TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma \vdash t\colon \tau$ and $\Gamma' \vdash \sigma\colon \Gamma$ then the derivation Δ' deriving $\Gamma' \vdash t\,[\sigma] : \tau$ satisfies:

$$\Delta' = \Delta \circ [\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M \tag{1}$$

This is proved by induction over the derivation of $\Gamma \vdash t\colon \tau$. We shall use σ to denote $[\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M$ where it is clear from the context.

0.0.1 Proof For Value Terms

Case Var By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var})\frac{\Gamma\,\texttt{Ok}}{\Gamma'', x : A \vdash x\colon A} \tag{2}$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v\colon A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{3}$$
$$\Delta = [\![\Gamma'', x : A \vdash x\colon A]\!]_M = \pi_2 \tag{4}$$
$$\tag{5}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{6}$$
$$= \Delta' \quad \text{By product property} \tag{7}$$

Case Weaken By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x: A}}{\Gamma'', y : B \vdash x\colon A} \tag{8}$$

Also by inversion of the well-formed-ness of $\Gamma' \vdash \sigma\colon \Gamma$, we have $\Gamma' \vdash \sigma'\colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle [\![\Gamma' \vdash \sigma\colon \Gamma'']\!]_M, [\![\Gamma' \vdash v\colon B]\!]_M \rangle \tag{9}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$()\frac{\Delta_1'}{\Gamma' \vdash x\,[\sigma] : A} \tag{10}$$

Hence

$$\Delta' = \Delta_1' \quad \text{By definition} \tag{11}$$
$$= \Delta_1 \circ \sigma' \quad \text{By induction} \tag{12}$$
$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v\colon B]\!]_M \rangle \quad \text{By product property} \tag{13}$$
$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.} \tag{14}$$

Case Constants The logic for all constant terms ($\texttt{true}, \texttt{false}, \texttt{()}\texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{15}$$

$$\begin{aligned}
\Delta' &= c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} && (16)\\
&= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} && (17)\\
&= \Delta \circ \sigma \quad \text{By definition} && (18)
\end{aligned}$$

Case Lambda By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C:\texttt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \texttt{M}_\epsilon B} \tag{19}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash (C[\sigma]):\texttt{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A.C)\,[\sigma] : A \to \texttt{M}_\epsilon B} \tag{20}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{21}$$

Hence:

$$\begin{aligned}
\Delta' &= \texttt{cur}(\Delta_1') \quad \text{By definition} && (22)\\
&= \texttt{cur}(\Delta_1 \circ (\sigma \times \texttt{Id}_A)) \quad \text{By induction and extension lemma.} && (23)\\
&= \texttt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (24)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (25)\\
& && (26)
\end{aligned}$$

Case Sub-type By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Gamma \vdash v:A} \quad A \leq: B}{\Gamma \vdash v : B} \tag{27}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma]:A} \quad A \leq: B}{\Gamma' \vdash v\,[\sigma] : B} \tag{28}$$

Hence,

$$\begin{aligned}
\Delta' &= [\![A \leq: B]\!]_M \circ \Delta_1' \quad \text{By definition} && (29)\\
&= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \sigma \quad \text{By induction} && (30)\\
&= \Delta \circ \sigma \quad \text{By definition} && (31)\\
& && (32)
\end{aligned}$$

0.0.2 Proof For Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \texttt{return} v \colon \texttt{M}_{\texttt{1}} A} \tag{33}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma] : A}}{\Gamma' \vdash (\texttt{return} v)\,[\sigma] : \texttt{M}_{\texttt{1}} A} \tag{34}$$

Hence,

$$\begin{aligned} \Delta' &= \eta_A \circ \Delta_1' \quad \text{By Definition} && (35)\\ &= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} && (36)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (37)\\ & && (38) \end{aligned}$$

Case Apply By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma \vdash v_1 : A \to \texttt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 \colon \texttt{M}_\epsilon B} \tag{39}$$

By induction we find Δ_1', Δ_2' such that

$$\begin{aligned} \Delta_1' &= \Delta_1 \circ \sigma && (40)\\ \Delta_2' &= \Delta_2 \circ \sigma && (41)\\ & && (42) \end{aligned}$$

And

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v_1[\sigma] : A \to \texttt{M}_\epsilon B} \quad ()\frac{\Delta_2'}{\Gamma' \vdash v_2[\sigma] : A}}{\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \texttt{M}_\epsilon B} \tag{43}$$

Hence

$$\begin{aligned} \Delta' &= \texttt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Definition} && (44)\\ &= \texttt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} && (45)\\ &= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} && (46)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (47)\\ & && (48) \end{aligned}$$

Case If By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma \vdash v : \texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma \vdash C_1 : \texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma \vdash C_2 : \texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon, A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 \colon \texttt{M}_\epsilon A} \tag{49}$$

By induction we find $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta_1' = \Delta_1 \circ \sigma \tag{50}$$
$$\Delta_2' = \Delta_2 \circ \sigma \tag{51}$$
$$\Delta_3' = \Delta_3 \circ \sigma \tag{52}$$
$$\tag{53}$$

And

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma]:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Gamma' \vdash C_1[\sigma]:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3'}{\Gamma' \vdash C_2[\sigma]:\texttt{M}_\epsilon A}}{\Gamma' \vdash (\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2)\,[\sigma] : \texttt{M}_\epsilon A} \tag{54}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\sigma = \texttt{cur}(\texttt{app} \circ (\texttt{Id}_{T_\epsilon A} \times w)) \tag{55}$$

. And hence, we have:

$$\texttt{cur}(f \circ (\texttt{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \texttt{cur}(f) \tag{56}$$

And so:

$$\begin{aligned}
\Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta_2' \circ \pi_2), \texttt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (57)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \texttt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (58)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma))] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (59)\\
&= \texttt{app} \circ (([(T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} && (60)\\
&= \texttt{app} \circ (((T_\epsilon A)^\sigma \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (61)\\
&= \texttt{app} \circ ((T_\epsilon A)^\sigma \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (62)\\
&= \texttt{app} \circ (\texttt{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (T_\epsilon A)^\sigma && (63)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (64)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \sigma \quad \text{By Definition of the diagonal morphism.} && (65)\\
&= \Delta \circ \sigma && (66)
\end{aligned}$$

Case Bind By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Gamma \vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_2}{\Gamma, x:A \vdash C_1:\texttt{M}_\epsilon B}}{\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{67}$$

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

By property 3,

$$(\Gamma', x : A) \vdash (\sigma, x := x\colon (\Gamma, x : A) \tag{68}$$

With denotation (extension lemma)

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x\colon (\Gamma, x : A)]\!]_M = \sigma \times \mathtt{Id}_A \tag{69}$$

By induction, we derive Δ_1', Δ_2' such that:

$$\Delta_1' = \Delta_1 \circ \sigma \tag{70}$$

$$\Delta_2' = \Delta_2 \circ (\sigma \times \mathtt{Id}_A) \quad \text{By Extension Lemma} \tag{71}$$

And:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma' \vdash C_1[\sigma]:\mathtt{M}_\epsilon A} \quad ()\frac{\Delta_2'}{\Gamma', x:A \vdash C_1[\sigma]:\mathtt{M}_\epsilon B}}{\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\,[\sigma] : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{72}$$

Hence:

$$\Delta' = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1' \rangle \quad \text{By Definition} \tag{73}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} \tag{74}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1} A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} \tag{75}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} \tag{76}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} \tag{77}$$

$$= \Delta \circ \sigma \quad \text{By Defintion} \tag{78}$$

$$\tag{79}$$

Case Subeffect By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-effect})\frac{()\frac{\Delta_1}{\Gamma \vdash C:\mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_2} B} \tag{80}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma' \vdash C[\sigma]:\mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_2} B} \tag{81}$$

Hence, Let

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{82}$$

$$g = [\![A \leq: B]\!]_M \tag{83}$$

$$
\begin{aligned}
\Delta' &= h_B \circ T_{\epsilon_1} g \circ \Delta'_1 \quad \text{By definition} && (84)\\
&= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \sigma \quad \text{By induction} && (85)\\
&= \Delta \circ \sigma \quad \text{By definition} && (86)\\
& && (87)
\end{aligned}
$$