Chapter 1

Adequacy of a Model of the Polymorphic Effect Calculus

1.1 Instantiation of the Polymorphic Effect Calculus

Let us instantiate the polymorphic effect calculus to be a language in which one can write programs which can create an output signal. The effect system of EC is then used to count an upper bound on the number of outputs that a program can make. This language shall be called

1.1.1 Ground Types

We simply use the basic ground types.

$$\gamma ::= \texttt{Bool} \mid \texttt{Unit} \tag{1.1}$$

1.1.2 Graded Monad

We grade index the base graded monad with a partially ordered monoid derived from the natural numbers.

$$E = (\mathbb{N}, 0, +, \leq) \tag{1.2}$$

This extended as described in the dissertation **TODO: Ref**, to symbolically include variables α, β, γ which range over the natural numbers.

This means that the $\texttt{do}\ x \leftarrow v\ \texttt{in}\ v'$ type rule adds together the upper bounds on the two expressions to give an upper bound on the number of outputs of the sequenced expression. The $\texttt{return}\ v$ type rule acknowledges that a pure expression does not have any output.

1.1.3 Constants

We extend the set of constant, built in expressions to include a `put` statement which makes a single output action.

$$\texttt{k}^{A} ::= \texttt{true}^{\texttt{Bool}} \mid \texttt{false}^{\texttt{Bool}} \mid \texttt{()}^{\texttt{Unit}} \mid \texttt{put}^{\texttt{M}_1\texttt{Unit}} \tag{1.3}$$

1.1.4 Subtyping

The ground subtyping relation is the trivial identity relation. This is extended using the subeffect and function subtyping rules given in **TODO: Ref**.

1.2 Instantiation of a Model of the Polymorphic Effect Calculus

Let us now instantiate a model of in the indexed category derived as in chapter **TODO: ref** from a model of the polymorphic version of in `Set`, the category of sets and functions.

1.2.1 Cartesian Closed Category

Is given by the usage of `Set`.

1.2.2 Graded Monad

The strong graded monad on `Set` is given by tagging values of the underlying type with the number of output operations required to compute that value.

$$\mathtt{T}_n^0 A = \{(n', a) \mid n' \leq n \wedge a \in A\} \tag{1.4}$$
$$\mu^0_{m,n,A} = (m', (n', a)) \mapsto (n' + m', a) \tag{1.5}$$
$$\eta^0_A = a \mapsto (0, a) \tag{1.6}$$
$$\mathtt{t}^0_{n,A,B} = (a, (n', b)) \mapsto (n', (a, b)) \tag{1.7}$$

1.2.3 Subeffecting Natural Transformations

These natural transformations are given by inclusion functions (identities), since $n \leq m \wedge (n', a) \in \mathtt{T}_n^0 A \implies (n' \leq n \leq m, a \in A) \implies (n', a) \in \mathtt{T}_m^0 A$. Other subtyping morphisms are generated using the usual method according to the subtype derivation.

1.2.4 Ground Denotations

We define denotations for ground types as follows:

$$[\![\Phi\mathtt{Unit}]\!] = \vec{\epsilon} \mapsto \{*\} \tag{1.8}$$
$$[\![\Phi\mathtt{Bool}]\!] = \vec{\epsilon} \mapsto \{\top, \bot\} \tag{1.9}$$

We then define denotations for the constant expressions, including the `put`operation.

$$[\![\mathtt{()}]\!] = \vec{\epsilon} \mapsto * \mapsto * \tag{1.10}$$
$$[\![\mathtt{true}]\!] = \vec{\epsilon} \mapsto * \mapsto \top \tag{1.11}$$
$$[\![\mathtt{false}]\!] = \vec{\epsilon} \mapsto * \mapsto \bot \tag{1.12}$$
$$[\![\mathtt{put}]\!] = \vec{\epsilon} \mapsto * \mapsto (1, *) \tag{1.13}$$
$$\tag{1.14}$$

1.2.5 Soundness

This category is now an Indexed-S-category and hence a sound model for when completed using the techniques in chapter **TODO: ref**.

1.2.6 Denotational Shorthands

In the remaining sections, I shall use $[\![vA]\!]$ to indicate $\vec{\epsilon} \mapsto ([\![\Phi \mid \diamond \vdash v\colon A]\!]\vec{\epsilon})(*)$.

Furthermore, I shall use some extra notation to aid manipulation of the dependently typed values.

Denotations of PEC terms are dependently typed functions $[\![\Phi \mid \Gamma \vdash v\colon A]\!] : \vec{\epsilon} : E^n \to \Gamma\vec{\epsilon} \to A\vec{\epsilon}$. However, we shall often want to talk about the result of applying such a term to a dependently typed argument. Therefore we introduce the notation $\_$ to apply the function in the functor-category domain.

$$\begin{aligned} d :&\quad \vec{\epsilon} : E^n \to A\vec{\epsilon} \to B\vec{\epsilon} \\ d :&\quad (\vec{\epsilon} : E^n \to A\vec{\epsilon}) \to (\vec{\epsilon} : E^n \to B\vec{\epsilon}) \\ \text{Let}\quad e :&\quad E^n \to A\vec{\epsilon} \\ de =&\quad \vec{\epsilon} \mapsto (d\vec{\epsilon})(e\vec{\epsilon}) \end{aligned}$$

Another piece of notation to introduce when dealing with dependent functions that return product types is to lift the dependent function into a product of dependent functions.

Definition 1.3.1.

$$\begin{aligned} d :&\quad \vec{\epsilon} : E^n \to (A\vec{\epsilon} \times B\vec{\epsilon}) \\ d :&\quad (E^n \to A\vec{\epsilon}) \times (E^n \to B\vec{\epsilon}) \\ d =&\quad (\vec{\epsilon} \mapsto \pi_1(d\vec{\epsilon}), \vec{\epsilon} \mapsto \pi_2(d\vec{\epsilon})) \end{aligned}$$

1.3 Programming With Put

Lemma 1.3.1 (Denotations of Powers of Put)**.** *Powers of put have a simple denotation.* $[\![\mathtt{put}^m M_m \mathtt{Unit}]\!] = \vec{\epsilon} \mapsto (m, *)$

This simple language now has some extra properties which the general EC does not have.

Definition 1.3.1 (Powers of Put as an Equational Equivalence Class)**.** *Define* $\mathtt{put}^n$ *as follows:*

Proof: By induction on m.

Case 0:

$$\mathtt{put}^0\ \mathtt{return}\ ()\ M_0\,\mathtt{Unit}$$

$$\mathtt{put}^{m+1}\ \mathtt{do}\ \_ \leftarrow \mathtt{put}^m\ \mathtt{in}\ \mathtt{put}\ M_{m+1}\,\mathtt{Unit}$$

$$[\![\mathtt{put}^0 \mathtt{M}_0 \mathtt{Unit}]\!] = \eta^n(*) = \vec{\epsilon} \mapsto (0, *) \tag{1.15}$$

Case m+1:

$$\begin{aligned} [\![\mathtt{put}^{m+1}\mathtt{M}_0\mathtt{Unit}]\!] &= (\mu^n \circ \mathtt{T}^n_m([\![\diamond \vdash \mathtt{put}\colon \mathtt{M}_1\mathtt{Unit}]\!] \circ \pi_1) \circ \mathtt{t}^n)(*, [\![\mathtt{put}^m\mathtt{M}_n\mathtt{Unit}]\!]) \\ &= \vec{\epsilon} \mapsto (m+1, *) \end{aligned}$$

1.4 Logical Relations

$$\lhd_{\Phi A} \in [\![\Phi A]\!] \times \tag{1.16}$$

1.4.1 Definition

Definition 1.4.1 (Logical Relation)**.**

$$
\begin{aligned}
d\mathit{Unit}v &\Leftrightarrow (d = * \wedge v\,\mathit{()Unit}) \\
d\mathit{Bool}v &\Leftrightarrow ((d = \top \wedge v\,\mathit{trueBool}) \vee (d = \bot \wedge v\,\mathit{falseBool})) \\
dA \to Bv &\Leftrightarrow (\forall e, u.eAu \implies d(e)B(v\ u)) \\
dM_n Av &\Leftrightarrow (d = (\vec{\epsilon} \mapsto n', d') \in T_n[\![A]\!] \\
&\qquad \wedge \exists v'.(d'Av' \wedge v'A \wedge v\,\mathit{do}\ \_ \leftarrow \mathit{put}^{n'}\ \mathit{in\ return}\ v'\ \ M_n A)) \\
d\forall a.Av &\Leftrightarrow \forall n \in E.\pi_n(d)A[n/\alpha]v\ n
\end{aligned}
$$

1.4.2 Subtyping

Theorem 1.4.1 (Logical Relation and Subtyping)**.** *If $A{\leq:}_\Phi B$ and dAv then dBv*

Proof: By induction on the derivation of $A{\leq:}_\Phi B$.

Case S-Ground: $A{\leq:}_\Phi B \implies A = B$, since ground subtyping is the identity relation.

Case S-Fn: $A{\leq:}_\Phi B \implies A = A_1 \to A_2, B = B_1 \to B_2$ where $B_1{\leq:}_\Phi A_1$ and $A_2{\leq:}_\Phi B_2$.
By the definition of the $\lhd_{\Phi A \to B}$ relation, $dA \to Bv \Leftrightarrow (\forall e, u.eAu \implies d(e)Bv\ u)$.
So

$$
\begin{aligned}
\forall e, u.eB_1u &\implies eA_1u && \text{By induction } B_1{\leq:}_\Phi A_1 \\
&\implies d(e)A_2u\ v && \text{By definition} \\
&\implies d(e)B_2u\ v && \text{By induction } A_2{\leq:}_\Phi B_2
\end{aligned}
$$

As required.

Case S-Effect: $\mathtt{M}_{n_1}A_1{\leq:}_\Phi\mathtt{M}_{n_2}A_2 \implies n_1 \leq n_2, A_1{\leq:}_\Phi A_2$

$$
\begin{aligned}
d\mathtt{M}_{n_1}A_1v &\implies d = (n_1', d') \wedge n_1' \leq n_1 \leq n_2 \wedge \exists v'.(d'A_1v' \wedge v\mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ \ \mathtt{M}_{n_1}A_1) \\
&\implies v_1'A_2 \wedge d'A_2v' \wedge v\mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in\ return}\ v'\ \ \mathtt{M}_{n_1}A_2 \\
&\implies d\mathtt{M}_{n_2}A_2v
\end{aligned}
$$

Case S-Quantification: $\forall\alpha.A_1{\leq:}_\Phi\forall\alpha.A_2 \implies A_1{\leq:}A_2$
So:

$$
\begin{aligned}
d\forall\alpha.A_1v &\implies \forall\epsilon.(d)A_1[\alpha/\epsilon]v[\alpha/\epsilon] \\
&\implies \forall\epsilon.(d)A_2[\alpha/\epsilon]v[\alpha/\epsilon] \\
&\implies d\forall\alpha.A_2v
\end{aligned}
$$

1.4.3 Fundamental Property

Let $\lhd_{\Phi \vdash \Gamma\ \texttt{Ok}} \in [\![\Gamma]\!] \times$ mean:

$$\rho \Gamma \sigma \Leftrightarrow \forall x. \rho(x) \Gamma(x) \sigma(x) \tag{1.17}$$

Theorem 1.4.2 (Fundamental Theorem). *If* $\rho\Gamma\sigma$ *then* $[\![\Phi \mid \Gamma \vdash v \colon A]\!]\rho A v[\sigma]$ *up to equational equivalence.*

Proof: By induction over the derivation of $\Phi \mid \Gamma \vdash v \colon A$

Case Variables:

$$[\![\Phi \mid \Gamma \vdash x \colon \Gamma(x)]\!]\rho = \rho(x)\Gamma(x)\sigma(x) \approx x[\sigma] \tag{1.18}$$

Case Constants:

$$\begin{aligned}
[\![\Phi \mid \Gamma \vdash \texttt{true} \colon \texttt{Bool}]\!]\rho &= \top \wedge \texttt{true}[\sigma]\texttt{trueBool} \quad \text{So } [\![\Phi \mid \Gamma \vdash \texttt{true} \colon \texttt{Bool}]\!]\rho\texttt{Booltrue}[\sigma] \\
[\![\Phi \mid \Gamma \vdash \texttt{false} \colon \texttt{Bool}]\!]\rho &= \bot \wedge \texttt{false}[\sigma]\texttt{falseBool} \quad \text{So } [\![\Phi \mid \Gamma \vdash \texttt{true} \colon \texttt{Bool}]\!]\rho\texttt{Boolfalse}[\sigma] \\
[\![\Phi \mid \Gamma \vdash \texttt{()} \colon \texttt{Unit}]\!]\rho &= * \wedge \texttt{()}[\sigma]\texttt{()Unit} \quad \text{So } [\![\Phi \mid \Gamma \vdash \texttt{()} \colon \texttt{Unit}]\!]\rho\texttt{Unit()}[\sigma] \\
[\![\Phi \mid \Gamma \vdash \texttt{put} \colon \texttt{M}_1\texttt{Unit}]\!]\rho &= (1, *) \wedge \texttt{putdo}\ _ \leftarrow \texttt{put}^1\ \texttt{in return ()}\ \ \texttt{M}_1\texttt{Unit}
\end{aligned}$$

Case Subtype:

$$[\![\Phi \mid \Gamma \vdash v \colon B]\!]\rho = [\![\Phi \mid \Gamma \vdash v \colon A]\!]\rho A v[\sigma] \tag{1.19}$$

Since $A {\leq}{:}_{\Phi} B \wedge dAv \implies dBv$, we have that $[\![\Phi \mid \Gamma \vdash v \colon B]\!]Bv[\sigma]$.

Case Fn: For all dAu,

$$\begin{aligned}
([\![\Phi \mid \Gamma \vdash \lambda x \colon A.v : A \to B]\!]\rho)d &= (\texttt{cur}([\![\Gamma, x \colon A \vdash v \colon B]\!])\rho)d \\
&= [\![\Gamma, x \colon A \vdash v \colon B]\!](\rho[x \mapsto d])
\end{aligned}$$

Since dAu, we have $(\rho[x \mapsto d])\Gamma, x \colon A(\sigma, x \colon= u)$, so by induction

$$\begin{aligned}
([\![\Phi \mid \Gamma \vdash \lambda x \colon A.v : A \to B]\!]\rho)d = [\![\Gamma, x \colon A \vdash v \colon B]\!](\rho[x \mapsto d])&Bv[\sigma, x \colon= u] \\
&Bv[\sigma][u/x] \\
&\approx (\lambda x \colon A.(v[\sigma])\ )\ u
\end{aligned}$$

Case Apply:

$$[\![\Phi \mid \Gamma \vdash v\ u \colon B]\!]\rho = ([\![\Phi \mid \Gamma \vdash v \colon A \to B]\!]\rho)([\![\Phi \mid \Gamma \vdash u \colon A]\!]\rho) \tag{1.20}$$

By induction $[\![\Phi \mid \Gamma \vdash v \colon A \to B]\!]\rho A \to Bv[\sigma]$ and $[\![\Phi \mid \Gamma \vdash u \colon A]\!]\rho A u[\sigma]$. So by the definition of $\lhd_{\Phi A \to B}$,

$$[\![\Phi \mid \Gamma \vdash v\ u \colon B]\!]\rho = ([\![\Phi \mid \Gamma \vdash v \colon A \to B]\!]\rho)([\![\Phi \mid \Gamma \vdash u \colon A]\!]\rho)B(v[\sigma])\ (u[\sigma]) \approx (v\ u)[\sigma]$$

Case Return:

$$[\![\Phi \mid \Gamma \vdash v\colon \mathtt{M}_0 A]\!]\rho = (\vec{\epsilon} \mapsto 0, [\![\Phi \mid \Gamma \vdash v\colon A]\!]\rho) \tag{1.21}$$

By induction, $[\![\Phi \mid \Gamma \vdash v\colon A]\!] A v[\sigma]$, so by picking $v' = v[\sigma]$

$$(\mathtt{return}\ v\ )[\sigma]\mathtt{return}\ (v[\sigma]) \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^0\ \mathtt{in}\ \mathtt{return}\ v' \quad \mathtt{M}_0 A \tag{1.22}$$

So $[\![\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_0 A]\!]\rho \mathtt{M}_0 A(\mathtt{return}\ v\ )[\sigma]$

Case Bind: By inversion, $([\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u : \mathtt{M}_{m+n} B]\!]\rho) = (\vec{\epsilon} \mapsto m' + n', d_u)$, where $(\vec{\epsilon} \mapsto n', d_u) = ([\![\Gamma, x\colon A \vdash u\colon \mathtt{M}_n B]\!](\rho[x \mapsto d_v]))$, and $(\vec{\epsilon} \mapsto n', d_v) = ([\![\Phi \mid \Gamma \vdash v\colon \mathtt{M}_m A]\!]\rho)$.

By induction, $(\vec{\epsilon} \mapsto m', d_v)\mathtt{M}_m A v[\sigma]$. So $\exists v'$ such that $v[\sigma]\mathtt{do}\ \_ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v' \quad \mathtt{M}_m A$. So $(\rho[x \mapsto d_v])\Gamma, x\colon A([\sigma], x\colon= v')$.

So by induction $[\![\Gamma, x\colon A \vdash u\colon \mathtt{M}_n B]\!](\rho[x \mapsto d_v])\mathtt{M}_n B u[\sigma, x\colon= v']$.

Hence, $\exists u'$ such that $u[\sigma, x\colon= v']$

$\mathtt{M}_{m+n} B$ and $d_u \mathtt{M}_n B u'$.

Hence,

$$\begin{aligned}
\mathtt{do}\ x \leftarrow v[\sigma]\ \mathtt{in}\ u[\sigma]\ \mathtt{do}\ x \leftarrow (\mathtt{do}\ _ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ \mathtt{return}\ v'\ \ )\ \mathtt{in}\ (u[\sigma])\ \mathtt{M}_{m+n} B \\
\approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{m'}\ \mathtt{in}\ u[\sigma, x\colon= v'] \\
\approx \mathtt{do}\ _ \leftarrow \mathtt{put}^{m'+n'}\ \mathtt{in}\ \mathtt{return}\ u'
\end{aligned}$$

So $[\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u : \mathtt{M}_{m+n} B]\!]\rho \mathtt{M}_{m+n} B(\mathtt{do}\ x \leftarrow v\ \mathtt{in}\ u\ )[\sigma]$.

Case If: By inversion, $[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!]\rho = \begin{cases} [\![\Phi \mid \Gamma \vdash v_1\colon A]\!]\rho & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!]\rho = \vec{\epsilon} \mapsto \top \\ [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]\rho & \text{If } [\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!]\rho = \vec{\epsilon} \mapsto \bot \end{cases}$.

By induction,

$$[\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!]\rho \mathtt{Bool} b[\sigma]$$
$$[\![\Phi \mid \Gamma \vdash v_1\colon A]\!]\rho A v_1[\sigma]$$
$$[\![\Phi \mid \Gamma \vdash v_2\colon A]\!]\rho A v_2[\sigma]$$

Case: $[\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!]\rho = \vec{\epsilon} \mapsto \top$ and $b\mathtt{true}\mathtt{Bool}$

$$[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!]\rho = [\![\Phi \mid \Gamma \vdash v_1\colon A]\!]\rho A v_1[\sigma] \approx (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ )[\sigma] \tag{1.23}$$

Case: $[\![\Phi \mid \Gamma \vdash b\colon \mathtt{Bool}]\!]\rho = \vec{\epsilon} \mapsto \bot$ and $b\mathtt{false}\mathtt{Bool}$

$$[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A]\!]\rho = [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]\rho A v_2[\sigma] \approx (\mathtt{if}_A\ b\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\ )[\sigma] \tag{1.24}$$

Case Effect-Gen: By inversion, $[\![\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A]\!] = \overline{[\![\Phi, \alpha \mid \Gamma \vdash v\colon A]\!]}$ and by induction, $\Phi, \alpha([\![\Phi, \alpha \mid \Gamma \vdash v\colon A]\!]\rho)Av[\sigma$

$$
\begin{aligned}
\forall\epsilon.(\llbracket \Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A \rrbracket\rho) &= \vec{\epsilon} \mapsto (\vec{\epsilon})(\langle \llbracket \Phi,\alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon},\epsilon')\rangle_{\epsilon' \in E}(\rho\vec{\epsilon}))\\
&= \vec{\epsilon} \mapsto (\llbracket \Phi,\alpha \mid \Gamma \vdash v\colon A \rrbracket(\vec{\epsilon},\epsilon)(\rho\vec{\epsilon}))\\
&= \vec{\epsilon} \mapsto (\langle \mathtt{Id}_I, \llbracket \Phi\epsilon \rrbracket\rangle^* \llbracket \Phi,\alpha \mid \Gamma \vdash v\colon A \rrbracket)\vec{\epsilon}(\rho\vec{\epsilon})\\
&= (\langle \mathtt{Id}_I, \llbracket \Phi\epsilon \rrbracket\rangle^* \llbracket \Phi,\alpha \mid \Gamma \vdash v\colon A \rrbracket)\rho\\
&= \llbracket \Phi \mid \Gamma \vdash v[\epsilon/\alpha]\colon A[\epsilon/\alpha] \rrbracket\\
&\lhd_{\Phi A[\epsilon/\alpha]}\, v[\epsilon/\alpha]\rho
\end{aligned}
$$

Case Effect-Spec: Required to prove $\llbracket \Phi \mid \Gamma \vdash v\ \epsilon\colon A[\epsilon/\alpha] \rrbracket \rho A[\epsilon/\alpha](v\ \epsilon)[\sigma] \approx (v[\sigma])\ \epsilon$.

By inversion and induction, $\llbracket \Phi \mid \Gamma \vdash v\colon \forall\alpha.A \rrbracket \forall\alpha.Av[\sigma]$. So,

$$\forall\epsilon.(\llbracket \Phi \mid \Gamma \vdash v\colon \forall\alpha.A \rrbracket\rho)A[\epsilon/\alpha](v[\sigma])\ \epsilon$$

$$
\begin{aligned}
\llbracket \Phi \mid \Gamma \vdash v\ \epsilon\colon A[\epsilon/\alpha] \rrbracket\rho &= \langle \mathtt{Id}, \llbracket \Phi\epsilon \rrbracket\rangle^*(\boldsymbol{\epsilon}_{\Phi,\beta A[\beta/\alpha]})(\llbracket \Phi \mid \Gamma \vdash v\colon \forall\alpha.A \rrbracket(\rho))\\
&= \vec{\epsilon} \mapsto (\llbracket \Phi \mid \Gamma \vdash v\colon \forall\alpha.A \rrbracket(\rho)\vec{\epsilon}) \qquad = (\llbracket \Phi \mid \Gamma \vdash v\colon \forall\alpha.A \rrbracket\rho)A[\epsilon/\alpha]v\ \epsilon
\end{aligned}
$$

1.5 Adequacy

Theorem 1.5.1 (Adequacy)**.** *For G defined as:*

$$G ::= \mathtt{Bool} \mid \mathtt{Unit} \mid \mathtt{M}_n G$$

Equality of denotations implies equational equality.

$$\llbracket vG \rrbracket = \llbracket uG \rrbracket \implies vuG \tag{1.25}$$

Proof: By induction on the structure of G, making use of the fundamental property **??**.

Case Boolean: Let $d = \llbracket v\mathtt{Bool} \rrbracket = \llbracket v\mathtt{Bool} \rrbracket \in \{\vec{\epsilon} \mapsto \top, \vec{\epsilon} \mapsto \bot\}$. By the fundamental property, $d\mathtt{Bool}v$ and $d\mathtt{Bool}v$.

Case: $d = \vec{\epsilon} \mapsto \top$ Then $v \approx \mathtt{true}u\mathtt{Bool}$

Case: $d = \vec{\epsilon} \mapsto \bot$ Then $v \approx \mathtt{false}u\mathtt{Bool}$

Case Unit: Let $* = \llbracket v\mathtt{Unit} \rrbracket = \llbracket v\mathtt{Unit} \rrbracket \in \{\vec{\epsilon} \mapsto *\}$. By the fundamental property, $d\mathtt{Unit}v$ and $d\mathtt{Unit}v$. Hence $v \approx \mathtt{()}u\mathtt{Unit}$.

Case T-Effect: Let $(\vec{\epsilon} \mapsto n', \vec{\epsilon} \mapsto d) = [\![v \mathtt{M}_n G]\!] = [\![u \mathtt{M}_n G]\!]$. By the fundamental property, $(\vec{\epsilon} \mapsto (n', d)) \mathtt{M}_n G v$ and $(\vec{\epsilon} \mapsto (n', d)) \mathtt{M}_n G u$. So there exists u', v' such that $d' G u'$ and $d' G u'$ and:

$$
\begin{aligned}
&v \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ v' \ \ \mathtt{M}_n G \\
&\quad \approx \mathtt{do}\ \_ \leftarrow \mathtt{put}^{n'}\ \mathtt{in}\ \mathtt{return}\ u' \\
&\quad \approx u
\end{aligned}
$$

□