

0.1 Terms

0.1.1 Value Terms

$$\begin{aligned}
v ::= & x \\
& | \lambda x : A. C \\
& | \mathbf{c}^A \\
& | () \\
& | \mathbf{true} \mid \mathbf{false} \\
& | \Lambda \alpha. v \\
& | v \epsilon
\end{aligned} \tag{1}$$

0.1.2 Computation Terms

$$\begin{aligned}
C ::= & \mathbf{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 \\
& | v_1 \ v_2 \\
& | \mathbf{do } x \leftarrow C_1 \text{ in } C_2 \\
& | \mathbf{return} v
\end{aligned} \tag{2}$$

0.2 Type System

0.2.1 Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, 1, \leq)$ with ground elements e , and denoted by metavariables ϵ , and in language-effect variables α

0.2.2 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Value Types

$$A, B, C ::= \gamma \mid A \rightarrow \mathbf{M}_\epsilon B \mid \forall \alpha. A$$

Computation Types Computation types are of the form $\mathbf{M}_\epsilon A$

0.2.3 Sub-typing

There exists a sub-typing pre-order relation \leq_γ over ground types that is:

- (Reflexive) $\overline{A \leq_\gamma A}$
- (Transitive) $\frac{A \leq_\gamma B \quad B \leq_\gamma C}{A \leq_\gamma C}$

We extend this relation with the function and effect-lambda sub-typing rules to yield the full sub-typing relation \leq :

- (ground) $\frac{A \leq_\gamma B}{A \leq B}$
- (Fn) $\frac{A \leq A' \quad B' \leq B \quad \epsilon \leq \epsilon'}{A' \rightarrow \mathbf{M}_{\epsilon'} B' \leq A \rightarrow \mathbf{M}_\epsilon B}$
- (All) $\frac{A \leq A'}{\forall \alpha. A \leq \forall a. A'}$

0.2.4 Type and Effect Environments

A type environment is a snoc-list of value-variable, type pairs, $G ::= \diamond \mid \Gamma, x : A$. An effect environment is a snoc-list of effect-variables.

$$\Phi ::= \diamond \mid \Phi, \alpha$$

Domain Function on Type Environments

- $\text{dom}(\diamond) = \emptyset$
- $\text{dom}(\Gamma, x : A) = \text{dom}(\Gamma) \cup \{x\}$

Membership of Effect Environments Informally, $\alpha \in \Phi$ if α appears in the list represented by Φ .

Ok Predicate On Effect Environments

- $(\text{Atom}) \frac{}{\diamond \text{Ok}}$
- $(A) \frac{\Phi \text{Ok} \quad \alpha \notin \Phi}{\Phi, \alpha \text{Ok}}$

Well-Formed-ness of effects We define a relation $\Phi \vdash \epsilon$.

- $(\text{Ground}) \frac{\Phi \text{Ok}}{\Phi \vdash_e}$
- $(\text{Var}) \frac{\Phi, \alpha \text{Ok}}{\Phi, \alpha \vdash \alpha}$
- $(\text{Weaken}) \frac{\Phi \vdash \alpha}{\Phi, \beta \vdash \alpha} \text{ (if } \alpha \neq \beta \text{)}$
- $(\text{Monoid Op}) \frac{\Phi \vdash \epsilon_1 \quad \Phi \vdash \epsilon_2}{\Phi \vdash \epsilon_1 \cdot \epsilon_2}$

Well-Formed-ness of Types We define a relation $\Phi \vdash \tau$ on types.

- $(\text{Ground}) \frac{}{\Phi \vdash \gamma}$
- $(\text{Lambda}) \frac{\Phi \vdash A \quad \Phi \vdash M_\epsilon B}{\Phi \vdash A \rightarrow M_\epsilon B}$
- $(\text{Computation}) \frac{\Phi \vdash A \quad \Phi \vdash \epsilon}{\Phi \vdash M_\epsilon A}$
- $(\text{For-All}) \frac{\Phi, \alpha \vdash A}{\Phi \vdash \forall \alpha. A}$

Ok Predicate on Type Environments We now define a predicate on type environments and effect environments: $\Phi \vdash \Gamma \text{Ok}$

- $(\text{Nil}) \frac{}{\Phi \vdash \diamond \text{Ok}}$
- $(\text{Var}) \frac{\Phi \vdash \Gamma \text{Ok} \quad \alpha \notin \text{dom}(\Gamma) \quad \Phi \vdash A}{\Phi \vdash \Gamma, x : A \text{Ok}}$

0.2.5 Type Rules

Value Typing Rules

- (Const) $\frac{\Phi \vdash \Gamma \mathbf{Ok} \quad \Phi \vdash A}{\Phi \mid \Gamma \vdash \mathbf{C}^A.A}$
- (Unit) $\frac{\Phi \vdash \Gamma \mathbf{Ok}}{\Phi \mid \Gamma \vdash ().\mathbf{Unit}}$
- (True) $\frac{\Phi \vdash \Gamma \mathbf{Ok}}{\Phi \mid \Gamma \vdash \mathbf{true}.\mathbf{Bool}}$
- (False) $\frac{\Phi \vdash \Gamma \mathbf{Ok}}{\Phi \mid \Gamma \vdash \mathbf{false}.\mathbf{Bool}}$
- (Var) $\frac{\Phi \vdash \Gamma, x:A \mathbf{Ok}}{\Phi \mid \Gamma, x:A \vdash x:A}$
- (Weaken) $\frac{\Phi \mid \Gamma \vdash x:A}{\Phi \mid \Gamma, y:B \vdash x:A} \text{ (if } x \neq y)$
- (Fn) $\frac{\Phi \mid \Gamma, x:A \vdash C:\mathbf{M}_\epsilon B}{\Phi \mid \Gamma \vdash \lambda x:A. C:A \rightarrow \mathbf{M}_\epsilon B}$
- (Sub) $\frac{\Phi \mid \Gamma \vdash v:A \quad A \leq B}{\Phi \mid \Gamma \vdash v:B}$
- (Effect-Abs) $\frac{\Phi, \alpha \mid \Gamma \vdash v:A}{\Phi \mid \Gamma \vdash \Lambda \alpha.v:\forall \alpha.A}$
- (Effect-apply) $\frac{\Phi \mid \Gamma \vdash v:\forall \alpha.A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v \epsilon:A[\epsilon/\alpha]}$

Computation typing rules

- (Return) $\frac{\Phi \mid \Gamma \vdash v:A}{\Phi \mid \Gamma \vdash \mathbf{return} v:\mathbf{M}_1 A}$
- (Apply) $\frac{\Phi \mid \Gamma \vdash v_1:A \rightarrow \mathbf{M}_\epsilon B \quad \Phi \mid \Gamma \vdash v_2:A}{\Phi \mid \Gamma \vdash v_1 v_2:\mathbf{M}_\epsilon B}$
- (if) $\frac{\Phi \mid \Gamma \vdash v:\mathbf{Bool} \quad \Phi \mid \Gamma \vdash C_1:\mathbf{M}_\epsilon A \quad \Phi \mid \Gamma \vdash C_2:\mathbf{M}_\epsilon A}{\Phi \mid \Gamma \vdash \mathbf{if}_{\epsilon,A} v \mathbf{then} C_1 \mathbf{else} C_2:\mathbf{M}_\epsilon A}$
- (Do) $\frac{\Phi \mid \Gamma \vdash C_1:\mathbf{M}_{\epsilon_1} A \quad \Phi \mid \Gamma, x:A \vdash C_2:\mathbf{M}_{\epsilon_2} B}{\Phi \mid \Gamma \vdash \mathbf{do} x \leftarrow C_1 \mathbf{in} C_2:\mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (Subeffect) $\frac{\Phi \mid \Gamma \vdash C:\mathbf{M}_{\epsilon_1} A \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Phi \mid \Gamma \vdash C:\mathbf{M}_{\epsilon_2} B}$

0.2.6 Ok Lemma

If $\Phi \mid \Gamma \vdash t:\tau$ then $\Phi \vdash \Gamma \mathbf{Ok}$.

Proof If $\Gamma, x:A \mathbf{Ok}$ then by inversion $\Gamma \mathbf{Ok}$. Only the type rule **Weaken** adds terms to the environment from its preconditions to its post-condition and it does so in an **Ok** preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Phi \vdash \Gamma \mathbf{Ok}$. And all non-axiom derivations preserve the **Ok** property.

0.3 Beta-Eta-Equivalence

0.3.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Phi|\Gamma, x:A \vdash C:\mathbb{M}_\epsilon B \quad \Phi|\Gamma \vdash v:A}{\Phi|\Gamma \vdash (\lambda x:A. C) v =_{\beta_\eta} C[x/v]:\mathbb{M}_\epsilon B}$
- (Lambda-Eta) $\frac{\Phi|\Gamma \vdash v:A \rightarrow \mathbb{M}_\epsilon B}{\Phi|\Gamma \vdash \lambda x:A. (v x) =_{\beta_\eta} v:A \rightarrow \mathbb{M}_\epsilon B}$
- (Left Unit) $\frac{\Phi|\Gamma \vdash v:A \quad \Phi|\Gamma, x:A \vdash C:\mathbb{M}_\epsilon B}{\Phi|\Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C =_{\beta_\eta} C[V/x]:\mathbb{M}_\epsilon B}$
- (Right Unit) $\frac{\Phi|\Gamma \vdash C:\mathbb{M}_\epsilon A}{\Phi|\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x =_{\beta_\eta} C:\mathbb{M}_\epsilon A}$
- (Associativity) $\frac{\Phi|\Gamma \vdash C_1:\mathbb{M}_{\epsilon_1} A \quad \Phi|\Gamma, x:A \vdash C_2:\mathbb{M}_{\epsilon_2} B \quad \Phi|\Gamma, y:B \vdash C_3:\mathbb{M}_{\epsilon_3} C}{\Phi|\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) =_{\beta_\eta} \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$
- (Unit) $\frac{\Phi|\Gamma \vdash v:\text{Unit}}{\Phi|\Gamma \vdash v =_{\beta_\eta} ():\text{Unit}}$
- (if-true) $\frac{\Phi|\Gamma \vdash C_1:\mathbb{M}_\epsilon A \quad \Phi|\Gamma \vdash C_2:\mathbb{M}_\epsilon A}{\Phi|\Gamma \vdash \text{if}_{\epsilon, A} \text{ true then } C_1 \text{ else } C_2 =_{\beta_\eta} C_1:\mathbb{M}_\epsilon A}$
- (if-false) $\frac{\Phi|\Gamma \vdash C_2:\mathbb{M}_\epsilon A \quad \Phi|\Gamma \vdash C_1:\mathbb{M}_\epsilon A}{\Phi|\Gamma \vdash \text{if}_{\epsilon, A} \text{ false then } C_1 \text{ else } C_2 =_{\beta_\eta} C_2:\mathbb{M}_\epsilon A}$
- (If-Eta) $\frac{\Phi|\Gamma, x:\text{Bool} \vdash C:\mathbb{M}_\epsilon A \quad \Phi|\Gamma \vdash v:\text{Bool}}{\Phi|\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C[\text{true}/x] \text{ else } C[\text{false}/x] =_{\beta_\eta} C[V/x]:\mathbb{M}_\epsilon A}$
- (Effect-beta) $\frac{\Phi \vdash \epsilon \quad \Phi, \alpha|\Gamma \vdash v:A}{\Phi|\Gamma \vdash \Lambda \alpha. v =_{\beta_\eta} v[\epsilon/\alpha]:A[\epsilon/\alpha]}$
- (Effect-eta) $\frac{\Phi, \alpha|\Gamma \vdash v:A \quad \Phi \vdash \beta}{\Phi|\Gamma \vdash \Lambda \alpha. (v \beta) =_{\beta_\eta} v[\beta/\alpha]:A[\beta/\alpha]}$

0.3.2 Equivalence Relation

- (Reflexive) $\frac{\Phi|\Gamma \vdash t:\tau}{\Phi|\Gamma \vdash t =_{\beta_\eta} t:\tau}$
- (Symmetric) $\frac{\Phi|\Gamma \vdash t_1 =_{\beta_\eta} t_2:\tau}{\Phi|\Gamma \vdash t_2 =_{\beta_\eta} t_1:\tau}$
- (Transitive) $\frac{\Phi|\Gamma \vdash t_1 =_{\beta_\eta} t_2:\tau \quad \Phi|\Gamma \vdash t_2 =_{\beta_\eta} t_3:\tau}{\Phi|\Gamma \vdash t_1 =_{\beta_\eta} t_3:\tau}$

0.3.3 Congruences

- (Effect-Abs) $\frac{\Phi, \alpha|\Gamma \vdash v_1 =_{\beta_\eta} v_2:A}{\Phi|\Gamma \vdash \Lambda \alpha. v_1 =_{\beta_\eta} \Lambda \alpha. v_2:\forall \alpha. A}$
- (Effect-Apply) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta_\eta} v_2:\forall \alpha. A \quad \Phi \vdash \epsilon}{\Phi|\Gamma \vdash v_1 \epsilon =_{\beta_\eta} v_2 \epsilon:A[\epsilon/\alpha]}$
- (Lambda) $\frac{\Phi|\Gamma, x:A \vdash C_1 =_{\beta_\eta} C_2:\mathbb{M}_\epsilon B}{\Phi|\Gamma \vdash \lambda x:A. C_1 =_{\beta_\eta} \lambda x:A. C_2:A \rightarrow \mathbb{M}_\epsilon B}$
- (Return) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta_\eta} v_2:A}{\Phi|\Gamma \vdash \text{return } v_1 =_{\beta_\eta} \text{return } v_2:\mathbb{M}_1 A}$
- (Apply) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta_\eta} v'_1:A \rightarrow \mathbb{M}_\epsilon B \quad \Phi|\Gamma \vdash v_2 =_{\beta_\eta} v'_2:A}{\Phi|\Gamma \vdash v_1 v_2 =_{\beta_\eta} v'_1 v'_2:\mathbb{M}_\epsilon B}$
- (Bind) $\frac{\Phi|\Gamma \vdash C_1 =_{\beta_\eta} C'_1:\mathbb{M}_{\epsilon_1} A \quad \Phi|\Gamma, x:A \vdash C_2 =_{\beta_\eta} C'_2:\mathbb{M}_{\epsilon_2} B}{\Phi|\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 =_{\beta_\eta} \text{do } x \leftarrow C'_1 \text{ in } C'_2:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (If) $\frac{\Phi|\Gamma \vdash v =_{\beta_\eta} v':\text{Bool} \quad \Phi|\Gamma \vdash C_1 =_{\beta_\eta} C'_1:\mathbb{M}_{\epsilon_1} A \quad \Phi|\Gamma \vdash C_2 =_{\beta_\eta} C'_2:\mathbb{M}_{\epsilon_2} A}{\Phi|\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 =_{\beta_\eta} \text{if}_{\epsilon, A} v \text{ then } C'_1 \text{ else } C'_2:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A}$

- (Subtype) $\frac{\Phi|\Gamma \vdash v =_{\beta\eta} v' : A \quad A \leq B}{\Phi|\Gamma \vdash v =_{\beta\eta} v' : B}$
- (Subeffect) $\frac{\Phi|\Gamma \vdash C =_{\beta\eta} C' : \mathbf{M}_{\epsilon_1} A \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Phi|\Gamma \vdash C =_{\beta\eta} C' : \mathbf{M}_{\epsilon_2} B}$