0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v\colon A$,there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Proof: We induct on the structure of terms.

Case Variables: To find the unique derivation of $\Phi \mid \Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$: Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is, if $A' \leq:_\Phi A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\dfrac{\Phi \vdash \Gamma', x : A' \mathtt{Ok}}{\Phi \mid \Gamma, x : A' \vdash x\colon A'} \qquad A' \leq: A}{\Phi \mid \Gamma', x : A' \vdash x\colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$: with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x\colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Phi \mid \Gamma' \vdash x\colon A'} \qquad A' \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A' \vdash x\colon A'}}{\Phi \mid \Gamma \vdash x\colon A'} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} \tag{3}$$

Case Constants: For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\dfrac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A\colon A} \qquad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_{\Phi} A$).

Case Lambda: The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v : A' \to B'$ is:

$$\text{(Subtype)}\dfrac{\text{(Lambda)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v\colon B}}{\Phi \mid \Gamma \vdash \lambda x : A.B : A \to B} \qquad A \to B \leq:_{\Phi} A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v : A' \to B'}$$

Where

$$\text{(Sub-Type)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v\colon B} \qquad B \leq:_{\Phi} B'}{\Phi \mid \Gamma, x : A \vdash v\colon B'} \tag{4}$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v\colon B'$ if it exists.

Case Return: The reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_{\epsilon}B$ is

$$\text{(Subtype)}\dfrac{\text{(Return)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon A}}{\Phi \mid \Gamma \vdash \mathtt{return}\ v : \mathtt{M}_{\mathtt{1}}A} \qquad \text{(Computation)}\dfrac{\mathtt{1} \leq_{\Phi} \epsilon \qquad A \leq:_{\Phi} B}{\mathtt{M}_{\mathtt{1}}A \leq:_{\Phi} \mathtt{M}_{\epsilon}B}}{\Phi \mid \Gamma \vdash \mathtt{return}\ v : B}$$

Where

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon A} \qquad A \leq: B}{\Phi \mid \Gamma \vdash v\colon B}$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v\colon B$

Case Apply: If

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1\colon A \to B} \qquad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1\colon A' \to B'}$$

and

$$\text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2\colon A''} \qquad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1\colon A' \to B'$ and $\Phi \mid \Gamma \vdash v_2\colon A'$

Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2\colon \mathtt{M}_{\epsilon'}B'$ as

$$\text{(Subtype)}\dfrac{\text{(Apply)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1\colon A \to B} \qquad \text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v\colon A''} \qquad A'' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash v\colon A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \qquad \text{(Computation)}\dfrac{\epsilon \leq_{\Phi} \epsilon' \qquad B \leq:_{\Phi} B'}{\mathtt{M}_{\epsilon}B \leq:_{\Phi} \mathtt{M}_{\epsilon'}B'}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon \mathtt{M}_{\epsilon'}B'}$$

Case If: Let

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon B} \qquad B \leq: \texttt{Bool}}{\Phi \mid \Gamma \vdash v\colon \texttt{Bool}} \tag{5}$$

$$\text{(Subtype)}\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_1\colon A'} \qquad A' \leq: A}{\Phi \mid \Gamma \vdash v_1\colon A} \tag{6}$$

$$\text{(Subtype)}\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2\colon A''} \qquad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2\colon A} \tag{7}$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v\colon \texttt{Bool}$, $\Phi \mid \Gamma \vdash v_1\colon A$, $\Phi \mid \Gamma \vdash v_2\colon A$.

Then the only reduced derivation of $\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A$ is:

TODO: Scale this properly

$$\text{(Subtype)}\frac{\text{(If)}\dfrac{\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon B} \quad B \leq: \texttt{Bool}}{\Phi \mid \Gamma \vdash v\colon \texttt{Bool}} \quad \text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_1\colon A'} \quad A' \leq: A}{\Phi \mid \Gamma \vdash v_1\colon A} \quad \text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v_2\colon A''} \quad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2\colon A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \qquad \texttt{M}_\epsilon A \leq:_\Phi \texttt{M}_\epsilon A}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 : A} \tag{8}$$

Case Bind: Let

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1} A} \qquad \text{(Computation)}\dfrac{A \leq:_\Phi A' \qquad \epsilon_1 \leq_\Phi \epsilon_1'}{\texttt{M}_{\epsilon_1} A \leq:_\Phi \texttt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1\colon \texttt{M}_{\epsilon_1'} A'} \tag{9}$$

$$\text{(Subtype)}\frac{\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v_2\colon \texttt{M}_{\epsilon_2} B} \qquad \text{(Computation)}\dfrac{B \leq:_\Phi B' \qquad \epsilon_2 \leq_\Phi \epsilon_2'}{\texttt{M}_{\epsilon_2} B \leq:_\Phi \texttt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A \vdash v_2\colon \texttt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms.

By weakening, $\Phi \vdash \iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2\colon B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2\colon B$.

$$\text{(Subtype)}\frac{\dfrac{\Delta''}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \texttt{M}_{\epsilon_2} B} \qquad \text{(Computation)}\dfrac{B \leq:_\Phi B' \qquad \epsilon_2 \leq_\Phi \epsilon_2'}{\texttt{M}_{\epsilon_2} B \leq:_\Phi \texttt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \texttt{M}_{\epsilon_2'} B'} \tag{11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v-2 : \texttt{M}_{\epsilon_1'\cdot\epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Subtype)}\dfrac{\text{(Bind)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v_1 : \texttt{M}_{\epsilon_1} A'} \qquad \dfrac{\Delta''}{\Phi \mid \Gamma \vdash \Gamma, x : A' : v_2 \texttt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \qquad \text{(Computation)}\dfrac{\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2' \qquad B \leq:_\Phi B'}{\texttt{M}_{\epsilon_1\cdot\epsilon_2} B \leq:_\Phi \texttt{M}_{\epsilon_1'\cdot\epsilon_2'} B'}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v-2 : \texttt{M}_{\epsilon_1'\cdot\epsilon_2'} B'} \tag{12}$$

Case Effect-Fn: The unique reduced derivation of $\Phi \mid \Gamma \vdash \Lambda\alpha.A : \forall\alpha.B$ is

$$\text{(Sub-type)}\dfrac{\text{(Effect-Fn)}\dfrac{\dfrac{\Delta}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\alpha.A} \qquad \forall\alpha.A \leq_\Phi \forall\alpha.B}{\Phi \mid \Gamma \vdash \Lambda\alpha.B : \forall\alpha.B} \tag{13}$$

Where

$$\text{(Sub-type)}\dfrac{\dfrac{\Delta}{\Phi, \alpha \mid \Gamma \vdash v : A} \qquad A \leq:_{\Phi,\alpha} B}{\Phi, \alpha \mid \Gamma \vdash v : B} \tag{14}$$

Is the unique reduced derivation of $\Phi, \alpha \mid \Gamma \vdash v : B$

Case Effect-App: The unique reduced derivation of $\Phi \mid \Gamma \vdash v\ \alpha : B'$ is

$$\text{(Subtype)}\dfrac{\text{(Effect-App)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : \forall\alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon : A\,[\epsilon/\alpha]} \qquad A\,[\epsilon/\alpha] \leq:_\Phi B'}{\Phi \mid \Gamma \vdash v\ \alpha : B'} \tag{15}$$

Where $B\,[\epsilon/\alpha] \leq:_\Phi B'$ and

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v : \forall\alpha.B} \qquad \text{(Quantification)}\dfrac{A \leq:_{\Phi,\alpha} B}{\forall\alpha.A \leq:_\Phi \forall\alpha.B}}{\Phi \mid \Gamma \vdash v : \forall\alpha.B} \tag{16}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v : A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

Case Constants: For the constants $\texttt{true}, \texttt{false}, \texttt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced.

$$reduce((\text{Const})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma \vdash \texttt{C}^A\colon A}) = (\text{Const})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma \vdash \texttt{C}^A\colon A}$$

Case Var:

$$reduce((\text{Var})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A}) = (\text{Var})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A} \tag{17}$$

Case Weaken:

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash x\colon A}}{\Phi \mid \Gamma, y : B \vdash x\colon A}) \tag{18}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x\colon A} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{19}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x\colon A'}}{\Phi \mid \Gamma, y : B \vdash x\colon A'} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma, y : B \vdash x\colon A} \tag{20}$$

Case Lambda:

reduce **definition** To find:

$$reduce((\text{Fn})\frac{\dfrac{\Delta}{\Phi \mid \Gamma, x : A \vdash v\colon B}}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to \epsilon_2 B}) \tag{21}$$

Let

$$(\text{Sub-type})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v\colon B'} \qquad B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A \vdash v\colon B} = reduce(\Delta) \tag{22}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\dfrac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v\colon \texttt{M}_{\epsilon_1} B'} \qquad A \to \epsilon_1 B' \leq:_\Phi A \to \epsilon_2 B}{\Phi \mid \Gamma \vdash \lambda x : A.v : A \to \epsilon_2 B} \tag{23}$$

Case Subtype:

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon A} \qquad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v\colon B}) \tag{24}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash x\colon A} \qquad A' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{25}$$

In

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v\colon A'} \qquad A' \leq:_{\Phi} A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v\colon B} \tag{26}$$

Case Return:

reduce **definition** To find:

$$reduce((\text{Return})\frac{\dfrac{\Delta}{\Phi \mid \Gamma \vdash v\colon A}}{\Phi \mid \Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}} A}) \tag{27}$$

Let

$$(\text{Sub-type})\frac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v\colon A'} \qquad A' \leq:_{\Phi} A}{\Phi \mid \Gamma \vdash v\colon A} = reduce(\Delta) \tag{28}$$

In

$$(\text{Sub-type})\frac{(\text{Return})\dfrac{\dfrac{\Delta'}{\Phi \mid \Gamma \vdash v\colon A'}}{\Phi \mid \Gamma \vdash \texttt{return}\ v\ \texttt{M}_{\texttt{1}} A'\colon} \qquad (\text{Computation})\dfrac{\texttt{1} \leq_{\Phi} \texttt{1} \qquad A' \leq:_{\Phi} A}{\texttt{M}_{\texttt{1}} A' \leq:_{\Phi} \texttt{M}_{\texttt{1}} A}}{\Phi \mid \Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}} A} \tag{29}$$

Case Apply:

reduce **definition** To find:

$$reduce((\text{Apply})\frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v_1\colon A \to B} \qquad \dfrac{\Delta_2}{\Phi \mid \Gamma \vdash v_2\colon A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B}) \tag{30}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1\colon A' \to B'} \qquad A' \to B' \leq:_\Phi A \to \epsilon B}{\Phi \mid \Gamma \vdash v_1\colon A \to B} = \mathit{reduce}(\Delta_1) \tag{31}$$

$$(\text{Subtype})\frac{\dfrac{\Delta_2'}{\Phi \mid \Gamma \vdash v\colon A'} \qquad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v_1\colon A} = \mathit{reduce}(\Delta_2) \tag{32}$$

In

$$(\text{Subtype})\frac{(\text{Apply})\dfrac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1\colon A' \to B'} \qquad (\text{Sub-type})\dfrac{\dfrac{\Delta_2'}{\Phi \mid \Gamma \vdash v_2\colon A''} \qquad A'' \leq:_\Phi A \leq:_\Phi A'}{\Phi \mid \Gamma \vdash v_2\colon A'}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon \mathtt{M}_{\epsilon'} B'} \qquad (\text{Computation})\dfrac{\epsilon' \leq_\Phi \epsilon \qquad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon'} B' \leq:_\Phi \mathtt{M}_\epsilon B}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{33}$$

Case If:

reduce **definition**

$$\mathit{reduce}((\text{If})\frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}} \qquad \dfrac{\Delta_2}{\Phi \mid \Gamma \vdash v_1\colon A} \qquad \dfrac{\Delta_3}{\Phi \mid \Gamma \vdash v_2\colon A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A}) = (\text{If})\frac{\dfrac{\mathit{reduce}(\Delta_1)}{\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}} \qquad \dfrac{\mathit{reduce}(\Delta_2)}{\Phi \mid \Gamma \vdash v_1\colon A} \qquad \dfrac{\mathit{reduce}(\Delta_3)}{\Phi \mid \Gamma \vdash v_2\colon A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A} \tag{34}$$

Case Bind:

reduce **definition** To find

$$\mathit{reduce}((\text{Bind})\frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A} \qquad \dfrac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}) \tag{35}$$

Let

$$(\text{Sub-Type})\frac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1'} A'} \qquad (\text{Computation})\dfrac{\epsilon_1' \leq_\Phi \epsilon_1 \qquad A' \leq:_\Phi A}{\mathtt{M}_{\epsilon_1'} A' \leq:_\Phi \mathtt{M}_{\epsilon_1} A}}{\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A} = \mathit{reduce}(\Delta_1) \tag{36}$$

Since $\Phi \vdash i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq:_\Phi A$, and by Δ_2, $\Phi \mid (\Gamma, x : A) \vdash v_2\colon \mathtt{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2\colon \mathtt{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-Type})\frac{\dfrac{\Delta_3'}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \mathtt{M}_{\epsilon_2'} B'} \qquad (\text{Computation})\dfrac{\epsilon_2' \leq_\Phi \epsilon_2 \qquad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon_2'} B' \leq:_\Phi \mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma, x : A' \vdash v_2\colon \mathtt{M}_{\epsilon_2} B} = \mathit{reduce}(\Delta_3) \tag{37}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$\text{(Sub-Type)}\frac{\text{(Bind)}\dfrac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \qquad \dfrac{\Delta_3'}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B} \qquad \text{(Computation)}\dfrac{\epsilon_1' \cdot \epsilon_2' \leq_\Phi \epsilon_1 \cdot \epsilon_2 \qquad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B' \leq:_\Phi \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{38}$$

Case Effect-Fn:

reduce **definition** To find

$$reduce(\text{(Effect-Lambda)}\frac{\dfrac{\Delta_1}{\Phi, \alpha \mid \Gamma \vdash v \colon A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A}) \tag{39}$$

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta_1'}{\Phi, \alpha \mid \Gamma \vdash v \colon A'} \qquad A' \leq:_\Phi A}{\Phi, \alpha \mid \Gamma \vdash v \colon A} = reduce(\Delta_1) \tag{40}$$

in

$$\text{(Subtype)}\frac{\text{(Effect-Fn)}\dfrac{\dfrac{\Delta_1'}{\Phi, \alpha \mid \Gamma \vdash v \colon A'}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A'} \qquad \text{(Quantification)}\dfrac{A' \leq:_{\Phi,\alpha}}{\forall\alpha.A' \leq:_\Phi \forall\alpha.A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{41}$$

Case Effect-Application:

reduce **definition** To find

$$reduce(\text{(Effect-App)}\frac{\dfrac{\Delta_1}{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]}) \tag{42}$$

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A'} \qquad \text{(Quantification)}\dfrac{A' \leq:_{\Phi,\alpha} A}{\forall\alpha.A' \leq:_\Phi \forall\alpha.A}}{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A} = reduce(\Delta_1) \tag{43}$$

In

$$
(\text{Subtype})\dfrac{(\text{E-app})\dfrac{\dfrac{\Delta_1'}{\Phi \mid \Gamma \vdash v \colon \forall\alpha.A} \qquad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \qquad A'\,[\epsilon/\alpha] \leq:_{\Phi} A\,[\epsilon/\alpha]}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{44}
$$

0.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v \colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!] = [\![reduce \Delta]\!] = [\![reduce \Delta']\!] = [\![\Delta']\!]$, hence the denotation $[\![\Phi \mid \Gamma \vdash v \colon A]\!]$ is unique.