We need to define substitutions of effects on effects, effects on types, effects on terms, terms on terms.

0.1 Effect Substitutions

Define a substitution, σ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \tag{1}$$

Define the free-effect Variables of σ:

$$\begin{aligned} fev(\diamond) &= \emptyset \\ fev(\sigma, \alpha := \epsilon) &= fev(\sigma) \cup fev(\epsilon) \end{aligned}$$

We define the property:

$$\alpha\#\sigma \Leftrightarrow \alpha \notin (\mathtt{dom}(\sigma) \cup fev(\sigma)) \tag{2}$$

0.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \tag{3}$$

$$\sigma(e) = e \tag{4}$$
$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \tag{5}$$
$$\diamond(\alpha) = \alpha \tag{6}$$
$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \tag{7}$$
$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \tag{8}$$

0.1.2 Action of Effect Substitution on Types

Define the effect of applying an effect substitution, σ to a type τ as:

$$\tau\,[\sigma]$$

Defined as so

$$\gamma\,[\sigma] = \gamma \tag{9}$$
$$(A \to \mathtt{M}_{\epsilon} B)\,[\sigma] = (A\,[\sigma]) \to \mathtt{M}_{\sigma(\epsilon)}(B\,[\sigma]) \tag{10}$$
$$(\mathtt{M}_{\epsilon} A)\,[\sigma] = \mathtt{M}_{\sigma(\epsilon)}(A\,[\sigma]) \tag{11}$$
$$(\forall \alpha. A)\,[\sigma] = \forall \alpha. (A\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{12}$$

0.1.3 Action of Effect-Substitution on Type Environments

Define the action of effect substitution on type environments:

$$\Gamma\,[\sigma]$$

Defined as so:

$$\begin{aligned} \diamond\,[\sigma] &= \diamond \\ (\Gamma, x : A)\,[\sigma] &= (\Gamma\,[\sigma]\,, x : (A\,[\sigma])) \end{aligned}$$

0.1.4 Action of Effect Substitution on Terms

Define the effect of effect-substitution on terms:

$$x\,[\sigma] = x \tag{13}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^{(A[\sigma])} \tag{14}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : (A\,[\sigma]).(C\,[\sigma]) \tag{15}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\sigma(\epsilon),(A[\sigma])}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{16}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{17}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \tag{18}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{19}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \sigma(\epsilon) \tag{20}$$
$$\tag{21}$$

0.1.5 Well-Formed-ness

For any two effect-environments, and a substitution, define the wellformedness relation:

$$\Phi' \vdash \sigma \colon \Phi \tag{22}$$

- (Nil) $\frac{\Phi'\mathtt{Ok}}{\Phi' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi' \vdash \sigma : \Phi \quad \Phi' \vdash \epsilon \quad \alpha \notin \Phi}{\Phi' \vdash \sigma, \alpha := \epsilon : (\Phi, \alpha)}$

0.1.6 Property 1

If $\Phi' \vdash \sigma \colon \Phi$ then $P'\mathtt{Ok}$ (By the Nil case) and $P\mathtt{Ok}$ Since each use of the extend case preserves $\mathtt{Ok}$.

0.1.7 Property 2

If $\Phi' \vdash \sigma \colon \Phi$ then $\omega : \Phi' \triangleright \Phi' \implies \Phi'' \vdash \sigma \colon \Phi$ since $\Phi' \vdash \epsilon \implies \Phi'' \vdash \epsilon$ and $\Phi'\mathtt{Ok} \implies \Phi''\mathtt{Ok}$

0.1.8 Property 3

If $\Phi' \vdash \sigma \colon \Phi$ then

$$\alpha \notin \wedge \alpha \notin \Phi' \implies (\Phi', \alpha) \vdash (\sigma, \alpha := \alpha) \colon (\Phi, \alpha) \tag{23}$$

Since $\iota\pi : \Phi', \alpha \triangleright \Phi'$ so $\Phi', \alpha \vdash \sigma \colon \Phi$ and $\Phi', \alpha \vdash \alpha$

0.2 Substitution Preserves the Well-formed-ness of Effects

I.e.

$$\Phi \vdash \epsilon \wedge \Phi' \vdash \iota \colon \Phi \implies \Phi' \vdash \sigma(\epsilon) \tag{24}$$

Proof:

Case Ground: $\sigma(e) = e$, so $\Phi' \vdash \sigma(\epsilon)$ holds.

Case Multiply: By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$ so $\Phi' \vdash \sigma(\epsilon_1)$ and $\Phi' \vdash \sigma(\epsilon_2)$ by induction and hence $\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)$

Case Var: By inversion, $\Phi = \Phi'', \alpha$ and $\Phi'', \alpha \texttt{Ok}$. Hence byu case splitting on ι, we see that $\sigma = \sigma', \alpha := \epsilon$.

So by inversion, $\sigma \vdash \epsilon$ so $\Phi' \vdash \sigma(\alpha) = \epsilon$

Case Weaken: By inversion$\Phi = \Phi'', \beta$ and $\Phi'' \vdash \alpha$, so $\sigma = \sigma'\beta := \epsilon$.

So $\Phi' \vdash \sigma' \colon \Phi''$.

hence by induction, $\Phi' \vdash \sigma'(a)$, so $\Phi' \vdash \sigma(\alpha)$ since $\alpha \neq \beta$)

0.2.1 Substitution preserves well-formed-ness of Types

$$\Phi' \vdash \sigma \colon \Phi \wedge \Phi \vdash A \implies \Phi' \vdash A[\sigma] \tag{25}$$

Proof:

Case Ground: $\Phi' \texttt{Ok}$ so $\Phi' \vdash \gamma$ and $\gamma[\sigma] = \gamma$.

Hence $\Phi' \vdash \gamma[\sigma]$.

Case Lambda: By inversion$\Phi \vdash A$ and $\Phi \vdash B$.

So by induction, $\Phi' \vdash A[\sigma]$ and $\Phi' \vdash B[\sigma]$.

So

$$\Phi' \vdash (A[\sigma]) \to (B[\sigma]) \tag{26}$$

So

$$\Phi' \vdash (A \to B)[\sigma] \tag{27}$$

Case Computation: By inversion, $\Phi \vdash \epsilon$ and $\Phi \vdash A$ so by induction and substitution of effect preserving effect-well-formed-ness,

$\Phi' \vdash \sigma(\epsilon)$ and $\Phi' \vdash A[\sigma]$ so $\Phi \vdash \texttt{M}_{\sigma(\epsilon)} A[\sigma]$ so $\Phi' \vdash (\texttt{M}_{\epsilon} A)[\sigma]$

Case For All: By inversion, $\Phi, \alpha \vdash A$. So by picking $\alpha \notin \Phi \wedge \alpha \notin \Phi'$ using α-equivalence, we have $(\Phi', \alpha) \vdash (\sigma\alpha := \alpha) \colon (\Phi, \alpha)$.

So by induction $(\Phi, \alpha) \vdash A[\sigma, \alpha := \alpha]$

So $(\Phi', \alpha) \vdash A[\sigma]$

So $\Phi' \vdash (\forall \alpha . A)[\sigma]$

0.3 Term-Term Substitutions

0.3.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{28}$$

0.3.2 Trivial Properties of substitutions

$\texttt{fv}(\sigma)$

$$\texttt{fv}(\diamond) = \emptyset \tag{29}$$

$$\texttt{fv}(\sigma, x := v) = \texttt{fv}(\sigma) \cup \texttt{fv}(v) \tag{30}$$

$\texttt{dom}(\sigma)$

$$\texttt{dom}(\diamond) = \emptyset \tag{31}$$

$$\texttt{dom}(\sigma, x := v) = \texttt{dom}(\sigma) \cup \{x\} \tag{32}$$

$x \# \sigma$

$$x \# \sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{33}$$

0.3.3 Action of substitutions

We define the effect of applying a substitution σ as

$$t\,[\sigma]$$

$$\begin{aligned}
x\,[\diamond] &= x & (34)\\
x\,[\sigma, x := v] &= v & (35)\\
x\,[\sigma, x' := v'] &= x\,[\sigma] \quad \text{If } x \neq x' & (36)\\
\mathtt{C}^A\,[\sigma] &= \mathtt{C}^A & (37)\\
(\lambda x : A.C)\,[\sigma] &= \lambda x : A.(C\,[\sigma]) \quad \text{If } x \# \sigma & (38)\\
(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] &= \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] & (39)\\
(v_1\ v_2)\,[\sigma] &= (v_1\,[\sigma])\ v_2\,[\sigma] & (40)\\
(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) &= \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x \# \sigma & (41)\\
(\Lambda \alpha.v)\,[\sigma] &= \Lambda \alpha.(v\,[\sigma]) & (42)\\
(v\ \epsilon)\,[\sigma] &= (v\,[\sigma])\ \epsilon & (43)\\
& & (44)
\end{aligned}$$

0.3.4 Well-Formed-ness

0.3.5 Simple Properties Of Substitution

If $\Gamma' \vdash \sigma \colon \Gamma$ then: **TODO: Number these**

Property 1: $\Gamma\mathtt{Ok}$ and $\Gamma'\mathtt{Ok}$ Since $\Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Gamma'' \vdash v \colon A$ holds if $\Gamma' \vdash v \colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ implies $(\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$(\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{45}$$

0.4 Substitution Preserves Typing

0.4.1 Variables

Case Var

Case Weaken

0.4.2 Other Value Terms

Case Lambda

Case Constants

0.4.3 Computation Terms

Case Return

Case Apply

Case If

Case Bind

0.4.4 Sub-typing and Sub-effecting

Case Sub-type

Case Sub-effect

0.5 Semantics of Substitution

0.5.1 Denotation of Substitutions

0.5.2 Extension Lemma

0.5.3 Substitution Theorem

0.5.4 Proof For Value Terms

Case Var

Case Weaken

Case Constants

Case Lambda

Case Sub-type

0.5.5 Proof For Computation Terms

Case Return

Case Apply

Case If

Case Bind

Case Subeffect

0.6 The Identity Substitution

0.6.1 Properties of the Identity Substitution

Property 1

Property 2