0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and subeffect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t : \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t : \tau$, there exists at most one reduced derivation of $\Gamma \vdash t : \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Variables

To find the unique derivation of $\Gamma \vdash x : A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x : A$ is, if $A' \leq: A$, as below:

$$(\text{Subtype})\frac{(\text{Var})\frac{\Gamma', x : A' \, \text{Ok}}{\Gamma, x : A' \vdash x : A'} \quad A' \leq: A}{\Gamma', x : A' \vdash x : A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.
   Hence, if $\Gamma \vdash x : A$ holds, then so must $\Gamma' \vdash x : A$.
   Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Gamma' \vdash x : A'} \quad A' \leq: A}{\Gamma' \vdash x : A} \tag{2}$$

Be the unique reduced derivation of $\Gamma' \vdash x : A$.
   Then the unique reduced derivation of $\Gamma \vdash x : A$ is:

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta}{\Gamma, x : A' \vdash x : A'}}{\Gamma \vdash x : A'} \quad A' \leq: A}{\Gamma \vdash x : A} \tag{3}$$

0.2.2 Constants

For each of the constants, (C^A, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c : A$ for a given A. I shall give examples using the case C^A

$$(\text{Subtype})\frac{(\text{Const})\frac{\Gamma \, \text{Ok}}{\Gamma \vdash \text{C}^A : A} \quad A \leq: B}{\Gamma \vdash \text{C}^A : B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

0.2.3 Value Terms

Case Lambda   The reduced derivation of $\Gamma \vdash \lambda x : A.C : A' \to \text{M}_{\epsilon'} B'$ is:

$$(\text{Subtype})\frac{(\text{Lambda})\frac{()\frac{\Delta}{\Gamma, x : A \vdash C : \text{M}_{\epsilon} B}}{\Gamma \vdash \lambda x : A.B : A \to \text{M}_{\epsilon} B} \quad A \to \text{M}_{\epsilon} B \leq: A' \to \text{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C : A' \to \text{M}_{\epsilon'} B'}$$

Where

$$(\text{Sub-Effect}) \frac{() \frac{\Delta}{\Gamma, x:A \vdash C : \mathtt{M}_\epsilon B} \quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma, x : A \vdash C : \mathtt{M}_{\epsilon'} B'} \tag{4}$$

is the reduced derivation of $\Gamma, x : A \vdash C : \mathtt{M}_\epsilon B$ if it exists.

Case Subtype   TODO: Do we need to write anything here? (Probably needs an explanation)

0.2.4   Computation Terms

Case Return   The reduced denotation of $\Gamma \vdash \mathtt{return}\, v : \mathtt{M}_\epsilon B$ is

$$(\text{Subtype}) \frac{(\text{Return}) \frac{() \frac{\Delta}{\Gamma \vdash v : A}}{\Gamma \vdash \mathtt{return}\, v : \mathtt{M}_1 A} \quad A \leq: B \quad 1 \leq \epsilon}{\Gamma \vdash \mathtt{return}\, v : \mathtt{M}_\epsilon B}$$

Where

$$(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v : A} \quad A \leq: B}{\Gamma \vdash v : B}$$

is the reduced derivation of $\Gamma \vdash v : B$

Case Apply   If

$$(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1 : A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$(\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash v_2 : A''} \quad A'' \leq: A'}{\Gamma \vdash v_2 : A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1 : A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2 : A'$

Then we can construct the reduced derivation of $\Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'$ as

$$(\text{Subeffect}) \frac{(\text{Apply}) \frac{() \frac{\Delta}{\Gamma \vdash v_1 : A \to \mathtt{M}_\epsilon B} \quad (\text{Subtype}) \frac{() \frac{\Delta'}{\Gamma \vdash v : A''} \quad A'' \leq: A}{\Gamma \vdash v : A}}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_\epsilon B} \quad B \leq: B' \quad \epsilon \leq \epsilon'}{\Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'}$$

Case If   Let

$$(\text{Subtype}) \frac{() \frac{\Delta}{\Gamma \vdash v : B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v : \mathtt{Bool}} \tag{5}$$

$$(\text{Subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon'} A'} \quad A' \leq: A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \tag{6}$$

$$(\text{Subeffect}) \frac{() \frac{\Delta''}{\Gamma \vdash C_2 : \mathtt{M}_{\epsilon''} A''} \quad A'' \leq: A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 : \mathtt{M}_\epsilon A} \tag{7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v : \mathtt{Bool}$, $\Gamma \vdash C_1 : \mathtt{M}_\epsilon A$, $\Gamma \vdash C_2 : \mathtt{M}_\epsilon A$.

Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_{\epsilon, A}\, v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A$ is:

TODO: Scale this properly

$$(\text{Subtype}) \frac{(\text{If}) \frac{(\text{Subtype}) \frac{0 \frac{\Delta}{\Gamma \vdash v : B} \quad B \leq: \mathtt{Bool}}{\Gamma \vdash v : \mathtt{Bool}} \quad (\text{Subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon'} A'} \quad A' \leq: A \quad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 : \mathtt{M}_\epsilon A} \quad (\text{Subeffect}) \frac{() \frac{\Delta''}{\Gamma \vdash C_2 : \mathtt{M}_{\epsilon''} A''} \quad A'' \leq: A \quad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 : \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\, v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A \quad \epsilon \leq \epsilon \quad A \leq: A}}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\, v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A} \tag{8}$$

Case Bind Let

$$\text{(Subeffect)} \frac{() \dfrac{\Delta}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A} \quad A \leq: A' \quad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$\text{(Subeffect)} \frac{() \dfrac{\Delta'}{\Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2} B} \quad B \leq: B' \quad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A \vdash C_2 : \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the subterms]

By weakening, $\iota \times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2 : \mathtt{M}_\epsilon B$, there's also one of $\Gamma, x : A \vdash C_2 : \mathtt{M}_\epsilon B$.

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Subeffect)} \frac{\text{(Bind)} \dfrac{\text{(Subeffect)} \frac{() \frac{\Delta}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1} A} \ A \leq: A' \ \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 : \mathtt{M}_{\epsilon_1'} A'} \quad \text{(Subeffect)} \frac{() \frac{\Delta'}{\Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2} B} \ B \leq: B' \ \epsilon_2 \leq \epsilon_2'}{\Gamma, x:A \vdash C_2 : \mathtt{M}_{\epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad B \leq: B' \ \epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{11}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t : \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

0.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$reduce((\text{Const}) \frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A}) = (\text{Const}) \frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A : A}$

0.3.2 Value Types

Var

$$reduce((\text{Var}) \frac{\Gamma \mathtt{Ok}}{\Gamma, x : A \vdash x : A}) = (\text{Var}) \frac{\Gamma \mathtt{Ok}}{\Gamma, x : A \vdash x : A} \tag{12}$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$reduce((\text{Weaken}) \frac{() \dfrac{\Delta}{\Gamma \vdash x : A}}{\Gamma, y : B \vdash x : A}) \tag{13}$$

Let

$$\text{(Subtype)} \frac{() \dfrac{\Delta'}{\Gamma \vdash x : A} \quad A' \leq: A}{\Gamma \vdash x : A} = reduce(\Delta) \tag{14}$$

In

$$\text{(Subtype)} \frac{(\text{Weaken}) \dfrac{() \frac{\Delta'}{\Gamma \vdash x : A'}}{\Gamma, y : B \vdash x : A'} \quad A' \leq: A}{\Gamma, y : B \vdash x : A} \tag{15}$$

3

Preserves Denotation   Using the construction of denotations, we can find the denotation of the original derivation to be:

$$\left[\!\!\left[(\text{Weaken})\dfrac{()\frac{\Delta}{\Gamma\vdash x:A}}{\Gamma, y:B\vdash x:A}\right]\!\!\right]_M = \Delta \circ \pi_1 \tag{16}$$

Similarly, the denotation of the reduced denotation is:

$$\left[\!\!\left[(\text{Subtype})\dfrac{(\text{Weaken})\frac{()\frac{\Delta'}{\Gamma\vdash x:A'}}{\Gamma,y:B\vdash x:A'}\quad A'\leq:A}{\Gamma, y:B\vdash x:A}\right]\!\!\right]_M = [\![A'\leq:A]\!]_M \circ \Delta' \circ \pi_1 \tag{17}$$

By induction on *reduce* preserving denotations and the reduction of Δ (14), we have:

$$\Delta = [\![A'\leq:A]\!]_M \circ \Delta' \tag{18}$$

So the denotations of the un-reduced and reduced derivations are equal.

Lambda

reduce **definition**   To find:

$$reduce\left((\text{Fn})\dfrac{()\frac{\Delta}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_2}B}}{\Gamma\vdash \lambda x:A.C:A\rightarrow \mathtt{M}_{\epsilon_2}B}\right) \tag{19}$$

Let

$$(\text{Sub-effect})\dfrac{()\frac{\Delta'}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_1}B'}\quad \epsilon_1\leq \epsilon_2\quad B'\leq:B}{\Gamma, x:A\vdash C:\mathtt{M}_{\epsilon_2}B} = reduce(\Delta) \tag{20}$$

In

$$(\text{Sub-type})\dfrac{(\text{Fn})\frac{\Delta'}{\Gamma,x:A\vdash C:\mathtt{M}_{\epsilon_1}B'}\quad A\rightarrow \mathtt{M}_{\epsilon_1}B'\leq:A\rightarrow \mathtt{M}_{\epsilon_2}B}{\Gamma\vdash \lambda x:A.C:A\rightarrow \mathtt{M}_{\epsilon_2}B} \tag{21}$$

Preserves Denotation   Let

$$f = [\![\mathtt{M}_{\epsilon_1}B'\leq:\mathtt{M}_{\epsilon_2}B]\!]_M = [\![\epsilon_1\leq\epsilon_2]\!]_{M,B}\circ T_{\epsilon_1}([\![B'\leq:B]\!]_M) \tag{22}$$

$$[\![A\rightarrow \mathtt{M}_{\epsilon_1}B'\leq:A\rightarrow \mathtt{M}_{\epsilon_2}B]\!]_M = f^A = \mathtt{cur}(f\circ \mathtt{app}) \tag{23}$$

Then

$$
\begin{aligned}
before &= \mathtt{cur}(\Delta)\quad \text{By definition} \tag{24}\\
&= \mathtt{cur}(f\circ \Delta')\quad \text{By reduction of }\Delta \tag{25}\\
&= f^A\circ \mathtt{cur}(\Delta')\quad \text{By the property of }f^X\circ \mathtt{cur}(g)=\mathtt{cur}(f\circ g) \tag{26}\\
&= after\quad \text{By definition} \tag{27}\\
&\tag{28}
\end{aligned}
$$

Subtype

reduce **definition**   To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Gamma \vdash v:A} \quad A \leq: B}{\Gamma \vdash v: B}) \tag{29}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash x:A} \quad A' \leq: A}{\Gamma \vdash x: A} = reduce(\Delta) \tag{30}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Gamma \vdash v:A'} \quad A' \leq: A \leq: B}{\Gamma \vdash v: B} \tag{31}$$

Preserves Denotation

$$before = [\![A \leq: B]\!]_M \circ \Delta \tag{32}$$
$$= [\![A \leq: B]\!]_M \circ ([\![A' \leq: A]\!]_M \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. \tag{33}$$
$$= [\![A' \leq: B]\!]_M \circ \Delta' \quad \text{Subtyping relations are unique} \tag{34}$$
$$= after \tag{35}$$
$$\tag{36}$$

0.3.3   Computation Types

Return

reduce **definition**   To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Gamma \vdash v:A}}{\Gamma \vdash \texttt{return} v: \texttt{M}_1 A}) \tag{37}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Gamma \vdash v:A'} \quad A' \leq: A}{\Gamma \vdash v: A} = reduce(\Delta) \tag{38}$$

In

$$(\text{Sub-effect})\frac{(\text{Return})\frac{\Delta'}{\Gamma \vdash v:A} \quad \texttt{1} \leq \texttt{1} \quad A' \leq: A}{\Gamma \vdash \texttt{return} v: \texttt{M}_1 A} \tag{39}$$

Then

$$before = \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} \tag{40}$$
$$= \eta_A \circ [\![A' \leq: A]\!]_M \circ \Delta' \quad \text{BY reduction of } \Delta \tag{41}$$
$$= T_1[\![A' \leq: A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta \tag{42}$$
$$= [\![\texttt{1} \leq \texttt{1}]\!]_{M,A} \circ T_1[\![A' \leq: A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{Since } [\![\texttt{1} \leq \texttt{1}]\!]_M \text{ is the identity Nat-Trans} \tag{43}$$
$$= after \quad \text{By definition} \tag{44}$$
$$\tag{45}$$

Apply

reduce **definition**   To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B} \quad ()\frac{\Delta_2}{\Gamma\vdash v_2:A}}{\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B}) \tag{46}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Gamma\vdash v_1:A'\to\mathtt{M}_{\epsilon'} B'} \quad A'\to\mathtt{M}_{\epsilon'}B' \le: A\to\mathtt{M}_\epsilon B}{\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B} = reduce(\Delta_1) \tag{47}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Gamma\vdash v:A'} \quad A'\le:A}{\Gamma\vdash v_1:A} = reduce(\Delta_2) \tag{48}$$

In

$$(\text{Sub-effect})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Gamma\vdash v_1:A'\to\mathtt{M}_{\epsilon'}B'} \quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Gamma\vdash v_2:A''} \quad A''\le:A\le:A'}{\Gamma\vdash v_2:A'}}{\Gamma\vdash v_1\ v_2:\mathtt{M}_{\epsilon'}B'} \quad \epsilon'\le\epsilon \ \ B'\le:B}{\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B} \tag{49}$$

Preserves Denotation   Let

$$f = [\![A \le: A']\!]_M : A \to A' \tag{50}$$
$$f' = [\![A'' \le: A]\!]_M : A'' \to A \tag{51}$$
$$g = [\![B' \le: B]\!]_M : B' \to B \tag{52}$$
$$h = [\![\epsilon' \le \epsilon]\!]_M : T_{\epsilon'} \to T_\epsilon \tag{53}$$

Hence

$$[\![A' \to \mathtt{M}_{e'}B' \le: A \to \mathtt{M}_\epsilon B]\!]_M = (h_B \circ T_{\epsilon'}g)^A \circ (T_{\epsilon'}B')^f \tag{54}$$
$$= \mathtt{cur}(h_B \circ T_{\epsilon'}g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f)) \tag{55}$$
$$= \mathtt{cur}(h_B \circ T_{\epsilon'}g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \tag{56}$$

Then

$$before = \mathtt{app} \circ \langle\Delta_1, \Delta_2\rangle \quad \text{By definition} \tag{57}$$
$$= \mathtt{app} \circ \langle\mathtt{cur}(h_B \circ T_{\epsilon'}g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \circ \Delta_1', f' \circ \Delta_2'\rangle \quad \text{By reductions of } \Delta_1, \Delta_2 \tag{58}$$
$$= \mathtt{app} \circ (\mathtt{cur}(h_B \circ T_{\epsilon'}g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \times \mathtt{Id}_A) \circ \langle\Delta_1', f' \circ \Delta_2'\rangle \quad \text{Factoring out} \tag{59}$$
$$= h_B \circ T_{\epsilon'}g \circ \mathtt{app} \circ (\mathtt{Id} \times f) \circ \langle\Delta_1', f' \circ \Delta_2'\rangle \quad \text{By the exponential property} \tag{60}$$
$$= h_B \circ T_{\epsilon'}g \circ \mathtt{app} \circ \langle\Delta_1', f \circ f' \circ \Delta_2'\rangle \tag{61}$$
$$= after \quad \text{By defintion} \tag{62}$$

If

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Gamma\vdash C_1:\mathtt{M}_\epsilon A} \quad ()\frac{\Delta_3}{\Gamma\vdash C_2:\mathtt{M}_\epsilon A}}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Gamma\vdash C_1:\mathtt{M}_\epsilon A} \quad ()\frac{reduce(\Delta_3)}{\Gamma\vdash C_2:\mathtt{M}_\epsilon A}}{\Gamma\vdash\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2:\mathtt{M}_\epsilon A} \tag{63}$$

Preserves Denotation   Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Bind

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reduce **definition**   To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1}A}\ ()\frac{\Delta_2}{\Gamma,x:A\vdash C_2:\texttt{M}_{\epsilon_2}B}}{\Gamma\vdash\ \texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{64}$$

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_1'}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1'}A'}\ \ \epsilon_1'\leq:\epsilon_1\ \ A'\leq:A}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1}A}=reduce(\Delta_1) \tag{65}$$

Since $i,\times\ :\ \Gamma,x\ :\ A'\rhd\Gamma,x\ :\ A$ if $A'\leq:A$, and by Δ_2, $(\Gamma,x\ :\ A)\vdash C_2:\texttt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $(\Gamma,x\ :\ A')\vdash C_2:\texttt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_3'}{\Gamma,x:A'\vdash C_2:\texttt{M}_{\epsilon_2'}B'}\ \ \epsilon_2'\leq:\epsilon_2\ \ B'\leq:B}{\Gamma,x\ :\ A'\vdash C_2:\texttt{M}_{\epsilon_2}B}=reduce(\Delta_3) \tag{66}$$

Since the effects monoid operation is monotone, if $\epsilon_1\leq\epsilon_1'$ and $\epsilon_2\leq\epsilon_2'$ then $\epsilon_1\cdot\epsilon_2\leq\epsilon_1'\cdot\epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-effect})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Gamma\vdash C_1:\texttt{M}_{\epsilon_1'}A'}\ ()\frac{\Delta_3'}{\Gamma,x:A'\vdash C_2:\texttt{M}_{\epsilon_2'}B'}}{\Gamma\vdash\texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1'\cdot\epsilon_2'}B}\ \ B'\leq:B\ \ \epsilon_1'\cdot\epsilon_2'\leq\epsilon_1\cdot\epsilon_2}{\Gamma\vdash\ \texttt{do}\ x\leftarrow C_1\ \texttt{in}\ C_2:\texttt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{67}$$

Preserves Denotation   Let

$$f=[\![A'\leq:A]\!]_M:A'\rightarrow A \tag{68}$$
$$g=[\![B'\leq:B]\!]_M:B'\rightarrow B \tag{69}$$
$$h_1=[\![\epsilon_1'\leq\epsilon_1]\!]_M:T_{\epsilon_1'}\rightarrow T_{\epsilon_1} \tag{70}$$
$$h_2=[\![\epsilon_2'\leq\epsilon_2]\!]_M:T_{\epsilon_2'}\rightarrow T_{\epsilon_2} \tag{71}$$
$$h=[\![\epsilon_1'\cdot\epsilon_2'\leq\epsilon_1\cdot\epsilon_2]\!]_M:T_{\epsilon_1'\cdot\epsilon_2'}\rightarrow T_{\epsilon_1\cdot\epsilon_2} \tag{72}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3=\Delta_2\circ(\texttt{Id}_\Gamma\times f) \tag{73}$$

And due to the reduction of Δ_3, we have

$$\Delta_3=h_{2,B}\circ T_{\epsilon_2'}g\circ\Delta_3' \tag{74}$$

So:

$$before = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \quad \text{By definition.} \tag{75}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon_1'} f \circ \Delta_1' \rangle \quad \text{By reduction of } \Delta_1. \tag{76}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \rangle \quad \text{Factor out } h_1 \tag{77}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma \times A)} \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \rangle \quad \text{Tensor strength and sub-effecting } h_1 \tag{78}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \rangle \quad \text{Naturality of } h_1 \tag{79}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1'} f) \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out pairing again} \tag{80}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Tensorstrength} \tag{81}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_3) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the definition of } \Delta_3 \tag{82}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3') \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the reduction of } \Delta_3 \tag{83}$$

$$= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'} h_{2,B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out the functor} \tag{84}$$

$$= h_B \circ \mu_{\epsilon_1',\epsilon_2',B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the } \mu \text{ and Sub-effect rule} \tag{85}$$

$$= h_B \circ T_{\epsilon_1' \cdot \epsilon_2'} g \circ \mu_{\epsilon_1',\epsilon_2',B'} \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By naturality of } \mu_{,,} \tag{86}$$

$$= after \quad \text{By definition} \tag{87}$$

Subeffect

reduce **definition**   To find:

$$reduce((\text{Subeffect}) \frac{() \frac{\Delta}{\Gamma \vdash C : \mathtt{M}_{\epsilon'} B'} \quad \epsilon' \le \epsilon \quad B' \le: B}{\Gamma \vdash C : \mathtt{M}_\epsilon B}) \tag{88}$$

Let

$$(\text{Subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C : \mathtt{M}_{\epsilon''} B''} \quad \epsilon'' \le \epsilon' \quad \mathtt{Bool}'' \le: B}{\Gamma \vdash C : \mathtt{M}_{\epsilon'} B} = reduce(\Delta) \tag{89}$$

in

$$(\text{subeffect}) \frac{() \frac{\Delta'}{\Gamma \vdash C : \mathtt{M}_{\epsilon''} B''} \quad \epsilon'' \le \epsilon \quad B'' \le: B}{\Gamma \vdash C : \mathtt{M}_\epsilon B} \tag{90}$$

Preserves Denotation   Let

$$f = [\![B' \le: B]\!]_M \tag{91}$$

$$g = [\![B'' \le: B']\!]_M \tag{92}$$

$$h_1 = [\![\epsilon' \le \epsilon]\!]_M \tag{93}$$

$$h_2 = [\![\epsilon' \le \epsilon']\!]_M \tag{94}$$

$$f \circ g = [\![B'' \le: B]\!]_M \tag{95}$$

$$h_1 \circ h_2 = [\![\epsilon'' \le \epsilon']\!]_M \tag{96}$$

$$\tag{97}$$

Hence we can find the denotation of the derivation before reduction.

$$before = h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} \tag{98}$$

$$= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta \tag{99}$$

$$= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \qquad = after \quad \text{By definition.} \tag{100}$$

0.4   Denotations are Equivalent

For each type relation instance $\Gamma \vdash t \colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\,\Delta]\!]_M = [\![reduce\,\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Gamma \vdash t \colon \tau]\!]_M$ is unique.