0.1 Helper Morphisms

0.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\tau_{A,B} : (A \times B) \to (B \times A) = \langle \pi_2, \pi_1 \rangle \tag{1}$$

$$\delta_A : A \to (A \times A) = \langle \mathtt{Id}_A, \mathtt{Id}_A \rangle \tag{2}$$

0.2 Denotations of Types

0.2.1 Denotation of Ground Types

The denotations of the default ground types, $\mathtt{Unit}, \mathtt{Bool}$ should be as follows:

$$[\![\mathtt{Unit}]\!]_M = 1 \tag{3}$$

$$[\![\mathtt{Bool}]\!]_M = 1 + 1 \tag{4}$$

The mapping $[\![\_]\!]_M$ should then map each other ground type γ to an object in $\mathbb{C}$.

0.2.2 Denotation of Computation Type

Given a function $[\![\_]\!]_M$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\mathtt{M}_\epsilon A$ as so:

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon [\![A]\!]_M$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon A$$

0.2.3 Denotation of Function Types

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to \mathtt{M}_\epsilon B$ as so:

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon [\![B]\!]_M)^{[\![A]\!]_M}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon B)^A$$

0.2.4 Denotation of Type Environments

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an $\mathtt{Ok}$ type environment Γ.

$$[\![\diamond]\!]_M = 1$$

$$[\![\Gamma, x : A]\!]_M = ([\![\Gamma]\!]_M \times [\![A]\!]_M)$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x : A]\!]_M = \Gamma \times A$$

0.3 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$[\![\Gamma \vdash t{:}\tau]\!]_M : \Gamma \to [\![\tau]\!]_M$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t{:}\tau$ are equal, we need not refer to the derivation that yielded each denotation.

0.3.1 Denotation of Value Terms

- (Unit) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma\vdash()\mathtt{:Unit}]\!]_M = \langle\rangle_\Gamma : \Gamma \to 1}$

- (Const) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma\vdash\mathsf{C}^A\mathtt{:}A]\!]_M = [\![\mathsf{C}^A]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]_M}$

- (True) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma\vdash\mathtt{true}\mathtt{:Bool}]\!]_M = \mathtt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = 1+1}$

- (False) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma\vdash\mathtt{false}\mathtt{:Bool}]\!]_M = \mathtt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = 1+1}$

- (Var) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma,x{:}A\vdash x{:}A]\!]_M = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Gamma\vdash x{:}A]\!]_M : \Gamma \to A}{[\![\Gamma,y{:}B\vdash x{:}A]\!]_M = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\dfrac{f = [\![\Gamma,x{:}A\vdash C{:}\mathsf{M}_\epsilon B]\!]_M : \Gamma \times A \to T_\epsilon B}{[\![\Gamma\vdash\lambda x{:}A.C{:}A\to\mathsf{M}_\epsilon B]\!]_M = \mathtt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\dfrac{f = [\![\Gamma\vdash v{:}A]\!]_M : \Gamma \to A \quad g = [\![A\leq{:}B]\!]_M}{[\![\Gamma\vdash v{:}B]\!]_M = g \circ f : \Gamma \to B}$

0.3.2 Denotation of Computation Terms

- (Return) $\dfrac{f = [\![\Gamma\vdash v{:}A]\!]_M}{[\![\Gamma\vdash\mathtt{return}\,v{:}\mathsf{M}_1 A]\!]_M = \eta_A \circ f}$

- (If) $\dfrac{f = [\![\Gamma\vdash v{:}\mathtt{Bool}]\!]_M : \Gamma \to 1+1 \quad g = [\![\Gamma\vdash C_1{:}\mathsf{M}_\epsilon A]\!]_M \quad h = [\![\Gamma\vdash C_2{:}\mathsf{M}_\epsilon A]\!]_M}{[\![\Gamma\vdash\mathtt{if}_{\epsilon,A}\,v\,\mathtt{then}\,C_1\,\mathtt{else}\,C_2{:}\mathsf{M}_\epsilon A]\!]_M = \mathtt{app}\circ(([\mathtt{cur}(g\circ\pi_2),\mathtt{cur}(h\circ\pi_2)]\circ f)\times\mathtt{Id}_\Gamma)\circ\delta_\Gamma : \Gamma \to T_\epsilon A}$

- (Bind) $\dfrac{f = [\![\Gamma\vdash C_1{:}\mathsf{M}_{\epsilon_1} A{:}\Gamma\to T_{\epsilon_1} A \quad g = [\![\Gamma,x{:}A\vdash C_2{:}\mathsf{M}_{\epsilon_2} B]\!]_M]\!]_M : \Gamma\times A\to T_{\epsilon_2} B}{[\![\Gamma\vdash\mathtt{do}\,x{\leftarrow}C_1\,\mathtt{in}\,C_2{:}\mathsf{M}_{\epsilon_1\cdot\epsilon_2}]\!]_M = \mu_{\epsilon_1,\epsilon_2,B}\circ T_{\epsilon_1} g\circ \mathtt{t}_{\Gamma,A,\epsilon_1}\circ\langle\mathtt{Id}_\Gamma,f\rangle : \Gamma\to T_{\epsilon_1\cdot\epsilon_2} B}$

- (Subeffect) $\dfrac{f = [\![\Gamma\vdash c{:}\mathsf{M}_{\epsilon_1} A]\!]_M : \Gamma\to T_{\epsilon_1} A \quad g = [\![A\leq{:}B]\!]_M \quad h = [\![\epsilon_1\leq\epsilon_2]\!]}{[\![\Gamma\vdash C{:}\mathsf{M}_{\epsilon_2} B]\!]_M = h_B\circ T_{\epsilon_1} g\circ f}$

- (Apply) $\dfrac{f = [\![\Gamma\vdash v_1{:}A\to\mathsf{M}_\epsilon B]\!]_M : \Gamma\to(T_\epsilon B)^A \quad g = [\![\Gamma\vdash v_2{:}A]\!]_M : \Gamma\to A}{[\![\Gamma\vdash v_1\,v_2{:}\mathsf{M}_\epsilon B]\!]_M = \mathtt{app}\circ\langle f,g\rangle : \Gamma\to T_\epsilon B}$