Chapter 1

Adequacy of a Model of the Polymorphic Effect Calculus

1.1 Instantiation of the Polymorphic Effect Calculus

Let us instantiate the polymorphic effect calculus to be a language in which one can write programs which can create an output signal. The effect system of EC is then used to count an upper bound on the number of outputs that a program can make. This language shall be called $\texttt{PEC}_{\texttt{put}}$

1.1.1 Ground Types

We simply use the basic ground types.

$$\gamma ::= \texttt{Bool} \mid \texttt{Unit} \tag{1.1}$$

1.1.2 Graded Monad

We grade index the base graded monad with a partially ordered monoid derived from the natural numbers.

$$E = (\mathbb{N}, 0, +, \leq) \tag{1.2}$$

This extended as described in the dissertation **TODO: Ref**, to symbolically include variables α, β, γ which range over the natural numbers.

This means that the $\texttt{do}\ x \leftarrow v\ \texttt{in}\ v'$ type rule adds together the upper bounds on the two expressions to give an upper bound on the number of outputs of the sequenced expression. The $\texttt{return}\ v$ type rule acknowledges that a pure expression does not have any output.

1.1.3 Constants

We extend the set of constant, built in expressions to include a `put` statement which makes a single output action.

$$\texttt{k}^{A} ::= \texttt{true}^{\texttt{Bool}} \mid \texttt{false}^{\texttt{Bool}} \mid \texttt{()}^{\texttt{Unit}} \mid \texttt{put}^{\texttt{M}_1\texttt{Unit}} \tag{1.3}$$

1.1.4 Subtyping

The ground subtyping relation is the trivial identity relation. This is extended using the subeffect and function subtyping rules given in **TODO: Ref**.

1.2 Instantiation of a Model of the Polymorphic Effect Calculus

Let us now instantiate a model of $\mathtt{PEC}_{\mathtt{put}}$ in the indexed category derived as in chapter **TODO: ref** from a model of the polymorphic version of $\mathtt{PEC}_{\mathtt{put}}$ in `Set`, the category of sets and functions.

1.2.1 Cartesian Closed Category

Is given by the usage of `Set`.

1.2.2 Graded Monad

The strong graded monad on `Set` is given by tagging values of the underlying type with the number of output operations required to compute that value.

$$\mathtt{T}^0_n A = \{(n', a) \mid n' \leq n \wedge a \in A\} \tag{1.4}$$
$$\mu^0_{m,n,A} = (m', (n', a)) \mapsto (n' + m', a) \tag{1.5}$$
$$\eta^0_A = a \mapsto (0, a) \tag{1.6}$$
$$\mathtt{t}^0_{n,A,B} = (a, (n', b)) \mapsto (n', (a, b)) \tag{1.7}$$

1.2.3 Subeffecting Natural Transformations

These natural transformations are given by inclusion functions (identities), since $n \leq m \wedge (n', a) \in \mathtt{T}^0_n A \implies (n' \leq n \leq m, a \in A) \implies (n', a) \in \mathtt{T}^0_m A$. Other subtyping morphisms are generated using the usual method according to the subtype derivation.

1.2.4 Ground Denotations

We define denotations for ground types as follows:

$$[\![\Phi \vdash \mathtt{Unit} \colon \mathtt{Type}]\!] = \vec{\epsilon} \mapsto \{*\} \tag{1.8}$$
$$[\![\Phi \vdash \mathtt{Bool} \colon \mathtt{Type}]\!] = \vec{\epsilon} \mapsto \{\top, \bot\} \tag{1.9}$$

We then define denotations for the constant expressions, including the `put`operation.

$$[\![\mathtt{()}]\!] = \vec{\epsilon} \mapsto * \mapsto * \tag{1.10}$$
$$[\![\mathtt{true}]\!] = \vec{\epsilon} \mapsto * \mapsto \top \tag{1.11}$$
$$[\![\mathtt{false}]\!] = \vec{\epsilon} \mapsto * \mapsto \bot \tag{1.12}$$
$$[\![\mathtt{put}]\!] = \vec{\epsilon} \mapsto * \mapsto (1, *) \tag{1.13}$$
$$\tag{1.14}$$

1.2.5 Soundness

This category is now an Indexed-S-category and hence a sound model for $\mathtt{PEC}_{\mathtt{put}}$ when completed using the techniques in chapter **TODO: ref**.

1.3 Programming With Put

This simple language now has some extra properties which the general EC does not have.

Definition 1.3.1 (Powers of Put as an Equational Equivalence Class). *Define put^n as follows:*

$$\Phi \mid \Gamma \vdash put^0 \approx return\ () : M_0\,Unit$$
$$\Phi \mid \Gamma \vdash put^{m+1} \approx do\ \_ \leftarrow put^n\ in\ put : M_{m+1}\,Unit$$

Lemma 1.3.1 (Denotations of Powers of Put). *Powers of put have a simple denotation.* $[\![\Phi \mid \Gamma \vdash put^m : M_m\,Unit]\!] = \vec{\epsilon} \mapsto \rho \mapsto (m, *)$

Proof: By induction on m.

Case 0:

$$[\![\Phi \mid \Gamma \vdash \mathtt{put}^0 : \mathtt{M}_0\mathtt{Unit}]\!](\vec{\epsilon})(\rho) = \eta^n(*) = (0, *) \tag{1.15}$$

Case m+1:

$$\begin{aligned}[\![\Phi \mid \Gamma \vdash \mathtt{put}^{m+1} : \mathtt{M}_0\mathtt{Unit}]\!](\vec{\epsilon})(\rho) &= (\mu^n \circ \mathtt{T}^n_m([\![\diamond \vdash \mathtt{put} : \mathtt{M}_1\mathtt{Unit}]\!] \circ \pi_1) \circ \mathtt{t}^n)(*, [\![\Phi \mid \Gamma \vdash \mathtt{put}^m : \mathtt{M}_n\mathtt{Unit}]\!](\vec{\epsilon})(\rho)) \\ &= (m+1, *)\end{aligned}$$

1.4 Logical Relations

We define a logical relation between elements of the image of a type denotation and closed $\mathtt{PEC}_{\mathtt{put}}$ terms.

$$\mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon} : E^n) \in \wp([\![\Phi \vdash A : \mathtt{Type}]\!]\vec{\epsilon} \times \mathtt{PEC}^{A[\epsilon/\alpha]}_{\mathtt{put}}) \tag{1.16}$$

1.4.1 Definition

Definition 1.4.1 (Logical Relation).

$$\begin{aligned}
(d, v) \in R_{\Phi \vdash \forall \alpha . A : Type}(\vec{\epsilon}) &\leftrightarrow \forall \epsilon . (\pi_\epsilon(d), v\ \epsilon) \in R_{\Phi \vdash A[\epsilon/\alpha] : Type}(\vec{\epsilon}) \\
(d, v) \in R_{\Phi \vdash Unit : Type}(\vec{\epsilon}) &\Leftrightarrow (d = * \wedge \vdash v \approx () : Unit) \\
(d, v) \in R_{\Phi \vdash Bool : Type}(\vec{\epsilon}) &\Leftrightarrow ((d = \top \wedge \vdash v \approx true : Bool) \\
&\quad \vee (d = \bot \wedge \vdash v \approx false : Bool)) \\
(d, v) \in R_{\Phi \vdash A \to B : Type}(\vec{\epsilon}) &\Leftrightarrow \forall e, u . (e, u) \in R_{\Phi \vdash A : Type}(\vec{\epsilon}) \implies (d(e), v\ u) \in R_{\Phi \vdash B : Type}(\vec{\epsilon}) \\
(d, v) \in R_{\Phi \vdash M_f A : Type}(\vec{\epsilon}) &\Leftrightarrow (d = (n, d') \\
&\quad \wedge \exists v' . ((d', v') \in R_{\Phi \vdash A : Type} \\
&\quad \wedge \vdash v \approx do\ _ \leftarrow put^{n'}\ in\ return\ v'\ : (M_f A)[\vec{\epsilon}/\vec{\alpha}]))
\end{aligned}$$

1.4.2 Subtyping

Theorem 1.4.1 (Logical Relation and Subtyping). *If $A \leq :_\Phi B$ and $(d, v) \in R_{\Phi \vdash A:\mathit{Type}}(\vec{\epsilon})$ then $(d, v) \in R_{\Phi \vdash B:\mathit{Type}}(\vec{\epsilon})$*

Proof: By induction on the derivation of $A \leq :_\Phi B$.

Case S-Ground: $A \leq :_\Phi B \implies A = B$, since ground subtyping is the identity relation.

Case S-Fn: $A \leq :_\Phi B \implies A = A_1 \to A_2, B = B_1 \to B_2$ where $B_1 \leq :_\Phi A_1$ and $A_2 \leq :_\Phi B_2$.

By the definition of the $\triangleleft_{\Phi \vdash A \to B:\mathtt{Type}}$ relation, $(d, v) \in \mathtt{R}_{\Phi \vdash A \to B:\mathtt{Type}}(\vec{\epsilon}) \Leftrightarrow (\forall e, u.(e, u) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon}) \implies (d(e), v\ u) \in \mathtt{R}_{\Phi \vdash B:\mathtt{Type}}(\vec{\epsilon}))$.

So

$$\begin{aligned}
\forall e, u.(e, u) \in \mathtt{R}_{\Phi \vdash B_1:\mathtt{Type}}(\vec{\epsilon}) &\implies (e, u) \in \mathtt{R}_{\Phi \vdash A_1:\mathtt{Type}}(\vec{\epsilon}) \quad \text{By induction } B_1 \leq :_\Phi A_1 \\
&\implies (d(e), u\ v) \in \mathtt{R}_{\Phi \vdash A_2:\mathtt{Type}}(\vec{\epsilon}) \quad \text{By definition} \\
&\implies (d(e), u\ v) \in \mathtt{R}_{\Phi \vdash B_2:\mathtt{Type}}(\vec{\epsilon}) \quad \text{By induction } A_2 \leq :_\Phi B_2
\end{aligned}$$

As required.

Case S-Effect: $\mathtt{M}_{n_1} A_1 \leq :_\Phi \mathtt{M}_{n_2} A_2 \implies n_1 \leq n_2, A_1 \leq :_\Phi A_2$

$$\begin{aligned}
(d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_{n_1} A_1:\mathtt{Type}}(\vec{\epsilon}) &\implies d = (n_1', d') \wedge n_1' \leq n_1 \leq n_2 \\
&\quad \wedge \exists v'.((d', v') \in \mathtt{R}_{\Phi \vdash A_1:\mathtt{Type}}(\vec{\epsilon}) \wedge \vdash v \approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' \ : \mathtt{M}_{n_1} A_1) \\
&\implies \vdash v_1' \colon A_2 \wedge (d', v') \in \mathtt{R}_{\Phi \vdash A_2:\mathtt{Type}}(\vec{\epsilon}) \wedge \vdash v \approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' \ : \mathtt{M}_{n_1} A_2 \\
&\implies (d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_{n_2} A_2:\mathtt{Type}}(\vec{\epsilon})
\end{aligned}$$

Case S-Quantification: $\forall \alpha.A_1 \leq :_\Phi \forall \alpha.A_2 \implies A_1 \leq : A_2$

So:

$$\begin{aligned}
(d, v) \in \mathtt{R}_{\Phi \vdash \forall \alpha.A_1:\mathtt{Type}}(\vec{\epsilon}) &\implies \forall \epsilon.(\pi_\epsilon(d), v\ \epsilon) \in \mathtt{R}_{\Phi \vdash A_1[\alpha/\epsilon]:\mathtt{Type}}(\vec{\epsilon}) \\
&\implies \forall \epsilon(\pi_\epsilon(d), v\ \epsilon) \in \mathtt{R}_{\Phi \vdash A_2[\alpha/\epsilon]:\mathtt{Type}}(\vec{\epsilon}) \\
&\implies (d, v) \in \mathtt{R}_{\Phi \vdash \forall \alpha.A_2:\mathtt{Type}}(\vec{\epsilon})
\end{aligned}$$

1.4.3 The Environment Lemma

TODO: Write this up

Lemma 1.4.2 (Environment Lemma).

$$R_{\Phi, \alpha \vdash A:\mathit{Type}}(\vec{\epsilon}, \epsilon) = R_{\Phi \vdash A[\alpha/\epsilon]:\mathit{Type}}(\vec{\epsilon})$$

Proof: By induction over the type A, proving that $(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash A:\mathtt{Type}}(\vec{\epsilon}, \epsilon) \Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon})$

Case Bool:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash \mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow (d = \top \wedge \vdash v \approx \mathtt{true}\colon \mathtt{Bool}) \\ &\quad \wedge (d = \bot \wedge \vdash v \approx \mathtt{false}\colon \mathtt{Bool}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash \mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case Unit:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash \mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow (d = * \wedge \vdash v \approx \mathtt{()}\colon \mathtt{Unit}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash \mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Fn:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash A\to B:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow \forall (e, u) \in \mathtt{R}_{\Phi,\alpha\vdash A:\mathtt{Type}}(\vec{\epsilon}, \epsilon).(d(e), v\ u) \in \mathtt{R}_{\Phi,\alpha\vdash B:\mathtt{Type}}(\vec{\epsilon}, \epsilon) \\ &\Leftrightarrow \forall (e, u) \in \mathtt{R}_{\Phi\vdash A[\alpha/\epsilon]:\mathtt{Type}}(\vec{\epsilon}).(d(e), v\ u) \in \mathtt{R}_{\Phi\vdash B[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]\to B[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Effect:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash \mathtt{M}_f A:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow d = (n, d') \\ &\quad \wedge \exists v'.(d', v') \in \mathtt{R}_{\Phi,\alpha\vdash A:\mathtt{Type}}(\vec{\epsilon}, \epsilon) \\ &\quad\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' \ \ \colon \mathtt{M}_f A[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha] \\ &\Leftrightarrow \exists v'.(d', v') \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\ &\quad\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' \ \ \colon \mathtt{M}_f A[\epsilon/\alpha][\vec{\epsilon}/\vec{\alpha}] \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Quantification:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash \forall\beta.A:\mathtt{Type}}(\vec{\epsilon}, \epsilon_1) &\Leftrightarrow \forall \epsilon_2.(\pi_{\epsilon_2}(d), v\ \epsilon_2) \in \mathtt{R}_{\Phi,\alpha\vdash A[\epsilon_2/\beta]:\mathtt{Type}}(\vec{\epsilon}, \epsilon_1) \\ &\Leftrightarrow \forall \epsilon_2.(\pi_{\epsilon_2}(d), v\ \epsilon_2) \in \mathtt{R}_{\Phi\vdash A[\epsilon_2/\beta][\epsilon_1/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \quad \Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash \forall\beta.A[\epsilon_1/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

1.4.4 Fundamental Property

Let $\mathtt{R}_{\Phi\vdash\Gamma\ \ \mathtt{Ok}}(\vec{\epsilon}) \in \wp(\llbracket\Gamma\rrbracket\vec{\epsilon}) \times \mathtt{PEC}^{\Gamma[\vec{\epsilon}/\vec{\alpha}]}_{\mathtt{put}})$ mean:

$$(\rho\sigma) \in \mathtt{R}_{\Phi\vdash\Gamma\ \ \mathtt{Ok}}(\vec{\epsilon}) \Leftrightarrow \forall x.(\rho(x), \sigma(x)) \in \mathtt{R}_{\Phi\vdash\Gamma(x):\mathtt{Type}}(\vec{\epsilon}) \tag{1.17}$$

Theorem 1.4.3 (Fundamental Theorem)**.** *If* $(\rho\sigma) \in R_{\Phi\vdash\Gamma\ \ Ok}(\vec{\epsilon})$ *then* $(\llbracket\Phi \mid \Gamma \vdash v\colon A\rrbracket(\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in R_{\Phi\vdash A:\,Type}(\vec{\epsilon})$ *up to equational equivalence.*

Proof: By induction over the derivation of $\Phi \mid \Gamma \vdash v\colon A$

Case Variables:

$$\begin{aligned} [\![\Phi \mid \Gamma \vdash x \colon \Gamma(x)]\!](\vec{\epsilon})(\rho) &= \rho(x) \\ x[\sigma][\vec{\epsilon}/\vec{\alpha}] &= \sigma(x) \end{aligned}$$

And $(\rho(x), \sigma(x)[\vec{\epsilon}/\vec{\alpha}]) \in \mathtt{R}_{\Phi \vdash \Gamma(x) : \mathtt{Type}}(\vec{\epsilon})$.

Case Constants:

$$[\![\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \top \wedge \vdash \mathtt{true}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{true} \colon \mathtt{Bool} \quad \text{So } ([\![\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho), \mathtt{true}[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{Boo}$$

$$[\![\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \bot \wedge \vdash \mathtt{false}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{false} \colon \mathtt{Bool} \quad \text{So } ([\![\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho), \mathtt{false}[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash}$$

$$[\![\Phi \mid \Gamma \vdash \mathtt{()} \colon \mathtt{Unit}]\!](\vec{\epsilon})(\rho) = * \wedge \vdash \mathtt{()}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{()} \colon \mathtt{Unit} \quad \text{So } ([\![\Phi \mid \Gamma \vdash \mathtt{()} \colon \mathtt{Unit}]\!](\vec{\epsilon})(\rho), \mathtt{()}[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{Unit} : \mathtt{Type}}(\vec{\epsilon})$$

$$[\![\Phi \mid \Gamma \vdash \mathtt{put} \colon \mathtt{M}_1\mathtt{Unit}]\!](\vec{\epsilon})(\rho) = (1, *) \wedge \vdash \mathtt{put} \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^1\ \mathtt{in\ return}\ \mathtt{()}\ :\ \mathtt{M}_1\mathtt{Unit} \quad \text{So } ([\![\Phi \mid \Gamma \vdash \mathtt{put} \colon \mathtt{M}_1 U]\!](\vec{\epsilon})(\rho), \mathtt{put}[\vec{\epsilon}$$

Case Subtype:

$$[\![\Phi \mid \Gamma \vdash v \colon B]\!](\vec{\epsilon})(\rho) = [\![\Phi \mid \Gamma \vdash v \colon A]\!](\vec{\epsilon})(\rho) \tag{1.18}$$

By induction, $([\![\Phi \mid \Gamma \vdash v \colon A]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon})$. Since, by the subtyping lemma (Lemma 1.4.1) $A \leq :_{\Phi} B \wedge (d, v) \in \mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon}) \implies (d, v) \in \mathtt{R}_{\Phi \vdash B : \mathtt{Type}}(\vec{\epsilon})$, we have that $([\![\Phi \mid \Gamma \vdash v \colon B]\!], v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash B : \mathtt{Type}}(\vec{\epsilon})$.

Case Fn: For all $(d, u) \in \mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon})$,

$$\begin{aligned} ([\![\Phi \mid \Gamma \vdash \lambda x \colon A.v : A \to B]\!](\vec{\epsilon})(\rho))d &= (\mathtt{cur}([\![\Gamma, x \colon A \vdash v \colon B]\!])(\vec{\epsilon})(\rho))d \\ &= [\![\Gamma, x \colon A \vdash v \colon B]\!]((\vec{\epsilon})(\rho[x \mapsto d])) \end{aligned}$$

Since $(d, u) \in \mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon})$, we have $((\rho[x \mapsto d])(\sigma, x \colon = u)) \in \mathtt{R}_{\Phi \vdash \Gamma, x : A}\ \mathtt{Ok}(\vec{\epsilon})$, so by induction

$$([\![\Gamma, x \colon A \vdash v \colon B]\!](\vec{\epsilon})(\rho[x \mapsto d]), v[\vec{\epsilon}/\vec{\alpha}][\sigma, x \colon = u]) \in \mathtt{R}_{\Phi \vdash B : \mathtt{Type}}(\vec{\epsilon})$$

We can see that $\vdash v[\vec{\epsilon}/\vec{\alpha}][\sigma, x \colon = u] \approx ((\lambda x \colon A.v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma])\ u \colon B[\vec{\epsilon}/\vec{\alpha}]$.

Hence $(([\![\Phi \mid \Gamma \vdash \lambda x \colon A.v : A \to B]\!](\vec{\epsilon})(\rho)), (\lambda x \colon A.v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A \to B : \mathtt{Type}}(\vec{\epsilon})$.

Case Apply:

$$[\![\Phi \mid \Gamma \vdash v\ u \colon B]\!](\vec{\epsilon})(\rho) = ([\![\Phi \mid \Gamma \vdash v \colon A \to B]\!](\vec{\epsilon})(\rho))([\![\Phi \mid \Gamma \vdash u \colon A]\!](\vec{\epsilon})(\rho)) \tag{1.19}$$

By induction $([\![\Phi \mid \Gamma \vdash v \colon A \to B]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A \to B : \mathtt{Type}}(\vec{\epsilon})$ and $([\![\Phi \mid \Gamma \vdash u \colon A]\!](\vec{\epsilon})(\rho), u[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A : \mathtt{Type}}(\vec{\epsilon})$. So by the definition of $\mathtt{R}_{\Phi \vdash A \to B : \mathtt{Type}}$,

$$(([\![\Phi \mid \Gamma \vdash v \colon A \to B]\!](\vec{\epsilon})(\rho))([\![\Phi \mid \Gamma \vdash u \colon A]\!](\vec{\epsilon})(\rho)), (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ (u[\vec{\epsilon}/\vec{\alpha}][\sigma]))) \in \mathtt{R}_{\Phi \vdash B : \mathtt{Type}}(\vec{\epsilon})$$

Where $\vdash (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ (u[\vec{\epsilon}/\vec{\alpha}][\sigma]) \approx (v\ u)[\vec{\epsilon}/\vec{\alpha}][\sigma] \colon B[\vec{\epsilon}/\vec{\alpha}]$.

So $([\![\Phi \mid \Gamma \vdash v\ u \colon B]\!](\vec{\epsilon})(\rho), (v\ u)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash B : \mathtt{Type}}(\vec{\epsilon})$.

Case Return:

$$[\![\Phi \mid \Gamma \vdash v\colon \mathtt{M}_0 A]\!](\vec{\epsilon})(\rho) = (0, [\![\Phi \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon})(\rho)) \tag{1.20}$$

By induction, $([\![\Phi \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon})$. So by picking $v' = v[\vec{\epsilon}/\vec{\alpha}][\sigma]$, we have

$$\vdash (\mathtt{return}\ v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{return}\ (v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^0\ \mathtt{in}\ \mathtt{return}\ v' \quad : \mathtt{M}_0 A[\vec{\epsilon}/\vec{\alpha}] \tag{1.21}$$

So $([\![\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_0 A]\!](\vec{\epsilon})(\rho), (\mathtt{return}\ v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_0 A:\mathtt{Type}}(\vec{\epsilon})$

Case Bind: By inversion, $([\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{m+n} B]\!](\vec{\epsilon})(\rho)) = (m' + n', d_2)$, where $(n', d_2) = ([\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_n B]\!]((\vec{\epsilon})(\rho[x \mapsto d_1])))$, and $(n', d_1) = ([\![\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_m A]\!](\vec{\epsilon})(\rho))$.

By induction, $((m', d_1), v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_m A:\mathtt{Type}}(\vec{\epsilon})$. So $\exists v_1'$ such that $\vdash v_1[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v_1' \ : \mathtt{M}_m A$. So $(((\vec{\epsilon})(\rho[x \mapsto d_1])), ([\sigma], x\colon= v_1')) \in \mathtt{R}_{\Phi \vdash \Gamma, x:A:\mathtt{Type}}(\vec{\epsilon})$.

So by induction $([\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_n B]\!]((\vec{\epsilon})(\rho[x \mapsto d_1])), v_2[\sigma, x\colon= v_1']) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_n B:\mathtt{Type}}(\vec{\epsilon})$.

Hence, $\exists v_2'$ such that $\vdash v_2[\sigma, x\colon= v_1'] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'} \mathtt{return}\ v_2'\ \ \mathtt{in}\ : \mathtt{M}_{m+n} B$ and $(d_2, v_2') \in \mathtt{R}_{\Phi \vdash \mathtt{M}_n B:\mathtt{Type}}(\vec{\epsilon})$.

Hence,

$$\begin{aligned}\vdash \mathtt{do}\ x \leftarrow v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]\ \mathtt{in}\ v_2[\vec{\epsilon}/\vec{\alpha}][\sigma]\ &\approx \mathtt{do}\ x \leftarrow (\mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v_1'\ \ )\ \mathtt{in}\ (v_2[\vec{\epsilon}/\vec{\alpha}][\sigma]) : \mathtt{M}_{m+n} B\\ &\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ v_2[\sigma, x\colon= v_1']\\ &\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'+n'}\ \mathtt{in}\ \mathtt{return}\ v_2'\end{aligned}$$

So $([\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{m+n} B]\!](\vec{\epsilon})(\rho), (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_{m+n} B:\mathtt{Type}}(\vec{\epsilon})$.

Case If: By inversion,

$$[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho) = \begin{cases} [\![\Phi \mid \Gamma \vdash v_1\colon A]\!](\vec{\epsilon})(\rho) & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \top \\ [\![\Phi \mid \Gamma \vdash v_2\colon A]\!](\vec{\epsilon})(\rho) & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \bot \end{cases}$$

.

By induction,

$$\begin{gathered}([\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho), b[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon})\\ ([\![\Phi \mid \Gamma \vdash v_1\colon A]\!](\vec{\epsilon})(\rho), v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon})\\ ([\![\Phi \mid \Gamma \vdash v_2\colon A]\!](\vec{\epsilon})(\rho), v_2[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon})\end{gathered}$$

Case: $[\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \top$ **and** $\vdash b[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{true}\colon \mathtt{Bool}$

$$[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho) = [\![\Phi \mid \Gamma \vdash v_1\colon A]\!](\vec{\epsilon})(\rho) \tag{1.22}$$

And

$$\vdash v_1[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]\colon A[\vec{\epsilon}/\vec{\alpha}] \tag{1.23}$$

So

$$([\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho), (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon}) \tag{1.24}$$

Case: $[\![\Phi \mid \Gamma \vdash b\colon \texttt{Bool}]\!](\vec{\epsilon})(\rho) = \bot$ **and** $\vdash b[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \texttt{false}\colon \texttt{Bool}$

$$[\![\Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho) = [\![\Phi \mid \Gamma \vdash v_2\colon A]\!](\vec{\epsilon})(\rho) \tag{1.25}$$

And

$$\vdash v_2[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]\colon A[\vec{\epsilon}/\vec{\alpha}] \tag{1.26}$$

So

$$([\![\Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho), (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\texttt{Type}}(\vec{\epsilon}) \tag{1.27}$$

Case Effect-Gen: For all $\epsilon \in E$:

$$\begin{aligned} \pi_\epsilon([\![\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A]\!](\vec{\epsilon})(\rho)) &= \pi_\epsilon(\langle [\![\Phi,\alpha \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon},\epsilon')\rangle_{\epsilon'\in E}(\rho)) &(1.28)\\ &= [\![\Phi,\alpha \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon},\epsilon)(\rho) &(1.29)\end{aligned}$$

By induction, we know that $([\![\Phi,\alpha \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon},\epsilon)(\rho), v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma]) \in \mathtt{R}_{\Phi,\alpha\vdash A:\texttt{Type}}(\vec{\epsilon},\epsilon)$.

By the environment lemma **TODO: Ref**, $\mathtt{R}_{\Phi,\alpha\vdash A:\texttt{Type}}(\vec{\epsilon},\epsilon) = \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$.

So $([\![\Phi,\alpha \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon},\epsilon)(\rho), v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma]) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$

Since α does not appear in σ (Since σ is well formed under $\Phi = \diamond$), we can commute the substitutions: $v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma] = v[\vec{\epsilon}/\vec{\alpha}][\sigma][\epsilon/\alpha]$.

Hence:

$$\vdash v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma] = v[\vec{\epsilon}/\vec{\alpha}][\sigma][\epsilon/\alpha] \approx \Lambda\alpha.(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon\colon A[\epsilon/\alpha][\vec{\epsilon}/\vec{\alpha}]$$

So $(\pi_\epsilon([\![\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A]\!](\vec{\epsilon})(\rho)), \Lambda\alpha.(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$.

So $([\![\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A]\!], (\Lambda\alpha.v)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash\forall\alpha.A:\texttt{Type}}(\vec{\epsilon})$.

Case Effect-Spec: Required to prove $([\![\Phi \mid \Gamma \vdash v\ \ \epsilon\colon A[\epsilon/\alpha]]\!](\vec{\epsilon})(\rho), (v\ \ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \ \epsilon) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$.

By inversion and induction, $([\![\Phi \mid \Gamma \vdash v\colon \forall\alpha.A]\!], v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash\forall\alpha.A:\texttt{Type}}(\vec{\epsilon})$. So,

$$\forall\epsilon.(\pi_\epsilon([\![\Phi \mid \Gamma \vdash v\colon \forall\alpha.A]\!](\vec{\epsilon})(\rho)), (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$$

Let $\epsilon' = [\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]\vec{\epsilon}$.

$$\begin{aligned} [\![\Phi \mid \Gamma \vdash v\ \epsilon\colon A[\epsilon/\alpha]]\!](\vec{\epsilon})(\rho) &= \langle \mathtt{Id}, [\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]\rangle^*(\boldsymbol{\epsilon}_{\Phi,\beta\vdash A[\beta/\alpha]:\texttt{Type}})(\vec{\epsilon})([\![\Phi \mid \Gamma \vdash v\colon \forall\alpha.A]\!](\vec{\epsilon})(\rho))\\ &= \pi_{\epsilon'}([\![\Phi \mid \Gamma \vdash v\colon \forall\alpha.A]\!](\vec{\epsilon})(\rho))\end{aligned}$$

Its also the case that

$$(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon' = (v\ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma] \tag{1.30}$$

So

$$([\![\Phi \mid \Gamma \vdash v\ \epsilon\colon A[\epsilon/\alpha]]\!](\vec{\epsilon})(\rho), (v\ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon}) \tag{1.31}$$

1.5 Adequacy

Theorem 1.5.1 (Adequacy)**.** *For G defined as:*

$$G ::= \texttt{Bool} \mid \texttt{Unit} \mid \texttt{M}_n G$$

Equality of denotations implies equational equality.

$$[\![\vdash v : G]\!] = [\![\vdash u : G]\!] \implies \vdash v \approx u : G \tag{1.32}$$

Proof: By induction on the structure of G, making use of the fundamental property 1.4.4.

Case Boolean: Let $d = [\![\vdash v : \texttt{Bool}]\!] = [\![\vdash v : \texttt{Bool}]\!] \in \{\top, \bot\}$. By the fundamental property, $(d, v) \in \texttt{R}_{\Phi \vdash \texttt{Bool} : \texttt{Type}}(\vec{\epsilon})$ and $(d, v) \in \texttt{R}_{\Phi \vdash \texttt{Bool} : \texttt{Type}}(\vec{\epsilon})$.

Case: $d = \top$ Then $\vdash v \approx \texttt{true} \approx u : \texttt{Bool}$

Case: $d = \bot$ Then $\vdash v \approx \texttt{false} \approx u : \texttt{Bool}$

Case Unit: Let $* = [\![\vdash v : \texttt{Unit}]\!] = [\![\vdash v : \texttt{Unit}]\!] \in \{*\}$. By the fundamental property, $(d, v) \in \texttt{R}_{\Phi \vdash \texttt{Unit} : \texttt{Type}}(\vec{\epsilon})$ and $(d, v) \in \texttt{R}_{\Phi \vdash \texttt{Unit} : \texttt{Type}}(\vec{\epsilon})$. Hence $\vdash v \approx \texttt{()} \approx u : \texttt{Unit}$.

Case T-Effect: Let $(n', d) = [\![\vdash v : \texttt{M}_n G]\!] = [\![\vdash u : \texttt{M}_n G]\!]$. By the fundamental property, $(((n', d)), v) \in \texttt{R}_{\Phi \vdash \texttt{M}_n G : \texttt{Type}}(\vec{\epsilon})$ and $(((n', d)), u) \in \texttt{R}_{\Phi \vdash \texttt{M}_n G : \texttt{Type}}(\vec{\epsilon})$. So there exists u', v' such that $(d', u') \in \texttt{R}_{\Phi \vdash G : \texttt{Type}}(\vec{\epsilon})$ and $(d', u') \in \texttt{R}_{\Phi \vdash G : \texttt{Type}}(\vec{\epsilon})$ and:

$$\begin{aligned} \vdash v &\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in return}\ v' \ : \texttt{M}_n G \\ &\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in return}\ u' \\ &\approx u \end{aligned}$$

□