Chapter 1

Adequacy of a Model of the Effect Calculus

1.1 Instantiation of the Effect Calculus

Let us instantiate the effect calculus to be a language in which one can write programs which can create an output signal. The effect system of EC is then used to count an upper bound on the number of outputs that a program can make. This language shall be called $\mathtt{EC}_{\mathtt{put}}$

1.1.1 Ground Types

We simply use the basic ground types.

$$\gamma ::= \mathtt{Bool} \mid \mathtt{Unit} \tag{1.1}$$

1.1.2 Graded Monad

We grade index the graded monad with a partially ordered monoid derived from the natural numbers.

$$E = (\mathbb{N}, 0, +, \leq) \tag{1.2}$$

This means that the $\mathtt{do}\ x \leftarrow v\ \mathtt{in}\ v'$ type rule adds together the upper bounds on the two expressions to give an upper bound on the number of outputs of the sequenced expression. The $\mathtt{return}\ v$ type rule acknowledges that a pure expression does not have any output.

1.1.3 Constants

We extend the set of constant, built in expressions to include a `put` statement which makes a single output action.

$$\mathtt{C}^{A} ::= \mathtt{true}^{\mathtt{Bool}} \mid \mathtt{false}^{\mathtt{Bool}} \mid ()^{\mathtt{Unit}} \mid \mathtt{put}^{\mathtt{M_1 Unit}} \tag{1.3}$$

1.1.4 Subtyping

The ground subtyping relation is the trivial identity relation. This is extended using the subeffect and function subtyping rules given in **TODO: Ref**.

1.2 Instantiation of a Model of the Effect Calculus

Let us now instantiate a model of $\mathtt{EC}_{\mathtt{put}}$ in `Set`, the category of sets and functions.

1.2.1 Cartesian Closed Category

Is given by the usage of `Set`.

1.2.2 Graded Monad

The strong graded monad is given by tagging values of the underlying type with the number of output operations required to compute that value.

$$T_n A = \{(n', a) \mid n' \leq n \wedge a \in A\} \tag{1.4}$$
$$\mu_{m,n,A} = (m', (n', a)) \mapsto (n' + m', a) \tag{1.5}$$
$$\eta_A = a \mapsto (0, a) \tag{1.6}$$
$$\mathtt{t}_{n,A,B} = (a, (n', b)) \mapsto (n', (a, b)) \tag{1.7}$$

1.2.3 Subeffecting Natural Transformations

These natural transformations are given by inclusion functions (identities), since $n \leq m \wedge (n', a) \in T_n A \implies (n' \leq n \leq m, a \in A) \implies (n', a) \in T_m A$. Other subtyping morphisms are generated using the usual method according to the subtype derivation.

1.2.4 Ground Denotations

We define denotations for ground types as follows:

$$[\![\mathtt{Unit}]\!] = \{*\} \tag{1.8}$$
$$[\![\mathtt{Bool}]\!] = \{\top, \bot\} \tag{1.9}$$

We then define denotations for the constant expressions, including the `put`operation.

$$[\![\mathtt{()}]\!] = * \mapsto * \tag{1.10}$$
$$[\![\mathtt{true}]\!] = * \mapsto \top \tag{1.11}$$
$$[\![\mathtt{false}]\!] = * \mapsto \bot \tag{1.12}$$
$$[\![\mathtt{put}]\!] = * \mapsto (1, *) \tag{1.13}$$
$$\tag{1.14}$$

1.2.5 Soundness

This category is now an S-category and hence a sound model for $\mathtt{EC}_{\mathtt{put}}$.

1.2.6 Denotational Shorthand

In the remaining sections I shall use $[\![\vdash v\colon A]\!]$ to indicate $[\![\diamond \vdash v\colon A]\!](*)$.

1.3 Programming With Put

This simple language now has some extra properties which the general EC does not have.

Definition 1.3.1 (Powers of Put as an Equational Equivalence Class)**.** *Define* put^n *as follows:*

$$\vdash \mathit{put}^0 \approx \mathit{return}\ () : M_0\,\mathit{Unit} \tag{1.15}$$

$$\vdash \mathit{put}^{n+1} \approx \mathit{do}\ \_ \leftarrow \mathit{put}^n\ \mathit{in}\ \mathit{put} : M_{n+1}\,\mathit{Unit} \tag{1.16}$$

Lemma 1.3.1 (Denotations of Powers of Put)**.** *Powers of put have a simple denotation.* $[\![\vdash \mathit{put}^n\colon M_n\,\mathit{Unit}]\!] = (n, *)$

Proof: By induction on n.

Case 0:

$$[\![\vdash \mathtt{put}^0\colon \mathtt{M}_0\mathtt{Unit}]\!] = \eta(*) = (0, *) \tag{1.17}$$

Case n+1:

$$[\![\vdash \mathtt{put}^{n+1}\colon \mathtt{M}_0\mathtt{Unit}]\!] = (\mu \circ T_n([\![\diamond \vdash \mathtt{put}\colon \mathtt{M}_1\mathtt{Unit}]\!] \circ \pi_1) \circ \mathtt{t})(*, [\![\vdash \mathtt{put}^n\colon \mathtt{M}_n\mathtt{Unit}]\!]) \tag{1.18}$$

$$= (n+1, *) \tag{1.19}$$

1.4 Logical Relations

$$\lhd_A \in [\![A]\!] \times \mathtt{EC}^A_{\mathtt{put}} \tag{1.20}$$

1.4.1 Definition

Definition 1.4.1 (Logical Relation)**.**

$$d \lhd_{\mathit{Unit}} v \Leftrightarrow (d = * \implies \vdash v \approx ()\colon \mathit{Unit}) \tag{1.21}$$

$$d \lhd_{\mathit{Bool}} v \Leftrightarrow ((d = \top \implies \vdash v \approx \mathit{true}\colon \mathit{Bool}) \wedge (d = \bot \implies \vdash v \approx \mathit{false}\colon \mathit{Bool})) \tag{1.22}$$

$$d \lhd_{A \to B} v \Leftrightarrow (\forall e, u. e \lhd_A u \implies d(e) \lhd_B v\ u) \tag{1.23}$$

$$d \lhd_{M_n A} v \Leftrightarrow (d = (n', d') \in T_n[\![A]\!] \implies \exists v'. d' \lhd_A v' \wedge \vdash v'\colon A \wedge \vdash v \approx \mathit{do}\ \_ \leftarrow \mathit{put}^{n'}\ \mathit{in}\ \mathit{return}\ v' : M_n A) \tag{1.24}$$

1.4.2 Subtyping

Theorem 1.4.1 (Logical Relation and Subtyping)**.** *If $A \leq: B$ and $d \lhd_A v$ then $d \lhd_B v$]*

Proof: By induction on the derivation of $A \leq: B$.

Case Ground: $A \leq: B \implies A = B$, since ground subtyping is the identity relation.

Case Fun: $A \leq: B \implies A = A_1 \to A_2, B = B_1 \to B_2$ where $B_1 \leq: A_1$ and $A_2 \leq: B_2$.
By the definition of the $\lhd_{A\to B}$ relation, $d \lhd_{A\to B} v \Leftrightarrow (\forall e, u. e \lhd_A u \implies d(e) \lhd_B v\ u)$.
So

$$\begin{aligned} \forall e, u. e \lhd_{B_1} u &\implies e \lhd_{A_1} u \quad \text{By induction } B_1 \leq: A_1 && (1.25)\\ &\implies d(e) \lhd_{A_2} u\ v \quad \text{By definition} && (1.26)\\ &\implies d(e) \lhd_{B_2} u\ v \quad \text{By induction } A_2 \leq: B_2 && (1.27)\end{aligned}$$

As required.

Case Effect: $\mathtt{M}_{n_1} A_1 \leq: \mathtt{M}_{n_2} A_2 \implies n_1 \leq n_2, A_1 \leq: A_2$

$$d \lhd_{\mathtt{M}_{n_1} A_1} v \implies d = (n_1', d') \wedge n_1' \leq n_1 \leq n_2 \wedge \exists v'. (d' \lhd_{A_1} v' \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' : \mathtt{M}_{n_1} A_1) \quad (1.28)$$

$$\implies \vdash v_1' : A_2 \wedge d' \lhd_{A_2} v' \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v' : \mathtt{M}_{n_1} A_2 \quad (1.29)$$

$$\implies d \lhd_{\mathtt{M}_{n_2} A_2} v \quad (1.30)$$

1.4.3 Fundamental Property

Let $\lhd_\Gamma \in [\![\Gamma]\!] \times \mathtt{EC}^G_{\mathtt{put}}$ mean:

$$\rho \lhd_\Gamma \sigma \Leftrightarrow \forall x. \rho(x) \lhd_{\Gamma(x)} \sigma(x) \quad (1.31)$$

Theorem 1.4.2 (Fundamental Theorem)**.** *If $\rho \lhd_\Gamma \sigma$ then $[\![\Gamma \vdash v : A]\!] \rho \lhd_A v\,[\sigma]$ up to equational equivalence.*

Proof: By induction over the derivation of $\Gamma \vdash v : A$

Case Variables:

$$[\![\Gamma \vdash x : \Gamma(x)]\!]\rho = \rho(x) \lhd_{\Gamma(x)} \sigma(x) \approx x\,[\sigma] \quad (1.32)$$

Case Constants:

$$\begin{aligned} [\![\Gamma \vdash \mathtt{true} : \mathtt{Bool}]\!]\rho = \top \wedge \vdash \mathtt{true}\,[\sigma] \approx \mathtt{true} : \mathtt{Bool} \quad \text{So } \top \lhd_{\mathtt{Bool}} \mathtt{true}\,[\sigma] && (1.33)\\ [\![\Gamma \vdash \mathtt{false} : \mathtt{Bool}]\!]\rho = \bot \wedge \vdash \mathtt{false}\,[\sigma] \approx \mathtt{false} : \mathtt{Bool} \quad \text{So } \top \lhd_{\mathtt{Bool}} \mathtt{false}\,[\sigma] && (1.34)\\ [\![\Gamma \vdash () : \mathtt{Unit}]\!]\rho = * \wedge \vdash ()\,[\sigma] \approx () : \mathtt{Unit} \quad \text{So } * \lhd_{\mathtt{Unit}} ()\,[\sigma] && (1.35)\\ [\![\Gamma \vdash \mathtt{put} : \mathtt{M}_1 \mathtt{Unit}]\!]\rho = (1, *) \wedge \vdash \mathtt{put} \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^1\ \mathtt{in\ return}\ () : \mathtt{M}_1 \mathtt{Unit} && (1.36)\\ && (1.37)\end{aligned}$$

Case Subtype:

$$[\![\Gamma \vdash v\colon B]\!]\rho = [\![\Gamma \vdash v\colon A]\!] \lhd_A v\,[\sigma] \tag{1.38}$$

Since $A \leq\colon B \wedge d \lhd_A v \implies d \lhd_B v$, we have that $[\![\Gamma \vdash v\colon B]\!] \lhd_B v\,[\sigma]$.

Case Fn: For all $d \lhd_A u$,

$$\begin{aligned}([\![\Gamma \vdash \lambda x\colon A.v : A \to B]\!]\rho)d &= (\mathtt{cur}([\![\Gamma, x\colon A \vdash v\colon B]\!])\rho)d && (1.39)\\ &= [\![\Gamma, x\colon A \vdash v\colon B\rho]\!](\rho[x \mapsto d]) && (1.40)\\ & && (1.41)\end{aligned}$$

Since $d \lhd_A u$, $(\rho[x \mapsto d]) \lhd_{\Gamma, x:A} (\sigma, x\colon = u)$, so by induction

$$\begin{aligned}([\![\Gamma \vdash \lambda x\colon A.v : A \to B]\!]\rho)d = [\![\Gamma, x\colon A \vdash v\colon B]\!](\rho[x \mapsto d]) &\lhd_B v\,[\sigma, x\colon = u] && (1.42)\\ &\lhd_B v\,[\sigma]\,[u/x] && (1.43)\\ &\approx (\lambda x\colon A.(v\,[\sigma])\ )\ u && (1.44)\end{aligned}$$

Case Apply:

$$[\![\Gamma \vdash v\ u\colon B]\!]\rho = ([\![\Gamma \vdash v\colon A \to B]\!]\rho)([\![\Gamma \vdash u\colon A]\!]\rho) \tag{1.45}$$

By induction $[\![\Gamma \vdash v\colon A \to B]\!]\rho \lhd_{A\to B} v\,[\sigma]$ and $[\![\Gamma \vdash u\colon A]\!]\rho \lhd_A u\,[\sigma]$. So by the definition of $\lhd_{A\to B}$,

$$[\![\Gamma \vdash v\ u\colon B]\!]\rho = ([\![\Gamma \vdash v\colon A \to B]\!]\rho)([\![\Gamma \vdash u\colon A]\!]\rho) \lhd_B (v\,[\sigma])\ (u\,[\sigma]) \approx (v\ u)\,[\sigma] \tag{1.46}$$

Case Return:

$$[\![\Gamma \vdash v\colon \mathtt{M}_0 A]\!]\rho = (0, [\![\Gamma \vdash v\colon A]\!]\rho) \tag{1.47}$$

By induction, $[\![\Gamma \vdash v\colon A]\!] \lhd_A v\,[\sigma]$, so by picking $v' = v\,[\sigma]$

$$\vdash (\mathtt{return}\ v\ )\,[\sigma] \approx \mathtt{return}\ (v\,[\sigma]) \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^0\ \mathtt{in}\ \mathtt{return}\ v' \ :\mathtt{M}_0 A \tag{1.48}$$

So $[\![\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_0 A]\!] \lhd_{\mathtt{M}_0 A \rho} (\mathtt{return}\ v\ )\,[\sigma]$

Case Bind: By inversion, $[\![\Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u : \mathtt{M}_{m+n} B]\!]\rho = (m' + n', d_u)$, where $(n', d_u) = [\![\Gamma, x\colon A \vdash u\colon \mathtt{M}_n B]\!](\rho[x \mapsto d_v])$, and $(n', d_v) = [\![\Gamma \vdash v\colon \mathtt{M}_m A]\!]\rho$.

By induction, $(m', d_v) \lhd_{\mathtt{M}_m A} v\,[\sigma]$. So $\exists v'$ such that $\vdash v\,[\sigma] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v' \ :\mathtt{M}_m A$. So $(\rho[x \mapsto d_v]) \lhd_{\Gamma, x:A} ([\sigma]\,, x\colon = v')$.

So by induction $[\![\Gamma, x\colon A \vdash u\colon \mathtt{M}_n B]\!](\rho[x \mapsto d_v]) \lhd_{\mathtt{M}_n B} u\,[\sigma, x\colon = v']$.

Hence, $\exists u'$ such that $\vdash u\,[\sigma, x\colon = v'] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'} \mathtt{return}\ u'\ \ \mathtt{in} : \mathtt{M}_{m+n} B$ and $d_u \lhd_{\mathtt{M}_n B} u'$.

Hence,

$$\begin{aligned}\vdash \mathtt{do}\ x \leftarrow v\,[\sigma]\ \mathtt{in}\ u\,[\sigma]\ &\approx \mathtt{do}\ x \leftarrow (\mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v'\ )\ \mathtt{in}\ (u\,[\sigma]) : \mathtt{M}_{m+n} B && (1.49)\\ &\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ u\,[\sigma, x\colon = v'] && (1.50)\\ &\approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'+n'}\ \mathtt{in}\ \mathtt{return}\ u' && (1.51)\end{aligned}$$

So $[\![\Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u : \mathtt{M}_{m+n} B]\!]\rho \lhd_{\mathtt{M}_{m+n} B} (\mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u\ )\,[\sigma]$.

Case If: By inversion, $[\![\Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A]\!]\rho = \begin{cases} [\![\Gamma \vdash v_1 \colon A]\!]\rho & \text{If } [\![\Gamma \vdash b \colon \texttt{Bool}]\!]\rho = \top \\ [\![\Gamma \vdash v_2 \colon A]\!]\rho & \text{If } [\![\Gamma \vdash b \colon \texttt{Bool}]\!]\rho = \bot \end{cases}$.

By induction,

$$[\![\Gamma \vdash b \colon \texttt{Bool}]\!]\rho \lhd_{\texttt{Bool}} b\,[\sigma] \tag{1.52}$$
$$[\![\Gamma \vdash v_1 \colon A]\!]\rho \lhd_A v_1\,[\sigma] \tag{1.53}$$
$$[\![\Gamma \vdash v_2 \colon A]\!]\rho \lhd_A v_2\,[\sigma] \tag{1.54}$$

Case: $[\![\Gamma \vdash b \colon \texttt{Bool}]\!]\rho = \top$

$$[\![\Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A]\!]\rho = [\![\Gamma \vdash v_1 \colon A]\!]\rho \lhd_A v_1\,[\sigma] \approx (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )\,[\sigma] \tag{1.55}$$

Case: $[\![\Gamma \vdash b \colon \texttt{Bool}]\!]\rho = \bot$

$$[\![\Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A]\!]\rho = [\![\Gamma \vdash v_2 \colon A]\!]\rho \lhd_A v_2\,[\sigma] \approx (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )\,[\sigma] \tag{1.56}$$

1.5 Adequacy

Theorem 1.5.1 (Adequacy)**.** *For G defined as:*

$$G ::= \mathit{Bool} \mid \mathit{Unit} \mid \mathtt{M}_n G \tag{1.57}$$

Equality of denotations implies equational equality.

$$[\![\vdash v \colon G]\!] = [\![\vdash u \colon G]\!] \implies \vdash v \approx u \colon G \tag{1.58}$$

Proof: By induction on the structure of G, making use of the fundamental property 1.4.3.

Case Boolean: Let $d = [\![\vdash v \colon \texttt{Bool}]\!] = [\![\vdash v \colon \texttt{Bool}]\!] \in \{\top, \bot\}$. By the fundamental property, $d \lhd_{\texttt{Bool}} v$ and $d \lhd_{\texttt{Bool}} v$.

Case: $d = \top$ Then $\vdash v \approx \texttt{true} \approx u \colon \texttt{Bool}$

Case: $d = \bot$ Then $\vdash v \approx \texttt{false} \approx u \colon \texttt{Bool}$

Case Unit: Let $* = [\![\vdash v \colon \texttt{Unit}]\!] = [\![\vdash v \colon \texttt{Unit}]\!] \in \{*\}$. By the fundamental property, $d \lhd_{\texttt{Unit}} v$ and $d \lhd_{\texttt{Unit}} v$. Hence $\vdash v \approx () \approx u \colon \texttt{Unit}$.

Case Effect: Let $(n', d) = [\![\vdash v \colon \mathtt{M}_n G]\!] = [\![\vdash u \colon \mathtt{M}_n G]\!]$. By the fundamental property, $(n', d) \lhd_{\mathtt{M}_n G} v$ and $(n', d) \lhd_{\mathtt{M}_n G} u$. So there exists u', v' such that $d' \lhd_G u'$ and $d' \lhd_G u'$ and:

$$\vdash v \approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ v' \ : \mathtt{M}_n G \tag{1.59}$$
$$\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ u' \tag{1.60}$$
$$\approx u \tag{1.61}$$

□