0.1 Denotations of Types

0.1.1 Denotation of Type Environments

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we can define the denotation of an $\mathtt{Ok}$ type environment Γ.

$$[\![\diamond]\!]_M = \mathtt{1}$$

$$[\![\Gamma, x : A]\!]_M = ([\![\Gamma]\!]_M \times [\![A]\!]_M)$$

For ease of notation, and since we normally only talk about one denotation function at a time, I shall typically drop the denotation notation when talking about the denotation of value types and type environments. Hence,

$$[\![\Gamma, x : A]\!]_M = \Gamma \times A$$

0.1.2 Denotation of Computation Type

Given a function $[\![\_]\!]_M$ mapping value types to objects in the category $\mathbb{C}$, we write the denotation of Computation types $\mathtt{M}_\epsilon A$ as so:

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon [\![A]\!]_M$$

Since we can infer the denotation function, we can include it implicitly an drop the denotation sign.

$$[\![\mathtt{M}_\epsilon A]\!]_M = T_\epsilon A$$

0.1.3 Denotation of Function Types

Given a function $[\![\_]\!]_M$ mapping types to objects in the category $\mathbb{C}$, we write the denotation of a function type $A \to \mathtt{M}_\epsilon B$ as so:

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon [\![B]\!]_M)^{[\![A]\!]_M}$$

Again, since we can infer the denotation function, Let us drop the notation.

$$[\![A \to \mathtt{M}_\epsilon B]\!]_M = (T_\epsilon B)^A$$

0.2 Denotation of Terms

Given the denotation of types and typing environments, we can now define denotations of well typed terms.

$$[\![\Gamma \vdash t\colon \tau]\!]_M : \Gamma \to [\![\tau]\!]_M$$

Denotations are defined recursively over the typing derivation of a term. Hence, they implicitly depend on the exact derivation used. Since, as proven in the chapter on the uniqueness of derivations, the denotations of all type derivations yielding the same type relation $\Gamma \vdash t\colon \tau$ are equal, we need not refer to the derivation that yielded each denotation.

0.2.1 Denotation of Value Terms

- (Unit) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{()} : \mathtt{Unit}]\!]_M = [\![\mathtt{()}]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Unit}]\!]_M}$

- (Const) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{C}^A : A]\!]_M = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]_M}$

- (True) $\dfrac{\Gamma \mathtt{Ok}}{[\![\Gamma \vdash \mathtt{true} : \mathtt{Bool}]\!]_M = [\![\mathtt{true}]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M}$

- (False) $\frac{\Gamma\mathtt{Ok}}{[\![\Gamma\vdash\mathtt{false}:\mathtt{Bool}]\!]_M=[\![\mathtt{false}]\!]_M\circ\langle\rangle_\Gamma:\Gamma\to[\![\mathtt{Bool}]\!]_M}$

- (Var) $\frac{\Gamma\mathtt{Ok}}{[\![\Gamma,x:A\vdash x:A]\!]_M=\pi_2:\Gamma\times A\to A}$

- (Weaken) $\frac{f=[\![\Gamma\vdash x:A]\!]_M:\Gamma\to A}{[\![\Gamma,y:B\vdash x:A]\!]_M=f\circ\pi_1:\Gamma\times B\to A}$

- (Lambda) $\frac{f=[\![\Gamma,x:A]\!]_M C\mathtt{M}_\epsilon B:\Gamma\times A\to T_\epsilon B}{[\![\Gamma\vdash\lambda x:A.C:A\to\mathtt{M}_\epsilon B]\!]_M=\mathtt{cur}(f):\Gamma\to(T_\epsilon B)^A}$

- (Subtype) $\frac{f=[\![\Gamma\vdash v:A]\!]_M:\Gamma\to A g=[\![A\leq:B]\!]_M}{[\![\Gamma\vdash v:B]\!]_M=g\circ f:\Gamma\to B}$

0.2.2 Denotation of Computation Terms

- (Return) $\frac{f=[\![\Gamma\vdash v:A]\!]_M}{[\![\Gamma\vdash\mathtt{return}v:\mathtt{M}_{\mathtt{1}}A]\!]_M=\eta_A\circ f}$

- (If) $\frac{f=[\![\Gamma\vdash v:\mathtt{Bool}]\!]_M g=[\![\Gamma\vdash C_1:\mathtt{M}_\epsilon A]\!]_M h=[\![\Gamma\vdash C_2:\mathtt{M}_\epsilon A]\!]_M}{[\![\Gamma\vdash\mathtt{if}_{\epsilon,A}v\mathtt{then}C_1\mathtt{else}C_2:\mathtt{M}_\epsilon A]\!]_M=\mathtt{If}_{\mathtt{M}_\epsilon B}\circ\langle f,\langle g,h\rangle\rangle:\Gamma\to T_\epsilon A}$

- (Bind) $\frac{f=[\![\Gamma\vdash C_1:\mathtt{M}_{\epsilon_1}A:\Gamma\to T_{\epsilon_1}A g=[\![\Gamma,x:A\vdash C_2:\mathtt{M}_{\epsilon_2}B]\!]_M]\!]_M:\Gamma\times A\to T_{\epsilon_2}B}{[\![\Gamma\vdash\mathtt{do}x\leftarrow C_1\mathtt{in}C_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}]\!]_M=\mu_{\epsilon_1,\epsilon_2,B}\circ T_{\epsilon_1}g\circ\mathtt{t}_{\Gamma,A,\epsilon_1}\circ\big\langle\mathtt{Id}_\Gamma,f\big\rangle:\Gamma\to T_{\epsilon_1\cdot\epsilon_2}B}$

- (Subeffect) $\frac{f=[\![\Gamma\vdash c:\mathtt{M}_{\epsilon_1}A]\!]_M:\Gamma\to T_{\epsilon_1}A g=[\![A\leq:B]\!]_M h=[\![\epsilon_1\leq\epsilon_2]\!]}{[\![\Gamma\vdash C:\mathtt{M}_{\epsilon_2}B]\!]_M=h_B\circ T_{\epsilon_1}g\circ f}$

- (Apply) $\frac{f=[\![\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B]\!]_M:\Gamma\to(T_\epsilon B)^A g=[\![\Gamma\vdash v_2:A]\!]_M:\Gamma\to A}{[\![\Gamma\vdash v_1v_2:\mathtt{M}_\epsilon B]\!]_M:\Gamma\to T_\epsilon B}$