

We need to define substitutions of effects on effects, effects on types, effects on terms, terms on terms.

0.1 Effect Substitutions

Define a substitution, σ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \quad (1)$$

Define the free-effect Variables of σ :

$$\begin{aligned} fev(\diamond) &= \emptyset \\ fev(\sigma, \alpha := \epsilon) &= fev(\sigma) \cup fev(\epsilon) \end{aligned}$$

We define the property:

$$\alpha \# \sigma \Leftrightarrow \alpha \notin (\text{dom}(\sigma) \cup fev(\sigma)) \quad (2)$$

0.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \quad (3)$$

$$\sigma(e) = e \quad (4)$$

$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \quad (5)$$

$$\diamond(\alpha) = \alpha \quad (6)$$

$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \quad (7)$$

$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \quad (8)$$

0.1.2 Action of Effect Substitution on Types

Define the action of applying an effect substitution, σ to a type A as:

$$A[\sigma]$$

Defined as so

$$\gamma[\sigma] = \gamma \quad (9)$$

$$(A \rightarrow B)[\sigma] = (A[\sigma]) \rightarrow (B[\sigma]) \quad (10)$$

$$(\mathbb{M}_\epsilon A)[\sigma] = \mathbb{M}_{\sigma(\epsilon)}(A[\sigma]) \quad (11)$$

$$(\forall \alpha. A)[\sigma] = \forall \alpha. (A[\sigma]) \quad \text{If } \alpha \# \sigma \quad (12)$$

0.1.3 Action of Effect-Substitution on Type Environments

Define the action of effect substitution on type environments:

$$\Gamma [\sigma]$$

Defined as so:

$$\begin{aligned} \diamond [\sigma] &= \diamond \\ (\Gamma, x:A) [\sigma] &= (\Gamma [\sigma], x : (A [\sigma])) \end{aligned}$$

0.1.4 Action of Effect Substitution on Terms

Define the action of effect-substitution on terms:

$$x [\sigma] = x \tag{13}$$

$$\mathbf{c}^A [\sigma] = \mathbf{c}^{(A [\sigma])} \tag{14}$$

$$(\lambda x:A.v) [\sigma] = \lambda x: (A [\sigma]).(v [\sigma]) \tag{15}$$

$$(\text{if}_A v \text{ then } v_1 \text{ else } v_2) [\sigma] = \text{if}_{(A [\sigma])} v [\sigma] \text{ then } v_1 [\sigma] \text{ else } v_2 [\sigma] \tag{16}$$

$$(v_1 v_2) [\sigma] = (v_1 [\sigma]) v_2 [\sigma] \tag{17}$$

$$(\text{do } x \leftarrow v_1 \text{ in } v_2) [\sigma] = \text{do } x \leftarrow (v_1 [\sigma]) \text{ in } (v_2 [\sigma]) \tag{18}$$

$$(\Lambda \alpha.v) [\sigma] = \Lambda \alpha.(v [\sigma]) \quad \text{If } \alpha \# \sigma \tag{19}$$

$$(v \epsilon) [\sigma] = (v [\sigma]) \sigma(\epsilon) \tag{20}$$

$$\tag{21}$$

0.1.5 Well-Formed-ness

For any two effect-environments, and a substitution, define the well-formed-ness relation:

$$\Phi' \vdash \sigma: \Phi \tag{22}$$

- (Nil) $\frac{\Phi' \text{ Ok}}{\Phi' \vdash \diamond: \diamond}$
- (Extend) $\frac{\Phi' \vdash \sigma: \Phi \quad \Phi' \vdash \epsilon \quad \alpha \notin \Phi}{\Phi' \vdash \sigma, \alpha: = \epsilon: (\Phi, \alpha)}$

0.1.6 Property 1

If $\Phi' \vdash \sigma: \Phi$ then $\Phi' \text{ Ok}$ (By the Nil case) and $\Phi \text{ Ok}$ Since each use of the extend case preserves Ok .

0.1.7 Property 2

If $\Phi' \vdash \sigma: \Phi$ then $\omega: \Phi' \triangleright \Phi' \implies \Phi'' \vdash \sigma: \Phi$ since $\Phi' \vdash \epsilon \implies \Phi'' \vdash \epsilon$ and $\Phi' \text{ Ok} \implies \Phi'' \text{ Ok}$

0.1.8 Property 3

If $\Phi' \vdash \sigma: \Phi$ then

$$\alpha \notin \Phi \wedge \alpha \notin \Phi' \implies (\Phi', \alpha) \vdash (\sigma, \alpha = \alpha): (\Phi, \alpha) \quad (23)$$

Since $\iota\pi: \Phi', \alpha \triangleright \Phi'$ so $\Phi', \alpha \vdash \sigma: \Phi$ and $\Phi', \alpha \vdash \alpha$

0.2 Substitution Preserves the Well-formed-ness of Effects

I.e.

$$\Phi \vdash \epsilon \wedge \Phi' \vdash \iota: \Phi \implies \Phi' \vdash \sigma(\epsilon) \quad (24)$$

Proof:

Case Ground: $\sigma(e) = e$, so $\Phi' \vdash \sigma(\epsilon)$ holds.

Case Multiply: By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$ so $\Phi' \vdash \sigma(\epsilon_1)$ and $\Phi' \vdash \sigma(\epsilon_2)$ by induction and hence $\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)$

Case Var: By inversion, $\Phi = \Phi'', \alpha$ and $\Phi'', \alpha \text{ Ok}$. Hence by case splitting on ι , we see that $\sigma = \sigma', \alpha: = \epsilon$.

So by inversion, $\sigma \vdash \epsilon$ so $\Phi' \vdash \sigma(\alpha) = \epsilon$

Case Weaken: By inversion $\Phi = \Phi'', \beta$ and $\Phi'' \vdash \alpha$, so $\sigma = \sigma' \beta: = \epsilon$.

So $\Phi' \vdash \sigma': \Phi''$.

hence by induction, $\Phi' \vdash \sigma'(a)$, so $\Phi' \vdash \sigma(\alpha)$ since $\alpha \neq \beta$

0.2.1 Effect Substitution preserves the sub-effect relation

If $\Phi' \vdash \sigma: \Phi$ and $\epsilon_1 \leq_\Phi \epsilon_2$, then $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

Proof: For any ground substitution σ' of Φ' , then $\sigma\sigma'$ (the substitution σ' applied after σ) is also a ground substitution.

So $\epsilon_1 [\sigma] [\sigma'] \leq \epsilon_2 [\sigma] [\sigma']$.

So $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

0.2.2 Substitution preserves well-formed-ness of Types

$$\Phi' \vdash \sigma: \Phi \wedge \Phi \vdash A \implies \Phi' \vdash A [\sigma] \quad (25)$$

Proof:

Case Ground: $\Phi' \text{ Ok}$ so $\Phi' \vdash \gamma$ and $\gamma[\sigma] = \gamma$.

Hence $\Phi' \vdash \gamma[\sigma]$.

Case Lambda: By inversion $\Phi \vdash A$ and $\Phi \vdash B$.

So by induction, $\Phi' \vdash A[\sigma]$ and $\Phi' \vdash B[\sigma]$.

So

$$\Phi' \vdash (A[\sigma]) \rightarrow (B[\sigma]) \quad (26)$$

So

$$\Phi' \vdash (A \rightarrow B)[\sigma] \quad (27)$$

Case Computation: By inversion, $\Phi \vdash \epsilon$ and $\Phi \vdash A$ so by induction and substitution of effect preserving effect-well-formed-ness,

$$\Phi' \vdash \sigma(\epsilon) \text{ and } \Phi' \vdash A[\sigma] \text{ so } \Phi \vdash \mathbf{M}_{\sigma(\epsilon)} A[\sigma] \text{ so } \Phi' \vdash (\mathbf{M}_\epsilon A)[\sigma]$$

Case For All: By inversion, $\Phi, \alpha \vdash A$. So by picking $\alpha \notin \Phi \wedge \alpha \notin \Phi'$ using α -equivalence, we have $(\Phi', \alpha) \vdash (\sigma\alpha = \alpha): (\Phi, \alpha)$.

So by induction $(\Phi', \alpha) \vdash A[\sigma, \alpha = \alpha]$

So $(\Phi', \alpha) \vdash A[\sigma]$

So $\Phi' \vdash (\forall\alpha.A)[\sigma]$

0.2.3 Substitution of effects preserves Sub-Typing Relation

If $\Phi' \vdash \sigma: \Phi$ and $A \leq_\Phi B$ then $A[\sigma] \leq_{\Phi'} B[\sigma]$

Proof: By induction on the sub-typing relation

Case Ground: By inversion, $A \leq_\gamma B$, so A, B are ground types. Hence $A[\sigma] = A$ and $B[\sigma] = B$. So $A[\sigma] \leq_{\Phi'} B[\sigma]$

Case Fn: By inversion, $A' \leq_\Phi A$ and $B \leq_\Phi B'$.

So by induction, $A'[\sigma] \leq_{\Phi'} A[\sigma]$ and $B[\sigma] \leq_{\Phi'} B'[\sigma]$.

So $(A[\sigma]) \rightarrow (B[\sigma]) \leq_{\Phi'} (A'[\sigma]) \rightarrow (B'[\sigma])$

So $(A \rightarrow B)[\sigma] \leq_{\Phi'} (A' \rightarrow B')[\sigma]$

Case Computation: By inversion, $A \leq_\Phi B$, $\epsilon_1 \leq_\Phi \epsilon_2$.

So by induction and substitution preserving the sub-effect relation,

$A[\sigma] \leq_{\Phi'} B[\sigma]$ and $\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)$

So $\mathbf{M}_{\sigma(\epsilon_1)}(A[\sigma]) \leq_{\Phi'} \mathbf{M}_{\sigma(\epsilon_2)}(B[\sigma])$

So $(\mathbf{M}_{\epsilon_1} A)[\sigma] \leq_{\Phi'} (\mathbf{M}_{\epsilon_2} B)[\sigma]$

0.2.4 Substitution preserves well-formed-ness of Type Environments

If $\Phi \vdash \Gamma \text{ Ok}$ and $\Phi' \vdash \sigma: \Phi$ then $\Phi' \vdash \Gamma[\sigma] \text{ Ok}$

Proof:

Case Nil: $\Phi \text{ Ok} \implies \Phi' \text{ Ok}$ so $\Phi' \vdash \diamond \text{ Ok}$ and $\diamond[\sigma] = \diamond$

Case Var: By inversion, $\Phi \vdash \Gamma \text{ Ok}$ and $\Phi \vdash A$.

By induction and substitution preserving well-formed-ness of types, $\Phi' \vdash \Gamma'[\sigma] \text{ Ok}$ and $\Phi' \vdash A[\sigma]$.

So $\Phi' \vdash (\Gamma'[\sigma], x : A[\sigma]) \text{ Ok}$.

Hence $\Phi' \vdash \Gamma, x : A[\sigma] \text{ Ok}$.

0.2.5 Effect-Substitution Preserves the Typing Relation

If $\Phi' \vdash \sigma : \Phi$ and $\Phi \mid \Gamma \vdash v : A$, then $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$

Proof:

Case Const: By inversion, $\Phi \vdash \Gamma \text{ Ok}$.

So $\Phi' \vdash \Gamma \text{ Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash \mathsf{C}^{A[\sigma]} : A[\sigma]$

Case True, False, Unit: The logic is the same for each of these cases, so we look at the case **true** only.

By inversion, $\Phi \vdash \Gamma \text{ Ok}$.

So $\Phi' \vdash \Gamma \text{ Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash \text{true} : \text{Bool}$

Since $\text{true}[\sigma] = \text{true}$ and $\text{Bool}[\sigma] = \text{Bool}$.

Case Var: By inversion $\Gamma = \Gamma', x : A$ and $\Phi \vdash \Gamma', x : A \text{ Ok}$.

So since substitution preserves well-formed-ness of type environments, $\Phi' \vdash \Gamma'[\sigma], x : A[\sigma] \text{ Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash x : A[\sigma]$

Since $x[\sigma] = x$

Case Weaken: By inversion $\Gamma = \Gamma', y : B$, $\Phi \vdash B$, and $\Phi \mid \Gamma' \vdash x : A$. $x \neq y$

By induction and the theorem that effect-substitution preserves type well-formed-ness, we have: $\Phi' \mid \Gamma'[\sigma] \vdash x : A[\sigma]$ and $\Phi' \vdash B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash x[\sigma] : A[\sigma]$

Since $x[\sigma] = x$, $\Gamma[\sigma] = (\Gamma'[\sigma], y : B[\sigma])$

Case Lambda: By inversion $\Phi \mid \Gamma, x : A \vdash v : B$.

So, by induction $\Phi' \mid (\Gamma, x : A)[\sigma] \vdash v[\sigma] : B[\sigma]$.

So, $\Phi \mid \Gamma[\sigma], x : A[\sigma] \vdash v[\sigma] : B[\sigma]$.

Hence by the lambda type rule,

$$\Phi' \mid \Gamma[\sigma] \vdash \lambda x: A[\sigma]. v[\sigma] : (A[\sigma] \rightarrow (B[\sigma]))$$

So

$$\Phi' \mid \Gamma[\sigma] \vdash (\lambda x: A.v)[\sigma] : (A \rightarrow B)[\sigma]$$

Case Apply: By inversion, $\Phi \mid \Gamma \vdash v_1: A \rightarrow B$, $\Phi \mid \Gamma \vdash V_2: A$.

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : (A[\sigma] \rightarrow (B[\sigma]))$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1[\sigma]) (v_2[\sigma]) : B[\sigma]$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1 v_2)[\sigma] : (A \rightarrow B)[\sigma]$

Case Subtype: By inversion, $\Phi \mid \Gamma \vdash v: A$ and $\Phi \vdash A \leq B$

So by induction and effect-substitution preserving sub-typing, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ and $\Phi' \vdash A[\sigma] \leq B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : B[\sigma]$

Case Return: By inversion, $\Phi \mid \Gamma \vdash v: A$

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash \text{return}(v[\sigma]) : \mathbf{M}_1(A[\sigma])$

Hence $\Phi' \mid \Gamma[\sigma] \vdash (\text{return } v)[\sigma] : (\mathbf{M}_1 A)[\sigma]$

Case Bind: By inversion, $\Phi \mid \Gamma \vdash v_1: \mathbf{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x: A \vdash v_2: \mathbf{M}_{\epsilon_2} B$.

So by induction: $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : \mathbf{M}_{\sigma(\epsilon_1)}(A[\sigma])$, and $\Phi' \mid \Gamma[\sigma], x: A[\sigma] \vdash v_2[\sigma] : \mathbf{M}_{\sigma(\epsilon_2)}(B[\sigma])$.

And so $\Phi' \mid \Gamma[\sigma] \vdash \text{do } x \leftarrow (v_1[\sigma]) \text{ in } (v_2[\sigma]) : \mathbf{M}_{\sigma(\epsilon_1) \cdot (\epsilon_2[\sigma])} B[\sigma]$

Case If: By inversion, $\Phi \mid \Gamma \vdash v: \text{Bool}$, $\Phi \mid \Gamma \vdash v_1: A$, and $\Phi \mid \Gamma \vdash v_2: A$

So by induction $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \text{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : A[\sigma]$, and $\Phi' \mid \Gamma[\sigma] \vdash v_2[\sigma] : A[\sigma]$. (Since $\text{Bool}[\sigma] = \text{Bool}$)

Hence:

$$\Phi' \mid \Gamma[\sigma] \vdash \text{if}_{A[\sigma]} v[\sigma] \text{ then } v_1[\sigma] \text{ else } v_2[\sigma] : A[\sigma]$$

So $\Phi' \mid \Gamma[\sigma] \vdash (\text{if}_A v \text{ then } v_1 \text{ else } v_2)[\sigma] : A[\sigma]$

Case Effect-lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v: A$.

So by the substitution property 3 (**TODO: Is this correct/reference correctly**), pick $\alpha \notin \Phi' \wedge \alpha \notin \Phi$ so we have:

$$(\Phi', \alpha) \vdash (\sigma, \alpha = \alpha) : (\Phi, \alpha)$$

So by induction, $\Phi', \alpha \mid \Gamma[\sigma, \alpha = \alpha] \vdash v[\sigma, \alpha = \alpha] : A[\sigma, \alpha = \alpha]$

So $\Phi', \alpha \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ since $\alpha \notin \Phi' \wedge \alpha \notin \Phi$.

So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : (\forall \alpha. A)[\sigma]$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v : \forall \alpha. A, \Phi \vdash \epsilon$.

So by induction and effect-substitution preserving well-formed-ness of effects: $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : (\forall \alpha. A)[\sigma]$ and $\Phi' \vdash \sigma(\epsilon)$

So $\Phi' \mid \Gamma[\sigma] \vdash (v[\sigma]) (\sigma(\epsilon)) : A[\sigma][\sigma(\epsilon)/\alpha]$.

Since $\alpha \# \sigma$, we can commute the applications of substitution. **TODO: Do I need to prove this?**

So, $\Phi' \mid \Gamma[\sigma] \vdash (v \epsilon)[\sigma] : A[\epsilon/\alpha][\sigma]$

0.3 The Identity Substitution on Effect Environments

For each type environment Φ , define the identity substitution I_Φ as so:

- $I_\diamond = \diamond$
- $I_{(\Phi, \alpha)} = (I_\Phi, \alpha := \alpha)$

0.3.1 Properties of the Identity Substitution

Property 1 If $\Phi \vdash \text{Ok}$ then $\Phi \vdash I_\Phi : \Phi$, proved trivially by induction over the \vdash relation.

Property 2 **TODO: The denotational property of id-substitution**

0.4 Single Substitution on Effect Environments

If $\Phi \vdash \epsilon$, let the single substitution $\Phi \vdash [\epsilon/\alpha] : \Phi, \alpha$, be defined as:

$$[x/\alpha] = (I_\Phi, \alpha := \epsilon) \quad (28)$$

0.5 Term-Term Substitutions

0.5.1 Substitutions as SNOG lists

$$\sigma ::= \diamond \mid \sigma, x := v \quad (29)$$

0.5.2 Trivial Properties of substitutions

$\text{fv}(\sigma)$

$$\text{fv}(\diamond) = \emptyset \quad (30)$$

$$\text{fv}(\sigma, x := v) = \text{fv}(\sigma) \cup \text{fv}(v) \quad (31)$$

$\text{dom}(\sigma)$

$$\text{dom}(\diamond) = \emptyset \quad (32)$$

$$\text{dom}(\sigma, x := v) = \text{dom}(\sigma) \cup \{x\} \quad (33)$$

$x \# \sigma$

$$x \# \sigma \Leftrightarrow x \notin (\text{fv}(\sigma) \cup \text{dom}(\sigma')) \quad (34)$$

0.5.3 Action of substitutions

We define the action of applying a substitution σ as

$$t[\sigma]$$

$$x[\diamond] = x \quad (35)$$

$$x[\sigma, x := v] = v \quad (36)$$

$$x[\sigma, x' := v'] = x[\sigma] \quad \text{If } x \neq x' \quad (37)$$

$$\mathbf{C}^A[\sigma] = \mathbf{C}^A \quad (38)$$

$$(\lambda x: A. v)[\sigma] = \lambda x: A. (v[\sigma]) \quad \text{If } x \# \sigma \quad (39)$$

$$(\text{if}_A v \text{ then } v_1 \text{ else } v_2)[\sigma] = \text{if}_A v[\sigma] \text{ then } v_1[\sigma] \text{ else } v_2[\sigma] \quad (40)$$

$$(v_1 v_2)[\sigma] = (v_1[\sigma]) v_2[\sigma] \quad (41)$$

$$(\text{do } x \leftarrow v_1 \text{ in } v_2) = \text{do } x \leftarrow (v_1[\sigma]) \text{ in } (v_2[\sigma]) \quad \text{If } x \# \sigma \quad (42)$$

$$(\Lambda \alpha. v)[\sigma] = \Lambda \alpha. (v[\sigma]) \quad (43)$$

$$(v \epsilon)[\sigma] = (v[\sigma]) \epsilon \quad (44)$$

$$(45)$$

0.5.4 Well-Formed-ness

Define the relation

$$\Phi \mid \Gamma' \vdash \sigma; \Gamma$$

by:

- (Nil) $\frac{\Phi \vdash \Gamma' \text{ Ok}}{\Phi \mid \Gamma' \vdash \diamond; \diamond}$
- (Extend) $\frac{\Phi \mid \Gamma' \vdash \sigma; \Gamma \quad x \notin \text{dom}(\Gamma) \quad \Phi \mid \Gamma' \vdash v; A}{\Phi \mid \Gamma' \vdash (\sigma, x := v); (\Gamma, x: A)}$

0.5.5 Simple Properties Of Substitution

If $\Phi \mid \Gamma' \vdash \sigma; \Gamma$ then: **TODO: Number these**

Property 1: $\Phi \vdash \Gamma \text{ Ok}$ and $\Phi \vdash \Gamma' \text{ Ok}$ Since $\Phi \vdash \Gamma' \text{ Ok}$ holds by the Nil-axiom. $\Phi \vdash \Gamma \text{ Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega: \Gamma'' \triangleright \Gamma'$ implies $\Phi \mid \Gamma'' \vdash \sigma; \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ , $\Phi \mid \Gamma'' \vdash v; A$ holds if $\Phi \mid \Gamma' \vdash v; A$ holds.

Property 3: $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma''))$ implies $\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x) : (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Phi \mid \Gamma', x : A \vdash \sigma : \Gamma$$

In addition, $\Phi \mid \Gamma', x : A \vdash x : A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := v) : (\Gamma, x : A) \quad (46)$$

0.5.6 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$(\Phi \mid \Gamma \vdash v : A) \wedge (\Phi \mid \Gamma' \vdash \sigma : \Gamma) \Rightarrow (\Phi \mid \Gamma' \vdash v[\sigma] : A) \quad (47)$$

Assuming $\Phi \mid \Gamma' \vdash \sigma : \Gamma$, we induct over the typing relation, proving $\Phi \mid \Gamma \vdash v : A \Rightarrow \Phi \mid \Gamma' \vdash v : A$

Proof:

Case Var: By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Phi \mid \Gamma'', x : A \vdash x : A \quad (48)$$

So by inversion, since $\Phi \mid \Gamma' \vdash \sigma : \Gamma'', x : A$,

$$\sigma = (\sigma', x := v) \wedge \Phi \mid \Gamma' \vdash v : A \quad (49)$$

By the definition of the effect of substitutions, $x[\sigma] = v$, So

$$\Phi \mid \Gamma' \vdash x[\sigma] : A \quad (50)$$

holds.

Case Weaken: By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$\text{(Weaken)} \frac{\Delta}{\Phi \mid \Gamma'' \vdash x : A} \quad (51)$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Phi \mid \Gamma' \vdash \sigma' : \Gamma'' \quad (52)$$

So by induction,

$$\Phi \mid \Gamma' \vdash x[\sigma'] : A \quad (53)$$

And so by definition of the effect of σ , $x[\sigma] = x[\sigma']$

$$\Phi \mid \Gamma' \vdash x[\sigma] : A \quad (54)$$

Case Lambda: By inversion, there exists Δ such that:

$$(\text{Fn}) \frac{\overline{\Delta}}{\overline{\Phi \mid \Gamma, x : A \vdash v : B}} \quad (55)$$

Using alpha equivalence, we pick $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma'))$ Hence, by property 3, we have

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x : x) : \Gamma, x : A \quad (56)$$

So by induction using $\sigma, x : x$, we have Δ' such that:

$$(\text{Fn}) \frac{\overline{\Delta'}}{\overline{\Phi \mid \Gamma' \vdash \lambda x : A. v [\sigma, x : x] : A \rightarrow B}} \quad (57)$$

Since $\lambda x : A. (v [\sigma, x : x]) = \lambda x : A. (v [\sigma]) = (\lambda x : A. v) [\sigma]$, we have a typing derivation for $\Phi \mid \Gamma' \vdash (\lambda x : A. v) [\sigma] : A \rightarrow B$.

Case Constants: We use the same logic for all constants, $()$, true , false , \mathbf{C}^A :

$\Phi \mid \Gamma \vdash \sigma : \Gamma \Rightarrow \Phi \vdash \Gamma' \text{ Ok}$ and:

$$\mathbf{C}^A [\sigma] = \mathbf{C}^A \quad (58)$$

So

$$(\text{Const}) \frac{\Phi \vdash \Gamma' \text{ Ok}}{\Phi \mid \Gamma' \vdash \mathbf{C}^A : A} \quad (59)$$

0.5.7 Computation Terms

Case Return: By inversion, we have Δ_1 such that:

$$(\text{Return}) \frac{\overline{\Delta_1}}{\overline{\Phi \mid \Gamma \vdash v : A}} \quad (60)$$

By induction, we have Δ'_1 such that

$$(\text{Return}) \frac{\overline{\Delta'_1}}{\overline{\Phi \mid \Gamma' \vdash v [\sigma] : A}} \quad (61)$$

Since $(\text{return } v) [\sigma] = \text{return } (v [\sigma])$, the type derivation above holds for $\Phi \mid \Gamma' \vdash (\text{return } v) [\sigma] : \mathbf{M}_1 A$.

Case Apply: By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply}) \frac{\overline{\Delta_1} \quad \overline{\Delta_2}}{\overline{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B \quad \Phi \mid \Gamma \vdash v_2 : A}} \quad (62)$$

By induction on Δ_1, Δ_2 , we have Δ'_1, Δ'_2 such that

$$\text{(Apply)} \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1[\sigma] : A \rightarrow B} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_2[\sigma] : A}}{\Phi \mid \Gamma' \vdash (v_1[\sigma]) (v_2[\sigma]) : B} \quad (63)$$

Since $(v_1 v_2)[\sigma] = (v_1[\sigma]) (v_2[\sigma])$, we the above derivation holds for $\Phi \mid \Gamma' \vdash (v_1 v_2)[\sigma] : B$

Case If: By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)} \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \text{Bool}} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_1 : A} \quad \frac{\Delta_3}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A} \quad (64)$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta'_1, \Delta'_2, \Delta'_3$ such that:

$$\text{(If)} \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v[\sigma] : \text{Bool}} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_1[\sigma] : A} \quad \frac{\Delta'_3}{\Phi \mid \Gamma' \vdash v_2[\sigma] : A}}{\Phi \mid \Gamma' \vdash \text{if}_A (v[\sigma]) \text{ then } (v_1[\sigma]) \text{ else } (v_2[\sigma]) : A} \quad (65)$$

Since $(\text{if}_A v \text{ then } v_1 \text{ else } v_2)[\sigma] = \text{if}_A (v[\sigma]) \text{ then } (v_1[\sigma]) \text{ else } (v_2[\sigma])$ The derivation above holds for $\Phi \mid \Gamma' \vdash (\text{if}_A v \text{ then } v_1 \text{ else } v_2)[\sigma] : A$

Case Bind: By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)} \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A} \quad \frac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (66)$$

Using alpha-equivalence, we pick $x \notin (\text{dom}(\Gamma) \cup \text{dom}(\Gamma'))$. Hence by property 3,

$$\Phi \mid (\Gamma, x : A) \vdash (\sigma, x = x) : (\Gamma, x : A)$$

By induction on Δ_1, Δ_2 , we have Δ'_1, Δ'_2 such that:

$$\text{(Bind)} \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1[\sigma] : \mathbf{M}_{\epsilon_1} A} \quad \frac{\Delta'_2}{\Phi \mid \Gamma', x : A \vdash v_2[\sigma, x = x] : \mathbf{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \text{do } x \leftarrow (v_1[\sigma]) \text{ in } (v_2[\sigma, x = x]) : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (67)$$

Since $(\text{do } x \leftarrow v_1 \text{ in } v_2)[\sigma] = \text{do } x \leftarrow (v_1[\sigma]) \text{ in } (v_2[\sigma]) = \text{do } x \leftarrow (v_1[\sigma]) \text{ in } (v_2[\sigma, x = x])$, the above derivation holds for $\Phi \mid \Gamma' \vdash (\text{do } x \leftarrow v_1 \text{ in } v_2)[\sigma] : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B$

Case Sub-type: By inversion, there exists Δ such that

$$\text{(sub-type)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v : A} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v : B} \quad (68)$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)} \frac{\frac{\Delta'}{\Phi \mid \Gamma' \vdash v[\sigma] : A} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v[\sigma] : B} \quad (69)$$

Case Effect-Lambda: By inversion, there exists Δ such that

$$\text{(Effect-abs)} \frac{\frac{\Delta}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \alpha. A} \quad (70)$$

It is also the case that $\iota\pi : \Phi, \alpha \triangleright \Phi$.

So $\Phi, \alpha \mid \Gamma' \vdash \sigma : \Gamma$

So by induction there exists Δ' ,

$$\text{(Effect-abs)} \frac{\frac{\Delta'}{\Phi, \alpha \mid \Gamma' \vdash v[\sigma] : A}}{\Phi \mid \Gamma' \vdash \Lambda \alpha. (v[\sigma]) : \forall \alpha. A} \quad (71)$$

Where $\Lambda \alpha. (v[\sigma]) = (\Lambda \alpha. v)[\sigma]$

Case Effect Application: By inversion $\Phi \vdash \epsilon$ and there exists Δ such that

$$\text{(Effect-App)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v \epsilon : A[\epsilon/\alpha]} \quad (72)$$

So by induction there exists Δ' such that:

$$\text{(Effect-App)} \frac{\frac{\Delta'}{\Phi \mid \Gamma' \vdash v[\sigma] : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash (v[\sigma]) \epsilon : A[\epsilon/\alpha]} \quad (73)$$

Where $(v[\sigma]) \epsilon = (v \epsilon)[\sigma]$

0.6 The Identity Substitution on Type Environments

For each type environment Γ , define the identity substitution I_{Γ} as so:

- $I_{\diamond} = \diamond$
- $I_{(\Gamma, x:A)} = (I_{\Gamma}, x = x)$

0.6.1 Properties of the Identity Substitution

Property 1 If $\Phi \vdash \Gamma$ Ok then $\Phi \mid \Gamma \vdash I_{\Gamma} : \Gamma$, proved trivially by induction over the well-formed-ness relation.

Property 2 **TODO:** The denotational property of id-substitution

0.7 Single Substitution on Type Environments

If $\Phi \mid \Gamma \vdash v : A$, let the single substitution $\Phi \mid \Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{74}$$