0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v \colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v \colon A$, there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v \colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Proof: We induct on the structure of terms.

Case Variables: To find the unique derivation of $\Phi \mid \Gamma \vdash x \colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$: Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x \colon A$ is, if $A' \leq:_\Phi A$, as below:

$$\text{(Subtype)} \frac{\text{(Var)} \dfrac{\Phi \vdash \Gamma', x : A' \;\; \mathtt{Ok}}{\Phi \mid \Gamma, x : A' \vdash x \colon A'} \qquad A' \leq: A}{\Phi \mid \Gamma', x : A' \vdash x \colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$: with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x \colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x \colon A$.

Let

$$\text{(Subtype)} \frac{\dfrac{\Delta}{\Phi \mid \Gamma' \vdash x \colon A'} \qquad A' \leq: A}{\Phi \mid \Gamma' \vdash x \colon A} \tag{2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x \colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x \colon A$ is:

$$\text{(Subtype)} \frac{\text{(Weaken)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A' \vdash x \colon A'}}{\Phi \mid \Gamma \vdash x \colon A'} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x \colon A} \tag{3}$$

Case Constants: For each of the constants, ($\mathtt{C}^A$, $\mathtt{true}$, $\mathtt{false}$, $\mathtt{()}$), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)} \frac{\text{(Const)} \dfrac{\Gamma \;\; \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A} \qquad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash \mathtt{C}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_\Phi A$).

Case Lambda: The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v : A' \to B'$ is:

$$\text{(Subtype)} \dfrac{\text{(Lambda)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A.B : A \to B} \qquad A \to B \leq:_\Phi A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v : A' \to B'}$$

Where

$$\text{(Sub-Type)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v : B} \qquad B \leq:_\Phi B'}{\Phi \mid \Gamma, x : A \vdash v : B'} \qquad\qquad (4)$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v : B'$ if it exists.

Case Return: The reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)} \dfrac{\text{(Return)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_1 A} \qquad subeffecttreep1A\epsilon B}{\Phi \mid \Gamma \vdash \mathtt{return}\ v : B}$$

Where

$$\text{(Subtype)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : A} \qquad A \leq: B}{\Phi \mid \Gamma \vdash v : B}$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v : B$

Case Apply: If

$$\text{(Subtype)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1 : A \to B} \qquad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1 : A' \to B'}$$

and

$$\text{(Subtype)} \dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \qquad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2 : A'}$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1 : A' \to B'$ and $\Phi \mid \Gamma \vdash v_2 : A'$

Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subtype)} \dfrac{\text{(Apply)} \dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1 : A \to B} \qquad \text{(Subtype)} \dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v : A''} \qquad A'' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash v_1\ v_2 : B} \qquad subeffecttreep\epsilon B\epsilon' B'}{\Phi \mid \Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'}$$

Case If: Let

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : B} \qquad B \leq: \texttt{Bool}}{\Phi \mid \Gamma \vdash v : \texttt{Bool}} \tag{5}$$

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \qquad A' \leq: A}{\Phi \mid \Gamma \vdash v_1 : A} \tag{6}$$

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \qquad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2 : A} \tag{7}$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v : \texttt{Bool}$, $\Phi \mid \Gamma \vdash v_1 : A$, $\Phi \mid \Gamma \vdash v_2 : A$.

Then the only reduced derivation of $\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A$ is:

TODO: Scale this properly

$$\text{(Subtype)}\,\dfrac{\text{(If)}\,\dfrac{\text{(Subtype)}\,\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : B} \quad B \leq: \texttt{Bool}}{\Phi \mid \Gamma \vdash v : \texttt{Bool}} \quad \text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \quad A' \leq: A}{\Phi \mid \Gamma \vdash v_1 : A} \quad \text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \quad \mathtt{M}_\epsilon A \leq:_\Phi \mathtt{M}_\epsilon A}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{8}$$

Case Bind: Let

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A} \qquad \text{(Computation)}\,\dfrac{A \leq:_\Phi A' \qquad \epsilon_1 \leq_\Phi \epsilon_1'}{\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B} \qquad \text{(Computation)}\,\dfrac{B \leq:_\Phi B' \qquad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq:_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms.

By weakening, $\Phi \vdash \iota\times : \Gamma, x : A \rhd \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2 : B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2 : B$.

$$\text{(Subtype)}\,\dfrac{\dfrac{\Delta''}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2} B} \qquad \text{(Computation)}\,\dfrac{B \leq:_\Phi B' \qquad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq:_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2'} B'} \tag{11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v-2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Subtype)} \cfrac{\text{(Bind)} \cfrac{\cfrac{\Delta}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A'} \qquad \cfrac{\Delta''}{\Phi \mid \Gamma \vdash \Gamma, x : A' : v_2 \mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \qquad subeffecttree \epsilon_1 \cdot \epsilon_2 B \epsilon_1' \cdot \epsilon_2' B'}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v-2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{12}$$

Case Effect-Fn: The unique reduced derivation of $\Phi \mid \Gamma \vdash \Lambda \alpha . A : \forall \alpha . B$

is

$$\text{(Sub-type)} \cfrac{\text{(Effect-Fn)} \cfrac{\cfrac{\Delta}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda \alpha . v : \forall \alpha . A} \qquad \forall \alpha . A \leq_\Phi \forall \alpha . B}{\Phi \mid \Gamma \vdash \Lambda \alpha . B : \forall \alpha . B} \tag{13}$$

Where

$$\text{(Sub-type)} \cfrac{\cfrac{\Delta}{\Phi, \alpha \mid \Gamma \vdash v : A} \qquad A \leq:_{\Phi, \alpha} B}{\Phi, \alpha \mid \Gamma \vdash v : B} \tag{14}$$

Is the unique reduced derivation of $\Phi, \alpha \mid \Gamma \vdash v : B$

Case Effect-App: The unique reduced derivation of $\Phi \mid \Gamma \vdash v\ \alpha : B'$

is

$$\text{(Subtype)} \cfrac{\text{(Effect-App)} \cfrac{\cfrac{\Delta}{\Phi \mid \Gamma \vdash v : \forall \alpha . A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon : A\,[\epsilon / \alpha]} \qquad A\,[\epsilon / \alpha] \leq:_\Phi B'}{\Phi \mid \Gamma \vdash v\ \alpha : B'} \tag{15}$$

Where $B\,[\epsilon / \alpha] \leq:_\Phi B'$ and

$$\text{(Subtype)} \cfrac{\cfrac{\Delta}{\Phi \mid \Gamma \vdash v : \forall \alpha . B} \qquad \text{(Quantification)} \cfrac{A \leq:_{\Phi, \alpha} B}{\forall \alpha . A \leq:_\Phi \forall \alpha . B}}{\Phi \mid \Gamma \vdash v : \forall \alpha . B} \tag{16}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v : A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

Case Constants: For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced.

$$reduce((\mathrm{Const})\frac{\Phi \vdash \Gamma \ \mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{C}^A : A}) = (\mathrm{Const})\frac{\Phi \vdash \Gamma \ \mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{C}^A : A}$$

Case Var:

$$reduce((\mathrm{Var})\frac{\Phi \vdash \Gamma \ \mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x : A}) = (\mathrm{Var})\frac{\Phi \vdash \Gamma \ \mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x : A} \tag{17}$$

Case Weaken:

reduce **definition**   To find:

$$reduce((\mathrm{Weaken})\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash x : A}}{\Phi \mid \Gamma, y : B \vdash x : A}) \tag{18}$$

Let

$$(\mathrm{Subtype})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x : A} \qquad A' \leq :_\Phi A}{\Phi \mid \Gamma \vdash x : A} = reduce(\Delta) \tag{19}$$

In

$$(\mathrm{Subtype})\frac{(\mathrm{Weaken})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x : A'}}{\Phi \mid \Gamma, y : B \vdash x : A'} \qquad A' \leq :_\Phi A}{\Phi \mid \Gamma, y : B \vdash x : A} \tag{20}$$

Case Lambda:

reduce **definition**   To find:

$$reduce((\mathrm{Fn})\frac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to \epsilon_2 B}) \tag{21}$$

Let

$$(\mathrm{Sub\text{-}type})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v : B'} \qquad B' \leq :_\Phi B}{\Phi \mid \Gamma, x : A \vdash v : B} = reduce(\Delta) \tag{22}$$

In

$$(\mathrm{Sub\text{-}type})\frac{(\mathrm{Fn})\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v : \mathtt{M}_{\epsilon_1} B'} \qquad A \to \epsilon_1 B' \leq :_\Phi A \to \epsilon_2 B}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to \epsilon_2 B} \tag{23}$$

Case Subtype:

reduce **definition**   To find:

$$reduce((\text{Subtype})\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v \colon A} \quad A \leq \colon_\Phi B}{\Phi \mid \Gamma \vdash v \colon B}) \tag{24}$$

Let

$$(\text{Subtype})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x \colon A} \quad A' \leq \colon_\Phi A}{\Phi \mid \Gamma \vdash x \colon A} = reduce(\Delta) \tag{25}$$

In

$$(\text{Subtype})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v \colon A'} \quad A' \leq \colon_\Phi A \leq \colon_\Phi B}{\Phi \mid \Gamma \vdash v \colon B} \tag{26}$$

Case Return:

reduce **definition**   To find:

$$reduce((\text{Return})\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v \colon A}}{\Phi \mid \Gamma \vdash \mathtt{return}\ v \colon \mathtt{M_1}\,A}) \tag{27}$$

Let

$$(\text{Sub-type})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v \colon A'} \quad A' \leq \colon_\Phi A}{\Phi \mid \Gamma \vdash v \colon A} = reduce(\Delta) \tag{28}$$

In

$$(\text{Sub-type})\dfrac{(\text{Return})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v \colon A'}}{subeffecttreep1A'1A}}{\Phi \mid \Gamma \vdash \mathtt{return}\ v \colon \mathtt{M_1}\,A} \tag{29}$$

Case Apply:

reduce **definition**   To find:

$$reduce((\text{Apply})\dfrac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 \colon A \to B} \quad \dfrac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 \colon A}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B}) \tag{30}$$

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Let

$$\text{(Subtype)}\ \frac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 \colon A' \to B'} \qquad A' \to B' \leq:_\Phi A \to \epsilon B}{\Phi \mid \Gamma \vdash v_1 \colon A \to B} = reduce(\Delta_1) \tag{31}$$

$$\text{(Subtype)}\ \frac{\dfrac{\Delta_2'}{\Phi \mid \Gamma \vdash v \colon A'} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v_1 \colon A} = reduce(\Delta_2) \tag{32}$$

In

$$\text{(Subtype)}\ \frac{\text{(Apply)}\ \dfrac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 \colon A' \to B'} \qquad \text{(Sub-type)}\ \dfrac{\dfrac{\Delta_2'}{\Phi \mid \Gamma \vdash v_2 \colon A''} \qquad A'' \leq:_\Phi A \leq:_\Phi A'}{\Phi \mid \Gamma \vdash v_2 \colon A'}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'} \qquad subeffecttree\epsilon'B'\epsilon B}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B} \tag{33}$$

Case If:

reduce **definition**

$$reduce(\text{(If)}\ \frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}} \quad \dfrac{\Delta_2}{\Phi \mid \Gamma \vdash v_1 \colon A} \quad \dfrac{\Delta_3}{\Phi \mid \Gamma \vdash v_2 \colon A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A}) = \text{(If)}\ \frac{\dfrac{reduce(\Delta_1)}{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}} \quad \dfrac{reduce(\Delta_2)}{\Phi \mid \Gamma \vdash v_1 \colon A} \quad \dfrac{reduce(\Delta_3)}{\Phi \mid \Gamma \vdash v_2 \colon A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{34}$$

Case Bind:

reduce **definition**   To find

$$reduce(\text{(Bind)}\ \frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}) \tag{35}$$

Let

$$\text{(Sub-Type)}\ \frac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \qquad subeffecttree\epsilon_1'A'\epsilon_1 A}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} = reduce(\Delta_1) \tag{36}$$

Since $\Phi \vdash i, \times : \Gamma, x : A' \rhd \Gamma, x : A$ if $A' \leq:_\Phi A$, and by Δ_2, $\Phi \mid (\Gamma, x : A) \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$\text{(Sub-Type)}\ \frac{\dfrac{\Delta_3'}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \qquad subeffecttree\epsilon_2'B'\epsilon_2 B}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B} = reduce(\Delta_3) \tag{37}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$
\text{(Sub-Type)} \cfrac{\text{(Bind)}\cfrac{\cfrac{\Delta'_1}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon'_1} A'} \qquad \cfrac{\Delta'_3}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon'_2} B'}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon'_1 \cdot \epsilon'_2} B} \qquad subeffecttreep\epsilon'_1 \cdot \epsilon'_2 B' \epsilon_1 \cdot \epsilon_2 B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}
$$

$$(38)$$

Case Effect-Fn:

reduce **definition** To find

$$
reduce\big(\text{(Effect-Lambda)} \cfrac{\cfrac{\Delta_1}{\Phi, \alpha \mid \Gamma \vdash v \colon A}}{\Phi \mid \Gamma \vdash \Lambda \alpha.v \colon \forall \alpha.A}\big)
$$

$$(39)$$

Let

$$
\text{(Subtype)} \cfrac{\cfrac{\Delta'_1}{\Phi, \alpha \mid \Gamma \vdash v \colon A'} \qquad A' \leq :_\Phi A}{\Phi, \alpha \mid \Gamma \vdash v \colon A} = reduce(\Delta_1)
$$

$$(40)$$

in

$$
\text{(Subtype)} \cfrac{\text{(Effect-Fn)}\cfrac{\cfrac{\Delta'_1}{\Phi, \alpha \mid \Gamma \vdash v \colon A'}}{\Phi \mid \Gamma \vdash \Lambda \alpha.v \colon \forall \alpha.A'} \qquad \text{(Quantification)}\cfrac{A' \leq :_{\Phi, \alpha}}{\forall \alpha.A' \leq :_\Phi \forall \alpha.A}}{\Phi \mid \Gamma \vdash \Lambda \alpha.v \colon \forall \alpha.A}
$$

$$(41)$$

Case Effect-Application:

reduce **definition** To find

$$
reduce\big(\text{(Effect-App)} \cfrac{\cfrac{\Delta_1}{\Phi \mid \Gamma \vdash v \colon \forall \alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]}\big)
$$

$$(42)$$

Let

$$
\text{(Subtype)} \cfrac{\cfrac{\Delta'_1}{\Phi \mid \Gamma \vdash v \colon \forall \alpha.A'} \qquad \text{(Quantification)}\cfrac{A' \leq :_{\Phi, \alpha} A}{\forall \alpha.A' \leq :_\Phi \forall \alpha.A}}{\Phi \mid \Gamma \vdash v \colon \forall \alpha.A} = reduce(\Delta_1)
$$

$$(43)$$

In

$$
\text{(Subtype)} \cfrac{\text{(E-app)}\cfrac{\cfrac{\Delta'_1}{\Phi \mid \Gamma \vdash v \colon \forall \alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \qquad A'\,[\epsilon/\alpha] \leq :_\Phi A\,[\epsilon/\alpha]}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]}
$$

$$(44)$$

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0.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v{:}A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!] = [\![reduce\,\Delta]\!] = [\![reduce\,\Delta']\!] = [\![\Delta']\!]$, hence the denotation $[\![\Phi \mid \Gamma \vdash v{:}A]\!]$ is unique.