We need to define substitutions of effects on effects, effects on types, effects on terms, terms on terms.

0.1 Effect Substitutions

Define a substitution, σ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \tag{1}$$

Define the free-effect Variables of σ:

$$\begin{aligned} fev(\diamond) &= \emptyset \\ fev(\sigma, \alpha := \epsilon) &= fev(\sigma) \cup fev(\epsilon) \end{aligned}$$

We define the property:

$$\alpha \# \sigma \Leftrightarrow \alpha \notin (\mathtt{dom}(\sigma) \cup fev(\sigma)) \tag{2}$$

0.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \tag{3}$$

$$\sigma(e) = e \tag{4}$$
$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \tag{5}$$
$$\diamond(\alpha) = \alpha \tag{6}$$
$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \tag{7}$$
$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \tag{8}$$

0.1.2 Action of Effect Substitution on Types

Define the action of applying an effect substitution, σ to a type τ as:

$$\tau\left[\sigma\right]$$

Defined as so

$$\gamma\left[\sigma\right] = \gamma \tag{9}$$
$$(A \to \mathtt{M}_\epsilon B)\left[\sigma\right] = (A\left[\sigma\right]) \to \mathtt{M}_{\sigma(\epsilon)}(B\left[\sigma\right]) \tag{10}$$
$$(\mathtt{M}_\epsilon A)\left[\sigma\right] = \mathtt{M}_{\sigma(\epsilon)}(A\left[\sigma\right]) \tag{11}$$
$$(\forall \alpha . A)\left[\sigma\right] = \forall \alpha . (A\left[\sigma\right]) \quad \text{If } \alpha \# \sigma \tag{12}$$

0.1.3 Action of Effect-Substitution on Type Environments

Define the action of effect substitution on type environments:

$$\Gamma\left[\sigma\right]$$

Defined as so:

$$\begin{aligned} \diamond\left[\sigma\right] &= \diamond \\ (\Gamma, x : A)\left[\sigma\right] &= (\Gamma\left[\sigma\right], x : (A\left[\sigma\right])) \end{aligned}$$

0.1.4 Action of Effect Substitution on Terms

Define the action of effect-substitution on terms:

$$x\,[\sigma] = x \tag{13}$$
$$\mathtt{C}^{A}\,[\sigma] = \mathtt{C}^{(A[\sigma])} \tag{14}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : (A\,[\sigma]).(C\,[\sigma]) \tag{15}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\sigma(\epsilon),(A[\sigma])}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{16}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{17}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \tag{18}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{19}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \sigma(\epsilon) \tag{20}$$
$$\tag{21}$$

0.1.5 Well-Formed-ness

For any two effect-environments, and a substitution, define the well-formed-ness relation:

$$\Phi' \vdash \sigma\colon \Phi \tag{22}$$

- (Nil) $\frac{\Phi'\mathtt{Ok}}{\Phi' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi' \vdash \sigma:\Phi \quad \Phi' \vdash \epsilon \quad \alpha \notin \Phi}{\Phi' \vdash \sigma, \alpha := \epsilon : (\Phi, \alpha)}$

0.1.6 Property 1

If $\Phi' \vdash \sigma\colon \Phi$ then $\Phi'\mathtt{Ok}$ (By the Nil case) and $\Phi\mathtt{Ok}$ Since each use of the extend case preserves $\mathtt{Ok}$.

0.1.7 Property 2

If $\Phi' \vdash \sigma\colon \Phi$ then $\omega : \Phi' \triangleright \Phi' \implies \Phi'' \vdash \sigma\colon \Phi$ since $\Phi' \vdash \epsilon \implies \Phi'' \vdash \epsilon$ and $\Phi'\mathtt{Ok} \implies \Phi''\mathtt{Ok}$

0.1.8 Property 3

If $\Phi' \vdash \sigma\colon \Phi$ then

$$\alpha \notin \Phi \wedge \alpha \notin \Phi' \implies (\Phi', \alpha) \vdash (\sigma, \alpha := \alpha)\colon (\Phi, \alpha) \tag{23}$$

Since $\iota\pi : \Phi', \alpha \triangleright \Phi'$ so $\Phi', \alpha \vdash \sigma\colon \Phi$ and $\Phi', \alpha \vdash \alpha$

0.2 Substitution Preserves the Well-formed-ness of Effects

I.e.

$$\Phi \vdash \epsilon \wedge \Phi' \vdash \iota\colon \Phi \implies \Phi' \vdash \sigma(\epsilon) \tag{24}$$

Proof:

Case Ground: $\sigma(e) = e$, so $\Phi' \vdash \sigma(\epsilon)$ holds.

Case Multiply: By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$ so $\Phi' \vdash \sigma(\epsilon_1)$ and $\Phi' \vdash \sigma(\epsilon_2)$ by induction and hence $\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)$

Case Var: By inversion, $\Phi = \Phi'', \alpha$ and $\Phi'', \alpha\mathtt{Ok}$. Hence byu case splitting on ι, we see that $\sigma = \sigma', \alpha := \epsilon$.

So by inversion, $\sigma \vdash \epsilon$ so $\Phi' \vdash \sigma(\alpha) = \epsilon$

Case Weaken: By inversion$\Phi = \Phi'', \beta$ and $\Phi'' \vdash \alpha$, so $\sigma = \sigma'\beta := \epsilon$.

So $\Phi' \vdash \sigma' \colon \Phi''$.

hence by induction, $\Phi' \vdash \sigma'(a)$, so $\Phi' \vdash \sigma(\alpha)$ since $\alpha \neq \beta$)

0.2.1 Effect Substitution preserves the sub-effect relation

If $\Phi' \vdash \sigma \colon \Phi$ and $\epsilon_1 \leq_\Phi \epsilon_2$, then $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

Proof: For any ground substitution σ' of Φ', then $\sigma\sigma'$ (the substitution σ' applied after σ) is also a ground substitution.

So $\epsilon_1 [\sigma] [\sigma'] \leq \epsilon_2 [\sigma] [\sigma']$.

So $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

0.2.2 Substitution preserves well-formed-ness of Types

$$\Phi' \vdash \sigma \colon \Phi \wedge \Phi \vdash A \implies \Phi' \vdash A [\sigma] \tag{25}$$

Proof:

Case Ground: $\Phi'\mathtt{Ok}$ so $\Phi' \vdash \gamma$ and $\gamma [\sigma] = \gamma$.

Hence $\Phi' \vdash \gamma [\sigma]$.

Case Lambda: By inversion$\Phi \vdash A$ and $\Phi \vdash B$.

So by induction, $\Phi' \vdash A [\sigma]$ and $\Phi' \vdash B [\sigma]$.

So

$$\Phi' \vdash (A [\sigma]) \to (B [\sigma]) \tag{26}$$

So

$$\Phi' \vdash (A \to B) [\sigma] \tag{27}$$

Case Computation: By inversion, $\Phi \vdash \epsilon$ and $\Phi \vdash A$ so by induction and substitution of effect preserving effect-well-formed-ness,

$\Phi' \vdash \sigma(\epsilon)$ and $\Phi' \vdash A [\sigma]$ so $\Phi \vdash \mathtt{M}_{\sigma(\epsilon)} A [\sigma]$ so $\Phi' \vdash (\mathtt{M}_\epsilon A) [\sigma]$

Case For All: By inversion, $\Phi, \alpha \vdash A$. So by picking $\alpha \notin \Phi \wedge \alpha \notin \Phi'$ using α-equivalence, we have $(\Phi', \alpha) \vdash (\sigma\alpha := \alpha) \colon (\Phi, \alpha)$.

So by induction $(\Phi, \alpha) \vdash A [\sigma, \alpha := \alpha]$

So $(\Phi', \alpha) \vdash A [\sigma]$

So $\Phi' \vdash (\forall\alpha.A) [\sigma]$

0.2.3 Substitution of effects preserves Sub-Typing Relation

If $\Phi' \vdash \sigma \colon \Phi$ and $A \leq:_\Phi B$ then $A [\sigma] \leq:_{\Phi'} B [\sigma]$

Proof: By induction on the sub-typing relation

Case Ground: By inversion, $A \leq:_\gamma B$, so A, B are ground types. Hence $A [\sigma] = A$ and $B [\sigma] = B$. So $A [\sigma] \leq:_{\Phi'} B [\sigma]$

Case Fn: By inversion, $A' \leq:_{\Phi} A$ and $B \leq:_{\Phi} B'$.
So by induction, $A'[\sigma] \leq:_{\Phi'} A[\sigma]$ and $B[\sigma] \leq:_{\Phi'} B'[\sigma]$.
So $(A[\sigma]) \to (B[\sigma]) \leq:_{\Phi'} (A'[\sigma]) \to (B'[\sigma])$
So $(A \to B)[\sigma] \leq:_{\Phi'} (A' \to B')[\sigma]$

Case Computation: By inversion, $A \leq:_{\Phi} B$, $\epsilon_1 \leq_{\Phi} \epsilon_2$.
So by induction and substitution preserving the sub-effect relation,
$A[\sigma] \leq:_{\Phi'} B[\sigma]$ and $\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)$
So $\mathtt{M}_{\sigma(\epsilon_1)}(A[\sigma]) \leq:_{\Phi'} \mathtt{M}_{\sigma(\epsilon_2)}(B[\sigma])$
So $(\mathtt{M}_{\epsilon_1} A)[\sigma] \leq:_{\Phi'} (\mathtt{M}_{\epsilon_2} B)[\sigma]$

0.2.4 Substitution preserves well-formed-ness of Type Environments

If $\Phi \vdash \Gamma \mathtt{Ok}$ and $\Phi' \vdash \sigma \colon \Phi$ then $\Phi' \vdash \Gamma[\sigma]\, \mathtt{Ok}$

Proof:

Case Nil: $\Phi \mathtt{Ok} \implies \Phi' \mathtt{Ok}$ so $\Phi' \vdash \diamond \mathtt{Ok}$ and $\diamond[\sigma] = \diamond$

Case Var: By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$ and $\Phi \vdash A$.
By induction and substitution preserving well-formed-ness of types, $\Phi' \vdash \Gamma'[\sigma]\, \mathtt{Ok}$ and $\Phi' \vdash A[\sigma]$.
So $\Phi' \vdash (\Gamma'[\sigma], x : A[\sigma])\mathtt{Ok}$.
Hence $\Phi' \vdash \Gamma, x : A[\sigma]\, \mathtt{Ok}$.

0.2.5 Effect-Polymorphism Preserves the Typing Relation

If $\Phi' \vdash \sigma \colon \Phi$ and $\Phi \mid \Gamma \vdash v \colon A$, then $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$

Proof:

Case Const: By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$.
So $\Phi' \vdash \Gamma \mathtt{Ok}$
So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{C}^{A[\sigma]} \colon A[\sigma]$

Case True, False, Unit: The logic is the same for each of these cases, so we look at the case `true` only.
By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$.
So $\Phi' \vdash \Gamma \mathtt{Ok}$
So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{true} \colon \mathtt{Bool}$
Since $\mathtt{true}[\sigma] = \mathtt{true}$ and $\mathtt{Bool}[\sigma] = \mathtt{Bool}$.

Case Var: By inversion$\Gamma = \Gamma', x : A$ and $\Phi \vdash \Gamma', x : A \mathtt{Ok}$.
So since substitution preserves well-formed-ness of type environments, $\Phi' \vdash \Gamma'[\sigma], x : A[\sigma]\, \mathtt{Ok}$
So $\Phi' \mid \Gamma[\sigma] \vdash x \colon A[\sigma]$
Since $x[\sigma] = x$

Case Weaken: By inversion$\Gamma = \Gamma', y : B$, $\Phi \vdash B$, and $\Phi \mid \Gamma' \vdash x \colon A$. $x \neq y$
By induction and the theorem that effect-substitution preserves type well-formed-ness, we have: $\Phi' \mid \Gamma'[\sigma] \vdash x \colon A[\sigma]$ and $\Phi' \vdash B[\sigma]$
So $\Phi' \mid \Gamma[\sigma] \vdash x[\sigma] : A[\sigma]$
Since $x[\sigma] = x$, $\Gamma[\sigma] = (\Gamma'[\sigma], y : B[\sigma])$

Case Lambda: By inversion$\Phi \mid \Gamma, x : A \vdash v\colon B$.
So, by induction $\Phi' \mid (\Gamma, x : A)[\sigma] \vdash v[\sigma] : B[\sigma]$.
So, $\Phi \mid \Gamma[\sigma], x : A[\sigma] \vdash v[\sigma] : B[\sigma]$.
Hence by the lambda type rule,
$\Phi' \mid \Gamma[\sigma] \vdash \lambda x : A[\sigma].v[\sigma] : (A[\sigma]) \to (B[\sigma])$
So
$\Phi' \mid \Gamma[\sigma] \vdash (\lambda x : A.v)[\sigma] : (A \to B)[\sigma])$

Case Apply: By inversion, $\Phi \mid \Gamma \vdash v_1\colon A \to B$, $\Phi \mid \Gamma \vdash V_2\colon A$.
So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : (A[\sigma]) \to (B[\sigma])$.
So $\Phi' \mid \Gamma[\sigma] \vdash (v_1[\sigma])\ (v_2[\sigma])\colon B[\sigma]$.
So $\Phi' \mid \Gamma[\sigma] \vdash (v_1\ v_2)[\sigma] : (A \to B)[\sigma]$

Case Subtype: By inversion, $\Phi \mid \Gamma \vdash v\colon A$ and $\Phi \vdash A \leq: B$
So by induction and effect-substitution preserving sub-typing, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ and $\Phi' \vdash A[\sigma] \leq: B[\sigma]$
So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : B[\sigma]$

Case Return: By inversion, $\Phi \mid \Gamma \vdash v\colon A$
So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$
So $\Phi' \mid \Gamma[\sigma] \vdash \texttt{return}(v[\sigma])\colon \texttt{M}_{\mathbf{1}}(A[\sigma])$
Hence $\Phi' \mid \Gamma[\sigma] \vdash (\texttt{return}v)[\sigma] : (\texttt{M}_{\mathbf{1}} A)[\sigma]$

Case Bind: By inversion, $\Phi \mid \Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2\colon \texttt{M}_{\epsilon_2} B$.
So by induction: $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : \texttt{M}_{\sigma(\epsilon_1)}(A[\sigma])$, and $\Phi' \mid \Gamma[\sigma], x : A[\sigma] \vdash v_2\colon \texttt{M}_{\sigma(\epsilon_2)}(B[\sigma])$.
And so $\Phi' \mid \Gamma[\sigma] \vdash \texttt{do}\ x \leftarrow (v_1[\sigma])\ \texttt{in}\ (v_2[\sigma])\colon \texttt{M}_{\sigma(\epsilon_1)\cdot(\epsilon_2[\sigma])} B[\sigma]$

Case If: By inversion, $\Phi \mid \Gamma \vdash v\colon \texttt{Bool}$, $\Phi \mid \Gamma \vdash v_1\colon A$, and $\Phi \mid \Gamma \vdash v_2\colon A$
So by induction $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \texttt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_1\colon A[\sigma]$, and $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \texttt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_2\colon A[\sigma]$. (Since $\texttt{Bool}[\sigma] = \texttt{Bool}$)
Hence:
$\Phi' \mid \Gamma[\sigma] \vdash \texttt{if}_{A[\sigma]}\ v[\sigma]\ \texttt{then}\ v_1[\sigma]\ \texttt{else}\ v_2[\sigma] : A[\sigma]$
So $\Phi' \mid \Gamma[\sigma] \vdash (\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2)[\sigma] : A[\sigma]$

Case Effect-lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v\colon A$.
So by the substitution property 3 (**TODO: Is this correct/reference correctly**), pick $\alpha \notin \Phi' \wedge \alpha \notin \Phi$ so we have:

$$(\Phi', \alpha) \vdash (\sigma, \alpha := \alpha)\colon (\Phi, \alpha)$$

So by induction, $\Phi', \alpha \mid \Gamma[\sigma, \alpha := \alpha] \vdash v[\sigma, \alpha := \alpha] : A[\sigma, \alpha := \alpha]$
So $\Phi', \alpha \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ since $\alpha \notin \Phi' \wedge \alpha \notin \Phi$.
So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : (\forall \alpha.A)[\sigma]$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v\colon \forall \alpha.A$, $\Phi \vdash \epsilon$.
So by induction and effect-substitution preserving well-formed-ness of effects: $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : (\forall \alpha.A)[\sigma]$ and $\Phi' \vdash \sigma(\epsilon)$
So $\Phi' \mid \Gamma[\sigma] \vdash (v[\sigma])\ (\sigma(\epsilon))\colon A[\sigma][\sigma(\epsilon)/\alpha]$.
Since $\alpha \# \sigma$, we can commute the applications of substitution. **TODO: Do I need to prove this?**
So, $\Phi' \mid \Gamma[\sigma] \vdash (v\ \epsilon)[\sigma] : A[\epsilon/\alpha][\sigma]$

0.3 The Identity Substitution on Effect Environments

For each type environment Φ, define the identity substitution I_Φ as so:

- $I_\diamond = \diamond$
- $I_{(\Phi,\alpha} = (I_\Phi, \alpha := \alpha)$

0.3.1 Properties of the Identity Substitution

Property 1 If $\Phi\mathtt{Ok}$ then $\Phi \vdash I_\Phi \colon \Phi$, proved trivially by induction over the $\mathtt{Ok}$ relation.

Property 2 TODO: The denotational property of id-substitution

0.4 Single Substitution on Effect Environments

If $\Phi \vdash \epsilon$, let the single substitution $\Phi \vdash [\epsilon/\alpha] : \Phi, \alpha$, be defined as:

$$[x/\alpha] = (I_\Phi, \alpha := \epsilon) \tag{28}$$

0.5 Term-Term Substitutions

0.5.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{29}$$

0.5.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{30}$$

$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{31}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{32}$$

$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{33}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{34}$$

0.5.3 Action of substitutions

We define the action of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{35}$$
$$x\,[\sigma, x := v] = v \tag{36}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{37}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{38}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{39}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{40}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{41}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{42}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \tag{43}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \epsilon \tag{44}$$
$$\tag{45}$$

0.5.4 Well-Formed-ness

Define the relation

$$\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$$

by:

- (Nil) $\frac{\Phi \vdash \Gamma' \mathtt{Ok}}{\Phi \mid \Gamma' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi \mid \Gamma' \vdash \sigma : \Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Phi \mid \Gamma' \vdash v : A}{\Phi \mid \Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

0.5.5 Simple Properties Of Substitution

If $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$ then: **TODO: Number these**

Property 1: $\Phi \vdash \Gamma\mathtt{Ok}$ **and** $\Phi \vdash \Gamma'\mathtt{Ok}$ Since $\Phi \vdash \Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Phi \vdash \Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ **implies** $\Phi \mid \Gamma'' \vdash \sigma\colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Phi \mid \Gamma'' \vdash v\colon A$ holds if $\Phi \mid \Gamma' \vdash v\colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ **implies** $\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x)\colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Phi \mid \Gamma', x : A \vdash \sigma\colon \Gamma$$

In addition, $\Phi \mid \Gamma', x : A \vdash x\colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := v)\colon (\Gamma, x : A) \tag{46}$$

0.5.6 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$(\Phi \mid \Gamma \vdash v\colon A) \wedge (\Phi \mid \Gamma' \vdash \sigma\colon \Gamma) \Rightarrow (\Phi \mid \Gamma' \vdash v\,[\sigma] : A) \tag{47}$$

Assuming $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$, we induct over the typing relation, proving $\Phi \mid \Gamma \vdash v\colon A \implies \Phi \mid \Gamma' \vdash v\colon A$

Proof:

Case Var: By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Phi \mid \Gamma'', x : A \vdash x \colon A \tag{48}$$

So by inversion, since $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma'', x : A$,

$$\sigma = (\sigma', x := v) \wedge \Phi \mid \Gamma' \vdash v \colon A \tag{49}$$

By the definition of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{50}$$

holds.

Case Weaken: By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma'', y : B \vdash x \colon A} \tag{51}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Phi \mid \Gamma' \vdash \sigma' \colon \Gamma'' \tag{52}$$

So by induction,

$$\Phi \mid \Gamma' \vdash x\,[\sigma'] : A \tag{53}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{54}$$

Case Lambda: By inversion, there exists Δ such that:

$$(\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A \to B} \tag{55}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x) \colon \Gamma, x : A \tag{56}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{()\frac{\Delta'}{\Phi|\Gamma',x:A\vdash v[\sigma,x:=v]:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\,[\sigma, x := x] : A \to B} \tag{57}$$

Since $\lambda x : A.(v\,[\sigma, x := x]) = \lambda x : A.(v\,[\sigma]) = (\lambda x : A.v)\,[\sigma]$, we have a typing derivation for $\Phi \mid \Gamma' \vdash (\lambda x : A.v)\,[\sigma] : A \to B$.

Case Constants: We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:
$\Phi \mid \Gamma \vdash \sigma \colon \Gamma \Rightarrow \Phi \vdash \Gamma' \mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{58}$$

So

$$(\text{Const})\frac{\Phi \vdash \Gamma' \mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A \colon A} \tag{59}$$

0.5.7 Computation Terms

Case Return: By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return} v \colon \mathtt{M_1} A} \tag{60}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \mathtt{return}(v\,[\sigma]) \colon \mathtt{M_1} A} \tag{61}$$

Since $(\mathtt{return} v)\,[\sigma] = \mathtt{return}(v\,[\sigma])$, the type derivation above holds for $\Phi \mid \Gamma' \vdash (\mathtt{return} v)\,[\sigma] : \mathtt{M_1} A$.

Case Apply: By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B} \tag{62}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (v_1\,[\sigma])\ (v_2\,[\sigma]) \colon B} \tag{63}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Phi \mid \Gamma' \vdash (v_1\ v_2)\,[\sigma] : B$

Case If: By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$(\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{64}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$(\text{If})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v[\sigma]:\mathtt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash \mathtt{if}_A\ (v\,[\sigma])\ \mathtt{then}\ (v_1\,[\sigma])\ \mathtt{else}\ (v_2\,[\sigma]) \colon A} \tag{65}$$

Since $(\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)\,[\sigma] = \mathtt{if}_A\ (v\,[\sigma])\ \mathtt{then}\ (v_1\,[\sigma])\ \mathtt{else}\ (v_2\,[\sigma])$ The derivation above holds for $\Phi \mid \Gamma' \vdash (\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)\,[\sigma] : A$

Case Bind: By inversion, there exist Δ_1, Δ_2 such that:

$$(\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{66}$$

Using alpha-equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$. Hence by property 3,

$$\Phi \mid (\Gamma, x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$(\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma',x:A\vdash v_2[\sigma,x:=x]:\mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \mathtt{do}\ x \leftarrow (v_1\,[\sigma])\ \mathtt{in}\ (v_2\,[\sigma, x := x]) \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{67}$$

Since $(\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2)\,[\sigma] = \mathtt{do}\ x \leftarrow (v_1\,[\sigma])\ \mathtt{in}\ (v_2\,[\sigma]) = \mathtt{do}\ x \leftarrow (v_1\,[\sigma])\ \mathtt{in}\ (v_2\,[\sigma, x := x])$, the above derivation holds for $\Phi \mid \Gamma' \vdash (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2)\,[\sigma] : \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B$

Case Sub-type: By inversion, there exists Δ such that

$$\text{(sub-type)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v\colon B} \tag{68}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\frac{()\frac{\Delta'}{\Phi|\Gamma'\vdash v[\sigma]:A} \quad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v\,[\sigma]: B} \tag{69}$$

Case Effect-Lambda: By inversion, there exists Δ such that

$$\text{(Effect-abs)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{70}$$

It is also the case that $\iota\pi : \Phi, \alpha \triangleright \Phi$.
So $\Phi, \alpha \mid \Gamma' \vdash \sigma\colon \Gamma$
So by induction there exists Δ',

$$\text{(Effect-abs)}\frac{()\frac{\Delta'}{\Phi,\alpha|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.(v\,[\sigma])\colon \forall\alpha.A} \tag{71}$$

Where $\Lambda\alpha.(v\,[\sigma]) = (\Lambda\alpha.v)\,[\sigma]$

Case Effect Application: By inversion$\Phi \vdash \epsilon$ and there exists Δ such that

$$\text{(Effect-App)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.A}}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{72}$$

So by induction there exists Δ' such that:

$$\text{(Effect-App)}\frac{()\frac{\Delta'}{\Phi|\Gamma'\vdash v[\sigma]:\forall\alpha.A}}{\Phi \mid \Gamma' \vdash (v\,[\sigma])\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{73}$$

Where $(v\,[\sigma])\ \epsilon = (v\ \epsilon)\,[\sigma]$

0.6 The Identity Substitution on Type Environments

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma,x:A} = (I_\Gamma, x := x)$

0.6.1 Properties of the Identity Substitution

Property 1 If $\Phi \vdash \Gamma\texttt{Ok}$ then $\Phi \mid \Gamma \vdash I_\Gamma\colon \Gamma$, proved trivially by induction over the well-formed-ness relation.

Property 2 TODO: The denotational property of id-substitution

0.7 Single Substitution on Type Environments

If $\Phi \mid \Gamma \vdash v\colon A$, let the single substitution $\Phi \mid \Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{74}$$