0.1 Terms

0.1.1 Value Terms

$$
\begin{aligned}
v ::= & x \\
& \mid \lambda x : A.C \\
& \mid \mathtt{C}^A \\
& \mid \mathtt{()} \\
& \mid \mathtt{true} \mid \mathtt{false} \\
& \mid \Lambda \alpha . v \\
& \mid v\ \epsilon
\end{aligned} \tag{1}
$$

0.1.2 Computation Terms

$$
\begin{aligned}
C ::= & \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \\
& \mid v_1\ v_2 \\
& \mid \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \\
& \mid \mathtt{return} v
\end{aligned} \tag{2}
$$

0.2 Type System

0.2.1 Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, \mathtt{1}, \leq)$ with ground elements e, and denoted by metavariables ϵ, and in language-effect variables α

0.2.2 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Value Types

$$A, B, C ::= \gamma \mid A \rightarrow \mathtt{M}_\epsilon B \mid \forall \alpha . A$$

Computation Types Computation types are of the form $\mathtt{M}_\epsilon A$

0.2.3 Sub-typing

There exists a sub-typing pre-order relation $\leq :_\gamma$ over ground types that is:

- (Reflexive)$\frac{}{A \leq :_\gamma A}$
- (Transitive)$\frac{A \leq :_\gamma B \quad B \leq :_\gamma C}{A \leq :_\gamma C}$

We extend this relation with the function and effect-lambda sub-typing rules to yield the full sub-typing relation $\leq :$

- (ground)$\frac{A \leq :_\gamma B}{A \leq : B}$
- (Fn)$\frac{A \leq : A' \quad B' \leq : B \quad \epsilon \leq \epsilon'}{A' \rightarrow \mathtt{M}_{\epsilon'} B' \leq : A \rightarrow \mathtt{M}_\epsilon B}$
- (All)$\frac{A \leq : A'}{\forall \alpha . A \leq : \forall a . A'}$

0.2.4 Type and Effect Environments

A type environment is a snoc-list of value-variable, type pairs, $G ::= \diamond \mid \Gamma, x : A$. An effect environment is a snoc-list of effect-variables.
$\Phi ::= \diamond \mid \Phi, \alpha$

Domain Function on Type Environments

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

Membership of Effect Environments Informally, $\alpha \in \Phi$ if α appears in the list represented by Φ.

$\mathtt{Ok}$ Predicate On Effect Environments

- (Atom) $\frac{}{\diamond \mathtt{Ok}}$
- (A) $\frac{\Phi \mathtt{Ok} \quad \alpha \notin \Phi}{\Phi, \alpha \mathtt{Ok}}$

Well-Formed-ness of effects We define a relation $\Phi \vdash \epsilon$.

- (Ground) $\frac{\Phi \mathtt{Ok}}{\Phi \vdash e}$
- (Var) $\frac{\Phi, \alpha \mathtt{Ok}}{\Phi, \alpha \vdash \alpha}$
- (Weaken) $\frac{\Phi \vdash \alpha}{\Phi, \beta \vdash \alpha}$ (if $\alpha \neq \beta$)
- (Monoid Op) $\frac{\Phi \vdash \epsilon_1 \quad \Phi \vdash \epsilon_2}{\Phi \vdash \epsilon_1 \cdot \epsilon_2}$

Well-Formed-ness of Types We define a relation $\Phi \vdash \tau$ on types.

- (Ground) $\frac{}{\Phi \vdash \gamma}$
- (Lambda) $\frac{\Phi \vdash A \quad \Phi \vdash \mathtt{M}_\epsilon B}{\Phi \vdash A \rightarrow \mathtt{M}_\epsilon B}$
- (Computation) $\frac{\Phi \vdash A \quad \Phi \vdash \epsilon}{\Phi \vdash \mathtt{M}_\epsilon A}$
- (For-All) $\frac{\Phi, \alpha \vdash A}{\Phi \vdash \forall \alpha . A}$

Ok Predicate on Type Environments We now define a predicate on type environments and effect environments: $\Phi \vdash \Gamma \mathtt{Ok}$

- (Nil) $\frac{}{\Phi \vdash \diamond \mathtt{Ok}}$
- (Var) $\frac{\Phi \vdash \Gamma \mathtt{Ok} \quad \times \notin \mathtt{dom}(\Gamma) \quad \Phi \vdash A}{\Phi \vdash \Gamma, x : A \mathtt{Ok}}$

0.2.5 Type Rules

Value Typing Rules

- (Const) $\frac{\Phi \vdash \Gamma \texttt{Ok} \quad \Phi \vdash A}{\Phi | \Gamma \vdash \texttt{C}^A : A}$
- (Unit) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{()} : \texttt{Unit}}$
- (True) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{true} : \texttt{Bool}}$
- (False) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{false} : \texttt{Bool}}$
- (Var) $\frac{\Phi \vdash \Gamma, x : A \texttt{Ok}}{\Phi | \Gamma, x : A \vdash x : A}$
- (Weaken) $\frac{\Phi | \Gamma \vdash x : A}{\Phi | \Gamma, y : B \vdash x : A}$ (if $x \neq y$)
- (Fn) $\frac{\Phi | \Gamma, x : A \vdash C : \texttt{M}_\epsilon B}{\Phi | \Gamma \vdash \lambda x : A . C : A \to \texttt{M}_\epsilon B}$
- (Sub) $\frac{\Phi | \Gamma \vdash v : A \quad A \leq : B}{\Phi | \Gamma \vdash v : B}$
- (Effect-Abs) $\frac{\Phi, \alpha | \Gamma \vdash v : A}{\Phi | \Gamma \vdash \Lambda \alpha . v : \forall \alpha . A}$
- (Effect-apply) $\frac{\Phi | \Gamma \vdash \Phi : \Gamma v \forall \alpha . A}{\Phi | \Gamma \vdash \Lambda \alpha . v : \forall \alpha . A}$

Computation typing rules

- (Return) $\frac{\Phi | \Gamma \vdash v : A}{\Phi | \Gamma \vdash \texttt{return} v : \texttt{M}_{\mathbf{1}} A}$
- (Apply) $\frac{\Phi | \Gamma \vdash v_1 : A \to \texttt{M}_\epsilon B \quad \Phi | \Gamma \vdash v_2 : A}{\Phi | \Gamma \vdash v_1 \ v_2 : \texttt{M}_\epsilon B}$
- (if) $\frac{\Phi | \Gamma \vdash v : \texttt{Bool} \quad \Phi | \Gamma \vdash C_1 : \texttt{M}_\epsilon A \quad \Phi | \Gamma \vdash C_2 : \texttt{M}_\epsilon A}{\Phi | \Gamma \vdash \texttt{if}_{\epsilon, A} \ V \ \texttt{then} \ C_1 \ \texttt{else} \ C_2 : \texttt{M}_\epsilon A}$
- (Do) $\frac{\Phi | \Gamma \vdash C_1 : \texttt{M}_{\epsilon_1} A \quad \Phi | \Gamma, x : A \vdash C_2 : \texttt{M}_{\epsilon_2} B}{\Phi | \Gamma \vdash \texttt{do} \ x \leftarrow C_1 \ \texttt{in} \ C_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (Subeffect) $\frac{\Phi | \Gamma \vdash C : \texttt{M}_{\epsilon_1} A \quad A \leq : B \quad \epsilon_1 \leq \epsilon_2}{\Phi | \Gamma \vdash C : \texttt{M}_{\epsilon_2} B}$

0.2.6 Ok Lemma

If $\Phi \mid \Gamma \vdash t : \tau$ then $\Phi \vdash \Gamma \texttt{Ok}$.

Proof If $\Gamma, x : A \texttt{Ok}$ then by inversion $\Gamma \texttt{Ok}$ Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Phi \vdash \Gamma \texttt{Ok}$. And all non-axiom derivations preserve the `Ok` property.

0.3 Beta-Eta-Equivalence

0.3.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Phi|\Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B \quad \Phi|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash(\lambda x{:}A.C)\ v=_{\beta\eta}C[x/v]{:}\mathtt{M}_\epsilon B}$
- (Lambda-Eta) $\frac{\Phi|\Gamma\vdash v{:}A\to\mathtt{M}_\epsilon B}{\Phi|\Gamma\vdash\lambda x{:}A.(v\ x)=_{\beta\eta}v{:}A\to\mathtt{M}_\epsilon B}$
- (Left Unit) $\frac{\Phi|\Gamma\vdash v{:}A \quad \Phi|\Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow\mathtt{return}v\ \mathtt{in}\ C=_{\beta\eta}C[V/x]{:}\mathtt{M}_\epsilon B}$
- (Right Unit) $\frac{\Phi|\Gamma\vdash C{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow C\ \mathtt{in}\ \mathtt{return}x=_{\beta\eta}C{:}\mathtt{M}_\epsilon A}$
- (Associativity) $\frac{\Phi|\Gamma\vdash C_1{:}\mathtt{M}_{\epsilon_1}A \quad \Phi|\Gamma,x{:}A\vdash C_2{:}\mathtt{M}_{\epsilon_2}B \quad \Phi|\Gamma,y{:}B\vdash C_3{:}\mathtt{M}_{\epsilon_3}C}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y\leftarrow C_2\ \mathtt{in}\ C_3)=_{\beta\eta}\mathtt{do}\ y\leftarrow(\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}C}$
- (Unit) $\frac{\Phi|\Gamma\vdash v{:}\mathtt{Unit}}{\Phi|\Gamma\vdash v=_{\beta\eta}\mathtt{()}{:}\mathtt{Unit}}$
- (if-true) $\frac{\Phi|\Gamma\vdash C_1{:}\mathtt{M}_\epsilon A \quad \Phi|\Gamma\vdash C_2{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{true}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}C_1{:}\mathtt{M}_\epsilon A}$
- (if-false) $\frac{\Phi|\Gamma\vdash C_2{:}\mathtt{M}_\epsilon A \quad \Phi|\Gamma\vdash C_1{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{false}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}C_2{:}\mathtt{M}_\epsilon A}$
- (If-Eta) $\frac{\Phi|\Gamma,x{:}\mathtt{Bool}\vdash C{:}\mathtt{M}_\epsilon A \quad \Phi|\Gamma\vdash v{:}\mathtt{Bool}}{\Phi|\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C[\mathtt{true}/x]\ \mathtt{else}\ C\big[\mathtt{false}/x\big]=_{\beta\eta}C[V/x]{:}\mathtt{M}_\epsilon A}$
- (Effect-beta) $\frac{\Phi\vdash\epsilon \quad \Phi,\alpha|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash\Lambda\alpha.v\ \epsilon v[\epsilon/\alpha]{:}A[\epsilon/\alpha]}$
- (Effect-eta) $\frac{\Phi|\Gamma\vdash v{:}\forall\alpha.A}{\Phi|\Gamma\vdash\Lambda\alpha.v\ \beta v[\epsilon/\alpha]{:}A[\epsilon/\alpha]}$

0.3.2 Equivalence Relation

- (Reflexive) $\frac{\Phi|\Gamma\vdash t{:}\tau}{\Phi|\Gamma\vdash t=_{\beta\eta}t{:}\tau}$
- (Symmetric) $\frac{\Phi|\Gamma\vdash t_1=_{\beta\eta}t_2{:}\tau}{\Phi|\Gamma\vdash t_2=_{\beta\eta}t_1{:}\tau}$
- (Transitive) $\frac{\Phi|\Gamma\vdash t_1=_{\beta\eta}t_2{:}\tau \quad \Phi|\Gamma\vdash t_2=_{\beta\eta}t_3{:}\tau}{\Phi|\Gamma\vdash t_1=_{\beta\eta}t_3{:}\tau}$

0.3.3 Congruences

- (Lambda) $\frac{\Phi|\Gamma,x{:}A\vdash C_1=_{\beta\eta}C_2{:}\mathtt{M}_\epsilon B}{\Phi|\Gamma\vdash\lambda x{:}A.C_1=_{\beta\eta}\lambda x{:}A.C_2{:}A\to\mathtt{M}_\epsilon B}$
- (Return) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2{:}A}{\Phi|\Gamma\vdash\mathtt{return}v_1=_{\beta\eta}\mathtt{return}v_2{:}\mathtt{M}_{\mathtt{1}}A}$
- (Apply) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_1'{:}A\to\mathtt{M}_\epsilon B \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_2'{:}A}{\Phi|\Gamma\vdash v_1\ v_2=_{\beta\eta}v_1'\ v_2'{:}\mathtt{M}_\epsilon B}$
- (Bind) $\frac{\Phi|\Gamma\vdash C_1=_{\beta\eta}C_1'{:}\mathtt{M}_{\epsilon_1}A \quad \Phi|\Gamma,x{:}A\vdash C_2=_{\beta\eta}C_2'{:}\mathtt{M}_{\epsilon_2}B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2=_{\beta\eta}\mathtt{do}\ c\leftarrow C_1'\ \mathtt{in}\ C_2'{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$
- (If) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v'{:}\mathtt{Bool} \quad \Phi|\Gamma\vdash C_1=_{\beta\eta}C_1'{:}\mathtt{M}_\epsilon A \quad \Phi|\Gamma\vdash C_2=_{\beta\eta}C_2'{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1'\ \mathtt{else}\ C_2'{:}\mathtt{M}_\epsilon A}$
- (Subtype) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v'{:}A \quad A\leq{:}B}{\Phi|\Gamma\vdash v=_{\beta\eta}v'{:}B}$
- (Subeffect) $\frac{\Phi|\Gamma\vdash C=_{\beta\eta}C'{:}\mathtt{M}_{\epsilon_1}A \quad A\leq{:}B \quad \epsilon_1\leq\epsilon_2}{\Phi|\Gamma\vdash C=_{\beta\eta}C'{:}\mathtt{M}_{\epsilon_2}B}$