdash \Gamma, x : A \mathbf{0k} \rrbracket_M = \sigma^* (\Box(\llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M)) \quad (3.64)$$

$$= \Box(\llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M) \circ \sigma \quad (3.65)$$

$$= \Box(\llbracket \Phi \vdash \Gamma \mathbf{0k} \rrbracket_M \circ \sigma, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \sigma) \quad (3.66)$$

$$= \Box(\llbracket \Phi' \vdash \Gamma[\sigma] \mathbf{0k} \rrbracket_M, \llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket_M) \quad (3.67)$$

$$= \llbracket \Phi' \vdash \Gamma[\sigma], x : A[\sigma] \mathbf{0k} \rrbracket_M \quad (3.68)$$

$$= \llbracket \Phi' \vdash (\Gamma, x : A)[\sigma] \mathbf{0k} \rrbracket_M \quad (3.69)$$

$$(3.70)$$

3.5 Terms

If

$$\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket_M \quad (3.71)$$

$$\Delta = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (3.72)$$

$$\Delta' = \llbracket \Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma] \rrbracket_M \quad (3.73)$$

$$(3.74)$$

Then

$$\Delta' = \sigma^*(\Delta) \quad (3.75)$$

Proof: By induction over the derivation of Δ . Using the S-Closure of σ^* . We use Γ_I to indicate $\llbracket \Phi \vdash \Gamma \text{Ok} \rrbracket_M$, an A_I to indicate $\llbracket \Phi \vdash A : \text{Effect} \rrbracket_M$

Case Unit:

$$\Delta = \langle \rangle_{\Gamma_I} \quad (3.76)$$

So

$$\sigma^*(\Delta) = \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.77)$$

Case True, False: Giving the case for true as false is the same but using **inr**

$$\Delta = \text{inl} \circ \langle \rangle_{\Gamma_I} \quad (3.78)$$

So

$$\sigma^*(\Delta) = \text{inl} \circ \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.79)$$

Since σ^* is S-closed.

Case Constant:

$$\Delta = \llbracket \mathbf{c}^A \rrbracket_M \circ \langle \rangle_{\Gamma_I} \quad (3.80)$$

So

$$\sigma^*(\Delta) = \sigma^* \llbracket \mathbf{c}^A \rrbracket_M \circ \langle \rangle_{\Gamma_I[\sigma]} = \llbracket \mathbf{c}^{A[\sigma]} \rrbracket_M \circ \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.81)$$

Since σ^* is S-closed.

Case Subtype: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (3.82)$$

Then

$$\Delta = \llbracket A \leq_{:\Phi} B \rrbracket_M \circ \Delta_1 \quad (3.83)$$

So

$$\sigma^*(\Delta) = \sigma^* \llbracket A \leq_{:\Phi} B \rrbracket_M \circ \sigma^* \Delta_1 \quad (3.84)$$

$$= \llbracket A[\sigma] \leq_{:\Phi'} B[\sigma] \rrbracket_M \circ \Delta'_1 \quad \text{By induction} \quad (3.85)$$

$$= D' \quad (3.86)$$

Case Lambda: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma, x : A \vdash v : B \rrbracket_M \quad (3.87)$$

Then

$$\Delta = \text{cur}((\Delta_1)) \quad (3.88)$$

So

$$\sigma^*(\Delta) = \sigma^*(\text{cur}(\Delta_1)) \quad (3.89)$$

$$= \text{cur}(\sigma^*(\Delta_1)) \quad \text{By S-closure} \quad (3.90)$$

$$= \text{cur}(\Delta'_1) \quad \text{By induction} \quad (3.91)$$

$$= \Delta' \quad (3.92)$$

Case Application: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rightarrow B \rrbracket_M \quad (3.93)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket_M \quad (3.94)$$

Then

$$\Delta = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad (3.95)$$

So

$$\sigma^* \Delta = \sigma^*(\text{app} \circ \langle \Delta_1, \Delta_2 \rangle) \quad (3.96)$$

$$= \text{app} \circ \langle \sigma^*(\Delta_1), \sigma^*(\Delta_2) \rangle \quad \text{By S-closure} \quad (3.97)$$

$$= \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Induction} \quad (3.98)$$

$$= \Delta' \quad (3.99)$$

Case Return: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (3.100)$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \quad (3.101)$$

So

$$\sigma^*(\Delta) = \sigma^*(\eta_{A_I} \circ \Delta_1) \quad (3.102)$$

$$= \eta_{A_{I'}} \circ \sigma^*(\Delta_1) \quad \text{By S-closure} \quad (3.103)$$

$$= \eta_{A_{I'}} \circ \Delta'_1 \quad (3.104)$$

$$= \Delta' \quad (3.105)$$

Case Bind: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M \quad (3.106)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M \quad (3.107)$$

Then

$$\Delta = \mathbf{M}_{\epsilon_1 \epsilon_2} A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_{\Gamma_I}, \Delta_1 \rangle \quad (3.108)$$

So

$$\sigma^*(\Delta) = \sigma^*(\mu_{\epsilon_1, \epsilon_2, A} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_{\Gamma}, \Delta_1 \rangle) \quad (3.109)$$

$$= \sigma^*(\mu_{\epsilon_1, \epsilon_2, A}) \circ \sigma^*(T_{\epsilon_1} \Delta_2) \circ \sigma^*(\mathbf{t}_{\epsilon_1, \Gamma, A}) \circ \langle \sigma^*(\mathbf{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (3.110)$$

$$= \mu_{\sigma(\epsilon_1), \sigma(\epsilon_2), A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \sigma^*(\Delta_2) \circ \mathbf{t}_{\sigma(\epsilon_1), \Gamma[\sigma], A[\sigma]} \circ \langle \sigma^*(\mathbf{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (3.111)$$

$$= \mu_{\sigma(\epsilon_1), \sigma(\epsilon_2), A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \Delta'_2 \circ \mathbf{t}_{\sigma(\epsilon_1), \Gamma[\sigma], A[\sigma]} \circ \langle \sigma^*(\mathbf{Id}_{\Gamma_I}), \Delta'_1 \rangle \quad \text{By Induction} \quad (3.112)$$

$$= \Delta' \quad (3.113)$$

$$(3.114)$$

Case If: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \mathbf{Bool} \rrbracket_M \quad (3.115)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket_M \quad (3.116)$$

$$\Delta_3 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket_M \quad (3.117)$$

$$(3.118)$$

Then

$$\Delta = \mathbf{app} \circ (([\mathbf{cur}(\Delta_2 \circ \pi_2), \mathbf{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathbf{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad (3.119)$$

So

$$\sigma^*(\Delta) = \sigma^*(\mathbf{app} \circ (([\mathbf{cur}(\Delta_2 \circ \pi_2), \mathbf{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathbf{Id}_{\Gamma}) \circ \delta_{\Gamma}) \quad (3.120)$$

$$= \mathbf{app} \circ (([\mathbf{cur}(\sigma^*(\Delta_2) \circ \pi_2), \mathbf{cur}(\sigma^*(\Delta_3) \circ \pi_2)] \circ \sigma^*(\Delta_1)) \times \mathbf{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By S-Closure} \quad (3.121)$$

$$= \mathbf{app} \circ (([\mathbf{cur}(\Delta'_2 \circ \pi_2), \mathbf{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \mathbf{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By Induction} \quad (3.122)$$

$$= \Delta' \quad (3.123)$$

$$(3.124)$$

Case Effect-Lambda: Let

$$\Delta_1 = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket_M \quad (3.125)$$

Then

$$\Delta = \widehat{\Delta_1} \quad (3.126)$$

And also

$$\sigma \times \text{Id} = \llbracket (\Phi', \alpha) \vdash (\sigma, \alpha := \epsilon) : (\Phi, \alpha) \rrbracket_M \quad (3.127)$$

So

$$\sigma^* \Delta = \sigma^* (\widehat{\Delta_1}) \quad (3.128)$$

$$= \overline{(\sigma \times \text{Id}_U)^* \Delta_1} \quad \text{By naturality} \quad (3.129)$$

$$= \widehat{\Delta'_1} \quad \text{By induction} \quad (3.130)$$

$$= \Delta' \quad (3.131)$$

Case Effect-Application: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \forall \alpha. A \rrbracket_M \quad (3.132)$$

$$h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \quad (3.133)$$

$$(3.134)$$

Then

$$\Delta = \langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1 \quad (3.135)$$

So Due to the substitution theorem on effects

$$h \circ \sigma = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \circ \sigma = \llbracket \Phi' \vdash \sigma(\epsilon) : \text{Effect} \rrbracket_M = h' \quad (3.136)$$

$$\sigma^* \Delta = \sigma^* (\langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1) \quad (3.137)$$

$$= (\langle \text{Id}_\Gamma, h \rangle \circ \sigma)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \sigma^* (\Delta_1) \quad (3.138)$$

$$= ((\sigma \times \text{Id}_U) \circ \langle \text{Id}_\Gamma, h \circ \sigma \rangle)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1' \quad (3.139)$$

$$= (\langle \text{Id}_\Gamma, h' \rangle)^* ((\sigma \times \text{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1' \quad (3.140)$$

$$(3.141)$$

Looking at the inner part of the functor application: Let

$$A = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M \quad (3.142)$$

$$(3.143)$$

$$(\sigma \times \text{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M} = (\sigma \times \text{Id}_U)^* \epsilon_A \quad (3.144)$$

$$= (\sigma \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}}) \quad (3.145)$$

$$= \overline{(\sigma \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By bijection} \quad (3.146)$$

$$= \overline{\sigma^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By naturality} \quad (3.147)$$

$$= \overline{\sigma^* (\text{Id}_{\forall_I(A)})} \quad \text{By bijection} \quad (3.148)$$

$$= \overline{\text{Id}_{\forall_{I'}(A \circ (\sigma \times \text{Id}_U))}} \quad \text{By S-Closure, naturality} \quad (3.149)$$

$$= \overline{\text{Id}_{\forall_{I'}(A[\sigma, \alpha := \alpha])}} \quad \text{By Substitution theorem} \quad (3.150)$$

$$= \epsilon_{A[\sigma]} \quad (3.151)$$

Going back to the original expression:

$$\sigma^* \Delta = (\langle \text{Id}_\Gamma, h' \rangle)^* (\epsilon_{A[\sigma]} \circ \Delta_1)' \quad (3.152)$$

$$= \Delta' \quad (3.153)$$

$$(3.154)$$

Chapter 4

Effect Weakening Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-weakening upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the weakened relation, $\Delta' = \omega^*(\Delta)$.

4.1 Effects

If $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M$ then $\Phi' \vdash \epsilon : \mathbf{Effect} = \omega^* \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M = \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M \circ \omega$

Proof: By induction on the derivation on $\llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket_M$

Case Ground:

$$\llbracket \Phi \vdash e : \mathbf{Effect} \rrbracket_M \circ \omega = \llbracket e \rrbracket_M \circ \langle \rangle_I \circ \omega \quad (4.1)$$

$$= \llbracket e \rrbracket_M \circ \langle \rangle_{I'} \quad (4.2)$$

$$= \llbracket \Phi' \vdash e : \mathbf{Type} \rrbracket_M \quad (4.3)$$

$$(4.4)$$

Case Var: Case split on ω .

Case: $\omega = \iota$ Then $\Phi' = \Phi$ and $\omega = \text{Id}_I$. So the theorem holds trivially.

Case: $\omega = \omega' \times$ Then

$$\llbracket \Phi, \alpha \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega = \pi_2 \circ (\omega' \times \text{Id}_U) \quad (4.5)$$

$$= \pi_2 \quad (4.6)$$

$$= \llbracket \Phi', \alpha \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.7)$$

Case: $\omega = \omega' \pi_1$ Then

$$\llbracket \Phi, \alpha \vdash \alpha : \mathbf{Effect} \rrbracket_M = \pi_2 \circ \omega' \circ \pi_1 \quad (4.8)$$

Where $\Phi' = \Phi, \beta$ and $\omega' : \Phi'' \triangleright \Phi$.

So

$$\pi_2 \circ \omega' = \llbracket \Phi'' \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.9)$$

$$\pi_2 \circ \omega' \circ \pi_1 = \llbracket \Phi'', \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M = \llbracket \Phi' \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.10)$$

Case Weaken:

$$\llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega = \llbracket \Phi \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \pi_1 \circ \omega \quad (4.11)$$

Case split of structure of w

Case: $\omega = \iota$ Then $\Phi' = \Phi, \beta$ so $\omega = \text{Id}_I$ So $\llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega = \llbracket \Phi' \vdash \alpha : \mathbf{Effect} \rrbracket_M$

Case: $\omega = \omega' \pi_1$ Then $\Phi' = \Phi'', \gamma$ and $\omega = \omega' \circ \pi_1$ Where $\omega' : \Phi'' \triangleright \Phi, \beta$. So

$$\llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega = \llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega' \circ \pi_1 \quad (4.12)$$

$$= \llbracket \Phi'' \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \pi_1 \quad (4.13)$$

$$= \llbracket \Phi'', \gamma \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.14)$$

$$= \llbracket \Phi' \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.15)$$

$$(4.16)$$

Case: $\omega = \omega' \times$ Then $\Phi' = \Phi'', \beta$ and $\omega' : \Phi' \triangleright \Phi$

So

$$\llbracket \Phi, \beta \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega = \llbracket \Phi \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \pi_1 \circ (\omega' \times \text{Id}_U) \quad (4.17)$$

$$= \llbracket \Phi \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \omega' \circ \pi_1 \quad (4.18)$$

$$= \llbracket \Phi'' \vdash \alpha : \mathbf{Effect} \rrbracket_M \circ \pi_1 \quad (4.19)$$

$$= \llbracket \Phi' \vdash \alpha : \mathbf{Effect} \rrbracket_M \quad (4.20)$$

$$(4.21)$$

Case Multiply:

$$\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \mathbf{Type} \rrbracket_M \circ \omega = \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \mathbf{Effect} \rrbracket_M, \llbracket \Phi \vdash \epsilon_2 : \mathbf{Effect} \rrbracket_M) \circ \omega \quad (4.22)$$

$$= \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \mathbf{Effect} \rrbracket_M \circ \omega, \llbracket \Phi \vdash \epsilon_2 : \mathbf{Effect} \rrbracket_M \circ \omega) \quad \text{By Naturality} \quad (4.23)$$

$$= \text{Mul}(\llbracket \Phi' \vdash \epsilon_1 : \mathbf{Effect} \rrbracket_M, \llbracket \Phi \vdash \epsilon_2 : \mathbf{Effect} \rrbracket_M) \quad (4.24)$$

$$= \llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \mathbf{Effect} \rrbracket_M \quad (4.25)$$

4.2 Types

If $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M$ then $\llbracket \Phi' \vdash A : \mathbf{Type} \rrbracket_M = \omega^* \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M = \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M \circ \omega$.

Proof: By induction on the derivation on $\llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$\llbracket \Phi \vdash \gamma : \text{Type} \rrbracket_M \circ \omega = \llbracket \gamma \rrbracket_M \circ \langle \rangle_I \circ \omega \quad (4.26)$$

$$= \llbracket \gamma \rrbracket_M \circ \langle \rangle_{I'} \quad (4.27)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket_M \quad (4.28)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket_M \quad (4.29)$$

Case Monad:

$$\llbracket \Phi \vdash \mathsf{M}_\epsilon A : \text{Type} \rrbracket_M \circ \omega = \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M) \circ \omega \quad (4.30)$$

$$= \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \circ \omega, \llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \omega) \quad \text{By naturality} \quad (4.31)$$

$$= \text{Eff}(\llbracket \Phi' \vdash \epsilon : \text{Effect} \rrbracket_M, \llbracket \Phi' \vdash A : \text{Type} \rrbracket_M) \quad (4.32)$$

$$= \llbracket \Phi' \vdash (\mathsf{M}_\epsilon A) : \text{Type} \rrbracket_M \quad (4.33)$$

Case Quantification: Note $\llbracket \omega \times : \Phi', \alpha \triangleright \Phi, \alpha \rrbracket_M = \omega \times \text{Id}_U$

$$\llbracket \Phi \vdash \forall \alpha. A : \text{Type} \rrbracket_M \circ \omega = \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket_M) \circ \omega \quad (4.34)$$

$$= \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket_M \circ (\omega \times \text{Id}_U)) \quad \text{By naturality} \quad (4.35)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A : \text{Type} \rrbracket_M) \quad \text{By induction} \quad (4.36)$$

$$= \llbracket \Phi' \vdash \forall \alpha. A : \text{Type} \rrbracket_M \quad (4.37)$$

$$= \llbracket \Phi' \vdash (\forall \alpha. A) : \text{Type} \rrbracket_M \quad (4.38)$$

$$(4.39)$$

Case Function:

$$\llbracket \Phi \vdash A \rightarrow B : \text{Type} \rrbracket_M \circ \omega = \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket_M, \llbracket \Phi \vdash B : \text{Type} \rrbracket_M) \circ \omega \quad (4.40)$$

$$= \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket_M \circ \omega, \llbracket \Phi \vdash B : \text{Type} \rrbracket_M \circ \omega) \quad \text{By Naturality} \quad (4.41)$$

$$= \diamond(\llbracket \Phi' \vdash A : \text{Type} \rrbracket_M, \llbracket \Phi' \vdash B : \text{Type} \rrbracket_M) \quad (4.42)$$

$$= \llbracket \Phi' \vdash (A \rightarrow B) : \text{Type} \rrbracket_M \quad (4.43)$$

$$(4.44)$$

4.3 Sub-typing

If $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M$ then $\llbracket A \leq_{\Phi'} B \rrbracket_M = \omega^* \llbracket A \leq_{\Phi} B \rrbracket_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $\llbracket A \leq_{\Phi} B \rrbracket_M$. Using S-closure of ω^*

Case Ground:

$$\omega^*(\gamma_1 \leq_{\gamma} \gamma_2) = (\gamma_1 \leq_{\gamma} \gamma_2) \quad (4.45)$$

Since ω^* is s-closed.

Case Monad:

$$\omega^*[\![\mathbb{M}_{\epsilon_1} A \leq_{:\Phi} \mathbb{M}_{\epsilon_2} B]\!]_M = \omega^*([\![\epsilon_1 \leq_{\Phi} \epsilon_2]\!]_M) \circ \omega^*(T_{\epsilon_1}([\![A \leq_{:\Phi} B]\!]_M)) \quad (4.46)$$

$$= [\![\epsilon_1 \leq_{\Phi'} \epsilon_2]\!]_M \circ T_{\epsilon_1}([\![A \leq_{:\Phi'} B]\!]_M) \quad \text{By S-Closure} \quad (4.47)$$

$$= [\![\mathbb{M}_{\epsilon_1} A \leq_{:\Phi'} \mathbb{M}_{\epsilon_2} B]\!]_M \quad (4.48)$$

$$= [\![\mathbb{M}_{\epsilon_1} A] \leq_{:\Phi'} \mathbb{M}_{\epsilon_2} B]\!]_M \quad (4.49)$$

$$(4.50)$$

Case For All: Note $[\![\omega \times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = (\omega \times \text{Id}_U)$

$$\omega^*[\![\forall \alpha. A \leq_{:\Phi} \forall \alpha. B]\!]_M = \omega^*(\forall_I([\![A \leq_{:\Phi, \alpha} B]\!]_M)) \quad (4.51)$$

$$= \forall_{I'}((\omega \times \text{Id}_U)^*([\![A \leq_{:\Phi, \alpha} B]\!]_M)) \quad (4.52)$$

$$= \forall_{I'}([\![A \leq_{:\Phi', \alpha} B]\!]_M) \quad (4.53)$$

$$= [\![\forall \alpha. A] \leq_{:\Phi'} \forall \alpha. B]\!]_M \quad (4.54)$$

$$(4.55)$$

Case Fn:

$$\omega^*[\![A \rightarrow B] \leq_{:\Phi} A' \rightarrow B']_M = \omega^*([\![B \leq_{:\Phi} B']_M^{A'} \circ B^{[A' \leq_{:\Phi} A]}_M]) \quad (4.56)$$

$$= \omega^*(\text{cur}([\![B \leq_{:\Phi} B']_M \circ \text{app}]) \circ \omega^*(\text{cur}(\text{app} \circ (\text{Id}_B \times [\![A' \leq_{:\Phi} A]\!]_M)))) \quad (4.57)$$

$$= \text{cur}(\omega^*([\![B \leq_{:\Phi} B']_M] \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_B \times \omega^*([\![A' \leq_{:\Phi} A]\!]_M)))) \quad (4.58)$$

$$= \text{cur}([\![B \leq_{:\Phi'} B']_M \circ \text{app}] \circ \text{cur}(\text{app} \circ (\text{Id}_B \times [\![A' \leq_{:\Phi'} A]\!]_M))) \quad (4.59)$$

$$= [\![A \rightarrow B] \leq_{:\Phi'} (A' \rightarrow B')]\!]_M \quad (4.60)$$

4.4 Type Environments

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma \text{Ok}]\!]_M = \omega^*[\![\Phi \vdash \Gamma \text{Ok}]\!]_M = [\![\Phi \vdash \Gamma \text{Ok}]\!]_M \circ \omega : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma \text{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$\omega^*[\![\Phi \vdash \diamond \text{Ok}]\!]_M = \langle \rangle_I \circ \omega \quad (4.61)$$

$$= \langle \rangle_{I'} \quad (4.62)$$

$$= [\![\Phi' \vdash \diamond \text{Ok}]\!]_M \quad (4.63)$$

$$(4.64)$$

Case Var:

$$\omega^* \llbracket \Phi \vdash \Gamma, x : A \mathbf{Ok} \rrbracket_M = \omega^*(\Box(\llbracket \Phi \vdash \Gamma \mathbf{Ok} \rrbracket_M, \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M)) \quad (4.65)$$

$$= \Box(\llbracket \Phi \vdash \Gamma \mathbf{Ok} \rrbracket_M, \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M) \circ \omega \quad (4.66)$$

$$= \Box(\llbracket \Phi \vdash \Gamma \mathbf{Ok} \rrbracket_M \circ \omega, \llbracket \Phi \vdash A : \mathbf{Type} \rrbracket_M \circ \omega) \quad (4.67)$$

$$= \Box(\llbracket \Phi' \vdash \Gamma \mathbf{Ok} \rrbracket_M, \llbracket \Phi' \vdash A : \mathbf{Type} \rrbracket_M) \quad (4.68)$$

$$= \llbracket \Phi' \vdash (\Gamma, x : A) \mathbf{Ok} \rrbracket_M \quad (4.69)$$

$$(4.70)$$

4.5 Terms

4.6 Terms

If

$$\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M \quad (4.71)$$

$$\Delta = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (4.72)$$

$$\Delta' = \llbracket \Phi' \mid \Gamma \vdash v : A \rrbracket_M \quad (4.73)$$

$$(4.74)$$

Then

$$\Delta' = \omega^*(\Delta) \quad (4.75)$$

Proof: By induction over the derivation of Δ . Using the S-Closure of ω^* . We use Γ_I to indicate $\llbracket \Phi \vdash \Gamma \mathbf{Ok} \rrbracket_M$, an A_I to indicate $\llbracket \Phi \vdash A : \mathbf{Effect} \rrbracket_M$

Case Unit:

$$\Delta = \langle \rangle_{\Gamma_I} \quad (4.76)$$

So

$$\omega^*(\Delta) = \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.77)$$

Case True, False: Giving the case for true as false is the same but using **inr**

$$\Delta = \mathbf{inl} \circ \langle \rangle_{\Gamma_I} \quad (4.78)$$

So

$$\omega^*(\Delta) = \mathbf{inl} \circ \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.79)$$

Since ω^* is S-closed.

Case Constant:

$$\Delta = \llbracket \mathbf{c}^A \rrbracket_M \circ \langle \rangle_{\Gamma_I} \quad (4.80)$$

So

$$\omega^*(\Delta) = \omega^* \llbracket \mathbf{c}^A \rrbracket_M \circ \langle \rangle_{\Gamma_{I'}} = \llbracket \mathbf{c}^{A_{I'}} \rrbracket_M \circ \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.81)$$

Since ω^* is S-closed.

Case Subtype: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (4.82)$$

Then

$$\Delta = \llbracket A \leq_{:\Phi} B \rrbracket_M \circ \Delta_1 \quad (4.83)$$

So

$$\omega^*(\Delta) = \omega^* \llbracket A \leq_{:\Phi} B \rrbracket_M \circ \omega^* \Delta_1 \quad (4.84)$$

$$= \llbracket A_{I'} \leq_{:\Phi'} B_{I'} \rrbracket_M \circ \Delta'_1 \quad \text{By induction} \quad (4.85)$$

$$= D' \quad (4.86)$$

Case Lambda: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma, x : A \vdash v : B \rrbracket_M \quad (4.87)$$

Then

$$\Delta = \text{cur}((\Delta_1)) \quad (4.88)$$

So

$$\omega^*(\Delta) = \omega^*(\text{cur}(\Delta_1)) \quad (4.89)$$

$$= \text{cur}(\omega^*(\Delta_1)) \quad \text{By S-closure} \quad (4.90)$$

$$= \text{cur}(\Delta'_1) \quad \text{By induction} \quad (4.91)$$

$$= \Delta' \quad (4.92)$$

Case Application: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rightarrow B \rrbracket_M \quad (4.93)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket_M \quad (4.94)$$

Then

$$\Delta = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad (4.95)$$

So

$$\omega^* \Delta = \omega^*(\text{app} \circ \langle \Delta_1, \Delta_2 \rangle) \quad (4.96)$$

$$= \text{app} \circ \langle \omega^*(\Delta_1), \omega^*(\Delta_2) \rangle \quad \text{By S-closure} \quad (4.97)$$

$$= \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Induction} \quad (4.98)$$

$$= \Delta' \quad (4.99)$$

Case Return: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \quad (4.100)$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \quad (4.101)$$

So

$$\omega^*(\Delta) = \omega^*(\eta_{A_I} \circ \Delta_1) \quad (4.102)$$

$$= \eta_{A_{I'}} \circ \omega^*(\Delta_1) \quad \text{By S-closure} \quad (4.103)$$

$$= \eta_{A_{I'}} \circ \Delta'_1 \quad (4.104)$$

$$= \Delta' \quad (4.105)$$

Case Bind: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M \quad (4.106)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M \quad (4.107)$$

Then

$$\Delta = \mathbf{M}_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_{\Gamma_I}, \Delta_1 \rangle \quad (4.108)$$

So

$$\omega^*(\Delta) = \omega^*(\mu_{\epsilon_1, \epsilon_2, A} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_{\Gamma_I}, \Delta_1 \rangle) \quad (4.109)$$

$$= \omega^*(\mu_{\epsilon_1, \epsilon_2, A}) \circ \omega^*(T_{\epsilon_1} \Delta_2) \circ \omega^*(\mathbf{t}_{\epsilon_1, \Gamma, A}) \circ \langle \omega^*(\mathbf{Id}_{\Gamma_I}), \omega^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (4.110)$$

$$= \mu_{\epsilon_1, \epsilon_2, A_{I'}} \circ T_{\epsilon_1} \omega^*(\Delta_2) \circ \mathbf{t}_{\epsilon_1, \Gamma_{I'}, A_{I'}} \circ \langle \omega^*(\mathbf{Id}_{\Gamma_I}), \omega^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (4.111)$$

$$= \mu_{\epsilon_1, \epsilon_2, A_{I'}} \circ T_{\epsilon_1} \Delta'_2 \circ \mathbf{t}_{\epsilon_1, \Gamma_{I'}, A_{I'}} \circ \langle \omega^*(\mathbf{Id}_{\Gamma_I}), \Delta'_1 \rangle \quad \text{By Induction} \quad (4.112)$$

$$= \Delta' \quad (4.113)$$

$$(4.114)$$

Case If: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \mathbf{Bool} \rrbracket_M \quad (4.115)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket_M \quad (4.116)$$

$$\Delta_3 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket_M \quad (4.117)$$

$$(4.118)$$

Then

$$\Delta = \mathbf{app} \circ (([\mathbf{cur}(\Delta_2 \circ \pi_2), \mathbf{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathbf{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad (4.119)$$

So

$$\omega^*(\Delta) = \omega^*(\text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_\Gamma) \circ \delta_\Gamma) \quad (4.120)$$

$$= \text{app} \circ (([\text{cur}(\omega^*(\Delta_2) \circ \pi_2), \text{cur}(\omega^*(\Delta_3) \circ \pi_2)] \circ \omega^*(\Delta_1)) \times \text{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By S-Closure} \quad (4.121)$$

$$= \text{app} \circ (([\text{cur}(\Delta'_2 \circ \pi_2), \text{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By Induction} \quad (4.122)$$

$$= \Delta' \quad (4.123)$$

$$(4.124)$$

Case Effect-Lambda: Let

$$\Delta_1 = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket_M \quad (4.125)$$

Then

$$\Delta = \widehat{\Delta_1} \quad (4.126)$$

And also

$$\omega \times \text{Id} = \llbracket \omega \times : (\Phi', \alpha) \triangleright (\Phi, \alpha) \rrbracket_M \quad (4.127)$$

So

$$\omega^* \Delta = \omega^*(\widehat{\Delta_1}) \quad (4.128)$$

$$= \overline{(\omega \times \text{Id}_U)^* \Delta_1} \quad \text{By naturality} \quad (4.129)$$

$$= \widehat{\Delta'_1} \quad \text{By induction} \quad (4.130)$$

$$= \Delta' \quad (4.131)$$

Case Effect-Application: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \forall \alpha. A \rrbracket_M \quad (4.132)$$

$$h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \quad (4.133)$$

$$(4.134)$$

Then

$$\Delta = \langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1 \quad (4.135)$$

So due to the substitution theorem on effects

$$h \circ \omega = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket_M \circ \omega = \llbracket \Phi' \vdash \epsilon : \text{Effect} \rrbracket_M = h' \quad (4.136)$$

Also note $(\omega \times \text{Id}_U) = \llbracket \omega \times : \Phi', \alpha \triangleright \Phi \rrbracket_M$

$$\omega^* \Delta = \omega^*(\langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1) \quad (4.137)$$

$$= (\langle \text{Id}_\Gamma, h \rangle \circ \omega)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \omega^*(\Delta_1) \quad (4.138)$$

$$= ((\omega \times \text{Id}_U) \circ \langle \text{Id}_\Gamma, h \circ \omega \rangle)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1' \quad (4.139)$$

$$= (\langle \text{Id}_\Gamma, h' \rangle)^* ((\omega \times \text{Id}_U)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket_M}) \circ \Delta_1)' \quad (4.140)$$

$$(4.141)$$

Looking at the inner part of the functor application: Let

$$A = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \mathbf{Type} \rrbracket_M \quad (4.142)$$

$$(4.143)$$

$$(\omega \times \mathbf{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \mathbf{Type} \rrbracket_M} = (\omega \times \mathbf{Id}_U)^* \epsilon_A \quad (4.144)$$

$$= (\omega \times \mathbf{Id}_U)^* (\widehat{\mathbf{Id}_{\forall_I(A)}}) \quad (4.145)$$

$$= \overline{(\omega \times \mathbf{Id}_U)^* (\widehat{\mathbf{Id}_{\forall_I(A)}})} \quad \text{By bijection} \quad (4.146)$$

$$= \overline{\omega^* (\widehat{\mathbf{Id}_{\forall_I(A)}})} \quad \text{By naturality} \quad (4.147)$$

$$= \overline{\omega^* (\mathbf{Id}_{\forall_I(A)})} \quad \text{By bijection} \quad (4.148)$$

$$= \overline{\mathbf{Id}_{\forall_{I'}(A \circ (\omega \times \mathbf{Id}_U))}} \quad \text{By S-Closure, naturality} \quad (4.149)$$

$$= \overline{\mathbf{Id}_{\forall_{I'}(A)}} \quad \text{By Substitution theorem} \quad (4.150)$$

$$= \epsilon_{A_{I'}} \quad (4.151)$$

Going back to the original expression:

$$\omega^* \Delta = (\langle \mathbf{Id}_\Gamma, h' \rangle)^* (\epsilon_{A_{I'}} \circ \Delta_1)' \quad (4.152)$$

$$= \Delta' \quad (4.153)$$

$$(4.154)$$

4.7 Term-Substitution

If $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M$, then $\llbracket \Phi' \mid \Gamma' \vdash \sigma : \Gamma \rrbracket_M = \omega^* \llbracket \Phi \mid \Gamma' \vdash \sigma : \Gamma \rrbracket_M$.

Proof: By induction on the structure of σ , making use of the weakening of term denotations above.

Case Nil: Then $\sigma = \langle \rangle_{\Gamma'}$, so $\omega^*(\sigma) = \langle \rangle_{\Gamma'_{I'}} = \llbracket \Phi' \mid \Gamma' \vdash \sigma : \Gamma \rrbracket_M$

Case Var: Then $\sigma = (\sigma', x := v)$

$$\omega^* \sigma = \omega * \langle \sigma', \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \rangle \quad (4.155)$$

$$= \langle \omega^* \sigma', \omega^* \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket_M \rangle \quad (4.156)$$

$$= \langle \llbracket \Phi' \mid \Gamma' \vdash \sigma' : \Gamma \rrbracket_M, \llbracket \Gamma' \mid \Phi' \vdash v : A \rrbracket_M \rangle \quad (4.157)$$

$$= \llbracket \Phi' \mid \Gamma' \vdash \sigma : \Gamma, x : A \rrbracket_M \quad (4.158)$$

4.8 Term-Weakening

If $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket_M$, then $\llbracket \Phi' \vdash \omega_1 : \Gamma' \triangleright \Gamma \rrbracket_M = \omega^* \llbracket \Phi \vdash \omega_1 : \Gamma' \triangleright \Gamma \rrbracket_M$.

Proof: By induction on the structure of ω_1 .

Case Id: Then $\omega_1 = \iota$, so its denotation is $\omega_1 = \text{Id}_{\Gamma_I}$

So

$$\omega^*(\text{Id}_{\Gamma_I}) = \text{Id}_{\Gamma_{I'}} = \llbracket \Phi' \vdash \iota : \Gamma \triangleright \Gamma \rrbracket_M \quad (4.159)$$

Case Project: Then $\omega_1 = \omega'_1 \pi$

$$(\text{Project}) \frac{\Phi \vdash \omega'_1 : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1 \pi : \Gamma', x : A \triangleright \Gamma} \quad (4.160)$$

So $\omega_1 = \omega'_1 \circ \pi_1$

Hence

$$\omega^*(\omega_1) = \omega^*(\omega'_1) \circ \omega^*(\pi_1) \quad (4.161)$$

$$= \llbracket \Phi' \vdash \omega'_1 : \Gamma' \triangleright \Gamma \rrbracket_M \circ \pi_1 \quad (4.162)$$

$$= \llbracket \Phi' \vdash \omega'_1 \pi : \Gamma', x : A \triangleright \Gamma \rrbracket_M \quad (4.163)$$

$$= \llbracket \Phi' \vdash \omega_1 : \Gamma', x : A \triangleright \Gamma \rrbracket_M \quad (4.164)$$

Case Extend: Then $\omega_1 = \omega'_1 \times$

$$(\text{Extend}) \frac{\Phi \vdash \omega'_1 : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1 \times : \Gamma', x : A \triangleright \Gamma, x : A} \quad (4.165)$$

So $\omega_1 = \omega'_1 \times \text{Id}_{A_I}$

Hence

$$\omega^*(\omega_1) = (\omega^*(\omega'_1) \times \omega^*(\text{Id}_{A_I})) \quad (4.166)$$

$$= (\llbracket \Phi' \vdash \omega'_1 : \Gamma' \triangleright \Gamma \rrbracket_M \times \text{Id}_{A_I}) \quad (4.167)$$

$$= \llbracket \Phi' : \omega_1 \triangleright \Gamma', x : A \triangleright \Gamma, x : A \rrbracket_M \quad (4.168)$$

Chapter 5

Value Substitution Theorem

Chapter 6

Type-Environment Weakening Theorem

Chapter 7

Unique Denotation Theorem

Chapter 8

Beta-Eta-Equivalence Theorem (Soundness)