0.1 Weakening Definition

0.1.1 Relation

We define the ternary weaking relation $w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Gamma \texttt{Ok}}{\iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma') \quad A \leq : B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B}$

0.1.2 Weakening Denotations

The denotation of a weakening relation is defined as follows:

$$[\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma \tag{1}$$

- $[\![\iota : \Gamma \triangleright \Gamma]\!]_M = \texttt{Id}_\Gamma : \Gamma \to \Gamma$
- (Project) $\frac{f = [\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma}{[\![\omega\pi : \Gamma, x : A \triangleright \Gamma]\!]_M = f \circ \pi_1 : \Gamma' \times A \to \Gamma}$
- (Extend) $\frac{f = [\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma \quad g = [\![A \leq : B]\!]_M : A \to B}{[\![w \times : \Gamma', x : A \triangleright \Gamma, x : B]\!]_M = (f \times g) : (\Gamma \times A) \to (\Gamma \times B)}$

0.2 Weakening Theorems

0.2.1 Domain Lemma

If $\omega : \Gamma' \triangleright \Gamma$, then $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$.

Proof

Case Id Then $\Gamma' = \Gamma$ and so $\texttt{dom}(\Gamma') = \texttt{dom}(\Gamma)$.

Case Project By inversion and induction, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma') \subseteq \texttt{dom}(\Gamma' \cup \{x\})$

Case Extend By inversion and induction, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$ so

$$\texttt{dom}(\Gamma, x : A) = \texttt{dom}(\Gamma) \cup \{x\} \subseteq \texttt{dom}(\Gamma') \cup \{x\} = \texttt{dom}(\Gamma', x : A)$$

0.2.2 Theorem 1

If $\omega : \Gamma' \triangleright \Gamma$ and $\Gamma \texttt{Ok}$ then $\Gamma' \texttt{Ok}$

Proof

Case Id

$$(\text{Id}) \frac{\Gamma \texttt{Ok}}{\iota : \Gamma \triangleright \Gamma}$$

By inversion, $\Gamma \texttt{Ok}$.

Case Project

$$(\text{Project})\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\omega : \Gamma' \triangleright \Gamma$ and $x \notin \mathtt{dom}(\Gamma')$.
Hence by induction $\Gamma'\mathtt{Ok}$, $\Gamma\mathtt{Ok}$. Since $x \notin \mathtt{dom}(\Gamma')$, we have $\Gamma', x : A\mathtt{Ok}$.

Case Extend $(\text{Extend})\frac{\omega:\Gamma'\triangleright\Gamma \quad x\notin\mathtt{dom}(\Gamma') \quad A\leq:B}{w\times:\Gamma',x:A\triangleright\Gamma,x:B}$,
By inversion, we have
$\omega : \Gamma' \triangleright \Gamma$, $x \notin \mathtt{dom}(\Gamma')$.
Hence we have $\Gamma\mathtt{Ok}$, $\Gamma'\mathtt{Ok}$, and by the domain Lemma, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$, hence $x \notin \mathtt{dom}(\Gamma)$. Hence, we have $\Gamma, x : A\mathtt{Ok}$ and $\Gamma', x : A\mathtt{Ok}$

0.2.3 Theorem 2

If $\Gamma \vdash t\colon \tau$ and $\omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Gamma' \vdash t\colon \tau$

Proof Proved in parallel with theorem 3 below

0.2.4 Theorem 3

If $\omega : \Gamma' \triangleright \Gamma$ and $\Delta = [\![\Gamma \vdash t\colon \tau]\!]_M$ and $\Delta' = [\![\Gamma' \vdash t\colon \tau]\!]_M$, derived using Theorem 2, then

$$\Delta \circ [\![\omega]\!]_M = \Delta' : \Gamma' \to [\![\tau]\!]_M$$

Proof Below

0.3 Proof of Theorems 2 and 3

We induct over the structure of typing derivations of $\Gamma \vdash t\colon \tau$, assuming $\omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash t\colon \tau$ and show that $\Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M = \Delta'$

0.3.1 Variable Terms

Case Var and Weaken We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\iota : \Gamma \triangleright \Gamma]\!]_M \tag{2}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x\colon A$, such that

$$\Delta_1 = \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \quad \text{By Induction} \tag{3}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Gamma'' \vdash x\colon A}{\Gamma'', x' : A' \vdash x\colon A} \tag{4}$$

This preserves denotations:

$$\begin{aligned} \Delta' &= \Delta_1 \circ \pi_1 \quad \text{By Definition} && (5)\\ &= \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \circ \pi_1 \quad \text{By induction} && (6)\\ &= \Delta \circ [\![\omega'\pi_1 : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By denotation of weakening} && (7)\end{aligned}$$

If $\omega = \omega' \times$ Then

$$\Gamma' = \Gamma''', x' : B \tag{8}$$
$$\Gamma = \Gamma'', x' : A' \tag{9}$$
$$B \leq: A \tag{10}$$

If $x = x'$ Then $A = A'$.
Then we derive the new deriviation, Δ' as so:

$$(\text{Sub-type})\frac{(\text{var})\frac{}{\Gamma''', x{:}B \vdash x{:}B} \quad B \leq: A}{\Gamma' \vdash x{:}\, A} \tag{11}$$

This preserves denotations:

$$\begin{aligned} \Delta' &= [\![B \leq: A]\!]_M \circ \pi_2 \quad \text{By Definition} && (12)\\ &= \pi_2 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![B \leq: A]\!]_M) \quad \text{By the properties of binary products} && (13)\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By Definition} && (14) \end{aligned}$$

Case $x \neq x'$ Then

$$\Delta = (\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x{:}A}}{\Gamma \vdash x{:}\, A} \tag{15}$$

By induction with $\omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x{:}\, A$
We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{()\frac{\Delta_1'}{\Gamma''' \vdash x{:}A}}{\Gamma' \vdash x{:}\, A} \tag{16}$$

This preserves denotations:
By induction, we have

$$\Delta_1' = \Delta_1 \circ [\![\omega : \Gamma''' \triangleright \Gamma'']\!]_M \tag{17}$$

So we have:

$$\begin{aligned} \Delta' &= \Delta_1' \circ \pi_1 \quad \text{By denotation definition} && (18)\\ &= \Delta_1 \circ [\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \quad \text{By induction} \circ \pi_1 && (19)\\ &= \Delta_1 \circ \pi_1 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![A' \leq: B]\!]_M) \quad \text{By product properties} && (20)\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By definition} && (21) \end{aligned}$$

0.3.2 Value Terms

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\omega : \Gamma' \triangleright \Gamma']\!]_M$, simply as ω.

Case Constant The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$(\text{Const})\frac{\Gamma\,\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A{:}\, A} \tag{22}$$

By inversion, we have $\Gamma\mathtt{Ok}$, so we have $\Gamma'\mathtt{Ok}$.
Hence

$$(\text{Const})\frac{\Gamma'\,\mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A{:}\, A} \tag{23}$$

Holds.

This preserves denotations:

$$
\begin{aligned}
\Delta' &= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma'} \quad \text{By definition} && (24)\\
&= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} && (25)\\
&= \Delta \quad \text{By Definition} && (26)\\
& && (27)
\end{aligned}
$$

Case Lambda By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C:\mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\colon A \to \mathtt{M}_\epsilon B} \tag{28}$$

Since $\omega : \Gamma' \triangleright \Gamma$, we have:

$$\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{29}$$

Hence, by induction, using $\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash C:\mathtt{M}_\epsilon B}}{\Gamma', x : A \vdash \lambda x : A.C\colon A \to \mathtt{M}_\epsilon B} \tag{30}$$

This preserves denotations:

$$
\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By Definition} && (31)\\
&= \mathtt{cur}(\Delta_1 \circ (\omega \times \mathtt{Id}_\Gamma)) \quad \text{By the denotation of } \omega\times && (32)\\
&= \mathtt{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} && (33)\\
&= \Delta \circ \omega \quad \text{By Definition} && (34)
\end{aligned}
$$

Case Sub-typing

$$(\text{Sub-type})\frac{\Gamma \vdash v\colon A \quad A \leq\colon B}{\Gamma \vdash v\colon B} \tag{35}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash v\colon A} \tag{36}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-type})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v:a} \quad A \leq\colon B}{\Gamma' \vdash v\colon B} \tag{37}$$

This preserves denotations:

$$
\begin{aligned}
\Delta' &= [\![A \leq\colon B]\!]_M \circ \Delta_1' \quad \text{By Definition} && (38)\\
&= [\![A \leq\colon B]\!]_M \circ \Delta_1 \circ \omega \quad \text{By induction} && (39)\\
&= \Delta \circ \omega \quad \text{By Definition} && (40)\\
& && (41)
\end{aligned}
$$

0.3.3 Computation Terms

Case Return We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma\vdash v:A}}{\Gamma \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{42}$$

Hence, by induction, with $\omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ'_1 such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta'_1}{\Gamma'\vdash v:A}}{\Gamma' \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{43}$$

This preserves denotations:

$$\Delta' = \eta_A \circ \Delta'_1 \quad \text{By definition} \tag{44}$$
$$= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1 \tag{45}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{46}$$

Case Apply By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1:A\to \texttt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma\vdash v_2:A}}{\Gamma \vdash v_1\ v_2\colon \texttt{M}_\epsilon B} \tag{47}$$

By induction, this gives us the respective derivations: Δ'_1, Δ'_2 such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta'_1}{\Gamma'\vdash v_1:A\to \texttt{M}_\epsilon B} \quad ()\frac{\Delta'_2}{\Gamma'\vdash v_2:A}}{\Gamma' \vdash v_1\ v_2\colon \texttt{M}_\epsilon B} \tag{48}$$

This preserves denotations:

$$\Delta' = \texttt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \tag{49}$$
$$= \texttt{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \tag{50}$$
$$= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \tag{51}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{52}$$

Case If By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2\colon \texttt{M}_\epsilon A} \tag{53}$$

By induction, this gives us the sub-derivations $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta'_1}{\Gamma'\vdash v:\texttt{Bool}} \quad ()\frac{\Delta'_2}{\Gamma'\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta'_3}{\Gamma'\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2\colon \texttt{M}_\epsilon A} \tag{54}$$

And

$$\Delta'_1 = \Delta_1 \circ \omega \tag{55}$$
$$\Delta'_3 = \Delta_2 \circ \omega \tag{56}$$
$$\Delta'_3 = \Delta_3 \circ \omega \tag{57}$$

This preserves denotations. Since $\omega : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\omega : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\omega = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{58}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \omega)) = (T_\epsilon A)^\omega \circ \mathtt{cur}(f) \tag{59}$$

$$\begin{aligned}
\Delta' &= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (60)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \omega \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (61)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2 \circ (\mathtt{Id}_{\mathbb{1}} \times \omega)), \mathtt{cur}(\Delta_3 \circ \pi_2 \circ (\mathtt{Id}_{\mathbb{1}} \times \omega))] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (62)\\
&= \mathtt{app} \circ (([(T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\omega \text{ property} && (63)\\
&= \mathtt{app} \circ (((T_\epsilon A)^\omega \circ [\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (64)\\
&= \mathtt{app} \circ ((T_\epsilon A)^\omega \times \mathtt{Id}_{\Gamma'}) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (65)\\
&= \mathtt{app} \circ (\mathtt{Id}_{(T_\epsilon A)} \times \omega) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \mathtt{app}, (T_\epsilon A)^\omega && (66)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (67)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \omega \quad \text{By Definition of the diagonal morphism.} && (68)\\
&= \Delta \circ \omega && (69)
\end{aligned}$$

Case Bind By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Gamma \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{70}$$

If $\omega : \Gamma' \triangleright \Gamma$ then $\omega \times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ_1', Δ_2' such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma' \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2'}{\Gamma', x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{71}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2' \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta_1' \rangle \quad \text{By definition} && (72)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2 \circ (\omega \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta_1', \Delta_2' && (73)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} && (74)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} && (75)\\
&= \Delta \quad \text{By definition} && (76)
\end{aligned}$$

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

Case Sub-effect

$$(\text{Sub-effect})\frac{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_1} A \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_2} B} \tag{77}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_1} A} \tag{78}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma' \vdash C:\mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C\colon \mathtt{M}_{\epsilon_2} B} \tag{79}$$

This preserves denotations:
Let

$$g = [\![A \leq: B]\!]_M : A \to B \tag{80}$$
$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M : T_{\epsilon_1} \to T_{\epsilon_2} \tag{81}$$

Then

$$\Delta' = h_B \circ T_{\epsilon_1} g \circ \Delta_1' \quad \text{By Definition} \tag{82}$$
$$= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \omega \quad \text{By Induction} \tag{83}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{84}$$