0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v \colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v \colon A$,there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v \colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Proof: We induct on the structure of terms.

Case Variables: To find the unique derivation of $\Phi \mid \Gamma \vdash x \colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$: Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x \colon A$ is, if $A' \leq:_\Phi A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\frac{\Phi\vdash\Gamma',x:A'\,\mathtt{Ok}}{\Phi|\Gamma,x:A'\vdash x:A'} \quad A' \leq: A}{\Phi \mid \Gamma', x : A' \vdash x \colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$: with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x \colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x \colon A$.

Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma'\vdash x:A'} \quad A' \leq: A}{\Phi \mid \Gamma' \vdash x \colon A} \tag{2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x \colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x \colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\frac{()\frac{\Delta}{\Phi|\Gamma,x:A'\vdash x:A'}}{\Phi|\Gamma\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x \colon A} \tag{3}$$

Case Constants: For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash \mathtt{C}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_\Phi A$).

Case Lambda: The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v \colon A' \to B'$ is:

$$
(\text{Subtype})\dfrac{(\text{Lambda})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A.B : A \to B} \quad A \to B \leq:_{\Phi} A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A' \to B'}
$$

Where

$$
(\text{Sub-Type})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma, x : A \vdash v : B} \quad B \leq:_{\Phi} B'}{\Phi \mid \Gamma, x : A \vdash v \colon B'} \tag{4}
$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v \colon B'$ if it exists.

Case Return: The reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{return} v \colon \mathtt{M}_{\epsilon} B$ is

$$
(\text{Subtype})\dfrac{(\text{Return})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}} A} \quad (\text{Computation})\dfrac{A \leq:_{\Phi} B \quad \mathtt{1} \leq_{\Phi} \epsilon}{\mathtt{M}_{\mathtt{1}} A \leq_{\Phi} \mathtt{M}_{\epsilon} B}}{\Phi \mid \Gamma \vdash \mathtt{return} v \colon B}
$$

Where

$$
(\text{Subtype})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v : A} \quad A \leq: B}{\Phi \mid \Gamma \vdash v \colon B}
$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v \colon B$

Case Apply: If

$$
(\text{Subtype})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : A \to B} \quad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1 \colon A' \to B'}
$$

and

$$
(\text{Subtype})\dfrac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2 \colon A'}
$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1 \colon A' \to B'$ and $\Phi \mid \Gamma \vdash v_2 \colon A'$

Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'$ as

$$
(\text{Subtype})\dfrac{(\text{Apply})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : A \to B} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v : A''} \quad A'' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash v_1\ v_2 : B} \quad (\text{Computation})\dfrac{B \leq:_{\Phi} B' \quad \epsilon \leq_{\Phi} \epsilon'}{\mathtt{M}_{\epsilon} B \leq:_{\Phi} \mathtt{M}_{\epsilon'} B}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'}
$$

Case If: Let

$$
(\text{Subtype})\dfrac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v : B} \quad B \leq: \mathtt{Bool}}{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}} \tag{5}
$$

$$
(\text{Subtype})\dfrac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \quad A' \leq: A}{\Phi \mid \Gamma \vdash v_1 \colon A} \tag{6}
$$

$$
(\text{Subtype})\dfrac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2 \colon A} \tag{7}
$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$.

Then the only reduced derivation of $\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A$ is:

TODO: Scale this properly

$$\text{(Subtype)}\frac{\text{(If)}\frac{\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:B} \quad B \leq: \texttt{Bool}}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_1:A'} \quad A' \leq: A}{\Phi|\Gamma\vdash v_1:A} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_2:A''} \quad A'' \leq: A}{\Phi|\Gamma\vdash v_2:A}}{\Phi|\Gamma\vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2:A \quad \epsilon \leq_\Phi \epsilon \quad A \leq:_\Phi A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{8}$$

Case Bind: Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad \text{(Computation)}\frac{A \leq:_\Phi A' \quad \epsilon_1 \leq_\Phi \epsilon_1'}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma, x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B \leq:_\Phi B' \quad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq:_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms.

By weakening, $\Phi \vdash \iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2 \colon B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.

$$\text{(Subtype)}\frac{()\frac{\Delta''}{\Phi|\Gamma, x:A'\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B \leq:_\Phi B' \quad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq:_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Type)}\frac{\text{(Bind)}\frac{\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad \text{(Computation)}\frac{A \leq:_\Phi A' \quad \epsilon_1 \leq_\Phi \epsilon_1'}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_1'} A'}}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'} A'} \quad \text{(Subtype)}\frac{()\frac{\Delta''}{\Phi|\Gamma, x:A'\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B \leq:_\Phi B' \quad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq:_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi|\Gamma, x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'} B'}}{\Phi|\Gamma\vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2:\mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{12}$$

Case Effect-Fn: The unique reduced derivation of $\Phi \mid \Gamma \vdash \Lambda\alpha.A \colon \forall\alpha.B$

is

$$\text{(Sub-type)}\frac{\text{(Effect-Fn)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi|\Gamma\vdash \Lambda\alpha.v:\forall\alpha.A} \quad \forall\alpha.A \leq_\Phi \forall\alpha.B}{\Phi \mid \Gamma \vdash \Lambda\alpha.B \colon \forall\alpha.B} \tag{13}$$

Where

$$\text{(Sub-type)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A} \quad A \leq:_{\Phi,\alpha} B}{\Phi, \alpha \mid \Gamma \vdash v \colon B} \tag{14}$$

Is the unique reduced derivation of $\Phi, \alpha \mid \Gamma \vdash v \colon B$

Case Effect-App: The unique reduced derivation of $\Phi \mid \Gamma \vdash v\ \alpha \colon B'$

is

$$(\text{Subtype})\frac{(\text{Effect-App})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A\,[\epsilon/\alpha] \leq:_{\Phi} B'}{\Phi \mid \Gamma \vdash v\ \alpha \colon B'} \tag{15}$$

Where $B\,[\epsilon/\alpha] \leq:_{\Phi} B'$ and

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.B} \quad (\text{Quantification})\frac{A\leq:_{\Phi,\alpha}B}{\forall\alpha.A\leq:_{\Phi}\forall\alpha.B}}{\Phi \mid \Gamma \vdash v\colon \forall\alpha.B} \tag{16}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v\colon A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

Case Constants: For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced.

$reduce((\text{Const})\frac{\Phi\vdash\Gamma\mathtt{Ok}}{\Phi|\Gamma\vdash\mathtt{C}^A:A}) = (\text{Const})\frac{\Phi\vdash\Gamma\mathtt{Ok}}{\Phi|\Gamma\vdash\mathtt{C}^A:A}$

Case Var:

$$reduce((\text{Var})\frac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A}) = (\text{Var})\frac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A} \tag{17}$$

Case Weaken:

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash x:A}}{\Phi \mid \Gamma, y : B \vdash x\colon A}) \tag{18}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{19}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A'}}{\Phi|\Gamma,y:B\vdash x:A'} \quad A' \leq:_{\Phi} A}{\Phi \mid \Gamma, y : B \vdash x\colon A} \tag{20}$$

Case Lambda:

reduce **definition** To find:

$$reduce((\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to \epsilon_2 B}) \tag{21}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:B'} \quad B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A \vdash v\colon B} = reduce(\Delta) \tag{22}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:\mathtt{M}_{\epsilon_1} B'} \quad A \to \epsilon_1 B' \leq:_\Phi A \to \epsilon_2 B}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to \epsilon_2 B} \tag{23}$$

Case Subtype:

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B}) \tag{24}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{25}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B} \tag{26}$$

Case Return:

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return} v\colon \mathtt{M}_1 A}) \tag{27}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v\colon A} = reduce(\Delta) \tag{28}$$

In

$$(\text{Sub-type})\frac{(\text{Return})\frac{\Delta'}{\Phi|\Gamma\vdash v:A} \quad (\text{Computation})\frac{\mathtt{1} \leq_\Phi \mathtt{1} \quad A' \leq:_\Phi A}{\mathtt{M}_1 A' \leq:_\Phi \mathtt{M}_1 A}}{\Phi \mid \Gamma \vdash \mathtt{return} v\colon \mathtt{M}_1 A} \tag{29}$$

Case Apply:

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B}\quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B}) \tag{30}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:A'\to B'}\quad A'\to B' \leq:_{\Phi} A\to \epsilon B}{\Phi \mid \Gamma \vdash v_1\colon A\to B} = reduce(\Delta_1) \tag{31}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Phi|\Gamma\vdash v:A'}\quad A' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash v_1\colon A} = reduce(\Delta_2) \tag{32}$$

In

$$(\text{Subtype})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:A'\to B'}\quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Phi|\Gamma\vdash v_2:A''}\quad A''\leq:_{\Phi}A\leq:_{\Phi}A'}{\Phi|\Gamma\vdash v_2:A'}}{\Phi|\Gamma\vdash v_1\ v_2:\mathtt{M}_{\epsilon'}B'}\quad (\text{Computation})\frac{\epsilon'\leq_{\Phi}\epsilon\quad B'\leq:_{\Phi}B}{\mathtt{M}_{\epsilon'}B'\leq:_{\Phi}\mathtt{M}_{\epsilon}B}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{33}$$

Case If:

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}}\quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A}\quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Phi|\Gamma\vdash v:\mathtt{Bool}}\quad ()\frac{reduce(\Delta_2)}{\Phi|\Gamma\vdash v_1:A}\quad ()\frac{reduce(\Delta_3)}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A} \tag{34}$$

Case Bind:

reduce **definition** To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A}\quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{35}$$

Let

$$(\text{Sub-Type})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'}A'}\quad (\text{Computation})\frac{\epsilon_1'\leq_{\Phi}\epsilon_1\quad A'\leq:_{\Phi}A}{\mathtt{M}_{\epsilon_1'}A'\leq:_{\Phi}\mathtt{M}_{\epsilon_1}A}}{\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1}A} = reduce(\Delta_1) \tag{36}$$

Since $\Phi \vdash i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq:_{\Phi} A$, and by Δ_2, $\Phi \mid (\Gamma, x : A) \vdash v_2\colon \mathtt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2\colon \mathtt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'}B'}\quad (\text{Computation})\frac{\epsilon_2'\leq_{\Phi}\epsilon_2\quad B'\leq:_{\Phi}B}{\mathtt{M}_{\epsilon_2'}B'\leq:_{\Phi}\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \mathtt{M}_{\epsilon_2}B} = reduce(\Delta_3) \tag{37}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_{\Phi} \epsilon_1'$ and $\epsilon_2 \leq_{\Phi} \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_{\Phi} \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-Type})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'}A'}\quad ()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'}B'}}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1'\cdot\epsilon_2'}B}\quad (\text{Computation})\frac{\epsilon_1'\cdot\epsilon_2'\leq_{\Phi}\epsilon_1\cdot\epsilon_2\quad B'\leq:_{\Phi}B}{\mathtt{M}_{\epsilon_1'\cdot\epsilon_2'}B'\leq:_{\Phi}\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{38}$$

Case Effect-Fn:

reduce **definition** To find

$$reduce((\text{Effect-Lambda})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A}) \tag{39}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi,\alpha|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi,\alpha \mid \Gamma \vdash v\colon A} = reduce(\Delta_1) \tag{40}$$

in

$$(\text{Subtype})\frac{(\text{Effect-Fn})\frac{()\frac{\Delta_1'}{\Phi,\alpha|\Gamma\vdash v:A'}}{\Phi|\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}}{\forall\alpha.A'\leq:_\Phi\forall\alpha.A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{41}$$

Case Effect-Application:

reduce **definition** To find

$$reduce((\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]}) \tag{42}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}A}{\forall\alpha.A'\leq:_\Phi\forall\alpha.A}}{\Phi \mid \Gamma \vdash v\colon \forall\alpha.A} = reduce(\Delta_1) \tag{43}$$

In

$$(\text{Subtype})\frac{(\text{E-app})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A'\,[\epsilon/\alpha] \leq:_\Phi A\,[\epsilon/\alpha]}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{44}$$

0.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v\colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\Delta]\!]_M = [\![reduce\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Phi \mid \Gamma \vdash v\colon A]\!]_M$ is unique.