0.1 CCC

The section should be a cartesian closed category. That is it should have:

- A Terminal object $\mathbf{1}$
- Binary products
- Exponentials

0.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endofunctor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot, \mathbf{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_\mathbf{1}$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2}, : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

0.2.1 Left Unit

$$
\begin{array}{ccc}
T_\epsilon A & \xrightarrow{T_\epsilon \eta_A} & T_\epsilon T_\mathbf{1} A \\
& \searrow{\scriptstyle \mathtt{Id}_{T_\epsilon A}} & \downarrow{\scriptstyle \mu_{\epsilon,\mathbf{1},A}} \\
& & T_\epsilon A
\end{array}
$$

0.2.2 Right Unit

$$
\begin{array}{ccc}
T_\epsilon A & \xrightarrow{\eta_{T_\epsilon A}} & T_\mathbf{1} T_\mathbf{1} A \\
& \searrow{\scriptstyle \mathtt{Id}_{T_\epsilon A}} & \downarrow{\scriptstyle \mu_{\mathbf{1},\epsilon,A}} \\
& & T_\epsilon A
\end{array}
$$

0.2.3 Associativity

$$
\begin{array}{ccc}
T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2, T_{\epsilon_3} A}} & T_{\epsilon_1 \cdot \epsilon_2} T_{\epsilon_3} A \\
\downarrow{\scriptstyle T_{\epsilon_1} \mu_{\epsilon_2, \epsilon_3, A}} & & \downarrow{\scriptstyle \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, A}} \\
T_{\epsilon_1} T_{\epsilon_2 \cdot \epsilon_3} A & \xrightarrow{\mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, A}} & T_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} A
\end{array}
$$

0.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon (A \times B)$$

Satisfying the following rules:

0.3.1 Left Naturality

$$A \times T_\epsilon B \xrightarrow{\mathrm{Id}_A \times T_\epsilon f} A \times T_\epsilon B'$$

$$\downarrow \mathbf{t}_{\epsilon,A,B} \qquad\qquad \downarrow \mathbf{t}_{\epsilon,A,B'}$$

$$T_\epsilon(A \times B) \xrightarrow{T_\epsilon(\mathrm{Id}_A \times f)} T_\epsilon(A \times B')$$

0.3.2 Right Naturality

$$A \times T_\epsilon B \xrightarrow{f \times \mathrm{Id}_{T_\epsilon B}} A' \times T_\epsilon B$$

$$\downarrow \mathbf{t}_{\epsilon,A,B} \qquad\qquad \downarrow \mathbf{t}_{\epsilon,A',B}$$

$$T_\epsilon(A \times B) \xrightarrow{T_\epsilon(f \times \mathrm{Id}_B)} T_\epsilon(A' \times B)$$

0.3.3 Unitor Law

$$1 \times T_\epsilon A \xrightarrow{\mathbf{t}_{\epsilon,1,A}} T_\epsilon(1 \times A)$$

$$\searrow^{\lambda_{T_\epsilon A}} \quad \downarrow^{T_\epsilon(\lambda_A)}$$

$$T_\epsilon A$$

Where $\lambda : \mathbf{1} \times \mathrm{Id} \to \mathrm{Id}$ is the left-unitor. $(\lambda = \pi_2)$

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutivity of π_2 with $\mathbf{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathbf{1},B} \circ (\langle\rangle_A \times \mathrm{Id}_B)$$

And $\pi_{2,\mathbf{1}}$ is the left unitor, so by tensorial strength:

$$
\begin{aligned}
T_\epsilon \pi_2 \circ \mathbf{t}_{\epsilon,A,B} &= T_\epsilon \pi_{2,\mathbf{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathrm{Id}_B) \circ \mathbf{t}_{\epsilon,A,B} \\
&= T_\epsilon \pi_{2,1,B} \circ \mathbf{t}_{\epsilon,\mathbf{1},B} \circ (\langle\rangle_A \times \mathrm{Id}_B) \\
&= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathrm{Id}_B) \\
&= \pi_2
\end{aligned}
\tag{1}
$$

So the following commutes:

$$A \times T_\epsilon B \xrightarrow{\mathbf{t}_{\epsilon,A,B}} T_\epsilon(A \times B)$$

$$\searrow^{\pi_2} \quad \downarrow^{T_\epsilon \pi_2}$$

$$T_\epsilon B$$

0.3.4 Commutativity with Join

$$A \times T_{\epsilon_1} T_{\epsilon_2} B \xrightarrow{\mathbf{t}_{\epsilon_1,A,T_{\epsilon_2}B}} T_{\epsilon_1}(A \times T_{\epsilon_2}B) \xrightarrow{T_{\epsilon_1}\mathbf{t}_{\epsilon_2,A,B}} T_{\epsilon_1}T_{\epsilon_2}(A \times B)$$

$$\searrow^{\mathrm{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}} \qquad\qquad \downarrow^{\mu_{\epsilon_1,\epsilon_2,A\times B}}$$

$$A \times T_{\epsilon_1 \cdot \epsilon_2} B \xrightarrow{\mathbf{t}_{\epsilon_1 \cdot \epsilon_2,A,B}} T_{\epsilon_1 \cdot \epsilon_2}(A \times B)$$

0.4 Commutivity with Unit

$$A \times B \xrightarrow{\mathtt{Id}_A \times \eta_B} A \times T_\epsilon B$$

$$\eta_{A \times B} \searrow \qquad \downarrow \mathtt{t}_{\epsilon, A, B}$$

$$T_\epsilon(A \times B)$$

0.5 Commutivity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$(A \times B) \times T_\epsilon C \xrightarrow{\qquad \mathtt{t}_{\epsilon,(A \times B),C} \qquad} T_\epsilon((A \times B) \times C)$$

$$\downarrow \alpha_{A,B,T_\epsilon C} \qquad\qquad\qquad\qquad \downarrow T_\epsilon \alpha_{A,B,C}$$

$$A \times (B \times T_\epsilon C) \xrightarrow{\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}} A \times T_\epsilon(B \times C) \xrightarrow{\mathtt{t}_{\epsilon,A,(B \times C)}} T_\epsilon(A \times (B \times C))$$

TODO: Needed?

0.6 Subeffecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

0.6.1 Subeffecting and Tensor Strength

$$A \times T_{\epsilon_1} B \xrightarrow{\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B} A \times T_{\epsilon_2} B$$

$$\downarrow \mathtt{t}_{\epsilon_1, A, B} \qquad\qquad\qquad \downarrow \mathtt{t}_{\epsilon_2, A, B}$$

$$T_{\epsilon_1}(A \times B) \xrightarrow{[\![\epsilon_1 \leq \epsilon_2]\!]_{A \times B}} T_{\epsilon_2}(A \times B)$$

0.7 Subtyping

The denotation of ground types $[\![\_]\!]_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function subtyping to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)} \frac{f = [\![A' \leq: A]\!]_M \qquad g = [\![B \leq: B']\!]_M \qquad h = [\![\epsilon_1 \leq \epsilon_2]\!]}{rhs = [\![A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B']\!]_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$
\begin{aligned}
rhs &= (h_{B'} \circ T_{\epsilon_1} g)^A \circ (T_{\epsilon_1} B)^f \\
&= \mathtt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f))
\end{aligned}
\tag{2}
$$

0.8 If natural transformation

There exists a natural transformation $\mathtt{If}_A : (\mathtt{Bool} \times (A \times A)) \to A$ Satisfying the following:

- $\mathtt{If}_A \circ \langle [\![\mathtt{true}]\!]_M \circ \langle \rangle_\Gamma, \langle t, f \rangle \rangle = t$

- $\mathtt{If}_A \circ \langle [\![\mathtt{false}]\!]_M \circ \langle \rangle_\Gamma, \langle t, f \rangle \rangle = f$