0.1 Beta and Eta Equivalence

0.1.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B \quad \Gamma\vdash v{:}A}{\Gamma\vdash(\lambda x{:}A.C)\ v=_{\beta\eta}C[v/x]{:}\mathtt{M}_\epsilon B}$
- (Lambda-Eta) $\frac{\Gamma\vdash v{:}A\to\mathtt{M}_\epsilon B}{\Gamma\vdash\lambda x{:}A.(v\ x)=_{\beta\eta}v{:}A\to\mathtt{M}_\epsilon B}$
- (Left Unit) $\frac{\Gamma\vdash v{:}A \quad \Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B}{\Gamma\vdash\mathtt{do}\ x\leftarrow\mathtt{return}v\ \mathtt{in}\ C=_{\beta\eta}C[v/x]{:}\mathtt{M}_\epsilon B}$
- (Right Unit) $\frac{\Gamma\vdash C{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{do}\ x\leftarrow C\ \mathtt{in}\ \mathtt{return}x=_{\beta\eta}C{:}\mathtt{M}_\epsilon A}$
- (Associativity) $\frac{\Gamma\vdash C_1{:}\mathtt{M}_{\epsilon_1}A \quad \Gamma,x{:}A\vdash C_2{:}\mathtt{M}_{\epsilon_2}B \quad \Gamma,y{:}B\vdash C_3{:}\mathtt{M}_{\epsilon_3}C}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y\leftarrow C_2\ \mathtt{in}\ C_3)=_{\beta\eta}\mathtt{do}\ y\leftarrow(\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}C}$
- (Unit) $\frac{\Gamma\vdash v{:}\mathtt{Unit}}{\Gamma\vdash v=_{\beta\eta}\mathtt{()}{:}\mathtt{Unit}}$
- (if-true) $\frac{\Gamma\vdash C_1{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash C_2{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{true}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}C_1{:}\mathtt{M}_\epsilon A}$
- (if-false) $\frac{\Gamma\vdash C_2{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash C_1{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{false}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}C_2{:}\mathtt{M}_\epsilon A}$
- (If-Eta) $\frac{\Gamma,x{:}\mathtt{Bool}\vdash C{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash v{:}\mathtt{Bool}}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C[\mathtt{true}/x]\ \mathtt{else}\ C\big[\mathtt{false}/x\big]=_{\beta\eta}C[v/x]{:}\mathtt{M}_\epsilon A}$

0.1.2 Equivalence Relation

- (Reflexive) $\frac{\Gamma\vdash t{:}\tau}{\Gamma\vdash t=_{\beta\eta}t{:}\tau}$
- (Symmetric) $\frac{\Gamma\vdash t_1=_{\beta\eta}t_2{:}\tau}{\Gamma\vdash t_2=_{\beta\eta}t_1{:}\tau}$
- (Transitive) $\frac{\Gamma\vdash t_1=_{\beta\eta}t_2{:}\tau \quad \Gamma\vdash t_2=_{\beta\eta}t_3{:}\tau}{\Gamma\vdash t_1=_{\beta\eta}t_3{:}\tau}$

0.1.3 Congruences

- (Lambda) $\frac{\Gamma,x{:}A\vdash C_1=_{\beta\eta}C_2{:}\mathtt{M}_\epsilon B}{\Gamma\vdash\lambda x{:}A.C_1=_{\beta\eta}\lambda x{:}A.C_2{:}A\to\mathtt{M}_\epsilon B}$
- (Return) $\frac{\Gamma\vdash v_1=_{\beta\eta}v_2{:}A}{\Gamma\vdash\mathtt{return}v_1=_{\beta\eta}\mathtt{return}v_2{:}\mathtt{M}_{\mathtt{1}}A}$
- (Apply) $\frac{\Gamma\vdash v_1=_{\beta\eta}v_1'{:}A\to\mathtt{M}_\epsilon B \quad \Gamma\vdash v_2=_{\beta\eta}v_2'{:}A}{\Gamma\vdash v_1\ v_2=_{\beta\eta}v_1'\ v_2'{:}\mathtt{M}_\epsilon B}$
- (Bind) $\frac{\Gamma\vdash C_1=_{\beta\eta}C_1'{:}\mathtt{M}_{\epsilon_1}A \quad \Gamma,x{:}A\vdash C_2=_{\beta\eta}C_2'{:}\mathtt{M}_{\epsilon_2}B}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2=_{\beta\eta}\mathtt{do}\ c\leftarrow C_1'\ \mathtt{in}\ C_2'{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$
- (If) $\frac{\Gamma\vdash v=_{\beta\eta}v'{:}\mathtt{Bool} \quad \Gamma\vdash C_1=_{\beta\eta}C_1'{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash C_2=_{\beta\eta}C_2'{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2=_{\beta\eta}\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1'\ \mathtt{else}\ C_2'{:}\mathtt{M}_\epsilon A}$
- (Subtype) $\frac{\Gamma\vdash v=_{\beta\eta}v'{:}A \quad A\le{:}B}{\Gamma\vdash v=_{\beta\eta}v'{:}B}$
- (Subeffect) $\frac{\Gamma\vdash C=_{\beta\eta}C'{:}\mathtt{M}_{\epsilon_1}A \quad A\le{:}B \quad \epsilon_1\le\epsilon_2}{\Gamma\vdash C=_{\beta\eta}C'{:}\mathtt{M}_{\epsilon_2}B}$

0.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

Each derivation of $\Gamma \vdash t =_{\beta\eta} t' \colon \tau$ can be converted to a derivation of $\Gamma \vdash t \colon \tau$ and $\Gamma \vdash t' \colon \tau$ by induction over the beta-eta equivalence relation derivation.

0.2.1 Equivalence Relations

Case Reflexive By inversion we have a derivation of $\Gamma \vdash t \colon \tau$.

Case Symmetric By inversion $\Gamma \vdash t' =_{\beta\eta} t \colon \tau$. Hence by induction, derivations of $\Gamma \vdash t' \colon \tau$ and $\Gamma \vdash t \colon \tau$ are given.

Case Transitive By inversion, there exists t_2 such that $\Gamma \vdash t_1 =_{\beta\eta} t_2 \colon \tau$ and $\Gamma \vdash t_2 =_{\beta\eta} t_3 \colon \tau$. Hence by induction, we have derivations of $\Gamma \vdash t_1 \colon \tau$ and $\Gamma \vdash t_3 \colon \tau$

0.2.2 Beta-Eta Conversions

Case Lambda By inversion, we have $\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B$ and $\Gamma \vdash v \colon A$. Hence by the typing rules, we have:

$$(\text{Apply})\frac{(\text{Lambda})\frac{\Gamma, x:A \vdash C:\mathtt{M}_\epsilon B}{\Gamma \vdash \lambda x:A.C : A \to \mathtt{M}_\epsilon B} \quad \Gamma \vdash v \colon A}{\Gamma \vdash (\lambda x : A.C)\ v \colon \mathtt{M}_\epsilon A}$$

By the substitution rule **TODO: which?**, we have

$$(\text{Substitution})\frac{\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B \quad \Gamma \vdash v \colon A}{\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B}$$

Case Left Unit By inversion, we have $\Gamma \vdash v \colon A$ and $\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B$

Hence we have:

$$(\text{Bind})\frac{(\text{Return})\frac{\Gamma \vdash v : A}{\Gamma \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}}\, A} \quad \Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B}{\Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return} v\ \mathtt{in}\ C \colon \mathtt{M}_{\mathtt{1} \cdot \epsilon} B = \mathtt{M}_\epsilon B} \tag{1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B \tag{2}$$

Case Right Unit By inversion, we have $\Gamma \vdash C \colon \mathtt{M}_\epsilon A$.

Hence we have:

$$(\text{Bind})\frac{\Gamma \vdash C \colon \mathtt{M}_\epsilon A \quad (\text{Return})\frac{(\text{var})\frac{}{\Gamma, x : A \vdash x : A}}{\Gamma, x:A \vdash \mathtt{return} v : \mathtt{M}_{\mathtt{1}}\, A}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C\ \mathtt{in}\ \mathtt{return} x \colon \mathtt{M}_{\epsilon \cdot \mathtt{1}} A = \mathtt{M}_\epsilon A} \tag{3}$$

Case Associativity By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$(\iota\pi\times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Gamma, x : A, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A \quad \text{(Bind)}\frac{\Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2} B \quad \Gamma, x:A, y:B \vdash C_3 : \mathtt{M}_{\epsilon_3} C}{\Gamma, x:A \vdash x C_2 C_3 : \mathtt{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow C_2\ \mathtt{in}\ C_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\frac{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{5}$$

Case Eta By inversion, we have $\Gamma \vdash v \colon A \to \mathtt{M}_{\epsilon} B$

By weakening, we have $\iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\frac{\text{(App)}\frac{(\Gamma, x:A) \vdash x:A \quad \text{(weakening)}\frac{\Gamma \vdash v : A \to \mathtt{M}_{\epsilon} B \quad \iota\pi : \Gamma, x:A \triangleright \Gamma}{\Gamma, x:A \vdash v : A \to \mathtt{M}_{\epsilon} B}}{\Gamma, x:A \vdash v\ x : \mathtt{M}_{\epsilon} B}}{\Gamma \vdash \lambda x : A.(v\ x) \colon A \to \mathtt{M}_{\epsilon} B} \tag{6}$$

Case If-True By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon} A$, $\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon} A$. Hence by the typing lemma **TODO: Which?**, we have $\Gamma \mathtt{Ok}$ so $\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \quad \Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon} A \quad \Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon} A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ \mathtt{true}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_{\epsilon} A} \tag{7}$$

Case If-False As above,

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \quad \Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon} A \quad \Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon} A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ \mathtt{false}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_{\epsilon} A} \tag{8}$$

Case If-Eta By inversion, we have:

$$\Gamma \vdash v \colon \mathtt{Bool} \tag{9}$$

and

$$\Gamma, x : \mathtt{Bool} \vdash C \colon \mathtt{M}_{\epsilon} A \tag{10}$$

Hence we also have $\Gamma \mathtt{Ok}$. Hence, the following also hold:

$\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$, and $\Gamma \vdash \mathtt{false} \colon \mathtt{Bool}$.

Hence by the substitution theorem, we have:

$$\text{(If)}\frac{\Gamma \vdash v \colon \mathtt{Bool} \quad \Gamma \vdash C\,[\mathtt{true}/x] : \mathtt{M}_{\epsilon} A \quad \Gamma \vdash C\,[\mathtt{false}/x] : \mathtt{M}_{\epsilon} A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C\,[\mathtt{true}/x]\ \mathtt{else}\ C\,[\mathtt{false}/x] : \mathtt{M}_{\epsilon} A} \tag{11}$$

and

$$\Gamma \vdash C\,[v/x] : \mathtt{M}_{\epsilon} A \tag{12}$$

0.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda By inversion, $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_\epsilon B$. Hence by induction $\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B$, and $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B$.

So

$$\Gamma \vdash \lambda x : A.C_1 \colon A \to \mathtt{M}_\epsilon B \tag{13}$$

and

$$\Gamma \vdash \lambda x : A.C_2 \colon A \to \mathtt{M}_\epsilon B \tag{14}$$

Hold.

Case Return By inversion, $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, so by induction

$$\Gamma \vdash v_1 \colon A$$

and

$$\Gamma \vdash v_2 \colon A$$

Hence we have

$$\Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_{\mathbf{1}} A$$

and

$$\Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_{\mathbf{1}} A$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \mathtt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$. Hence we have by induction $\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B$, $\Gamma \vdash v_2 \colon A$, $\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B$, and $\Gamma \vdash v_2' \colon A$.

So we have:

$$\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B \tag{15}$$

and

$$\Gamma \vdash v_1'\ v_2' \colon \mathtt{M}_\epsilon B \tag{16}$$

Case Bind By inversion, we have: $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{17}$$

$$\Gamma \vdash \mathtt{do}\ x \leftarrow C_1'\ \mathtt{in}\ C_2' \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{18}$$

Case If By inversion, we have: $\Gamma \vdash v =_{\beta\eta} v' \colon \texttt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \texttt{M}_\epsilon A$, and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \texttt{M}_\epsilon A$.

Hence by induction, we have:

$\Gamma \vdash v \colon \texttt{Bool}$, $\Gamma \vdash v' \colon \texttt{Bool}$,

$\Gamma \vdash C_1 \colon \texttt{M}_\epsilon A$, $\Gamma \vdash C_1' \colon \texttt{M}_\epsilon A$,

$\Gamma \vdash C_2 \colon \texttt{M}_\epsilon A$, and $\Gamma \vdash C_2' \colon \texttt{M}_\epsilon A$.

So

$$\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 \colon \texttt{M}_\epsilon A \tag{19}$$

and

$$\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1'\ \texttt{else}\ C_2' \colon \texttt{M}_\epsilon A \tag{20}$$

Hold.

Case Subtype By inversion, we have $A \leq: B$ and $\Gamma \vdash v =_{\beta\eta} v' \colon A$. By induction, we therefore have $\Gamma \vdash v \colon A$ and $\Gamma \vdash v' \colon A$.

Hence we have

$$\Gamma \vdash v \colon B \tag{21}$$

$$\Gamma \vdash v' \colon B \tag{22}$$

Case subeffect By inversion we have: $A \leq: B$, $\epsilon_1 \leq \epsilon_2$, and $\Gamma \vdash C =_{\beta\eta} C' \colon \texttt{M}_{\epsilon_1} A$.

Hence by inductive hypothesis, we have $\Gamma \vdash C \colon \texttt{M}_{\epsilon_1} A$ and $\Gamma \vdash C' \colon \texttt{M}_{\epsilon_1} A$.

Hence,

$$\Gamma \vdash C \colon \texttt{M}_{\epsilon_2} B \tag{23}$$

and

$$\Gamma \vdash C' \colon \texttt{M}_{\epsilon_2} B \tag{24}$$

hold.

0.3 Beta-Eta equivalent terms have equal denotations

If $t \vdash t' =_{\beta\eta} \tau$: then $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$

By induction over Beta-eta equivalence relation.

0.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

Case Reflexive Equality is reflexive, so if $\Gamma \vdash t \colon \tau$ then $[\![\Gamma \vdash t \colon \tau]\!]_M$ is equal to itself.

Case Symmetric By inversion, if $\Gamma \vdash t =_{\beta\eta} t' \colon \tau$ then $\Gamma \vdash t' =_{\beta\eta} t \colon \tau$, so by induction $[\![\Gamma \vdash t' \colon \tau]\!]_M = [\![\Gamma \vdash t \colon \tau]\!]_M$ and hence $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$

Case Transitive There must exist t_2 such that $\Gamma \vdash t_1 =_{\beta\eta} t_2 : \tau$ and $\Gamma \vdash t_2 =_{\beta\eta} t_3 : \tau$, so by induction, $[\![\Gamma \vdash t_1 : \tau]\!]_M = [\![\Gamma \vdash t_2 : \tau]\!]_M$ and $[\![\Gamma \vdash t_2 : \tau]\!]_M = [\![\Gamma \vdash t_3 : \tau]\!]_M$. Hence by transitivity of equality, $[\![\Gamma \vdash t_1 : \tau]\!]_M = [\![\Gamma \vdash t_3 : \tau]\!]_M$

0.3.2 Beta-Eta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

Case Lambda Let $f = [\![\Gamma, x : A \vdash C : \mathtt{M}_\epsilon B]\!]_M : (\Gamma \times A) \to T_\epsilon B$

Let $g = [\![\Gamma \vdash v : A]\!]_M : \Gamma \to A$

By the substitution denotation,

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g\rangle$$

We have

$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g\rangle$$

and hence

$$\begin{aligned}
[\![\Gamma \vdash (\lambda x : A.C)\; v : \mathtt{M}_\epsilon B]\!]_M &= \mathtt{app} \circ \langle \mathtt{cur}(f), g\rangle \\
&= \mathtt{app} \circ (\mathtt{cur}(f) \times \mathtt{Id}_A) \circ \langle \mathtt{Id}_\Gamma, g\rangle \\
&= f \circ \langle \mathtt{Id}_\Gamma, g\rangle \\
&= [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M
\end{aligned} \tag{25}$$

Case Left Unit Let $f = [\![\Gamma, x : A \vdash C : \mathtt{M}_\epsilon B]\!]_M$

Let $g = [\![\Gamma \vdash v : A]\!]_M : \Gamma \to A$

By the substitution denotation,

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g\rangle$$

We have

$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g\rangle$$

And hence

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{do}\; x \leftarrow \mathtt{return} v\; \mathtt{in}\; C : \mathtt{M}_\epsilon B]\!]_M =& \mu_{\mathtt{1},\epsilon,B} \circ T_{\mathtt{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \eta_A \circ g\rangle \\
=& \mu_{\mathtt{1},\epsilon,B} \circ T_{\mathtt{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times \eta_A) \circ \langle \mathtt{Id}_\Gamma, g\rangle \\
=& \mu_{\mathtt{1},\epsilon,B} \circ T_{\mathtt{1}} f \circ \eta_{(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By Tensor strength + unit} \\
=& \mu_{\mathtt{1},\epsilon,B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By Naturality of } \eta \\
=& f \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By left unit law} \\
=& [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M
\end{aligned} \tag{26}$$

Case Right Unit Let $f = [\![\Gamma \vdash C : \mathtt{M}_\epsilon A]\!]_M$

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{do}\; x \leftarrow C\; \mathtt{in}\; \mathtt{return} x : \mathtt{M}_\epsilon A]\!]_M &= \mu_{\epsilon,\mathtt{1},A} \circ T_\epsilon(\eta_A \circ \pi_2) \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= \pi_2 \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= f
\end{aligned} \tag{27}$$

Case Associative Let

$$f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M \tag{28}$$
$$g = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M \tag{29}$$
$$h = [\![\Gamma, y : B \vdash C_3 \colon \mathtt{M}_\epsilon C]\!]_M \tag{30}$$

We also have the weakening:

$$\iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \tag{31}$$

With denotation:

$$[\![\iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B]\!]_M = (\pi_1 \times \mathtt{Id}_B) \tag{32}$$

We need to prove that the following are equal

$$lhs = [\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow C_2\ \mathtt{in}\ C_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2}]\!]_M \tag{33}$$
$$= \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon_2, (\Gamma \times A), B} \circ \big\langle \mathtt{Id}_{(\Gamma \times A)}, g \big\rangle) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \tag{34}$$
$$rhs = [\![\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2}]\!]_M \tag{35}$$
$$= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \tag{36}$$
$$\tag{37}$$

Let's look at fragment F of rhs.

$$F = \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle) \rangle \tag{38}$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \tag{39}$$

$$\begin{aligned} F &= \mathtt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathtt{t}_{\epsilon_2, \Gamma, B} \circ \mathtt{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\mathtt{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{By } \textbf{TODO: ref: mu+tstrength} \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{By naturality of t-strength} \end{aligned} \tag{40}$$

Since $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F$,

$$\begin{aligned} rhs =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\ =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\ =& \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \end{aligned} \tag{41}$$

Let's now look at the fragment G of rhs

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \rangle \tag{42}$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2, \Gamma, B}) \circ G \tag{43}$$

By folding out the $\langle ..., ... \rangle$, we have

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1,\Gamma,\Gamma\times A} \circ (\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle \tag{44}$$

From the rule **TODO: Ref** showing the commutativity of tensor strength with α, the following commutes

$$\begin{tikzcd}
\Gamma \arrow[r, "{\langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle}"] & \Gamma \times (\Gamma \times T_{\epsilon_1} A) \arrow[d, "\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1,\Gamma,A}"] & (\Gamma\times\Gamma)\times T_{\epsilon_1}A \arrow[l, "\alpha_{\Gamma,\Gamma,(T_{\epsilon_1}A)}"'] \arrow[d, "\mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A}"] \\
& \Gamma \times T_{\epsilon_1}(\Gamma\times A) \arrow[d, "\mathtt{t}_{\epsilon_1,\Gamma,\Gamma\times A}"] & T_{\epsilon_1}((\Gamma\times\Gamma)\times A) \arrow[dl, "T_{\epsilon_1}\alpha_{\Gamma,\Gamma,A}"] \\
& T_{\epsilon_1}(\Gamma\times(\Gamma\times A)) &
\end{tikzcd}$$

Where $\alpha : ((\_ \times \_) \times \_) \to (\_ \times (\_ \times \_))$ is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle \tag{45}$$
$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2\rangle, \pi_2 \circ \pi_2\rangle \tag{46}$$

So:

$$\begin{aligned}
G =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ \alpha^{-1}_{\Gamma,\Gamma,(T_{\epsilon_1}A)} \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By definition of } \alpha \text{ and products} \\
=& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By tensor strength's left-naturality} \\
=& T_{\epsilon_1}((\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle
\end{aligned} \tag{47}$$

Since

$$rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{48}$$

We Have

$$\begin{aligned}
rhs &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \mathtt{Id}_B)) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By Left-Tensor Stre} \\
&= lhs \quad \text{Woohoo!}
\end{aligned} \tag{49}$$

Case Eta Let

$$f = [\![\Gamma \vdash v \colon A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \tag{50}$$

By weakening, we have

$$[\![\Gamma, x : A \vdash v \colon A \to \mathtt{M}_\epsilon B]\!]_M = f \circ \pi_1 : \Gamma \times A \to (T_\epsilon B)^A \tag{51}$$
$$[\![\Gamma, x : A \vdash v\ x \colon \mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle \tag{52}$$
$$\tag{53}$$

Hence, we have

$$\begin{aligned} [\![\Gamma \vdash \lambda x : A.(v\ x) \colon A \to \mathtt{M}_\epsilon B]\!]_M &= \mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \\ \mathtt{app} \circ ([\![\Gamma \vdash \lambda x : A.(v\ x) \colon A \to \mathtt{M}_\epsilon B]\!]_M \times \mathtt{Id}_A) &= \mathtt{app} \circ (\mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \times \mathtt{Id}_A) \\ &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \\ &= \mathtt{app} \circ (f \times \mathtt{Id}_A) \end{aligned} \tag{54}$$

Hence, by the fact that $\mathtt{cur}(f)$ is unique in a cartesian closed category,

$$[\![\Gamma \vdash \lambda x : A.(v\ x) \colon A \to \mathtt{M}_\epsilon B]\!]_M = f = [\![\Gamma \vdash v \colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{55}$$

Case If-True Let

$$f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M \tag{56}$$
$$g = [\![\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A]\!]_M \tag{57}$$
$$\tag{58}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{true},A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inl} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ ((\mathtt{cur}(f \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ (\mathtt{cur}(f \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= f \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\ &= f \\ &= [\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{59}$$

Case If-False Let

$$f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M \tag{60}$$
$$g = [\![\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A]\!]_M \tag{61}$$
$$\tag{62}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{true},A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inr} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ ((\mathtt{cur}(g \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= \mathtt{app} \circ (\mathtt{cur}(g \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= g \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\ &= g \\ &= [\![\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{63}$$

0.3.3 Case If-Eta

Let

$$f = [\![\Gamma \vdash v \colon \mathtt{Bool}]\!]_M \tag{64}$$
$$g = [\![\Gamma, x : \mathtt{Bool} \vdash C \colon \mathtt{M}_\epsilon A]\!]_M \tag{65}$$
$$\tag{66}$$

Then by the substitution theorem,

$$[\![\Gamma \vdash C\,[\mathtt{true}/x] : \mathtt{M}_\epsilon A]\!]_M = g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_{\mathtt{1}} \circ \langle\rangle_\Gamma \rangle \tag{67}$$
$$[\![\Gamma \vdash C\,[\mathtt{false}/x] : \mathtt{M}_\epsilon A]\!]_M = g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_{\mathtt{1}} \circ \langle\rangle_\Gamma \rangle \tag{68}$$
$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon A]\!]_M = g \circ \langle \mathtt{Id}_\Gamma, f \rangle \tag{69}$$

Hence we have (Using the diagonal and twist morphisms):

$$[\![\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C\,[\mathtt{true}/x]\ \mathtt{else}\ C\,[\mathtt{false}/x] : \mathtt{M}_\epsilon A]\!]_M \tag{70}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_{\mathtt{1}} \circ \langle\rangle_\Gamma\rangle \circ \pi_2), \mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_{\mathtt{1}} \circ \langle\rangle_\Gamma\rangle \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{71}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_2\rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_2\rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Pairing property} \tag{72}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_1\rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_1\rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Terminal is unique} \tag{73}$$
$$= \mathtt{app} \circ (\left(\left[\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inl}_{\mathtt{1}} \circ \langle\rangle_{\mathtt{1}})) \circ \tau_{\mathtt{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inr}_{\mathtt{1}} \circ \langle\rangle_{\mathtt{1}})) \circ \tau_{\mathtt{1},\Gamma})\right] \circ f\right) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of t} \tag{74}$$
$$= \mathtt{app} \circ (\left(\left[\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inl}_{\mathtt{1}}) \circ \tau_{\mathtt{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inr}_{\mathtt{1}}) \circ \tau_{\mathtt{1},\Gamma})\right] \circ f\right) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Identity} = \mathtt{Id}_{\mathtt{1}} \tag{75}$$
$$= \mathtt{app} \circ (\left(\left[\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (\mathtt{inl}_{\mathtt{1}} \times \mathtt{Id}_\Gamma)), \mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (\mathtt{inr}_{\mathtt{1}} \times \mathtt{Id}_\Gamma))\right] \circ f\right) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Twist commutivity} \tag{76}$$
$$= \mathtt{app} \circ (\left(\left[\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ \mathtt{inl}_{\mathtt{1}}, \mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ \mathtt{inr}_{\mathtt{1}}\right] \circ f\right) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Exponential property} \tag{77}$$
$$= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ [\mathtt{inl}_{\mathtt{1}}, \mathtt{inr}_{\mathtt{1}}] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring out } \mathtt{cur}(..) \tag{78}$$
$$= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Since } [\mathtt{inl}, \mathtt{inr}]\, \textit{istheidentity} \tag{79}$$
$$= \mathtt{app} \circ (\mathtt{cur}(g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma}) \times \mathtt{Id}_\Gamma) \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring} \tag{80}$$
$$= g \circ \tau_{\mathtt{1}+\mathtt{1},\Gamma} \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of } \mathtt{app}, \mathtt{cur}(..) \tag{81}$$
$$= g \circ (\mathtt{Id}_\Gamma \times f) \circ \tau_{\Gamma,\Gamma} \circ \delta_\Gamma \quad \text{Twist commutivity} \tag{82}$$
$$= g \circ (\mathtt{Id}_\Gamma \times f) \circ \langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \quad \text{Twist, diagonal defintions} \tag{83}$$
$$= g \circ \langle \mathtt{Id}_\Gamma, f \rangle \tag{84}$$
$$= [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon A]\!]_M \tag{85}$$
$$\tag{86}$$

0.3.4 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of sub-expressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

Case Lambda By inversion, we have $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 : \mathtt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma, x : A \vdash C_1 : \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 : \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma, x : A \vdash C_1 : \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 : \mathtt{M}_\epsilon B]\!]_M \tag{87}$$

And so

$$[\![\Gamma \vdash \lambda x : A.C_1 : A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) = [\![\Gamma \vdash \lambda x : A.C_2 : A \to \mathtt{M}_\epsilon B]\!]_M \tag{88}$$

Case Return By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M \tag{89}$$

And so

$$[\![\Gamma \vdash \texttt{return} v_1 \colon \texttt{M}_{\mathbf{1}} A]\!]_M = \eta_A \circ f = [\![\Gamma \vdash \texttt{return} v_2 \colon \texttt{M}_{\mathbf{1}} A]\!]_M \tag{90}$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \texttt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A \to \texttt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \texttt{M}_\epsilon B]\!]_M$ and $[\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A \to \texttt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \texttt{M}_\epsilon B]\!]_M \tag{91}$$

$$g = [\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M \tag{92}$$

And so

$$[\![\Gamma \vdash v_1\ v_2 \colon \texttt{M}_\epsilon A]\!]_M = \texttt{app} \circ \langle f, g \rangle = [\![\Gamma \vdash v_1'\ v_2' \colon \texttt{M}_\epsilon A]\!]_M \tag{93}$$

Case Bind By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \texttt{M}_\epsilon A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \texttt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \texttt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \texttt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \texttt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \texttt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash C_1 \colon \texttt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \texttt{M}_{\epsilon_1} A]\!]_M \tag{94}$$

$$g = [\![\Gamma, x : A \vdash C_2 \colon \texttt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \texttt{M}_{\epsilon_2} B]\!]_M \tag{95}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \\ &= [\![\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{96}$$

Case If By inversion, we have $\Gamma \vdash v =_{\beta\eta} v' \colon \texttt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \texttt{M}_\epsilon A$ and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \texttt{M}_\epsilon A$ By induction, we therefore have $[\![\Gamma \vdash v \colon \texttt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M$, $[\![\Gamma \vdash C_1 \colon \texttt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \texttt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \texttt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \texttt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash v \colon \texttt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M \tag{97}$$

$$g = [\![\Gamma \vdash C_1 \colon \texttt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \texttt{M}_{\epsilon_1} A]\!]_M \tag{98}$$

$$h = [\![\Gamma, x : A \vdash C_2 \colon \texttt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \texttt{M}_{\epsilon_2} B]\!]_M \tag{99}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \texttt{if}_{\epsilon, A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 \colon]\!]_{\mathbf{M}} &= \texttt{app} \circ (([\texttt{cur}(g \circ \pi_2), \texttt{cur}(h \circ \pi_2)] \circ f) \times \texttt{Id}_\Gamma) \circ \delta_\Gamma \\ &= [\![\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{100}$$

Case Subtype By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, and $A \leq: B$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{101}$$

$$g = [\![A \leq: B]\!]_M \tag{102}$$

And so

$$[\![\Gamma \vdash v_1 \colon B]\!]_M = g \circ f = [\![\Gamma \vdash v_1 \colon B]\!]_M \tag{103}$$

Case subeffect By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_{\epsilon_1} A$, and $A \leq: B$ and $\epsilon_1 \leq \epsilon_2$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon_1} A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{104}$$

$$g = [\![A \leq: B]\!]_M \tag{105}$$

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{106}$$

And so

$$[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = h_B \circ T_{\epsilon_1} g \circ f = [\![\Gamma \vdash C_2 \mathtt{M}_{\epsilon_2} B \colon]\!]_{\mathbf{M}} \tag{107}$$

$\square$