0.1 Introduce Substitutions

0.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{1}$$

0.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{2}$$

$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{4}$$

$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6}$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{7}$$

$$x\,[\sigma, x := v] = v \tag{8}$$

$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{9}$$

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{10}$$

$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{11}$$

$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{12}$$

$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{13}$$

$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{14}$$

$$\tag{15}$$

0.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil)$\frac{\Gamma'\mathtt{Ok}}{\Gamma' \vdash \diamond : \diamond}$
- (Extend)$\frac{\Gamma' \vdash \sigma:\Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Gamma' \vdash v:A}{\Gamma' \vdash (\sigma, x:=v):(\Gamma, x:A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma \colon \Gamma$ then:

Property 1: ΓOk and Γ'Ok Since $\Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ **implies** $\Gamma'' \vdash \sigma\colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v\colon A$ holds if $\Gamma' \vdash v\colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ **implies** $(\Gamma', x : A) \vdash (\sigma, x := x)\colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma\colon \Gamma$$

In addition, $\Gamma', x : A \vdash x\colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v)\colon (\Gamma, x : A) \tag{16}$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g\colon \tau \wedge \Gamma' \vdash \sigma\colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{17}$$

Assuming $\Gamma' \vdash \sigma\colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t\colon \tau \rightarrow \Gamma' \vdash t\colon \tau$

0.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x\colon A \tag{18}$$

So by inversion, since $\Gamma' \vdash \sigma\colon \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v\colon A \tag{19}$$

By the defintion of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Gamma' \vdash x\,[\sigma] : A \tag{20}$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{()\frac{\Delta}{\Gamma'' \vdash x:A}}{\Gamma'', y : B \vdash x\colon A} \tag{21}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma'\colon \Gamma'' \tag{22}$$

So by induction,

$$\Gamma' \vdash x\,[\sigma'] : A \tag{23}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Gamma' \vdash x\,[\sigma] : A \tag{24}$$

0.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn})\frac{()\frac{\Delta}{\Gamma,x:A\vdash C:\mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\colon A \to \mathtt{M}_\epsilon B} \tag{25}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma', x : A) \vdash (\sigma, x := x)\colon \Gamma, x : A \tag{26}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{()\frac{\Delta'}{\Gamma',x:A\vdash C[\sigma,x:=v]:\mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\,[\sigma, x := x] : A \to \mathtt{M}_\epsilon B} \tag{27}$$

Since $\lambda x : A.(C\,[\sigma, x := x]) = \lambda x : A.(C\,[\sigma]) = (\lambda x : A.C)\,[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x : A.C)\,[\sigma] : A \to \mathtt{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:
$\Gamma \vdash \sigma\colon \Gamma \Rightarrow \Gamma'\mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{28}$$

So

$$(\text{Const})\frac{\Gamma'\mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A\colon A} \tag{29}$$

0.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{()\frac{\Delta_1}{\Gamma\vdash v:A}}{\Gamma \vdash \mathtt{return}v\colon \mathtt{M}_{\mathtt{1}} A} \tag{30}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v[\sigma]:A}}{\Gamma' \vdash \mathtt{return}(v\,[\sigma])\colon \mathtt{M}_{\mathtt{1}} A} \tag{31}$$

Since $(\mathtt{return}v)\,[\sigma] = \mathtt{return}(v\,[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\mathtt{return}v)\,[\sigma] : \mathtt{M}_{\mathtt{1}} A$.

Case Apply By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma\vdash v_2:A}}{\Gamma \vdash v_1\ v_2\colon \mathtt{M}_\epsilon B} \tag{32}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v_1[\sigma]:A\to\mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2'}{\Gamma'\vdash v_2[\sigma]:A}}{\Gamma' \vdash (v_1\,[\sigma])\ (v_2\,[\sigma])\colon \mathtt{M}_\epsilon B} \tag{33}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \mathtt{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$(\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 \colon \texttt{M}_\epsilon A} \tag{34}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$(\text{If})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v[\sigma]:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Gamma'\vdash C_1[\sigma]:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3'}{\Gamma'\vdash C_2[\sigma]:\texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A}\ (v\,[\sigma])\ \texttt{then}\ (C_1\,[\sigma])\ \texttt{else}\ (C_2\,[\sigma]) \colon \texttt{M}_\epsilon A} \tag{35}$$

Since $(\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2)\,[\sigma] = \texttt{if}_{\epsilon,A}\ (v\,[\sigma])\ \texttt{then}\ (C_1\,[\sigma])\ \texttt{else}\ (C_2\,[\sigma])$ The derivation above holds for $\Gamma' \vdash (\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2)\,[\sigma] : \texttt{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$(\text{Bind})\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma,x:A\vdash C_2:\texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{36}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$(\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma'\vdash C_1[\sigma]:\texttt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma',x:A\vdash C_2[\sigma,x:=x]:\texttt{M}_{\epsilon_2} B}}{\Gamma' \vdash \texttt{do}\ x \leftarrow (C_1\,[\sigma])\ \texttt{in}\ (C_2\,[\sigma, x := x]) \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{37}$$

Since $(\texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2)\,[\sigma] = \texttt{do}\ x \leftarrow (C_1\,[\sigma])\ \texttt{in}\ (C_2\,[\sigma]) = \texttt{do}\ x \leftarrow (C_1\,[\sigma])\ \texttt{in}\ (C_2\,[\sigma, x := x])$, the above derivation holds for $\Gamma' \vdash (\texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2)\,[\sigma] : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B$

0.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$(\text{sub-type})\frac{()\frac{\Delta}{\Gamma\vdash v:A} \quad A \leq: B}{\Gamma \vdash v \colon B} \tag{38}$$

By induction on Δ we derive Δ' such that:

$$(\text{sub-type})\frac{()\frac{\Delta'}{\Gamma'\vdash v[\sigma]:A} \quad A \leq: B}{\Gamma \vdash v\,[\sigma] : B} \tag{39}$$

Case Sub-effect By inversion, there exists Δ such that

$$(\text{sub-effect})\frac{()\frac{\Delta}{\Gamma\vdash C:\texttt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma \vdash C \colon \texttt{M}_{\epsilon_2} B} \tag{40}$$

By induction on Δ we derive Δ' such that:

$$(\text{sub-effect})\frac{()\frac{\Delta'}{\Gamma'\vdash C[\sigma]:\texttt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq: \epsilon_2}{\Gamma' \vdash C\,[\sigma] : \texttt{M}_{\epsilon_2} B} \tag{41}$$

0.3 Semantics of Substitution

0.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M : \Gamma' \to \Gamma \tag{42}$$

- (Nil) $\frac{\Gamma' \mathtt{Ok}}{[\![\Gamma' \vdash \diamond \colon \diamond]\!]_M = \langle\rangle_{\Gamma'}}$
- (Extend) $\frac{f = [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \quad g = [\![\Gamma' \vdash v \colon A]\!]_M}{[\![\Gamma' \vdash (\sigma, x := v \colon (\Gamma, x \colon A)]\!]_M = \langle f, g \rangle \colon \Gamma' \to (\Gamma \times A)}$

0.3.2 Extension Lemma

If $\Gamma' \vdash \sigma \colon \Gamma$ and $x \notin (\mathtt{dom}(\Gamma') \cup \mathtt{dom}(\Gamma))$ then the substitution in property 3 has denotation:

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)]\!]_M = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \times \mathtt{Id}_A) \tag{43}$$

This holds since

$$[\![\Gamma', x : A \vdash x \colon A]\!]_M = \pi_2 \tag{44}$$

And $\iota\pi : (\Gamma', x : A) \triangleright \Gamma'$

$$[\![\iota\pi : (\Gamma', x : A) \triangleright \Gamma']\!]_M = \pi_1 \tag{45}$$

So for each denotation $[\![\Gamma' \vdash v \colon B]\!]_M$ of each $y := v$ in σ, we can prepend the denotation with the weakening denotation to yield:

$$[\![\Gamma', x : A \vdash v \colon B]\!]_M = [\![\Gamma' \vdash v \colon B]\!]_M \circ \pi_1 \tag{46}$$

Since π_1 appears in every branch of $[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!]_M$, it can be factored out to yield:

$$[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!]_M = [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \circ \pi_1 \tag{47}$$

Hence,

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x) \colon \Gamma, x : A]\!]_M = \langle [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \circ \pi_1, \pi_2 \rangle = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \times \mathtt{Id}_A) \tag{48}$$

0.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma \vdash t \colon \tau$ and $\Gamma' \vdash \sigma \colon \Gamma$ then the derivation Δ' deriving $\Gamma' \vdash t\,[\sigma] : \tau$ satisfies:

$$\Delta' = \Delta \circ [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M \tag{49}$$

This is proved by induction over the derivation of $\Gamma \vdash t \colon \tau$. We shall use σ to denote $[\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M$ where it is clear from the context.

0.3.4 Proof For Value Terms

Case Var By inversion $\Gamma = \Gamma'', x : A$

$$(\mathrm{Var})\frac{\Gamma \mathtt{Ok}}{\Gamma'', x : A \vdash x \colon A} \tag{50}$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v \colon A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{51}$$
$$\Delta = [\![\Gamma'', x : A \vdash x \colon A]\!]_M = \pi_2 \tag{52}$$
$$\tag{53}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{54}$$
$$= \Delta' \quad \text{By product property} \tag{55}$$

Case Weaken By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x \colon A} \tag{56}$$

Also by inversion of the well-formedness of $\Gamma' \vdash \sigma \colon \Gamma$, we have $\Gamma' \vdash \sigma' \colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma \colon \Gamma]\!]_M = \langle [\![\Gamma' \vdash \sigma \colon \Gamma'']\!]_M, [\![\Gamma' \vdash v \colon B]\!]_M \rangle \tag{57}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$()\frac{\Delta_1'}{\Gamma' \vdash x\,[\sigma] : A} \tag{58}$$

Hence

$$\Delta' = \Delta_1' \quad \text{By definition} \tag{59}$$
$$= \Delta_1 \circ \sigma' \quad \text{By induction} \tag{60}$$
$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v \colon B]\!]_M \rangle \quad \text{By product property} \tag{61}$$
$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.} \tag{62}$$

Case Constants The logic for all constant terms ($\texttt{true}, \texttt{false}, \texttt{()}\texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{63}$$

$$\Delta' = c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} \tag{64}$$
$$= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} \tag{65}$$
$$= \Delta \circ \sigma \quad \text{By definition} \tag{66}$$

Case Lambda By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C : \texttt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C \colon A \to \texttt{M}_\epsilon B} \tag{67}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash (C[\sigma]) : \texttt{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A.C)\,[\sigma] : A \to \texttt{M}_\epsilon B} \tag{68}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{69}$$

Hence:

$$\begin{aligned}
\Delta' &= \texttt{cur}(\Delta_1') \quad \text{By definition} && (70)\\
&= \texttt{cur}(\Delta_1 \circ (\sigma \times \texttt{Id}_A)) \quad \text{By induction and extension lemma.} && (71)\\
&= \texttt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (72)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (73)\\
& && (74)
\end{aligned}$$

Case Subtype By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Gamma \vdash v : A} \quad A \leq: B}{\Gamma \vdash v : B} \tag{75}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma] : A} \quad A \leq: B}{\Gamma' \vdash v\,[\sigma] : B} \tag{76}$$

0.3.5 Proof For Computation Terms

Case Return

Case Apply

Case If

Case Bind

Case Subeffect

0.4 Single Substitution