Chapter 1

Adequacy

An important result, though often not proved in presentations of denotational semantics, is that of adequacy. A denotational semantics is adequate if two terms having equal denotations implies that the terms are equationally equivalent. In this chapter, I shall introduce an instantiation of PEC, $\texttt{PEC}_{\texttt{put}}$ with a suitable denotational semantics, generated using the methods described in Chapter **TODO: Ref**. Following this, I shall introduce a set of logical relation between the denotations and closed terms. Using these relations, I shall prove that closed terms with the same denotations are equivalent under the equational equivalence given in **TODO: ref**.

1.1 Instantiation of the Polymorphic Effect Calculus

Let us instantiate the polymorphic effect calculus to be a language in which one can write programs which can create an output signal using the statement `put`. The effect system of EC is then used to count an upper bound on the number of outputs that a program can make. This language shall be called $\texttt{PEC}_{\texttt{put}}$.

1.1.1 Ground Types

We simply use the basic ground types.

$$\gamma ::= \texttt{Bool} \mid \texttt{Unit} \tag{1.1}$$

1.1.2 Graded Monad

We index the base graded monad with a partially ordered monoid derived from the natural numbers. This base monoid is extended as described in section **TODO: Ref** to symbolically include variables α, β, γ which range over the natural numbers. The abstract effect of a term therefore computes an upper bound on the number of outputs produced. This means that the $\texttt{do}\ x \leftarrow v\ \texttt{in}\ v'$ type rule adds together the upper bounds on the two expressions to give an upper bound on the number of outputs of the sequenced expression. The $\texttt{return}\ v$ type rule acknowledges that a pure expression does not have any output. Subtyping allows the type system to compute a looser bound.

$$E = (\mathbb{N}, 0, +, \leq) \tag{1.2}$$

1.1.3 Constants

We extend the set of constant, built in expressions to include a `put` statement which performs a single output action.

$$\mathtt{k}^{A} ::= \mathtt{true}^{\mathtt{Bool}} \mid \mathtt{false}^{\mathtt{Bool}} \mid ()^{\mathtt{Unit}} \mid \mathtt{put}^{\mathtt{M_1 Unit}} \tag{1.3}$$

1.1.4 Subtyping

The ground subtyping relation is the identity relation, meaning the `Unit` and `Bool` types are subtypes only of themselves. This is extended using the subeffect and function subtyping rules given in **TODO: Ref** to give a full subtyping relation.

1.2 Instantiation of a Model of the Polymorphic Effect Calculus

Let us now instantiate a model of $\mathtt{PEC}_{\mathtt{put}}$ in the indexed category derived as in Chapter **TODO: ref** from a model of the non-polymorphic version of $\mathtt{PEC}_{\mathtt{put}}$ in `Set`, the category of sets and functions. To do this, we build an S-category for $\mathtt{PEC}_{\mathtt{put}}$. Since `Set` is already a CCC and has co-products, we simply need to define a strong graded monad, a denotation for `put`, and the subeffecting natural transformations.

1.2.1 Non-Polymorphic Model

Graded Monad The strong graded monad on `Set` is given by tagging values of the underlying type with the number of output operations required to compute that value.

$$\mathtt{T}^0_n A = \{(n', a) \mid n' \leq n \wedge a \in A\} \tag{1.4}$$
$$\mu^0_{m,n,A} = (m', (n', a)) \mapsto (n' + m', a) \tag{1.5}$$
$$\eta^0_A = a \mapsto (0, a) \tag{1.6}$$
$$\mathtt{t}^0_{n,A,B} = (a, (n', b)) \mapsto (n', (a, b)) \tag{1.7}$$

Subeffecting Natural Transformations These natural transformations are given by inclusion functions (identities), since $n \leq m \wedge (n', a) \in \mathtt{T}^0_n A \implies (n' \leq n \leq m, a \in A) \implies (n', a) \in \mathtt{T}^0_m A$. Other subtyping morphisms are generated using the usual method according to the subtype derivation (**TODO: Ref**).

Ground Denotations We define denotations for ground types an ground terms as in Figure **TODO: Figurify + put horizontal**

$$[\![\mathtt{Unit}]\!] = \{*\}$$
$$[\![\mathtt{Bool}]\!] = \{\top, \bot\}$$

$$
\begin{aligned}
[\![()]\!] &= * \mapsto * \\
[\![\texttt{true}]\!] &= * \mapsto \top \\
[\![\texttt{false}]\!] &= * \mapsto \bot \\
[\![\texttt{put}]\!] &= * \mapsto (1, *)
\end{aligned}
$$

1.2.2 Making the Model Polymorphic

Using the methods explained in Chapter **TODO: ref**, we can lift this model into an indexed S-Category using the functor categories $[E^n, \texttt{Set}]$. Hence the model can be proved sound for $\texttt{PEC}_{\texttt{put}}$.

1.3 Programming with Put

We can now introduce some syntactic sugar for $\texttt{PEC}_{\texttt{put}}$. Specifically, we introduce a notion of powers of $\texttt{put}$ in Definition 1.3.1. These powers have simple denotations in our indexed category. Due to soundness, all terms in the equivalence class have the same denotation.

Definition 1.3.1 (Powers of Put as an Equational Equivalence Class)**.** *Define put^n as follows:*

$$
\begin{aligned}
\Phi \mid \Gamma \vdash put^0 &\approx return\ () : M_0\, Unit \\
\Phi \mid \Gamma \vdash put^{m+1} &\approx do\ \_ \leftarrow put^n\ in\ put : M_{m+1}\, Unit
\end{aligned}
$$

Lemma 1.3.1 (Denotations of Powers of Put)**.** $[\![\Phi \mid \Gamma \vdash put^m : M_m\, Unit]\!] = \vec{\epsilon} \mapsto \rho \mapsto (m, *)$

Proof: By induction on m.

Case 0:

$$
[\![\Phi \mid \Gamma \vdash \texttt{put}^0 : \texttt{M}_0\texttt{Unit}]\!](\vec{\epsilon})(\rho) = \eta^n(*) = (0, *) \tag{1.8}
$$

Case m+1:

$$
\begin{aligned}
[\![\Phi \mid \Gamma \vdash \texttt{put}^{m+1} : \texttt{M}_0\texttt{Unit}]\!](\vec{\epsilon})(\rho) &= (\mu^n \circ \texttt{T}^n_m([\![\diamond \vdash \texttt{put} : \texttt{M}_1\texttt{Unit}]\!] \circ \pi_1) \circ \texttt{t}^n) \\
&\quad (*, [\![\Phi \mid \Gamma \vdash \texttt{put}^m : \texttt{M}_n\texttt{Unit}]\!](\vec{\epsilon})(\rho)) \\
&= (m + 1, *)
\end{aligned}
$$

1.4 Logical Relations

It is difficult to directly prove adequacy. This is because we cannot directly perform induction over morphisms. So instead we use a system of logical relations which relate elements of denotations of types to closed terms of the same type (Definition 1.4.1). Specifically we define a relation for a given type, effect environment, and tuple of ground effects. Since they are defined structurally with respect the type, we can induct over the structure to prove properties. We then show that these logical relations are preserved by subtyping (**TODO: Ref**) and that they behave well with respect to single effect-variable substitutions (**TODO: Ref**). Finally, we prove the fundamental property (**TODO: Ref**) which shows that the denotation of a term relates to the term itself. In **TODO: ref**, we finally use the logical relations to prove adequacy.

Definition 1.4.1 (Logical Relation)**.**

$$R_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon} : E^n) \in \wp(\llbracket \Phi \vdash A : \mathtt{Type} \rrbracket \vec{\epsilon} \times PEC^{A[\epsilon/\alpha]}_{\mathtt{put}}) \tag{1.9}$$

$$\begin{aligned}
(d, v) \in R_{\Phi \vdash \forall \alpha.A:\mathtt{Type}}(\vec{\epsilon}) &\leftrightarrow \forall \epsilon.(\pi_\epsilon(d), v\ \epsilon) \in R_{\Phi \vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\
(d, v) \in R_{\Phi \vdash \mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon}) &\Leftrightarrow (d = * \wedge \vdash v \approx () : \mathtt{Unit}) \\
(d, v) \in R_{\Phi \vdash \mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon}) &\Leftrightarrow ((d = \top \wedge \vdash v \approx \mathtt{true} : \mathtt{Bool}) \\
&\qquad \vee (d = \bot \wedge \vdash v \approx \mathtt{false} : \mathtt{Bool})) \\
(d, v) \in R_{\Phi \vdash A \to B:\mathtt{Type}}(\vec{\epsilon}) &\Leftrightarrow \forall e, u.(e, u) \in R_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon}) \implies (d(e), v\ u) \in R_{\Phi \vdash B:\mathtt{Type}}(\vec{\epsilon}) \\
(d, v) \in R_{\Phi \vdash M_f A:\mathtt{Type}}(\vec{\epsilon}) &\Leftrightarrow (d = (n, d') \\
&\qquad \wedge \exists v'.((d', v') \in R_{\Phi \vdash A:\mathtt{Type}} \\
&\qquad \wedge \vdash v \approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ v' \ : (M_f A)[\vec{\epsilon}/\vec{\alpha}]))
\end{aligned}$$

Theorem 1.4.1 (Logical Relation and Subtyping)**.** *If* $A {\leq:}_\Phi B$ *and* $(d, v) \in R_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon})$ *then* $(d, v) \in R_{\Phi \vdash B:\mathtt{Type}}(\vec{\epsilon})$

Proof: By induction on the derivation of $A {\leq:}_\Phi B$.

Case S-Ground: $A {\leq:}_\Phi B \implies A = B$, since ground subtyping is the identity relation.

Case S-Fn: $A {\leq:}_\Phi B \implies A = A_1 \to A_2, B = B_1 \to B_2$ where $B_1 {\leq:}_\Phi A_1$ and $A_2 {\leq:}_\Phi B_2$.

By the definition of the $\lhd_{\Phi \vdash A \to B:\mathtt{Type}}$ relation, $(d, v) \in \mathtt{R}_{\Phi \vdash A \to B:\mathtt{Type}}(\vec{\epsilon}) \Leftrightarrow (\forall e, u.(e, u) \in \mathtt{R}_{\Phi \vdash A:\mathtt{Type}}(\vec{\epsilon}) \implies (d(e), v\ u) \in \mathtt{R}_{\Phi \vdash B:\mathtt{Type}}(\vec{\epsilon}))$.

So

$$
\begin{aligned}
\forall e, u.(e,u) \in \mathtt{R}_{\Phi \vdash B_1 : \mathtt{Type}}(\vec{\epsilon}) &\implies (e,u) \in \mathtt{R}_{\Phi \vdash A_1 : \mathtt{Type}}(\vec{\epsilon}) \quad \text{By induction } B_1 \leq :_\Phi A_1 \\
&\implies (d(e), u\ v) \in \mathtt{R}_{\Phi \vdash A_2 : \mathtt{Type}}(\vec{\epsilon}) \quad \text{By definition} \\
&\implies (d(e), u\ v) \in \mathtt{R}_{\Phi \vdash B_2 : \mathtt{Type}}(\vec{\epsilon}) \quad \text{By induction } A_2 \leq :_\Phi B_2
\end{aligned}
$$

As required.

Case S-Effect: $\mathtt{M}_{n_1} A_1 \leq :_\Phi \mathtt{M}_{n_2} A_2 \implies n_1 \leq n_2, A_1 \leq :_\Phi A_2$

$$
\begin{aligned}
(d,v) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_{n_1} A_1 : \mathtt{Type}}(\vec{\epsilon}) &\implies d = (n_1', d') \wedge n_1' \leq n_1 \leq n_2 \\
&\quad \wedge \exists v'.((d', v') \in \mathtt{R}_{\Phi \vdash A_1 : \mathtt{Type}}(\vec{\epsilon}) \\
&\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ v' \ : \mathtt{M}_{n_1} A_1) \\
&\implies \vdash v_1' : A_2 \wedge (d', v') \in \mathtt{R}_{\Phi \vdash A_2 : \mathtt{Type}}(\vec{\epsilon}) \\
&\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ v' \ : \mathtt{M}_{n_1} A_2 \\
&\implies (d,v) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_{n_2} A_2 : \mathtt{Type}}(\vec{\epsilon})
\end{aligned}
$$

Case S-Quantification: $\forall \alpha. A_1 \leq :_\Phi \forall \alpha. A_2 \implies A_1 \leq : A_2$

So:

$$
\begin{aligned}
(d,v) \in \mathtt{R}_{\Phi \vdash \forall \alpha. A_1 : \mathtt{Type}}(\vec{\epsilon}) &\implies \forall \epsilon.(\pi_\epsilon(d), v\ \epsilon) \in \mathtt{R}_{\Phi \vdash A_1[\alpha/\epsilon] : \mathtt{Type}}(\vec{\epsilon}) \\
&\implies \forall \epsilon(\pi_\epsilon(d), v\ \epsilon) \in \mathtt{R}_{\Phi \vdash A_2[\alpha/\epsilon] : \mathtt{Type}}(\vec{\epsilon}) \\
&\implies (d,v) \in \mathtt{R}_{\Phi \vdash \forall \alpha. A_2 : \mathtt{Type}}(\vec{\epsilon})
\end{aligned}
$$

Lemma 1.4.2 (Environment Lemma)**.**

$$
R_{\Phi, \alpha \vdash A : Type}(\vec{\epsilon}, \epsilon) = R_{\Phi \vdash A[\alpha/\epsilon] : Type}(\vec{\epsilon})
$$

Proof: By induction over the type A, proving that $(d,v) \in \mathtt{R}_{\Phi, \alpha \vdash A : \mathtt{Type}}(\vec{\epsilon}, \epsilon) \Leftrightarrow (d,v) \in \mathtt{R}_{\Phi \vdash A[\epsilon/\alpha] : \mathtt{Type}}(\vec{\epsilon})$

Case Bool:

$$
\begin{aligned}
(d,v) \in \mathtt{R}_{\Phi, \alpha \vdash \mathtt{Bool} : \mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow (d = \top \wedge \vdash v \approx \mathtt{true} : \mathtt{Bool}) \\
&\quad \wedge (d = \bot \wedge \vdash v \approx \mathtt{false} : \mathtt{Bool}) \\
&\Leftrightarrow (d,v) \in \mathtt{R}_{\Phi \vdash \mathtt{Bool} : \mathtt{Type}}(\vec{\epsilon})
\end{aligned}
$$

Case Unit:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash\mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow (d = * \wedge \vdash v \approx \mathtt{()}:\mathtt{Unit}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash\mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Fn:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash A\to B:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow \forall(e, u) \in \mathtt{R}_{\Phi,\alpha\vdash A:\mathtt{Type}}(\vec{\epsilon}, \epsilon).(d(e), v\ u) \in \mathtt{R}_{\Phi,\alpha\vdash B:\mathtt{Type}}(\vec{\epsilon}, \epsilon) \\ &\Leftrightarrow \forall(e, u) \in \mathtt{R}_{\Phi\vdash A[\alpha/\epsilon]:\mathtt{Type}}(\vec{\epsilon}).(d(e), v\ u) \in \mathtt{R}_{\Phi\vdash B[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]\to B[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Effect:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash\mathtt{M}_f A:\mathtt{Type}}(\vec{\epsilon}, \epsilon) &\Leftrightarrow d = (n, d') \\ &\quad \wedge \exists v'.(d', v') \in \mathtt{R}_{\Phi,\alpha\vdash A:\mathtt{Type}}(\vec{\epsilon}, \epsilon) \\ &\quad\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ : \mathtt{M}_f A[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha] \\ &\Leftrightarrow \exists v'.(d', v') \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\ &\quad\quad \wedge \vdash v \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ : \mathtt{M}_f A[\epsilon/\alpha][\vec{\epsilon}/\vec{\alpha}] \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case T-Quantification:

$$\begin{aligned}(d, v) \in \mathtt{R}_{\Phi,\alpha\vdash\forall\beta.A:\mathtt{Type}}(\vec{\epsilon}, \epsilon_1) &\Leftrightarrow \forall\epsilon_2.(\pi_{\epsilon_2}(d), v\ \epsilon_2) \in \mathtt{R}_{\Phi,\alpha\vdash A[\epsilon_2/\beta]:\mathtt{Type}}(\vec{\epsilon}, \epsilon_1) \\ &\Leftrightarrow \forall\epsilon_2.(\pi_{\epsilon_2}(d), v\ \epsilon_2) \in \mathtt{R}_{\Phi\vdash A[\epsilon_2/\beta][\epsilon_1/\alpha]:\mathtt{Type}}(\vec{\epsilon}) \\ &\Leftrightarrow (d, v) \in \mathtt{R}_{\Phi\vdash\forall\beta.A[\epsilon_1/\alpha]:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

1.4.1 Fundamental Property

Theorem 1.4.3 (Fundamental Property). *Let* $R_{\Phi\vdash\Gamma\ \ \mathtt{Ok}}(\vec{\epsilon}) \in \wp([\![\Gamma]\!]\vec{\epsilon}) \times PEC_{\mathtt{put}}^{\Gamma[\vec{\epsilon}/\vec{\alpha}]})$ *mean:*

$$(\rho\sigma) \in R_{\Phi\vdash\Gamma\ \ \mathtt{Ok}}(\vec{\epsilon}) \Leftrightarrow \forall x.(\rho(x), \sigma(x)) \in R_{\Phi\vdash\Gamma(x):\mathtt{Type}}(\vec{\epsilon}) \tag{1.10}$$

If $(\rho\sigma) \in R_{\Phi\vdash\Gamma\ \ \mathtt{Ok}}(\vec{\epsilon})$ *then* $([\![\Phi \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in R_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$ *up to equational equivalence.*

Proof: By induction over the derivation of $\Phi \mid \Gamma \vdash v\colon A$

Case Variables:

$$\begin{aligned}\llbracket \Phi \mid \Gamma \vdash x\colon \Gamma(x) \rrbracket(\vec{\epsilon})(\rho) &= \rho(x)\\ x[\sigma][\vec{\epsilon}/\vec{\alpha}] &= \sigma(x)\end{aligned}$$

And $(\rho(x), \sigma(x)[\vec{\epsilon}/\vec{\alpha}]) \in \mathtt{R}_{\Phi\vdash\Gamma(x):\mathtt{Type}}(\vec{\epsilon})$.

Case Constants:

$$\begin{aligned}\llbracket \Phi \mid \Gamma \vdash \mathtt{true}\colon \mathtt{Bool} \rrbracket(\vec{\epsilon})(\rho) &= \top \wedge \vdash \mathtt{true}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{true}\colon \mathtt{Bool}\\ \llbracket \Phi \mid \Gamma \vdash \mathtt{false}\colon \mathtt{Bool} \rrbracket(\vec{\epsilon})(\rho) &= \bot \wedge \vdash \mathtt{false}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{false}\colon \mathtt{Bool}\\ \llbracket \Phi \mid \Gamma \vdash \mathtt{()}\colon \mathtt{Unit} \rrbracket(\vec{\epsilon})(\rho) &= * \wedge \vdash \mathtt{()}[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{()}\colon \mathtt{Unit}\\ \llbracket \Phi \mid \Gamma \vdash \mathtt{put}\colon \mathtt{M_1Unit} \rrbracket(\vec{\epsilon})(\rho) &= (1, *) \wedge \vdash \mathtt{put} \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^1\ \mathtt{in\ return\ ()}\ :\ \mathtt{M_1Unit}\end{aligned}$$

$$\begin{aligned}\text{So } (\llbracket \Phi \mid \Gamma \vdash \mathtt{true}\colon \mathtt{Bool} \rrbracket(\vec{\epsilon})(\rho), \mathtt{true}[\vec{\epsilon}/\vec{\alpha}][\sigma]) &\in \mathtt{R}_{\Phi\vdash\mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon})\\ \text{So } (\llbracket \Phi \mid \Gamma \vdash \mathtt{true}\colon \mathtt{Bool} \rrbracket(\vec{\epsilon})(\rho), \mathtt{false}[\vec{\epsilon}/\vec{\alpha}][\sigma]) &\in \mathtt{R}_{\Phi\vdash\mathtt{Bool}:\mathtt{Type}}(\vec{\epsilon})\\ \text{So } (\llbracket \Phi \mid \Gamma \vdash \mathtt{()}\colon \mathtt{Unit} \rrbracket(\vec{\epsilon})(\rho), \mathtt{()}[\vec{\epsilon}/\vec{\alpha}][\sigma]) &\in \mathtt{R}_{\Phi\vdash\mathtt{Unit}:\mathtt{Type}}(\vec{\epsilon})\\ \text{So } (\llbracket \Phi \mid \Gamma \vdash \mathtt{put}\colon \mathtt{M_1}U \rrbracket(\vec{\epsilon})(\rho), \mathtt{put}[\vec{\epsilon}/\vec{\alpha}][\sigma]) &\in \mathtt{R}_{\Phi\vdash\mathtt{M_1Unit}:\mathtt{Type}}(\vec{\epsilon})\end{aligned}$$

Case Subtype:

$$\llbracket \Phi \mid \Gamma \vdash v\colon B \rrbracket(\vec{\epsilon})(\rho) = \llbracket \Phi \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon})(\rho) \tag{1.11}$$

By induction, $(\llbracket \Phi \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$. Since, by the subtyping lemma (Lemma 1.4.1) $A \leq:_\Phi B \wedge (d, v) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon}) \implies (d, v) \in \mathtt{R}_{\Phi\vdash B:\mathtt{Type}}(\vec{\epsilon})$, we have that $(\llbracket \Phi \mid \Gamma \vdash v\colon B \rrbracket, v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash B:\mathtt{Type}}(\vec{\epsilon})$.

Case Fn: For all $(d, u) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$,

$$\begin{aligned}(\llbracket \Phi \mid \Gamma \vdash \lambda x\colon A.v : A \to B \rrbracket(\vec{\epsilon})(\rho))d &= (\mathtt{cur}(\llbracket \Gamma, x\colon A \vdash v\colon B \rrbracket)(\vec{\epsilon})(\rho))d\\ &= \llbracket \Gamma, x\colon A \vdash v\colon B \rrbracket((\vec{\epsilon})(\rho[x \mapsto d]))\end{aligned}$$

Since $(d, u) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$, we have $((\rho[x \mapsto d])(\sigma, x\colon= u)) \in \mathtt{R}_{\Phi\vdash\Gamma,x:A}\ \mathtt{Ok}(\vec{\epsilon})$, so by induction

$$(\llbracket \Gamma, x\colon A \vdash v\colon B \rrbracket(\vec{\epsilon})(\rho[x \mapsto d]), v[\vec{\epsilon}/\vec{\alpha}][\sigma, x\colon= u]) \in \mathtt{R}_{\Phi\vdash B:\mathtt{Type}}(\vec{\epsilon})$$

We can see that $\vdash v[\vec{\epsilon}/\vec{\alpha}][\sigma, x\colon= u] \approx ((\lambda x\colon A.v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma])\ u\colon B[\vec{\epsilon}/\vec{\alpha}]$.
Hence $((\llbracket \Phi \mid \Gamma \vdash \lambda x\colon A.v : A \to B \rrbracket(\vec{\epsilon})(\rho)), (\lambda x\colon A.v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A\to B:\mathtt{Type}}(\vec{\epsilon})$.

Case Apply:

$$[\![\Phi \mid \Gamma \vdash v\ u\colon B]\!](\vec{\epsilon})(\rho) = ([\![\Phi \mid \Gamma \vdash v\colon A \to B]\!](\vec{\epsilon})(\rho))([\![\Phi \mid \Gamma \vdash u\colon A]\!](\vec{\epsilon})(\rho)) \tag{1.12}$$

By induction $([\![\Phi \mid \Gamma \vdash v\colon A \to B]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A\to B:\mathtt{Type}}(\vec{\epsilon})$ and $([\![\Phi \mid \Gamma \vdash u\colon A]\!](\vec{\epsilon})(\rho), u[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$. So by the definition of $\mathtt{R}_{\Phi\vdash A\to B:\mathtt{Type}}$,

$$(([\![\Phi \mid \Gamma \vdash v\colon A \to B]\!](\vec{\epsilon})(\rho))([\![\Phi \mid \Gamma \vdash u\colon A]\!](\vec{\epsilon})(\rho)), (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ (u[\vec{\epsilon}/\vec{\alpha}][\sigma]))) \in \mathtt{R}_{\Phi\vdash B:\mathtt{Type}}(\vec{\epsilon})$$

Where $\vdash (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ (u[\vec{\epsilon}/\vec{\alpha}][\sigma]) \approx (v\ u)[\vec{\epsilon}/\vec{\alpha}][\sigma]\colon B[\vec{\epsilon}/\vec{\alpha}]$.

So $([\![\Phi \mid \Gamma \vdash v\ u\colon B]\!](\vec{\epsilon})(\rho), (v\ u)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash B:\mathtt{Type}}(\vec{\epsilon})$.

Case Return:

$$[\![\Phi \mid \Gamma \vdash v\colon \mathtt{M}_0 A]\!](\vec{\epsilon})(\rho) = (0, [\![\Phi \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon})(\rho)) \tag{1.13}$$

By induction, $([\![\Phi \mid \Gamma \vdash v\colon A]\!](\vec{\epsilon})(\rho), v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\mathtt{Type}}(\vec{\epsilon})$. So by picking $v' = v[\vec{\epsilon}/\vec{\alpha}][\sigma]$, we have

$$\vdash (\mathtt{return}\ v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{return}\ (v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^0\ \mathtt{in\ return}\ v' \quad : \mathtt{M}_0 A[\vec{\epsilon}/\vec{\alpha}] \tag{1.14}$$

So $([\![\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_0 A]\!](\vec{\epsilon})(\rho), (\mathtt{return}\ v\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash \mathtt{M}_0 A:\mathtt{Type}}(\vec{\epsilon})$

Case Bind: By inversion, $([\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{m+n} B]\!](\vec{\epsilon})(\rho)) = (m' + n', d_2)$, where $(n', d_2) = ([\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_n B]\!]((\vec{\epsilon})(\rho[x \mapsto d_1])))$, and $(n', d_1) = ([\![\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_m A]\!](\vec{\epsilon})(\rho))$.

By induction, $((m', d_1), v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash \mathtt{M}_m A:\mathtt{Type}}(\vec{\epsilon})$. So $\exists v_1'$ such that $\vdash v_1[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in\ return}\ v_1' \ : \mathtt{M}_m A$. So $(((\vec{\epsilon})(\rho[x \mapsto d_1])), ([\sigma], x\colon= v_1')) \in \mathtt{R}_{\Phi\vdash\Gamma,x:A:\mathtt{Type}}(\vec{\epsilon})$.

So by induction $([\![\Gamma, x\colon A \vdash v_2\colon \mathtt{M}_n B]\!]((\vec{\epsilon})(\rho[x \mapsto d_1])), v_2[\sigma, x\colon= v_1']) \in \mathtt{R}_{\Phi\vdash \mathtt{M}_n B:\mathtt{Type}}(\vec{\epsilon})$.

Hence, $\exists v_2'$ such that $\vdash v_2[\sigma, x\colon= v_1'] \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\mathtt{return}\ v_2' \ \ \mathtt{in} : \mathtt{M}_{m+n} B$ and $(d_2, v_2') \in \mathtt{R}_{\Phi\vdash \mathtt{M}_n B:\mathtt{Type}}(\vec{\epsilon})$.

Hence,

$$\begin{aligned}
\vdash \mathtt{do}\ x \leftarrow v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]\ \mathtt{in}\ v_2[\vec{\epsilon}/\vec{\alpha}][\sigma] \ &\approx \mathtt{do}\ x \leftarrow (\mathtt{do}\ _ \leftarrow \mathtt{put}^{m'}\ \mathtt{in\ return}\ v_1' \ )\ \mathtt{in}\ (v_2[\vec{\epsilon}/\vec{\alpha}][\sigma]) : \mathtt{M}_{m+n} B \\
&\approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ v_2[\sigma, x\colon= v_1'] \\
&\approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{m'+n'}\ \mathtt{in\ return}\ v_2'
\end{aligned}$$

So $([\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{m+n} B]\!](\vec{\epsilon})(\rho), (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash \mathtt{M}_{m+n} B:\mathtt{Type}}(\vec{\epsilon})$.

Case If: By inversion,

$$[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!](\vec{\epsilon})(\rho) = \begin{cases} [\![\Phi \mid \Gamma \vdash v_1\colon A]\!](\vec{\epsilon})(\rho) & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \top \\ [\![\Phi \mid \Gamma \vdash v_2\colon A]\!](\vec{\epsilon})(\rho) & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!](\vec{\epsilon})(\rho) = \bot \end{cases}$$

.

By induction,

$$(\llbracket \Phi \mid \Gamma \vdash b\colon \texttt{Bool} \rrbracket(\vec{\epsilon})(\rho), b[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash\texttt{Bool}:\texttt{Type}}(\vec{\epsilon})$$
$$(\llbracket \Phi \mid \Gamma \vdash v_1\colon A \rrbracket(\vec{\epsilon})(\rho), v_1[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\texttt{Type}}(\vec{\epsilon})$$
$$(\llbracket \Phi \mid \Gamma \vdash v_2\colon A \rrbracket(\vec{\epsilon})(\rho), v_2[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\texttt{Type}}(\vec{\epsilon})$$

Case: $\llbracket \Phi \mid \Gamma \vdash b\colon \texttt{Bool} \rrbracket(\vec{\epsilon})(\rho) = \top$ **and** $\vdash b[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \texttt{true}\colon \texttt{Bool}$

$$\llbracket \Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A \rrbracket(\vec{\epsilon})(\rho) = \llbracket \Phi \mid \Gamma \vdash v_1\colon A \rrbracket(\vec{\epsilon})(\rho) \tag{1.15}$$

And

$$\vdash v_1[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]\colon A[\vec{\epsilon}/\vec{\alpha}] \tag{1.16}$$

So

$$(\llbracket \Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A \rrbracket(\vec{\epsilon})(\rho), (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\texttt{Type}}(\vec{\epsilon}) \tag{1.17}$$

Case: $\llbracket \Phi \mid \Gamma \vdash b\colon \texttt{Bool} \rrbracket(\vec{\epsilon})(\rho) = \bot$ **and** $\vdash b[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx \texttt{false}\colon \texttt{Bool}$

$$\llbracket \Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A \rrbracket(\vec{\epsilon})(\rho) = \llbracket \Phi \mid \Gamma \vdash v_2\colon A \rrbracket(\vec{\epsilon})(\rho) \tag{1.18}$$

And

$$\vdash v_2[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]\colon A[\vec{\epsilon}/\vec{\alpha}] \tag{1.19}$$

So

$$(\llbracket \Phi \mid \Gamma \vdash \texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A \rrbracket(\vec{\epsilon})(\rho), (\texttt{if}_A\ b\ \texttt{then}\ v_1\ \texttt{else}\ v_2\ )[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash A:\texttt{Type}}(\vec{\epsilon}) \tag{1.20}$$

Case Effect-Gen: For all $\epsilon \in E$:

$$\pi_\epsilon(\llbracket \Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A \rrbracket(\vec{\epsilon})(\rho)) = \pi_\epsilon(\langle \llbracket \Phi, \alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon}, \epsilon') \rangle_{\epsilon' \in E}(\rho)) \tag{1.21}$$
$$= \llbracket \Phi, \alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon}, \epsilon)(\rho) \tag{1.22}$$

By induction, we know that $(\llbracket \Phi, \alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon}, \epsilon)(\rho), v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma]) \in \mathtt{R}_{\Phi,\alpha\vdash A:\texttt{Type}}(\vec{\epsilon}, \epsilon)$.

By the environment lemma **TODO: Ref**, $\mathtt{R}_{\Phi,\alpha\vdash A:\texttt{Type}}(\vec{\epsilon}, \epsilon) = \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$.

So $(\llbracket \Phi, \alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon}, \epsilon)(\rho), v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma]) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$

Since α does not appear in σ (Since σ is well formed under $\Phi = \diamond$), we can commute the substitutions: $v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma] = v[\vec{\epsilon}/\vec{\alpha}][\sigma][\epsilon/\alpha]$.

Hence:

$$\vdash v[\vec{\epsilon}/\vec{\alpha}][\epsilon/\alpha][\sigma] = v[\vec{\epsilon}/\vec{\alpha}][\sigma][\epsilon/\alpha] \approx \Lambda\alpha.(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon\colon A[\epsilon/\alpha][\vec{\epsilon}/\vec{\alpha}]$$

So $(\pi_\epsilon(\llbracket \Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A \rrbracket(\vec{\epsilon})(\rho)), \Lambda\alpha.(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon) \in \mathtt{R}_{\Phi\vdash A[\epsilon/\alpha]:\texttt{Type}}(\vec{\epsilon})$.

So $(\llbracket \Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A \rrbracket, (\Lambda\alpha.v)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi\vdash\forall\alpha.A:\texttt{Type}}(\vec{\epsilon})$.

Case Effect-Spec: Required to prove $(\llbracket \Phi \mid \Gamma \vdash v\ \epsilon \colon A[\epsilon/\alpha] \rrbracket(\vec{\epsilon})(\rho), (v\ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma] \approx (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon) \in \mathtt{R}_{\Phi \vdash A[\epsilon/\alpha] : \mathtt{Type}}(\vec{\epsilon})$.

By inversion and induction, $(\llbracket \Phi \mid \Gamma \vdash v \colon \forall\alpha.A \rrbracket, v[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash \forall\alpha.A : \mathtt{Type}}(\vec{\epsilon})$. So,

$$\forall \epsilon.(\pi_\epsilon(\llbracket \Phi \mid \Gamma \vdash v \colon \forall\alpha.A \rrbracket(\vec{\epsilon})(\rho)), (v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon) \in \mathtt{R}_{\Phi \vdash A[\epsilon/\alpha] : \mathtt{Type}}(\vec{\epsilon})$$

Let $\epsilon' = \llbracket \Phi \vdash \epsilon \colon \mathtt{Effect} \rrbracket \vec{\epsilon}$.

$$\begin{aligned}\llbracket \Phi \mid \Gamma \vdash v\ \epsilon \colon A[\epsilon/\alpha] \rrbracket(\vec{\epsilon})(\rho) &= \langle \mathtt{Id}, \llbracket \Phi \vdash \epsilon \colon \mathtt{Effect} \rrbracket \rangle^*(\boldsymbol{\epsilon}_{\Phi,\beta \vdash A[\beta/\alpha] : \mathtt{Type}})(\vec{\epsilon})(\llbracket \Phi \mid \Gamma \vdash v \colon \forall\alpha.A \rrbracket(\vec{\epsilon})(\rho)) \\ &= \pi_{\epsilon'}(\llbracket \Phi \mid \Gamma \vdash v \colon \forall\alpha.A \rrbracket(\vec{\epsilon})(\rho))\end{aligned}$$

Its also the case that

$$(v[\vec{\epsilon}/\vec{\alpha}][\sigma])\ \epsilon' = (v\ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma] \tag{1.23}$$

So

$$(\llbracket \Phi \mid \Gamma \vdash v\ \epsilon \colon A[\epsilon/\alpha] \rrbracket(\vec{\epsilon})(\rho), (v\ \epsilon)[\vec{\epsilon}/\vec{\alpha}][\sigma]) \in \mathtt{R}_{\Phi \vdash A[\epsilon/\alpha] : \mathtt{Type}}(\vec{\epsilon}) \tag{1.24}$$

1.5 Adequacy

Theorem 1.5.1 (Adequacy). *For types G defined as: $G ::= \mathtt{Bool} \mid \mathtt{Unit} \mid \mathtt{M}_n G$, equality of denotations implies equational equality.*

$$\llbracket \vdash v \colon G \rrbracket = \llbracket \vdash u \colon G \rrbracket \implies \vdash v \approx u \colon G \tag{1.25}$$

Proof: By induction on the structure of G, making use of the fundamental Property 1.4.1.

Case Boolean: Let $d = \llbracket \vdash v \colon \mathtt{Bool} \rrbracket = \llbracket \vdash v \colon \mathtt{Bool} \rrbracket \in \{\top, \bot\}$. By the fundamental property, $(d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{Bool} : \mathtt{Type}}(\vec{\epsilon})$ and $(d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{Bool} : \mathtt{Type}}(\vec{\epsilon})$.

Case: $d = \top$ Then $\vdash v \approx \mathtt{true} \approx u \colon \mathtt{Bool}$

Case: $d = \bot$ Then $\vdash v \approx \mathtt{false} \approx u \colon \mathtt{Bool}$

Case Unit: Let $* = \llbracket \vdash v \colon \mathtt{Unit} \rrbracket = \llbracket \vdash v \colon \mathtt{Unit} \rrbracket \in \{*\}$. By the fundamental property, $(d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{Unit} : \mathtt{Type}}(\vec{\epsilon})$ and $(d, v) \in \mathtt{R}_{\Phi \vdash \mathtt{Unit} : \mathtt{Type}}(\vec{\epsilon})$. Hence $\vdash v \approx () \approx u \colon \mathtt{Unit}$.

Case T-Effect: Let $(n', d) = \llbracket \vdash v \colon \mathtt{M}_n G \rrbracket = \llbracket \vdash u \colon \mathtt{M}_n G \rrbracket$. By the fundamental property, $(((n', d)), v) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_n G : \mathtt{Type}}(\vec{\epsilon})$ and $(((n', d)), u) \in \mathtt{R}_{\Phi \vdash \mathtt{M}_n G : \mathtt{Type}}(\vec{\epsilon})$. So there exists u', v' such that $(d', u') \in \mathtt{R}_{\Phi \vdash G : \mathtt{Type}}(\vec{\epsilon})$ and $(d', u') \in \mathtt{R}_{\Phi \vdash G : \mathtt{Type}}(\vec{\epsilon})$ and:

$$
\begin{aligned}
\vdash v &\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ v' \ \ : \mathtt{M}_n G \\
&\approx \texttt{do}\ \_ \leftarrow \texttt{put}^{n'}\ \texttt{in}\ \texttt{return}\ u' \\
&\approx u
\end{aligned}
$$

□