0.1 Introduce Substitutions

0.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{1}$$

0.1.2 Trivial Properties of substitutions

$\texttt{fv}(\sigma)$

$$\texttt{fv}(\diamond) = \emptyset \tag{2}$$
$$\texttt{fv}(\sigma, x := v) = \texttt{fv}(\sigma) \cup \texttt{fv}(v) \tag{3}$$

$\texttt{dom}(\sigma)$

$$\texttt{dom}(\diamond) = \emptyset \tag{4}$$
$$\texttt{dom}(\sigma, x := v) = \texttt{dom}(\sigma) \cup \{x\} \tag{5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\texttt{fv}(\sigma) \cup \texttt{dom}(\sigma`)) \tag{6}$$

0.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{7}$$
$$x\,[\sigma, x := v] = v \tag{8}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{9}$$
$$\texttt{C}^A\,[\sigma] = \texttt{C}^A \tag{10}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{11}$$
$$(\texttt{if}_{\epsilon,A} \quad v \quad \texttt{then} \quad C_1 \quad \texttt{else} \quad C_2)\,[\sigma] = \texttt{if}_{\epsilon,A} \quad v\,[\sigma] \quad \texttt{then} \quad C_1\,[\sigma] \quad \texttt{else} \quad C_2\,[\sigma] \tag{12}$$
$$(v_1 \quad v_2)\,[\sigma] = (v_1\,[\sigma]) \quad v_2\,[\sigma] \tag{13}$$
$$(\texttt{do} \quad x \leftarrow C_1 \quad \texttt{in} \quad C_2) = \texttt{do} \quad x \leftarrow (C_1\,[\sigma]) \quad \texttt{in} \quad (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{14}$$
$$\tag{15}$$

0.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma : \Gamma$$

by:

- (Nil)$\frac{\Gamma' \texttt{Ok}}{\Gamma' \vdash \diamond : \diamond}$
- (Extend)$\frac{\Gamma' \vdash \sigma : \Gamma x \notin \texttt{dom}(\Gamma) \Gamma' \vdash v : A}{\Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

0.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma : \Gamma$ then: **TODO: Number these**

Γ `Ok` and Γ' `Ok` Since Γ' `Ok` holds by the Nil-axiom. Γ `Ok` holds by induction on the well-formed-ness relation.

$\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v \colon A$ holds if $\Gamma' \vdash v \colon A$ holds.

$x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{16}$$

0.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g \colon \tau \wedge \Gamma' \vdash \sigma \colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{17}$$

TODO: Proof by induction over type relation Assuming $\Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t \colon \tau \to \Gamma' \vdash t \colon \tau$

0.2.1 Variables

Case Var **TODO: The more difficult case. case split on the structure of σ**

Case Weaken **TODO:**

0.2.2 Other Value Terms

Case Lambda **TODO:**

Case Constants **TODO:**

Case Unit **TODO:**

Case True **TODO:**

False **TODO:**

0.2.3 Computation Terms

Case Return **TODO: Induct using preconditions, then construct new tree**

Case Apply **TODO:**

Case If **TODO:**

Case Bind **TODO:**

0.2.4 Sub-typing and Sub-effecting

Case Sub-type **TODO:**

Case Sub-effect **TODO:**

0.3 Semantics of Substitution

0.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma' \vdash \sigma : \Gamma]\!]_M : \Gamma' \to \Gamma \tag{18}$$

- (Nil) $\frac{\Gamma' \texttt{Ok}}{[\![\Gamma' \vdash \diamond : \diamond]\!]_M = \langle\rangle_{\Gamma'}}$
- (Extend) $\frac{f = [\![\Gamma' \vdash \sigma : \Gamma]\!]_M \, g = [\![\Gamma' \vdash v : A]\!]_M}{[\![\Gamma' \vdash (\sigma, x := v : (\Gamma, x : A)]\!]_M = \langle f, g \rangle : \Gamma' \to (\Gamma \times A)}$

0.3.2 Lemma

TODO: Fill in from p98

0.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If $\Gamma \vdash t : \tau$ and $\Gamma' \vdash \sigma : \Gamma$ then

0.4 Single Substitution