

0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, **lambda** or **apply** rule.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t : \tau$, there exists at most one reduced derivation of $\Gamma \vdash t : \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Constants

For each of the constants, (\mathbb{C}^A , **true**, **false**, $()$), there is exactly one possible derivation for $\Gamma \vdash c : A$ for a given A . I shall give examples using the case \mathbb{C}^A

$$(\text{Subtype}) \frac{(\text{Const}) \frac{}{\Gamma \vdash \mathbb{C}^A : A} \quad A \leq B}{\Gamma \vdash \mathbb{C}^A : B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq A$).

0.2.2 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C : A' \rightarrow \mathbb{M}_{\epsilon'} B'$ is:

$$(\text{Subtype}) \frac{(\text{Lambda}) \frac{() \frac{\Delta}{\Gamma, x : A \vdash C : \mathbb{M}_{\epsilon} B}}{\Gamma \vdash \lambda x : A.B : A \rightarrow \mathbb{M}_{\epsilon} B} \quad A \rightarrow \mathbb{M}_{\epsilon} B \leq A' \rightarrow \mathbb{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C : A' \rightarrow \mathbb{M}_{\epsilon'} B'}$$

Where Δ is the reduced derivation of $\Gamma, x : A \vdash C : \mathbb{M}_{\epsilon} B$ if it exists.

Case Subtype **TODO: Do we need to write anything here? (Probably needs an explanation)**

0.2.3 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \text{return } v : \mathbb{M}_{\epsilon} B$ is

$$(\text{Subtype}) \frac{(\text{Return}) \frac{() \frac{\Delta}{\Gamma \vdash v : A}}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A} \quad A \leq A' \leq B \quad 1 \leq \epsilon}{\Gamma \vdash \text{return } v : \mathbb{M}_{\epsilon} B}$$

Where

$$(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v : A} \quad A \leq A'}{\Gamma \vdash v : A'}$$

is the reduced derivation of $\Gamma \vdash v : A'$

Case Apply If

$$(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \rightarrow \mathbb{M}_{\epsilon} B B} \quad A \rightarrow \mathbb{M}_{\epsilon} B B \leq A' \rightarrow \mathbb{M}_{\epsilon'} B'}{\Gamma \vdash v_1 : A' \rightarrow \mathbb{M}_{\epsilon'} B'}$$

and

$$(\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash v_2 : A''} \quad A'' \leq A'}{\Gamma \vdash v_2 : A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1 : A' \rightarrow \mathbb{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2 : A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1 \quad v_2 : \mathbb{M}_{\epsilon'} B'$ as

$$\begin{array}{c}
\text{(Subeffect)} \frac{
\begin{array}{c}
\text{(Apply)} \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \rightarrow M_\epsilon B}}{\Gamma \vdash v_1 : A \rightarrow M_\epsilon B} \quad
\text{(Subtype)} \frac{() \frac{\Delta'}{\Gamma \vdash v_2 : A''} \quad A'' \leq A}{\Gamma \vdash v_2 : A} \\
\Gamma \vdash v_1 \quad v_2 : M_\epsilon B
\end{array}
}{\Gamma \vdash v_1 \quad v_2 : M_{\epsilon'} B'} \quad B \leq B' \quad \epsilon \leq \epsilon'
\end{array}$$

Case If

Case Bind

Case Subeffect

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed. **TODO: Fill in these cases with actual maths**

0.3.1 Constants

TODO: *reduce* just appends the identity subtype rule to the derivation, trivially preserves denotation

0.3.2 Value Types

Lambda **TODO:** Recursively call *reduce* on C then push subtyping through using currying

Subtype **TODO:** Recursively call *reduce* then merge subtypes

0.3.3 Computation Types

Return **TODO:** Recursively call *reduce* then use naturality to push subtyping into subeffect

Apply **TODO:** Recursively call *reduce*, then construct the reduced apply as in the proof of uniqueness

If **TODO:** Recursively call *reduce*, then leave tree otherwise unchanged.

Bind **TODO:** Recursively call *reduce* then push subtyping rules through the bind

Subeffect **TODO:** Recursively call *reduce*, then merge subeffecting rules

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t : \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $\llbracket \Delta \rrbracket_M = \llbracket \text{reduce} \Delta \rrbracket_M = \llbracket \text{reduce} \Delta' \rrbracket_M = \llbracket \Delta' \rrbracket_M$, hence the denotation $\llbracket \Gamma \vdash t : \tau \rrbracket_M$ is unique.