Abstract

This document contains a terse explanation of the semantics of the Effect Calculus in an S-Category.

Contents

Chapter 1

Language Definition

1.1 Terms

$$
\begin{aligned}
v ::= &\mid x \\
&\mid \lambda x : A . v \\
&\mid \mathtt{k}^{A} \\
&\mid \mathtt{()} \\
&\mid \mathtt{true} \mid \mathtt{false} \\
&\mid v_1 \; v_2 \\
&\mid \mathtt{if}_{A} \; v \; \mathtt{then} \; v_1 \; \mathtt{else} \; v_2 \\
&\mid \mathtt{do} \; x \leftarrow v_1 \; \mathtt{in} \; v_2 \\
&\mid \mathtt{return} \; v
\end{aligned}
\tag{1.1}
$$

1.2 Type System

1.2.1 Effects

The effects should form a monotonous, partially-ordered monoid $(E, \cdot, \mathtt{1}, \leq)$ with elements ϵ

1.2.2 Types

Ground Types There exists a set γ of ground types, including $\mathtt{Unit}$, $\mathtt{Bool}$

Types

$$A, B, C ::= \gamma \mid \mathtt{M}_{\epsilon} A \mid A \to B$$

1.2.3 Subtyping

There exists a subtyping partial-order relation $\leq:_{\gamma}$ over ground types that is:

- (S-Reflexive) $\dfrac{}{A \leq:_{\gamma} A}$

- (S-Transitive) $\dfrac{A \leq:_\gamma B \qquad B \leq:_\gamma C}{A \leq:_\gamma C}$

We extend this relation with the function subtyping rule to yield the full subtyping relation $\leq:$

- (S-Ground) $\dfrac{A \leq:_\gamma B}{A \leq: B}$
- (S-Fn) $\dfrac{A \leq: A' \qquad B' \leq: B}{A' \to B' \leq: A \to B}$
- (S-Effect) $\dfrac{A \leq: A' \qquad \epsilon \leq \epsilon'}{\mathtt{M}_\epsilon A \leq: \mathtt{M}_{\epsilon'} A'}$

1.2.4 Type Environments

An environment, $G ::= \diamond \mid \Gamma, x \colon A$

Domain Function

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x \colon A) = \mathtt{dom}(\Gamma) \cup \{x\}$

Ok Predicate

- (Env-Nil) $\dfrac{}{\diamond \ \mathtt{Ok}}$
- (Env-Extend) $\dfrac{\Gamma \ \mathtt{Ok} \qquad x \notin \mathtt{dom}(\Gamma)}{\Gamma, x \colon A \ \mathtt{Ok}}$

1.2.5 Type Rules

Typing Rules

- (Const) $\dfrac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{k}^A \colon A}$
- (Unit) $\dfrac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{()} \colon \mathtt{Unit}}$
- (True) $\dfrac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}}$
- (False) $\dfrac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{false} \colon \mathtt{Bool}}$
- (Var) $\dfrac{\Gamma, x \colon A \ \mathtt{Ok}}{\Gamma, x \colon A \vdash x \colon A}$

- (Weaken) $\dfrac{\Gamma \vdash x\colon A}{\Gamma, y\colon B \vdash x\colon A}$ (if $x \neq y$)
- (Fn) $\dfrac{\Gamma, x\colon A \vdash v\colon B}{\Gamma \vdash \lambda x\colon A.v : A \to B}$
- (Subtype) $\dfrac{\Gamma \vdash v\colon A \qquad A {\leq:} B}{\Gamma \vdash v\colon B}$
- (Return) $\dfrac{\Gamma \vdash v\colon A}{\Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}} A}$
- (Apply) $\dfrac{\Gamma \vdash v_1\colon A \to B \qquad \Gamma \vdash v_2\colon A}{\Gamma \vdash v_1\ v_2\colon B}$
- (If) $\dfrac{\Gamma \vdash v\colon \texttt{Bool} \qquad \Gamma \vdash v_1\colon A \qquad \Gamma \vdash v_2\colon A}{\Gamma \vdash \texttt{if}_A\ V\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A}$
- (Bind) $\dfrac{\Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1} A \qquad \Gamma, x\colon A \vdash v_2\colon \texttt{M}_{\epsilon_2} B}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

1.2.6 Ok Lemma

Lemma 1.2.1 (Ok Lemma)**.** *If* $\Gamma \vdash v\colon A$ *then* Γ *`Ok`.*

Proof: If $\Gamma, x\colon A$ `Ok` then by inversion Γ `Ok` Only the type rule *Weaken* adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require Γ `Ok`. And all non-axiom derivations preserve the `Ok` property.

Chapter 2

Category Requirements

$\mathbb{C}$ should be an S-Category instantiated with the relevant subtyping and subeffecting morphisms and natural transformations.

Chapter 3

Denotations

3.1 Helper Morphisms

3.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\begin{aligned} \tau_{A,B} &: (A \times B) \to (B \times A) = \langle \pi_2, \pi_1 \rangle \\ \delta_A &: A \to (A \times A) = \langle \mathtt{Id}_A, \mathtt{Id}_A \rangle \end{aligned}$$

3.2 Denotations of Types

3.2.1 Denotation of Ground Types

The denotations of the default ground types, $\mathtt{Unit}, \mathtt{Bool}$ should be as follows:

$$[\![\mathtt{Unit}]\!] = \mathtt{1} \tag{3.1}$$

$$[\![\mathtt{Bool}]\!] = \mathtt{1} + \mathtt{1} \tag{3.2}$$

The mapping $[\![\_]\!]$ should then map each other ground type γ to an object $[\![\gamma]\!]$ in $\mathbb{C}$.

3.2.2 Denotation of Computation Types

Given a function $[\![\_]\!]$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\mathtt{M}_\epsilon A$ as so:

$$[\![\mathtt{M}_\epsilon A]\!] = T_\epsilon [\![A]\!]$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\mathtt{M}_\epsilon A]\!] = T_\epsilon A$$

3.2.3 Denotation of Function Types

Given a function $[\![\_]\!]$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to B$ as so:

$$[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$$

Again, since we can infer the denotation function, Let us drop the denotation syntax.

$$[\![A \to B]\!] = (B)^A$$

3.2.4 Denotation of Type Environments

Given a function $[\![\_]\!]$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an `Ok` type environment Γ.

$$[\![\diamond]\!] = \mathtt{1}$$

$$[\![\Gamma, x\colon A]\!] = ([\![\Gamma]\!] \times [\![A]\!])$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x\colon A]\!] = \Gamma \times A$$

3.3 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$[\![\Gamma \vdash v\colon A]\!] : \Gamma \to A$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash v\colon A$ are equal, we need not refer to the derivation that yielded each denotation.

3.3.1 Denotation of Terms

- (Unit) $\dfrac{\Gamma\ \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{()}\colon \mathtt{Unit}]\!] = \langle\rangle_\Gamma : \Gamma \to \mathtt{1}}$

- (Const) $\dfrac{\Gamma\ \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{k}^A\colon A]\!] = [\![\mathtt{k}^A]\!] \circ \langle\rangle_\Gamma : \Gamma \to A}$

- (True) $\dfrac{\Gamma\ \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{true}\colon \mathtt{Bool}]\!] = \mathtt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!] = \mathtt{1} + \mathtt{1}}$

- (False) $\dfrac{\Gamma\ \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{false}\colon \mathtt{Bool}]\!] = \mathtt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!] = \mathtt{1} + \mathtt{1}}$

- (Var) $\dfrac{\Gamma\ \mathtt{Ok}}{[\![\Gamma, x\colon A \vdash x\colon A]\!] = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Gamma \vdash x\colon A]\!] : \Gamma \to A}{[\![\Gamma, y\colon B \vdash x\colon A]\!] = f \circ \pi_1 : \Gamma \times B \to A}$

- (Fn) $\dfrac{f = [\![\Gamma, x\colon A \vdash v\colon B]\!] : \Gamma \times A \to B}{[\![\Gamma \vdash \lambda x\colon A.v : A \to B]\!] = \mathtt{cur}(f) : \Gamma \to (B)^A}$

- (Subtype) $\dfrac{f = [\![\Gamma \vdash v\colon A]\!] : \Gamma \to A \qquad g = [\![A \leq\colon B]\!]}{[\![\Gamma \vdash v\colon B]\!] = g \circ f : \Gamma \to B}$

- (Return) $\dfrac{f = [\![\Gamma \vdash v\colon A]\!]}{[\![\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_1 A]\!] = \eta_A \circ f}$

- (If) $\dfrac{f = [\![\Gamma \vdash v\colon \mathtt{Bool}]\!] : \Gamma \to \mathtt{1} + \mathtt{1} \qquad g = [\![\Gamma \vdash v_1\colon A]\!] \qquad h = [\![\Gamma \vdash v_2\colon A]\!]}{[\![\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!] = \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \to A}$

- (Bind) $\dfrac{f = [\![\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A : \Gamma \to T_{\epsilon_1} A]\!] \qquad g = [\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B]\!] : \Gamma \times A \to T_{\epsilon_2} B}{[\![\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B]\!] = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma, A, \epsilon_1} \circ \langle \mathtt{Id}_\Gamma, f \rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$

- (Apply) $\dfrac{f = [\![\Gamma \vdash v_1\colon A \to B]\!] : \Gamma \to (B)^A \qquad g = [\![\Gamma \vdash v_2\colon A]\!] : \Gamma \to A}{[\![\Gamma \vdash v_1\ v_2\colon B]\!] = \mathtt{app} \circ \langle f, g \rangle : \Gamma \to B}$

Chapter 4

Unique Denotations

4.1 Reduced Type Derivation

A reduced type derivation is one where instances of the subtype rule must, and may only, occur at the root or directly above an *If*, or *Apply* rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash v\colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

4.2 Reduced Type Derivations are Unique

Theorem 4.2.1 (Reduced Type Derivations are Unique)**.** *For each instance of the relation $\Gamma \vdash v\colon A$,there exists at most one reduced derivation of $\Gamma \vdash v\colon A$.*

Proof: This is proved by induction over the typing rules on the bottom rule used in each derivation.

4.2.1 Variables

To find the unique derivation of $\Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$: Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is, if $A' \leq: A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\dfrac{\Gamma', x : A' \;\; \texttt{Ok}}{\Gamma, x : A' \vdash x\colon A'} \qquad A' \leq: A}{\Gamma', x : A' \vdash x\colon A} \tag{4.1}$$

Case $\Gamma = \Gamma', y : B$: with $y \neq x$.

Hence, if $\Gamma \vdash x\colon A$ holds, then so must $\Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Gamma' \vdash x \colon A'} \qquad A' \leq: A}{\Gamma' \vdash x \colon A} \tag{4.2}$$

Be the unique reduced derivation of $\Gamma' \vdash x \colon A$.

Then the unique reduced derivation of $\Gamma \vdash x \colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\dfrac{\dfrac{\Delta}{\Gamma, x : A' \vdash x \colon A'}}{\Gamma \vdash x \colon A'} \qquad A' \leq: A}{\Gamma \vdash x \colon A} \tag{4.3}$$

4.2.2 Constants

For each of the constants, ($\mathtt{k}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{k}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\dfrac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{k}^A \colon A} \qquad A \leq: B}{\Gamma \vdash \mathtt{k}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

Case Fn: The reduced derivation of $\Gamma \vdash \lambda x \colon A.v : A' \to B'$ is:

$$\text{(Subtype)}\frac{\text{(Fn)}\dfrac{\dfrac{\Delta}{\Gamma, x \colon A \vdash v \colon B}}{\Gamma \vdash \lambda x \colon A.B : A \to B} \qquad A \to B \leq: A' \to B'}{\Gamma \vdash \lambda x \colon A.v : A' \to B'}$$

Where

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Gamma, x \colon A \vdash v \colon B} \qquad B \leq: B'}{\Gamma, x \colon A \vdash v \colon B'} \tag{4.4}$$

is the reduced derivation of $\Gamma, x \colon A \vdash v \colon \mathtt{M}_{\epsilon'} B$ if it exists.

Case Return: The reduced denotation of $\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v \colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_1 A} \qquad \text{(T-Effect)}\dfrac{1 \leq \epsilon \qquad A \leq: B}{\mathtt{M}_1 A \leq: \mathtt{M}_\epsilon B}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_\epsilon B}$$

Where

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Gamma \vdash v \colon A} \qquad A \leq: B}{\Gamma \vdash v \colon B}$$

is the reduced derivation of $\Gamma \vdash v\colon B$

Case Apply: If

$$\text{(Subtype)}\,\frac{\dfrac{\Delta}{\Gamma \vdash v_1\colon A \to B} \qquad A \to B \leq: A' \to B'}{\Gamma \vdash v_1\colon A' \to B'}$$

and

$$\text{(Subtype)}\,\frac{\dfrac{\Delta'}{\Gamma \vdash v_2\colon A''} \qquad A'' \leq: A'}{\Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1\colon A' \to B'$ and $\Gamma \vdash v_2\colon A'$

Then we can construct the reduced derivation of $\Gamma \vdash v_1\ v_2\colon B'$ as

$$\text{(Subtype)}\,\frac{\text{(Apply)}\,\dfrac{\dfrac{\Delta}{\Gamma \vdash v_1\colon A \to B} \qquad \text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Gamma \vdash v\colon A''} \qquad A'' \leq: A}{\Gamma \vdash v\colon A}}{\Gamma \vdash v_1\ v_2\colon B} \qquad B \leq: B'}{\Gamma \vdash v_1\ v_2\colon B'}$$

Case If: Let

$$\text{(Subtype)}\,\frac{\dfrac{\Delta}{\Gamma \vdash v\colon B} \qquad B \leq: \mathtt{Bool}}{\Gamma \vdash v\colon \mathtt{Bool}} \tag{4.5}$$

$$\text{(Subeffect)}\,\frac{\dfrac{\Delta'}{\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon'} A'} \qquad \text{(T-Effect)}\,\dfrac{\epsilon' \leq \epsilon \qquad A' \leq: A}{\mathtt{M}_{\epsilon'} A' \leq: \mathtt{M}_{\epsilon} A}}{\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon} A} \tag{4.6}$$

$$\text{(Subtype)}\,\frac{\dfrac{\Delta''}{\Gamma \vdash v_2\colon \mathtt{M}_{\epsilon''} A''} \qquad \text{(T-Effect)}\,\dfrac{\epsilon'' \leq \epsilon \qquad A'' \leq: A}{\mathtt{M}_{\epsilon''} A'' \leq: \mathtt{M}_{\epsilon} A}}{\Gamma \vdash v_2\colon \mathtt{M}_{\epsilon} A} \tag{4.7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v\colon \mathtt{Bool}$, $\Gamma \vdash v_1\colon A$, $\Gamma \vdash v_2\colon A$.

Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : B$ is:

$$\text{(Subtype)}\,\frac{\text{(If)}\,\dfrac{\text{(Subtype)}\,\dfrac{\dfrac{\Delta}{\Gamma \vdash v\colon B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v\colon \mathtt{Bool}} \quad \text{(Subeffect)}\,\dfrac{\dfrac{\Delta'}{\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon'} A'} \quad \text{(T-Effect)}\,\dfrac{\epsilon' \leq \epsilon \quad A' \leq: A}{\mathtt{M}_{\epsilon'} A' \leq: \mathtt{M}_{\epsilon} A}}{\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon} A} \quad \text{(Subtype)}\,\dfrac{\dfrac{\Delta''}{\Gamma \vdash v_2\colon \mathtt{M}_{\epsilon''} A''} \quad \text{(T-Effect)}\,\dfrac{\epsilon'' \leq \epsilon \quad A'' \leq: A}{\mathtt{M}_{\epsilon''} A'' \leq: \mathtt{M}_{\epsilon} A}}{\Gamma \vdash v_2\colon \mathtt{M}_{\epsilon} A}}{\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \qquad A \leq: B}{\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : B} \tag{4.8}$$

Case Bind: Let

$$(\text{Subeffect})\dfrac{\dfrac{\Delta}{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad (\text{T-Effect})\dfrac{\epsilon_1 \leq \epsilon_1' \qquad A \leq: A'}{\mathtt{M}_{\epsilon_1} A \leq: \mathtt{M}_{\epsilon_1'} A'}}{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{4.9}$$

$$(\text{Subeffect})\dfrac{\dfrac{\Delta'}{\Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad (\text{T-Effect})\dfrac{\epsilon_2 \leq \epsilon_2' \qquad B \leq: B'}{\mathtt{M}_{\epsilon_2} B \leq: \mathtt{M}_{\epsilon_2'} B'}}{\Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{4.10}$$

Be the respective unique reduced type derivations of the subterms]

By weakening, $\iota \times \colon \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, there's also one of $\Gamma, x \colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$.

$$(\text{Subtype})\dfrac{\dfrac{\Delta''}{\Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad (\text{T-Effect})\dfrac{\epsilon_2 \leq \epsilon_2' \qquad B \leq: B'}{\mathtt{M}_{\epsilon_2} B \leq: \mathtt{M}_{\epsilon_2'} B'}}{\Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{4.11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

$$(\text{Subtype})\dfrac{(\text{Bind})\dfrac{(\text{Subeffect})\dfrac{\dfrac{\Delta}{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad (\text{T-Effect})\dfrac{\epsilon_1 \leq \epsilon_1' \qquad A \leq: A'}{\mathtt{M}_{\epsilon_1} A \leq: \mathtt{M}_{\epsilon_1'} A'}}{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \qquad (\text{Subtype})\dfrac{\dfrac{\Delta''}{\Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad (\text{T-Effect})\dfrac{\epsilon_2 \leq \epsilon_2' \qquad B \leq: B'}{\mathtt{M}_{\epsilon_2} B \leq: \mathtt{M}_{\epsilon_2'} B'}}{\Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \qquad (\text{T-Effect})\dfrac{\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2' \qquad B \leq: B'}{\mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B \leq: \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{4.12}$$

4.3 Each Type Derivation has a Reduced Equivalent with the Same Denotation

Theorem 4.3.1 (Each Type Derivation has a Reduced Equivalent with the Same Denotation)**.** *We introduce a function, reduce that maps each valid type derivation of* $\Gamma \vdash v \colon A$ *to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.*

Proof:

4.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{k}^{A}$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$$reduce((\text{Const})\dfrac{\Gamma\ \mathtt{Ok}}{\Gamma \vdash \mathtt{k}^{A} \colon A}) = (\text{Const})\dfrac{\Gamma\ \mathtt{Ok}}{\Gamma \vdash \mathtt{k}^{A} \colon A}$$

Case Var:

$$reduce\left((\text{Var})\frac{\Gamma\ \texttt{Ok}}{\Gamma, x\colon A \vdash x\colon A}\right) = (\text{Var})\frac{\Gamma\ \texttt{Ok}}{\Gamma, x\colon A \vdash x\colon A} \tag{4.13}$$

Preserves denotation trivially.

Case Weaken:

reduce **definition** To find:

$$reduce\left((\text{Weaken})\frac{\dfrac{\Delta}{\Gamma \vdash x\colon A}}{\Gamma, y\colon B \vdash x\colon A}\right) \tag{4.14}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Gamma \vdash x\colon A} \qquad A' \leq: A}{\Gamma \vdash x\colon A} = reduce(\Delta) \tag{4.15}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\dfrac{\dfrac{\Delta'}{\Gamma \vdash x\colon A'}}{\Gamma, y\colon B \vdash x\colon A'} \qquad A' \leq: A}{\Gamma, y\colon B \vdash x\colon A} \tag{4.16}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$\left[\!\left[(\text{Weaken})\frac{\dfrac{\Delta}{\Gamma \vdash x\colon A}}{\Gamma, y\colon B \vdash x\colon A}\right]\!\right] = \Delta \circ \pi_1 \tag{4.17}$$

Similarly, the denotation of the reduced denotation is:

$$\left[\!\left[(\text{Subtype})\frac{(\text{Weaken})\dfrac{\dfrac{\Delta'}{\Gamma \vdash x\colon A'}}{\Gamma, y\colon B \vdash x\colon A'} \qquad A' \leq: A}{\Gamma, y\colon B \vdash x\colon A}\right]\!\right] = [\![A' \leq: A]\!] \circ \Delta' \circ \pi_1 \tag{4.18}$$

By induction on *reduce* preserving denotations and the reduction of Δ (4.15), we have:

$$\Delta = [\![A' \leq: A]\!] \circ \Delta' \tag{4.19}$$

So the denotations of the un-reduced and reduced derivations are equal.

Case Fn:

reduce **definition** To find:

$$reduce((\text{Fn})\frac{\frac{\Delta}{\Gamma, x\colon A \vdash v\colon \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \lambda x\colon A.v : A \to B}) \tag{4.20}$$

Let

$$(\text{Subtype})\frac{\frac{\Delta'}{\Gamma, x\colon A \vdash v\colon B'} \qquad B{\leq:}B'}{\Gamma, x\colon A \vdash v\colon B} = reduce(\Delta) \tag{4.21}$$

In

$$(\text{Subtype})\frac{(\text{Fn})\frac{\Delta'}{\Gamma, x\colon A \vdash v\colon \mathtt{M}_{\epsilon_1} B'} \qquad A \to B'{\leq:}A \to B}{\Gamma \vdash \lambda x\colon A.v : A \to B} \tag{4.22}$$

Preserves Denotation Let

$$f = [\![\mathtt{M}_{\epsilon_1} B'{\leq:}\mathtt{M}_{\epsilon_2} B]\!] = [\![\epsilon_1 {\leq} \epsilon_2]\!]_B \circ T_{\epsilon_1}([\![B'{\leq:}B]\!])$$
$$[\![A \to B'{\leq:}A \to B]\!] = f^A = \mathtt{cur}(f \circ \mathtt{app})$$

Then

$$\begin{aligned}
before &= \mathtt{cur}(\Delta) && \text{By definition}\\
&= \mathtt{cur}(f \circ \Delta') && \text{By reduction of } \Delta\\
&= f^A \circ \mathtt{cur}(\Delta') && \text{By the property of } f^X \circ \mathtt{cur}(g) = \mathtt{cur}(f \circ g)\\
&= after && \text{By definition}
\end{aligned}$$

Case Subtype:

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{\frac{\Delta}{\Gamma \vdash v\colon A} \qquad A{\leq:}B}{\Gamma \vdash v\colon B}) \tag{4.23}$$

Let

$$(\text{Subtype})\frac{\frac{\Delta'}{\Gamma \vdash x\colon A} \qquad A'{\leq:}A}{\Gamma \vdash x\colon A} = reduce(\Delta) \tag{4.24}$$

In

$$(\text{Subtype})\frac{\frac{\Delta'}{\Gamma \vdash v\colon A'} \qquad A'{\leq:}A{\leq:}B}{\Gamma \vdash v\colon B} \tag{4.25}$$

Preserves Denotation

$$\begin{aligned} before &= [\![A{\leq:}B]\!] \circ \Delta \\ &= [\![A{\leq:}B]\!] \circ ([\![A'{\leq:}A]\!] \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. \\ &= [\![A'{\leq:}B]\!] \circ \Delta' \quad \text{Subtyping relations are unique} \\ &= after \end{aligned}$$

Case Return:

reduce **definition** To find:

$$reduce((\text{Return})\frac{\dfrac{\Delta}{\Gamma \vdash v\colon A}}{\Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}}\, A}) \tag{4.26}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Gamma \vdash v\colon A'} \qquad A'{\leq:}A}{\Gamma \vdash v\colon A} = reduce(\Delta) \tag{4.27}$$

In

$$(\text{Subtype})\frac{(\text{Return})\dfrac{\Delta'}{\Gamma \vdash v\colon A'} \qquad (\text{T-Effect})\dfrac{\texttt{1}{\leq}\texttt{1} \qquad A'{\leq:}A}{\texttt{M}_{\texttt{1}}\, A'{\leq:}\texttt{M}_{\texttt{1}}\, A}}{\Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}}\, A} \tag{4.28}$$

Then

$$\begin{aligned} before &= \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} \\ &= \eta_A \circ [\![A'{\leq:}A]\!] \circ \Delta' \quad \text{BY reduction of } \Delta \\ &= T_{\texttt{1}} [\![A'{\leq:}A]\!] \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta \\ &= [\![\texttt{1}{\leq}\texttt{1}]\!]_A \circ T_{\texttt{1}} [\![A'{\leq:}A]\!] \circ \eta_{A'} \circ \Delta' \quad \text{Since } [\![\texttt{1}{\leq}\texttt{1}]\!] \text{ is the identity Nat-Trans} \\ &= after \quad \text{By definition} \end{aligned}$$

Case Apply:

reduce **definition** To find:

$$reduce((\text{Apply})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1\colon A \to B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2\colon A}}{\Gamma \vdash v_1\ v_2\colon B}) \tag{4.29}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta_1'}{\Gamma \vdash v_1 \colon A' \to B'} \qquad A' \to B' \leq: A \to B}{\Gamma \vdash v_1 \colon A \to B} = \mathit{reduce}(\Delta_1)$$

$$(\text{Subtype})\frac{\dfrac{\Delta_2'}{\Gamma \vdash v \colon A'} \qquad A' \leq: A}{\Gamma \vdash v_1 \colon A} = \mathit{reduce}(\Delta_2)$$

In

$$(\text{Subtype})\frac{(\text{Apply})\dfrac{\dfrac{\Delta_1'}{\Gamma \vdash v_1 \colon A' \to B'} \qquad (\text{Subtype})\dfrac{\dfrac{\Delta_2'}{\Gamma \vdash v_2 \colon A''} \qquad A'' \leq: A \leq: A'}{\Gamma \vdash v_2 \colon A'}}{\Gamma \vdash v_1\ v_2 \colon B'} \qquad B' \leq: B}{\Gamma \vdash v_1\ v_2 \colon B} \tag{4.30}$$

Preserves Denotation Let

$$\begin{aligned} f &= [\![A \leq: A']\!] : A \to A' \\ f' &= [\![A'' \leq: A]\!] : A'' \to A \\ g &= [\![B' \leq: B]\!] : B' \to B \\ h &= [\![\epsilon' \leq \epsilon]\!] : T_{\epsilon'} \to T_\epsilon \end{aligned}$$

Hence

$$\begin{aligned} [\![A' \to B' \leq: A \to B]\!] &= (h_B \circ T_{\epsilon'} g)^A \circ (T_{\epsilon'} B')^f \\ &= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f)) \\ &= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \end{aligned}$$

Then

$$\begin{aligned} \mathit{before} &= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} \\ &= \mathtt{app} \circ \langle \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \circ \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 \\ &= \mathtt{app} \circ (\mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \times \mathtt{Id}_A) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{Factoring out} \\ &= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By the exponential property} \\ &= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ \langle \Delta_1', f \circ f' \circ \Delta_2' \rangle \\ &= \mathit{after} \quad \text{By defintion} \end{aligned}$$

Case If:

reduce **definition**

$$\text{reduce}((\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v \colon \mathtt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_1 \colon A} \qquad \dfrac{\Delta_3}{\Gamma \vdash v_2 \colon A}}{\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A}) = (\text{If})\frac{\dfrac{\mathit{reduce}(\Delta_1)}{\Gamma \vdash v \colon \mathtt{Bool}} \qquad \dfrac{\mathit{reduce}(\Delta_2)}{\Gamma \vdash v_1 \colon A} \qquad \dfrac{\mathit{reduce}(\Delta_3)}{\Gamma \vdash v_2 \colon A}}{\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{4.31}$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Case Bind:

reduce **definition** To find

$$reduce((\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}) \tag{4.32}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta_1'}{\Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1'} A'} \qquad (\text{T-Effect})\dfrac{\epsilon_1' \leq \epsilon_1 \qquad A' \leq: A}{\mathtt{M}_{\epsilon_1'} A' \leq: \mathtt{M}_{\epsilon_1} A}}{\Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A} = reduce(\Delta_1) \tag{4.33}$$

Since $i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq: A$, and by Δ_2, $(\Gamma, x : A) \vdash v_2 : \mathtt{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $(\Gamma, x : A') \vdash v_2 : \mathtt{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Subtype) rule below all instances of the (Var) rule.

Let

$$(\text{Subtype})\frac{\dfrac{\Delta_3'}{\Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2'} B'} \qquad (\text{T-Effect})\dfrac{\epsilon_2' \leq \epsilon_2 \qquad B' \leq: B}{\mathtt{M}_{\epsilon_2'} B' \leq: \mathtt{M}_{\epsilon_2} B}}{\Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2} B} = reduce(\Delta_3) \tag{4.34}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Subtype})\frac{(\text{Bind})\dfrac{\dfrac{\Delta_1'}{\Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1'} A'} \qquad \dfrac{\Delta_3'}{\Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B} \qquad (\text{T-Effect})\dfrac{\epsilon_1' \cdot \epsilon_2' \leq \epsilon_1 \cdot \epsilon_3 \qquad B' \leq: B}{\mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B' \leq: \mathtt{M}_{\epsilon_1 \cdot \epsilon_3} B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{4.35}$$

Preserves Denotation Let

$$\begin{aligned}
f &= [\![A' \leq: A]\!] : A' \to A \\
g &= [\![B' \leq: B]\!] : B' \to B \\
h_1 &= [\![\epsilon_1' \leq \epsilon_1]\!] : T_{\epsilon_1'} \to T_{\epsilon_1} \\
h_2 &= [\![\epsilon_2' \leq \epsilon_2]\!] : T_{\epsilon_2'} \to T_{\epsilon_2} \\
h &= [\![\epsilon_1' \cdot \epsilon_2' \leq \epsilon_1 \cdot \epsilon_2]\!] : T_{\epsilon_1' \cdot \epsilon_2'} \to T_{\epsilon_1 \cdot \epsilon_2}
\end{aligned}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3 = \Delta_2 \circ (\mathtt{Id}_\Gamma \times f) \tag{4.36}$$

And due to the reduction of Δ_3, we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3 \tag{4.37}$$

So:

$$\begin{aligned}
before &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1\rangle \quad \text{By definition.} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon'_1} f \circ \Delta'_1\rangle \quad \text{By reduction of } \Delta_1. \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1\rangle \quad \text{Factor out } h_1 \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma\times A)} \circ \mathtt{t}_{\epsilon'_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1\rangle \quad \text{Tensor strength and subeffecting } h_1 \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1}\Delta_2 \circ \mathtt{t}_{\epsilon'_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon'_1} f \circ \Delta'_1\rangle \quad \text{Naturality of } h_1 \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1}\Delta_2 \circ \mathtt{t}_{\epsilon'_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon'_1} f) \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{Factor out pairing again} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{Tensorstrength} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1}(\Delta_3) \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{By the definition of } \Delta_3 \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1}(h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3) \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{By the reduction of } \Delta_3 \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon'_1} h_{2,B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1}\Delta'_3 \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{Factor out the functor} \\
&= h_B \circ \mu_{\epsilon'_1,\epsilon'_2,B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1}\Delta'_3 \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{By the } \mu \text{ and Subtype rule} \\
&= h_B \circ T_{\epsilon'_1 \cdot \epsilon'_2} g \circ \mu_{\epsilon'_1,\epsilon'_2,B'} \circ T_{\epsilon'_1}\Delta'_3 \circ \mathtt{t}_{\epsilon'_1,\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta'_1\rangle \quad \text{By naturality of } \mu_{,,} \\
&= after \quad \text{By definition}
\end{aligned}$$

4.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash v \colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!] = [\![reduce\Delta]\!] = [\![reduce\Delta']\!] = [\![\Delta']\!]$, hence the denotation $[\![\Gamma \vdash v \colon A]\!]$ is unique.

Chapter 5

Weakening

5.1 Weakening Definition

5.1.1 Relation

We define the ternary weakening relation $w\colon \Gamma' \triangleright \Gamma$ using the following rules.

- (T-Id) $\dfrac{\Gamma \;\; \texttt{Ok}}{\iota\colon \Gamma \triangleright \Gamma}$
- (T-Project) $\dfrac{\omega\colon \Gamma' \triangleright \Gamma \qquad x \notin \texttt{dom}(\Gamma')}{\omega\pi\colon \Gamma, x : A \triangleright \Gamma}$
- (T-Extend) $\dfrac{\omega\colon \Gamma' \triangleright \Gamma \qquad x \notin \texttt{dom}(\Gamma') \qquad A \leq: B}{w\times\colon \Gamma', x : A \triangleright \Gamma, x : B}$

5.1.2 Weakening Denotations

The denotation of a weakening relation is defined as follows:

$$[\![\omega\colon \Gamma' \triangleright \Gamma]\!] : \Gamma' \to \Gamma \tag{5.1}$$

- $[\![\iota\colon \Gamma \triangleright \Gamma]\!] = \texttt{Id}_\Gamma : \Gamma \to \Gamma$
- (T-Project) $\dfrac{f = [\![\omega\colon \Gamma' \triangleright \Gamma]\!] : \Gamma' \to \Gamma}{[\![\omega\pi\colon \Gamma, x : A \triangleright \Gamma]\!] = f \circ \pi_1 : \Gamma' \times A \to \Gamma}$
- (T-Extend) $\dfrac{f = [\![\omega\colon \Gamma' \triangleright \Gamma]\!] : \Gamma' \to \Gamma \qquad g = [\![A \leq: B]\!] : A \to B}{[\![w\times\colon \Gamma', x : A \triangleright \Gamma, x : B]\!] = (f \times g) : (\Gamma \times A) \to (\Gamma \times B)}$

5.2 Weakening Theorems

Lemma 5.2.1 (Domain Lemma). *If $\omega\colon \Gamma' \triangleright \Gamma$, then $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(\Gamma')$.*

Proof:

Case T-Id: Then $\Gamma' = \Gamma$ and so $\mathtt{dom}(\Gamma') = \mathtt{dom}(\Gamma)$.

Case T-Project: By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma') \subseteq \mathtt{dom}(\Gamma' \cup \{x\})$

Case T-Extend: By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$ so

$$\mathtt{dom}(\Gamma, x{:}\, A) = \mathtt{dom}(\Gamma) \cup \{x\} \subseteq \mathtt{dom}(\Gamma') \cup \{x\} = \mathtt{dom}(\Gamma', x : A)$$

Theorem 5.2.2 (Ok Preservation)**.** *If $\omega{:}\, \Gamma' \triangleright \Gamma$ and Γ $\mathtt{Ok}$ then Γ' $\mathtt{Ok}$*

Proof:

Case T-Id:

$$(\text{T-Id})\frac{\Gamma\ \mathtt{Ok}}{\iota{:}\, \Gamma \triangleright \Gamma}$$

By inversion, Γ $\mathtt{Ok}$.

Case T-Project:

$$(\text{T-Project})\frac{\omega{:}\, \Gamma' \triangleright \Gamma \qquad x \notin \mathtt{dom}(\Gamma')}{\omega\pi{:}\, \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\omega{:}\, \Gamma' \triangleright \Gamma$ and $x \notin \mathtt{dom}(\Gamma')$.

Hence by induction Γ' $\mathtt{Ok}$, Γ $\mathtt{Ok}$. Since $x \notin \mathtt{dom}(\Gamma')$, we have $\Gamma', x : A$ $\mathtt{Ok}$.

Case T-Extend: $$(\text{T-Extend})\frac{\omega{:}\, \Gamma' \triangleright \Gamma \qquad x \notin \mathtt{dom}(\Gamma') \qquad A {\leq:} B}{w\times{:}\, \Gamma', x : A \triangleright \Gamma, x : B},$$

By inversion, we have

$\omega{:}\, \Gamma' \triangleright \Gamma$, $x \notin \mathtt{dom}(\Gamma')$.

Hence we have Γ $\mathtt{Ok}$, Γ' $\mathtt{Ok}$, and by the domain Lemma, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$, hence $x \notin \mathtt{dom}(\Gamma)$. Hence, we have $\Gamma, x{:}\, A$ $\mathtt{Ok}$ and $\Gamma', x : A$ $\mathtt{Ok}$

Theorem 5.2.3 (Weakening Preserves Type Relation)**.** *If $\Gamma \vdash v{:}\, A$ and $\omega{:}\, \Gamma' \triangleright \Gamma$ then there is a derivation of $\Gamma' \vdash v{:}\, A$*

Proof: Proved in parallel with theorem 5.2.4 below

Theorem 5.2.4 (Weakening and Denotations)**.** *If $\omega{:}\, \Gamma' \triangleright \Gamma$ and $\Delta = [\![\Gamma \vdash v{:}\, A]\!]$ and $\Delta' = [\![\Gamma' \vdash v{:}\, A]\!]$, derived using theorem 5.2.3, then*

$$\Delta \circ [\![\omega]\!] = \Delta' : \Gamma' \to A$$

Proof: We induct over the structure of typing derivations of $\Gamma \vdash v\colon A$, assuming $\omega\colon \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash v\colon A$ and show that $\Delta \circ [\![\omega\colon \Gamma' \triangleright \Gamma]\!] = \Delta'$

5.2.1 Variable Terms

Case Var, Weaken: We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\iota\colon \Gamma \triangleright \Gamma]\!] \tag{5.2}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega'\colon \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x\colon A$, such that

$$\Delta_1 = \Delta \circ [\![\omega'\colon \Gamma'' \triangleright \Gamma]\!] \quad \text{By Induction} \tag{5.3}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Gamma'' \vdash x\colon A}{\Gamma'', x' : A' \vdash x\colon A} \tag{5.4}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \Delta_1 \circ \pi_1 \quad \text{By Definition}\\ &= \Delta \circ [\![\omega'\colon \Gamma'' \triangleright \Gamma]\!] \circ \pi_1 \quad \text{By induction}\\ &= \Delta \circ [\![\omega'\pi_1\colon \Gamma' \triangleright \Gamma]\!] \quad \text{By denotation of weakening}\end{aligned}$$

If $\omega = \omega'\times$ Then

$$\begin{aligned}\Gamma' &= \Gamma''', x' : B\\ \Gamma &= \Gamma'', x' : A'\\ B &\leq: A\end{aligned}$$

If $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$(\text{Subtype})\frac{(\text{Var})\dfrac{}{\Gamma''', x : B \vdash x\colon B} \qquad B \leq: A}{\Gamma' \vdash x\colon A} \tag{5.5}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= [\![B \leq: A]\!] \circ \pi_2 \quad \text{By Definition}\\ &= \pi_2 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!] \times [\![B \leq: A]\!]) \quad \text{By the properties of binary products}\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!] \quad \text{By Definition}\end{aligned}$$

Case $x \neq x'$: Then

$$\Delta = (\text{Weaken})\frac{\dfrac{\Delta_1}{\Gamma'' \vdash x : A}}{\Gamma \vdash x : A} \tag{5.6}$$

By induction with $\omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x : A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{\dfrac{\Delta_1'}{\Gamma''' \vdash x : A}}{\Gamma' \vdash x : A} \tag{5.7}$$

This preserves denotations:

By induction, we have

$$\Delta_1' = \Delta_1 \circ [\![\omega : \Gamma''' \triangleright \Gamma'']\!] \tag{5.8}$$

So we have:

$$\begin{aligned}\Delta' &= \Delta_1' \circ \pi_1 \quad \text{By denotation definition}\\ &= \Delta_1 \circ [\![\omega' : \Gamma''' \triangleright \Gamma'']\!] \quad \text{By induction} \circ \pi_1\\ &= \Delta_1 \circ \pi_1 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!] \times [\![A' \leq: B]\!]) \quad \text{By product properties}\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!] \quad \text{By definition}\end{aligned}$$

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\omega : \Gamma' \triangleright \Gamma']\!]$, simply as ω.

Case Const: The constant typing rules, `()`, `true`, `false`, $\mathtt{k}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{k}^A$.

$$(\text{Const})\frac{\Gamma \ \mathtt{Ok}}{\Gamma \vdash \mathtt{k}^A : A} \tag{5.9}$$

By inversion, we have $\Gamma \ \mathtt{Ok}$, so we have $\Gamma' \ \mathtt{Ok}$.

Hence

$$(\text{Const})\frac{\Gamma' \ \mathtt{Ok}}{\Gamma' \vdash \mathtt{k}^A : A} \tag{5.10}$$

Holds.

This preserves denotations:

$$
\begin{aligned}
\Delta' &= [\![\mathtt{k}^{A}]\!] \circ \langle\rangle_{\Gamma'} \quad \text{By definition} \\
&= [\![\mathtt{k}^{A}]\!] \circ \langle\rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} \\
&= \Delta \quad \text{By Definition}
\end{aligned}
$$

Case Fn: By inversion, we have a derivation Δ_1 giving

$$
\Delta = (\mathrm{Fn})\frac{\dfrac{\Delta_1}{\Gamma, x\colon A \vdash v\colon B}}{\Gamma \vdash \lambda x\colon A.v : A \to B} \tag{5.11}
$$

Since $\omega\colon \Gamma' \triangleright \Gamma$, we have:

$$
\omega\times\colon (\Gamma, x : A) \triangleright (\Gamma, x\colon A) \tag{5.12}
$$

Hence, by induction, using $\omega\times\colon (\Gamma, x : A) \triangleright (\Gamma, x\colon A)$, we derive Δ_1':

$$
\Delta' = (\mathrm{Fn})\frac{\dfrac{\Delta_1'}{\Gamma', x : A \vdash v\colon B}}{\Gamma', x : A \vdash \lambda x\colon A.v : A \to B} \tag{5.13}
$$

This preserves denotations:

$$
\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By Definition} \\
&= \mathtt{cur}(\Delta_1 \circ (\omega \times \mathtt{Id}_{\Gamma})) \quad \text{By the denotation of } \omega\times \\
&= \mathtt{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} \\
&= \Delta \circ \omega \quad \text{By Definition}
\end{aligned}
$$

Case Subtype:

$$
(\mathrm{Subtype})\frac{\Gamma \vdash v\colon A \qquad A{\leq\colon}B}{\Gamma \vdash v\colon B} \tag{5.14}
$$

by inversion, we have a derivation Δ_1

$$
\frac{\Delta_1}{\Gamma \vdash v\colon A} \tag{5.15}
$$

So by induction, we have a derivation Δ_1' such that:

$$
(\mathrm{Subtype})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v\colon a} \qquad A{\leq\colon}B}{\Gamma' \vdash v\colon B} \tag{5.16}
$$

This preserves denotations:

$$\begin{aligned}\Delta' &= [\![A {\leq:} B]\!] \circ \Delta'_1 \quad \text{By Definition}\\ &= [\![A {\leq:} B]\!] \circ \Delta_1 \circ \omega \quad \text{By induction}\\ &= \Delta \circ \omega \quad \text{By Definition}\end{aligned}$$

Case Return: We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\dfrac{\dfrac{\Delta_1}{\Gamma \vdash v \colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M_1}\, A} \tag{5.17}$$

Hence, by induction, with $\omega \colon \Gamma' \triangleright \Gamma$, we find the derivation Δ'_1 such that:

$$\Delta' = (\text{Return})\dfrac{\dfrac{\Delta'_1}{\Gamma' \vdash v \colon A}}{\Gamma' \vdash \mathtt{return}\ v : \mathtt{M_1}\, A} \tag{5.18}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \eta_A \circ \Delta'_1 \quad \text{By definition}\\ &= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1\\ &= \Delta \circ \omega \quad \text{By Definition}\end{aligned}$$

Case Apply: By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\dfrac{\dfrac{\Delta_1}{\Gamma \vdash v_1 \colon A \to B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2 \colon A}}{\Gamma \vdash v_1\ v_2 \colon B} \tag{5.19}$$

By induction, this gives us the respective derivations: Δ'_1, Δ'_2 such that

$$\Delta' = (\text{Apply})\dfrac{\dfrac{\Delta'_1}{\Gamma' \vdash v_1 \colon A \to B} \qquad \dfrac{\Delta'_2}{\Gamma' \vdash v_2 \colon A}}{\Gamma' \vdash v_1\ v_2 \colon B} \tag{5.20}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \mathsf{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition}\\ &= \mathsf{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2\\ &= \mathsf{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega\\ &= \Delta \circ \omega \quad \text{By Definition}\end{aligned}$$

Case If: By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v\colon \texttt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_1\colon A} \qquad \dfrac{\Delta_3}{\Gamma \vdash v_2\colon A}}{\Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{5.21}$$

By induction, this gives us the sub-derivations $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\Delta' = (\text{If})\frac{\dfrac{\Delta'_1}{\Gamma' \vdash v\colon \texttt{Bool}} \qquad \dfrac{\Delta'_2}{\Gamma' \vdash v_1\colon A} \qquad \dfrac{\Delta'_3}{\Gamma' \vdash v_2\colon A}}{\Gamma' \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{5.22}$$

And

$$\begin{aligned} \Delta'_1 &= \Delta_1 \circ \omega \\ \Delta'_3 &= \Delta_2 \circ \omega \\ \Delta'_3 &= \Delta_3 \circ \omega \end{aligned}$$

This preserves denotations. Since $\omega : \Gamma' \to \Gamma$,
Let $(A)^\omega : A^\Gamma \to A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(A)^\omega = \texttt{cur}(\texttt{app} \circ (\texttt{Id}_A \times w))$$

. And hence, we have:

$$\texttt{cur}(f \circ (\texttt{Id} \times \omega)) = (A)^\omega \circ \texttt{cur}(f)$$

$$\begin{aligned} \Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta'_2 \circ \pi_2), \texttt{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} \\ &= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \omega \circ \pi_2), \texttt{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta'_1) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} \\ &= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_1 \times \omega)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_1 \times \omega))] \circ \Delta_1 \circ \omega) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} \\ &= \texttt{app} \circ (([(A)^\omega \circ \texttt{cur}(\Delta_2 \circ \pi_2), (A)^\omega \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (A)^\omega \text{ property} \\ &= \texttt{app} \circ (((A)^\omega \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} \\ &= \texttt{app} \circ ((A)^\omega \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\omega \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} \\ &= \texttt{app} \circ (\texttt{Id}_{(A)} \times \omega) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\omega \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (A)^\omega \\ &= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} \\ &= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \omega \quad \text{By Definition of the diagonal morphism.} \\ &= \Delta \circ \omega \end{aligned}$$

Case Bind: By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1\colon \texttt{M}_{\mathbb{E}_1} A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash v_2\colon \texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.23}$$

If $\omega\colon \Gamma' \triangleright \Gamma$ then $\omega \times\colon \Gamma', x : A \triangleright \Gamma, x\colon A$, so by induction, we can derive Δ'_1, Δ'_2 such that:

$$\Delta' = (\text{Bind})\dfrac{\dfrac{\Delta_1'}{\Gamma' \vdash v_1 : \mathtt{M}_{\mathbb{E}_1} A} \qquad \dfrac{\Delta_2'}{\Gamma', x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.24}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} \Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{G'}, \Delta_1' \rangle \quad \text{By definition} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} (\Delta_2 \circ (\omega \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta_1', \Delta_2' \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} \\
&= \Delta \quad \text{By definition}
\end{aligned}$$

Chapter 6

Substitution

6.1 Introduce Substitutions

6.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{6.1}$$

6.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset$$
$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v)$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6.2}$$

6.1.3 Action of Substitutions

We define the action of applying a substitution σ as

$$v[\sigma]$$

$$
\begin{aligned}
x[\diamond] &= x \\
x[\sigma, x{:}= v] &= v \\
x[\sigma, x'{:}= v'] &= x[\sigma] \quad \text{If } x \neq x' \\
\mathtt{k}^{A}[\sigma] &= \mathtt{k}^{A} \\
(\lambda x{:}\, A.v\ )[\sigma] &= \lambda x{:}\, A.(v[\sigma]) \quad \text{If } x\#\sigma \\
(\mathtt{if}_{A}\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ )[\sigma] &= \mathtt{if}_{A}\ v[\sigma]\ \mathtt{then}\ v_1[\sigma]\ \mathtt{else}\ v_2[\sigma] \\
(v_1\ v_2)[\sigma] &= (v_1[\sigma])\ v_2[\sigma] \\
(\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ ) &= \mathtt{do}\ x \leftarrow (v_1[\sigma])\ \mathtt{in}\ (v_2[\sigma]) \quad \text{If } x\#\sigma
\end{aligned}
$$

6.1.4 Wellformedness

Define the relation

$$\Gamma' \vdash \sigma{:}\,\Gamma$$

by:

- (Nil) $\dfrac{\Gamma'\ \mathtt{Ok}}{\Gamma' \vdash \diamond{:}\,\diamond}$
- (T-Extend) $\dfrac{\Gamma' \vdash \sigma{:}\,\Gamma \qquad x \notin \mathtt{dom}(\Gamma) \qquad \Gamma' \vdash v{:}\, A}{\Gamma' \vdash (\sigma, x{:}= v){:}\,(\Gamma, x{:}\, A)}$

6.1.5 Simple Properties Of Substitution

Property 6.1.1 (Ok Relation)**.** *If $\Gamma' \vdash \sigma{:}\,\Gamma$ then Γ $\mathtt{Ok}$ and Γ' $\mathtt{Ok}$. Since Γ' $\mathtt{Ok}$ holds by the Nil-axiom. Γ $\mathtt{Ok}$ holds by induction on the wellformedness relation.*

Property 6.1.2 (Weakening)**.** *If $\Gamma' \vdash \sigma{:}\,\Gamma$ then $\omega{:}\,\Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma{:}\,\Gamma$. This holds by induction over wellformedness relation. For each $x{:}= v$ in σ, $\Gamma'' \vdash v{:}\, A$ holds if $\Gamma' \vdash v{:}\, A$ holds.*

Property 6.1.3 (Extension)**.** *If $\Gamma' \vdash \sigma{:}\,\Gamma$ then $x \notin (\mathit{dom}(\Gamma) \cup \mathit{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x{:}= x){:}\,(\Gamma, x{:}\, A)$. This occurs since $\iota\pi{:}\,\Gamma', x : A \triangleright \Gamma'$, so by property 6.1.2, $\Gamma', x : A \vdash \sigma{:}\,\Gamma$. In addition, $\Gamma', x : A \vdash x{:}\, A$ trivially, so by the rule T-Extend, wellformedness holds for*

$$(\Gamma', x : A) \vdash (\sigma, x{:}= x){:}\,(\Gamma, x{:}\, A) \tag{6.3}$$

6.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

Theorem 6.2.1 (Substitution Preserves Typing Relation)**.** *If $\Gamma \vdash g{:}\, A$ and $\Gamma' \vdash \sigma{:}\,\Gamma$ then $\Gamma' \vdash v[\sigma]{:}\, A$.*

Proof: Assuming $\Gamma' \vdash \sigma{:}\,\Gamma$, we induct over the typing relation, proving $\Gamma \vdash v{:}\, A \rightarrow \Gamma' \vdash v{:}\, A$

Case Var: By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x\colon A \tag{6.4}$$

So by inversion, since $\Gamma' \vdash \sigma\colon \Gamma'', x : A$,

$$\sigma = \sigma', x\colon= v \wedge \Gamma' \vdash v\colon A \tag{6.5}$$

By the definition of the action of substitutions, $x[\sigma] = v$, So

$$\Gamma' \vdash x[\sigma]\colon A \tag{6.6}$$

holds.

Case Weaken: By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{\dfrac{\Delta}{\Gamma'' \vdash x\colon A}}{\Gamma'', y : B \vdash x\colon A} \tag{6.7}$$

By inversion, $\sigma = \sigma', y\colon= v$ and:

$$\Gamma' \vdash \sigma'\colon \Gamma'' \tag{6.8}$$

So by induction,

$$\Gamma' \vdash x[\sigma']\colon A \tag{6.9}$$

And so by definition of the action of σ, $x[\sigma] = x[\sigma']$

$$\Gamma' \vdash x[\sigma]\colon A \tag{6.10}$$

Case Fn: By inversion, there exists Δ such that:

$$(\text{Fn})\frac{\dfrac{\Delta}{\Gamma, x : A \vdash v\colon B}}{\Gamma \vdash \lambda x\colon A.v : A \to B} \tag{6.11}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 6.1.3, we have

$$(\Gamma', x : A) \vdash (\sigma, x\colon= x)\colon \Gamma, x\colon A \tag{6.12}$$

So by induction using $\sigma, x\colon= x$, we have Δ' such that:

$$(\text{Fn})\frac{\dfrac{\Delta'}{\Gamma', x : A \vdash v[\sigma, x\colon= v]\colon B}}{\Gamma \vdash \lambda x\colon A.v[\sigma, x\colon= x] : A \to B} \tag{6.13}$$

Since $\lambda x\colon A.(v[\sigma, x\colon= x]) = \lambda x\colon A.(v[\sigma]) = (\lambda x\colon A.v\ )[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x\colon A.v\ )[\sigma]\colon A \to B$.

Case Const: We use the same logic for all constants, $\texttt{()}, \texttt{true}, \texttt{false}, \texttt{k}^{A}$:

$\Gamma \vdash \sigma : \Gamma \Rightarrow \Gamma'$ $\texttt{Ok}$ and:

$$\texttt{k}^{A}[\sigma] = \texttt{k}^{A} \tag{6.14}$$

So

$$(\text{Const})\frac{\Gamma' \ \texttt{Ok}}{\Gamma' \vdash \texttt{k}^{A} : A} \tag{6.15}$$

Case Return: By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{\dfrac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}}\, A} \tag{6.16}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v[\sigma] : A}}{\Gamma' \vdash \texttt{return}\ (v[\sigma]) : \texttt{M}_{\texttt{1}}\, A} \tag{6.17}$$

Since $(\texttt{return}\ v\ )[\sigma] = \texttt{return}\ (v[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\texttt{return}\ v\ )[\sigma] : \texttt{M}_{\texttt{1}}\, A$.

Case Apply: By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1 : A \to B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 : B} \tag{6.18}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v_1[\sigma] : A \to B} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash v_2[\sigma] : A}}{\Gamma' \vdash (v_1[\sigma])\ (v_2[\sigma]) : B} \tag{6.19}$$

Since $(v_1\ v_2)[\sigma] = (v_1[\sigma])\ (v_2[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1\ v_2)[\sigma] : B$

Case If: By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$(\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v : \texttt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_1 : A} \qquad \dfrac{\Delta_3}{\Gamma \vdash v_2 : A}}{\Gamma \vdash \texttt{if}_{A}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{6.20}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$\text{(If)}\,\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v[\sigma] \colon \texttt{Bool}} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash v_1[\sigma] \colon A} \qquad \dfrac{\Delta_3'}{\Gamma' \vdash v_2[\sigma] \colon A}}{\Gamma' \vdash \texttt{if}_A\ (v[\sigma])\ \texttt{then}\ (v_1[\sigma])\ \texttt{else}\ (v_2[\sigma]) : A} \tag{6.21}$$

Since $(\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\sigma] = \texttt{if}_A\ (v[\sigma])\ \texttt{then}\ (v_1[\sigma])\ \texttt{else}\ (v_2[\sigma])$ The derivation above holds for $\Gamma' \vdash (\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\sigma] \colon A$

Case Bind: By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)}\,\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.22}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 6.1.3,

$$(\Gamma, x : A) \vdash (\sigma, x \colon= x) \colon (\Gamma, x \colon A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\,\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v_1[\sigma] \colon \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma', x : A \vdash v_2[\sigma, x \colon= x] \colon \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \texttt{do}\ x \leftarrow (v_1[\sigma])\ \texttt{in}\ (v_2[\sigma, x \colon= x]) : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.23}$$

Since $(\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2\ )[\sigma] = \texttt{do}\ x \leftarrow (v_1[\sigma])\ \texttt{in}\ (v_2[\sigma])\ = \texttt{do}\ x \leftarrow (v_1[\sigma])\ \texttt{in}\ (v_2[\sigma, x \colon= x])$, the above derivation holds for $\Gamma' \vdash (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2\ )[\sigma] \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B$

Case Subtype: By inversion, there exists Δ such that

$$\text{(Subtype)}\,\frac{\dfrac{\Delta}{\Gamma \vdash v \colon A} \qquad A \leq: B}{\Gamma \vdash v \colon B} \tag{6.24}$$

By induction on Δ we derive Δ' such that:

$$\text{(Subtype)}\,\frac{\dfrac{\Delta'}{\Gamma' \vdash v[\sigma] \colon A} \qquad A \leq: B}{\Gamma \vdash v[\sigma] \colon B} \tag{6.25}$$

6.2.1 Extension Lemma

Lemma 6.2.2 (Extension Denotation). *If $\Gamma' \vdash \sigma \colon \Gamma$ and $x \notin (\mathit{dom}(\Gamma') \cup \mathit{dom}(\Gamma))$ then the substitution in property 6.1.3 has denotation:*

$$[\![(\Gamma', x : A) \vdash (\sigma, x \colon= x) \colon (\Gamma, x \colon A)]\!] = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!] \times \mathit{Id}_A) \tag{6.26}$$

Proof: This holds since

$$[\![\Gamma', x : A \vdash x \colon A]\!] = \pi_2 \tag{6.27}$$

And $\iota\pi \colon (\Gamma', x : A) \triangleright \Gamma'$

$$[\![\iota\pi \colon (\Gamma', x : A) \triangleright \Gamma']\!] = \pi_1 \tag{6.28}$$

So for each denotation $[\![\Gamma' \vdash v \colon B]\!]$ of each $y \colon = v$ in σ, we can pre-pend the denotation with the weakening denotation to yield:

$$[\![\Gamma', x : A \vdash v \colon B]\!] = [\![\Gamma' \vdash v \colon B]\!] \circ \pi_1 \tag{6.29}$$

Since π_1 appears in every branch of $[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!]$, it can be factored out to yield:

$$[\![\Gamma', x : A \vdash \sigma \colon \Gamma]\!] = [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \circ \pi_1 \tag{6.30}$$

Hence,

$$[\![(\Gamma', x : A) \vdash (\sigma, x \colon = x) \colon \Gamma, x \colon A]\!] = \langle [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \circ \pi_1, \pi_2 \rangle = ([\![\Gamma' \vdash \sigma \colon \Gamma]\!] \times \mathtt{Id}_A) \tag{6.31}$$

6.2.2 Substitution Theorem

Theorem 6.2.3 (Substitutions and Denotations)**.** *If* Δ *derives* $\Gamma \vdash v \colon A$ *and* $\Gamma' \vdash \sigma \colon \Gamma$ *then the derivation* Δ' *deriving* $\Gamma' \vdash v[\sigma] \colon A$ *satisfies:*

$$\Delta' = \Delta \circ [\![\Gamma' \vdash \sigma \colon \Gamma]\!] \tag{6.32}$$

$$\begin{tikzcd}
\Gamma' \arrow[r, "{[\![\Gamma' \vdash \sigma : \Gamma]\!]}"] \arrow[dr, "{[\![\Gamma' \vdash v[\sigma] : A]\!]}"'] & \Gamma \arrow[d, "{[\![\Gamma \vdash v : A]\!]}"] \\
& [\![T]\!]
\end{tikzcd}$$

Proof: This is proved by induction over the derivation of $\Gamma \vdash v \colon A$. We shall use σ to denote $[\![\Gamma' \vdash \sigma \colon \Gamma]\!]$ where it is clear from the context.

Case Var: By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var})\frac{\Gamma \;\; \mathtt{Ok}}{\Gamma'', x : A \vdash x \colon A} \tag{6.33}$$

By inversion, $\sigma = \sigma', x \colon = v$ and $\Gamma' \vdash v \colon A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma \colon \Gamma]\!] = \langle \sigma', \Delta' \rangle$$
$$\Delta = [\![\Gamma'', x : A \vdash x \colon A]\!] = \pi_2$$

$$\begin{aligned}\Delta \circ \sigma &= \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \\ &= \Delta' \quad \text{By product property}\end{aligned}$$

Case Weaken: By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y \colon= v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{\dfrac{\Delta_1}{\Gamma'' \vdash x \colon A}}{\Gamma'', y : B \vdash x \colon A} \tag{6.34}$$

Also by inversion of the wellformedness of $\Gamma' \vdash \sigma \colon \Gamma$, we have $\Gamma' \vdash \sigma' \colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma \colon \Gamma]\!] = \langle [\![\Gamma' \vdash \sigma \colon \Gamma'']\!], [\![\Gamma' \vdash v \colon B]\!] \rangle \tag{6.35}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$\frac{\Delta_1'}{\Gamma' \vdash x[\sigma] \colon A} \tag{6.36}$$

Hence

$$\begin{aligned}\Delta' &= \Delta_1' \quad \text{By definition} \\ &= \Delta_1 \circ \sigma' \quad \text{By induction} \\ &= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v \colon B]\!] \rangle \quad \text{By product property} \\ &= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.}\end{aligned}$$

Case Const: The logic for all constant terms ($\mathtt{true}, \mathtt{false}, \mathtt{()}\mathtt{k}^A$) is the same. Let

$$c = [\![\mathtt{k}^A]\!] \tag{6.37}$$

$$\begin{aligned}\Delta' &= c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} \\ &= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} \\ &= \Delta \circ \sigma \quad \text{By definition}\end{aligned}$$

Case Fn: By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{\dfrac{\Delta_1}{\Gamma, x : A \vdash v \colon B}}{\Gamma \vdash \lambda x \colon A.v : A \to B} \tag{6.38}$$

By induction of Δ_1 we have Δ'_1 such that

$$\Delta' = (\text{Fn})\frac{\dfrac{\Delta'_1}{\Gamma', x : A \vdash (v[\sigma])\colon B}}{\Gamma \vdash (\lambda x\colon A.v\ )[\sigma]\colon A \to B} \tag{6.39}$$

By induction and the extension lemma, we have:

$$\Delta'_1 = \Delta_1 \circ (\sigma \times \mathtt{Id}_A) \tag{6.40}$$

Hence:

$$\begin{aligned}\Delta' &= \mathtt{cur}(\Delta'_1) \quad \text{By definition}\\ &= \mathtt{cur}(\Delta_1 \circ (\sigma \times \mathtt{Id}_A)) \quad \text{By induction and extension lemma.}\\ &= \mathtt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)}\\ &= \Delta \circ \sigma \quad \text{By Definition}\end{aligned}$$

Case Subtype: By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Subtype})\frac{\dfrac{\Delta_1}{\Gamma \vdash v\colon A} \qquad A{\leq:}B}{\Gamma \vdash v\colon B} \tag{6.41}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Subtype})\frac{\dfrac{\Delta'_1}{\Gamma' \vdash v[\sigma]\colon A} \qquad A{\leq:}B}{\Gamma' \vdash v[\sigma]\colon B} \tag{6.42}$$

Hence,

$$\begin{aligned}\Delta' &= [\![A{\leq:}B]\!] \circ \Delta'_1 \quad \text{By definition}\\ &= [\![A{\leq:}B]\!] \circ \Delta_1 \circ \sigma \quad \text{By induction}\\ &= \Delta \circ \sigma \quad \text{By definition}\end{aligned}$$

Case Return: By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{\dfrac{\Delta_1}{\Gamma \vdash v\colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_1 A} \tag{6.43}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\dfrac{\dfrac{\Delta'_1}{\Gamma' \vdash v[\sigma] \colon A}}{\Gamma' \vdash (\texttt{return}\ v\ )[\sigma] \colon \texttt{M}_1 A} \tag{6.44}$$

Hence,

$$\begin{aligned} \Delta' &= \eta_A \circ \Delta'_1 \quad \text{By Definition} \\ &= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} \\ &= \Delta \circ \sigma \quad \text{By Definition} \end{aligned}$$

Case Apply: By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\dfrac{\dfrac{\Delta_1}{\Gamma \vdash v_1 \colon A \to B} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_2 \colon A}}{\Gamma \vdash v_1\ v_2 \colon B} \tag{6.45}$$

By induction we find Δ'_1, Δ'_2 such that

$$\begin{aligned} \Delta'_1 &= \Delta_1 \circ \sigma \\ \Delta'_2 &= \Delta_2 \circ \sigma \end{aligned}$$

And

$$\Delta' = (\text{Apply})\dfrac{\dfrac{\Delta'_1}{\Gamma' \vdash v_1[\sigma] \colon A \to B} \qquad \dfrac{\Delta'_2}{\Gamma' \vdash v_2[\sigma] \colon A}}{\Gamma' \vdash (v_1\ v_2)[\sigma] \colon B} \tag{6.46}$$

Hence

$$\begin{aligned} \Delta' &= \texttt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \\ &= \texttt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} \\ &= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} \\ &= \Delta \circ \sigma \quad \text{By Definition} \end{aligned}$$

Case If: By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If})\dfrac{\dfrac{\Delta_1}{\Gamma \vdash v \colon \texttt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash v_1 \colon A} \qquad \dfrac{\Delta_3}{\Gamma \vdash v_2 \colon A}}{\Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{6.47}$$

By induction we find $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\begin{aligned} \Delta'_1 &= \Delta_1 \circ \sigma \\ \Delta'_2 &= \Delta_2 \circ \sigma \\ \Delta'_3 &= \Delta_3 \circ \sigma \end{aligned}$$

And

$$\Delta' = (\text{If})\frac{\dfrac{\Delta_1'}{\Gamma' \vdash v[\sigma]\colon \texttt{Bool}} \qquad \dfrac{\Delta_2'}{\Gamma' \vdash v_1[\sigma]\colon A} \qquad \dfrac{\Delta_3'}{\Gamma' \vdash v_2[\sigma]\colon A}}{\Gamma' \vdash (\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\sigma]\colon A} \tag{6.48}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(A)^\sigma : A^\Gamma \to A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(A)^\sigma = \texttt{cur}(\texttt{app} \circ (\texttt{Id}_A \times w))$$

. And hence, we have:

$$\texttt{cur}(f \circ (\texttt{Id} \times \sigma)) = (A)^\sigma \circ \texttt{cur}(f)$$

And so:

$$\begin{aligned}
\Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta_2' \circ \pi_2), \texttt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition}\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \texttt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction}\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma))] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property}\\
&= \texttt{app} \circ (([(A)^\sigma \circ \texttt{cur}(\Delta_2 \circ \pi_2), (A)^\sigma \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (A)^\sigma \text{ property}\\
&= \texttt{app} \circ (((A)^\sigma \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation}\\
&= \texttt{app} \circ ((A)^\sigma \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs}\\
&= \texttt{app} \circ (\texttt{Id}_{(A)} \times \sigma) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (A)^\sigma\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs}\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \sigma \quad \text{By Definition of the diagonal morphism.}\\
&= \Delta \circ \sigma
\end{aligned}$$

Case Bind: By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Gamma, x : A \vdash v_2\colon \texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.49}$$

By property 6.1.3,

$$(\Gamma', x : A) \vdash (\sigma, x\colon = x\colon (\Gamma, x\colon A) \tag{6.50}$$

With denotation (extension lemma)

$$[\![(\Gamma', x : A) \vdash (\sigma, x\colon = x\colon (\Gamma, x\colon A)]\!] = \sigma \times \texttt{Id}_A \tag{6.51}$$

By induction, we derive Δ_1', Δ_2' such that:

$$\begin{aligned}
\Delta_1' &= \Delta_1 \circ \sigma\\
\Delta_2' &= \Delta_2 \circ (\sigma \times \texttt{Id}_A) \quad \text{By Extension Lemma}
\end{aligned}$$

And:

$$\Delta' = (\text{Bind})\frac{\dfrac{\Delta'_1}{\Gamma' \vdash v_1[\sigma]\colon \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta'_2}{\Gamma', x : A \vdash v_1[\sigma]\colon \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )[\sigma]\colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.52}$$

Hence:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta'_2 \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1\rangle \quad \text{By Definition} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma\rangle \quad \text{By Induction using the extension lemma} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma\rangle \quad \text{By Tensor Strength} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma\rangle \quad \text{By Product rule} \\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1\rangle \circ \sigma \quad \text{By Product rule} \\
&= \Delta \circ \sigma \quad \text{By Defintion}
\end{aligned}$$

6.3 The Identity Substitution

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma,x:A} = (I_\Gamma, x\colon= x)$

6.3.1 Properties of the Identity Substitution

Property 6.3.1 (Wellformedness)**.** *If Γ Ok then $\Gamma \vdash I_\Gamma\colon \Gamma$, proved trivially by induction over the wellformedness relation.*

Property 6.3.2 (Denotation)**.** $[\![\Gamma \vdash I_\Gamma\colon \Gamma]\!] = \mathit{Id}_\Gamma$*, proved trivially by induction over the definition of* I_Γ

6.4 Single Substitution

If $\Gamma \vdash v\colon A$, let the single substitution $\Gamma \vdash [v/x]\colon \Gamma, x\colon A$, be defined as:

$$[v/x] = (I_\Gamma, x\colon= v) \tag{6.53}$$

Then by properties 6.3.1, 6.3.2 of the identity substitution, we have:

$$[\![\Gamma \vdash [v/x]\colon \Gamma, x\colon A]\!] = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v\colon A]\!]\rangle : \Gamma \to (\Gamma \times A) \tag{6.54}$$

6.4.1 The Semantics of Single Substitution

The following diagram commutes:

$$[\![\Gamma \vdash v_1[v/x] \colon A]\!] = [\![\Gamma, x \colon A \vdash v_1 \colon A]\!] \circ \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!] \rangle$$

$$\begin{tikzcd}
\Gamma \arrow[r, "{\langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v : A]\!] \rangle}"] \arrow[dr, "{[\![\Gamma \vdash v_1[v/x] : A]\!]}"] & \Gamma \times A \arrow[d, "{[\![\Gamma, x : A \vdash v_1 : A]\!]}"] \\
& A
\end{tikzcd}$$

Since $[\![\Gamma \vdash (I_\Gamma, x \colon= v) \colon (\Gamma, x \colon A)]\!] = \langle \mathtt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!] \rangle$ And $v_1[v/x] = v_1[I_\Gamma, x \colon= v]$

Chapter 7

Soundness

7.1 Equational Equivalence

7.1.1 Reduction Conversions

- (Eq-Lambda-Beta) $\dfrac{\Gamma, x\colon A \vdash v_1\colon B \qquad \Gamma \vdash v_2\colon A}{\Gamma \vdash (\lambda x\colon A.v_1\ )\ v_2 \approx v_1[v_2/x]\colon B}$

- (Eq-Lambda-Eta) $\dfrac{\Gamma \vdash v\colon A \to B}{\Gamma \vdash \lambda x\colon A.(v\ x\ ) \approx v\colon A \to B}$

- (Eq-Left-Unit) $\dfrac{\Gamma \vdash v_1\colon A \qquad \Gamma, x\colon A \vdash v_2\colon \mathtt{M}_\epsilon B}{\Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return}\ v_1\ \ \mathtt{in}\ v_2\ \approx v_2[v_1/x]\colon \mathtt{M}_\epsilon B}$

- (Eq-Right-Unit) $\dfrac{\Gamma \vdash v\colon \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return}\ x\ \ \approx v\colon \mathtt{M}_\epsilon A}$

- (Eq-Associativity) $\dfrac{\Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A \qquad \Gamma, x\colon A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B \qquad \Gamma, y\colon B \vdash v_3\colon \mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3\ )\ \approx \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )\ \mathtt{in}\ v_3 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$

- (Eq-Unit) $\dfrac{\Gamma \vdash v\colon \mathtt{Unit}}{\Gamma \vdash v \approx \mathtt{()}\colon \mathtt{Unit}}$

- (Eq-If-True) $\dfrac{\Gamma \vdash v_1\colon A \qquad \Gamma \vdash v_2\colon A}{\Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ \ \approx v_1\colon A}$

- (Eq-If-False) $\dfrac{\Gamma \vdash v_2\colon A \qquad \Gamma \vdash v_1\colon A}{\Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ \ \approx v_2\colon A}$

- (Eq-If-Eta) $\dfrac{\Gamma, x : \mathtt{Bool} \vdash v_1\colon A \qquad \Gamma \vdash v_2\colon \mathtt{Bool}}{\Gamma \vdash \mathtt{if}_A\ v_2\ \mathtt{then}\ v_1[\mathtt{true}/x]\ \mathtt{else}\ v_1[\mathtt{false}/x]\ \ \approx v_1[v_2/x]\colon A}$

7.1.2 Equivalence Relation

- (Eq-Reflexive) $\dfrac{\Gamma \vdash v\colon A}{\Gamma \vdash v \approx v\colon A}$

- (Eq-Symmetric) $\dfrac{\Gamma \vdash v_1 \approx v_2 \colon A}{\Gamma \vdash v_2 \approx v_1 \colon A}$

- (Eq-Transitive) $\dfrac{\Gamma \vdash v_1 \approx v_2 \colon A \qquad \Gamma \vdash v_2 \approx v_3 \colon A}{\Gamma \vdash v_1 \approx v_3 \colon A}$

7.1.3 Congruences

- (Eq-Fn) $\dfrac{\Gamma, x \colon A \vdash v_1 \approx v_2 \colon B}{\Gamma \vdash \lambda x \colon A.v_1 \ \approx \lambda x \colon A.v_2 : A \to B}$

- (Eq-Return) $\dfrac{\Gamma \vdash v_1 \approx v_2 \colon A}{\Gamma \vdash \texttt{return}\ v_1 \ \approx \texttt{return}\ v_2 : \texttt{M}_1 A}$

- (Eq-Apply) $\dfrac{\Gamma \vdash v_1 \approx v_1' \colon A \to B \qquad \Gamma \vdash v_2 \approx v_2' \colon A}{\Gamma \vdash v_1\ v_2 \approx v_1'\ v_2' \colon B}$

- (Eq-Bind) $\dfrac{\Gamma \vdash v_1 \approx v_1' \colon \texttt{M}_{\epsilon_1} A \qquad \Gamma, x \colon A \vdash v_2 \approx v_2' \colon \texttt{M}_{\epsilon_2} B}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \ \approx \texttt{do}\ c \leftarrow v_1'\ \texttt{in}\ v_2' : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

- (Eq-If) $\dfrac{\Gamma \vdash v \approx v' \colon \texttt{Bool} \qquad \Gamma \vdash v_1 \approx v_1' \colon A \qquad \Gamma \vdash v_2 \approx v_2' \colon A}{\Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \ \approx \texttt{if}_A\ v\ \texttt{then}\ v_1'\ \texttt{else}\ v_2' : A}$

- (Eq-Subtype) $\dfrac{\Gamma \vdash v \approx v' \colon A \qquad A \leq: B}{\Gamma \vdash v \approx v' \colon B}$

7.2 Equational Equivalence Implies Both Sides Have the Same Type

Theorem 7.2.1 (Equational Equivalence Implies Both Sides Have the Same Type)**.** *Each derivation of $\Gamma \vdash v \approx v' \colon A$ can be converted to a derivation of $\Gamma \vdash v \colon A$ and $\Gamma \vdash v' \colon A$ by induction over the beta-eta equivalence relation derivation.*

Proof:

7.2.1 Equivalence Relations

Case Eq-Reflexive: By inversion we have a derivation of $\Gamma \vdash v \colon A$.

Case Eq-Symmetric: By inversion $\Gamma \vdash v' \approx v \colon A$. Hence by induction, derivations of $\Gamma \vdash v' \colon A$ and $\Gamma \vdash v \colon A$ are given.

Case Eq-Transitive: By inversion, there exists v_2 such that $\Gamma \vdash v_1 \approx v_2 \colon A$ and $\Gamma \vdash v_2 \approx v_3 \colon A$. Hence by induction, we have derivations of $\Gamma \vdash v_1 \colon A$ and $\Gamma \vdash v_3 \colon A$

7.2.2 Reduction Conversions

Case Fn: By inversion, we have $\Gamma, x{:}\, A \vdash v{:}\, B$ and $\Gamma \vdash v{:}\, A$. Hence by the typing rules, we have:

$$\text{(Apply)}\dfrac{\text{(Fn)}\dfrac{\Gamma, x{:}\, A \vdash v_1{:}\, B}{\Gamma \vdash \lambda x{:}\, A.v_1 : A \to B} \qquad \Gamma \vdash v_2{:}\, A}{\Gamma \vdash (\lambda x{:}\, A.v_1\ )\ v_2{:}\, B}$$

By the substitution typing theorem 6.2.1, we have

$$\text{(Substitution)}\dfrac{\Gamma, x{:}\, A \vdash v_1{:}\, B \qquad \Gamma \vdash v_2{:}\, A}{\Gamma \vdash v_1[v_2/x]{:}\, B}$$

Case Eq-Left-Unit: By inversion, we have $\Gamma \vdash v_1{:}\, A$ and $\Gamma, x{:}\, A \vdash v_2{:}\, B$

Hence we have:

$$\text{(Bind)}\dfrac{\text{(Return)}\dfrac{\Gamma \vdash v_1{:}\, A}{\Gamma \vdash \texttt{return}\ v_1 : \texttt{M}_{\mathbf{1}} A} \qquad \Gamma, x{:}\, A \vdash v_2{:}\, \texttt{M}_\epsilon B}{\Gamma \vdash \texttt{do}\ x \leftarrow \texttt{return}\ v_1\ \ \texttt{in}\ v_2 : \texttt{M}_{\mathbf{1}\cdot\epsilon} B = \texttt{M}_\epsilon B} \tag{7.1}$$

And by the substitution typing theorem 6.2.1 we have:

$$\Gamma \vdash v_2[v_1/x]{:}\, \texttt{M}_\epsilon B \tag{7.2}$$

Case Eq-Right-Unit: By inversion, we have $\Gamma \vdash v{:}\, \texttt{M}_\epsilon A$.

Hence we have:

$$\text{(Bind)}\dfrac{\Gamma \vdash v{:}\, \texttt{M}_\epsilon A \qquad \text{(Return)}\dfrac{\text{(Var)}\dfrac{\Gamma\ \ \texttt{Ok}}{\Gamma, x{:}\, A \vdash x{:}\, A}}{\Gamma, x{:}\, A \vdash \texttt{return}\ v : \texttt{M}_{\mathbf{1}} A}}{\Gamma \vdash \texttt{do}\ x \leftarrow v\ \texttt{in return}\ x\ \ : \texttt{M}_{\epsilon\cdot\mathbf{1}} A = \texttt{M}_\epsilon A} \tag{7.3}$$

Case Eq-Associativity: By inversion, we have $\Gamma \vdash v_1{:}\, \texttt{M}_{\epsilon_1} A$, $\Gamma, x{:}\, A \vdash v_2{:}\, \texttt{M}_{\epsilon_2} B$, and $\Gamma, y{:}\, B \vdash v_3{:}\, \texttt{M}_{\epsilon_3} C$.

$$(\iota\pi\times){:}\, (\Gamma, x{:}\, A, y : B) \triangleright (\Gamma, y{:}\, B)$$

So by the weakening property 5.2.3, $\Gamma, x{:}\, A, y : B \vdash v_3{:}\, \texttt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\dfrac{\Gamma \vdash v_1{:}\, \texttt{M}_{\epsilon_1} A \qquad \text{(Bind)}\dfrac{\Gamma, x{:}\, A \vdash v_2{:}\, \texttt{M}_{\epsilon_2} B \qquad \Gamma, x{:}\, A, y : B \vdash v_3{:}\, \texttt{M}_{\epsilon_3} C}{\Gamma, x{:}\, A \vdash x v_2 v_3{:}\, \texttt{M}_{\epsilon_2\cdot\epsilon_3} C}}{\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ (\texttt{do}\ y \leftarrow v_2\ \texttt{in}\ v_3\ ) : \texttt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C} \tag{7.4}$$

and

$$
(\text{Bind})\,\cfrac{(\text{Bind})\,\cfrac{\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \qquad \Gamma, x\colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \qquad \Gamma, y\colon B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )\ \mathtt{in}\ v_3 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.5}
$$

Case Eq-Lambda-Eta: By inversion, we have $\Gamma \vdash v \colon A \to B$

By weakening, we have $\iota\pi \colon (\Gamma, x\colon A) \triangleright \Gamma$ Hence, we have

$$
(\text{Fn})\,\cfrac{(\text{Apply})\,\cfrac{(\Gamma, x\colon A) \vdash x \colon A \qquad (\text{Weaken})\,\cfrac{\Gamma \vdash v \colon A \to B \qquad \iota\pi \colon \Gamma, x\colon A \triangleright \Gamma}{\Gamma, x\colon A \vdash v \colon A \to B}}{\Gamma, x\colon A \vdash v\ x \colon B}}{\Gamma \vdash \lambda x \colon A.(v\ x) : A \to B} \tag{7.6}
$$

Case Eq-If-True: By inversion, we have $\Gamma \vdash v_1 \colon A$, $\Gamma \vdash v_2 \colon A$. Hence by the typing ok lemma 1.2.1, we have $\Gamma\ \mathtt{Ok}$ so $\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$
(\text{If})\,\cfrac{\Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \qquad \Gamma \vdash v_1 \colon A \qquad \Gamma \vdash v_2 \colon A}{\Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{7.7}
$$

Case Eq-If-False: As above,

Hence

$$
(\text{If})\,\cfrac{\Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \qquad \Gamma \vdash v_1 \colon A \qquad \Gamma \vdash v_2 \colon A}{\Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{7.8}
$$

Case Eq-If-Eta: By inversion, we have:

$$
\Gamma \vdash v_2 \colon \mathtt{Bool} \tag{7.9}
$$

and

$$
\Gamma, x : \mathtt{Bool} \vdash v_1 \colon A \tag{7.10}
$$

Hence we also have $\Gamma\ \mathtt{Ok}$. Hence, the following also hold:

$\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$, and $\Gamma \vdash \mathtt{false} \colon \mathtt{Bool}$.

Hence by the substitution theorem, we have:

$$
(\text{If})\,\cfrac{\Gamma \vdash v_2 \colon \mathtt{Bool} \qquad \Gamma \vdash v_1[\mathtt{true}/x] \colon A \qquad \Gamma \vdash v_1[\mathtt{false}/x] \colon A}{\Gamma \vdash \mathtt{if}_A\ v_2\ \mathtt{then}\ v_1[\mathtt{true}/x]\ \mathtt{else}\ v_1[\mathtt{false}/x] : A} \tag{7.11}
$$

and

$$
\Gamma \vdash v_1[v_2/x] \colon A \tag{7.12}
$$

7.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Eq-Fn: By inversion, $\Gamma, x\colon A \vdash v_1 \approx v_2\colon B$. Hence by induction $\Gamma, x\colon A \vdash v_1\colon B$, and $\Gamma, x\colon A \vdash v_2\colon B$.

So

$$\Gamma \vdash \lambda x\colon A.v_1 : A \to B \tag{7.13}$$

and

$$\Gamma \vdash \lambda x\colon A.v_2 : A \to B \tag{7.14}$$

Hold.

Case Eq-Return: By inversion, $\Gamma \vdash v_1 \approx v_2\colon A$, so by induction

$$\Gamma \vdash v_1\colon A$$

and

$$\Gamma \vdash v_2\colon A$$

Hence we have

$$\Gamma \vdash \texttt{return}\ v_1 : \texttt{M}_{\texttt{1}} A$$

and

$$\Gamma \vdash \texttt{return}\ v_2 : \texttt{M}_{\texttt{1}} A$$

Case Eq-Apply: By inversion, we have $\Gamma \vdash v_1 \approx v_1'\colon A \to B$ and $\Gamma \vdash v_2 \approx v_2'\colon A$. Hence we have by induction $\Gamma \vdash v_1\colon A \to B$, $\Gamma \vdash v_2\colon A$, $\Gamma \vdash v_1'\colon A \to B$, and $\Gamma \vdash v_2'\colon A$.

So we have:

$$\Gamma \vdash v_1\ v_2\colon B \tag{7.15}$$

and

$$\Gamma \vdash v_1'\ v_2'\colon B \tag{7.16}$$

Case Eq-Bind: By inversion, we have: $\Gamma \vdash v_1 \approx v_1'\colon \texttt{M}_{\epsilon_1} A$ and $\Gamma, x\colon A \vdash v_2 \approx v_2'\colon \texttt{M}_{\epsilon_2} B$. Hence by induction, we have $\Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1} A$, $\Gamma \vdash v_1'\colon \texttt{M}_{\epsilon_1} A$, $\Gamma, x\colon A \vdash v_2\colon \texttt{M}_{\epsilon_2} B$, and $\Gamma, x\colon A \vdash v_2'\colon \texttt{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.17}$$

$$\Gamma \vdash \texttt{do}\ x \leftarrow v_1'\ \texttt{in}\ v_2' : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.18}$$

Case Eq-If: By inversion, we have: $\Gamma \vdash v \approx v'\colon \texttt{Bool}$, $\Gamma \vdash v_1 \approx v_1'\colon A$, and $\Gamma \vdash v_2 \approx v_2'\colon A$.

Hence by induction, we have:

$\Gamma \vdash v\colon \texttt{Bool}$, $\Gamma \vdash v'\colon \texttt{Bool}$,

$\Gamma \vdash v_1\colon A$, $\Gamma \vdash v_1'\colon A$,

$\Gamma \vdash v_2\colon A$, and $\Gamma \vdash v_2'\colon A$.

So

$$\Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A \tag{7.19}$$

and

$$\Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1'\ \texttt{else}\ v_2' : A \tag{7.20}$$

Hold.

Case Eq-Subtype: By inversion, we have $A \leq: B$ and $\Gamma \vdash v \approx v' \colon A$. By induction, we therefore have $\Gamma \vdash v \colon A$ and $\Gamma \vdash v' \colon A$.

Hence we have

$$\Gamma \vdash v \colon B \tag{7.21}$$

$$\Gamma \vdash v' \colon B \tag{7.22}$$

7.3 Equationally Equivalent Terms Have Equal Denotations

Theorem 7.3.1 (Equationally Equivalent Terms Have Equal Denotations)**.** *If* $\Gamma \vdash v \approx v' \colon A$ *then* $[\![\Gamma \vdash v \colon A]\!] = [\![\Gamma \vdash v' \colon A]\!]$

Proof: By induction over Beta-eta equivalence relation.

7.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

Case Eq-Reflexive: Equality is reflexive, so if $\Gamma \vdash v \colon A$ then $[\![\Gamma \vdash v \colon A]\!]$ is equal to itself.

Case Eq-Symmetric: By inversion, if $\Gamma \vdash v \approx v' \colon A$ then $\Gamma \vdash v' \approx v \colon A$, so by induction $[\![\Gamma \vdash v' \colon A]\!] = [\![\Gamma \vdash v \colon A]\!]$ and hence $[\![\Gamma \vdash v \colon A]\!] = [\![\Gamma \vdash v' \colon A]\!]$

Case Eq-Transitive: There must exist v_2 such that $\Gamma \vdash v_1 \approx v_2 \colon A$ and $\Gamma \vdash v_2 \approx v_3 \colon A$, so by induction, $[\![\Gamma \vdash v_1 \colon A]\!] = [\![\Gamma \vdash v_2 \colon A]\!]$ and $[\![\Gamma \vdash v_2 \colon A]\!] = [\![\Gamma \vdash v_3 \colon A]\!]$. Hence by transitivity of equality, $[\![\Gamma \vdash v_1 \colon A]\!] = [\![\Gamma \vdash v_3 \colon A]\!]$

7.3.2 Reduction Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

Case Eq-Lambda-Beta: Let $f = [\![\Gamma, x\colon A \vdash v_1\colon B]\!] : (\Gamma \times A) \to B$

Let $g = [\![\Gamma \vdash v_2\colon A]\!] : \Gamma \to A$

By the substitution denotation,

$$[\![\Gamma \vdash [v_2/x]\colon \Gamma, x\colon A]\!] : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Gamma \vdash v_1[v_2/x]\colon B]\!] = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned} [\![\Gamma \vdash (\lambda x\colon A.v_1\ )\ v_2\colon B]\!] &= \mathtt{app} \circ \langle \mathtt{cur}(f), g \rangle \\ &= \mathtt{app} \circ (\mathtt{cur}(f) \times \mathtt{Id}_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\ &= f \circ \langle \mathtt{Id}_\Gamma, g \rangle \\ &= [\![\Gamma \vdash v_1[v_2/x]\colon B]\!] \end{aligned} \tag{7.23}$$

Case Eq-Left-Unit: Let $f = [\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_\epsilon B]\!]$

Let $g = [\![\Gamma \vdash v_1\colon A]\!] : \Gamma \to A$

By the substitution denotation,

$$[\![\Gamma \vdash [v_1/x]\colon \Gamma, x\colon A]\!] : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Gamma \vdash v_2[v_1/x]\colon \mathtt{M}_\epsilon B]\!] = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned} [\![\Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return}\ v_1\ \ \mathtt{in}\ v_2 : \mathtt{M}_\epsilon B]\!] =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \eta_A \circ g \rangle \\ =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times \eta_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\ =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \eta_{(\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\ =& \mu_{\mathbf{1},\epsilon,B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\ =& f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\ =& [\![\Gamma \vdash v_2[v_1/x]\colon \mathtt{M}_\epsilon B]\!] \end{aligned} \tag{7.24}$$

Case Eq-Right-Unit: Let $f = [\![\Gamma \vdash v\colon \mathtt{M}_\epsilon A]\!]$

$$\begin{aligned} [\![\Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return}\ x\ \ : \mathtt{M}_\epsilon A]\!] &= \mu_{\epsilon,\mathbf{1},A} \circ T_\epsilon(\eta_A \circ \pi_2) \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= \pi_2 \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= f \end{aligned} \tag{7.25}$$

Case Eq-Associativity: Let

$$\begin{aligned} f &= [\![\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!] \\ g &= [\![\Gamma, x \colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!] \\ h &= [\![\Gamma, y \colon B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C]\!] \end{aligned}$$

We also have the weakening:

$$\iota\pi\times \colon \Gamma, x \colon A, y : B \triangleright \Gamma, y \colon B \tag{7.26}$$

With denotation:

$$[\![\iota\pi\times \colon \Gamma, x \colon A, y : B \triangleright \Gamma, y \colon B]\!] = (\pi_1 \times \mathtt{Id}_B) \tag{7.27}$$

We need to prove that the following are equal

$$\begin{aligned} lhs &= [\![\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3\ ) : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C]\!] \\ &= \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \mathtt{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ rhs &= [\![\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )\ \mathtt{in}\ v_3 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2} C]\!] \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \end{aligned}$$

Let's look at fragment F of rhs.

$$F = \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \tag{7.28}$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \tag{7.29}$$

$$\begin{aligned} F &= \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathtt{t}_{\epsilon_2, \Gamma, B} \circ \mathtt{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\mathtt{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle && \text{By commutativity of bind and tensor-strength} \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle && \text{By naturality of t-strength} \end{aligned}$$

Since $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F$,

$$\begin{aligned} rhs =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\ =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle && \text{Naturality of } \mu \\ =& \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \end{aligned}$$

Let's now look at the fragment G of rhs

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \tag{7.30}$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B}) \circ G \tag{7.31}$$

By folding out the $\langle ..., ... \rangle$, we have

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ (\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1, \Gamma, A}) \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f \rangle \rangle \tag{7.32}$$

Using the commutativity of tensor strength with the associativity natural transformation α, we have:

$$
\begin{tikzcd}
\Gamma \arrow[r, "\langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle"] & \Gamma \times (\Gamma \times T_{\epsilon_1} A) \arrow[d, "\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1,\Gamma,A}"] & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \arrow[l, "\alpha_{\Gamma,\Gamma,(T_{\epsilon_1} A)}"'] \arrow[d, "\mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A}"] \\
& \Gamma \times T_{\epsilon_1}(\Gamma \times A) \arrow[d, "\mathtt{t}_{\epsilon_1,\Gamma,\Gamma\times A}"] & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \arrow[dl, "T_{\epsilon_1}\alpha_{\Gamma,\Gamma,A}"] \\
& T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) &
\end{tikzcd}
$$

Where $\alpha : ((\_ \times \_) \times \_) \to (\_ \times (\_ \times \_))$ is a natural isomorphism.

$$
\begin{aligned}
\alpha &= \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle \\
\alpha^{-1} &= \langle\langle \pi_1, \pi_1 \circ \pi_2\rangle, \pi_2 \circ \pi_2\rangle
\end{aligned}
$$

So:

$$
\begin{aligned}
G =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ \alpha^{-1}_{\Gamma,\Gamma,(T_{\epsilon_1} A)} \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_{T_{\epsilon_1} A}) \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By definition of } \alpha \text{ and products} \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By tensor strength's left-naturality} \\
=& T_{\epsilon_1}((\pi_1 \times \mathtt{Id}_{T_{\epsilon_2} B}) \circ \langle \mathtt{Id}_{(\Gamma\times A)}, g\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle
\end{aligned}
$$

Since

$$
rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{7.33}
$$

We have

$$
\begin{aligned}
rhs &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\pi_1 \times \mathtt{Id}_{T_{\epsilon_2} B}) \circ \langle \mathtt{Id}_{(\Gamma\times A)}, g\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \mathtt{Id}_B)) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \langle \mathtt{Id}_{(\Gamma\times A)}, g\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By Left-Tensor Strength} \\
&= lhs \quad \text{Woohoo!}
\end{aligned}
$$

Case Eq-Lambda-Eta: Let

$$
f = [\![\Gamma \vdash v \colon A \to B]\!] : \Gamma \to (B)^A \tag{7.34}
$$

By weakening, we have

$$
\begin{aligned}
[\![\Gamma, x \colon A \vdash v \colon A \to B]\!] &= f \circ \pi_1 : \Gamma \times A \to (B)^A \\
[\![\Gamma, x \colon A \vdash v\ x \colon B]\!] &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
[\![\Gamma \vdash \lambda x \colon A.(v\ x) : A \to B]\!] &= \mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) & (7.35) \\
\mathtt{app} \circ ([\![\Gamma \vdash \lambda x \colon A.(v\ x) : A \to B]\!] \times \mathtt{Id}_A) &= \mathtt{app} \circ (\mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) \times \mathtt{Id}_A) & (7.36) \\
&= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle & (7.37) \\
&= \mathtt{app} \circ (f \times \mathtt{Id}_A) & (7.38)
\end{aligned}
$$

Hence, by the fact that $\mathtt{cur}(f)$ is unique in a cartesian closed category,

$$[\![\Gamma \vdash \lambda x{:}\, A.(v\ x) : A \to B]\!] = f = [\![\Gamma \vdash v{:}\, A \to B]\!] \tag{7.39}$$

Case Eq-If-True: Let

$$\begin{aligned} f =& [\![\Gamma \vdash v_1{:}\, A]\!] \\ g =& [\![\Gamma \vdash v_2{:}\, A]\!] \end{aligned}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!] &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inl} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ ((\mathtt{cur}(f \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ (\mathtt{cur}(f \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= f \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\ &= f \\ &= [\![\Gamma \vdash v_1{:}\, A]\!] \end{aligned} \tag{7.40}$$

Case Eq-If-False: Let

$$\begin{aligned} f =& [\![\Gamma \vdash v_1{:}\, A]\!] \\ g =& [\![\Gamma \vdash v_2{:}\, A]\!] \end{aligned}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!] &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inr} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ ((\mathtt{cur}(g \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ (\mathtt{cur}(g \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= g \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\ &= g \\ &= [\![\Gamma \vdash v_2{:}\, A]\!] \end{aligned} \tag{7.41}$$

7.3.3 Case If-Eta

Let

$$\begin{aligned} f &= [\![\Gamma \vdash v_2{:}\, \mathtt{Bool}]\!] \\ g &= [\![\Gamma, x : \mathtt{Bool} \vdash v_1{:}\, A]\!] \end{aligned}$$

Then by the substitution theorem,

$$\begin{aligned} [\![\Gamma \vdash v_1[\mathtt{true}/x]{:}\, A]\!] &= g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \rangle \\ [\![\Gamma \vdash v_1[\mathtt{false}/x]{:}\, A]\!] &= g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \rangle \\ [\![\Gamma \vdash v_1[v_2/x]{:}\, A]\!] &= g \circ \langle \mathtt{Id}_\Gamma, f \rangle \end{aligned}$$

Hence we have (Using the diagonal and twist morphisms):

$$
\begin{aligned}
&[\![\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1[\mathtt{true}/x]\ \mathtt{else}\ v_1[\mathtt{false}/x] : A]\!]\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_1 \circ \langle\rangle_\Gamma\rangle \circ \pi_2), \mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_1 \circ \langle\rangle_\Gamma\rangle \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_2\rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_2\rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Pairing property}\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_1\rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_1\rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Terminal is unique}\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inl}_1 \circ \langle\rangle_{\mathtt{1}})) \circ \tau_{\mathtt{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inr}_1 \circ \langle\rangle_{\mathtt{1}})) \circ \tau_{\mathtt{1},\Gamma})] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of the twist morphism}\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inl}_1) \circ \tau_{\mathtt{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inr}_1) \circ \tau_{\mathtt{1},\Gamma})] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Identity} = \mathtt{Id}_{\mathtt{1}}\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (\mathtt{inl}_1 \times \mathtt{Id}_\Gamma)), \mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (\mathtt{inr}_1 \times \mathtt{Id}_\Gamma))] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Twist commutivity}\\
&= \mathtt{app} \circ (([\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ \mathtt{inl}_1, \mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ \mathtt{inr}_1] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Exponential property}\\
&= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ [\mathtt{inl}_1, \mathtt{inr}_1] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring out } \mathtt{cur}(..)\\
&= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Since } [\mathtt{inl}, \mathtt{inr}] is the identity\\
&= \mathtt{app} \circ (\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \times \mathtt{Id}_\Gamma) \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring}\\
&= g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of } \mathtt{app}, \mathtt{cur}(..)\\
&= g \circ (\mathtt{Id}_\Gamma \times f) \circ \tau_{\Gamma,\Gamma} \circ \delta_\Gamma \quad \text{Twist commutivity}\\
&= g \circ (\mathtt{Id}_\Gamma \times f) \circ \langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \quad \text{Twist, diagonal defintions}\\
&= g \circ \langle \mathtt{Id}_\Gamma, f \rangle\\
&= [\![\Gamma \vdash v_1[v_2/x] \colon A]\!]
\end{aligned}
$$

7.3.4 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of subexpressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

Case Eq-Fn: By inversion, we have $\Gamma, x\colon A \vdash v_1 \approx v_2\colon B$ By induction, we therefore have $[\![\Gamma, x\colon A \vdash v_1\colon B]\!] = [\![\Gamma, x\colon A \vdash v_2\colon B]\!]$

Then let

$$f = [\![\Gamma, x\colon A \vdash v_1\colon B]\!] = [\![\Gamma, x\colon A \vdash v_2\colon B]\!] \tag{7.42}$$

And so

$$[\![\Gamma \vdash \lambda x\colon A.v_1 : A \to B]\!] = \mathtt{cur}(f) = [\![\Gamma \vdash \lambda x\colon A.v_2 : A \to B]\!] \tag{7.43}$$

Case Eq-Return: By inversion, we have $\Gamma \vdash v_1 \approx v_2\colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1\colon A]\!] = [\![\Gamma \vdash v_2\colon A]\!]$

Then let

$$f = [\![\Gamma \vdash v_1\colon A]\!] = [\![\Gamma \vdash v_2\colon A]\!] \tag{7.44}$$

And so

$$[\![\Gamma \vdash \mathtt{return}\ v_1 : \mathtt{M}_{\mathtt{1}} A]\!] = \eta_A \circ f = [\![\Gamma \vdash \mathtt{return}\ v_2 : \mathtt{M}_{\mathtt{1}} A]\!] \tag{7.45}$$

Case Eq-Apply: By inversion, we have $\Gamma \vdash v_1 \approx v_1'\colon A \to B$ and $\Gamma \vdash v_2 \approx v_2'\colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1\colon A \to B]\!] = [\![\Gamma \vdash v_1'\colon A \to B]\!]$ and $[\![\Gamma \vdash v_2\colon A]\!] = [\![\Gamma \vdash v_2'\colon A]\!]$

Then let

$$
\begin{aligned}
f &= [\![\Gamma \vdash v_1\colon A \to B]\!] = [\![\Gamma \vdash v_1'\colon A \to B]\!]\\
g &= [\![\Gamma \vdash v_2\colon A]\!] = [\![\Gamma \vdash v_2'\colon A]\!]
\end{aligned}
$$

And so

$$[\![\Gamma \vdash v_1\ v_2\colon \mathtt{M}_\epsilon A]\!] = \mathtt{app} \circ \langle f, g \rangle = [\![\Gamma \vdash v_1'\ v_2'\colon \mathtt{M}_\epsilon A]\!] \tag{7.46}$$

Case Eq-Bind: By inversion, we have $\Gamma \vdash v_1 \approx v_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma, x\colon A \vdash v_2 \approx v_2' \colon \mathtt{M}_{\epsilon_2} B$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!] = [\![\Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A]\!]$ and $[\![\Gamma, x\colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_1} A]\!] = [\![\Gamma, x\colon A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]$

Then let

$$f = [\![\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!] = [\![\Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A]\!]$$
$$g = [\![\Gamma, x\colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!] = [\![\Gamma, x\colon A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!] &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= [\![\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!] \end{aligned} \tag{7.47}$$

Case Eq-If: By inversion, we have $\Gamma \vdash v \approx v' \colon \mathtt{Bool}$, $\Gamma \vdash v_1 \approx v_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma \vdash v_2 \approx v_2' \colon \mathtt{M}_{\epsilon_2} B$ By induction, we therefore have $[\![\Gamma \vdash v \colon \mathtt{Bool}]\!] = [\![\Gamma \vdash v' \colon B]\!]$, $[\![\Gamma \vdash v_1 \colon A]\!] = [\![\Gamma \vdash v_1' \colon A]\!]$ and $[\![\Gamma, x\colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!] = [\![\Gamma, x\colon A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]$

Then let

$$f = [\![\Gamma \vdash v \colon \mathtt{Bool}]\!] = [\![\Gamma \vdash v' \colon B]\!]$$
$$g = [\![\Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!] = [\![\Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A]\!]$$
$$h = [\![\Gamma, x\colon A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!] = [\![\Gamma, x\colon A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 :]\!] &= \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= [\![\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!] \end{aligned} \tag{7.48}$$

Case Eq-Subtype: By inversion, we have $\Gamma \vdash v_1 \approx v_2 \colon A$, and $A {\leq:} B$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!] = [\![\Gamma \vdash v_2 \colon A]\!]$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!] = [\![\Gamma \vdash v_2 \colon A]\!]$$
$$g = [\![A {\leq:} B]\!]$$

And so

$$[\![\Gamma \vdash v_1 \colon B]\!] = g \circ f = [\![\Gamma \vdash v_1 \colon B]\!] \tag{7.49}$$

□