0.1 Helper Morphisms

0.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\tau_{A,B} : (A \times B) \to (B \times A) = \langle \pi_2, \pi_1 \rangle \tag{1}$$
$$\delta_A : A \to (A \times A) = \langle \mathtt{Id}_A, \mathtt{Id}_A \rangle \tag{2}$$

0.2 Denotations of Types

0.2.1 Denotation of Ground Types

The denotations of the default ground types, $\mathtt{Unit}, \mathtt{Bool}$ should be as follows:

$$[\![\mathtt{Unit}]\!] = \mathtt{1} \tag{3}$$

$$[\![\mathtt{Bool}]\!] = \mathtt{1} + \mathtt{1} \tag{4}$$

The mapping $[\![\_]\!]$ should then map each other ground type γ to an object in $\mathbb{C}$.

0.2.2 Denotation of Computation Type

Given a function $[\![\_]\!]$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\mathtt{M}_\epsilon A$ as so:

$$[\![\mathtt{M}_\epsilon A]\!] = T_\epsilon [\![A]\!]$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\mathtt{M}_\epsilon A]\!] = T_\epsilon A$$

0.2.3 Denotation of Function Types

Given a function $[\![\_]\!]$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to \mathtt{M}_\epsilon B$ as so:

$$[\![A \to \mathtt{M}_\epsilon B]\!] = (T_\epsilon [\![B]\!])^{[\![A]\!]}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$[\![A \to \mathtt{M}_\epsilon B]\!] = (T_\epsilon B)^A$$

0.2.4 Denotation of Type Environments

Given a function $[\![\_]\!]$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an $\mathtt{Ok}$ type environment Γ.

$$[\![\, \diamond \,]\!] = \mathtt{1}$$

$$[\![\Gamma, x : A]\!] = ([\![\Gamma]\!] \times [\![A]\!])$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x : A]\!] = \Gamma \times A$$

0.3 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$[\![\Gamma \vdash t \colon \tau]\!] : \Gamma \to [\![\tau]\!]$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t \colon \tau$ are equal, we need not refer to the derivation that yielded each denotation.

0.3.1 Denotation of Value Terms

- (Unit) $\dfrac{\Gamma \texttt{Ok}}{[\![\Gamma \vdash \texttt{()} \colon \texttt{Unit}]\!] = \langle\rangle_\Gamma : \Gamma \to \texttt{1}}$

- (Const) $\dfrac{\Gamma \texttt{Ok}}{[\![\Gamma \vdash \texttt{C}^A \colon A]\!] = [\![\texttt{C}^A]\!] \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]}$

- (True) $\dfrac{\Gamma \texttt{Ok}}{[\![\Gamma \vdash \texttt{true} \colon \texttt{Bool}]\!] = \texttt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\texttt{Bool}]\!] = \texttt{1} + \texttt{1}}$

- (False) $\dfrac{\Gamma \texttt{Ok}}{[\![\Gamma \vdash \texttt{false} \colon \texttt{Bool}]\!] = \texttt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\texttt{Bool}]\!] = \texttt{1} + \texttt{1}}$

- (Var) $\dfrac{\Gamma \texttt{Ok}}{[\![\Gamma, x : A \vdash x \colon A]\!] = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Gamma \vdash x \colon A]\!] : \Gamma \to A}{[\![\Gamma, y : B \vdash x \colon A]\!] = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\dfrac{f = [\![\Gamma, x : A \vdash C \colon \texttt{M}_\epsilon B]\!] : \Gamma \times A \to T_\epsilon B}{[\![\Gamma \vdash \lambda x : A.C : A \to \texttt{M}_\epsilon B]\!] = \texttt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\dfrac{f = [\![\Gamma \vdash v \colon A]\!] : \Gamma \to A \qquad g = [\![A \leq \colon B]\!]}{[\![\Gamma \vdash v \colon B]\!] = g \circ f : \Gamma \to B}$

0.3.2 Denotation of Computation Terms

- (Return) $\dfrac{f = [\![\Gamma \vdash v \colon A]\!]}{[\![\Gamma \vdash \texttt{return}\ v : \texttt{M}_1 A]\!] = \eta_A \circ f}$

- (If) $$\frac{f = [\![\Gamma \vdash v \colon \mathtt{Bool}]\!] : \Gamma \to \mathtt{1}+\mathtt{1} \qquad g = [\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!] \qquad h = [\![\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A]\!]}{[\![\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A]\!] = \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \to T_\epsilon A}$$

- (Bind) $$\frac{f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A : \Gamma \to T_{\epsilon_1} A]\!] \qquad g = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B]\!] : \Gamma \times A \to T_{\epsilon_2} B}{[\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2}]\!] = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma,A,\epsilon_1} \circ \langle \mathtt{Id}_\Gamma, f \rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$$

- (Subeffect) $$\frac{f = [\![\Gamma \vdash c \colon \mathtt{M}_{\epsilon_1} A]\!] : \Gamma \to T_{\epsilon_1} A \qquad g = [\![A \leq: B]\!]}{h = [\![\epsilon_1 \leq \epsilon_2]\!]} [\![\Gamma \vdash C \colon \mathtt{M}_{\epsilon_2} B]\!] = h_B \circ T_{\epsilon_1} g \circ f$$

- (Apply) $$\frac{f = [\![\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B]\!] : \Gamma \to (T_\epsilon B)^A \qquad g = [\![\Gamma \vdash v_2 \colon A]\!] : \Gamma \to A}{[\![\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B]\!] = \mathtt{app} \circ \langle f, g \rangle : \Gamma \to T_\epsilon B}$$