0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v\colon A$,there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Proof: We induct on the structure of terms.

Case Variables To find the unique derivation of $\Phi \mid \Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is, if $A' \leq:_\Phi A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\frac{\Gamma', x:A'\,\texttt{Ok}}{\Phi \mid \Gamma, x:A' \vdash x:A'} \quad A' \leq: A}{\Phi \mid \Gamma', x : A' \vdash x\colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x\colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi \mid \Gamma' \vdash x:A'} \quad A' \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\frac{()\frac{\Delta}{\Phi \mid \Gamma, x:A' \vdash x:A'}}{\Phi \mid \Gamma \vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} \tag{3}$$

Case Constants For each of the constants, ($\texttt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\texttt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma\texttt{Ok}}{\Gamma \vdash \texttt{C}^A:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash \texttt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_\Phi A$).

Case Lambda The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v\colon A' \to B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Phi \mid \Gamma, x:A \vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x:A.B:A \to B} \quad A \to B \leq:_\Phi A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A' \to B'}$$

Where

$$\text{(Sub-Type)}\frac{()\frac{\Delta}{\Phi \mid \Gamma, x:A \vdash v:B} \quad B \leq:_\Phi B'}{\Phi \mid \Gamma, x : A \vdash v\colon B'} \tag{4}$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v \colon B'$ if it exists.

0.2.1 Computation Terms

Case Return The reduced derivation of $\Phi \mid \Gamma \vdash \texttt{return} v \colon \texttt{M}_{\epsilon} B$ is

$$(\text{Subtype})\frac{(\text{Return})\dfrac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A}}{\Phi|\Gamma\vdash\texttt{return}v:\texttt{M}_{\texttt{1}}A} \quad (\text{Computation})\dfrac{A\leq:_{\Phi}B \quad \texttt{1}\leq_{\Phi}\epsilon}{\texttt{M}_{\texttt{1}}A\leq_{\Phi}\texttt{M}_{\epsilon}B}}{\Phi \mid \Gamma \vdash \texttt{return} v \colon B}$$

Where

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq: B}{\Phi \mid \Gamma \vdash v \colon B}$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v \colon B$

Case Apply If

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:A\to B} \quad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1 \colon A' \to B'}$$

and

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_2:A''} \quad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2 \colon A'}$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1 \colon A' \to B'$ and $\Phi \mid \Gamma \vdash v_2 \colon A'$
Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2 \colon \texttt{M}_{\epsilon'} B'$ as

$$(\text{Subeffect})\frac{(\text{Apply})\dfrac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:A\to B} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A''} \quad A''\leq:_{\Phi}A}{\Phi|\Gamma\vdash v:A}}{\Phi|\Gamma\vdash v_1\ v_2:B} \quad B \leq:_{\Phi} B' \quad \epsilon \leq_{\Phi} \epsilon'}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon \texttt{M}_{\epsilon'} B'}$$

Case If Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:B} \quad B \leq: \texttt{Bool}}{\Phi \mid \Gamma \vdash v \colon \texttt{Bool}} \tag{5}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_1:A'} \quad A' \leq: A}{\Phi \mid \Gamma \vdash v_1 \colon A} \tag{6}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_2:A''} \quad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2 \colon A} \tag{7}$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v \colon \texttt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$.
Then the only reduced derivation of $\Phi \mid \Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A$ is:
TODO: Scale this properly

$$(\text{Subtype})\frac{(\text{If})\dfrac{(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:B} \quad B\leq:\texttt{Bool}}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_1:A'} \quad A'\leq:A}{\Phi|\Gamma\vdash v_1:A} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_2:A''} \quad A''\leq:A}{\Phi|\Gamma\vdash v_2:A}}{\Phi|\Gamma\vdash\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2:A \quad \epsilon\leq_{\Phi}\epsilon \quad A\leq:_{\Phi}A}}{\Phi \mid \Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{8}$$

Case Bind Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A} \quad (\text{Computation})\frac{A\leq:_\Phi A' \quad \epsilon_1\leq_\Phi\epsilon_1'}{\mathtt{M}_{\epsilon_1}A\leq:_\Phi\mathtt{M}_{\epsilon_1'}A'}}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B} \quad (\text{Computation})\frac{B\leq:_\Phi B' \quad \epsilon_2\leq_\Phi\epsilon_2'}{\mathtt{M}_{\epsilon_2}B\leq:_\Phi\mathtt{M}_{\epsilon_2'}B'}}{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms]

By weakening, $\iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2 \colon B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$(\text{Subeffect})\frac{(\text{Bind})\frac{(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A} \quad A\leq:A' \quad \epsilon_1\leq\epsilon_1'}{\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'}A'} \quad (\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B} \quad B\leq:B' \quad \epsilon_2\leq\epsilon_2'}{\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2'}B'}}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \quad B \leq:_\Phi B' \quad \epsilon_1\cdot\epsilon_2 \leq_\Phi \epsilon_1'\cdot\epsilon_2'}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B'} \tag{11}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v \colon A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

0.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced.

$\mathit{reduce}((\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A}) = (\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A}$

0.3.2 Value Types

Var

$$\mathit{reduce}((\text{Var})\frac{\Gamma\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A}) = (\text{Var})\frac{\Gamma\mathtt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A} \tag{12}$$

Weaken

reduce **definition** To find:

$$\mathit{reduce}((\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash x:A}}{\Phi \mid \Gamma, y : B \vdash x \colon A}) \tag{13}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x \colon A} = \mathit{reduce}(\Delta) \tag{14}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A'}}{\Phi|\Gamma,y:B\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma, y : B \vdash x \colon A} \tag{15}$$

Lambda

reduce **definition** To find:

$$reduce((\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to \epsilon_2 B}) \tag{16}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:\mathtt{M}_{\epsilon_1}B'} \quad \epsilon_1 \leq_\Phi \epsilon_2 \quad B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A \vdash v\colon \mathtt{M}_{\epsilon_2}B} = reduce(\Delta) \tag{17}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:\mathtt{M}_{\epsilon_1}B'} \quad A \to \epsilon_1 B' \leq:_\Phi A \to \epsilon_2 B}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to \epsilon_2 B} \tag{18}$$

Subtype

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B}) \tag{19}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{20}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B} \tag{21}$$

0.3.3 Computation Types

Return

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return}v\colon \mathtt{M}_{\mathtt{1}}A}) \tag{22}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v\colon A} = reduce(\Delta) \tag{23}$$

In

$$(\text{Sub-effect})\frac{(\text{Return})\frac{\Delta'}{\Phi|\Gamma\vdash v:A} \quad \mathtt{1} \leq_\Phi \mathtt{1} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash \mathtt{return}v\colon \mathtt{M}_{\mathtt{1}}A} \tag{24}$$

Apply

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B}) \tag{25}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:A'\to B'} \quad A'\to B' \leq:_\Phi A\to \epsilon B}{\Phi \mid \Gamma \vdash v_1\colon A\to B} = reduce(\Delta_1) \tag{26}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v_1\colon A} = reduce(\Delta_2) \tag{27}$$

In

$$(\text{Sub-effect})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:A'\to B'} \quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Phi|\Gamma\vdash v_2:A''} \quad A''\leq:_\Phi A \leq:_\Phi A'}{\Phi|\Gamma\vdash v_2:A'}}{\Phi|\Gamma\vdash v_1\ v_2:\mathtt{M}_{\epsilon'}B'} \quad \epsilon' \leq_\Phi \epsilon \quad B' \leq:_\Phi B}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{28}$$

If

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{reduce(\Delta_3)}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A} \tag{29}$$

Bind

reduce **definition** To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{30}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'}A'} \quad \epsilon_1' \leq:_\Phi \epsilon_1 \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1}A} = reduce(\Delta_1) \tag{31}$$

Since $i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq:_\Phi A$, and by Δ_2, $\Phi \mid (\Gamma, x : A) \vdash v_2\colon \mathtt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2\colon \mathtt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'}B'} \quad \epsilon_2' \leq:_\Phi \epsilon_2 \quad B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \mathtt{M}_{\epsilon_2}B} = reduce(\Delta_3) \tag{32}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-effect})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'}A'} \quad ()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'}B'}}{\Phi|\Gamma\vdash \mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1'\cdot\epsilon_2'}B} \quad B' \leq:_\Phi B \quad \epsilon_1'\cdot\epsilon_2' \leq_\Phi \epsilon_1\cdot\epsilon_2}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{33}$$

0.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v\colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\Delta]\!]_M = [\![reduce\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Phi \mid \Gamma \vdash v\colon A]\!]_M$ is unique.