

0.1 Beta and Eta Equivalence

0.1.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Phi|\Gamma, x:A \vdash v_2:B \quad \Phi|\Gamma \vdash v_1:A}{\Phi|\Gamma \vdash (\lambda x:A. v_1) \ v_2 =_{\beta\eta} v_1 [v_2/x]:B}$
- (Lambda-Eta) $\frac{\Phi|\Gamma \vdash v:A \rightarrow B}{\Phi|\Gamma \vdash \lambda x:A. (v \ x) =_{\beta\eta} v:A \rightarrow B}$
- (Left Unit) $\frac{\Phi|\Gamma \vdash v_1:A \quad \Phi|\Gamma, x:A \vdash v_2:\mathbf{M}_\epsilon B}{\Phi|\Gamma \vdash \mathbf{do} \ x \leftarrow \mathbf{return} \ v_1 \ \mathbf{in} \ v_2 =_{\beta\eta} v_2 [v_1/x]:\mathbf{M}_\epsilon B}$
- (Right Unit) $\frac{\Phi|\Gamma \vdash v:\mathbf{M}_\epsilon A}{\Phi|\Gamma \vdash \mathbf{do} \ x \leftarrow v \ \mathbf{in} \ \mathbf{return} \ x =_{\beta\eta} v:\mathbf{M}_\epsilon A}$
- (Associativity) $\frac{\Phi|\Gamma \vdash v_1:\mathbf{M}_{\epsilon_1} A \quad \Phi|\Gamma, x:A \vdash v_2:\mathbf{M}_{\epsilon_2} B \quad \Phi|\Gamma, y:B \vdash v_3:\mathbf{M}_{\epsilon_3} C}{\Phi|\Gamma \vdash \mathbf{do} \ x \leftarrow v_1 \ \mathbf{in} \ (\mathbf{do} \ y \leftarrow v_2 \ \mathbf{in} \ v_3) =_{\beta\eta} \mathbf{do} \ y \leftarrow (\mathbf{do} \ x \leftarrow v_1 \ \mathbf{in} \ v_2) \ \mathbf{in} \ v_3:\mathbf{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$
- (Unit) $\frac{\Phi|\Gamma \vdash v:\mathbf{Unit}}{\Phi|\Gamma \vdash v =_{\beta\eta} ():\mathbf{Unit}}$
- (if-true) $\frac{\Phi|\Gamma \vdash v_1:A \quad \Phi|\Gamma \vdash v_2:A}{\Phi|\Gamma \vdash \mathbf{if}_A \ \mathbf{true} \ \mathbf{then} \ v_1 \ \mathbf{else} \ v_2 =_{\beta\eta} v_1:A}$
- (if-false) $\frac{\Phi|\Gamma \vdash v_2:A \quad \Phi|\Gamma \vdash v_1:A}{\Phi|\Gamma \vdash \mathbf{if}_A \ \mathbf{false} \ \mathbf{then} \ v_1 \ \mathbf{else} \ v_2 =_{\beta\eta} v_2:A}$
- (If-Eta) $\frac{\Phi|\Gamma, x:\mathbf{Bool} \vdash v_2:A \quad \Phi|\Gamma \vdash v_1:\mathbf{Bool}}{\Phi|\Gamma \vdash \mathbf{if}_A \ v_1 \ \mathbf{then} \ v_2[\mathbf{true}/x] \ \mathbf{else} \ v_2[\mathbf{false}/x] =_{\beta\eta} v_2[v_1/x]:A}$
- (Effect-beta) $\frac{\Phi \vdash \epsilon \quad \Phi, \alpha|\Gamma \vdash v:A}{\Phi|\Gamma \vdash (\Lambda \alpha. v \ \epsilon) =_{\beta\eta} v[\epsilon/\alpha]:A[\epsilon/\alpha]}$
- (Effect-eta) $\frac{\Phi|\Gamma \vdash v:\forall \alpha. A}{\Phi|\Gamma \vdash \Lambda \alpha. (v \ \alpha) =_{\beta\eta} v:\forall \alpha. A}$

0.1.2 Equivalence Relation

- (Reflexive) $\frac{\Phi|\Gamma \vdash v:A}{\Phi|\Gamma \vdash v =_{\beta\eta} v:A}$
- (Symmetric) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v_2:A}{\Phi|\Gamma \vdash v_2 =_{\beta\eta} v_1:A}$
- (Transitive) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v_2:A \quad \Phi|\Gamma \vdash v_2 =_{\beta\eta} v_3:A}{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v_3:A}$

0.1.3 Congruences

- (Effect-Abs) $\frac{\Phi, \alpha|\Gamma \vdash v_1 =_{\beta\eta} v_2:A}{\Phi|\Gamma \vdash \Lambda \alpha. v_1 =_{\beta\eta} \Lambda \alpha. v_2:\forall \alpha. A}$
- (Effect-Apply) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v_2:\forall \alpha. A \quad \Phi \vdash \epsilon}{\Phi|\Gamma \vdash v_1 \ \epsilon =_{\beta\eta} v_2 \ \epsilon:A[\epsilon/\alpha]}$
- (Lambda) $\frac{\Phi|\Gamma, x:A \vdash v_1 =_{\beta\eta} v_2:B}{\Phi|\Gamma \vdash \lambda x:A. v_1 =_{\beta\eta} \lambda x:A. v_2:A \rightarrow B}$
- (Return) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v_2:A}{\Phi|\Gamma \vdash \mathbf{return} \ v_1 =_{\beta\eta} \mathbf{return} \ v_2:\mathbf{M}_1 A}$
- (Apply) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v'_1:A \rightarrow B \quad \Phi|\Gamma \vdash v_2 =_{\beta\eta} v'_2:A}{\Phi|\Gamma \vdash v_1 \ v_2 =_{\beta\eta} v'_1 \ v'_2:B}$
- (Bind) $\frac{\Phi|\Gamma \vdash v_1 =_{\beta\eta} v'_1:\mathbf{M}_{\epsilon_1} A \quad \Phi|\Gamma, x:A \vdash v_2 =_{\beta\eta} v'_2:\mathbf{M}_{\epsilon_2} B}{\Phi|\Gamma \vdash \mathbf{do} \ x \leftarrow v_1 \ \mathbf{in} \ v_2 =_{\beta\eta} \mathbf{do} \ c \leftarrow v'_1 \ \mathbf{in} \ v'_2:\mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (If) $\frac{\Phi|\Gamma \vdash v =_{\beta\eta} v':\mathbf{Bool} \quad \Phi|\Gamma \vdash v_1 =_{\beta\eta} v'_1:A \quad \Phi|\Gamma \vdash v_2 =_{\beta\eta} v'_2:A}{\Phi|\Gamma \vdash \mathbf{if}_A \ v \ \mathbf{then} \ v_1 \ \mathbf{else} \ v_2 =_{\beta\eta} \mathbf{if}_A \ v' \ \mathbf{then} \ v'_1 \ \mathbf{else} \ v'_2:A}$
- (Subtype) $\frac{\Phi|\Gamma \vdash v =_{\beta\eta} v':A \quad A \leq_\Phi B}{\Phi|\Gamma \vdash v =_{\beta\eta} v':B}$

0.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

If $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' : A$ then each derivation of $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' : A$ can be converted to a derivation of $\Phi \mid \Gamma \vdash v : A$ and $\Phi \mid \Gamma \vdash v' : A$ by induction over the beta-eta equivalence relation derivation.

0.2.1 Equivalence Relations

Case Reflexive: By inversion we have a derivation of $\Phi \mid \Gamma \vdash v : A$.

Case Symmetric: By inversion $\Phi \mid \Gamma \vdash v' =_{\beta\eta} v : A$. Hence by induction, derivations of $\Phi \mid \Gamma \vdash v' : A$ and $\Phi \mid \Gamma \vdash v : A$ are given.

Case Transitive: By inversion, there exists v_2 such that $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 : A$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_3 : A$. Hence by induction, we have derivations of $\Phi \mid \Gamma \vdash v_1 : A$ and $\Phi \mid \Gamma \vdash v_3 : A$.

0.2.2 Beta-Eta conversions

Case Lambda: By inversion, we have $\Phi \mid \Gamma, x : A \vdash v_1 : B$ and $\Phi \mid \Gamma \vdash v_2 : A$. Hence by the typing rules, we have:

$$\text{(Apply)} \frac{(\text{Lambda}) \frac{\Phi \mid \Gamma, x : A \vdash v_1 : B}{\Phi \mid \Gamma \vdash \lambda x : A. v_1 : A \rightarrow B} \quad \Phi \mid \Gamma \vdash v_2 : A}{\Phi \mid \Gamma \vdash (\lambda x : A. v_1) v_2 : A}$$

By the substitution rule **TODO: which?**, we have

$$\text{(Substitution)} \frac{\Phi \mid \Gamma, x : A \vdash v_1 : B \quad \Phi \mid \Gamma \vdash v_2 : A}{\Phi \mid \Gamma \vdash v_1 [v_2/x] : B}$$

Case Left Unit: By inversion, we have $\Phi \mid \Gamma \vdash v_1 : A$ and $\Phi \mid \Gamma, x : A \vdash v_2 : M_\epsilon B$. Hence we have:

$$\text{(Bind)} \frac{(\text{Return}) \frac{\Phi \mid \Gamma \vdash v_1 : A}{\Phi \mid \Gamma \vdash \text{return } v_1 : M_1 A} \quad \Phi \mid \Gamma, x : A \vdash v_2 : M_\epsilon B}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow \text{return } v_1 \text{ in } v_2 : M_{1.\epsilon} B = M_\epsilon B} \quad (1)$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Phi \mid \Gamma \vdash v_2 [v_1/x] : M_\epsilon B \quad (2)$$

Case Right Unit: By inversion, we have $\Phi \mid \Gamma \vdash v : M_\epsilon A$.

Hence we have:

$$\text{(Bind)} \frac{\Phi \mid \Gamma \vdash v : M_\epsilon A \quad (\text{Return}) \frac{(\text{var}) \frac{\Phi \mid \Gamma, x : A \vdash x : A}{\Phi \mid \Gamma, x : A \vdash \text{return } x : M_1 A}}{\Phi \mid \Gamma, x : A \vdash \text{return } v : M_1 A}}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v \text{ in return } x : M_{\epsilon.1} A = M_\epsilon A} \quad (3)$$

Case Associativity: By inversion, we have $\Phi \mid \Gamma \vdash v_1 : M_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 : M_{\epsilon_2} B$, and $\Phi \mid \Gamma, y : B \vdash v_3 : M_{\epsilon_3} C$.

$$\Phi \vdash (\iota\pi \times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Phi \mid \Gamma, x : A, y : B \vdash v_3 : M_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)} \frac{\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \quad \text{(Bind)} \frac{\Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \quad \Phi \mid \Gamma, x : A, y : B \vdash v_3 : \mathbf{M}_{\epsilon_3} C}{\Phi \mid \Gamma, x : A \vdash x v_2 v_3 : \mathbf{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } (\text{do } y \leftarrow v_2 \text{ in } v_3) : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \quad (4)$$

and

$$\text{(Bind)} \frac{\text{(Bind)} \frac{\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \quad \Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \Phi \mid \Gamma, y : B \vdash v_3 : \mathbf{M}_{\epsilon_3} C}{\Phi \mid \Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow v_1 \text{ in } v_2) \text{ in } v_3 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \quad (5)$$

Case Eta: By inversion, we have $\Phi \mid \Gamma \vdash v : A \rightarrow B$

By weakening, we have $\Phi \vdash \iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)} \frac{\text{(App)} \frac{\Phi \mid (\Gamma, x : A) \vdash x : A \quad \text{(weakening)} \frac{\Phi \mid \Gamma \vdash v : A \rightarrow B \quad \Phi \vdash \iota\pi : \Gamma, x : A \triangleright \Gamma}{\Phi \mid \Gamma, x : A \vdash v : A \rightarrow B}}{\Phi \mid \Gamma, x : A \vdash v x : B}}{\Phi \mid \Gamma \vdash \lambda x : A. (v x) : A \rightarrow B} \quad (6)$$

Case If-True: By inversion, we have $\Phi \mid \Gamma \vdash v_1 : A$, $\Phi \mid \Gamma \vdash v_2 : A$. Hence by the typing lemma **TODO: Which?**, we have $\Phi \vdash \Gamma \mathbf{0k}$ so $\Phi \mid \Gamma \vdash \text{true} : \text{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)} \frac{\Phi \mid \Gamma \vdash \text{true} : \text{Bool} \quad \Phi \mid \Gamma \vdash v_1 : A \quad \Phi \mid \Gamma \vdash v_2 : A}{\Phi \mid \Gamma \vdash \text{if}_A \text{ true then } v_1 \text{ else } v_2 : A} \quad (7)$$

Case If-False: As above,

Hence

$$\text{(If)} \frac{\Phi \mid \Gamma \vdash \text{false} : \text{Bool} \quad \Phi \mid \Gamma \vdash v_1 : A \quad \Phi \mid \Gamma \vdash v_2 : A}{\Phi \mid \Gamma \vdash \text{if}_A \text{ false then } v_1 \text{ else } v_2 : A} \quad (8)$$

Case If-Eta: By inversion, we have:

$$\Phi \mid \Gamma \vdash v_1 : \text{Bool} \quad (9)$$

and

$$\Phi \mid \Gamma, x : \text{Bool} \vdash v_2 : A \quad (10)$$

Hence we also have $\Phi \vdash \Gamma \mathbf{0k}$. Hence, the following also hold:

$\Phi \mid \Gamma \vdash \text{true} : \text{Bool}$, and $\Phi \mid \Gamma \vdash \text{false} : \text{Bool}$.

Hence by the substitution theorem, we have:

$$\text{(If)} \frac{\Phi \mid \Gamma \vdash v_1 : \text{Bool} \quad \Phi \mid \Gamma \vdash v_2 [\text{true}/x] : A \quad \Phi \mid \Gamma \vdash v_2 [\text{false}/x] : A}{\Phi \mid \Gamma \vdash \text{if}_A v_1 \text{ then } v_2 [\text{true}/x] \text{ else } v_2 [\text{false}/x] : A} \quad (11)$$

and

$$\Phi \mid \Gamma \vdash v_2 [v_1/x] : A \quad (12)$$

Case Effect-Beta: By inversion, $\Phi, \alpha \mid \Gamma \vdash v : A$ and $\Phi \vdash \epsilon$.

Then we have the following type derivation:

$$\text{(Effect-App)} \frac{\text{(Effect-Fn)} \frac{\Phi, \alpha \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash \Lambda \alpha. v \epsilon : A [\epsilon/\alpha]} \quad (13)$$

And we can construct the single-effect-substitution:

$$\text{(Single Substitution)} \frac{\Phi \vdash \epsilon}{\Phi \vdash [\epsilon/\alpha] : (\Phi, \alpha)} \quad (14)$$

Hence by the substitution theorem,

$$\Phi \mid \Gamma \vdash v[\epsilon/\alpha] : A[\epsilon/\alpha] \quad (15)$$

Case Effect-Eta: By inversion $\Phi \mid \Gamma \vdash v : \forall \alpha. A$

So the following derivation holds:

$$\text{(Effect-Fn)} \frac{\text{(Effect-App)} \frac{\text{(Effect-weakening)} \frac{\Phi \mid \Gamma \vdash v : \forall \alpha. A}{\Phi, \alpha \mid \Gamma \vdash v : \forall \alpha. A} \quad \Phi, \alpha \vdash \alpha}{\Phi, \alpha \mid \Gamma \vdash v \alpha : A[\alpha/\alpha] = A}}{\Phi \mid \Gamma \vdash \Lambda \alpha. (v \alpha) : \forall \alpha. A} \quad (16)$$

And

$$\Phi \mid \Gamma \vdash v : \forall \alpha. A \quad (17)$$

0.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda: By inversion, $\Phi \mid \Gamma, x : A \vdash v_1 =_{\beta\eta} v_2 : B$. Hence by induction $\Phi \mid \Gamma, x : A \vdash v_1 : B$, and $\Phi \mid \Gamma, x : A \vdash v_2 : B$.

So

$$\Phi \mid \Gamma \vdash \lambda x : A. v_1 : A \rightarrow B \quad (18)$$

and

$$\Phi \mid \Gamma \vdash \lambda x : A. v_2 : A \rightarrow B \quad (19)$$

Hold.

Case Return: By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 : A$, so by induction

$$\Phi \mid \Gamma \vdash v_1 : A$$

and

$$\Phi \mid \Gamma \vdash v_2 : A$$

Hence we have

$$\Phi \mid \Gamma \vdash \text{return} v_1 : \mathbf{M}_1 A$$

and

$$\Phi \mid \Gamma \vdash \text{return} v_2 : \mathbf{M}_1 A$$

Case Apply: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v'_1 : A \rightarrow B$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$. Hence we have by induction $\Phi \mid \Gamma \vdash v_1 : A \rightarrow B$, $\Phi \mid \Gamma \vdash v_2 : A$, $\Phi \mid \Gamma \vdash v'_1 : A \rightarrow B$, and $\Phi \mid \Gamma \vdash v'_2 : A$.

So we have:

$$\Phi \mid \Gamma \vdash v_1 v_2 : B \quad (20)$$

and

$$\Phi \mid \Gamma \vdash v'_1 v'_2 : B \quad (21)$$

Case Bind: By inversion, we have: $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v'_1 : \mathbb{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2 =_{\beta\eta} v'_2 : \mathbb{M}_{\epsilon_2} B$. Hence by induction, we have $\Phi \mid \Gamma \vdash v_1 : \mathbb{M}_{\epsilon_1} A$, $\Phi \mid \Gamma \vdash v'_1 : \mathbb{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 : \mathbb{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, x : A \vdash v'_2 : \mathbb{M}_{\epsilon_2} B$.

Hence we have

$$\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A \quad (22)$$

$$\Phi \mid \Gamma \vdash \text{do } x \leftarrow v'_1 \text{ in } v'_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A \quad (23)$$

Case If: By inversion, we have: $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' : \text{Bool}$, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v'_1 : A$, and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$.

Hence by induction, we have:

$\Phi \mid \Gamma \vdash v : \text{Bool}$, $\Phi \mid \Gamma \vdash v' : \text{Bool}$,

$\Phi \mid \Gamma \vdash v_1 : A$, $\Phi \mid \Gamma \vdash v'_1 : A$,

$\Phi \mid \Gamma \vdash v_2 : A$, and $\Phi \mid \Gamma \vdash v'_2 : A$.

So

$$\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A \quad (24)$$

and

$$\Phi \mid \Gamma \vdash \text{if}_A v' \text{ then } v'_1 \text{ else } v'_2 : A \quad (25)$$

hold.

Case Subtype: By inversion, we have $A \leq_{\Phi} B$ and $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' : A$. By induction, we therefore have $\Phi \mid \Gamma \vdash v : A$ and $\Phi \mid \Gamma \vdash v' : A$.

Hence we have

$$\Phi \mid \Gamma \vdash v : B \quad (26)$$

$$\Phi \mid \Gamma \vdash v' : B \quad (27)$$

Case Effect-Lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 : A$. So

$$(\text{Effect-Lambda}) \frac{\Phi, \alpha \mid \Gamma \vdash v_1 : A}{\Phi \mid \Gamma \vdash \Lambda \alpha. v_2 : \forall \alpha. A} \quad (28)$$

and

$$(\text{Effect-Lambda}) \frac{\Phi, \alpha \mid \Gamma \vdash v_2 : A}{\Phi \mid \Gamma \vdash \Lambda \alpha. v_2 : \forall \alpha. A} \quad (29)$$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 : \forall \alpha. A$ and $\Phi \vdash \epsilon$.

So

$$(\text{Effect-App}) \frac{\Phi \mid \Gamma \vdash v_1 : \forall \alpha. A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_1 : A [\alpha/\epsilon]} \quad (30)$$

and

$$(\text{Effect-App}) \frac{\Phi \mid \Gamma \vdash v_2 : \forall \alpha. A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_2 : A [\alpha/\epsilon]} \quad (31)$$