

Given a set based S-Category \mathbb{C} which is a model of the non-polymorphic effect calculus, we generate an indexed category capable of modelling the polymorphic effect calculus.

0.1 The Non-Polymorphic Model

Since \mathbb{C} is a model of the non-polymorphic calculus,

- \mathbb{C} is cartesian closed.
- \mathbb{C} has a strong graded monad: $T^0 : (E, \cdot, \leq_0, 1) \rightarrow [\mathbb{C}, \mathbb{C}]$
- \mathbb{C} has a co-product on the terminal object 1.

In addition, we require that

- \mathbb{C} should be complete (e.g a sub-category of **Set**)
- E should be small.

0.2 Base Category

We construct the base category, **Eff** as follows:

- $U = E$, the set of ground effects in the non-polymorphic language.
- 1 is a singleton set.
- $U^n = E^n$, set of n -wide tuples of effects, \vec{e}

Hence when we treat effects that are well formed in Φ as morphisms, $E^n \rightarrow E$ in **Eff**, we should treat them as functions $f : E^n \rightarrow E$. Ground effects become point functions: $e : 1 \rightarrow E$, so the denotation of a ground effect is the constant value function: $\llbracket \Phi \vdash e : \mathbf{Effect} \rrbracket_M = \vec{e} \mapsto e$

We extend the multiplication of ground effects to multiplication on effect functions, giving us our **Mul** operation

$$\mathbf{Mul}(f, g) = f \cdot g \tag{1}$$

$$(f \cdot g)(\vec{e}) = (f\vec{e}) \cdot (g\vec{e}) \tag{2}$$

$$\tag{3}$$

This satisfies naturality of **Mul**.

$$((f \cdot g) \circ \theta)\vec{e} = (f(\theta\vec{e})) \cdot (g(\theta\vec{e})) = ((f \circ \theta) \cdot (g \circ \theta))\vec{e} \tag{4}$$

0.3 S-Categories

The semantic category, \mathbb{C}_0 of the effect-environment \diamond is \mathbb{C} .

Since each effect-environment is alpha equivalent to a natural number, the semantic category for Φ shall be represented as $\mathbb{C}_\Phi = \mathbb{C}_n = [E^n, \mathbb{C}]$, the category of functions $E^n \rightarrow \mathbb{C}$.

Objects in $[E^n, \mathbb{C}]$ are functions and we describe them by their actions on a generic vector of ground effects, $\vec{\epsilon}$.

Morphisms in $[E^n, \mathbb{C}]$ are natural transformations between the functions. So:

$$m : A \rightarrow B \quad \text{In } [E^n, \mathbb{C}] \quad (5)$$

$$m\vec{\epsilon} : A\vec{\epsilon} \rightarrow B\vec{\epsilon} \quad \text{In } \mathbb{C} \quad (6)$$

$$(f \circ g)\vec{\epsilon} = (f\vec{\epsilon}) \circ (g\vec{\epsilon}) \quad (7)$$

$$1(\vec{\epsilon}) = 1 \quad (8)$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in \mathbb{C} .

0.3.1 Each S-Category is a CCC

Since \mathbb{C} is complete and a CCC, and E^n is small, since E is small, $[E^n, \mathbb{C}]$ is a CCC.

$$(A \times B)\vec{\epsilon} = (A\vec{\epsilon}) \times (B\vec{\epsilon}) \quad (9)$$

$$1\vec{\epsilon} = 1 \quad (10)$$

$$(B^A)\vec{\epsilon} = (B\vec{\epsilon})^{(A\vec{\epsilon})} \quad (11)$$

$$\pi_1\vec{\epsilon} = \pi_1 \quad (12)$$

$$\pi_2\vec{\epsilon} = \pi_2 \quad (13)$$

$$\mathbf{app}\vec{\epsilon} = \mathbf{app} \quad (14)$$

$$\mathbf{cur}(f)\vec{\epsilon} = \mathbf{cur}(f\vec{\epsilon}) \quad (15)$$

$$\langle f, g \rangle \vec{\epsilon} = \langle f\vec{\epsilon}, g\vec{\epsilon} \rangle \quad (16)$$

$$(17)$$

0.3.2 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation $\llbracket \gamma \rrbracket_M \in \mathbf{obj} \ \mathbb{C}$. The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$\llbracket \gamma \rrbracket_M : E^n \rightarrow \mathbf{obj} \ \mathbb{C} \quad (18)$$

$$\vec{\epsilon} \mapsto \llbracket \gamma \rrbracket_M \quad (19)$$

$$(20)$$

Each constant term \mathbf{C}^A in the non-polymorphic calculus has a fixed denotation $\llbracket \mathbf{C}^A \rrbracket_M \in \mathbb{C}(1, A)$.

So the morphism $\llbracket \mathbf{C}^A \rrbracket_M$ in $[E^n, \mathbb{C}]$ is the corresponding constant dependently typed morphism.

$$\llbracket \mathbf{C}^A \rrbracket_M : [E^n, \mathbb{C}](1, A) \quad (21)$$

$$\vec{\epsilon} \mapsto \llbracket \mathbf{C}^A \rrbracket_M \quad (22)$$

0.3.3 Graded Monad

Given the strong graded monad $(T^0, \eta^0, \mu^0, \tau^0)$ on \mathbb{C} we can construct an appropriate graded monad on $[E^n, \mathbb{C}]$.

$$\mathbf{T}^n : (E^n, \cdot, \leq_n, 1_n) \rightarrow [[E^n, \mathbb{C}], [E^n, \mathbb{C}]] \quad (23)$$

$$(\mathbf{T}_f^n A)\vec{\epsilon} = \mathbf{T}_{(f\vec{\epsilon})}^0 A\vec{\epsilon} \quad (24)$$

$$(\eta_A^n)\vec{\epsilon} = \eta_{A\vec{\epsilon}}^0 \quad (25)$$

$$(\mu_{f,g,A}^n)\vec{\epsilon} = \mu_{(f\vec{\epsilon}), (g\vec{\epsilon}), (A\vec{\epsilon})}^0 \quad (26)$$

$$(\mathbf{t}_{f,A,B}^n)\vec{\epsilon} = \mathbf{t}_{(f\vec{\epsilon}), (A\vec{\epsilon}), (B\vec{\epsilon})}^0 \quad (27)$$

Through some mechanical proof and the naturality of the \mathbb{C} strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in $[E^n, \mathbb{C}]$

Naturality

$$\begin{array}{ccc} A\vec{\epsilon} & \xrightarrow{\eta_{(A\vec{\epsilon})}^0} & \mathbf{T}_1^0(A\vec{\epsilon}) \\ \downarrow f\vec{\epsilon} & & \downarrow \mathbf{T}_1^0(f\vec{\epsilon}) \\ B\vec{\epsilon} & \xrightarrow{\eta_{(B\vec{\epsilon})}^0} & \mathbf{T}_1^0(B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} \mathbf{T}_{(f\vec{\epsilon})}^0 \mathbf{T}_{(g\vec{\epsilon})}^0(A\vec{\epsilon}) & \xrightarrow{\mu_{f\vec{\epsilon}, g\vec{\epsilon}, (B\vec{\epsilon})}^0} & \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0(A\vec{\epsilon}) \\ \downarrow \mathbf{T}_{f\vec{\epsilon}}^0 \mathbf{T}_{g\vec{\epsilon}}^0 m\vec{\epsilon} & & \downarrow \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0 m\vec{\epsilon} \\ \mathbf{T}_{(f\vec{\epsilon})}^0 \mathbf{T}_{(g\vec{\epsilon})}^0(B\vec{\epsilon}) & \xrightarrow{\mu_{f\vec{\epsilon}, g\vec{\epsilon}, (B\vec{\epsilon})}^0} & \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0(B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B\vec{\epsilon}) \\ \downarrow (m\vec{\epsilon} \times \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 B}) & & \downarrow \mathbf{T}_{(f\vec{\epsilon})}^0(m\vec{\epsilon} \times \text{Id}_{B\vec{\epsilon}}) \\ A'\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A'\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A'\vec{\epsilon} \times B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B\vec{\epsilon}) \\ \downarrow (\text{Id}_{A\vec{\epsilon}} \times \mathbf{T}_{f\vec{\epsilon}}^0(m\vec{\epsilon})) & & \downarrow \mathbf{T}_{(f\vec{\epsilon})}^0(\text{Id}_{A\vec{\epsilon}} \times m\vec{\epsilon}) \\ A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B'\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B'\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B'\vec{\epsilon}) \end{array}$$

Monad Laws

Left Unit

$$(\mu_{f,1,A}^n \circ \mathbf{T}_f^n \eta_A^n)\vec{\epsilon} = \mu_{(f\vec{\epsilon}), 1, (A\vec{\epsilon})}^0 \circ \mathbf{T}_{f\vec{\epsilon}}^0(\eta_{A\vec{\epsilon}}^0) \quad (28)$$

$$= \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 A\vec{\epsilon}} \quad (29)$$

$$= (\text{Id}_{\mathbf{T}_f^n A})\vec{\epsilon} \quad (30)$$

Right Unit

$$(\mu_{1,g,A}^n \circ \eta_{\mathbf{T}_f^n A}^n)\vec{\epsilon} = \mu_{1, (f\vec{\epsilon}), (A\vec{\epsilon})}^0 \circ (\eta_{\mathbf{T}_{f\vec{\epsilon}}^0 A\vec{\epsilon}}^0) \quad (31)$$

$$= \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 A\vec{\epsilon}} \quad (32)$$

$$= (\text{Id}_{\mathbf{T}_f^n A})\vec{\epsilon} \quad (33)$$

Monad Associativity

$$((\mu_{f,(g \cdot h),A}^n) \circ \mathbf{T}_f^n(\mu_{g,h,A}^n))\vec{e} = \mu_{(f\vec{e}),((g\vec{e}) \cdot (h\vec{e})),(A\vec{e})}^0 \circ \mathbf{T}_{f\vec{e}}^0 \mu_{(h\vec{e}), (g\vec{e}), A\vec{e}}^0 \quad (34)$$

$$= \mu_{((f\vec{e}) \cdot (g\vec{e})), (h\vec{e}), (A\vec{e})}^0 \circ \mu_{(f\vec{e}), (g\vec{e}), (\mathbf{T}_{h\vec{e}}^0(A\vec{e}))}^0 \quad (35)$$

$$= (\mu_{f \cdot g, h, A}^n \circ \mu_{f, g, \mathbf{T}_h^0 A}^n) \vec{e} \quad (36)$$

Tensorial Strength

Unitor Law

$$(\mathbf{T}_f^n \pi_2) \vec{e} = \mathbf{T}_{(f\vec{e})}^0 (\pi_2 \vec{e}) \quad (37)$$

$$= \mathbf{T}_{(f\vec{e})}^0 (\pi_2) \quad (38)$$

$$= \pi_2 \quad (39)$$

$$= \pi_2 \vec{e} \quad (40)$$

Bind Law

$$\begin{array}{ccc} A \times \mathbf{T}_f^n \mathbf{T}_g^n B & \xrightarrow{\mathbf{t}_{f,A, \mathbf{T}_g^n B}} & \mathbf{T}_f^n(A \times \mathbf{T}_g^n B) \xrightarrow{\mathbf{T}_f^n \mathbf{t}_{g,A,B}} \mathbf{T}_f^n \mathbf{T}_g^n(A \times B) \\ & \searrow \text{Id}_A \times \mu_{f,g,B}^n & \downarrow \mu_{f,g,A \times B}^n \\ & A \times \mathbf{T}_{f \cdot g}^n B & \xrightarrow{\mathbf{t}_{f \cdot g, A, B}} \mathbf{T}_{f \cdot g}^n(A \times B) \end{array}$$

$$(\mathbf{t}_{(f \cdot g), A, B}^n \circ (\text{Id}_A \times \mu_{f,g,B}^n)) \vec{e} = (\mathbf{t}_{((f\vec{e}) \cdot (g\vec{e})), (A\vec{e}), (B\vec{e})}^0 \circ (\text{Id}_{A\vec{e}} \times \mu_{(f\vec{e}), (g\vec{e}), (B\vec{e})}^n)) \quad (41)$$

$$= \mu_{(f\vec{e}), (g\vec{e}), (A \times B)\vec{e}}^0 \circ \mathbf{T}_{f\vec{e}}^0 (\mathbf{t}_{(g\vec{e}), (A\vec{e}), (B\vec{e})}^0) \circ \mathbf{t}_{(f\vec{e}), (A\vec{e}), \mathbf{T}_{g\vec{e}}^0(B\vec{e})}^0 \quad (42)$$

$$= (\mu_{f,g, (A \times B)}^n \circ \mathbf{T}_f^n(\mathbf{t}_{g,A,B}^n) \circ \mathbf{t}_{f,A, \mathbf{T}_g^n(B)}^n) \vec{e} \quad (43)$$

Commutativity with Unit

$$\begin{array}{ccc} A \times B & \xrightarrow{\text{Id}_A \times \eta_B} & A \times \mathbf{T}_1 B \\ & \searrow \eta_{A \times B} & \downarrow \mathbf{t}_{1,A,B} \\ & & \mathbf{T}_1^n(A \times B) \end{array}$$

$$(\mathbf{t}_{1,A,B}^n \circ (\text{Id}_A \times \eta_A^n)) \vec{e} = \mathbf{t}_{1, (A\vec{e}), (B\vec{e})}^0 \circ (\text{Id}_{A\vec{e}} \times \eta_{A\vec{e}}^0) \quad (44)$$

$$= \eta_{A\vec{e} \times B\vec{e}}^0 \quad (45)$$

$$= (\eta_{A \times B}^n) \vec{e} \quad (46)$$

Commutativity with α Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \rightarrow (A \times (B \times C))$

$$\begin{array}{ccc} (A \times B) \times \mathbf{T}_\epsilon^n C & \xrightarrow{\mathbf{t}_{\epsilon, (A \times B), C}} & \mathbf{T}_\epsilon^n((A \times B) \times C) \\ \downarrow \alpha_{A,B, \mathbf{T}_\epsilon^n C} & & \downarrow \mathbf{T}_\epsilon^n \alpha_{A,B,C} \\ A \times (B \times \mathbf{T}_\epsilon^n C) & \xrightarrow{\text{Id}_A \times \mathbf{t}_{\epsilon, B, C}} A \times \mathbf{T}_\epsilon^n(B \times C) \xrightarrow{\mathbf{t}_{\epsilon, A, (B \times C)}} \mathbf{T}_\epsilon^n(A \times (B \times C)) \end{array}$$

$$(\mathbf{T}_f^n \alpha_{A,B,C} \circ \mathbf{t}_{f,A \times B,C}^n) \vec{\epsilon} = \mathbf{T}_{f\vec{\epsilon}}^0 \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathbf{t}_{(f\vec{\epsilon}), (A \times B)\vec{\epsilon}, (C\vec{\epsilon})}^0 \quad (47)$$

$$= \mathbf{t}_{(f\vec{\epsilon}), (A\vec{\epsilon}), (B\vec{\epsilon} \times C\vec{\epsilon})}^0 \circ (\mathbf{Id}_{A\vec{\epsilon}} \times \mathbf{t}_{(f\vec{\epsilon}), (B\vec{\epsilon}), (C\vec{\epsilon})}^0) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \quad (48)$$

$$= (\mathbf{t}_{f,A,(B \times C)}^n \circ (\mathbf{Id}_A \times \mathbf{t}_{f,B,C}^n) \circ \alpha_{A,B,C}) \vec{\epsilon} \quad (49)$$

$$(50)$$

0.3.4 Sub-Effecting

Given a collection of sub-effecting natural transformation in \mathbb{C} ,

$$\llbracket \epsilon_1 \leq_0 \epsilon_2 \rrbracket_M : \mathbf{T}_{\epsilon_1}^0 \rightarrow \mathbf{T}_{\epsilon_2}^0 \quad (51)$$

We can form sub-effect natural transformations in $[E^n, \mathbb{C}]$:

$$\llbracket f \leq_n g \rrbracket_M : \mathbf{T}_f^n \rightarrow \mathbf{T}_g^n \quad (52)$$

$$\llbracket f \leq_n g \rrbracket_M A\vec{\epsilon} : \mathbf{T}_{f\vec{\epsilon}}^n(A\vec{\epsilon}) \rightarrow \mathbf{T}_{g\vec{\epsilon}}^n(B\vec{\epsilon}) \quad (53)$$

$$= \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M A\vec{\epsilon} \quad (54)$$

Naturality

$$\begin{array}{ccc} \mathbf{T}_{f\vec{\epsilon}}^0 A\vec{\epsilon} & \xrightarrow{\llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M} & \mathbf{T}_{g\vec{\epsilon}}^0 A\vec{\epsilon} \\ \downarrow \mathbf{T}_{f\vec{\epsilon}}^0 m\vec{\epsilon} & & \downarrow \mathbf{T}_{g\vec{\epsilon}}^0 m\vec{\epsilon} \\ \mathbf{T}_{f\vec{\epsilon}}^0 B\vec{\epsilon} & \xrightarrow{\llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M} & \mathbf{T}_{g\vec{\epsilon}}^0 B\vec{\epsilon} \end{array}$$

Commutes With Tensor Strength

$$\begin{array}{ccc} A \times \mathbf{T}_f^n B & \xrightarrow{\mathbf{Id}_A \times \llbracket f \leq_n g \rrbracket_B} & A \times \mathbf{T}_g^n B \\ \downarrow \mathbf{t}_{f,A,B}^n & & \downarrow \mathbf{t}_{g,A,B}^n \\ \mathbf{T}_f^n(A \times B) & \xrightarrow{\llbracket f \leq_n g \rrbracket_{A \times B}} & \mathbf{T}_g^n(A \times B) \end{array}$$

$$(\mathbf{t}_{g,A,B}^n \circ (\mathbf{Id}_A \times \llbracket f \leq_n g \rrbracket_B)) \vec{\epsilon} = \mathbf{t}_{(g\vec{\epsilon}), (A\vec{\epsilon}), (B\vec{\epsilon})}^0 \circ (\mathbf{Id}_{A\vec{\epsilon}} \times \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_{B\vec{\epsilon}}) \quad (55)$$

$$= \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_{(A \times B)\vec{\epsilon}} \circ \mathbf{t}_{(f\vec{\epsilon}), (A\vec{\epsilon}), (B\vec{\epsilon})}^0 \quad (56)$$

$$= (\llbracket f \leq_n g \rrbracket_{(A \times B)} \circ \mathbf{t}_{f,A,B}^n) \vec{\epsilon} \quad (57)$$

$$(58)$$

Commutes with Join

$$\begin{array}{ccc} \mathbf{T}_f^n \mathbf{T}_g^n & \xrightarrow{\mathbf{T}_f^n \llbracket g \leq_n g' \rrbracket_M} & \mathbf{T}_f^n \mathbf{T}_{g'}^n \xrightarrow{\llbracket f \leq_n f' \rrbracket_M, \mathbf{T}_{g'}^n} \mathbf{T}_{f'}^n \mathbf{T}_{g'}^n \\ \downarrow \mu_{f,g}^n & & \downarrow \mu_{f',g'}^n \\ \mathbf{T}_{f \cdot g}^n & \xrightarrow{\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_M} & \mathbf{T}_{f' \cdot g'}^n \end{array}$$

$$(\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_A \circ \mu_{f,g,A}^n) \vec{\epsilon} = \llbracket (f \vec{\epsilon}) \cdot (g \vec{\epsilon}) \leq_0 (f' \vec{\epsilon}) \cdot (g \vec{\epsilon}) \rrbracket_{A \vec{\epsilon}} \circ \mu_{(f \vec{\epsilon}), (g \vec{\epsilon}), (A \vec{\epsilon})}^0 \quad (59)$$

$$= \mu_{(f \vec{\epsilon}), (g \vec{\epsilon}), (A \vec{\epsilon})}^0 \circ \llbracket f \vec{\epsilon} \leq_0 f' \vec{\epsilon} \rrbracket_{\mathbf{T}_{g' \vec{\epsilon}}^0(A \vec{\epsilon})} \circ \mathbf{T}_{f \vec{\epsilon}}^0 \llbracket g \vec{\epsilon} \leq_0 g' \vec{\epsilon} \rrbracket_{(A \vec{\epsilon})} \quad (60)$$

$$= \mu_{f,g,A}^n \circ \llbracket f \leq_n f' \rrbracket_{\mathbf{T}_{g',A}^n} \circ \mathbf{T}_f^n \llbracket g \leq_n g' \rrbracket_A \quad (61)$$

0.3.5 Sub-Typing

Sub-typing in $[E^n, \mathbb{C}]$ holds via sub-typing in \mathbb{C}

$$\llbracket A \leq_n B \rrbracket_M : A \rightarrow B \quad (62)$$

$$\llbracket A \leq_n B \rrbracket_M \vec{\epsilon} = \llbracket A \vec{\epsilon} \leq_0 B \vec{\epsilon} \rrbracket_M \quad (63)$$

So the subtyping relation $A \leq B$ forms a morphism in $[E^n, \mathbb{C}]$

0.4 Functors Between S-Categories

For a function $\theta : E^m \rightarrow E^n$, the re-indexing functor θ^* is defined as follows:

$$\theta^* : [E^n, \mathbb{C}] \rightarrow [E^m, \mathbb{C}] \quad (64)$$

$$\theta^*(A) \epsilon_m^\rightarrow = A(\theta(\epsilon_m^\rightarrow)) \quad (65)$$

$$f : A \rightarrow B \in [E^n, \mathbb{C}] \quad (66)$$

$$\theta^*(f) \epsilon_m^\rightarrow = f(\theta(\epsilon_m^\rightarrow)) : A(\theta(\epsilon_m^\rightarrow)) \rightarrow B(\theta(\epsilon_m^\rightarrow)) \quad (67)$$

0.4.1 θ^* is S-closed.

CCC

$$(\theta^*(A \times B)) \vec{\epsilon} = (A \times B)(\theta \vec{\epsilon}) \quad (68)$$

$$= (A(\theta \vec{\epsilon}) \times B(\theta \vec{\epsilon})) \quad (69)$$

$$= (\theta^* A \times \theta^* B) \vec{\epsilon} \quad (70)$$

$$(71)$$

$$(\theta^* \pi_1) \vec{\epsilon} = \pi_1(\theta \vec{\epsilon}) \quad (72)$$

$$= \pi_1 \quad \text{Constant function} \quad (73)$$

$$= \pi_1 \vec{\epsilon} \quad (74)$$

$$(\theta^* \pi_2) \vec{\epsilon} = \pi_2(\theta \vec{\epsilon}) \quad (75)$$

$$= \pi_2 \quad \text{Constant function} \quad (76)$$

$$= \pi_2 \vec{\epsilon}$$

$$(\theta^* \langle f, g \rangle) \vec{\epsilon} = (\langle f, g \rangle)(\theta \vec{\epsilon}) \quad (77)$$

$$= \langle f(\theta \vec{\epsilon}), g(\theta \vec{\epsilon}) \rangle \quad (78)$$

$$= \langle \theta^* f, \theta^* g \rangle \vec{\epsilon} \quad (79)$$

$$(\theta^* (A^B)) \vec{\epsilon} = (A^B)(\theta \vec{\epsilon}) \quad (80)$$

$$= (A(\theta \vec{\epsilon}))^{(B(\theta \vec{\epsilon}))} \quad (81)$$

$$= (\theta^* A)^{(\theta^* B)} \vec{\epsilon} \quad (82)$$

$$(83)$$

$$(\theta^* \mathbf{app}) \vec{\epsilon} = \mathbf{app}(\theta \vec{\epsilon}) \quad (84)$$

$$= \mathbf{app} \quad \text{Constant fn} \quad (85)$$

$$= \mathbf{app} \vec{\epsilon} \quad (86)$$

$$(\theta^* \mathbf{cur}(f)) \vec{\epsilon} = \mathbf{cur}(f)(\theta \vec{\epsilon}) \quad (87)$$

$$= \mathbf{cur}(f(\theta \vec{\epsilon})) \quad (88)$$

$$= \mathbf{cur}(\theta^* f) \quad (89)$$

$$(\theta^* \mathbf{1}) \vec{\epsilon} = \mathbf{1}(\theta \vec{\epsilon}) \quad (90)$$

$$= \mathbf{1} \quad (91)$$

$$= \mathbf{1}'_I \vec{\epsilon} \quad (92)$$

$$(93)$$

$$(\theta^* \langle \rangle_A) \vec{\epsilon} = \langle \rangle_A (\theta \vec{\epsilon}) \quad (94)$$

$$= \langle \rangle_{A(\theta \vec{\epsilon})} \quad (95)$$

$$= \langle \rangle_{\theta^* A} \vec{\epsilon} \quad (96)$$

Co-Product

$$(\theta^* (\mathbf{1} + \mathbf{1})) \vec{\epsilon} = (\mathbf{1} + \mathbf{1})(\theta \vec{\epsilon}) \quad (97)$$

$$= (\mathbf{1} + \mathbf{1}) \quad \text{Constant function} \quad (98)$$

$$= (\mathbf{1} + \mathbf{1}) \vec{\epsilon} \quad (99)$$

$$(\theta^* \mathbf{inl}) \vec{\epsilon} = \mathbf{inl}(\theta \vec{\epsilon}) \quad (100)$$

$$= \mathbf{inl} \quad \text{Constant Fn} \quad (101)$$

$$= \mathbf{inl} \vec{\epsilon} \quad (102)$$

$$(\theta^* \mathbf{inr})\vec{\epsilon} = \mathbf{inr}(\theta\vec{\epsilon}) \quad (103)$$

$$= \mathbf{inr} \quad \text{Constant Fn} \quad (104)$$

$$= \mathbf{inr}\vec{\epsilon} \quad (105)$$

$$(\theta^*[f, g])\vec{\epsilon} = [f, g](\theta\vec{\epsilon}) \quad (106)$$

$$= [f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})] \quad (107)$$

$$= [\theta^*f, \theta^*g]\vec{\epsilon} \quad (108)$$

$$(109)$$

Strong Graded Monad

$$(\theta^* \mathbf{T}_f^n A)\vec{\epsilon} = \mathbf{T}_f^n A(\theta\vec{\epsilon}) \quad (110)$$

$$= \mathbf{T}_{(f(\theta\vec{\epsilon}))}^0(A(\theta\vec{\epsilon})) \quad (111)$$

$$= (\mathbf{T}_{(f \circ \theta)}^m \theta^* A)\vec{\epsilon} \quad (112)$$

$$(113)$$

$$(\theta^* \eta_A^n)\vec{\epsilon} = \eta_A^n(\theta\vec{\epsilon}) \quad (114)$$

$$= \eta_{A(\theta\vec{\epsilon})}^0 \quad (115)$$

$$= \eta_{\theta^* A}^m \vec{\epsilon} \quad (116)$$

$$(117)$$

$$(\theta^* \mathbf{t}_{f, A, B}^n)\vec{\epsilon} = \mathbf{t}_{f, A, B}^n(\theta\vec{\epsilon}) \quad (118)$$

$$= \mathbf{t}_{(f(\theta\vec{\epsilon})), (A(\theta\vec{\epsilon})), (B(\theta\vec{\epsilon}))}^0 \quad (119)$$

$$= \mathbf{t}_{f \circ \theta, \theta^* A, \theta^* B} \vec{\epsilon} \quad (120)$$

$$(121)$$

Sub-Effecting

$$(\theta^* (\llbracket f \leq_n g \rrbracket A))\vec{\epsilon} = (\llbracket f \leq_n g \rrbracket A)(\theta\vec{\epsilon}) \quad (122)$$

$$= (\llbracket f(\theta\vec{\epsilon}) \leq_n g(\theta\vec{\epsilon}) \rrbracket (A(\theta\vec{\epsilon}))) \quad (123)$$

$$= (\theta^* f \theta^* g(\theta^* A))\vec{\epsilon} \quad (124)$$

Ground Sub-Typing

$$\theta^* (\llbracket A \leq_\gamma B \rrbracket_M)\vec{\epsilon} = \llbracket A \leq_\gamma B \rrbracket_M(\theta\vec{\epsilon}) \quad (125)$$

$$= \llbracket A \leq B \rrbracket_M \quad \text{Constant Function} \quad (126)$$

$$= \llbracket A \leq B \rrbracket_M \vec{\epsilon} \quad (127)$$

$$(128)$$

0.4.2 Quantification

We need to define $\forall_{E^n} : [E^{n+1}, \mathbb{C}] \rightarrow [E^n, \mathbb{C}]$

So

$$(\forall_{E^n} A) \vec{\epsilon}_n = \Pi_{\epsilon \in E} A(\vec{\epsilon}_n, \epsilon) \quad (129)$$

$$m : A \rightarrow B \quad (130)$$

$$(\forall_{E^n} m) : \forall_{E^n} A \rightarrow \forall_{E^n} B \quad (131)$$

$$(\forall_{E^n} m) \vec{\epsilon}_n = \Pi_{\epsilon \in E} m(\vec{\epsilon}_n, \epsilon) \quad (132)$$

$$(133)$$

0.4.3 Adjunction

It is the case that:

$$\pi_1^* \dashv \forall_{E^n}$$

With unit:

$$\eta_A : A \rightarrow \forall_{E^n} \pi_1^* A \quad (134)$$

$$\eta_A(\vec{\epsilon}_n) = \langle \text{Id}_{A(\vec{\epsilon}_n, e)} \rangle_{\epsilon \in E} \quad (135)$$

And co-unit

$$\epsilon_B : \pi_1^* \forall_{E^n} B \rightarrow B \quad (136)$$

$$\epsilon_B(\vec{\epsilon}_n, \epsilon) = \pi_\epsilon : \Pi_{e \in E} B(\vec{\epsilon}_n, e) \rightarrow \Pi_{e \in E} B(\vec{\epsilon}_n, \epsilon) \quad (137)$$

We then define the natural bi-jection as so:

$$\overline{(-)} : [E^{n+1}, \mathbb{C}](\pi_1^* A, B) \leftrightarrow [E^n, \mathbb{C}](A, \forall_{E^n} B) : \widehat{(-)} \quad (138)$$

$$m : \pi_1^* A \rightarrow B \quad (139)$$

$$\overline{m} : A \rightarrow \forall_{E^n} B \quad (140)$$

$$\overline{m}(\vec{\epsilon}_n) = \langle m(\vec{\epsilon}_n, \epsilon) \rangle_{\epsilon \in E} \quad (141)$$

$$n : A \rightarrow \forall_{E^n} B \quad (142)$$

$$\widehat{n} : \pi_1^* A \rightarrow B \quad (143)$$

$$\widehat{n}(\vec{\epsilon}_n, \epsilon_{n+1}) = \pi_\epsilon \circ g(\vec{\epsilon}_n) \quad (144)$$

This is an Adjunction

For any $g : \pi_1^* A \rightarrow B$,

$$(\epsilon_B \circ \pi_1^*(\overline{g}))(\vec{\epsilon}_n, \epsilon_{n+1}) = \pi_{\epsilon_{n+1}} \circ \langle g(\vec{\epsilon}_n, \epsilon') \rangle_{\epsilon' \in E} \quad (145)$$

$$= g(\vec{\epsilon}_n, \epsilon_{n+1}) \quad (146)$$

0.4.4 Beck-Chevalley Condition

For $\theta : E^m \rightarrow E^n$:

$$((\theta^* \circ \forall_{E^n})A)\vec{\epsilon}_n = \theta^*(\forall_{E^n}A)\vec{\epsilon}_n \quad (147)$$

$$= (\forall_{E^n}A)(\theta(\vec{\epsilon}_n)) \quad (148)$$

$$= \Pi_{\epsilon \in E}(A(\theta(\vec{\epsilon}_n), \epsilon)) \quad (149)$$

$$= \Pi_{\epsilon \in E}(((\theta \times \text{Id}_U)^*A)(\vec{\epsilon}_n, \epsilon)) \quad (150)$$

$$= \forall_{E^m}((\theta \times \text{Id}_E)^*A)\vec{\epsilon}_n \quad (151)$$

$$= ((\forall_{E^m} \circ (\theta \times \text{Id}_E)^*)A)\vec{\epsilon}_n \quad (152)$$

And $\overline{(\theta \times \text{Id}_U)^*}\epsilon = \text{Id}_{\theta^* \circ \forall_I}$.

$$\overline{(\theta \times \text{Id}_U)^*}\epsilon_A \vec{\epsilon} = \langle (\theta \times \text{Id}_U)^* \epsilon_A(\vec{\epsilon}, \epsilon) \rangle_{\epsilon \in E} \quad (153)$$

$$= \langle \epsilon_A(\theta \vec{\epsilon}, \epsilon) \rangle_{\epsilon \in E} \quad (154)$$

$$= \langle \pi_\epsilon \rangle_{\epsilon \in E} : \Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon) \rightarrow \Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon) \quad (155)$$

$$= \text{Id}_{\Pi_{\epsilon \in E} A(\theta \vec{\epsilon}, \epsilon)} \quad (156)$$

$$= \text{Id}_{\forall_{I'} \circ (\theta \times \text{Id}_U)^*A} \vec{\epsilon} \quad (157)$$

$$= \text{Id}_{\theta^* \circ \forall_I} \quad (158)$$