0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule. **TODO: No-lambda?**

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t\colon \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t\colon \tau$,there exists at most one reduced derivation of $\Gamma \vdash t\colon \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Variables

To find the unique derivation of $\Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is, if $A' \leq: A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\frac{\Gamma', x:A' \,\mathtt{Ok}}{\Gamma, x:A' \vdash x:A'} \qquad A' \leq: A}{\Gamma', x : A' \vdash x\colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Gamma \vdash x\colon A$ holds, then so must $\Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma' \vdash x:A'} \qquad A' \leq: A}{\Gamma' \vdash x\colon A} \tag{2}$$

Be the unique reduced derivation of $\Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\frac{()\frac{\Delta}{\Gamma, x:A' \vdash x:A'}}{\Gamma \vdash x:A'} \qquad A' \leq: A}{\Gamma \vdash x\colon A} \tag{3}$$

0.2.2 Constants

For each of the constants, ($\mathtt{C}^A$, $\mathtt{true}$, $\mathtt{false}$, $\mathtt{()}$), there is exactly one possible derivation for $\Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A} \qquad A \leq: B}{\Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

0.2.3 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Gamma, x:A \vdash C:\mathtt{M}_{\epsilon} B}}{\Gamma \vdash \lambda x:A.B : A \to \mathtt{M}_{\epsilon} B} \qquad A \to \mathtt{M}_{\epsilon} B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'}$$

Where

$$\text{(Sub-Effect)}\dfrac{()\frac{\Delta}{\Gamma, x{:}A \vdash C{:}\mathtt{M}_\epsilon B} \qquad B \leq: B' \qquad \epsilon \leq \epsilon'}{\Gamma, x : A \vdash C{:}\,\mathtt{M}_{\epsilon'} B'} \tag{4}$$

is the reduced derivation of $\Gamma, x : A \vdash C{:}\,\mathtt{M}_\epsilon B$ if it exists.

Case Subtype **TODO: Do we need to write anything here? (Probably needs an explanation)**

0.2.4 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return} v{:}\,\mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\dfrac{\text{(Return)}\dfrac{()\frac{\Delta}{\Gamma\vdash v:A}}{\Gamma\vdash \mathtt{return} v{:}\mathtt{M}_{\mathtt{1}} A} \qquad A \leq: B \qquad \mathtt{1} \leq \epsilon}{\Gamma \vdash \mathtt{return} v{:}\,\mathtt{M}_\epsilon B}$$

Where

$$\text{(Subtype)}\dfrac{()\frac{\Delta}{\Gamma\vdash v:A} \qquad A \leq: B}{\Gamma \vdash v{:}\,B}$$

is the reduced derivation of $\Gamma \vdash v{:}\,B$

Case Apply If

$$\text{(Subtype)}\dfrac{()\frac{\Delta}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B} \qquad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1{:}\,A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)}\dfrac{()\frac{\Delta'}{\Gamma\vdash v_2:A''} \qquad A'' \leq: A'}{\Gamma \vdash v_2{:}\,A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1{:}\,A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2{:}\,A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1 \quad v_2{:}\,\mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subeffect)}\dfrac{\text{(Apply)}\dfrac{()\frac{\Delta}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B} \qquad \text{(Subtype)}\dfrac{()\frac{\Delta'}{\Gamma\vdash v:A''} \qquad A''\leq:A}{\Gamma\vdash v:A}}{\Gamma\vdash v_1 \quad v_2{:}\mathtt{M}_\epsilon B} \qquad B \leq: B' \qquad \epsilon \leq \epsilon'}{\Gamma \vdash v_1 \quad v_2{:}\,\mathtt{M}_{\epsilon'} B'}$$

Case If Let

$$\text{(Subtype)}\dfrac{()\frac{\Delta}{\Gamma\vdash v:B} \qquad B \leq: \mathtt{Bool}}{\Gamma \vdash v{:}\,\mathtt{Bool}} \tag{5}$$

$$\text{(Subeffect)}\dfrac{()\frac{\Delta'}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon'} A'} \qquad A' \leq: A \qquad \epsilon' \leq \epsilon}{\Gamma \vdash C_1{:}\,\mathtt{M}_\epsilon A} \tag{6}$$

$$\text{(Subeffect)}\dfrac{()\frac{\Delta''}{\Gamma\vdash C_2:\mathtt{M}_{\epsilon''} A''} \qquad A'' \leq: A \qquad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2{:}\,\mathtt{M}_\epsilon A} \tag{7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v{:}\,\mathtt{Bool}$, $\Gamma \vdash C_1{:}\,\mathtt{M}_\epsilon A$, $\Gamma \vdash C_2{:}\,\mathtt{M}_\epsilon A$.
Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2{:}\,\mathtt{M}_\epsilon A$ is:
TODO: Scale this properly

$$\text{(Subtype)}\dfrac{\text{(If)}\dfrac{\text{(Subtype)}\dfrac{()\frac{\Delta}{\Gamma\vdash v:B} \qquad B\leq:\mathtt{Bool}}{\Gamma\vdash v:\mathtt{Bool}} \qquad \text{(Subeffect)}\dfrac{()\frac{\Delta'}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon'} A'} \qquad A'\leq:A \qquad \epsilon'\leq\epsilon}{\Gamma\vdash C_1:\mathtt{M}_\epsilon A} \qquad \text{(Subeffect)}\dfrac{()\frac{\Delta'}{\Gamma\vdash C_2:\mathtt{M}}}{}}{\Gamma\vdash \mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2:\mathtt{M}_\epsilon A} \qquad \epsilon\leq\epsilon \qquad A\leq:A}{\Gamma \vdash \mathtt{if}_{\epsilon,A} \quad v \quad \mathtt{then} \quad C_1 \quad \mathtt{else} \quad C_2{:}\,\mathtt{M}_\epsilon A} \tag{8}$$

Case Bind Let

$$(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1 : \mathtt{M}_{\epsilon_1} A} \qquad A \leq: A' \qquad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma, x : A\vdash C_2 : \mathtt{M}_{\epsilon_2} B} \qquad B \leq: B' \qquad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the subterms]

By weakening, $\iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2 : \mathtt{M}_\epsilon B$, there's also one of $\Gamma, x : A \vdash C_2 : \mathtt{M}_\epsilon B$.

Since the monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Makle this and the other smaller

$$(\text{Subeffect})\frac{(\text{Bind})\frac{(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1 : \mathtt{M}_{\epsilon_1} A} \qquad A \leq: A' \qquad \epsilon_1 \leq \epsilon_1'}{\Gamma\vdash C_1 : \mathtt{M}_{\epsilon_1'} A'} \qquad (\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma, x : A\vdash C_2 : \mathtt{M}_{\epsilon_2} B} \qquad B \leq: B' \qquad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A\vdash C_2 : \mathtt{M}_{\epsilon_2'} B'}}{\Gamma\vdash \mathtt{do} \quad x\leftarrow C_1 \quad \mathtt{in} \quad C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}}{\Gamma \vdash \mathtt{do} \quad x \leftarrow C_1 \quad \mathtt{in} \quad C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{11}$$

Case Subeffect **TODO: Do I want to talk about this?**

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed. **TODO: Fill in these cases with actual maths**

0.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$\mathit{reduce}((\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash \mathtt{C}^A : A}) = (\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash \mathtt{C}^A : A}$

0.3.2 Value Types

Var

$$\mathit{reduce}((\text{Var})\frac{\Gamma\mathtt{Ok}}{\Gamma, x : A \vdash x : A}) = (\text{Var})\frac{\Gamma\mathtt{Ok}}{\Gamma, x : A \vdash x : A} \tag{12}$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$\mathit{reduce}((\text{Weaken})\frac{()\frac{\Delta}{\Gamma\vdash x : A}}{\Gamma, y : B \vdash x : A}) \tag{13}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash x : A} \qquad A' \leq: A}{\Gamma \vdash x : A} = \mathit{reduce}(\Delta) \tag{14}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma \vdash x: A'}}{\Gamma, y: B \vdash x: A'} \qquad A' \leq: A}{\Gamma, y : B \vdash x \colon A} \tag{15}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$[\![(\text{Weaken})\frac{()\frac{\Delta}{\Gamma \vdash x: A}}{\Gamma, y : B \vdash x \colon A}]\!]_M = \Delta \circ \pi_1 \tag{16}$$

Similarly, the denotation of the reduced denotation is:

$$[\![(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma \vdash x: A'}}{\Gamma, y: B \vdash x: A'} \qquad A' \leq: A}{\Gamma, y : B \vdash x \colon A}]\!]_M = [\![A' \leq: A]\!]_M \circ \Delta' \circ \pi_1 \tag{17}$$

By induction on *reduce* preserving denotations and the reduction of Δ (14), we have:

$$\Delta = [\![A' \leq: A]\!]_M \circ \Delta' \tag{18}$$

So the denotations of the unreduced and reduced derivations are equal.

Lambda

reduce **definition**

Preserves Denotation **TODO: Recursively call reduce on C then push subtyping through using currying**

Subtype

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Gamma \vdash v: A} \qquad A \leq: B}{\Gamma \vdash v \colon B}) \tag{19}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash x: A} \qquad A' \leq: A}{\Gamma \vdash x \colon A} = reduce(\Delta) \tag{20}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash v: A'} \qquad A' \leq: A \leq: B}{\Gamma \vdash v \colon B} \tag{21}$$

Preserves Denotation

$$\begin{aligned} before &= [\![A \leq: B]\!]_M \circ \Delta & (22)\\ &= [\![A \leq: B]\!]_M \circ ([\![A' \leq: A]\!]_M \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. & (23)\\ &= [\![A' \leq: B]\!]_M \circ \Delta' \quad \text{Subtyping relations are unique} & (24)\\ &= after & (25)\\ & & (26) \end{aligned}$$

0.3.3 Computation Types

Return

reduce **definition**

Preserves Denotation TODO: Recursively call reduce then use naturality to push subtyping into subeffect

Apply

reduce **definition**

Preserves Denotation TODO: Recursively call reduce, then construct the reduced apply as in the proof of uniqueness

If

reduce **definition**

Preserves Denotation TODO: Recursively call reduce, then leave tree otherwise unchanged.

Bind

reduce **definition**

Preserves Denotation TODO: Recursively call reduce then push subtyping rules through the bind

Subeffect

reduce **definition** To find:

$$reduce((\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C:\mathtt{M}_{\epsilon'}B'} \qquad \epsilon' \leq \epsilon \qquad B' \leq: B}{\Gamma \vdash C : \mathtt{M}_{\epsilon} B}) \tag{27}$$

Let

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \qquad \epsilon'' \leq \epsilon' \qquad \mathtt{Bool}'' \leq: B}{\Gamma \vdash C : \mathtt{M}_{\epsilon'} B} = reduce(\Delta) \tag{28}$$

in

$$(\text{subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \qquad \epsilon'' \leq \epsilon \qquad B'' \leq: B}{\Gamma \vdash C : \mathtt{M}_{\epsilon} B} \tag{29}$$

Preserves Denotation Let

$$
\begin{align}
f &= [\![B' \leq: B]\!]_M \tag{30}\\
g &= [\![B'' \leq: B']\!]_M \tag{31}\\
h_1 &= [\![\epsilon' \leq \epsilon]\!]_M \tag{32}\\
h_2 &= [\![\epsilon' \leq \epsilon']\!]_M \tag{33}\\
f \circ g &= [\![B'' \leq: B]\!]_M \tag{34}\\
h_1 \circ h_2 &= [\![\epsilon'' \leq \epsilon']\!]_M \tag{35}\\
& \tag{36}
\end{align}
$$

Hence we can find the denotation of the derivation before reduction.

$$
\begin{align}
before &= h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} \tag{37}\\
&= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta \tag{38}\\
&= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \qquad = after \quad \text{By definition.} \tag{39}
\end{align}
$$

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t\colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\Delta]\!]_M = [\![reduce\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t\colon \tau]\!]_M$ is unique.