0.1 Weakening Definition

0.1.1 Relation

We define the ternary weakening relation $w : \Gamma' \rhd \Gamma$ using the following rules.

- (Id)$\dfrac{\Gamma\mathtt{Ok}}{\iota:\Gamma\rhd\Gamma}$

- (Project)$\dfrac{\omega:\Gamma'\rhd\Gamma \quad x\notin\mathtt{dom}(\Gamma')}{\omega\pi:\Gamma,x:A\rhd\Gamma}$

- (Extend)$\dfrac{\omega:\Gamma'\rhd\Gamma \quad x\notin\mathtt{dom}(\Gamma') \quad A\leq:B}{w\times:\Gamma',x:A\rhd\Gamma,x:B}$

0.1.2 Weakening Denotations

The denotation of a weakening relation is defined as follows:

$$[\![\omega : \Gamma' \rhd \Gamma]\!]_M : \Gamma' \to \Gamma \tag{1}$$

- $[\![\iota : \Gamma \rhd \Gamma]\!]_M = \mathtt{Id}_\Gamma : \Gamma \to \Gamma$

- (Project)$\dfrac{f=[\![\omega:\Gamma'\rhd\Gamma]\!]_M:\Gamma'\to\Gamma}{[\![\omega\pi:\Gamma,x:A\rhd\Gamma]\!]_M=f\circ\pi_1:\Gamma'\times A\to\Gamma}$

- (Extend)$\dfrac{f=[\![\omega:\Gamma'\rhd\Gamma]\!]_M:\Gamma'\to\Gamma \quad g=[\![A\leq:B]\!]_M:A\to B}{[\![w\times:\Gamma',x:A\rhd\Gamma,x:B]\!]_M=(f\times g):(\Gamma\times A)\to(\Gamma\times B)}$

0.2 Weakening Theorems

0.2.1 Domain Lemma

If $\omega : \Gamma' \rhd \Gamma$, then $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$.

Proof

Case Id Then $\Gamma' = \Gamma$ and so $\mathtt{dom}(\Gamma') = \mathtt{dom}(\Gamma)$.

Case Project By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma') \subseteq \mathtt{dom}(\Gamma' \cup \{x\})$

Case Extend By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$ so

$$\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\} \subseteq \mathtt{dom}(\Gamma') \cup \{x\} = \mathtt{dom}(\Gamma', x : A)$$

0.2.2 Theorem 1

If $\omega : \Gamma' \rhd \Gamma$ and $\Gamma\mathtt{Ok}$ then $\Gamma'\mathtt{Ok}$

Proof

Case Id

$$(\mathrm{Id})\dfrac{\Gamma\mathtt{Ok}}{\iota : \Gamma \rhd \Gamma}$$

By inversion, $\Gamma\mathtt{Ok}$.

Case Project

$$(\text{Project})\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\omega : \Gamma' \triangleright \Gamma$ and $x \notin \mathtt{dom}(\Gamma')$.
Hence by induction $\Gamma'\mathtt{Ok}$, $\Gamma\mathtt{Ok}$. Since $x \notin \mathtt{dom}(\Gamma')$, we have $\Gamma', x : A\mathtt{Ok}$.

Case Extend $(\text{Extend})\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma') \quad A \leq : B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B}$,
By inversion, we have
$\omega : \Gamma' \triangleright \Gamma$, $x \notin \mathtt{dom}(\Gamma')$.
Hence we have $\Gamma\mathtt{Ok}$, $\Gamma'\mathtt{Ok}$, and by the domain Lemma, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$, hence $x \notin \mathtt{dom}(\Gamma)$. Hence, we have $\Gamma, x : A\mathtt{Ok}$ and $\Gamma', x : A\mathtt{Ok}$

0.2.3 Theorem 2

If $\Gamma \vdash t : \tau$ and $\omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Gamma' \vdash t : \tau$

Proof Proved in parallel with theorem 3 below

0.2.4 Theorem 3

If $\omega : \Gamma' \triangleright \Gamma$ and $\Delta = [\![\Gamma \vdash t : \tau]\!]_M$ and $\Delta' = [\![\Gamma' \vdash t : \tau]\!]_M$, derived using Theorem 2, then

$$\Delta \circ [\![\omega]\!]_M = \Delta' : \Gamma' \to [\![\tau]\!]_M$$

Proof Below

0.3 Proof of Theorems 2 and 3

We induct over the structure of typing derivations of $\Gamma \vdash t : \tau$, assuming $\omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash t : \tau$ and show that $\Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M = \Delta'$

0.3.1 Variable Terms

Case Var and Weaken We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x : A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\iota : \Gamma \triangleright \Gamma]\!]_M \tag{2}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x : A$, such that

$$\Delta_1 = \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \quad \text{By Induction} \tag{3}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Gamma'' \vdash x : A}{\Gamma'', x' : A' \vdash x : A} \tag{4}$$

This preserves denotations:

$$\Delta' = \Delta_1 \circ \pi_1 \quad \text{By Definition} \tag{5}$$
$$= \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \circ \pi_1 \quad \text{By induction} \tag{6}$$
$$= \Delta \circ [\![\omega'\pi_1 : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By denotation of weakening} \tag{7}$$

If $\omega = \omega' \times$ Then

$$\Gamma' = \Gamma''', x' : B \tag{8}$$

$$\Gamma = \Gamma'', x' : A' \tag{9}$$

$$B \leq: A \tag{10}$$

If $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$\text{(Sub-type)} \frac{\text{(var)} \overline{\Gamma''', x:B \vdash x:B} \quad B \leq: A}{\Gamma' \vdash x: A} \tag{11}$$

This preserves denotations:

$$\Delta' = [\![B \leq: A]\!]_M \circ \pi_2 \quad \text{By Definition} \tag{12}$$

$$= \pi_2 \circ ([\![\omega' : \Gamma''' \rhd \Gamma'']\!]_M \times [\![B \leq: A]\!]_M) \quad \text{By the properties of binary products} \tag{13}$$

$$= \Delta \circ [\![\omega : \Gamma' \rhd \Gamma]\!]_M \quad \text{By Definition} \tag{14}$$

Case $x \neq x'$ Then

$$\Delta = \text{(Weaken)} \frac{() \frac{\Delta_1}{\Gamma'' \vdash x:A}}{\Gamma \vdash x: A} \tag{15}$$

By induction with $\omega : \Gamma''' \rhd \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x: A$

We have the weakened derivation:

$$\Delta' = \text{(Weaken)} \frac{() \frac{\Delta'_1}{\Gamma''' \vdash x:A}}{\Gamma' \vdash x: A} \tag{16}$$

This preserves denotations:

By induction, we have

$$\Delta'_1 = \Delta_1 \circ [\![\omega : \Gamma''' \rhd \Gamma'']\!]_M \tag{17}$$

So we have:

$$\Delta' = \Delta'_1 \circ \pi_1 \quad \text{By denotation definition} \tag{18}$$

$$= \Delta_1 \circ [\![\omega' : \Gamma''' \rhd \Gamma'']\!]_M \quad \text{By induction} \circ \pi_1 \tag{19}$$

$$= \Delta_1 \circ \pi_1 \circ ([\![\omega' : \Gamma''' \rhd \Gamma'']\!]_M \times [\![A' \leq: B]\!]_M) \quad \text{By product properties} \tag{20}$$

$$= \Delta \circ [\![\omega : \Gamma' \rhd \Gamma]\!]_M \quad \text{By definition} \tag{21}$$

0.3.2   Value Terms

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\omega : \Gamma' \rhd \Gamma']\!]_M$, simply as ω.

Case Constant   The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$\text{(Const)} \frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A} \tag{22}$$

By inversion, we have $\Gamma \mathtt{Ok}$, so we have $\Gamma' \mathtt{Ok}$.

Hence

$$\text{(Const)} \frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A : A} \tag{23}$$

3

Holds.

This preserves denotations:

$$\Delta' = [\![\mathsf{c}^A]\!]_M \circ \langle \rangle_{\Gamma'} \quad \text{By definition} \tag{24}$$

$$= [\![\mathsf{c}^A]\!]_M \circ \langle \rangle_\Gamma \circ \omega \quad \text{By the terminal property} \tag{25}$$

$$= \Delta \quad \text{By Definition} \tag{26}$$

$$\tag{27}$$

Case Lambda   By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn}) \frac{() \frac{\Delta_1}{\Gamma, x:A \vdash C : \mathsf{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \mathsf{M}_\epsilon B} \tag{28}$$

Since $\omega : \Gamma' \triangleright \Gamma$, we have:

$$\omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{29}$$

Hence, by induction, using $\omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = (\text{Fn}) \frac{() \frac{\Delta_1'}{\Gamma', x:A \vdash C : \mathsf{M}_\epsilon B}}{\Gamma', x : A \vdash \lambda x : A.C : A \to \mathsf{M}_\epsilon B} \tag{30}$$

This preserves denotations:

$$\Delta' = \mathsf{cur}(\Delta_1') \quad \text{By Definition} \tag{31}$$

$$= \mathsf{cur}(\Delta_1 \circ (\omega \times \mathsf{Id}_\Gamma)) \quad \text{By the denotation of } \omega \times \tag{32}$$

$$= \mathsf{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} \tag{33}$$

$$= \Delta \circ \omega \quad \text{By Definition} \tag{34}$$

Case Sub-typing

$$(\text{Sub-type}) \frac{\Gamma \vdash v{:}A \quad A \leq: B}{\Gamma \vdash v{:}B} \tag{35}$$

by inversion, we have a derivation Δ_1

$$() \frac{\Delta_1}{\Gamma \vdash v{:}A} \tag{36}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-type}) \frac{() \frac{\Delta_1'}{\Gamma' \vdash v{:}a} \quad A \leq: B}{\Gamma' \vdash v{:}B} \tag{37}$$

This preserves denotations:

$$\Delta' = [\![A \leq: B]\!]_M \circ \Delta_1' \quad \text{By Definition} \tag{38}$$

$$= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \omega \quad \text{By induction} \tag{39}$$

$$= \Delta \circ \omega \quad \text{By Definition} \tag{40}$$

$$\tag{41}$$

0.3.3 Computation Terms

Case Return We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma\vdash v:A}}{\Gamma \vdash \texttt{return}\, v: \texttt{M}_1 A} \tag{42}$$

Hence, by induction, with $\omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ_1' such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v:A}}{\Gamma' \vdash \texttt{return}\, v: \texttt{M}_1 A} \tag{43}$$

This preserves denotations:

$$\Delta' = \eta_A \circ \Delta_1' \quad \text{By definition} \tag{44}$$
$$= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta_1' \tag{45}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{46}$$

Case Apply By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1:A\to\texttt{M}_\epsilon B} \; ()\frac{\Delta_2}{\Gamma\vdash v_2:A}}{\Gamma \vdash v_1\ v_2: \texttt{M}_\epsilon B} \tag{47}$$

By induction, this gives us the respective derivations: Δ_1', Δ_2' such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v_1:A\to\texttt{M}_\epsilon B} \; ()\frac{\Delta_2'}{\Gamma'\vdash v_2:A}}{\Gamma' \vdash v_1\ v_2: \texttt{M}_\epsilon B} \tag{48}$$

This preserves denotations:

$$\Delta' = \texttt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Definition} \tag{49}$$
$$= \texttt{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \tag{50}$$
$$= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \tag{51}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{52}$$

Case If By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \; ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \; ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2: \texttt{M}_\epsilon A} \tag{53}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v:\texttt{Bool}} \; ()\frac{\Delta_2'}{\Gamma'\vdash C_1:\texttt{M}_\epsilon A} \; ()\frac{\Delta_3'}{\Gamma'\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2: \texttt{M}_\epsilon A} \tag{54}$$

And

$$\Delta_1' = \Delta_1 \circ \omega \tag{55}$$
$$\Delta_3' = \Delta_2 \circ \omega \tag{56}$$
$$\Delta_3' = \Delta_3 \circ \omega \tag{57}$$

This preserves denotations. Since $\omega : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\omega : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\omega = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{58}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \omega)) = (T_\epsilon A)^\omega \circ \mathtt{cur}(f) \tag{59}$$

$$\Delta' = \mathtt{app} \circ (([\mathtt{cur}(\Delta'_2 \circ \pi_2), \mathtt{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} \tag{60}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \omega \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta'_1) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} \tag{61}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2 \circ (\mathtt{Id_1} \times \omega)), \mathtt{cur}(\Delta_3 \circ \pi_2 \circ (\mathtt{Id_1} \times \omega))] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} \tag{62}$$

$$= \mathtt{app} \circ ((([(T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\omega \text{ property} \tag{63}$$

$$= \mathtt{app} \circ (((T_\epsilon A)^\omega \circ [\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} \tag{64}$$

$$= \mathtt{app} \circ ((T_\epsilon A)^\omega \times \mathtt{Id}_{\Gamma'}) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} \tag{65}$$

$$= \mathtt{app} \circ (\mathtt{Id}_{(T_\epsilon A)} \times \omega) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \mathtt{app}, (T_\epsilon A)^\omega \tag{66}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} \tag{67}$$

$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \circ \omega \quad \text{By Definition of the diagonal morphism.} \tag{68}$$

$$= \Delta \circ \omega \tag{69}$$

Case Bind By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind}) \frac{() \frac{\Delta_1}{\Gamma \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad () \frac{\Delta_2}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \mathtt{do} \; x \leftarrow C_1 \; \mathtt{in} \; C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{70}$$

If $\omega : \Gamma' \rhd \Gamma$ then $\omega \times : \Gamma', x : A \rhd \Gamma, x : A$, so by induction, we can derive Δ'_1, Δ'_2 such that:

$$\Delta' = (\text{Bind}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad () \frac{\Delta'_2}{\Gamma', x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathtt{do} \; x \leftarrow C_1 \; \mathtt{in} \; C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{71}$$

This preserves denotations:

$$\Delta' = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta'_1 \rangle \quad \text{By definition} \tag{72}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1}(\Delta_2 \circ (\omega \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta'_1, \Delta'_2 \tag{73}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} \tag{74}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \circ \omega \quad \text{By product property} \tag{75}$$

$$= \Delta \quad \text{By definition} \tag{76}$$

---

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

Case Sub-effect

$$(\text{Sub-effect}) \frac{\Gamma \vdash C : \mathtt{M}_{\epsilon_1} A \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \mathtt{M}_{\epsilon_2} B} \tag{77}$$

by inversion, we have a derivation Δ_1

$$() \frac{\Delta_1}{\Gamma \vdash C : \mathtt{M}_{\epsilon_1} A} \tag{78}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-effect}) \frac{() \frac{\Delta_1'}{\Gamma' \vdash C : \mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C : \mathtt{M}_{\epsilon_2} B} \tag{79}$$

This preserves denotations:
Let

$$g = [\![A \leq: B]\!]_M : A \to B \tag{80}$$
$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M : T_{\epsilon_1} \to T_{\epsilon_2} \tag{81}$$

Then

$$\Delta' = h_B \circ T_{\epsilon_1} g \circ \Delta_1' \quad \text{By Definition} \tag{82}$$
$$= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \omega \quad \text{By Induction} \tag{83}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{84}$$