Contents

Chapter 1

Language Definition

1.1 Terms

1.1.1 Value Terms

$$
\begin{aligned}
v ::=& x \\
& \mid \lambda x : A.C \\
& \mid \mathtt{C}^{A} \\
& \mid \mathtt{()} \\
& \mid \mathtt{true} \mid \mathtt{false}
\end{aligned} \tag{1.1}
$$

1.1.2 Computation Terms

$$
\begin{aligned}
C ::=& \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \\
& \mid v_1\ v_2 \\
& \mid \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \\
& \mid \mathtt{return} v
\end{aligned} \tag{1.2}
$$

1.2 Type System

1.2.1 Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, \mathtt{1}, \leq)$ with elements ϵ

1.2.2 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Value Types

$$A, B, C ::= \gamma \mid A \to \mathtt{M}_{\epsilon} B$$

Computation Types Computation types are of the form $\mathtt{M}_{\epsilon} A$

1.2.3 Sub-typing

There exists a sub-typing pre-order relation $\leq:_\gamma$ over ground types that is:

- (Reflexive)$\frac{}{A \leq:_\gamma A}$
- (Transitive)$\frac{A \leq:_\gamma B \quad B \leq:_\gamma C}{A \leq:_\gamma C}$

We extend this relation with the function sub-typing rule to yield the full sub-typing relation $\leq:$

- (ground)$\frac{A \leq:_\gamma B}{A \leq: B}$
- (Fn)$\frac{A \leq: A' \quad B' \leq: B \quad \epsilon \leq \epsilon'}{A' \to \mathtt{M}_{\epsilon'} B' \leq: A \to \mathtt{M}_\epsilon B}$

1.2.4 Type Environments

An environment, $G ::= \diamond \mid \Gamma, x : A$

Domain Function

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

Ok Predicate

- (Atom)$\frac{}{\diamond \mathtt{Ok}}$
- (Var)$\frac{\Gamma \mathtt{Ok} \quad x \notin \mathtt{dom}(\Gamma)}{\Gamma, x : A \mathtt{Ok}}$

1.2.5 Type Rules

Value Typing Rules

- (Const)$\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A}$
- (Unit)$\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{()} : \mathtt{Unit}}$
- (True)$\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{true} : \mathtt{Bool}}$
- (False)$\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{false} : \mathtt{Bool}}$
- (Var)$\frac{\Gamma, x : A \mathtt{Ok}}{\Gamma, x : A \vdash X : A}$
- (Weaken)$\frac{\Gamma \vdash x : A}{\Gamma, y : B \vdash X : A}$(if $x \neq y$)
- (Fn)$\frac{\Gamma, x : A \vdash C : \mathtt{M}_\epsilon B}{\Gamma \vdash \lambda x : A . C : A \to \mathtt{M}_\epsilon B}$
- (Sub)$\frac{\Gamma \vdash v : A \quad A \leq: B}{\Gamma \vdash v : B}$

Computation typing rules

- (Return) $\frac{\Gamma\vdash v:A}{\Gamma\vdash\mathtt{return}\,v:\mathtt{M}_1 A}$
- (Apply) $\frac{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B \quad \Gamma\vdash v_2:A}{\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B}$
- (if) $\frac{\Gamma\vdash v:\mathtt{Bool} \quad \Gamma\vdash C_1:\mathtt{M}_\epsilon A \quad \Gamma\vdash C_2:\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ V\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A}$
- (Do) $\frac{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1} A \quad \Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2} B}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2} B}$
- (Subeffect) $\frac{\Gamma\vdash C:\mathtt{M}_{\epsilon_1} A \quad A\leq:B \quad \epsilon_1\leq\epsilon_2}{\Gamma\vdash C:\mathtt{M}_{e_2} B}$

1.2.6 Ok Lemma

If $\Gamma \vdash t{:}\,\tau$ then $\Gamma\mathtt{Ok}$.

Proof If $\Gamma, x : A\mathtt{Ok}$ then by inversion $\Gamma\mathtt{Ok}$ Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Gamma\mathtt{Ok}$. And all non-axiom derivations preserve the `Ok` property.

Chapter 2

Category Requirements

2.1 CCC

The section should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

2.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endofunctor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot\,\mathtt{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1,\epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

2.2.1 Left Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon,\mathtt{1},A}"] \\
& T_\epsilon A
\end{tikzcd}$$

2.2.2 Right Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1},\epsilon,A}"] \\
& T_\epsilon A
\end{tikzcd}$$

2.2.3 Associativity

$$\begin{tikzcd}
T_{\epsilon_1}T_{\epsilon_2}T_{\epsilon_3}A \arrow[r, "\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3}A}"] \arrow[d, "T_{\epsilon_1}\mu_{\epsilon_2,\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2}T_{\epsilon_3}A \arrow[d, "\mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,A}"] \\
T_{\epsilon_1}T_{\epsilon_2\cdot\epsilon_3}A \arrow[r, "\mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}A
\end{tikzcd}$$

2.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$$

Satisfying the following rules:

2.3.1 Left Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{Id}_A \times T_\epsilon f"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A \times T_\epsilon B' \arrow[d, "\mathtt{t}_{\epsilon,A,B'}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(\mathtt{Id}_A \times f)"] & T_\epsilon(A \times B')
\end{tikzcd}$$

2.3.2 Right Naturality

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "f \times \mathtt{Id}_{T_\epsilon B}"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A' \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A',B}"] \\
T_\epsilon(A \times B) \arrow[r, "T_\epsilon(f \times \mathtt{Id}_B)"] & T_\epsilon(A' \times B)
\end{tikzcd}$$

2.3.3 Unitor Law

$$\begin{tikzcd}
\mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\
& T_\epsilon A
\end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutivity of π_2 with $\mathtt{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B)$$

And $\pi_{2,\mathtt{1}}$ is the left unitor, so by tensorial strength:

$$\begin{aligned}
T_\epsilon\pi_2 \circ \mathtt{t}_{\epsilon,A,B} &= T_\epsilon\pi_{2,\mathtt{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon,A,B} \\
&= T_\epsilon\pi_{2,1,B} \circ \mathtt{t}_{\epsilon,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_2
\end{aligned} \tag{2.1}$$

So the following commutes:

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{t}_{\epsilon,A,B}"] \arrow[dr, "\pi_2"] & T_\epsilon(A \times B) \arrow[d, "T_\epsilon \pi_2"] \\
& T_\epsilon B
\end{tikzcd}$$

2.3.4 Commutativity with Join

$$\begin{tikzcd}
A \times T_{\epsilon_1} T_{\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1} T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A \times B}"] \\
& A \times T_{\epsilon_1 \cdot \epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1 \cdot \epsilon_2,A,B}"] & T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
\end{tikzcd}$$

2.4 Commutivity with Unit

$$\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"] & A \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] \\
& T_\epsilon(A \times B)
\end{tikzcd}$$

2.5 Commutivity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times T_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A \times B),C}"] \arrow[d, "\alpha_{A,B,T_\epsilon C}"] & & T_\epsilon((A \times B) \times C) \arrow[d, "T_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times T_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times T_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B \times C)}"] & T_\epsilon(A \times (B \times C))
\end{tikzcd}$$

TODO: Needed?

2.6 Subeffecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $\llbracket \epsilon_1 \leq \epsilon_2 \rrbracket : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

2.6.1 Subeffecting and Tensor Strength

$$\begin{tikzcd}
A \times T_{\epsilon_1} B \arrow[r, "\mathtt{Id}_A \times \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2} B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "\llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_{A \times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}$$

2.6.2 Sub-effecting and Monadic Join

Since the monoid operation on effects is monotone, we can introduce the following diagram.

$$\begin{tikzcd}
T_{\epsilon_1} T_{\epsilon_2} \arrow[r, "T_{\epsilon_1} \llbracket \epsilon_2 \leq \epsilon'_2 \rrbracket_M"] \arrow[d, "\mu_{\epsilon_1,\epsilon_2,}"] & T_{\epsilon_1} T_{\epsilon'_2} \arrow[r, "\llbracket \epsilon_1 \leq \epsilon'_1 \rrbracket_{M,T_{\epsilon'_2}}"] & T_{\epsilon'_1} T_{\epsilon'_2} \arrow[d, "\mu_{\epsilon'_1,\epsilon'_2,}"] \\
T_{\epsilon_1 \cdot \epsilon_2} \arrow[rr, "\llbracket \epsilon_1 \cdot \epsilon_2 \leq \epsilon'_1 \epsilon'_2 \rrbracket_M"] & & T_{\epsilon'_1 \cdot \epsilon'_2}
\end{tikzcd}$$

2.7 Subtyping

The denotation of ground types $[\![\_]\!]_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function subtyping to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)}\frac{f = [\![A' \leq: A]\!]_M \quad g = [\![B \leq: B']\!]_M \quad h = [\![\epsilon_1 \leq \epsilon_2]\!]}{rhs = [\![A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B']\!]_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned} rhs =& (h_{B'} \circ T_{\epsilon_1} g)^{A'} \circ (T_{\epsilon_1} B)^f \\ =& \mathtt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_{\epsilon_1} B^{A'}} \times f)) \end{aligned} \tag{2.2}$$

2.8 If natural transformation

There exists a natural transformation $\mathtt{If}_A : (\mathtt{Bool} \times (A \times A)) \to A$ Satisfying the following:

- $\mathtt{If}_A \circ \langle [\![\mathtt{true}]\!]_M \circ \langle\rangle_\Gamma, \langle t, f\rangle\rangle = t$
- $\mathtt{If}_A \circ \langle [\![\mathtt{false}]\!]_M \circ \langle\rangle_\Gamma, \langle t, f\rangle\rangle = f$

Chapter 3

Denotations

3.1 Denotations of Types

3.1.1 Denotation of Type Environments

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an `Ok` type environment Γ.

$$[\![\diamond]\!]_M = \mathtt{1}$$

$$[\![\Gamma, x : A]\!]_M = ([\![\Gamma]\!]_M \times [\![A]\!]_M)$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x : A]\!]_M = \Gamma \times A$$

3.1.2 Denotation of Computation Type

Given a function $[\![\_]\!]_M$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\mathtt{M}_\epsilon A$ as so:

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon [\![A]\!]_M$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon A$$

3.1.3 Denotation of Function Types

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to \mathtt{M}_\epsilon B$ as so:

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon [\![B]\!]_M)^{[\![A]\!]_M}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon B)^A$$

3.2 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$[\![\Gamma \vdash t \colon \tau]\!]_M : \Gamma \to [\![\tau]\!]_M$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t \colon \tau$ are equal, we need not refer to the derivation that yielded each denotation.

3.2.1 Denotation of Value Terms

- (Unit) $\dfrac{\Gamma\mathtt{Ok}}{[\![\Gamma\vdash\mathtt{()}:\mathtt{Unit}]\!]_M=[\![\mathtt{()}]\!]_M\circ\langle\rangle_\Gamma:\Gamma\to[\![\mathtt{Unit}]\!]_M}$

- (Const) $\dfrac{\Gamma\mathtt{Ok}}{[\![\Gamma\vdash\mathtt{C}^A:A]\!]_M=[\![\mathtt{C}^A]\!]_M\circ\langle\rangle_\Gamma:\Gamma\to[\![A]\!]_M}$

- (True) $\dfrac{\Gamma\mathtt{Ok}}{[\![\Gamma\vdash\mathtt{true}:\mathtt{Bool}]\!]_M=[\![\mathtt{true}]\!]_M\circ\langle\rangle_\Gamma:\Gamma\to[\![\mathtt{Bool}]\!]_M}$

- (False) $\dfrac{\Gamma\mathtt{Ok}}{[\![\Gamma\vdash\mathtt{false}:\mathtt{Bool}]\!]_M=[\![\mathtt{false}]\!]_M\circ\langle\rangle_\Gamma:\Gamma\to[\![\mathtt{Bool}]\!]_M}$

- (Var) $\dfrac{\Gamma\mathtt{Ok}}{[\![\Gamma,x:A\vdash x:A]\!]_M=\pi_2:\Gamma\times A\to A}$

- (Weaken) $\dfrac{f=[\![\Gamma\vdash x:A]\!]_M:\Gamma\to A}{[\![\Gamma,y:B\vdash x:A]\!]_M=f\circ\pi_1:\Gamma\times B\to A}$

- (Lambda) $\dfrac{f=[\![\Gamma,x:A]\!]_M C\mathtt{M}_\epsilon B:\Gamma\times A\to T_\epsilon B}{[\![\Gamma\vdash\lambda x:A.C:A\to\mathtt{M}_\epsilon B]\!]_M=\mathtt{cur}(f):\Gamma\to(T_\epsilon B)^A}$

- (Subtype) $\dfrac{f=[\![\Gamma\vdash v:A]\!]_M:\Gamma\to A\quad g=[\![A\leq:B]\!]_M}{[\![\Gamma\vdash v:B]\!]_M=g\circ f:\Gamma\to B}$

3.2.2 Denotation of Computation Terms

- (Return) $\dfrac{f=[\![\Gamma\vdash v:A]\!]_M}{[\![\Gamma\vdash\mathtt{return}v:\mathtt{M}_\mathtt{1}A]\!]_M=\eta_A\circ f}$

- (If) $\dfrac{f=[\![\Gamma\vdash v:\mathtt{Bool}]\!]_M\quad g=[\![\Gamma\vdash C_1:\mathtt{M}_\epsilon A]\!]_M\quad h=[\![\Gamma\vdash C_2:\mathtt{M}_\epsilon A]\!]_M}{[\![\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A]\!]_M=\mathtt{If}_{\mathtt{M}_\epsilon B}\circ\langle f,\langle g,h\rangle\rangle:\Gamma\to T_\epsilon A}$

- (Bind) $\dfrac{f=[\![\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1}A:\Gamma\to T_{\epsilon_1}A\quad g=[\![\Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2}B]\!]_M]\!]_M:\Gamma\times A\to T_{\epsilon_2}B}{[\![\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}]\!]_M=\mu_{\epsilon_1,\epsilon_2,B}\circ T_{\epsilon_1}g\circ\mathtt{t}_{\Gamma,A,\epsilon_1}\circ\left\langle\mathtt{Id}_\Gamma,f\right\rangle:\Gamma\to T_{\epsilon_1\cdot\epsilon_2}B}$

- (Subeffect) $\dfrac{f=[\![\Gamma\vdash c:\mathtt{M}_{\epsilon_1}A]\!]_M:\Gamma\to T_{\epsilon_1}A\quad g=[\![A\leq:B]\!]_M\quad h=[\![\epsilon_1\leq\epsilon_2]\!]}{[\![\Gamma\vdash C:\mathtt{M}_{\epsilon_2}B]\!]_M=h_B\circ T_{\epsilon_1}g\circ f}$

- (Apply) $\dfrac{f=[\![\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B]\!]_M:\Gamma\to(T_\epsilon B)^A\quad g=[\![\Gamma\vdash v_2:A]\!]_M:\Gamma\to A}{[\![\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B]\!]_M=\mathtt{app}\circ\langle f,g\rangle:\Gamma\to T_\epsilon B}$

Chapter 4

Unique Denotations

4.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t\colon\tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

4.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t\colon\tau$,there exists at most one reduced derivation of $\Gamma \vdash t\colon\tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

4.2.1 Variables

To find the unique derivation of $\Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is, if $A' \leq: A$, as below:

$$(\text{Subtype})\frac{(\text{Var})\frac{\Gamma', x{:}A' \, \mathtt{Ok}}{\Gamma, x{:}A' \vdash x{:}A'} \quad A' \leq: A}{\Gamma', x : A' \vdash x\colon A} \tag{4.1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Gamma \vdash x\colon A$ holds, then so must $\Gamma' \vdash x\colon A$.

Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Gamma' \vdash x{:}A'} \quad A' \leq: A}{\Gamma' \vdash x\colon A} \tag{4.2}$$

Be the unique reduced derivation of $\Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is:

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta}{\Gamma, x{:}A' \vdash x{:}A'}}{\Gamma \vdash x{:}A'} \quad A' \leq: A}{\Gamma \vdash x\colon A} \tag{4.3}$$

4.2.2 Constants

For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma\,\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A\colon A}\quad A \leq: B}{\Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

4.2.3 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Gamma,x:A\vdash C:\mathtt{M}_\epsilon B}}{\Gamma\vdash\lambda x:A.B:A\to\mathtt{M}_\epsilon B}\quad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C\colon A' \to \mathtt{M}_{\epsilon'} B'}$$

Where

$$\text{(Sub-Effect)}\frac{()\frac{\Delta}{\Gamma,x:A\vdash C:\mathtt{M}_\epsilon B}\quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon'} B'} \tag{4.4}$$

is the reduced derivation of $\Gamma, x : A \vdash C\colon \mathtt{M}_\epsilon B$ if it exists.

Case Subtype TODO: Do we need to write anything here? (Probably needs an explanation)

4.2.4 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return}v\colon \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\frac{()\frac{\Delta}{\Gamma\vdash v:A}}{\Gamma\vdash\mathtt{return}v:\mathtt{M}_{\mathtt{1}} A}\quad A \leq: B \quad \mathtt{1} \leq \epsilon}{\Gamma \vdash \mathtt{return}v\colon \mathtt{M}_\epsilon B}$$

Where

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma\vdash v:A}\quad A \leq: B}{\Gamma \vdash v\colon B}$$

is the reduced derivation of $\Gamma \vdash v\colon B$

Case Apply If

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B}\quad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1\colon A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Gamma\vdash v_2:A''}\quad A'' \leq: A'}{\Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1\colon A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2\colon A'$
Then we can construct the reduced derivation of $\Gamma \vdash v_1\ v_2\colon \mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subeffect)}\frac{\text{(Apply)}\frac{()\frac{\Delta}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B}\quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Gamma\vdash v:A''}\quad A''\leq:A}{\Gamma\vdash v:A}}{\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B}\quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma \vdash v_1\ v_2\colon \mathtt{M}_{\epsilon'} B'}$$

Case If Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Gamma\vdash v\colon B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v\colon \mathtt{Bool}} \tag{4.5}$$

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C_1\colon \mathtt{M}_{\epsilon'} A'} \quad A' \leq: A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A} \tag{4.6}$$

$$(\text{Subeffect})\frac{()\frac{\Delta''}{\Gamma\vdash C_2\colon \mathtt{M}_{\epsilon''} A''} \quad A'' \leq: A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A} \tag{4.7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v\colon \mathtt{Bool}$, $\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A$.

Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\colon \mathtt{M}_\epsilon A$ is:

TODO: Scale this properly

$$(\text{Subtype})\frac{(\text{If})\frac{(\text{Subtype})\frac{()\frac{\Delta}{\Gamma\vdash v:B} \quad B\leq:\mathtt{Bool}}{\Gamma\vdash v:\mathtt{Bool}} \quad (\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon'} A'} \quad A'\leq:A \quad \epsilon'\leq\epsilon}{\Gamma\vdash C_1:\mathtt{M}_\epsilon A} \quad (\text{Subeffect})\frac{()\frac{\Delta''}{\Gamma\vdash C_2:\mathtt{M}_{\epsilon''} A''} \quad A''\leq:A \quad \epsilon''\leq\epsilon}{\Gamma\vdash C_2:\mathtt{M}_\epsilon A}}{\Gamma\vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A \quad \epsilon\leq\epsilon \quad A\leq:A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\colon \mathtt{M}_\epsilon A} \tag{4.8}$$

Case Bind Let

$$(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1\colon \mathtt{M}_{\epsilon_1} A} \quad A \leq: A' \quad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1\colon \mathtt{M}_{\epsilon_1'} A'} \tag{4.9}$$

$$(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma, x:A\vdash C_2\colon \mathtt{M}_{\epsilon_2} B} \quad B \leq: B' \quad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A \vdash C_2\colon \mathtt{M}_{\epsilon_2'} B'} \tag{4.10}$$

Be the respective unique reduced type derivations of the subterms]

By weakening, $\iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2\colon \mathtt{M}_\epsilon B$, there's also one of $\Gamma, x : A \vdash C_2\colon \mathtt{M}_\epsilon B$.

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2\colon \mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$(\text{Subeffect})\frac{(\text{Bind})\frac{(\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1} A} \quad A\leq:A' \quad \epsilon_1\leq\epsilon_1'}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1'} A'} \quad (\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2} B} \quad B\leq:B' \quad \epsilon_2\leq\epsilon_2'}{\Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2'} B'}}{\Gamma\vdash \mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \quad B \leq: B' \quad \epsilon_1\cdot\epsilon_2 \leq \epsilon_1'\cdot\epsilon_2'}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2\colon \mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B'} \tag{4.11}$$

4.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t\colon \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

4.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$reduce((\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A}) = (\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A}$

4.3.2 Value Types

Var

$$reduce((\text{Var})\frac{\Gamma\,\texttt{Ok}}{\Gamma, x : A \vdash x \colon A}) = (\text{Var})\frac{\Gamma\,\texttt{Ok}}{\Gamma, x : A \vdash x \colon A} \tag{4.12}$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{()\frac{\Delta}{\Gamma\vdash x:A}}{\Gamma, y : B \vdash x \colon A}) \tag{4.13}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma\vdash x:A} \quad A' \leq: A}{\Gamma \vdash x \colon A} = reduce(\Delta) \tag{4.14}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma\vdash x:A'}}{\Gamma,y:B\vdash x:A'} \quad A' \leq: A}{\Gamma, y : B \vdash x \colon A} \tag{4.15}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$[\![(\text{Weaken})\frac{()\frac{\Delta}{\Gamma\vdash x:A}}{\Gamma, y : B \vdash x \colon A}]\!]_M = \Delta \circ \pi_1 \tag{4.16}$$

Similarly, the denotation of the reduced denotation is:

$$[\![(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma\vdash x:A'}}{\Gamma,y:B\vdash x:A'} \quad A' \leq: A}{\Gamma, y : B \vdash x \colon A}]\!]_M = [\![A' \leq: A]\!]_M \circ \Delta' \circ \pi_1 \tag{4.17}$$

By induction on *reduce* preserving denotations and the reduction of Δ (4.14), we have:

$$\Delta = [\![A' \leq: A]\!]_M \circ \Delta' \tag{4.18}$$

So the denotations of the un-reduced and reduced derivations are equal.

Lambda

reduce **definition** To find:

$$reduce((\text{Fn})\frac{()\frac{\Delta}{\Gamma,x:A\vdash C:\texttt{M}_{\epsilon_2}B}}{\Gamma \vdash \lambda x : A.C \colon A \to \texttt{M}_{\epsilon_2}B}) \tag{4.19}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta'}{\Gamma,x:A\vdash C:\texttt{M}_{\epsilon_1}B'} \quad \epsilon_1 \leq \epsilon_2 \quad B' \leq: B}{\Gamma, x : A \vdash C \colon \texttt{M}_{\epsilon_2}B} = reduce(\Delta) \tag{4.20}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{\Delta'}{\Gamma,x:A\vdash C:\texttt{M}_{\epsilon_1}B'} \quad A \to \texttt{M}_{\epsilon_1}B' \leq: A \to \texttt{M}_{\epsilon_2}B}{\Gamma \vdash \lambda x : A.C \colon A \to \texttt{M}_{\epsilon_2}B} \tag{4.21}$$

Preserves Denotation Let

$$f = [\![\mathtt{M}_{\epsilon_1} B' \leq: \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\epsilon_1 \leq \epsilon_2]\!]_{M,B} \circ T_{\epsilon_1}([\![B' \leq: B]\!]_M) \tag{4.22}$$

$$[\![A \to \mathtt{M}_{\epsilon_1} B' \leq: A \to \mathtt{M}_{\epsilon_2} B]\!]_M = f^A = \mathtt{cur}(f \circ \mathtt{app}) \tag{4.23}$$

Then

$$\begin{aligned}
before &= \mathtt{cur}(\Delta) \quad \text{By definition} && (4.24)\\
&= \mathtt{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta && (4.25)\\
&= f^A \circ \mathtt{cur}(\Delta') \quad \text{By the property of } f^X \circ \mathtt{cur}(g) = \mathtt{cur}(f \circ g) && (4.26)\\
&= after \quad \text{By definition} && (4.27)\\
& && (4.28)
\end{aligned}$$

Subtype

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Gamma \vdash v : A} \quad A \leq: B}{\Gamma \vdash v : B}) \tag{4.29}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash x : A} \quad A' \leq: A}{\Gamma \vdash x : A} = reduce(\Delta) \tag{4.30}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash v : A'} \quad A' \leq: A \leq: B}{\Gamma \vdash v : B} \tag{4.31}$$

Preserves Denotation

$$\begin{aligned}
before &= [\![A \leq: B]\!]_M \circ \Delta && (4.32)\\
&= [\![A \leq: B]\!]_M \circ ([\![A' \leq: A]\!]_M \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. && (4.33)\\
&= [\![A' \leq: B]\!]_M \circ \Delta' \quad \text{Subtyping relations are unique} && (4.34)\\
&= after && (4.35)\\
& && (4.36)
\end{aligned}$$

4.3.3 Computation Types

Return

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Gamma \vdash v : A}}{\Gamma \vdash \mathtt{return} v : \mathtt{M}_1 A}) \tag{4.37}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Gamma \vdash v : A'} \quad A' \leq: A}{\Gamma \vdash v : A} = reduce(\Delta) \tag{4.38}$$

In

$$(\text{Sub-effect})\frac{(\text{Return})\frac{\Delta'}{\Gamma\vdash v:A}\quad \mathtt{1}\leq\mathtt{1}\quad A'\leq: A}{\Gamma\vdash \mathtt{return}v\colon \mathtt{M}_{\mathtt{1}}A} \tag{4.39}$$

Then

$$\begin{aligned}
before &= \eta_A\circ\Delta \quad \text{By definition} \quad \text{By defintion} && (4.40)\\
&= \eta_A\circ[\![A'\leq: A]\!]_M\circ\Delta' \quad \text{BY reduction of } \Delta && (4.41)\\
&= T_{\mathtt{1}}[\![A'\leq: A]\!]_M\circ\eta_{A'}\circ\Delta' \quad \text{By naturality of } \eta && (4.42)\\
&= [\![\mathtt{1}\leq\mathtt{1}]\!]_{M,A}\circ T_{\mathtt{1}}[\![A'\leq: A]\!]_M\circ\eta_{A'}\circ\Delta' \quad \text{Since } [\![\mathtt{1}\leq\mathtt{1}]\!]_M \text{ is the identity Nat-Trans} && (4.43)\\
&= after \quad \text{By definition} && (4.44)\\
& && (4.45)
\end{aligned}$$

Apply

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1\colon A\to\mathtt{M}_\epsilon B}\quad ()\frac{\Delta_2}{\Gamma\vdash v_2\colon A}}{\Gamma\vdash v_1\ v_2\colon \mathtt{M}_\epsilon B}) \tag{4.46}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Gamma\vdash v_1\colon A'\to\mathtt{M}_{\epsilon'}B'}\quad A'\to\mathtt{M}_{\epsilon'}B'\leq: A\to\mathtt{M}_\epsilon B}{\Gamma\vdash v_1\colon A\to\mathtt{M}_\epsilon B} = reduce(\Delta_1) \tag{4.47}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Gamma\vdash v\colon A'}\quad A'\leq: A}{\Gamma\vdash v_1\colon A} = reduce(\Delta_2) \tag{4.48}$$

In

$$(\text{Sub-effect})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma\vdash v_1\colon A'\to\mathtt{M}_{\epsilon'}B'}\quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Gamma\vdash v_2\colon A''}\quad A''\leq: A\leq: A'}{\Gamma\vdash v_2\colon A'}}{\Gamma\vdash v_1\ v_2\colon\mathtt{M}_{\epsilon'}B'}\quad \epsilon'\leq\epsilon\quad B'\leq: B}{\Gamma\vdash v_1\ v_2\colon \mathtt{M}_\epsilon B} \tag{4.49}$$

Preserves Denotation Let

$$\begin{aligned}
f &= [\![A\leq: A']\!]_M : A\to A' && (4.50)\\
f' &= [\![A''\leq: A]\!]_M : A''\to A && (4.51)\\
g &= [\![B'\leq: B]\!]_M : B'\to B && (4.52)\\
h &= [\![\epsilon'\leq\epsilon]\!]_M : T_{\epsilon'}\to T_\epsilon && (4.53)
\end{aligned}$$

Hence

$$\begin{aligned}
[\![A'\to\mathtt{M}_{\epsilon'}B'\leq: A\to\mathtt{M}_\epsilon B]\!]_M &= (h_B\circ T_{\epsilon'}g)^A\circ(T_{\epsilon'}B')^f && (4.54)\\
&= \mathtt{cur}(h_B\circ T_{\epsilon'}g\circ\mathtt{app})\circ\mathtt{cur}(\mathtt{app}\circ(\mathtt{Id}\times f)) && (4.55)\\
&= \mathtt{cur}(h_B\circ T_{\epsilon'}g\circ\mathtt{app}\circ(\mathtt{Id}\times f)) && (4.56)
\end{aligned}$$

Then

$$\begin{aligned}
before &= \mathtt{app}\circ\langle\Delta_1,\Delta_2\rangle \quad \text{By definition} && (4.57)\\
&= \mathtt{app}\circ\langle\mathtt{cur}(h_B\circ T_{\epsilon'}g\circ\mathtt{app}\circ(\mathtt{Id}\times f))\circ\Delta_1', f'\circ\Delta_2'\rangle \quad \text{By reductions of } \Delta_1,\Delta_2 && (4.58)\\
&= \mathtt{app}\circ(\mathtt{cur}(h_B\circ T_{\epsilon'}g\circ\mathtt{app}\circ(\mathtt{Id}\times f))\times\mathtt{Id}_A)\circ\langle\Delta_1', f'\circ\Delta_2'\rangle \quad \text{Factoring out} && (4.59)\\
&= h_B\circ T_{\epsilon'}g\circ\mathtt{app}\circ(\mathtt{Id}\times f)\circ\langle\Delta_1', f'\circ\Delta_2'\rangle \quad \text{By the exponential property} && (4.60)\\
&= h_B\circ T_{\epsilon'}g\circ\mathtt{app}\circ\langle\Delta_1', f\circ f'\circ\Delta_2'\rangle && (4.61)\\
&= after \quad \text{By defintion} && (4.62)
\end{aligned}$$

If

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma\vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 : \texttt{M}_\epsilon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{reduce(\Delta_3)}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma\vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 : \texttt{M}_\epsilon A} \tag{4.63}$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Bind

reduce **definition** To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Gamma,x:A\vdash C_2:\texttt{M}_{\epsilon_2}B}}{\Gamma\vdash \texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{4.64}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1'}A'} \quad \epsilon_1' \leq: \epsilon_1 \quad A' \leq: A}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1}A} = reduce(\Delta_1) \tag{4.65}$$

Since $i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq: A$, and by Δ_2, $(\Gamma, x : A) \vdash C_2{:}\texttt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $(\Gamma, x : A') \vdash C_2{:}\texttt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_3'}{\Gamma,x:A'\vdash C_2:\texttt{M}_{\epsilon_2'}B'} \quad \epsilon_2' \leq: \epsilon_2 \quad B' \leq: B}{\Gamma, x : A' \vdash C_2:\texttt{M}_{\epsilon_2}B} = reduce(\Delta_3) \tag{4.66}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-effect})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1'}A'} \quad ()\frac{\Delta_3'}{\Gamma,x:A'\vdash C_2:\texttt{M}_{\epsilon_2'}B'}}{\Gamma\vdash\texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1'\cdot\epsilon_2'}B} \quad B' \leq: B \quad \epsilon_1'\cdot\epsilon_2' \leq \epsilon_1\cdot\epsilon_2}{\Gamma\vdash \texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{4.67}$$

Preserves Denotation Let

$$f = [\![A' \leq: A]\!]_M : A' \to A \tag{4.68}$$
$$g = [\![B' \leq: B]\!]_M : B' \to B \tag{4.69}$$
$$h_1 = [\![\epsilon_1' \leq \epsilon_1]\!]_M : T_{\epsilon_1'} \to T_{\epsilon_1} \tag{4.70}$$
$$h_2 = [\![\epsilon_2' \leq \epsilon_2]\!]_M : T_{\epsilon_2'} \to T_{\epsilon_2} \tag{4.71}$$
$$h = [\![\epsilon_1'\cdot\epsilon_2' \leq \epsilon_1\cdot\epsilon_2]\!]_M : T_{\epsilon_1'\cdot\epsilon_2'} \to T_{\epsilon_1\cdot\epsilon_2} \tag{4.72}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3 = \Delta_2 \circ (\texttt{Id}_\Gamma \times f) \tag{4.73}$$

And due to the reduction of Δ_3, we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3' \tag{4.74}$$

So:

$$
\begin{aligned}
before &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \quad \text{By definition.} && (4.75)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{By reduction of } \Delta_1. && (4.76)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Factor out } h_1 && (4.77)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma\times A)} \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Tensor strength and sub-effecting } h_1 && (4.78)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Naturality of } h_1 && (4.79)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1'} f) \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out pairing again} && (4.80)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Tensorstrength} && (4.81)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_3) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the definition of } \Delta_3 && (4.82)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3') \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the reduction of } \Delta_3 && (4.83)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'} h_{2,B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'} \Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out the functor} && (4.84)\\
&= h_B \circ \mu_{\epsilon_1',\epsilon_2',B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'} \Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the } \mu \text{ and Sub-effect rule} && (4.85)\\
&= h_B \circ T_{\epsilon_1'\cdot\epsilon_2'} g \circ \mu_{\epsilon_1',\epsilon_2',B'} \circ T_{\epsilon_1'} \Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By naturality of } \mu_{,,} && (4.86)\\
&= after \quad \text{By definition} && (4.87)
\end{aligned}
$$

Subeffect

reduce **definition** To find:

$$
reduce((\text{Subeffect})\frac{()\frac{\Delta}{\Gamma\vdash C:\mathtt{M}_{\epsilon'}B'} \quad \epsilon' \le \epsilon \quad B' \le: B}{\Gamma \vdash C: \mathtt{M}_\epsilon B}) \tag{4.88}
$$

Let

$$
(\text{Subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \quad \epsilon'' \le \epsilon' \quad \mathtt{Bool}'' \le: B}{\Gamma \vdash C: \mathtt{M}_{\epsilon'} B} = reduce(\Delta) \tag{4.89}
$$

in

$$
(\text{subeffect})\frac{()\frac{\Delta'}{\Gamma\vdash C:\mathtt{M}_{\epsilon''}B''} \quad \epsilon'' \le \epsilon \quad B'' \le: B}{\Gamma \vdash C: \mathtt{M}_\epsilon B} \tag{4.90}
$$

Preserves Denotation Let

$$
\begin{aligned}
f &= [\![B' \le: B]\!]_M && (4.91)\\
g &= [\![B'' \le: B']\!]_M && (4.92)\\
h_1 &= [\![\epsilon' \le \epsilon]\!]_M && (4.93)\\
h_2 &= [\![\epsilon' \le \epsilon']\!]_M && (4.94)\\
f \circ g &= [\![B'' \le: B]\!]_M && (4.95)\\
h_1 \circ h_2 &= [\![\epsilon'' \le \epsilon']\!]_M && (4.96)\\
& && (4.97)
\end{aligned}
$$

Hence we can find the denotation of the derivation before reduction.

$$before = h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} \tag{4.98}$$
$$= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta \tag{4.99}$$
$$= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \qquad = after \quad \text{By definition.} \tag{4.100}$$

4.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t \colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce \Delta]\!]_M = [\![reduce \Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t \colon \tau]\!]_M$ is unique.

Chapter 5

Weakening

5.1 Weakening Definition

5.1.1 Relation

We define the ternary weaking relation $w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Gamma \, \texttt{Ok}}{\iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma') \quad A \leq : B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B}$

5.1.2 Weakening Denotations

The denotation of a weakening relation is defined as follows:

$$[\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma \tag{5.1}$$

- $[\![\iota : \Gamma \triangleright \Gamma]\!]_M = \texttt{Id}_\Gamma : \Gamma \to \Gamma$
- (Project) $\frac{f = [\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma}{[\![\omega\pi : \Gamma, x : A \triangleright \Gamma]\!]_M = f \circ \pi_1 : \Gamma' \times A \to \Gamma}$
- (Extend) $\frac{f = [\![\omega : \Gamma' \triangleright \Gamma]\!]_M : \Gamma' \to \Gamma \quad g = [\![A \leq : B]\!]_M : A \to B}{[\![w \times : \Gamma', x : A \triangleright \Gamma, x : B]\!]_M = (f \times g) : (\Gamma \times A) \to (\Gamma \times B)}$

5.2 Weakening Theorems

5.2.1 Domain Lemma

If $\omega : \Gamma' \triangleright \Gamma$, then $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$.

Proof

Case Id Then $\Gamma' = \Gamma$ and so $\texttt{dom}(\Gamma') = \texttt{dom}(\Gamma)$.

Case Project By inversion and induction, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma') \subseteq \texttt{dom}(\Gamma' \cup \{x\})$

Case Extend By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$ so

$$\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\} \subseteq \mathtt{dom}(\Gamma') \cup \{x\} = \mathtt{dom}(\Gamma', x : A)$$

5.2.2 Theorem 1

If $\omega : \Gamma' \triangleright \Gamma$ and $\Gamma\mathtt{Ok}$ then $\Gamma'\mathtt{Ok}$

Proof

Case Id

$$(\text{Id})\frac{\Gamma\mathtt{Ok}}{\iota : \Gamma \triangleright \Gamma}$$

By inversion, $\Gamma\mathtt{Ok}$.

Case Project

$$(\text{Project})\frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\omega : \Gamma' \triangleright \Gamma$ and $x \notin \mathtt{dom}(\Gamma')$.
Hence by induction $\Gamma'\mathtt{Ok}$, $\Gamma\mathtt{Ok}$. Since $x \notin \mathtt{dom}(\Gamma')$, we have $\Gamma', x : A\mathtt{Ok}$.

Case Extend $(\text{Extend})\frac{\omega:\Gamma'\triangleright\Gamma \quad x\notin\mathtt{dom}(\Gamma') \quad A\leq:B}{w\times:\Gamma',x:A\triangleright\Gamma,x:B}$,
By inversion, we have
$\omega : \Gamma' \triangleright \Gamma$, $x \notin \mathtt{dom}(\Gamma')$.
Hence we have $\Gamma\mathtt{Ok}$, $\Gamma'\mathtt{Ok}$, and by the domain Lemma, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$, hence $x \notin \mathtt{dom}(\Gamma)$. Hence, we have $\Gamma, x : A\mathtt{Ok}$ and $\Gamma', x : A\mathtt{Ok}$

5.2.3 Theorem 2

If $\Gamma \vdash t\colon\tau$ and $\omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Gamma' \vdash t\colon\tau$

Proof Proved in parallel with theorem 3 below

5.2.4 Theorem 3

If $\omega : \Gamma' \triangleright \Gamma$ and $\Delta = [\![\Gamma \vdash t\colon\tau]\!]_M$ and $\Delta' = [\![\Gamma' \vdash t\colon\tau]\!]_M$, derived using Theorem 2, then

$$\Delta \circ [\![\omega]\!]_M = \Delta' : \Gamma' \to [\![\tau]\!]_M$$

Proof Below

5.3 Proof of Theorems 2 and 3

We induct over the structure of typing derivations of $\Gamma \vdash t\colon\tau$, assuming $\omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash t\colon\tau$ and show that $\Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M = \Delta'$

5.3.1 Variable Terms

Case Var and Weaken We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\iota : \Gamma \triangleright \Gamma]\!]_M \tag{5.2}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x\colon A$, such that

$$\Delta_1 = \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \quad \text{By Induction} \tag{5.3}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Gamma'' \vdash x\colon A}{\Gamma'', x' : A' \vdash x\colon A} \tag{5.4}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \Delta_1 \circ \pi_1 \quad \text{By Definition} && (5.5)\\ &= \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \circ \pi_1 \quad \text{By induction} && (5.6)\\ &= \Delta \circ [\![\omega'\pi_1 : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By denotation of weakening} && (5.7)\end{aligned}$$

If $\omega = \omega'\times$ Then

$$\begin{aligned}\Gamma' &= \Gamma''', x' : B && (5.8)\\ \Gamma &= \Gamma'', x' : A' && (5.9)\\ B &\leq\colon A && (5.10)\end{aligned}$$

If $x = x'$ Then $A = A'$.
Then we derive the new deriviation, Δ' as so:

$$(\text{Sub-type})\frac{(\text{var})\frac{}{\Gamma''', x:B \vdash x:B} \quad B \leq\colon A}{\Gamma' \vdash x\colon A} \tag{5.11}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= [\![B \leq\colon A]\!]_M \circ \pi_2 \quad \text{By Definition} && (5.12)\\ &= \pi_2 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![B \leq\colon A]\!]_M) \quad \text{By the properties of binary products} && (5.13)\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By Definition} && (5.14)\end{aligned}$$

Case $x \neq x'$ Then

$$\Delta = (\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x:A}}{\Gamma \vdash x\colon A} \tag{5.15}$$

By induction with $\omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x\colon A$
We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{()\frac{\Delta_1'}{\Gamma''' \vdash x:A}}{\Gamma' \vdash x\colon A} \tag{5.16}$$

This preserves denotations:
By induction, we have

$$\Delta_1' = \Delta_1 \circ [\![\omega : \Gamma''' \triangleright \Gamma'']\!]_M \tag{5.17}$$

So we have:

$$\begin{aligned}\Delta' &= \Delta_1' \circ \pi_1 \quad \text{By denotation definition} && (5.18)\\ &= \Delta_1 \circ [\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \quad \text{By induction} \circ \pi_1 && (5.19)\\ &= \Delta_1 \circ \pi_1 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![A' \leq\colon B]\!]_M) \quad \text{By product properties} && (5.20)\\ &= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By definition} && (5.21)\end{aligned}$$

5.3.2 Value Terms

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\omega : \Gamma' \triangleright \Gamma']\!]_M$, simply as ω.

Case Constant The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$(\text{Const})\frac{\Gamma\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A} \tag{5.22}$$

By inversion, we have $\Gamma\mathtt{Ok}$, so we have $\Gamma'\mathtt{Ok}$.
Hence

$$(\text{Const})\frac{\Gamma'\mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A \colon A} \tag{5.23}$$

Holds.
This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma'} \quad \text{By definition} && (5.24)\\
&= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} && (5.25)\\
&= \Delta \quad \text{By Definition} && (5.26)\\
& && (5.27)
\end{aligned}$$

Case Lambda By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C \colon A \to \mathtt{M}_\epsilon B} \tag{5.28}$$

Since $\omega : \Gamma' \triangleright \Gamma$, we have:

$$\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{5.29}$$

Hence, by induction, using $\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash C : \mathtt{M}_\epsilon B}}{\Gamma', x : A \vdash \lambda x : A.C \colon A \to \mathtt{M}_\epsilon B} \tag{5.30}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By Definition} && (5.31)\\
&= \mathtt{cur}(\Delta_1 \circ (\omega \times \mathtt{Id}_\Gamma)) \quad \text{By the denotation of } \omega\times && (5.32)\\
&= \mathtt{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} && (5.33)\\
&= \Delta \circ \omega \quad \text{By Definition} && (5.34)
\end{aligned}$$

Case Sub-typing

$$(\text{Sub-type})\frac{\Gamma \vdash v \colon A \quad A \leq : B}{\Gamma \vdash v \colon B} \tag{5.35}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash v \colon A} \tag{5.36}$$

So by induction, we have a derivation Δ'_1 such that:

$$\text{(Sub-type)}\frac{()\frac{\Delta'_1}{\Gamma' \vdash v : a} \quad A \leq: B}{\Gamma' \vdash v : B} \tag{5.37}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= [\![A \leq: B]\!]_M \circ \Delta'_1 \quad \text{By Definition} && (5.38)\\ &= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \omega \quad \text{By induction} && (5.39)\\ &= \Delta \circ \omega \quad \text{By Definition} && (5.40)\\ & && (5.41)\end{aligned}$$

5.3.3 Computation Terms

Case Return We have the sub-derivation Δ_1 such that

$$\Delta = \text{(Return)}\frac{()\frac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \mathtt{return} v : \mathtt{M_1} A} \tag{5.42}$$

Hence, by induction, with $\omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ'_1 such that:

$$\Delta' = \text{(Return)}\frac{()\frac{\Delta'_1}{\Gamma' \vdash v : A}}{\Gamma' \vdash \mathtt{return} v : \mathtt{M_1} A} \tag{5.43}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \eta_A \circ \Delta'_1 \quad \text{By definition} && (5.44)\\ &= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1 && (5.45)\\ &= \Delta \circ \omega \quad \text{By Definition} && (5.46)\end{aligned}$$

Case Apply By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = \text{(Apply)}\frac{()\frac{\Delta_1}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B} \tag{5.47}$$

By induction, this gives us the respective derivations: Δ'_1, Δ'_2 such that

$$\Delta' = \text{(Apply)}\frac{()\frac{\Delta'_1}{\Gamma' \vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta'_2}{\Gamma' \vdash v_2 : A}}{\Gamma' \vdash v_1\ v_2 : \mathtt{M}_\epsilon B} \tag{5.48}$$

This preserves denotations:

$$\begin{aligned}\Delta' &= \mathtt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} && (5.49)\\ &= \mathtt{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 && (5.50)\\ &= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega && (5.51)\\ &= \Delta \circ \omega \quad \text{By Definition} && (5.52)\end{aligned}$$

Case If By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 : \texttt{M}_\epsilon A} \tag{5.53}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Gamma'\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Gamma'\vdash C_1:\texttt{M}_\epsilon A} \quad ()\frac{\Delta_3'}{\Gamma'\vdash C_2:\texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2 : \texttt{M}_\epsilon A} \tag{5.54}$$

This preserves denotations

$$\begin{aligned}
\Delta' &= \texttt{If}_{T_\epsilon A} \circ \langle \Delta_1', \langle \Delta_2', \Delta_3' \rangle\rangle \quad \text{By Definition} && (5.55)\\
&= \texttt{If}_{T_\epsilon A} \circ \langle \Delta_1 \circ \omega, \langle \Delta_2 \circ \omega, \Delta_3 \circ \omega \rangle\rangle \quad \text{By induction} && (5.56)\\
&= \texttt{If}_{T_\epsilon A} \circ \langle \Delta_1, \langle \Delta_2, \Delta_3 \rangle\rangle \circ \omega \quad \text{By Product property} && (5.57)\\
&= \Delta \circ \omega \quad \text{By Definition} && (5.58)\\
& && (5.59)
\end{aligned}$$

Case Bind By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2}{\Gamma, x:A\vdash C_2:\texttt{M}_{\epsilon_2} B}}{\Gamma \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{5.60}$$

If $\omega : \Gamma' \triangleright \Gamma$ then $\omega\times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ_1', Δ_2' such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma'\vdash C_1:\texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2'}{\Gamma', x:A\vdash C_2:\texttt{M}_{\epsilon_2} B}}{\Gamma' \vdash \texttt{do}\ x \leftarrow C_1\ \texttt{in}\ C_2 : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{5.61}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \texttt{Id}_{G'}, \Delta_1' \rangle \quad \text{By definition} && (5.62)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\omega \times \texttt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \texttt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta_1', \Delta_2' && (5.63)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} && (5.64)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \texttt{Id}_\Gamma, \Delta_1 \rangle \circ \omega \quad \text{By product property} && (5.65)\\
&= \Delta \quad \text{By definition} && (5.66)
\end{aligned}$$

Case Sub-effect

$$(\text{Sub-effect})\frac{\Gamma \vdash C : \texttt{M}_{\epsilon_1} A \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \texttt{M}_{\epsilon_2} B} \tag{5.67}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash C : \texttt{M}_{\epsilon_1} A} \tag{5.68}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma'\vdash C:\texttt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C : \texttt{M}_{\epsilon_2} B} \tag{5.69}$$

This preserves denotations:

Let

$$g = [\![A \leq: B]\!]_M : A \to B \tag{5.70}$$
$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M : T_{\epsilon_1} \to T_{\epsilon_2} \tag{5.71}$$

Then

$$\Delta' = h_B \circ T_{\epsilon_1} g \circ \Delta_1' \quad \text{By Definition} \tag{5.72}$$
$$= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \omega \quad \text{By Induction} \tag{5.73}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{5.74}$$

Chapter 6

Substitution

6.1 Introduce Substitutions

6.1.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{6.1}$$

6.1.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{6.2}$$

$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{6.3}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{6.4}$$

$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{6.5}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6.6}$$

6.1.3 Effect of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{6.7}$$

$$x\,[\sigma, x := v] = v \tag{6.8}$$

$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{6.9}$$

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.10}$$

$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.11}$$

$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{6.12}$$

$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{6.13}$$

$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.14}$$

$$\tag{6.15}$$

6.1.4 Well Formedness

Define the relation

$$\Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil) $\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \diamond \colon \diamond}$
- (Extend) $\frac{\Gamma' \vdash \sigma \colon \Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Gamma' \vdash v \colon A}{\Gamma' \vdash (\sigma, x := v) \colon (\Gamma, x \colon A)}$

6.1.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma \colon \Gamma$ then: **TODO: Number these**

Property 1: $\Gamma \mathtt{Ok}$ and $\Gamma' \mathtt{Ok}$ Since $\Gamma' \mathtt{Ok}$ holds by the Nil-axiom. $\Gamma \mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v \colon A$ holds if $\Gamma' \vdash v \colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{6.16}$$

6.2 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$\Gamma \vdash g \colon \tau \wedge \Gamma' \vdash \sigma \colon \Gamma \Rightarrow \Gamma' \vdash t\,[\sigma] : \tau \tag{6.17}$$

Assuming $\Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Gamma \vdash t \colon \tau \rightarrow \Gamma' \vdash t \colon \tau$

6.2.1 Variables

Case Var By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Gamma'', x : A \vdash x \colon A \tag{6.18}$$

So by inversion, since $\Gamma' \vdash \sigma \colon \Gamma'', x : A$,

$$\sigma = \sigma', x := v \wedge \Gamma' \vdash v \colon A \tag{6.19}$$

By the defintion of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Gamma' \vdash x\,[\sigma] : A \tag{6.20}$$

holds.

Case Weaken By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken})\frac{()\frac{\Delta}{\Gamma'' \vdash x : A}}{\Gamma'', y : B \vdash x : A} \tag{6.21}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Gamma' \vdash \sigma' : \Gamma'' \tag{6.22}$$

So by induction,

$$\Gamma' \vdash x\,[\sigma'] : A \tag{6.23}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Gamma' \vdash x\,[\sigma] : A \tag{6.24}$$

6.2.2 Other Value Terms

Case Lambda By inversion, there exists Δ such that:

$$(\text{Fn})\frac{()\frac{\Delta}{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_\epsilon B} \tag{6.25}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$(\Gamma', x : A) \vdash (\sigma, x := x) : \Gamma, x : A \tag{6.26}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{()\frac{\Delta'}{\Gamma', x:A \vdash C\,[\sigma, x:=v] : \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C\,[\sigma, x := x] : A \to \mathtt{M}_\epsilon B} \tag{6.27}$$

Since $\lambda x : A.(C\,[\sigma, x := x]) = \lambda x : A.(C\,[\sigma]) = (\lambda x : A.C)\,[\sigma]$, we have a typing derivation for $\Gamma' \vdash (\lambda x : A.C)\,[\sigma] : A \to \mathtt{M}_\epsilon B$.

Case Constants We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:
$\Gamma \vdash \sigma : \Gamma \Rightarrow \Gamma' \mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.28}$$

So

$$(\text{Const})\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A : A} \tag{6.29}$$

6.2.3 Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v : A}}{\Gamma \vdash \mathtt{return} v : \mathtt{M_1} A} \tag{6.30}$$

By induction, we have Δ'_1 such that

$$(\text{Return})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v[\sigma] : A}}{\Gamma' \vdash \mathtt{return}(v\,[\sigma]) : \mathtt{M_1} A} \tag{6.31}$$

Since $(\mathtt{return} v)\,[\sigma] = \mathtt{return}(v\,[\sigma])$, the type derivation above holds for $\Gamma' \vdash (\mathtt{return} v)\,[\sigma] : \mathtt{M_1} A$.

Case Apply By inversion, we have Δ_1, Δ_2 such that:

$$\text{(Apply)}\frac{()\frac{\Delta_1}{\Gamma\vdash v_1 : A\to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma\vdash v_2 : A}}{\Gamma\vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B} \tag{6.32}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$\text{(Apply)}\frac{()\frac{\Delta_1'}{\Gamma'\vdash v_1[\sigma] : A\to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2'}{\Gamma'\vdash v_2[\sigma] : A}}{\Gamma'\vdash (v_1\,[\sigma])\ (v_2\,[\sigma]) \colon \mathtt{M}_\epsilon B} \tag{6.33}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \mathtt{M}_\epsilon B$

Case If By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1}{\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\mathtt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\mathtt{M}_\epsilon A}}{\Gamma\vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A} \tag{6.34}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1'}{\Gamma'\vdash v[\sigma]:\mathtt{Bool}} \quad ()\frac{\Delta_2'}{\Gamma'\vdash C_1[\sigma]:\mathtt{M}_\epsilon A} \quad ()\frac{\Delta_3'}{\Gamma'\vdash C_2[\sigma]:\mathtt{M}_\epsilon A}}{\Gamma'\vdash \mathtt{if}_{\epsilon,A}\ (v\,[\sigma])\ \mathtt{then}\ (C_1\,[\sigma])\ \mathtt{else}\ (C_2\,[\sigma]) \colon \mathtt{M}_\epsilon A} \tag{6.35}$$

Since $(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ (v\,[\sigma])\ \mathtt{then}\ (C_1\,[\sigma])\ \mathtt{else}\ (C_2\,[\sigma])$ The derivation above holds for $\Gamma' \vdash (\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] : \mathtt{M}_\epsilon A$

Case Bind By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma, x:A\vdash C_2:\mathtt{M}_{\epsilon_2} B}}{\Gamma\vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{6.36}$$

Using alpha-equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$. Hence by property 3,

$$(\Gamma, x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1'}{\Gamma'\vdash C_1[\sigma]:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Gamma', x:A\vdash C_2[\sigma, x:=x]:\mathtt{M}_{\epsilon_2} B}}{\Gamma'\vdash \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma, x := x]) \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{6.37}$$

Since $(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\,[\sigma] = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma, x := x])$, the above derivation holds for $\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\,[\sigma] : \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B$

6.2.4 Sub-typing and Sub-effecting

Case Sub-type By inversion, there exists Δ such that

$$\text{(sub-type)}\frac{()\frac{\Delta}{\Gamma\vdash v:A} \quad A \leq: B}{\Gamma\vdash v \colon B} \tag{6.38}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\frac{()\frac{\Delta'}{\Gamma'\vdash v[\sigma]:A} \quad A \leq: B}{\Gamma\vdash v\,[\sigma] : B} \tag{6.39}$$

Case Sub-effect By inversion, there exists Δ such that

$$\text{(sub-effect)}\frac{()\frac{\Delta}{\Gamma\vdash C{:}\mathtt{M}_{\epsilon_1}A} \quad A\leq: B \quad \epsilon_1\leq:\epsilon_2}{\Gamma\vdash C{:}\,\mathtt{M}_{\epsilon_2}B} \tag{6.40}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-effect)}\frac{()\frac{\Delta'}{\Gamma'\vdash C[\sigma]{:}\mathtt{M}_{\epsilon_1}A} \quad A\leq: B \quad \epsilon_1\leq:\epsilon_2}{\Gamma'\vdash C\,[\sigma]:\mathtt{M}_{\epsilon_2}B} \tag{6.41}$$

6.3 Semantics of Substitution

6.3.1 Denotation of Substitutions

We define the denotation of a well-formed-substitution as so:

$$[\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M:\Gamma'\to\Gamma \tag{6.42}$$

- (Nil)$\frac{\Gamma'\,\mathtt{Ok}}{[\![\Gamma'\vdash\diamond{:}\diamond]\!]_M=\langle\rangle_{\Gamma'}}$
- (Extend)$\frac{f=[\![\Gamma'\vdash\sigma:\Gamma]\!]_M \quad g=[\![\Gamma'\vdash v:A]\!]_M}{[\![\Gamma'\vdash(\sigma,x:=v:(\Gamma,x:A)]\!]_M=\langle f,g\rangle:\Gamma'\to(\Gamma\times A)}$

6.3.2 Extension Lemma

If $\Gamma'\vdash\sigma{:}\,\Gamma$ and $x\notin(\mathtt{dom}(\Gamma')\cup\mathtt{dom}(\Gamma))$ then the substitution in property 3 has denotation:

$$[\![(\Gamma',x:A)\vdash(\sigma,x:=x){:}\,(\Gamma,x:A)]\!]_M=([\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M\times\mathtt{Id}_A) \tag{6.43}$$

This holds since

$$[\![\Gamma',x:A\vdash x{:}\,A]\!]_M=\pi_2 \tag{6.44}$$

And $\iota\pi:(\Gamma',x:A)\triangleright\Gamma'$

$$[\![\iota\pi:(\Gamma',x:A)\triangleright\Gamma']\!]_M=\pi_1 \tag{6.45}$$

So for each denotation $[\![\Gamma'\vdash v{:}\,B]\!]_M$ of each $y:=v$ in σ, we can prepend the denotation with the weakening denotation to yield:

$$[\![\Gamma',x:A\vdash v{:}\,B]\!]_M=[\![\Gamma'\vdash v{:}\,B]\!]_M\circ\pi_1 \tag{6.46}$$

Since π_1 appears in every branch of $[\![\Gamma',x:A\vdash\sigma{:}\,\Gamma]\!]_M$, it can be factored out to yield:

$$[\![\Gamma',x:A\vdash\sigma{:}\,\Gamma]\!]_M=[\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M\circ\pi_1 \tag{6.47}$$

Hence,

$$[\![(\Gamma',x:A)\vdash(\sigma,x:=x){:}\,\Gamma,x:A]\!]_M=\langle[\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M\circ\pi_1,\pi_2\rangle=([\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M\times\mathtt{Id}_A) \tag{6.48}$$

6.3.3 Substitution Theorem

TODO: There is Tikz code here to draw the Substitution Theorem diagram, but it compiles v slowly If Δ derives $\Gamma\vdash t{:}\,\tau$ and $\Gamma'\vdash\sigma{:}\,\Gamma$ then the derivation Δ' deriving $\Gamma'\vdash t\,[\sigma]:\tau$ satisfies:

$$\Delta'=\Delta\circ[\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M \tag{6.49}$$

This is proved by induction over the derivation of $\Gamma\vdash t{:}\,\tau$. We shall use σ to denote $[\![\Gamma'\vdash\sigma{:}\,\Gamma]\!]_M$ where it is clear from the context.

6.3.4 Proof For Value Terms

Case Var By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var})\frac{\Gamma\,\texttt{Ok}}{\Gamma'', x : A \vdash x\colon A} \tag{6.50}$$

By inversion, $\sigma = \sigma', x := v$ and $\Gamma' \vdash v\colon A$.

Let

$$\sigma = [\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{6.51}$$
$$\Delta = [\![\Gamma'', x : A \vdash x\colon A]\!]_M = \pi_2 \tag{6.52}$$
$$\tag{6.53}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{6.54}$$
$$= \Delta' \quad \text{By product property} \tag{6.55}$$

Case Weaken By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x: A}}{\Gamma'', y : B \vdash x\colon A} \tag{6.56}$$

Also by inversion of the well-formedness of $\Gamma' \vdash \sigma\colon \Gamma$, we have $\Gamma' \vdash \sigma'\colon \Gamma''$ and

$$[\![\Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle [\![\Gamma' \vdash \sigma\colon \Gamma'']\!]_M, [\![\Gamma' \vdash v\colon B]\!]_M \rangle \tag{6.57}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$()\frac{\Delta_1'}{\Gamma' \vdash x\,[\sigma] : A} \tag{6.58}$$

Hence

$$\Delta' = \Delta_1' \quad \text{By definition} \tag{6.59}$$
$$= \Delta_1 \circ \sigma' \quad \text{By induction} \tag{6.60}$$
$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Gamma' \vdash v\colon B]\!]_M \rangle \quad \text{By product property} \tag{6.61}$$
$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma \qquad = \Delta \circ \sigma \quad \text{By defintion.} \tag{6.62}$$

Case Constants The logic for all constant terms ($\texttt{true}, \texttt{false}, \texttt{()}\texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{6.63}$$

$$\Delta' = c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} \tag{6.64}$$
$$= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} \tag{6.65}$$
$$= \Delta \circ \sigma \quad \text{By definition} \tag{6.66}$$

Case Lambda By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C: \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C: A \to \mathtt{M}_\epsilon B} \tag{6.67}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash (C[\sigma]): \mathtt{M}_\epsilon B}}{\Gamma \vdash (\lambda x : A.C)\,[\sigma] : A \to \mathtt{M}_\epsilon B} \tag{6.68}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \mathtt{Id}_A) \tag{6.69}$$

Hence:

$$\begin{aligned} \Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By definition} && (6.70)\\ &= \mathtt{cur}(\Delta_1 \circ (\sigma \times \mathtt{Id}_A)) \quad \text{By induction and extension lemma.} && (6.71)\\ &= \mathtt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (6.72)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (6.73)\\ & && (6.74) \end{aligned}$$

Case Sub-type By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Gamma \vdash v: A} \quad A \leq: B}{\Gamma \vdash v: B} \tag{6.75}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma]: A} \quad A \leq: B}{\Gamma' \vdash v\,[\sigma] : B} \tag{6.76}$$

Hence,

$$\begin{aligned} \Delta' &= [\![A \leq: B]\!]_M \circ \Delta_1' \quad \text{By definition} && (6.77)\\ &= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \sigma \quad \text{By induction} && (6.78)\\ &= \Delta \circ \sigma \quad \text{By definition} && (6.79)\\ & && (6.80) \end{aligned}$$

6.3.5 Proof For Computation Terms

Case Return By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v: A}}{\Gamma \vdash \mathtt{return} v: \mathtt{M_1} A} \tag{6.81}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v[\sigma]: A}}{\Gamma' \vdash (\mathtt{return} v)\,[\sigma] : \mathtt{M_1} A} \tag{6.82}$$

Hence,

$$\begin{aligned}\Delta' &= \eta_A \circ \Delta'_1 \quad \text{By Definition} && (6.83)\\ &= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} && (6.84)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (6.85)\\ & && (6.86)\end{aligned}$$

Case Apply By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma\vdash v_2 : A}}{\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B} \tag{6.87}$$

By induction we find Δ'_1, Δ'_2 such that

$$\begin{aligned}\Delta'_1 &= \Delta_1 \circ \sigma && (6.88)\\ \Delta'_2 &= \Delta_2 \circ \sigma && (6.89)\\ & && (6.90)\end{aligned}$$

And

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta'_1}{\Gamma'\vdash v_1[\sigma] : A \to \mathtt{M}_\epsilon B} \quad ()\frac{\Delta'_2}{\Gamma'\vdash v_2[\sigma] : A}}{\Gamma' \vdash (v_1\ v_2)\,[\sigma] : \mathtt{M}_\epsilon B} \tag{6.91}$$

Hence

$$\begin{aligned}\Delta' &= \mathtt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} && (6.92)\\ &= \mathtt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} && (6.93)\\ &= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} && (6.94)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (6.95)\\ & && (6.96)\end{aligned}$$

Case If By inversion, we find Δ_1, Δ_2, D_3 such that

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v : \mathtt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1 : \mathtt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2 : \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A} \tag{6.97}$$

By induction we find $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\begin{aligned}\Delta'_1 &= \Delta_1 \circ \sigma && (6.98)\\ \Delta'_2 &= \Delta_2 \circ \sigma && (6.99)\\ \Delta'_3 &= \Delta_3 \circ \sigma && (6.100)\\ & && (6.101)\end{aligned}$$

And

$$\Delta' = (\text{If})\frac{()\frac{\Delta'_1}{\Gamma'\vdash v[\sigma] : \mathtt{Bool}} \quad ()\frac{\Delta'_2}{\Gamma'\vdash C_1[\sigma] : \mathtt{M}_\epsilon A} \quad ()\frac{\Delta'_3}{\Gamma'\vdash C_2[\sigma] : \mathtt{M}_\epsilon A}}{\Gamma' \vdash (\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] : \mathtt{M}_\epsilon A} \tag{6.102}$$

Hence

$$\begin{aligned}\Delta' &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle \Delta'_1, \langle \Delta'_2, \Delta'_3 \rangle\rangle \quad \text{By Definition} && (6.103)\\ &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle \Delta_1 \circ \sigma, \langle \Delta_2 \circ \sigma, \Delta_3 \circ \sigma \rangle\rangle \quad \text{By induction} && (6.104)\\ &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle \Delta_1, \langle \Delta_2, \Delta_3 \rangle\rangle \circ \sigma \quad \text{By Product Property} && (6.105)\\ &= \Delta \circ \sigma \quad \text{By Definition} && (6.106)\\ & && (6.107)\end{aligned}$$

Case Bind By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \quad ()\frac{\Delta_2}{\Gamma, x : A \vdash C_1 : \mathtt{M}_\epsilon B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.108}$$

By property 3,

$$(\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A) \tag{6.109}$$

With denotation (extension lemma)

$$[\![(\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A)]\!]_M = \sigma \times \mathtt{Id}_A \tag{6.110}$$

By induction, we derive Δ'_1, Δ'_2 such that:

$$\Delta'_1 = \Delta_1 \circ \sigma \tag{6.111}$$

$$\Delta'_2 = \Delta_2 \circ (\sigma \times \mathtt{Id}_A) \quad \text{By Extension Lemma} \tag{6.112}$$

And:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta'_1}{\Gamma' \vdash C_1[\sigma] : \mathtt{M}_\epsilon A} \quad ()\frac{\Delta'_2}{\Gamma', x : A \vdash C_1[\sigma] : \mathtt{M}_\epsilon B}}{\Gamma' \vdash (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\,[\sigma] : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{6.113}$$

Hence:

$$\Delta' = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} \tag{6.114}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1}(\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using ht e extension lemma} \tag{6.115}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1} A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} \tag{6.116}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} \tag{6.117}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1}(\Delta_2) \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} \tag{6.118}$$

$$= \Delta \circ \sigma \quad \text{By Defintion} \tag{6.119}$$

$$\tag{6.120}$$

Case Subeffect By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-effect})\frac{()\frac{\Delta_1}{\Gamma \vdash C : \mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \mathtt{M}_{\epsilon_2} B} \tag{6.121}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-effect})\frac{()\frac{\Delta'_1}{\Gamma' \vdash C[\sigma] : \mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C\,[\sigma] : \mathtt{M}_{\epsilon_2} B} \tag{6.122}$$

Hence, Let

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{6.123}$$

$$g = [\![A \leq: B]\!]_M \tag{6.124}$$

$$
\begin{aligned}
\Delta' &= h_B \circ T_{\epsilon_1} g \circ \Delta_1' \quad \text{By definition} && (6.125)\\
&= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \sigma \quad \text{By induction} && (6.126)\\
&= \Delta \circ \sigma \quad \text{By definition} && (6.127)\\
& && (6.128)
\end{aligned}
$$

6.4 The Identity Substitution

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma, x:A} = (I_\Gamma, x := x)$

6.4.1 Properties of the Identity Substitution

Property 1 If $\Gamma \texttt{Ok}$ then $\Gamma \vdash I_\Gamma \colon \Gamma$, proved trivially by induction over the well formedness relation.

Property 2 $[\![\Gamma \vdash I_\Gamma \colon \Gamma]\!]_M = \texttt{Id}_\Gamma$, proved trivially by induction over the definition of I_Γ

6.5 Single Substitution

If $\Gamma \vdash v \colon A$, let the single substitution $\Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{6.129}$$

Then by properties 1, 2 of the identity substitution, we have:

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M = \langle \texttt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle : \Gamma \to (\Gamma \times A) \tag{6.130}$$

6.5.1 The Semantics of Single Substitution

The following diagram commutes:

$$[\![\Gamma \vdash t\,[v/x] : \tau]\!]_M = [\![\Gamma, x : A \vdash t \colon \tau]\!]_M \circ \langle \texttt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle \tag{6.131}$$

TODO: Again, there is code here to draw a Commutative diagram, but for some reason pdflatex hangs when compiling it Since $[\![\Gamma \vdash (I_\Gamma, x := v) \colon (\Gamma, x : A)]\!]_M = \langle \texttt{Id}_\Gamma, [\![\Gamma \vdash v \colon A]\!]_M \rangle$ And $\texttt{true}\,[v/x] = \texttt{true}\,[I_\Gamma, x := v]$

Chapter 7

Beta Eta Equivalence (Soundness)

7.1 Beta and Eta Equivalence

7.1.1 Beta conversions

- (Lambda) $\frac{\Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B \quad \Gamma\vdash v{:}A}{\Gamma\vdash(\lambda x{:}A.C)\ v{=}_{\beta\eta}C[x/v]{:}\mathtt{M}_\epsilon B}$

- (Left Unit) $\frac{\Gamma\vdash v{:}A \quad \Gamma,x{:}A\vdash C{:}\mathtt{M}_\epsilon B}{\Gamma\vdash\mathtt{do}\ x\leftarrow\mathtt{return}v\ \mathtt{in}\ C{=}_{\beta\eta}C[V/x]{:}\mathtt{M}_\epsilon B}$

- (Right Unit) $\frac{\Gamma\vdash C{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{do}\ x\leftarrow C\ \mathtt{in}\ \mathtt{return}x{=}_{\beta\eta}C{:}\mathtt{M}_\epsilon A}$

- (Associativity) $\frac{\Gamma\vdash C_1{:}\mathtt{M}_{\epsilon_1}A \quad \Gamma,x{:}A\vdash C_2{:}\mathtt{M}_{\epsilon_2}B \quad \Gamma,y{:}B\vdash C_3{:}\mathtt{M}_{\epsilon_3}C}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y\leftarrow C_2\ \mathtt{in}\ C_3){=}_{\beta\eta}\mathtt{do}\ y\leftarrow(\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}C}$

- (Eta) $\frac{\Gamma\vdash v{:}A\to\mathtt{M}_\epsilon B}{\Gamma\vdash\lambda x{:}A.(v\ x){=}_{\beta\eta}v{:}A\to\mathtt{M}_\epsilon B}$

- (if-true) $\frac{\Gamma\vdash C_1{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash C_2{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{true}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{=}_{\beta\eta}C_1{:}\mathtt{M}_\epsilon A}$

- (if-false) $\frac{\Gamma\vdash C_2{:}\mathtt{M}_\epsilon A \quad \Gamma\vdash C_1{:}\mathtt{M}_\epsilon A}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ \mathtt{false}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{=}_{\beta\eta}C_2{:}\mathtt{M}_\epsilon A}$

7.1.2 Equivalence Relation

- (Reflexive) $\frac{\Gamma\vdash t{:}\tau}{\Gamma\vdash t{=}_{\beta\eta}t{:}\tau}$

- (Symmetric) $\frac{\Gamma\vdash t_1{=}_{\beta\eta}t_2{:}\tau}{\Gamma\vdash t_2{=}_{\beta\eta}t_1{:}\tau}$

- (Transitive) $\frac{\Gamma\vdash t_1{=}_{\beta\eta}t_2{:}\tau \quad \Gamma\vdash t_2{=}_{\beta\eta}t_3{:}\tau}{\Gamma\vdash t_1{=}_{\beta\eta}t_3{:}\tau}$

7.1.3 Congruences

- (Lambda) $\frac{\Gamma,x{:}A\vdash C_1{=}_{\beta\eta}C_2{:}\mathtt{M}_\epsilon B}{\Gamma\vdash\lambda x{:}A.C_1{=}_{\beta\eta}\lambda x{:}A.C_2{:}A\to\mathtt{M}_\epsilon B}$

- (Return) $\frac{\Gamma\vdash v_1{=}_{\beta\eta}v_2{:}A}{\Gamma\vdash\mathtt{return}v_1{=}_{\beta\eta}\mathtt{return}v_2{:}\mathtt{M}_1 A}$

- (Apply) $\frac{\Gamma\vdash v_1{=}_{\beta\eta}v_1'{:}A\to\mathtt{M}_\epsilon B \quad \Gamma\vdash v_2{=}_{\beta\eta}v_2'{:}A}{\Gamma\vdash v_1\ v_2{=}_{\beta\eta}v_1'\ v_2'{:}\mathtt{M}_\epsilon B}$

- (Bind) $\frac{\Gamma\vdash C_1{=}_{\beta\eta}C_1'{:}\mathtt{M}_{\epsilon_1}A \quad \Gamma,x{:}A\vdash C_2{=}_{\beta\eta}C_2'{:}\mathtt{M}_{\epsilon_2}B}{\Gamma\vdash\mathtt{do}\ x\leftarrow C_1\ \mathtt{in}\ C_2{=}_{\beta\eta}\mathtt{do}\ c\leftarrow C_1'\ \mathtt{in}\ C_2'{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$

- (If) $\frac{\Gamma\vdash v=_{\beta\eta}v':\texttt{Bool}\ \ \Gamma\vdash C_1=_{\beta\eta}C_1':\texttt{M}_\epsilon A\ \ \Gamma\vdash C_2=_{\beta\eta}C_2':\texttt{M}_\epsilon A}{\Gamma\vdash\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1\ \texttt{else}\ C_2=_{\beta\eta}\texttt{if}_{\epsilon,A}\ v\ \texttt{then}\ C_1'\ \texttt{else}\ C_2':\texttt{M}_\epsilon A}$

- (Subtype) $\frac{\Gamma\vdash v=_{\beta\eta}v':A\ \ A\leq:B}{\Gamma\vdash v=_{\beta\eta}v':B}$

- (Subeffect) $\frac{\Gamma\vdash C=_{\beta\eta}C':\texttt{M}_{\epsilon_1}A\ \ A\leq:B\ \ \epsilon_1\leq\epsilon_2}{\Gamma\vdash C=_{\beta\eta}C':\texttt{M}_{\epsilon_2}B}$

7.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

Each derivation of $\Gamma\vdash t=_{\beta\eta}t':\tau$ can be converted to a derivation of $\Gamma\vdash t:\tau$ and $\Gamma\vdash t':\tau$ by induction over the beta-eta equivalence relation derivation.

7.2.1 Equivalence Relations

Case Reflexive By inversion we have a derivation of $\Gamma\vdash t:\tau$.

Case Symmetric By inversion $\Gamma\vdash t'=_{\beta\eta}t:\tau$. Hence by induction, derivations of $\Gamma\vdash t':\tau$ and $\Gamma\vdash t:\tau$ are given.

Case Transitive By inversion, there exists t_2 such that $\Gamma\vdash t_1=_{\beta\eta}t_2:\tau$ and $\Gamma\vdash t_2=_{\beta\eta}t_3:\tau$. Hence by induction, we have derivations of $\Gamma\vdash t_1:\tau$ and $\Gamma\vdash t_3:\tau$

7.2.2 Beta conversions

Case Lambda By inversion, we have $\Gamma,x:A\vdash C:\texttt{M}_\epsilon B$ and $\Gamma\vdash v:A$. Hence by the typing rules, we have:

$$(\text{Apply})\frac{(\text{Lambda})\frac{\Gamma,x:A\vdash C:\texttt{M}_\epsilon B}{\Gamma\vdash\lambda x:A.C:A\to\texttt{M}_\epsilon B}\quad\Gamma\vdash v:A}{\Gamma\vdash(\lambda x:A.C)\ v:\texttt{M}_\epsilon A}$$

By the substitution rule **TODO: which?**, we have

$$(\text{Substitution})\frac{\Gamma,x:A\vdash C:\texttt{M}_\epsilon B\ \ \Gamma\vdash v:A}{\Gamma\vdash C\,[v/x]:\texttt{M}_\epsilon B}$$

Case Left Unit By inversion, we have $\Gamma\vdash v:A$ and $\Gamma,x:A\vdash C:\texttt{M}_\epsilon B$

Hence we have:

$$(\text{Bind})\frac{(\text{Return})\frac{\Gamma\vdash v:A}{\Gamma\vdash\texttt{return}v:\texttt{M}_1 A}\quad\Gamma,x:A\vdash C:\texttt{M}_\epsilon B}{\Gamma\vdash\texttt{do}\ x\leftarrow\texttt{return}v\ \texttt{in}\ C:\texttt{M}_{1\cdot\epsilon}B=\texttt{M}_\epsilon B}\tag{7.1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Gamma\vdash C\,[v/x]:\texttt{M}_\epsilon B\tag{7.2}$$

Case Right Unit By inversion, we have $\Gamma\vdash C:\texttt{M}_\epsilon A$.

Hence we have:

$$(\text{Bind})\frac{\Gamma\vdash C:\texttt{M}_\epsilon A\quad(\text{Return})\frac{(\text{var})\frac{}{\Gamma,x:A\vdash x:A}}{\Gamma,x:A\vdash\texttt{return}v:\texttt{M}_1 A}}{\Gamma\vdash\texttt{do}\ x\leftarrow C\ \texttt{in}\ \texttt{return}x:\texttt{M}_{\epsilon\cdot 1}A=\texttt{M}_\epsilon A}\tag{7.3}$$

Case Associativity By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$(\iota\pi\times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Gamma, x : A, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A \quad \text{(Bind)}\frac{\Gamma, x{:}A \vdash C_2 {:} \mathtt{M}_{\epsilon_2} B \quad \Gamma, x{:}A, y{:}B \vdash C_3 {:} \mathtt{M}_{\epsilon_3} C}{\Gamma, x{:}A \vdash x C_2 C_3 {:} \mathtt{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow C_2\ \mathtt{in}\ C_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\frac{\Gamma \vdash C_1 {:} \mathtt{M}_{\epsilon_1} A \quad \Gamma, x{:}A \vdash C_2 {:} \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 {:} \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \Gamma, y : B \vdash C_3 \colon \mathtt{M}_{\epsilon_3} C}{\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.5}$$

Case Eta By inversion, we have $\Gamma \vdash v \colon A \to \mathtt{M}_\epsilon B$

By weakening, we have $\iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\frac{\text{(App)}\frac{(\Gamma, x{:}A) \vdash x{:}A \quad \text{(weakening)}\frac{\Gamma \vdash v{:}A \to \mathtt{M}_\epsilon B \quad \iota\pi{:}\Gamma, x{:}A \triangleright \Gamma}{\Gamma, x{:}A \vdash v{:}A \to \mathtt{M}_\epsilon B}}{\Gamma, x{:}A \vdash v\ x {:} \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.(v\ x) \colon A \to \mathtt{M}_\epsilon B} \tag{7.6}$$

Case If True By inversion, we have $\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A$. Hence by the typing lemma **TODO: Which?**, we have $\Gamma \mathtt{Ok}$ so $\Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \quad \Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A \quad \Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ \mathtt{true}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A} \tag{7.7}$$

Case If False As above,

Hence

$$\text{(If)}\frac{\Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \quad \Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A \quad \Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ \mathtt{false}\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A} \tag{7.8}$$

7.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda By inversion, $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_\epsilon B$. Hence by induction $\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B$, and $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B$.

So

$$\Gamma \vdash \lambda x : A.C_1 \colon A \to \mathtt{M}_\epsilon B \tag{7.9}$$

and

$$\Gamma \vdash \lambda x : A.C_2 \colon A \to \mathtt{M}_\epsilon B \tag{7.10}$$

Hold.

Case Return By inversion, $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, so by induction

$$\Gamma \vdash v_1 \colon A$$

and

$$\Gamma \vdash v_2 \colon A$$

Hence we have

$$\Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_{\mathbf{1}} A$$

and

$$\Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_{\mathbf{1}} A$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \mathtt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$. Hence we have by induction $\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B$, $\Gamma \vdash v_2 \colon A$, $\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B$, and $\Gamma \vdash v_2' \colon A$.

So we have:

$$\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B \tag{7.11}$$

and

$$\Gamma \vdash v_1'\ v_2' \colon \mathtt{M}_\epsilon B \tag{7.12}$$

Case Bind By inversion, we have: $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.13}$$

$$\Gamma \vdash \mathtt{do}\ x \leftarrow C_1'\ \mathtt{in}\ C_2' \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.14}$$

Case If By inversion, we have: $\Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$, and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon A$.

Hence by induction, we have:
$\Gamma \vdash v \colon \mathtt{Bool}$, $\Gamma \vdash v' \colon \mathtt{Bool}$,
$\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A$,
$\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A$, and $\Gamma \vdash C_2' \colon \mathtt{M}_\epsilon A$.
So

$$\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon \mathtt{M}_\epsilon A \tag{7.15}$$

and

$$\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1'\ \mathtt{else}\ C_2' \colon \mathtt{M}_\epsilon A \tag{7.16}$$

Hold.

Case Subtype By inversion, we have $A \leq: B$ and $\Gamma \vdash v =_{\beta\eta} v' \colon A$. By induction, we therefore have $\Gamma \vdash v \colon A$ and $\Gamma \vdash v' \colon A$.

Hence we have

$$\Gamma \vdash v \colon B \tag{7.17}$$

$$\Gamma \vdash v' \colon B \tag{7.18}$$

Case subeffect By inversion we have: $A \leq: B$, $\epsilon_1 \leq \epsilon_2$, and $\Gamma \vdash C =_{\beta\eta} C' \colon \mathtt{M}_{\epsilon_1} A$.
Hence by inductive hypothesis, we have $\Gamma \vdash C \colon \mathtt{M}_{\epsilon_1} A$ and $\Gamma \vdash C' \colon \mathtt{M}_{\epsilon_1} A$.
Hence,

$$\Gamma \vdash C \colon \mathtt{M}_{\epsilon_2} B \tag{7.19}$$

and

$$\Gamma \vdash C' \colon \mathtt{M}_{\epsilon_2} B \tag{7.20}$$

hold.

7.3 Beta-Eta equivalent terms have equal denotations

If $t \vdash t' =_{\beta\eta} \tau$: then $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$
By induction over Beta-eta equivalence relation.

7.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

Case Reflexive Equality is reflexive, so if $\Gamma \vdash t \colon \tau$ then $[\![\Gamma \vdash t \colon \tau]\!]_M$ is equal to itself.

Case Symmetric By inversion, if $\Gamma \vdash t =_{\beta\eta} t' \colon \tau$ then $\Gamma \vdash t' =_{\beta\eta} t \colon \tau$, so by induction $[\![\Gamma \vdash t' \colon \tau]\!]_M = [\![\Gamma \vdash t \colon \tau]\!]_M$ and hence $[\![\Gamma \vdash t \colon \tau]\!]_M = [\![\Gamma \vdash t' \colon \tau]\!]_M$

Case Transitive There must exist t_2 such that $\Gamma \vdash t_1 =_{\beta\eta} t_2 \colon \tau$ and $\Gamma \vdash t_2 =_{\beta\eta} t_3 \colon \tau$, so by induction, $[\![\Gamma \vdash t_1 \colon \tau]\!]_M = [\![\Gamma \vdash t_2 \colon \tau]\!]_M$ and $[\![\Gamma \vdash t_2 \colon \tau]\!]_M = [\![\Gamma \vdash t_3 \colon \tau]\!]_M$. Hence by transitivity of equality, $[\![\Gamma \vdash t_1 \colon \tau]\!]_M = [\![\Gamma \vdash t_3 \colon \tau]\!]_M$

7.3.2 Beta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

Case Lambda Let $f = [\![\Gamma, x : A \vdash C \colon \mathtt{M}_\epsilon B]\!]_M : (\Gamma \times A) \to T_\epsilon B$
Let $g = [\![\Gamma \vdash v \colon A]\!]_M : \Gamma \to A$
By the substitution denotation,

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned}
[\![\Gamma \vdash (\lambda x : A.C)\ v \colon \mathtt{M}_\epsilon B]\!]_M &= \mathtt{app} \circ \langle \mathtt{cur}(f), g \rangle \\
&= \mathtt{app} \circ (\mathtt{cur}(f) \times \mathtt{Id}_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
&= f \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
&= [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M
\end{aligned} \tag{7.21}$$

Case Left Unit Let $f = [\![\Gamma, x : A \vdash C{:}\, \mathtt{M}_\epsilon B]\!]_M$

Let $g = [\![\Gamma \vdash v{:}\, A]\!]_M : \Gamma \to A$

By the substitution denotation,

$$[\![\Gamma \vdash [v/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g\rangle$$

We have

$$[\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g\rangle$$

And hence

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return} v\ \mathtt{in}\ C{:}\,\mathtt{M}_\epsilon B]\!]_M =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \eta_A \circ g\rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times \eta_A) \circ \langle \mathtt{Id}_\Gamma, g\rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \eta_{(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By Tensor strength + unit} \\
=& \mu_{\mathbf{1},\epsilon,B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By Naturality of } \eta \\
=& f \circ \langle \mathtt{Id}_\Gamma, g\rangle \quad \text{By left unit law} \\
=& [\![\Gamma \vdash C\,[v/x] : \mathtt{M}_\epsilon B]\!]_M
\end{aligned} \tag{7.22}$$

Case Right Unit Let $f = [\![\Gamma \vdash C{:}\, \mathtt{M}_\epsilon A]\!]_M$

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{do}\ x \leftarrow C\ \mathtt{in}\ \mathtt{return} x{:}\,\mathtt{M}_\epsilon A]\!]_M &= \mu_{\epsilon,\mathbf{1},A} \circ T_\epsilon(\eta_A \circ \pi_2) \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= \pi_2 \circ \langle \mathtt{Id}_\Gamma, f\rangle \\
&= f
\end{aligned} \tag{7.23}$$

Case Associative Let

$$f = [\![\Gamma \vdash C_1{:}\, \mathtt{M}_\epsilon A]\!]_M \tag{7.24}$$

$$g = [\![\Gamma, x : A \vdash C_2{:}\, \mathtt{M}_\epsilon B]\!]_M \tag{7.25}$$

$$h = [\![\Gamma, y : B \vdash C_3{:}\, \mathtt{M}_\epsilon C]\!]_M \tag{7.26}$$

We also have the weakening:

$$\iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \tag{7.27}$$

With denotation:

$$[\![\iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B]\!]_M = (\pi_1 \times \mathtt{Id}_B) \tag{7.28}$$

We need to prove that the following are equal

$$lhs = [\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow C_2\ \mathtt{in}\ C_3){:}\, \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_2}]\!]_M \tag{7.29}$$

$$= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \big\langle \mathtt{Id}_{(\Gamma\times A)}, g\big\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \tag{7.30}$$

$$rhs = [\![\Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2)\ \mathtt{in}\ C_3{:}\, \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_2}]\!]_M \tag{7.31}$$

$$= \mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_1\cdot\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_1\cdot\epsilon_2,\Gamma,B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle)\rangle \tag{7.32}$$

$$\tag{7.33}$$

Let's look at fragment F of rhs.

$$F = \mathtt{t}_{\epsilon_1\cdot\epsilon_2,\Gamma,B} \circ \langle \mathtt{Id}_\Gamma, (\mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle)\rangle \tag{7.34}$$

So

$$rhs = \mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_1\cdot\epsilon_2}(h) \circ F \tag{7.35}$$

$$\begin{aligned}
F &= \mathtt{t}_{\epsilon_1\cdot\epsilon_2,\Gamma,B} \circ (\mathtt{Id}_\Gamma \times \mu_{\epsilon_1,\epsilon_2,B}) \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \\
&= \mu_{\epsilon_1,\epsilon_2,(\Gamma\times B)} \circ T_{\epsilon_1}\mathtt{t}_{\epsilon_2,\Gamma,B} \circ \mathtt{t}_{\epsilon_1,\Gamma,(T_{\epsilon_2}B)} \circ (\mathtt{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \quad \text{By } \textbf{TODO: ref: mu+tstrength} \\
&= \mu_{\epsilon_1,\epsilon_2,(\Gamma\times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1,\Gamma,(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \quad \text{By naturality of t-strength}
\end{aligned} \tag{7.36}$$

Since $rhs = \mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_1\cdot\epsilon_2}(h) \circ F$,

$$\begin{aligned}
rhs =&\mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_1\cdot\epsilon_2}(h) \circ \mu_{\epsilon_1,\epsilon_2,(\Gamma\times B))} \circ T_{\epsilon_1}(\mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1,\Gamma,(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \\
=&\mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,C} \circ \mu_{\epsilon_1,\epsilon_2,(T_{\epsilon_3}C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1,\Gamma,(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \quad \text{Naturality of } \mu \\
=&\mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\mathtt{Id}_\Gamma \times g)) \circ \mathtt{t}_{\epsilon_1,\Gamma,(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle
\end{aligned} \tag{7.37}$$

Let's now look at the fragment G of rhs

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1,\Gamma,(\Gamma\times A)} \circ \langle \mathtt{Id}_\Gamma, \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle\rangle \tag{7.38}$$

So

$$rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{7.39}$$

By folding out the $\langle ..., ...\rangle$, we have

$$G = T_{\epsilon_1}(\mathtt{Id}_\Gamma \times g) \circ \mathtt{t}_{\epsilon_1,\Gamma,\Gamma\times A} \circ (\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle \tag{7.40}$$

From the rule **TODO: Ref** showing the commutivity of tensor strength with α, the following commutes

$$\begin{tikzcd}
\Gamma \arrow[r, "{\langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle}"] & \Gamma \times (\Gamma \times T_{\epsilon_1} A) \arrow[d, "\mathtt{Id}_\Gamma \times \mathtt{t}_{\epsilon_1,\Gamma,A}"] & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \arrow[l, "{\alpha_{\Gamma,\Gamma,(T_{\epsilon_1}A)}}"'] \arrow[d, "{\mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A}}"] \\
& \Gamma \times T_{\epsilon_1}(\Gamma \times A) \arrow[d, "{\mathtt{t}_{\epsilon_1,\Gamma,\Gamma\times A}}"] & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \arrow[dl, "{T_{\epsilon_1}\alpha_{\Gamma,\Gamma,A}}"] \\
& T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) &
\end{tikzcd}$$

Where $\alpha : ((\_ \times \_) \times \_) \to (\_ \times (\_ \times \_))$ is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle \tag{7.41}$$

$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2\rangle, \pi_2 \circ \pi_2\rangle \tag{7.42}$$

So:

$$\begin{aligned}
G =&T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ \alpha^{-1}_{\Gamma,\Gamma,(T_{\epsilon_1}A)} \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f\rangle\rangle \\
=&T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By definition of } \alpha \text{ and products} \\
=&T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma\rangle \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By tensor strength's left-naturality} \\
=&T_{\epsilon_1}((\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle
\end{aligned} \tag{7.43}$$

Since

$$rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{7.44}$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\ &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \mathtt{Id}_B)) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g\right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \quad \text{By Left-Tensor Stre} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \tag{7.45}$$

Case Eta Let

$$f = [\![\Gamma \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \tag{7.46}$$

By weakening, we have

$$[\![\Gamma, x : A \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M = f \circ \pi_1 : \Gamma \times A \to (T_\epsilon B)^A \tag{7.47}$$

$$[\![\Gamma, x : A \vdash v\ x\colon \mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle \tag{7.48}$$

$$\tag{7.49}$$

Hence, we have

$$\begin{aligned} [\![\Gamma \vdash \lambda x : A.(v\ x)\colon A \to \mathtt{M}_\epsilon B]\!]_M &= \mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) \\ \mathtt{app} \circ ([\![\Gamma \vdash \lambda x : A.(v\ x)\colon A \to \mathtt{M}_\epsilon B]\!]_M \times \mathtt{Id}_A) &= \mathtt{app} \circ (\mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle) \times \mathtt{Id}_A) \\ &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2\rangle \\ &= \mathtt{app} \circ (f \times \mathtt{Id}_A) \end{aligned} \tag{7.50}$$

Hence, by the fact that $\mathtt{cur}(f)$ is unique in a cartesian closed category,

$$[\![\Gamma \vdash \lambda x : A.(v\ x)\colon A \to \mathtt{M}_\epsilon B]\!]_M = f = [\![\Gamma \vdash v\colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.51}$$

Case If-True Let

$$f = [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.52}$$

$$g = [\![\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.53}$$

$$\tag{7.54}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{true},A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle [\![\mathtt{true}]\!]_M \circ \langle\rangle_\Gamma, \langle f, g\rangle\rangle \\ &= f \\ &= [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{7.55}$$

Case If-False Let

$$f = [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.56}$$

$$g = [\![\Gamma \vdash C_2\colon \mathtt{M}_\epsilon A]\!]_M \tag{7.57}$$

$$\tag{7.58}$$

Then

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\mathtt{false},A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\colon \mathtt{M}_\epsilon A]\!]_M &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle [\![\mathtt{false}]\!]_M \circ \langle\rangle_\Gamma, \langle f, g\rangle\rangle \\ &= g \\ &= [\![\Gamma \vdash C_1\colon \mathtt{M}_\epsilon A]\!]_M \end{aligned} \tag{7.59}$$

7.3.3 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of subexpressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

Case Lambda By inversion, we have $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma, x : A \vdash C_1 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M \tag{7.60}$$

And so

$$[\![\Gamma \vdash \lambda x : A.C_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) = [\![\Gamma \vdash \lambda x : A.C_2 \colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.61}$$

Case Return By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M \tag{7.62}$$

And so

$$[\![\Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_{\mathbf{1}} A]\!]_M = \eta_A \circ f = [\![\Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_{\mathbf{1}} A]\!]_M \tag{7.63}$$

Case Apply By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to \mathtt{M}_\epsilon B$ and $\Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B]\!]_M$ and $[\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma \vdash v_1' \colon A \to \mathtt{M}_\epsilon B]\!]_M \tag{7.64}$$

$$g = [\![\Gamma \vdash v_2 \colon A]\!]_M = [\![\Gamma \vdash v_2' \colon A]\!]_M \tag{7.65}$$

And so

$$[\![\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon A]\!]_M = \mathtt{app} \circ \langle f, g \rangle = [\![\Gamma \vdash v_1'\ v_2' \colon \mathtt{M}_\epsilon A]\!]_M \tag{7.66}$$

Case Bind By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$ and $\Gamma, x : A \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon B$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{7.67}$$

$$g = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{7.68}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= [\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{7.69}$$

Case If By inversion, we have $\Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Gamma \vdash C_1 =_{\beta\eta} C_1' \colon \mathtt{M}_\epsilon A$ and $\Gamma \vdash C_2 =_{\beta\eta} C_2' \colon \mathtt{M}_\epsilon A$ By induction, we therefore have $[\![\Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M$, $[\![\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_\epsilon A]\!]_M$ and $[\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_\epsilon B]\!]_M$

Then let

$$f = [\![\Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Gamma \vdash v' \colon B]\!]_M \tag{7.70}$$

$$g = [\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_1' \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{7.71}$$

$$h = [\![\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Gamma, x : A \vdash C_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{7.72}$$

And so

$$\begin{aligned} [\![\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \colon]\!]_{\mathbf{M}} &= \mathtt{If}_{\mathtt{M}_\epsilon A} \circ \langle f, \langle g, h \rangle \rangle \\ &= [\![\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{7.73}$$

Case Subtype By inversion, we have $\Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, and $A \leq: B$ By induction, we therefore have $[\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{7.74}$$

$$g = [\![A \leq: B]\!]_M \tag{7.75}$$

And so

$$[\![\Gamma \vdash v_1 \colon B]\!]_M = g \circ f = [\![\Gamma \vdash v_1 \colon B]\!]_M \tag{7.76}$$

Case subeffect By inversion, we have $\Gamma \vdash C_1 =_{\beta\eta} C_2 \colon \mathtt{M}_{\epsilon_1} A$, and $A \leq: B$ and $\epsilon_1 \leq \epsilon_2$ By induction, we therefore have $[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon_1} A]\!]_M$

Then let

$$f = [\![\Gamma \vdash v_1 \colon A]\!]_M = [\![\Gamma \vdash v_2 \colon B]\!]_M \tag{7.77}$$

$$g = [\![A \leq: B]\!]_M \tag{7.78}$$

$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M \tag{7.79}$$

And so

$$[\![\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = h_B \circ T_{\epsilon_1} g \circ f = [\![\Gamma \vdash C_2 \mathtt{M}_{\epsilon_2} B \colon]\!]_{\mathbf{M}} \tag{7.80}$$

□