

0.1 CCC

The section should be a cartesian closed category. That is it should have:

- A Terminal object 1
- Binary products
- Exponentials

Further more, it should have a co-product of the terminal object 1 . This is required for the beta-eta equivalence of **if-then-else** terms.

$$1 \xrightarrow{inl} A \xleftarrow{inr} 1$$

For each:

$$1 \xrightarrow{f} A \xleftarrow{g} 1$$

There exists unique $[f, g] : 1 + 1 \rightarrow A$ such that:

$$\begin{array}{ccccc} & & A & & \\ & f \nearrow & \uparrow [f,g] & \nwarrow g & \\ 1 & \xrightarrow{inl} & 1 + 1 & \xleftarrow{inr} & 1 \end{array}$$

0.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endo-functor indexed by the po-monoid on effects: $T : (\mathbb{E}, \cdot, 1, \leq) \rightarrow \mathbf{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathbf{Id} \rightarrow T_1$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2, \cdot} : T_{\epsilon_1} T_{\epsilon_2} \rightarrow T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

0.2.1 Left Unit

$$\begin{array}{ccc} T_\epsilon A & \xrightarrow{T_\epsilon \eta_A} & T_\epsilon T_1 A \\ & \searrow \text{Id}_{T_\epsilon A} & \downarrow \mu_{\epsilon, 1, A} \\ & & T_\epsilon A \end{array}$$

0.2.2 Right Unit

$$\begin{array}{ccc} T_\epsilon A & \xrightarrow{\eta_{T_\epsilon A}} & T_1 T_\epsilon A \\ & \searrow \text{Id}_{T_\epsilon A} & \downarrow \mu_{1, \epsilon, A} \\ & & T_\epsilon A \end{array}$$

0.2.3 Associativity

$$\begin{array}{ccc}
T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2, T_{\epsilon_3} A}} & T_{\epsilon_1 \cdot \epsilon_2} T_{\epsilon_3} A \\
\downarrow T_{\epsilon_1} \mu_{\epsilon_2, \epsilon_3, A} & & \downarrow \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, A} \\
T_{\epsilon_1} T_{\epsilon_2 \cdot \epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, A}} & T_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} A
\end{array}$$

0.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathbf{t}_{\epsilon, A, B} : A \times T_{\epsilon} B \rightarrow T_{\epsilon}(A \times B)$$

Satisfying the following rules:

0.3.1 Left Naturality

$$\begin{array}{ccc}
A \times T_{\epsilon} B & \xrightarrow{\text{Id}_A \times T_{\epsilon} f} & A \times T_{\epsilon} B' \\
\downarrow \mathbf{t}_{\epsilon, A, B} & & \downarrow \mathbf{t}_{\epsilon, A, B'} \\
T_{\epsilon}(A \times B) & \xrightarrow{T_{\epsilon}(\text{Id}_A \times f)} & T_{\epsilon}(A \times B')
\end{array}$$

0.3.2 Right Naturality

$$\begin{array}{ccc}
A \times T_{\epsilon} B & \xrightarrow{f \times \text{Id}_{T_{\epsilon} B}} & A' \times T_{\epsilon} B \\
\downarrow \mathbf{t}_{\epsilon, A, B} & & \downarrow \mathbf{t}_{\epsilon, A', B} \\
T_{\epsilon}(A \times B) & \xrightarrow{T_{\epsilon}(f \times \text{Id}_B)} & T_{\epsilon}(A' \times B)
\end{array}$$

0.3.3 Unitor Law

$$\begin{array}{ccc}
1 \times T_{\epsilon} A & \xrightarrow{\mathbf{t}_{\epsilon, 1, A}} & T_{\epsilon}(1 \times A) \\
& \searrow \lambda_{T_{\epsilon} A} & \downarrow T_{\epsilon}(\lambda_A) \\
& & T_{\epsilon} A
\end{array}
\quad \text{Where } \lambda : 1 \times \text{Id} \rightarrow \text{Id} \text{ is the left-unitor. } (\lambda = \pi_2)$$

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutativity of π_2 with $\mathbf{t}_{\epsilon, \cdot, \cdot}$,

$$\pi_{2, A, B} = \pi_{2, 1, B} \circ (\langle \rangle_A \times \text{Id}_B)$$

And $\pi_{2, 1}$ is the left unitor, so by tensorial strength:

$$\begin{aligned}
T_{\epsilon} \pi_2 \circ \mathbf{t}_{\epsilon, A, B} &= T_{\epsilon} \pi_{2, 1, B} \circ T_{\epsilon}(\langle \rangle_A \times \text{Id}_B) \circ \mathbf{t}_{\epsilon, A, B} \\
&= T_{\epsilon} \pi_{2, 1, B} \circ \mathbf{t}_{\epsilon, 1, B} \circ (\langle \rangle_A \times \text{Id}_B) \\
&= \pi_{2, 1, B} \circ (\langle \rangle_A \times \text{Id}_B) \\
&= \pi_2
\end{aligned} \tag{1}$$

So the following commutes:

$$\begin{array}{ccc}
 A \times T_\epsilon B & \xrightarrow{\mathfrak{t}_{\epsilon, A, B}} & T_\epsilon(A \times B) \\
 & \searrow \pi_2 & \downarrow T_\epsilon \pi_2 \\
 & & T_\epsilon B
 \end{array}$$

0.3.4 Commutativity with Join

$$\begin{array}{ccc}
 A \times T_{\epsilon_1} T_{\epsilon_2} B & \xrightarrow{\mathfrak{t}_{\epsilon_1, A, T_{\epsilon_2} B}} & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \xrightarrow{T_{\epsilon_1} \mathfrak{t}_{\epsilon_2, A, B}} T_{\epsilon_1} T_{\epsilon_2}(A \times B) \\
 & \searrow \text{Id}_A \times \mu_{\epsilon_1, \epsilon_2, B} & \downarrow \mu_{\epsilon_1, \epsilon_2, A \times B} \\
 & & A \times T_{\epsilon_1 \cdot \epsilon_2} B \xrightarrow{\mathfrak{t}_{\epsilon_1 \cdot \epsilon_2, A, B}} T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
 \end{array}$$

0.4 Commutativity with Unit

$$\begin{array}{ccc}
 A \times B & \xrightarrow{\text{Id}_A \times \eta_B} & A \times T_1 B \\
 & \searrow \eta_{A \times B} & \downarrow \mathfrak{t}_{1, A, B} \\
 & & T_1(A \times B)
 \end{array}$$

0.5 Commutativity with α

Let $\alpha_{A, B, C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \rightarrow (A \times (B \times C))$

$$\begin{array}{ccc}
 (A \times B) \times T_\epsilon C & \xrightarrow{\mathfrak{t}_{\epsilon, (A \times B), C}} & T_\epsilon((A \times B) \times C) \\
 \downarrow \alpha_{A, B, T_\epsilon C} & & \downarrow T_\epsilon \alpha_{A, B, C} \\
 A \times (B \times T_\epsilon C) & \xrightarrow{\text{Id}_A \times \mathfrak{t}_{\epsilon, B, C}} A \times T_\epsilon(B \times C) \xrightarrow{\mathfrak{t}_{\epsilon, A, (B \times C)}} & T_\epsilon(A \times (B \times C))
 \end{array}$$

TODO: Needed?

0.6 Sub-Effecting

For each instance of the pre-order (\mathbb{E}, \leq) , $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $\llbracket \epsilon_1 \leq \epsilon_2 \rrbracket : T_{\epsilon_1} \rightarrow T_{\epsilon_2}$ that commutes with $\mathfrak{t}_{\epsilon, \cdot}$:

0.6.1 Sub-Effecting and Tensor Strength

$$\begin{array}{ccc}
 A \times T_{\epsilon_1} B & \xrightarrow{\text{Id}_A \times \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_B} & A \times T_{\epsilon_2} B \\
 \downarrow \mathfrak{t}_{\epsilon_1, A, B} & & \downarrow \mathfrak{t}_{\epsilon_2, A, B} \\
 T_{\epsilon_1}(A \times B) & \xrightarrow{\llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_{A \times B}} & T_{\epsilon_2}(A \times B)
 \end{array}$$

0.6.2 Sub-effecting and Monadic Join

Since the monoid operation on effects is monotone, we can introduce the following diagram.

$$\begin{array}{ccccc}
T_{\epsilon_1} T_{\epsilon_2} & \xrightarrow{T_{\epsilon_1} \llbracket \epsilon_2 \leq \epsilon'_2 \rrbracket_M} & T_{\epsilon_1} T_{\epsilon'_2} & \xrightarrow{\llbracket \epsilon_1 \leq \epsilon'_1 \rrbracket_{M, T_{\epsilon'_2}}} & T_{\epsilon'_1} T_{\epsilon'_2} \\
\downarrow \mu_{\epsilon_1, \epsilon_2,} & & & & \downarrow \mu_{\epsilon'_1, \epsilon'_2,} \\
T_{\epsilon_1 \cdot \epsilon_2} & \xrightarrow{\llbracket \epsilon_1 \cdot \epsilon_2 \leq \epsilon'_1 \epsilon'_2 \rrbracket_M} & & & T_{\epsilon'_1 \cdot \epsilon'_2}
\end{array}$$

0.7 Sub-typing

The denotation of ground types $\llbracket - \rrbracket_M$ is a functor from the pre-order category of ground types (γ, \leq_γ) to \mathbb{C} . This pre-ordered sub-category of \mathbb{C} is extended with the rule for function sub-typing to form a larger pre-ordered sub-category of \mathbb{C} .

$$\text{(Function Subtyping)} \frac{f = \llbracket A' \leq : A \rrbracket_M \quad g = \llbracket B \leq : B' \rrbracket_M \quad h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket}{rhs = \llbracket A \rightarrow M_{\epsilon_1} B \leq : A' \rightarrow M_{\epsilon_2} B' \rrbracket_M : (T_{\epsilon_1} B)^A \rightarrow (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned}
rhs &= (h_{B'} \circ T_{\epsilon_1} g)^{A'} \circ (T_{\epsilon_1} B)^f \\
&= \text{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_{T_{\epsilon_1} B^{A'}} \times f))
\end{aligned} \tag{2}$$