

0.1 Helper Morphisms

0.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\tau_{A,B} : (A \times B) \rightarrow (B \times A) = \langle \pi_2, \pi_1 \rangle \quad (1)$$

$$\delta_A : A \rightarrow (A \times A) = \langle \text{Id}_A, \text{Id}_A \rangle \quad (2)$$

0.2 Denotations of Types

0.2.1 Denotation of Ground Types

The denotations of the default ground types, `Unit`, `Bool` should be as follows:

$$\llbracket \text{Unit} \rrbracket = 1 \quad (3)$$

$$\llbracket \text{Bool} \rrbracket = 1 + 1 \quad (4)$$

The mapping $\llbracket _ \rrbracket$ should then map each other ground type γ to an object in \mathbb{C} .

0.2.2 Denotation of Computation Type

Given a function $\llbracket _ \rrbracket$ mapping value types to objects in the category \mathbb{C} , we write the denotation of Computation types $\mathbb{M}_\epsilon A$ as so:

$$\llbracket \mathbb{M}_\epsilon A \rrbracket = T_\epsilon \llbracket A \rrbracket$$

Since we can infer the denotation function, we can include it implicitly and drop the denotation sign.

$$\llbracket \mathbb{M}_\epsilon A \rrbracket = T_\epsilon A$$

0.2.3 Denotation of Function Types

Given a function $\llbracket _ \rrbracket$ mapping types to objects in the category \mathbb{C} , we write the denotation of a function type $A \rightarrow \mathbb{M}_\epsilon B$ as so:

$$\llbracket A \rightarrow \mathbb{M}_\epsilon B \rrbracket = (T_\epsilon \llbracket B \rrbracket)^{\llbracket A \rrbracket}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$\llbracket A \rightarrow \mathbb{M}_\epsilon B \rrbracket = (T_\epsilon B)^A$$

0.2.4 Denotation of Type Environments

Given a function $\llbracket _ \rrbracket$ mapping types to objects in the category \mathbb{C} , we can define the denotation of an Ok type environment Γ .

$$\llbracket \diamond \rrbracket = 1$$

$$\llbracket \Gamma, x : A \rrbracket = (\llbracket \Gamma \rrbracket \times \llbracket A \rrbracket)$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$\llbracket \Gamma, x : A \rrbracket = \Gamma \times A$$

0.3 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$\llbracket \Gamma \vdash t : \tau \rrbracket : \Gamma \rightarrow \llbracket \tau \rrbracket$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t : \tau$ are equal, we need not refer to the derivation that yielded each denotation.

0.3.1 Denotation of Value Terms

- (Unit) $\frac{\Gamma 0k}{\llbracket \Gamma \vdash () : \mathbf{Unit} \rrbracket = \langle \rangle_{\Gamma} : \Gamma \rightarrow 1}$
- (Const) $\frac{\Gamma 0k}{\llbracket \Gamma \vdash c^A : A \rrbracket = \llbracket c^A \rrbracket \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket A \rrbracket}$
- (True) $\frac{\Gamma 0k}{\llbracket \Gamma \vdash \mathbf{true} : \mathbf{Bool} \rrbracket = \mathbf{inl} \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket \mathbf{Bool} \rrbracket = 1 + 1}$
- (False) $\frac{\Gamma 0k}{\llbracket \Gamma \vdash \mathbf{false} : \mathbf{Bool} \rrbracket = \mathbf{inr} \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket \mathbf{Bool} \rrbracket = 1 + 1}$
- (Var) $\frac{\Gamma 0k}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \Gamma \times A \rightarrow A}$
- (Weaken) $\frac{f = \llbracket \Gamma \vdash x : A \rrbracket : \Gamma \rightarrow A}{\llbracket \Gamma, y : B \vdash x : A \rrbracket = f \circ \pi_1 : \Gamma \times B \rightarrow A}$
- (Lambda) $\frac{f = \llbracket \Gamma, x : A \vdash C : \mathbf{M}_{\epsilon} B \rrbracket : \Gamma \times A \rightarrow T_{\epsilon} B}{\llbracket \Gamma \vdash \lambda x : A. C : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket = \mathbf{cur}(f) : \Gamma \rightarrow (T_{\epsilon} B)^A}$
- (Subtype) $\frac{f = \llbracket \Gamma \vdash v : A \rrbracket : \Gamma \rightarrow A \quad g = \llbracket A \leq B \rrbracket}{\llbracket \Gamma \vdash v : B \rrbracket = g \circ f : \Gamma \rightarrow B}$

0.3.2 Denotation of Computation Terms

- (Return) $\frac{f = \llbracket \Gamma \vdash v : A \rrbracket}{\llbracket \Gamma \vdash \mathbf{return} v : \mathbf{M}_1 A \rrbracket = \eta_A \circ f}$

- (If)
$$\frac{f = \llbracket \Gamma \vdash v : \text{Bool} \rrbracket : \Gamma \rightarrow 1 + 1 \quad g = \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket \quad h = \llbracket \Gamma \vdash C_2 : \mathbb{M}_\epsilon A \rrbracket}{\llbracket \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 : \mathbb{M}_\epsilon A \rrbracket = \text{app} \circ ((\text{cur}(g \circ \pi_2), \text{cur}(h \circ \pi_2)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \rightarrow T_\epsilon A}$$
- (Bind)
$$\frac{f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A : \Gamma \rightarrow T_{\epsilon_1} A \rrbracket \quad g = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B \rrbracket : \Gamma \times A \rightarrow T_{\epsilon_2} B}{\llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B \rrbracket = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathfrak{t}_{\Gamma, A, \epsilon_1} \circ \langle \text{Id}_\Gamma, f \rangle : \Gamma \rightarrow T_{\epsilon_1 \cdot \epsilon_2} B}$$
- (Subeffect)
$$\frac{f = \llbracket \Gamma \vdash c : \mathbb{M}_{\epsilon_1} A \rrbracket : \Gamma \rightarrow T_{\epsilon_1} A \quad g = \llbracket A \leq : B \rrbracket}{h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket} \llbracket \Gamma \vdash C : \mathbb{M}_{\epsilon_2} B \rrbracket = h_B \circ T_{\epsilon_1} g \circ f$$
- (Apply)
$$\frac{f = \llbracket \Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket : \Gamma \rightarrow (T_\epsilon B)^A \quad g = \llbracket \Gamma \vdash v_2 : A \rrbracket : \Gamma \rightarrow A}{\llbracket \Gamma \vdash v_1 v_2 : \mathbb{M}_\epsilon B \rrbracket = \text{app} \circ \langle f, g \rangle : \Gamma \rightarrow T_\epsilon B}$$