

## 0.1 Beta and Eta Equivalence

### 0.1.1 Beta-Eta conversions

- (Lambda-Beta)  $\frac{\Gamma, x:A \vdash C:\mathbb{M}_\epsilon B \quad \Gamma \vdash v:A}{\Gamma \vdash (\lambda x:A. C) v =_{\beta\eta} C[x/v]:\mathbb{M}_\epsilon B}$
- (Lambda-Eta)  $\frac{\Gamma \vdash v:A \rightarrow \mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x:A. (v x) =_{\beta\eta} v:A \rightarrow \mathbb{M}_\epsilon B}$
- (Left Unit)  $\frac{\Gamma \vdash v:A \quad \Gamma, x:A \vdash C:\mathbb{M}_\epsilon B}{\Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C =_{\beta\eta} C[V/x]:\mathbb{M}_\epsilon B}$
- (Right Unit)  $\frac{\Gamma \vdash C:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x =_{\beta\eta} C:\mathbb{M}_\epsilon A}$
- (Associativity)  $\frac{\Gamma \vdash C_1:\mathbb{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2:\mathbb{M}_{\epsilon_2} B \quad \Gamma, y:B \vdash C_3:\mathbb{M}_{\epsilon_3} C}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) =_{\beta\eta} \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$
- (Unit)  $\frac{\Gamma \vdash v:\text{Unit}}{\Gamma \vdash v =_{\beta\eta} ():\text{Unit}}$
- (if-true)  $\frac{\Gamma \vdash C_1:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_2:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ true then } C_1 \text{ else } C_2 =_{\beta\eta} C_1:\mathbb{M}_\epsilon A}$
- (if-false)  $\frac{\Gamma \vdash C_2:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_1:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ false then } C_1 \text{ else } C_2 =_{\beta\eta} C_2:\mathbb{M}_\epsilon A}$
- (If-Eta)  $\frac{\Gamma, x:\text{Bool} \vdash C:\mathbb{M}_\epsilon A \quad \Gamma \vdash v:\text{Bool}}{\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C[\text{true}/x] \text{ else } C[\text{false}/x] =_{\beta\eta} C[V/x]:\mathbb{M}_\epsilon A}$

### 0.1.2 Equivalence Relation

- (Reflexive)  $\frac{\Gamma \vdash t:\tau}{\Gamma \vdash t =_{\beta\eta} t:\tau}$
- (Symmetric)  $\frac{\Gamma \vdash t_1 =_{\beta\eta} t_2:\tau}{\Gamma \vdash t_2 =_{\beta\eta} t_1:\tau}$
- (Transitive)  $\frac{\Gamma \vdash t_1 =_{\beta\eta} t_2:\tau \quad \Gamma \vdash t_2 =_{\beta\eta} t_3:\tau}{\Gamma \vdash t_1 =_{\beta\eta} t_3:\tau}$

### 0.1.3 Congruences

- (Lambda)  $\frac{\Gamma, x:A \vdash C_1 =_{\beta\eta} C_2:\mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x:A. C_1 =_{\beta\eta} \lambda x:A. C_2:A \rightarrow \mathbb{M}_\epsilon B}$
- (Return)  $\frac{\Gamma \vdash v_1 =_{\beta\eta} v_2:A}{\Gamma \vdash \text{return } v_1 =_{\beta\eta} \text{return } v_2:\mathbb{M}_1 A}$
- (Apply)  $\frac{\Gamma \vdash v_1 =_{\beta\eta} v'_1:A \rightarrow \mathbb{M}_\epsilon B \quad \Gamma \vdash v_2 =_{\beta\eta} v'_2:A}{\Gamma \vdash v_1 v_2 =_{\beta\eta} v'_1 v'_2:\mathbb{M}_\epsilon B}$
- (Bind)  $\frac{\Gamma \vdash C_1 =_{\beta\eta} C'_1:\mathbb{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2 =_{\beta\eta} C'_2:\mathbb{M}_{\epsilon_2} B}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 =_{\beta\eta} \text{do } x \leftarrow C'_1 \text{ in } C'_2:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B}$
- (If)  $\frac{\Gamma \vdash v =_{\beta\eta} v':\text{Bool} \quad \Gamma \vdash C_1 =_{\beta\eta} C'_1:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_2 =_{\beta\eta} C'_2:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 =_{\beta\eta} \text{if}_{\epsilon, A} v \text{ then } C'_1 \text{ else } C'_2:\mathbb{M}_\epsilon A}$
- (Subtype)  $\frac{\Gamma \vdash v =_{\beta\eta} v':A \quad A \leq B}{\Gamma \vdash v =_{\beta\eta} v':B}$
- (Subeffect)  $\frac{\Gamma \vdash C =_{\beta\eta} C':\mathbb{M}_{\epsilon_1} A \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C =_{\beta\eta} C':\mathbb{M}_{\epsilon_2} B}$

## 0.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

Each derivation of  $\Gamma \vdash t =_{\beta\eta} t':\tau$  can be converted to a derivation of  $\Gamma \vdash t:\tau$  and  $\Gamma \vdash t':\tau$  by induction over the beta-eta equivalence relation derivation.

### 0.2.1 Equivalence Relations

**Case Reflexive** By inversion we have a derivation of  $\Gamma \vdash t : \tau$ .

**Case Symmetric** By inversion  $\Gamma \vdash t' =_{\beta\eta} t : \tau$ . Hence by induction, derivations of  $\Gamma \vdash t' : \tau$  and  $\Gamma \vdash t : \tau$  are given.

**Case Transitive** By inversion, there exists  $t_2$  such that  $\Gamma \vdash t_1 =_{\beta\eta} t_2 : \tau$  and  $\Gamma \vdash t_2 =_{\beta\eta} t_3 : \tau$ . Hence by induction, we have derivations of  $\Gamma \vdash t_1 : \tau$  and  $\Gamma \vdash t_3 : \tau$

### 0.2.2 Beta conversions

**Case Lambda** By inversion, we have  $\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B$  and  $\Gamma \vdash v : A$ . Hence by the typing rules, we have:

$$(\text{Apply}) \frac{(\text{Lambda}) \frac{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x : A. C : A \rightarrow \mathbb{M}_\epsilon B} \quad \Gamma \vdash v : A}{\Gamma \vdash (\lambda x : A. C) v : \mathbb{M}_\epsilon A}$$

By the substitution rule **TODO: which?**, we have

$$(\text{Substitution}) \frac{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B \quad \Gamma \vdash v : A}{\Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B}$$

**Case Left Unit** By inversion, we have  $\Gamma \vdash v : A$  and  $\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B$

Hence we have:

$$(\text{Bind}) \frac{(\text{Return}) \frac{\Gamma \vdash v : A}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A} \quad \Gamma, x : A \vdash C : \mathbb{M}_\epsilon B}{\Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C : \mathbb{M}_{1.\epsilon} B = \mathbb{M}_\epsilon B} \quad (1)$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B \quad (2)$$

**Case Right Unit** By inversion, we have  $\Gamma \vdash C : \mathbb{M}_\epsilon A$ .

Hence we have:

$$(\text{Bind}) \frac{\Gamma \vdash C : \mathbb{M}_\epsilon A \quad (\text{Return}) \frac{(\text{var}) \frac{\Gamma, x : A \vdash x : A}{\Gamma, x : A \vdash \text{return } x : \mathbb{M}_1 A}}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_{\epsilon.1} A = \mathbb{M}_\epsilon A}}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_{\epsilon.1} A = \mathbb{M}_\epsilon A} \quad (3)$$

**Case Associativity** By inversion, we have  $\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A$ ,  $\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B$ , and  $\Gamma, y : B \vdash C_3 : \mathbb{M}_{\epsilon_3} C$ .

$$(\iota\pi \times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**,  $\Gamma, x : A, y : B \vdash C_3 : \mathbb{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$(\text{Bind}) \frac{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A \quad (\text{Bind}) \frac{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B \quad \Gamma, x : A, y : B \vdash C_3 : \mathbb{M}_{\epsilon_3} C}{\Gamma, x : A \vdash x C_2 C_3 : \mathbb{M}_{\epsilon_2.\epsilon_3} C}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) : \mathbb{M}_{\epsilon_1.\epsilon_2.\epsilon_3} C} \quad (4)$$

and

$$(\text{Bind}) \frac{(\text{Bind}) \frac{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A \quad \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1.\epsilon_2} B} \quad \Gamma, y : B \vdash C_3 : \mathbb{M}_{\epsilon_3} C}{\Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3 : \mathbb{M}_{\epsilon_1.\epsilon_2.\epsilon_3} C} \quad (5)$$

**Case Eta** By inversion, we have  $\Gamma \vdash v : A \rightarrow \mathbf{M}_\epsilon B$

By weakening, we have  $\iota\pi : (\Gamma, x : A) \triangleright \Gamma$  Hence, we have

$$(\text{Fn}) \frac{(\text{App}) \frac{(\Gamma, x:A) \vdash x:A \quad (\text{weakening}) \frac{\Gamma \vdash v:A \rightarrow \mathbf{M}_\epsilon B \quad \iota\pi : \Gamma, x:A \triangleright \Gamma}{\Gamma, x:A \vdash v:A \rightarrow \mathbf{M}_\epsilon B}}{\Gamma, x:A \vdash v \ x : \mathbf{M}_\epsilon B}}{\Gamma \vdash \lambda x : A. (v \ x) : A \rightarrow \mathbf{M}_\epsilon B} \quad (6)$$

**Case If-True** By inversion, we have  $\Gamma \vdash C_1 : \mathbf{M}_\epsilon A$ ,  $\Gamma \vdash C_2 : \mathbf{M}_\epsilon A$ . Hence by the typing lemma **TODO: Which?**, we have  $\Gamma \vdash \mathbf{true} : \mathbf{Bool}$  by the axiom typing rule.

Hence

$$(\text{If}) \frac{\Gamma \vdash \mathbf{true} : \mathbf{Bool} \quad \Gamma \vdash C_1 : \mathbf{M}_\epsilon A \quad \Gamma \vdash C_2 : \mathbf{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \ \mathbf{true} \ \text{then } C_1 \ \text{else } C_2 : \mathbf{M}_\epsilon A} \quad (7)$$

**Case If-False** As above,

Hence

$$(\text{If}) \frac{\Gamma \vdash \mathbf{false} : \mathbf{Bool} \quad \Gamma \vdash C_1 : \mathbf{M}_\epsilon A \quad \Gamma \vdash C_2 : \mathbf{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \ \mathbf{false} \ \text{then } C_1 \ \text{else } C_2 : \mathbf{M}_\epsilon A} \quad (8)$$

**Case If-Eta** By inversion, we have:

$$\Gamma \vdash v : \mathbf{Bool} \quad (9)$$

and

$$\Gamma, x : \mathbf{Bool} \vdash C : \mathbf{M}_\epsilon A \quad (10)$$

Hence we also have  $\Gamma \vdash \mathbf{true} : \mathbf{Bool}$ . Hence, the following also hold:

$\Gamma \vdash \mathbf{true} : \mathbf{Bool}$ , and  $\Gamma \vdash \mathbf{false} : \mathbf{Bool}$ .

Hence by the substitution theorem, we have:

$$(\text{If}) \frac{\Gamma \vdash v : \mathbf{Bool} \quad \Gamma \vdash C[\mathbf{true}/x] : \mathbf{M}_\epsilon A \quad \Gamma \vdash C[\mathbf{false}/x] : \mathbf{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \ v \ \text{then } C[\mathbf{true}/x] \ \text{else } C[\mathbf{false}/x] : \mathbf{M}_\epsilon A} \quad (11)$$

and

$$\Gamma \vdash C[v/x] : \mathbf{M}_\epsilon A \quad (12)$$

### 0.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

**Case Lambda** By inversion,  $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 : \mathbf{M}_\epsilon B$ . Hence by induction  $\Gamma, x : A \vdash C_1 : \mathbf{M}_\epsilon B$ , and  $\Gamma, x : A \vdash C_2 : \mathbf{M}_\epsilon B$ .

So

$$\Gamma \vdash \lambda x : A. C_1 : A \rightarrow \mathbf{M}_\epsilon B \quad (13)$$

and

$$\Gamma \vdash \lambda x : A. C_2 : A \rightarrow \mathbf{M}_\epsilon B \quad (14)$$

Hold.

**Case Return** By inversion,  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$ , so by induction

$$\Gamma \vdash v_1 : A$$

and

$$\Gamma \vdash v_2 : A$$

Hence we have

$$\Gamma \vdash \mathbf{return} v_1 : \mathbf{M}_1 A$$

and

$$\Gamma \vdash \mathbf{return} v_2 : \mathbf{M}_1 A$$

**Case Apply** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v'_1 : A \rightarrow \mathbf{M}_\epsilon B$  and  $\Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$ . Hence we have by induction  $\Gamma \vdash v_1 : A \rightarrow \mathbf{M}_\epsilon B$ ,  $\Gamma \vdash v_2 : A$ ,  $\Gamma \vdash v'_1 : A \rightarrow \mathbf{M}_\epsilon B$ , and  $\Gamma \vdash v'_2 : A$ .

So we have:

$$\Gamma \vdash v_1 \ v_2 : \mathbf{M}_\epsilon B \quad (15)$$

and

$$\Gamma \vdash v'_1 \ v'_2 : \mathbf{M}_\epsilon B \quad (16)$$

**Case Bind** By inversion, we have:  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbf{M}_{\epsilon_1} A$  and  $\Gamma, x : A \vdash C_2 =_{\beta\eta} C'_2 : \mathbf{M}_{\epsilon_2} B$ . Hence by induction, we have  $\Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A$ ,  $\Gamma \vdash C'_1 : \mathbf{M}_{\epsilon_1} A$ ,  $\Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B$ , and  $\Gamma, x : A \vdash C'_2 : \mathbf{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \mathbf{do} \ x \leftarrow C_1 \ \mathbf{in} \ C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \quad (17)$$

$$\Gamma \vdash \mathbf{do} \ x \leftarrow C'_1 \ \mathbf{in} \ C'_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \quad (18)$$

**Case If** By inversion, we have:  $\Gamma \vdash v =_{\beta\eta} v' : \mathbf{Bool}$ ,  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbf{M}_\epsilon A$ , and  $\Gamma \vdash C_2 =_{\beta\eta} C'_2 : \mathbf{M}_\epsilon A$ .

Hence by induction, we have:

$$\begin{aligned} &\Gamma \vdash v : \mathbf{Bool}, \Gamma \vdash v' : \mathbf{Bool}, \\ &\Gamma \vdash C_1 : \mathbf{M}_\epsilon A, \Gamma \vdash C'_1 : \mathbf{M}_\epsilon A, \\ &\Gamma \vdash C_2 : \mathbf{M}_\epsilon A, \text{ and } \Gamma \vdash C'_2 : \mathbf{M}_\epsilon A. \end{aligned}$$

So

$$\Gamma \vdash \mathbf{if}_{\epsilon, A} \ v \ \mathbf{then} \ C_1 \ \mathbf{else} \ C_2 : \mathbf{M}_\epsilon A \quad (19)$$

and

$$\Gamma \vdash \mathbf{if}_{\epsilon, A} \ v \ \mathbf{then} \ C'_1 \ \mathbf{else} \ C'_2 : \mathbf{M}_\epsilon A \quad (20)$$

Hold.

**Case Subtype** By inversion, we have  $A \leq B$  and  $\Gamma \vdash v =_{\beta\eta} v' : A$ . By induction, we therefore have  $\Gamma \vdash v : A$  and  $\Gamma \vdash v' : A$ .

Hence we have

$$\Gamma \vdash v : B \quad (21)$$

$$\Gamma \vdash v' : B \quad (22)$$

**Case subeffect** By inversion we have:  $A \leq B$ ,  $\epsilon_1 \leq \epsilon_2$ , and  $\Gamma \vdash C =_{\beta\eta} C' : \mathbf{M}_{\epsilon_1} A$ .

Hence by inductive hypothesis, we have  $\Gamma \vdash C : \mathbf{M}_{\epsilon_1} A$  and  $\Gamma \vdash C' : \mathbf{M}_{\epsilon_1} A$ .

Hence,

$$\Gamma \vdash C : \mathbf{M}_{\epsilon_2} B \quad (23)$$

and

$$\Gamma \vdash C' : \mathbf{M}_{\epsilon_2} B \quad (24)$$

hold.

### 0.3 Beta-Eta equivalent terms have equal denotations

If  $t \vdash t' =_{\beta\eta} \tau$ : then  $\llbracket \Gamma \vdash t : \tau \rrbracket_M = \llbracket \Gamma \vdash t' : \tau \rrbracket_M$

By induction over Beta-eta equivalence relation.

#### 0.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

**Case Reflexive** Equality is reflexive, so if  $\Gamma \vdash t : \tau$  then  $\llbracket \Gamma \vdash t : \tau \rrbracket_M$  is equal to itself.

**Case Symmetric** By inversion, if  $\Gamma \vdash t =_{\beta\eta} t' : \tau$  then  $\Gamma \vdash t' =_{\beta\eta} t : \tau$ , so by induction  $\llbracket \Gamma \vdash t' : \tau \rrbracket_M = \llbracket \Gamma \vdash t : \tau \rrbracket_M$  and hence  $\llbracket \Gamma \vdash t : \tau \rrbracket_M = \llbracket \Gamma \vdash t' : \tau \rrbracket_M$

**Case Transitive** There must exist  $t_2$  such that  $\Gamma \vdash t_1 =_{\beta\eta} t_2 : \tau$  and  $\Gamma \vdash t_2 =_{\beta\eta} t_3 : \tau$ , so by induction,  $\llbracket \Gamma \vdash t_1 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_2 : \tau \rrbracket_M$  and  $\llbracket \Gamma \vdash t_2 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_3 : \tau \rrbracket_M$ . Hence by transitivity of equality,  $\llbracket \Gamma \vdash t_1 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_3 : \tau \rrbracket_M$

#### 0.3.2 Beta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

**Case Lambda** Let  $f = \llbracket \Gamma, x : A \vdash C : \mathbf{M}_\epsilon B \rrbracket_M : (\Gamma \times A) \rightarrow T_\epsilon B$

Let  $g = \llbracket \Gamma \vdash v : A \rrbracket_M : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Gamma \vdash [v/x] : \Gamma, x : A \rrbracket_M : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Gamma \vdash C [v/x] : \mathbf{M}_\epsilon B \rrbracket_M = f \circ \langle \text{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned} \llbracket \Gamma \vdash (\lambda x : A. C) v : \mathbf{M}_\epsilon B \rrbracket_M &= \text{app} \circ \langle \text{cur}(f), g \rangle \\ &= \text{app} \circ (\text{cur}(f) \times \text{Id}_A) \circ \langle \text{Id}_\Gamma, g \rangle \\ &= f \circ \langle \text{Id}_\Gamma, g \rangle \\ &= \llbracket \Gamma \vdash C [v/x] : \mathbf{M}_\epsilon B \rrbracket_M \end{aligned} \quad (25)$$

**Case Left Unit** Let  $f = \llbracket \Gamma, x : A \vdash C : \mathbb{M}_\epsilon B \rrbracket_M$

Let  $g = \llbracket \Gamma \vdash v : A \rrbracket_M : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Gamma \vdash [v/x] : \Gamma, x : A \rrbracket_M : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B \rrbracket_M = f \circ \langle \text{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C : \mathbb{M}_\epsilon B \rrbracket_M &= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathbf{t}_{1, \Gamma, A} \circ \langle \text{Id}_\Gamma, \eta_A \circ g \rangle \\ &= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathbf{t}_{1, \Gamma, A} \circ (\text{Id}_\Gamma \times \eta_A) \circ \langle \text{Id}_\Gamma, g \rangle \\ &= \mu_{1, \epsilon, B} \circ T_1 f \circ \eta_{(\Gamma \times A)} \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\ &= \mu_{1, \epsilon, B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\ &= f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\ &= \llbracket \Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B \rrbracket_M \end{aligned} \tag{26}$$

**Case Right Unit** Let  $f = \llbracket \Gamma \vdash C : \mathbb{M}_\epsilon A \rrbracket_M$

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_\epsilon A \rrbracket_M &= \mu_{\epsilon, 1, A} \circ T_\epsilon (\eta_A \circ \pi_2) \circ \mathbf{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= T_\epsilon \pi_2 \circ \mathbf{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \pi_2 \circ \langle \text{Id}_\Gamma, f \rangle \\ &= f \end{aligned} \tag{27}$$

**Case Associative** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M \tag{28}$$

$$g = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M \tag{29}$$

$$h = \llbracket \Gamma, y : B \vdash C_3 : \mathbb{M}_\epsilon C \rrbracket_M \tag{30}$$

We also have the weakening:

$$\iota\pi \times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \tag{31}$$

With denotation:

$$\llbracket \iota\pi \times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \rrbracket_M = (\pi_1 \times \text{Id}_B) \tag{32}$$

We need to prove that the following are equal

$$lhs = \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} \rrbracket_M \tag{33}$$

$$= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \text{Id}_B) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \tag{34}$$

$$rhs = \llbracket \Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} \rrbracket_M \tag{35}$$

$$= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2} (h) \circ \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \tag{36}$$

$$\tag{37}$$

Let's look at fragment  $F$  of  $rhs$ .

$$F = \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \tag{38}$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \quad (39)$$

$$\begin{aligned} F &= \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\mathbf{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\mathbf{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathbf{t}_{\epsilon_2, \Gamma, B} \circ \mathbf{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\mathbf{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \quad \text{By \textbf{TODO: ref: mu+tstrength}} \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} (\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\mathbf{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \quad \text{By naturality of t-strength} \end{aligned} \quad (40)$$

$$\text{Since } rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F,$$

$$\begin{aligned} rhs &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1 \cdot \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} (\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\mathbf{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1 \cdot \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1} (T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\mathbf{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\ &= \mu_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\mathbf{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \end{aligned} \quad (41)$$

Let's now look at the fragment  $G$  of  $rhs$

$$G = T_{\epsilon_1}(\mathbf{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \mathbf{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \rangle \quad (42)$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (43)$$

By folding out the  $\langle \dots, \dots \rangle$ , we have

$$G = T_{\epsilon_1}(\mathbf{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ \langle \mathbf{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A} \rangle \circ \langle \mathbf{Id}_\Gamma, \langle \mathbf{Id}_\Gamma, f \rangle \rangle \quad (44)$$

From the rule **TODO: Ref** showing the commutivity of tensor strength with  $\alpha$ , the following commutes

$$\begin{array}{ccc} \Gamma \xrightarrow{\langle \mathbf{Id}_\Gamma, \langle \mathbf{Id}_\Gamma, f \rangle \rangle} \Gamma \times (\Gamma \times T_{\epsilon_1} A) & \xleftarrow{\alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}} & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \\ \downarrow \mathbf{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A} & & \downarrow \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \\ \Gamma \times T_{\epsilon_1}(\Gamma \times A) & & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \\ \downarrow \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} & \swarrow T_{\epsilon_1} \alpha_{\Gamma, \Gamma, A} & \\ T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) & & \end{array}$$

Where  $\alpha : ((- \times -) \times -) \rightarrow (- \times (- \times -))$  is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle \quad (45)$$

$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle \quad (46)$$

So:

$$\begin{aligned} G &= T_{\epsilon_1}((\mathbf{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}^{-1} \circ \langle \mathbf{Id}_\Gamma, \langle \mathbf{Id}_\Gamma, f \rangle \rangle \\ &= T_{\epsilon_1}((\mathbf{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \langle \langle \mathbf{Id}_\Gamma, \mathbf{Id}_\Gamma \rangle \times \mathbf{Id}_{T_{\epsilon_1} A} \rangle \circ \langle \mathbf{Id}_\Gamma, f \rangle \quad \text{By definition of } \alpha \text{ and products} \\ &= T_{\epsilon_1}((\mathbf{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A} \circ \langle \langle \mathbf{Id}_\Gamma, \mathbf{Id}_\Gamma \rangle \times \mathbf{Id}_A \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \quad \text{By tensor strength's left-naturality} \\ &= T_{\epsilon_1}((\pi_1 \times \mathbf{Id}_{T_{\epsilon_2} B}) \circ \langle \mathbf{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathbf{Id}_\Gamma, f \rangle \end{aligned} \quad (47)$$

Since

$$rhs = \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (48)$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\pi_1 \times \text{Id}_{T_{\epsilon_2} B}) \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, f \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \text{Id}_B)) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, f \rangle \quad \text{By Left-Tensor Stre} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \quad (49)$$

**Case Eta** Let

$$f = \llbracket \Gamma \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M : \Gamma \rightarrow (T_{\epsilon} B)^A \quad (50)$$

By weakening, we have

$$\llbracket \Gamma, x : A \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M = f \circ \pi_1 : \Gamma \times A \rightarrow (T_{\epsilon} B)^A \quad (51)$$

$$\llbracket \Gamma, x : A \vdash v x : \mathbf{M}_{\epsilon} B \rrbracket_M = \mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \quad (52)$$

$$(53)$$

Hence, we have

$$\begin{aligned} \llbracket \Gamma \vdash \lambda x : A. (v x) : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M &= \mathbf{cur}(\mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \\ \mathbf{app} \circ (\llbracket \Gamma \vdash \lambda x : A. (v x) : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M \times \text{Id}_A) &= \mathbf{app} \circ (\mathbf{cur}(\mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \times \text{Id}_A) \\ &= \mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \\ &= \mathbf{app} \circ (f \times \text{Id}_A) \end{aligned} \quad (54)$$

Hence, by the fact that  $\mathbf{cur}(f)$  is unique in a cartesian closed category,

$$\llbracket \Gamma \vdash \lambda x : A. (v x) : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M = f = \llbracket \Gamma \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B \rrbracket_M \quad (55)$$

**Case If-True** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A \rrbracket_M \quad (56)$$

$$g = \llbracket \Gamma \vdash C_2 : \mathbf{M}_{\epsilon} A \rrbracket_M \quad (57)$$

$$(58)$$

Then

$$\begin{aligned} \llbracket \Gamma \vdash \text{if}_{\text{true}, A} v \text{ then } C_1 \text{ else } C_2 : \mathbf{M}_{\epsilon} A \rrbracket_M &= \mathbf{app} \circ (([\mathbf{cur}(f \circ \pi_2), \mathbf{cur}(g \circ \pi_2)] \circ \text{inl} \circ \langle \rangle_{\Gamma}) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \\ &= \mathbf{app} \circ ((\mathbf{cur}(f \circ \pi_2) \circ \langle \rangle_{\Gamma}) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \\ &= \mathbf{app} \circ (\mathbf{cur}(f \circ \pi_2) \times \text{Id}_{\Gamma}) \circ (\langle \rangle_{\Gamma} \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \\ &= f \circ \pi_2 \circ \langle \rangle_{\Gamma}, \text{Id}_{\Gamma} \rangle \\ &= f \\ &= \llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A \rrbracket_M \end{aligned} \quad (59)$$



**Case If-False** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbf{M}_\epsilon A \rrbracket_M \quad (60)$$

$$g = \llbracket \Gamma \vdash C_2 : \mathbf{M}_\epsilon A \rrbracket_M \quad (61)$$

$$(62)$$

Then

$$\begin{aligned} \llbracket \Gamma \vdash \text{if}_{\text{true}, A} v \text{ then } C_1 \text{ else } C_2 : \mathbf{M}_\epsilon A \rrbracket_M &= \text{app} \circ (([\text{cur}(f \circ \pi_2), \text{cur}(g \circ \pi_2)] \circ \text{inr} \circ \langle \rangle_\Gamma) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ ((\text{cur}(g \circ \pi_2) \circ \langle \rangle_\Gamma) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ (\text{cur}(g \circ \pi_2) \times \text{Id}_\Gamma) \circ (\langle \rangle_\Gamma \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= g \circ \pi_2 \circ \langle \rangle_\Gamma, \text{Id}_\Gamma \rangle \\ &= g \\ &= \llbracket \Gamma \vdash C_2 : \mathbf{M}_\epsilon A \rrbracket_M \end{aligned} \quad (63)$$

### 0.3.3 Case If-Eta

Let

$$f = \llbracket \Gamma \vdash v : \text{Bool} \rrbracket_M \quad (64)$$

$$g = \llbracket \Gamma, x : \text{Bool} \vdash C : \mathbf{M}_\epsilon A \rrbracket_M \quad (65)$$

$$(66)$$

Then by the substitution theorem,

$$\llbracket \Gamma \vdash C [\text{true}/x] : \mathbf{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \quad (67)$$

$$\llbracket \Gamma \vdash C [\text{false}/x] : \mathbf{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \quad (68)$$

$$\llbracket \Gamma \vdash C [v/x] : \mathbf{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, f \rangle \quad (69)$$

Hence we have (Using the diagonal and twist morphisms):

$$\llbracket \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C [\text{true}/x] \text{ else } C [\text{false}/x] : \mathbb{M}_{\epsilon} A \rrbracket_M \quad (70)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \text{Id}_{\Gamma}, \text{inl}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_2), \text{cur}(g \circ \langle \text{Id}_{\Gamma}, \text{inr}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_2)) \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad (71)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_2), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_2)) \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Pairing property} \quad (72)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_1), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_{\Gamma} \rangle \circ \pi_1)) \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Terminal is unique} \quad (73)$$

$$= \text{app} \circ ((([\text{cur}(g \circ (\text{Id}_{\Gamma} \times (\text{inl}_1 \circ \langle \rangle_1)) \circ \tau_{1, \Gamma}), \text{cur}(g \circ (\text{Id}_{\Gamma} \times (\text{inr}_1 \circ \langle \rangle_1)) \circ \tau_{1, \Gamma})] \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Definition of } \tau \quad (74)$$

$$= \text{app} \circ ((([\text{cur}(g \circ (\text{Id}_{\Gamma} \times \text{inl}_1) \circ \tau_{1, \Gamma}), \text{cur}(g \circ (\text{Id}_{\Gamma} \times \text{inr}_1) \circ \tau_{1, \Gamma})] \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Identity} = \text{Id}_1 \quad (75)$$

$$= \text{app} \circ ((([\text{cur}(g \circ \tau_{1+1, \Gamma} \circ (\text{inl}_1 \times \text{Id}_{\Gamma})), \text{cur}(g \circ \tau_{1+1, \Gamma} \circ (\text{inr}_1 \times \text{Id}_{\Gamma}))] \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Twist commutivity} \quad (76)$$

$$= \text{app} \circ ((([\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ \text{inl}_1, \text{cur}(g \circ \tau_{1+1, \Gamma}) \circ \text{inr}_1] \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Exponential property} \quad (77)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ [\text{inl}_1, \text{inr}_1] \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Factoring out } \text{cur}(\cdot) \quad (78)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Since } [\text{inl}, \text{inr}] \text{ is the identity} \quad (79)$$

$$= \text{app} \circ (\text{cur}(g \circ \tau_{1+1, \Gamma}) \times \text{Id}_{\Gamma}) \circ (f \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Factoring} \quad (80)$$

$$= g \circ \tau_{1+1, \Gamma} \circ (f \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad \text{Definition of } \text{app}, \text{cur}(\cdot) \quad (81)$$

$$= g \circ (\text{Id}_{\Gamma} \times f) \circ \tau_{\Gamma, \Gamma} \circ \delta_{\Gamma} \quad \text{Twist commutivity} \quad (82)$$

$$= g \circ (\text{Id}_{\Gamma} \times f) \circ \langle \text{Id}_{\Gamma}, \text{Id}_{\Gamma} \rangle \quad \text{Twist, diagonal definitions} \quad (83)$$

$$= g \circ \langle \text{Id}_{\Gamma}, f \rangle \quad (84)$$

$$= \llbracket \Gamma \vdash C [v/x] : \mathbb{M}_{\epsilon} A \rrbracket_M \quad (85)$$

$$(86)$$

### 0.3.4 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of subexpressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

**Case Lambda** By inversion, we have  $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 : \mathbb{M}_{\epsilon} B$  By induction, we therefore have  $\llbracket \Gamma, x : A \vdash C_1 : \mathbb{M}_{\epsilon} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon} B \rrbracket_M$

Then let

$$f = \llbracket \Gamma, x : A \vdash C_1 : \mathbb{M}_{\epsilon} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon} B \rrbracket_M \quad (87)$$

And so

$$\llbracket \Gamma \vdash \lambda x : A. C_1 : A \rightarrow \mathbb{M}_{\epsilon} B \rrbracket_M = \text{cur}(f) = \llbracket \Gamma \vdash \lambda x : A. C_2 : A \rightarrow \mathbb{M}_{\epsilon} B \rrbracket_M \quad (88)$$

**Case Return** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$  By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M \quad (89)$$

And so

$$\llbracket \Gamma \vdash \text{return } v_1 : \mathbb{M}_1 A \rrbracket_M = \eta_A \circ f = \llbracket \Gamma \vdash \text{return } v_2 : \mathbb{M}_1 A \rrbracket_M \quad (90)$$

**Case Apply** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v'_1 : A \rightarrow \mathbb{M}_\epsilon B$  and  $\Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$ . By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma \vdash v'_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M$  and  $\llbracket \Gamma \vdash v_2 : A \rrbracket_M = \llbracket \Gamma \vdash v'_2 : A \rrbracket_M$ .

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma \vdash v'_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M \quad (91)$$

$$g = \llbracket \Gamma \vdash v_2 : A \rrbracket_M = \llbracket \Gamma \vdash v'_2 : A \rrbracket_M \quad (92)$$

And so

$$\llbracket \Gamma \vdash v_1 \ v_2 : \mathbb{M}_\epsilon A \rrbracket_M = \text{app} \circ \langle f, g \rangle = \llbracket \Gamma \vdash v'_1 \ v'_2 : \mathbb{M}_\epsilon A \rrbracket_M \quad (93)$$

**Case Bind** By inversion, we have  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbb{M}_\epsilon A$  and  $\Gamma, x : A \vdash C_2 =_{\beta\eta} C'_2 : \mathbb{M}_\epsilon B$ . By induction, we therefore have  $\llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbb{M}_\epsilon A \rrbracket_M$  and  $\llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbb{M}_\epsilon B \rrbracket_M$ .

Then let

$$f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbb{M}_{\epsilon_1} A \rrbracket_M \quad (94)$$

$$g = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbb{M}_{\epsilon_2} B \rrbracket_M \quad (95)$$

And so

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathfrak{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M \end{aligned} \quad (96)$$

**Case If** By inversion, we have  $\Gamma \vdash v =_{\beta\eta} v' : \text{Bool}$ ,  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbb{M}_\epsilon A$  and  $\Gamma \vdash C_2 =_{\beta\eta} C'_2 : \mathbb{M}_\epsilon A$ . By induction, we therefore have  $\llbracket \Gamma \vdash v : \text{Bool} \rrbracket_M = \llbracket \Gamma \vdash v' : \text{Bool} \rrbracket_M$ ,  $\llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbb{M}_\epsilon A \rrbracket_M$  and  $\llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbb{M}_\epsilon B \rrbracket_M$ .

Then let

$$f = \llbracket \Gamma \vdash v : \text{Bool} \rrbracket_M = \llbracket \Gamma \vdash v' : \text{Bool} \rrbracket_M \quad (97)$$

$$g = \llbracket \Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbb{M}_{\epsilon_1} A \rrbracket_M \quad (98)$$

$$h = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbb{M}_{\epsilon_2} B \rrbracket_M \quad (99)$$

And so

$$\begin{aligned} \llbracket \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 : \rrbracket_M &= \text{app} \circ (([\text{cur}(g \circ \pi_2), \text{cur}(h \circ \pi_2)] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M \end{aligned} \quad (100)$$

**Case Subtype** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$ , and  $A \leq B$ . By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M$ .

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M \quad (101)$$

$$g = \llbracket A \leq B \rrbracket_M \quad (102)$$

And so

$$\llbracket \Gamma \vdash v_1 : B \rrbracket_M = g \circ f = \llbracket \Gamma \vdash v_1 : B \rrbracket_M \quad (103)$$

**Case subeffect** By inversion, we have  $\Gamma \vdash C_1 =_{\beta\eta} C_2 : \mathbb{M}_{\epsilon_1} A$ , and  $A \leq B$  and  $\epsilon_1 \leq \epsilon_2$ . By induction, we therefore have  $\llbracket \Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C_2 : \mathbb{M}_{\epsilon_1} A \rrbracket_M$ .

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M \quad (104)$$

$$g = \llbracket A \leq B \rrbracket_M \quad (105)$$

$$h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_M \quad (106)$$

And so

$$\llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_2} B \rrbracket_M = h_B \circ T_{\epsilon_1} g \circ f = \llbracket \Gamma \vdash C_2 \mathbf{M}_{\epsilon_2} B : \rrbracket_{\mathbf{M}} \quad (107)$$

□