

- CCC with strong graded monad

- Strong Graded Monad Laws:

- Unitor Law

$$\begin{array}{ccc}
1 \times T_\epsilon A & \xrightarrow{\mathfrak{t}_{\epsilon, 1, A}} & T_\epsilon(1 \times A) \\
& \searrow \lambda_{T_\epsilon A} & \downarrow T_\epsilon(\lambda_A) \\
& & T_\epsilon A
\end{array}$$

- left-right unit

$$\begin{array}{ccc}
T_\epsilon A & \xrightarrow{T_\epsilon \eta_A} & T_\epsilon T_1 A \\
& \searrow \text{Id}_{T_\epsilon A} & \downarrow \mu_{\epsilon, 1, A} \\
& & T_\epsilon A
\end{array}$$

$$\begin{array}{ccc}
T_\epsilon A & \xrightarrow{\eta_{T_\epsilon A}} & T_1 T_1 A \\
& \searrow \text{Id}_{T_\epsilon A} & \downarrow \mu_{1, \epsilon, A} \\
& & T_\epsilon A
\end{array}$$

- mu

$$\begin{array}{ccc}
T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2, T_{\epsilon_3} A}} & T_{\epsilon_1 \cdot \epsilon_2} T_{\epsilon_3} A \\
\downarrow T_{\epsilon_1} \mu_{\epsilon_2, \epsilon_3, A} & & \downarrow \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, A} \\
T_{\epsilon_1} T_{\epsilon_2 \cdot \epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, A}} & T_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} A
\end{array}$$

- t and mu

$$\begin{array}{ccc}
A \times T_{\epsilon_1} T_{\epsilon_2} B & \xrightarrow{\mathfrak{t}_{\epsilon_1, A, T_{\epsilon_2} B}} & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \xrightarrow{T_{\epsilon_1} \mathfrak{t}_{\epsilon_2, A, B}} T_{\epsilon_1} T_{\epsilon_2}(A \times B) \\
& \searrow \text{Id}_A \times \mu_{\epsilon_1, \epsilon_2, B} & \downarrow \mu_{\epsilon_1 \cdot \epsilon_2, A \times B} \\
& & A \times T_{\epsilon_1 \cdot \epsilon_2} B \xrightarrow{\mathfrak{t}_{\epsilon_1 \cdot \epsilon_2, A, B}} T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
\end{array}$$

- t and sub-effecting

$$\begin{array}{ccc}
A \times T_{\epsilon_1} B & \xrightarrow{\text{Id}_A \times \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_{\mathbb{M}, B}} & A \times T_{\epsilon_2} B \\
\downarrow \mathfrak{t}_{\epsilon_1, A, B} & & \downarrow \mathfrak{t}_{\epsilon_2, A, B} \\
T_{\epsilon_1}(A \times B) & \xrightarrow{\llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_{\mathbb{M}, A} \times \mathfrak{t}} & T_{\epsilon_2}(A \times B)
\end{array}$$

- t and id times f

$$\begin{array}{ccc}
A \times T_\epsilon B & \xrightarrow{\text{Id}_A \times T_\epsilon f} & A \times T_\epsilon B' \\
\downarrow \mathfrak{t}_{\epsilon, A, B} & & \downarrow \mathfrak{t}_{\epsilon, A, B'} \\
T_\epsilon(A \times B) & \xrightarrow{T_\epsilon(\text{Id}_A \times f)} & T_\epsilon(A \times B')
\end{array}$$

$$\begin{array}{ccc}
A \times T_\epsilon B & \xrightarrow{f \times \text{Id}_{T_\epsilon B}} & A' \times T_\epsilon B \\
\downarrow \mathfrak{t}_{\epsilon, A, B} & & \downarrow \mathfrak{t}_{\epsilon, A', B} \\
T_\epsilon(A \times B) & \xrightarrow{T_\epsilon(f \times \text{Id}_B)} & T_\epsilon(A' \times B)
\end{array}$$