We induct over the structure of typing derivations of $\Gamma \vdash t\colon \tau$, assuming $\omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash t\colon \tau$ and show that $\Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M = \Delta'$

0.0.1 Variable Terms

Case Var and Weaken We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\iota : \Gamma \triangleright \Gamma]\!]_M \tag{1}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x\colon A$, such that

$$\Delta_1 = \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \quad \text{By Induction} \tag{2}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Gamma'' \vdash x\colon A}{\Gamma'', x' : A' \vdash x\colon A} \tag{3}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \Delta_1 \circ \pi_1 \quad \text{By Definition} && (4)\\
&= \Delta \circ [\![\omega' : \Gamma'' \triangleright \Gamma]\!]_M \circ \pi_1 \quad \text{By induction} && (5)\\
&= \Delta \circ [\![\omega'\pi_1 : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By denotation of weakening} && (6)
\end{aligned}$$

If $\omega = \omega'\times$ Then

$$\begin{aligned}
\Gamma' &= \Gamma''', x' : B && (7)\\
\Gamma &= \Gamma'', x' : A' && (8)\\
B &\leq: A && (9)
\end{aligned}$$

If $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$(\text{Sub-type})\frac{(\text{var})\frac{}{\Gamma''', x{:}B \vdash x{:}B} \quad B \leq: A}{\Gamma' \vdash x\colon A} \tag{10}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![B \leq: A]\!]_M \circ \pi_2 \quad \text{By Definition} && (11)\\
&= \pi_2 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![B \leq: A]\!]_M) \quad \text{By the properties of binary products} && (12)\\
&= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By Definition} && (13)
\end{aligned}$$

Case $x \neq x'$ Then

$$\Delta = (\text{Weaken})\frac{()\frac{\Delta_1}{\Gamma'' \vdash x : A}}{\Gamma \vdash x : A} \tag{14}$$

By induction with $\omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x : A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{()\frac{\Delta_1'}{\Gamma''' \vdash x : A}}{\Gamma' \vdash x : A} \tag{15}$$

This preserves denotations:

By induction, we have

$$\Delta_1' = \Delta_1 \circ [\![\omega : \Gamma''' \triangleright \Gamma'']\!]_M \tag{16}$$

So we have:

$$\begin{aligned}
\Delta' &= \Delta_1' \circ \pi_1 \quad \text{By denotation definition} && (17)\\
&= \Delta_1 \circ [\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \quad \text{By induction} \circ \pi_1 && (18)\\
&= \Delta_1 \circ \pi_1 \circ ([\![\omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![A' \leq: B]\!]_M) \quad \text{By product properties} && (19)\\
&= \Delta \circ [\![\omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By definition} && (20)
\end{aligned}$$

0.0.2 Value Terms

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\omega : \Gamma' \triangleright \Gamma']\!]_M$, simply as ω.

Case Constant The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$(\text{Const})\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A} \tag{21}$$

By inversion, we have $\Gamma\mathtt{Ok}$, so we have $\Gamma'\mathtt{Ok}$.

Hence

$$(\text{Const})\frac{\Gamma' \mathtt{Ok}}{\Gamma' \vdash \mathtt{C}^A : A} \tag{22}$$

Holds.

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma'} \quad \text{By definition} && (23)\\
&= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} && (24)\\
&= \Delta \quad \text{By Definition} && (25)\\
& && (26)
\end{aligned}$$

Case Lambda By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Gamma, x:A \vdash C: \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C \colon A \to \mathtt{M}_\epsilon B} \tag{27}$$

Since $\omega : \Gamma' \triangleright \Gamma$, we have:

$$\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{28}$$

Hence, by induction, using $\omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Gamma', x:A \vdash C: \mathtt{M}_\epsilon B}}{\Gamma', x : A \vdash \lambda x : A.C \colon A \to \mathtt{M}_\epsilon B} \tag{29}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By Definition} && (30)\\
&= \mathtt{cur}(\Delta_1 \circ (\omega \times \mathtt{Id}_\Gamma)) \quad \text{By the denotation of } \omega\times && (31)\\
&= \mathtt{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} && (32)\\
&= \Delta \circ \omega \quad \text{By Definition} && (33)
\end{aligned}$$

Case Sub-typing

$$(\text{Sub-type})\frac{\Gamma \vdash v \colon A \quad A \leq: B}{\Gamma \vdash v \colon B} \tag{34}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash v \colon A} \tag{35}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-type})\frac{()\frac{\Delta_1'}{\Gamma' \vdash v:a} \quad A \leq: B}{\Gamma' \vdash v \colon B} \tag{36}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![A \leq: B]\!]_M \circ \Delta_1' \quad \text{By Definition} && (37)\\
&= [\![A \leq: B]\!]_M \circ \Delta_1 \circ \omega \quad \text{By induction} && (38)\\
&= \Delta \circ \omega \quad \text{By Definition} && (39)\\
& && (40)
\end{aligned}$$

0.0.3 Computation Terms

Case Return We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Gamma \vdash v:A}}{\Gamma \vdash \mathtt{return} v \colon \mathtt{M}_1 A} \tag{41}$$

Hence, by induction, with $\omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ'_1 such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v : A}}{\Gamma' \vdash \texttt{return} v \colon \texttt{M}_1 A} \tag{42}$$

This preserves denotations:

$$\begin{align} \Delta' &= \eta_A \circ \Delta'_1 \quad \text{By definition} \tag{43}\\ &= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1 \tag{44}\\ &= \Delta \circ \omega \quad \text{By Definition} \tag{45} \end{align}$$

Case Apply By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Gamma \vdash v_1 : A \to \texttt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma \vdash v_2 : A}}{\Gamma \vdash v_1 \; v_2 \colon \texttt{M}_\epsilon B} \tag{46}$$

By induction, this gives us the respective derivations: Δ'_1, Δ'_2 such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v_1 : A \to \texttt{M}_\epsilon B} \quad ()\frac{\Delta'_2}{\Gamma' \vdash v_2 : A}}{\Gamma' \vdash v_1 \; v_2 \colon \texttt{M}_\epsilon B} \tag{47}$$

This preserves denotations:

$$\begin{align} \Delta' &= \texttt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \tag{48}\\ &= \texttt{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \tag{49}\\ &= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \tag{50}\\ &= \Delta \circ \omega \quad \text{By Definition} \tag{51} \end{align}$$

Case If By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Gamma \vdash v : \texttt{Bool}} \quad ()\frac{\Delta_2}{\Gamma \vdash C_1 : \texttt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma \vdash C_2 : \texttt{M}_\epsilon A}}{\Gamma \vdash \texttt{if}_{\epsilon,A} \; v \; \texttt{then} \; C_1 \; \texttt{else} \; C_2 \colon \texttt{M}_\epsilon A} \tag{52}$$

By induction, this gives us the sub-derivations $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta'_1}{\Gamma' \vdash v : \texttt{Bool}} \quad ()\frac{\Delta'_2}{\Gamma' \vdash C_1 : \texttt{M}_\epsilon A} \quad ()\frac{\Delta'_3}{\Gamma' \vdash C_2 : \texttt{M}_\epsilon A}}{\Gamma' \vdash \texttt{if}_{\epsilon,A} \; v \; \texttt{then} \; C_1 \; \texttt{else} \; C_2 \colon \texttt{M}_\epsilon A} \tag{53}$$

And

$$\begin{align} \Delta'_1 &= \Delta_1 \circ \omega \tag{54}\\ \Delta'_3 &= \Delta_2 \circ \omega \tag{55}\\ \Delta'_3 &= \Delta_3 \circ \omega \tag{56} \end{align}$$

This preserves denotations. Since $\omega : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\omega : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

$$(T_\epsilon A)^\omega = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{57}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \omega)) = (T_\epsilon A)^\omega \circ \mathtt{cur}(f) \tag{58}$$

$$\begin{aligned}
\Delta' &= \mathtt{app} \circ (([\mathtt{cur}(\Delta'_2 \circ \pi_2), \mathtt{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (59)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \omega \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta'_1) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (60)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2 \circ (\mathtt{Id}_{\mathbf{1}} \times \omega)), \mathtt{cur}(\Delta_3 \circ \pi_2 \circ (\mathtt{Id}_{\mathbf{1}} \times \omega))] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (61)\\
&= \mathtt{app} \circ (([(T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\omega \text{ property} && (62)\\
&= \mathtt{app} \circ (((T_\epsilon A)^\omega \circ [\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (63)\\
&= \mathtt{app} \circ ((T_\epsilon A)^\omega \times \mathtt{Id}_{\Gamma'}) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (64)\\
&= \mathtt{app} \circ (\mathtt{Id}_{(T_\epsilon A)} \times \omega) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \mathtt{app}, (T_\epsilon A)^\omega && (65)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (66)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \omega \quad \text{By Definition of the diagonal morphism.} && (67)\\
&= \Delta \circ \omega && (68)
\end{aligned}$$

Case Bind By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind}) \frac{() \frac{\Delta_1}{\Gamma \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad () \frac{\Delta_2}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{69}$$

If $\omega : \Gamma' \triangleright \Gamma$ then $\omega \times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ'_1, Δ'_2 such that:

$$\Delta' = (\text{Bind}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash C_1 : \mathtt{M}_{\mathbb{E}_1} A} \quad () \frac{\Delta'_2}{\Gamma', x : A \vdash C_2 : \mathtt{M}_{\epsilon_2} B}}{\Gamma' \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{70}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta'_1 \rangle \quad \text{By definition} && (71)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2 \circ (\omega \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1, \Gamma', A} \circ \langle \mathtt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta'_1, \Delta'_2 && (72)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} && (73)\\
&= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} && (74)\\
&= \Delta \quad \text{By definition} && (75)
\end{aligned}$$

Case Sub-effect

$$(\text{Sub-effect})\frac{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_1} A \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_2} B} \tag{76}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon_1} A} \tag{77}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma' \vdash C\colon \mathtt{M}_{\epsilon_1} A} \quad A \leq: B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C\colon \mathtt{M}_{\epsilon_2} B} \tag{78}$$

This preserves denotations:

Let

$$g = [\![A \leq: B]\!]_M : A \to B \tag{79}$$
$$h = [\![\epsilon_1 \leq \epsilon_2]\!]_M : T_{\epsilon_1} \to T_{\epsilon_2} \tag{80}$$

Then

$$\Delta' = h_B \circ T_{\epsilon_1} g \circ \Delta_1' \quad \text{By Definition} \tag{81}$$
$$= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \omega \quad \text{By Induction} \tag{82}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{83}$$