0.1 Beta and Eta Equivalence

0.1.1 Beta-Eta conversions

- (Lambda-Beta) $\dfrac{\Phi \mid \Gamma, x : A \vdash v_2 \colon B \qquad \Phi \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash (\lambda x : A.v_1\ )\ v_2 \approx v_1\,[v_2/x] : B}$

- (Lambda-Eta) $\dfrac{\Phi \mid \Gamma \vdash v \colon A \to B}{\Phi \mid \Gamma \vdash \lambda x : A.(v\ x\ ) \approx v \colon A \to B}$

- (Left Unit) $\dfrac{\Phi \mid \Gamma \vdash v_1 \colon A \qquad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return}\ v_1\ \ \mathtt{in}\ v_2\ \approx v_2\,[v_1/x] : \mathtt{M}_\epsilon B}$

- (Right Unit) $\dfrac{\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return}\ x\ \ \approx v \colon \mathtt{M}_\epsilon A}$

- (Associativity) $\dfrac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B \qquad \Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3\ )\ \approx \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )\ \mathtt{in}\ v_3 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$

- (Unit) $\dfrac{\Phi \mid \Gamma \vdash v \colon \mathtt{Unit}}{\Phi \mid \Gamma \vdash v \approx \mathtt{()} \colon \mathtt{Unit}}$

- (if-true) $\dfrac{\Phi \mid \Gamma \vdash v_1 \colon A \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ \approx v_1 \colon A}$

- (if-false) $\dfrac{\Phi \mid \Gamma \vdash v_2 \colon A \qquad \Phi \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ \approx v_2 \colon A}$

- (If-Eta) $\dfrac{\Phi \mid \Gamma, x : \mathtt{Bool} \vdash v_2 \colon A \qquad \Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v_1\ \mathtt{then}\ v_2\,[\mathtt{true}/x]\ \mathtt{else}\ v_2\,[\mathtt{false}/x]\ \approx v_2\,[v_1/x] : A}$

- (Effect-beta) $\dfrac{\Phi \vdash \epsilon \qquad \Phi, \alpha \mid \Gamma \vdash v \colon A}{\Phi \mid \Gamma \vdash (\Lambda\alpha.v\ \epsilon) \approx v\,[\epsilon/\alpha] : A\,[\epsilon/\alpha]}$

- (Effect-eta) $\dfrac{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A}{\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha) \approx v \colon \forall\alpha.A}$

0.1.2 Equivalence Relation

- (Reflexive) $\dfrac{\Phi \mid \Gamma \vdash v \colon A}{\Phi \mid \Gamma \vdash v \approx v \colon A}$

- (Symmetric) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon A}{\Phi \mid \Gamma \vdash v_2 \approx v_1 \colon A}$

- (Transitive) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon A \qquad \Phi \mid \Gamma \vdash v_2 \approx v_3 \colon A}{\Phi \mid \Gamma \vdash v_1 \approx v_3 \colon A}$

0.1.3 Congruences

- (Effect-Abs) $\dfrac{\Phi, \alpha \mid \Gamma \vdash v_1 \approx v_2 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_1 \approx \Lambda\alpha.v_2 \colon \forall\alpha.A}$

- (Effect-Apply) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon \forall\alpha.A \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_1\ \epsilon \approx v_2\ \epsilon \colon A\,[\epsilon/\alpha]}$

- (Lambda) $\dfrac{\Phi \mid \Gamma, x : A \vdash v_1 \approx v_2 \colon B}{\Phi \mid \Gamma \vdash \lambda x : A.v_1\ \approx \lambda x : A.v_2 : A \to B}$

- (Return) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{return}\, v_1\ \approx \mathtt{return}\, v_2 : \mathtt{M}_{\mathtt{1}} A}$

- (Apply) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_1' \colon A \to B \qquad \Phi \mid \Gamma \vdash v_2 \approx v_2' \colon A}{\Phi \mid \Gamma \vdash v_1\ v_2 \approx v_1'\ v_2' \colon B}$

- (Bind) $\dfrac{\Phi \mid \Gamma \vdash v_1 \approx v_1' \colon \mathtt{M}_{\epsilon_1} A \qquad \Phi \mid \Gamma, x : A \vdash v_2 \approx v_2' \colon \mathtt{M}_{\epsilon_2} B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ \approx \mathtt{do}\ c \leftarrow v_1'\ \mathtt{in}\ v_2' : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

- (If) $\dfrac{\Phi \mid \Gamma \vdash v \approx v' \colon \mathtt{Bool} \qquad \Phi \mid \Gamma \vdash v_1 \approx v_1' \colon A \qquad \Phi \mid \Gamma \vdash v_2 \approx v_2' \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ \approx \mathtt{if}_A\ v\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2' : A}$

- (Subtype) $\dfrac{\Phi \mid \Gamma \vdash v \approx v' \colon A \qquad A \leq \colon_{\Phi} B}{\Phi \mid \Gamma \vdash v \approx v' \colon B}$

0.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

If $\Phi \mid \Gamma \vdash v \approx v' \colon A$ then each derivation of $\Phi \mid \Gamma \vdash v \approx v' \colon A$ can be converted to a derivation of $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$ by induction over the beta-eta equivalence relation derivation.

0.2.1 Equivalence Relations

Case Reflexive: By inversion we have a derivation of $\Phi \mid \Gamma \vdash v \colon A$.

Case Symmetric: By inversion $\Phi \mid \Gamma \vdash v' \approx v \colon A$. Hence by induction, derivations of $\Phi \mid \Gamma \vdash v' \colon A$ and $\Phi \mid \Gamma \vdash v \colon A$ are given.

Case Transitive: By inversion, there exists v_2 such that $\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon A$ and $\Phi \mid \Gamma \vdash v_2 \approx v_3 \colon A$. Hence by induction, we have derivations of $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma \vdash v_3 \colon A$

0.2.2 Beta-Eta conversions

Case Lambda: By inversion, we have $\Phi \mid \Gamma, x : A \vdash v_1 \colon B$ and $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing rules, we have:

$$\text{(Apply)}\dfrac{\text{(Lambda)}\dfrac{\Phi \mid \Gamma, x : A \vdash v_1 \colon B}{\Phi \mid \Gamma \vdash \lambda x : A.v_1 : A \to B} \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash (\lambda x : A.v_1\ )\ v_2 \colon A}$$

By the substitution rule **TODO: which?**, we have

$$\text{(Substitution)}\frac{\Phi \mid \Gamma, x : A \vdash v_1 \colon B \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash v_1\,[v_2/x] \colon B}$$

Case Left Unit: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B$

Hence we have:

$$\text{(Bind)}\frac{\text{(Return)}\dfrac{\Phi \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash \mathtt{return}\ v_1 : \mathtt{M}_{\mathbf{1}} A} \qquad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return}\ v_1\ \ \mathtt{in}\ v_2 : \mathtt{M}_{\mathbf{1}\cdot\epsilon} B = \mathtt{M}_\epsilon B} \tag{1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Phi \mid \Gamma \vdash v_2\,[v_1/x] \colon \mathtt{M}_\epsilon B \tag{2}$$

Case Right Unit: By inversion, we have $\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A$.

Hence we have:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A \qquad \text{(Return)}\dfrac{\text{(var)}\dfrac{\Phi \vdash \Gamma, x : A\,\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A}}{\Phi \mid \Gamma, x : A \vdash \mathtt{return}\ v : \mathtt{M}_{\mathbf{1}} A}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return}\ x\ \ : \mathtt{M}_{\epsilon\cdot\mathbf{1}} A = \mathtt{M}_\epsilon A} \tag{3}$$

Case Associativity: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$\Phi \vdash (\iota\pi\times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Phi \mid \Gamma, x : A, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \text{(Bind)}\dfrac{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B \qquad \Phi \mid \Gamma, x : A, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Phi \mid \Gamma, x : A \vdash x v_2 v_3 \colon \mathtt{M}_{\epsilon_2\cdot\epsilon_3} C}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3\ ) : \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C} \tag{4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\dfrac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \qquad \Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Phi \mid \Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )\ \mathtt{in}\ v_3 : \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C} \tag{5}$$

Case Eta: By inversion, we have $\Phi \mid \Gamma \vdash v \colon A \to B$

By weakening, we have $\Phi \vdash \iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\dfrac{\text{(App)}\dfrac{\Phi \mid (\Gamma, x : A) \vdash x \colon A \qquad \text{(weakening)}\dfrac{\Phi \mid \Gamma \vdash v \colon A \to B \qquad \Phi \vdash \iota\pi : \Gamma, x : A \triangleright \Gamma}{\Phi \mid \Gamma, x : A \vdash v \colon A \to B}}{\Phi \mid \Gamma, x : A \vdash v\ x \colon B}}{\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) : A \to B} \tag{6}$$

Case If-True: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing lemma **TODO: Which?**, we have $\Phi \vdash \Gamma\mathtt{Ok}$ so $\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\dfrac{\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \qquad \Phi \mid \Gamma \vdash v_1 \colon A \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{7}$$

Case If-False: As above,

Hence

$$\text{(If)}\dfrac{\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \qquad \Phi \mid \Gamma \vdash v_1 \colon A \qquad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{8}$$

Case If-Eta: By inversion, we have:

$$\Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool} \tag{9}$$

and

$$\Phi \mid \Gamma, x : \mathtt{Bool} \vdash v_2 \colon A \tag{10}$$

Hence we also have $\Phi \vdash \Gamma\mathtt{Ok}$. Hence, the following also hold:

$\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$, and $\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool}$.

Hence by the substitution theorem, we have:

$$\text{(If)}\dfrac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool} \qquad \Phi \mid \Gamma \vdash v_2\,[\mathtt{true}/x] : A \qquad \Phi \mid \Gamma \vdash v_2\,[\mathtt{false}/x] : A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v_1\ \mathtt{then}\ v_2\,[\mathtt{true}/x]\ \mathtt{else}\ v_2\,[\mathtt{false}/x]\ : A} \tag{11}$$

and

$$\Phi \mid \Gamma \vdash v_2\,[v_1/x] : A \tag{12}$$

Case Effect-Beta: By inversion, $\Phi, \alpha \mid \Gamma \vdash v \colon A$ and $\Phi \vdash \epsilon$.

Then we have the following type derivation:

$$\text{(Effect-App)}\dfrac{\text{(Effect-Fn)}\dfrac{\Phi, \alpha \mid \Gamma \vdash v \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{13}$$

And we can construct the single-effect-substitution:

$$\text{(Single Substitution)}\dfrac{\Phi \vdash \epsilon}{\Phi \vdash [\epsilon/\alpha] : (\Phi, \alpha)} \tag{14}$$

Hence by the substitution theorem,

$$\Phi \mid \Gamma \vdash v\,[\epsilon/\alpha] : A\,[\epsilon/\alpha] \tag{15}$$

Case Effect-Eta: By inversion$\Phi \mid \Gamma \vdash v\colon \forall\alpha.A$

So the following derivation holds:

$$\dfrac{\dfrac{\dfrac{\Phi \mid \Gamma \vdash v\colon \forall\alpha.A}{\Phi, \alpha \mid \Gamma \vdash v\colon \forall\alpha.A}\text{(Effect-weakening)} \qquad \Phi, \alpha \vdash \alpha}{\Phi, \alpha \mid \Gamma \vdash v\ \alpha\colon A\,[\alpha/\alpha] = A}\text{(Effect-App)}}{\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha)\colon \forall\alpha.A}\text{(Effect-Fn)} \tag{16}$$

And

$$\Phi \mid \Gamma \vdash v\colon \forall\alpha.A \tag{17}$$

0.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda: By inversion, $\Phi \mid \Gamma, x : A \vdash v_1 \approx v_2\colon B$. Hence by induction $\Phi \mid \Gamma, x : A \vdash v_1\colon B$, and $\Phi \mid \Gamma, x : A \vdash v_2\colon B$.

So

$$\Phi \mid \Gamma \vdash \lambda x : A.v_1 : A \to B \tag{18}$$

and

$$\Phi \mid \Gamma \vdash \lambda x : A.v_2 : A \to B \tag{19}$$

Hold.

Case Return: By inversion, $\Phi \mid \Gamma \vdash v_1 \approx v_2\colon A$, so by induction

$$\Phi \mid \Gamma \vdash v_1\colon A$$

and

$$\Phi \mid \Gamma \vdash v_2\colon A$$

Hence we have

$$\Phi \mid \Gamma \vdash \texttt{return}\, v_1 : \texttt{M}_{\texttt{1}} A$$

and

$$\Phi \mid \Gamma \vdash \texttt{return}\, v_2 : \texttt{M}_{\texttt{1}} A$$

Case Apply: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \approx v_1'\colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 \approx v_2'\colon A$. Hence we have by induction $\Phi \mid \Gamma \vdash v_1\colon A \to B$, $\Phi \mid \Gamma \vdash v_2\colon A$, $\Phi \mid \Gamma \vdash v_1'\colon A \to B$, and $\Phi \mid \Gamma \vdash v_2'\colon A$.

So we have:

$$\Phi \mid \Gamma \vdash v_1\ v_2\colon B \tag{20}$$

and

$$\Phi \mid \Gamma \vdash v_1'\ v_2'\colon B \tag{21}$$

Case Bind: By inversion, we have: $\Phi \mid \Gamma \vdash v_1 \approx v_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2 \approx v_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, x : A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{22}$$

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1'\ \mathtt{in}\ v_2' : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{23}$$

Case If: By inversion, we have: $\Phi \mid \Gamma \vdash v \approx v' \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \approx v_1' \colon A$, and $\Phi \mid \Gamma \vdash v_2 \approx v_2' \colon A$.

Hence by induction, we have:

$\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v' \colon \mathtt{Bool}$,

$\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_1' \colon A$,

$\Phi \mid \Gamma \vdash v_2 \colon A$, and $\Phi \mid \Gamma \vdash v_2' \colon A$.

So

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A \tag{24}$$

and

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v'\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2' : A \tag{25}$$

hold.

Case Subtype: By inversion, we have $A \leq:_\Phi B$ and $\Phi \mid \Gamma \vdash v \approx v' \colon A$. By induction, we therefore have $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$.

Hence we have

$$\Phi \mid \Gamma \vdash v \colon B \tag{26}$$

$$\Phi \mid \Gamma \vdash v' \colon B \tag{27}$$

Case Effect-Lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v_1 \approx v_2 \colon A$. So

$$(\text{Effect-Lambda}) \frac{\Phi, \alpha \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{28}$$

and

$$(\text{Effect-Lambda}) \frac{\Phi, \alpha \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{29}$$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v_1 \approx v_2 \colon \forall\alpha.A$ and $\Phi \vdash \epsilon$.

So

$$(\text{Effect-App}) \frac{\Phi \mid \Gamma \vdash v_1 \colon \forall\alpha.A \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_1\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{30}$$

and

$$(\text{Effect-App}) \frac{\Phi \mid \Gamma \vdash v_2 \colon \forall\alpha.A \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_2\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{31}$$