Contents

Chapter 1

Preliminaries

1.1 Base Category Requirements

There are 3 distinct objects in the base category, $\mathbb{C}$:

- U - The kind of `Effect`
- W - The kind of `Type`
- `1` - A terminal object

And we have finite products on U.

- $U^0 = \mathtt{1}$
- $U^{n+1} = U^n \times U$

We also have the following natural operations on morphisms in $\mathbb{C}$.
Let $I = U^n$.

- $\diamond : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates exponential types.
- $\square : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates products of types.
- $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I, W)$ - generates quantified types.
- $\mathtt{Eff} : \mathbb{C}(I, U) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - generates monad types.
- $\mathtt{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \to \mathbb{C}(I, U)$ - Generates multiplication of effects.

With naturality conditions which mean, for $\theta : \mathtt{Unit}^m \to \mathtt{Unit}^n (I' \to I)$,

- $\diamond(\phi, \psi) \circ \theta = \diamond(\phi \circ \theta, \psi \circ \theta)$
- $\square(\phi, \psi) \circ \theta = \square(\phi \circ \theta, \psi \circ \theta)$
- $\forall_I(\phi) \circ \theta = \forall_{I'}(\phi \circ (\theta \times \mathtt{Id}_U))$
- $\mathtt{Eff}(\phi, \psi) \circ \theta = \mathtt{Eff}(\phi \circ \theta, \psi \circ \theta)$
- $\mathtt{Mul}(\phi, \psi) \circ \theta = \mathtt{Mul}(\phi \circ \theta, \psi \circ \theta)$

1.2 Well-Formed-ness

Each instance of the well-formed-ness relation on effects, $\Phi \vdash \epsilon$ has a denotation in $\mathbb{C}$:

$$[\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M : I \to U \tag{1.1}$$

Each instance of the well-formed-ness relation on types, $\Phi \vdash A$ has a denotation in $\mathbb{C}$:

$$[\![P \vdash A \colon \mathtt{Type}]\!]_M : I \to W \tag{1.2}$$

It should also be the case that

$$\mathtt{Mul}([\![\Phi \vdash \epsilon_1 \colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2 \colon \mathtt{Effect}]\!]_M) = [\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \mathtt{Effect}]\!]_M \in \mathbb{C}(I, U) \tag{1.3}$$

That is, Mul should be have identity $[\![\Phi \vdash \mathtt{1} \colon \mathtt{Effect}]\!]_M$ and be associative.

1.3 Substitution and Weakening of the Effect Environment

For each instance of the well-formed-ness relation on substitution of effects $\Phi' \vdash \sigma \colon \Phi$, there exists a denotation in $\mathbb{C}$:

$$[\![\Phi' \vdash \sigma \colon \Phi]\!]_M : I' \to I \tag{1.4}$$

For each instance of the well-formed weakening relation on effect-environments, $\omega : \Phi' \triangleright \Phi$ there exists a denotation in $\mathbb{C}$:

$$[\![\omega : \Phi' \triangleright \Phi]\!]_M : I' \to I \tag{1.5}$$

.

1.4 Fibre Categories

Each set of morphisms $\mathbb{C}(I, W)$ forms the objects of a semantic-closed (S-closed) category. That is, a category satisfying all the properties needed for the non-polymorphic language:

- Cartesian Closed
- Co-product of the terminal object with itself ($\mathtt{1} + \mathtt{1}$)
- Ground morphisms for each ground constant ($\mathtt{C}^A : \mathtt{1} \to A$)
- Partial order morphisms on ground types ($[\![A \leq\colon_\gamma]\!]_M B$)
- A strong, monad, graded by the po-monoid $(E_\Phi, \cdot_\Phi, \leq_\Phi, \mathtt{1})$.

1.5 Re-indexing Functors

For each morphism $f : I' \to I$ in $\mathbb{C}$, there should be a co-variant, re-indexing functor $f^* : \mathbb{C}(I, W) \to \mathbb{C}(I', W)$, which is S-closed. That is, it preserves the S-closed properties of $\mathbb{C}(I, W)$. (See below).

$(-)^*$ should be a contra-variant functor in its $\mathbb{C}$ argument and co-variant in its right argument.

- $(g \circ f)^*(a) = f^*(\gamma^*(a))$
- $\mathtt{Id}_I^*(a) = a$
- $f^*(\mathtt{Id}_A) = \mathtt{Id}_{f^*(A)}$
- $f^*(a \circ b) = f^*(a) \circ f^*(b)$

1.5.1 f^* Preserves Products

If $\langle g, h\rangle : \mathbb{C}(I, W)(Z, X \times Y)$ Then

$$f^*(X \times Y) = f^*(X) \times f^*(Y) \tag{1.6}$$
$$f^*(\langle g, h\rangle) = \langle f^*(g), f^*h\rangle \qquad : \mathbb{C}(I', W)(f^*Z, f^*(X) \times f^*(Y)) \tag{1.7}$$
$$f^*(\pi_1) = \pi_1 \qquad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(X)) \tag{1.8}$$
$$f^*(\pi_2) = \pi_2 \qquad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(Y)) \tag{1.9}$$

1.5.2 f^* Preserves Terminal Object

If $\mathtt{Id}_A : \mathbb{C}(I, W)(A, \mathtt{1})$ Then

$$f^*(\mathtt{1}) = \mathtt{1} \tag{1.10}$$
$$f^*(\langle\rangle_A) = \langle\rangle_{f^*(A)} \qquad : \mathbb{C}(I', W)(f^*A, \mathtt{1}) \tag{1.11}$$
$$\tag{1.12}$$

1.5.3 f^* Preserves Exponentials

$$f^*(Z^X) = (f^*(Z))^{(f^*(X))} \tag{1.13}$$
$$f^*(\mathtt{app}) = \mathtt{app} \qquad : \mathbb{C}(I', W)(f^*(Z^X) \times f^*(X), f^*(Z)) \tag{1.14}$$
$$f^*(\mathtt{cur}(g)) = \mathtt{cur}(f^*(g)) \qquad : \mathbb{C}(I', W)(f^*(Y) \times f^*(X), f^*(Z)^{f^*(X)}) \tag{1.15}$$

1.5.4 f^* Preserves Co-product on Terminal

$$f^*(\mathtt{1} + \mathtt{1}) = \mathtt{1} + \mathtt{1} \tag{1.16}$$
$$f^*(\mathtt{inl}) = \mathtt{inl} \qquad : \mathbb{C}(I', W)(\mathtt{1}, \mathtt{1} + \mathtt{1}) \tag{1.17}$$
$$f^*(\mathtt{inr}) = \mathtt{inr} \qquad : \mathbb{C}(I', W)(\mathtt{1}, \mathtt{1} + \mathtt{1}) \tag{1.18}$$
$$f^*([g, h]) = [f^*(g), f^*(h)] \qquad : \mathbb{C}(I', W)(\mathtt{1} + \mathtt{1}, f^*(Z)) \tag{1.19}$$

1.5.5 f^* Preserves Graded Monad

$$f^*(T_\epsilon A) = T_{f^*(\epsilon)} f^*(A) \qquad : \mathbb{C}(I', W) \tag{1.20}$$
$$f^*(\mathtt{1}) = \mathtt{1} \quad \text{The unit effect} \tag{1.21}$$
$$f^*(\eta_A) = \eta_{f^*(A)} \qquad : \mathbb{C}(I', W)(f^*(A), f^*(T_{\mathtt{1}} A)) \tag{1.22}$$
$$f^*(\mu_{\epsilon_1, \epsilon_2, A}) = \mu_{f^*(\epsilon_1), f^*(\epsilon_2), f^*(A)} \qquad : \mathbb{C}(I', W)(f^*(T_{\epsilon_1} T_{\epsilon_2} A), f^*(T_{f^*(\epsilon_1) \cdot f^*(\epsilon_2)} f^*(A))) \tag{1.23}$$
$$f^*(\epsilon_1 \cdot \epsilon_2) = f^*(\epsilon_1) \cdot f^*(\epsilon_2) \tag{1.24}$$
$$\tag{1.25}$$

1.5.6 f^* Preserves Tensor Strength

$$f^*(\mathtt{t}_{\epsilon,A,B}) = \mathtt{t}_{f^*(\epsilon),f^*(A),f^*(B)} \qquad : \mathbb{C}(I',W)(f^*(A \times T_\epsilon B), f^*(T_\epsilon(A \times B))) \tag{1.26}$$

1.5.7 f^* Preserves Ground Constants

For each ground constant $[\![\mathtt{C}^A]\!]_M$ in $\mathbb{C}(I,W)$,

$$f^*([\![\mathtt{C}^A]\!]_M) = \mathtt{C}^{f^*(A)} : \mathbb{C}(I',W)(\mathtt{1}, f^*(A)) \tag{1.27}$$

1.5.8 f^* Preserves Ground Sub-effecting

For ground effects e_1, e_2 such that $e_1 \leq e_2$

$$f^*(e) = e : \mathbb{C}(I',U) \tag{1.28}$$

$$f^*[\![\epsilon_1 \leq e_2]\!]_A = [\![e_1 \leq e_2]\!]_{f^*(A)} : \mathbb{C}(I',W)f^*(T_{e_1}A), f^*(T_{e_2}A) \tag{1.29}$$

$$\tag{1.30}$$

1.5.9 f^* Preserves Ground Sub-typing

For ground types γ_1, γ_2 such that $\gamma_1 \leq:_\gamma \gamma_2$

$$f^*\gamma = \gamma : \mathbb{C}(I',W)(\mathtt{1}, \gamma) \tag{1.31}$$

$$f^*([\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M) = [\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M \qquad : \mathbb{C}(I',W)(\gamma_1, \gamma_2) \tag{1.32}$$

$$\tag{1.33}$$

1.5.10 Action on Objects

It follows that the action of f^* on an object A in $\mathbb{C}(I,W)$ (i.e. a morphism $I \to U$ in $\mathbb{C}$) is:

$$f^*(A) = A \circ f : I' \to I \to W \tag{1.34}$$

1.6 Naturality Properties

1.7 The $\forall_I$ functor

We expand $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I,W)$ to be a functor which is right adjoint to the re-indexing functor π_1^*.

$$\overline{(\_)} : \mathbb{C}(I \times U, W)(\pi_1^* A, B) \leftrightarrow \mathbb{C}(I,W)(A, \forall_I B) : \widehat{(\_)} \tag{1.35}$$

For $A : \mathbb{C}(I,W)$, $B : \mathbb{C}(I \times U, W)$.

Hence the action of $\forall_I$ on a morphism $l : A \to A'$ is as follows:

$$\forall_I(l) = \overline{l \circ \epsilon_A} \tag{1.36}$$

Where $\epsilon_A : \mathbb{C}(I \times U, W)(\pi_1^* \forall_I A \to A)$ is the co-unit of the adjunction.

1.8 Naturality Corollaries

Here are some simple corollaries of the adjunction between π_1^* and $\forall_I$.

1.8.1 Naturality

By the definition of an adjunction:

$$\overline{f \circ \pi_1^*(n)} = \overline{f} \circ n \tag{1.37}$$

1.8.2 $\overline{(-)}$ and Re-indexing Functors

TODO: Why does this occur? it comes from page 222 of Crole?

$$\begin{aligned}
\theta^*(\overline{f}) &= (\pi_1 \circ (\theta \times \mathtt{Id}_U))^*(\overline{f}) && (1.38)\\
&= (\theta \times \mathtt{Id}_U)^*(\pi_1^*(\overline{f})) && (1.39)\\
& && (1.40)\\
& && (1.41)\\
&= \overline{(\theta \times \mathtt{Id}_U)^* f} && (1.42)\\
& && (1.43)\\
& && (1.44)
\end{aligned}$$

1.8.3 $(\hat{-})$ and Re-Indexing Functors

$$\begin{aligned}
\theta^*(\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m})) &= (\langle \mathtt{Id}_I, \rho\rangle \circ \theta)^*(\widehat{m}) && (1.45)\\
&= ((\theta \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_I, \rho\rangle)^*(\widehat{m}) && (1.46)\\
&= \langle \mathtt{Id}_I, \rho \circ \theta\rangle^* (\theta \times \mathtt{Id}_U)^*(\widehat{m}) && (1.47)\\
&= \langle \mathtt{Id}_I, \theta^*\rho\rangle^* (\theta^*(\widehat{m})) && (1.48)
\end{aligned}$$

1.8.4 Pushing Morphisms into f^*

$$\begin{aligned}
\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ n &= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ \langle \mathtt{Id}_I, \rho\rangle^* \pi_1^*(n) && (1.49)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m} \circ \pi_1^*(n)) && (1.50)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m \circ n}) && (1.51)
\end{aligned}$$

Chapter 2

Denotations

2.1 Effects

For each instance of the well-formed-ness relation on effects, we define a morphism $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M : \mathbb{C}(I, U)$

- $[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M = [\![\epsilon]\!]_M \circ \langle\rangle_I :\to U$
- $[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M = \pi_2 : I \times U \to U$
- $[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 : I \times U \to U$
- $[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Effect}]\!]_M = \texttt{Mul}([\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M) : I \to U$

2.2 Types

For each instance of the well-formed-ness relation on types, we define a morphism $[\![\Phi \vdash A \colon \texttt{Type}]\!]_M : \mathbb{C}(I, W)$.

$[\![\texttt{Unit}]\!]_M$ is the morphism generating the terminal object of $\mathbb{C}(I, W)$. $\texttt{Bool}$ is the morphism generating the co-product of this terminal object, $\mathtt{1} + \mathtt{1}$.

- $[\![\Phi \vdash \texttt{Unit} \colon \texttt{Type}]\!]_M = [\![\texttt{Unit}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \texttt{Bool} \colon \texttt{Type}]\!]_M = [\![\texttt{Bool}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \gamma \colon \texttt{Type}]\!]_M = [\![\gamma]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash A \to B \colon \texttt{Type}]\!]_M = \diamond([\![\Phi \vdash A \colon \texttt{Type}]\!]_M, [\![\Phi \vdash B \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \texttt{M}_\epsilon A \colon \texttt{Type}]\!]_M = \texttt{Eff}([\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash A \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \forall \alpha . A \colon \texttt{Type}]\!]_M = \forall_I([\![\Phi, \alpha \vdash A \colon \texttt{Type}]\!]_M) : I \to W$

2.3 Effect Substitution

For each effect-substitution well-formed-ness-relation, define a denotation morphism, $[\![\Phi' \vdash \sigma \colon \Phi]\!]_M : \mathbb{C}(I', I)$

- $[\![\Phi' \vdash \diamond : \diamond]\!]_M = \langle\rangle_I : \mathbb{C}(I', \mathtt{1})$
- $[\![\Phi' \vdash (\sigma, \alpha := \epsilon) : \Phi, \alpha]\!]_M = \langle [\![\Phi' \vdash \sigma : \Phi]\!]_M, [\![\Phi \vdash \epsilon : \mathtt{Effect}]\!]_M \rangle : \mathbb{C}(I', I \times U)$

2.4 Effect Weakening

For each instance of the effect-environment weakening relation, define a denotation morphism: $[\![\omega : \Phi' \triangleright P]\!]_M : \mathbb{C}(I', I)$

- $[\![\iota : \Phi \triangleright \Phi]\!]_M = \mathtt{Id}_I : I \to I$
- $[\![w\pi : \Phi', \alpha \triangleright \Phi]\!]_M = [\![\omega : \Phi' \triangleright \Phi]\!]_M \circ \pi_1 : I' \times U \to I$
- $[\![w\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = ([\![\omega : \Phi' \triangleright \Phi]\!]_M \times \mathtt{Id}_U) : I' \times U \to I \times U$

2.5 Sub-Typing

For each instance of the sub-typing relation with respect to an effect environment, there exists a denotation, $[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I, W)(A, B)$.

- $[\![\gamma_1 \leq:_\Phi \gamma_2]\!]_M = [\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M : \mathbb{C}(I, W)(\gamma_1, \gamma_2)$
- $[\![A \to B \leq:_\Phi A' \to B']\!]_M = [\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}$
- $[\![\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M \circ T_{\epsilon_1} [\![A \leq:_\Phi B]\!]_M$
- $[\![\forall \alpha . A \leq:_\Phi \forall \alpha . B]\!]_M = \forall_I [\![A \leq:_{\Phi, \alpha} B]\!]_M$

2.6 Type-Environments

For each instance of the well-formed relation on type environments, define an object in $[\![I \vdash W \mathtt{Ok}]\!]_M \in \mathbb{C}(I, W)$.

- $[\![\Phi \vdash \diamond \mathtt{Ok}]\!]_M = \mathtt{1} : \mathbb{C}(I, W)$
- $[\![\Phi \vdash \Gamma, x : A \mathtt{Ok}]\!]_M = \square([\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M, [\![\Phi \vdash A : \mathtt{Type}]\!]_M)$

2.7 Terms

For each instance of the typing relation, define a denotation morphism: $[\![\Phi \mid \Gamma \vdash v : A]\!]_M : \mathbb{C}(I, W)(\Gamma_I, A_I)$. Writing Γ_I and A_I for $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$ and $[\![\Phi \vdash A : \mathtt{Type}]\!]_M$.

For each ground constant, $\mathtt{C}^A$, there exists $c : \mathtt{1} \to A_I$ in $\mathbb{C}(I, W)$.

- (Unit) $\frac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{()} : \mathtt{Unit}]\!]_M = \langle\rangle_\Gamma : \Gamma_I \to \mathtt{1}}$
- (Const) $\frac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{C}^A : A]\!]_M = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]_M}$
- (True) $\frac{\Phi \vdash \Gamma \mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{true} : \mathtt{Bool}]\!]_M = \mathtt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = \mathtt{1} + \mathtt{1}}$

- (False) $\dfrac{\Phi \vdash \Gamma\, \mathtt{Ok}}{[\![\Phi|\Gamma \vdash \mathtt{false}{:}\mathtt{Bool}]\!]_M = \mathtt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = \mathtt{1}+\mathtt{1}}$

- (Var) $\dfrac{\Phi \vdash \Gamma\, \mathtt{Ok}}{[\![\Phi|\Gamma, x{:}A \vdash x{:}A]\!]_M = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Phi|\Gamma \vdash x{:}A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma, y{:}B \vdash x{:}A]\!]_M = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\dfrac{f = [\![\Phi|\Gamma, x{:}A \vdash C{:}\mathtt{M}_\epsilon B]\!]_M : \Gamma \times A \to T_\epsilon B}{[\![\Phi|\Gamma \vdash \lambda x{:}A.C{:}A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}A]\!]_M : \Gamma \to A \quad g = [\![A \leq{:}_\Phi B]\!]_M}{[\![\Phi|\Gamma \vdash v{:}B]\!]_M = g \circ f : \Gamma \to B}$

- (Return) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{return}\, v{:}\mathtt{M}_{\mathtt{1}} A]\!]_M = \eta_A \circ f}$

- (If) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}\mathtt{Bool}]\!]_M : \Gamma \to \mathtt{1}+\mathtt{1} \quad g = [\![\Phi|\Gamma \vdash C_1{:}\mathtt{M}_\epsilon A]\!]_M \quad h = [\![\Phi|\Gamma \vdash C_2{:}\mathtt{M}_\epsilon A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{:}\mathtt{M}_\epsilon A]\!]_M = \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \to T_\epsilon A}$

- (Bind) $\dfrac{f = [\![\Phi|\Gamma \vdash C_1{:}\mathtt{M}_{\epsilon_1} A{:}\Gamma \to T_{\epsilon_1} A \quad g = [\![\Phi|\Gamma, x{:}A \vdash C_2{:}\mathtt{M}_{\epsilon_2} B]\!]_M]\!]_M : \Gamma \times A \to T_{\epsilon_2} B}{[\![\Phi|\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2{:}\mathtt{M}_{\epsilon_1 \cdot \epsilon_2}]\!]_M = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma,A,\epsilon_1} \circ \left\langle \mathtt{Id}_\Gamma, f \right\rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$

- (Apply) $\dfrac{f = [\![\Phi|\Gamma \vdash v_1{:}A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \quad g = [\![\Phi|\Gamma \vdash v_2{:}A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma \vdash v_1\ v_2{:}\mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f, g\rangle : \Gamma \to T_\epsilon B}$

- (Effect-Lambda) $\dfrac{f = [\![\Phi, \alpha|\Gamma \vdash v{:}A]\!]_M : \mathbb{C}(I \times U, W)(\Gamma, A)}{[\![\Phi|\Gamma \vdash \Lambda\alpha.A{:}\forall\epsilon.A]\!]_M = \overline{f} : \mathbb{C}(I, W)(\Gamma, \forall_I(A))}$

- (Effect-App) $\dfrac{g = [\![\Phi|\Gamma \vdash v{:}\forall\alpha.A]\!]_M : \mathbb{C}(I, W)(\Gamma, \forall_I(A)) \quad h = [\![\Phi \vdash \epsilon{:}\mathtt{Effect}]\!]_M : \mathbb{C}(I, U)}{[\![\Phi|\Gamma \vdash v\ \epsilon{:}A[\epsilon/\alpha]]\!]_M = \left\langle \mathtt{Id}_I, h \right\rangle^* (\epsilon_{[\![\Phi,\beta \vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M}) \circ g : \mathbb{C}(I, W)(\Gamma, A[\epsilon/\alpha])}$

Chapter 3

Effect Substitution Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-variable substitution upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the substituted relation, $\Delta' = \sigma^*(\Delta)$.

3.1 Effects

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash \sigma(\epsilon) \colon \texttt{Effect}]\!]_M = \sigma^*[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M$

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M \circ \sigma &= [\![e]\!]_M \circ \langle\rangle_I \circ \sigma && (3.1)\\
&= [\![e]\!]_M \circ \langle\rangle_{I'} && (3.2)\\
&= [\![\Phi' \vdash e \colon \texttt{Type}]\!]_M && (3.3)\\
& && (3.4)
\end{aligned}$$

Case Var:

$$\begin{aligned}
[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \sigma' &= \pi_2 \circ \langle \sigma, [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle \quad \text{By inversion } \sigma' = (\sigma, \alpha := \epsilon) && (3.5)\\
&= [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M && (3.6)\\
&= [\![\Phi' \vdash \sigma'(\alpha) \colon \texttt{Effect}]\!]_M && (3.7)\\
& && (3.8)
\end{aligned}$$

Case Weaken:

$$
\begin{aligned}
[\![\Phi, \beta \vdash \alpha\colon \mathtt{Type}]\!]_M \circ \sigma' &= [\![\Phi \vdash \alpha\colon \mathtt{Type}]\!]_M \circ \pi_1 \circ \langle \sigma, [\![\Phi' \vdash \epsilon\colon \mathtt{Effect}]\!]_M \rangle \quad \text{By inversion, } \sigma' = (\sigma, \beta := \epsilon) && (3.9)\\
&= [\![\Phi \vdash \alpha\colon \mathtt{Type}]\!]_M \circ \sigma && (3.10)\\
&= [\![\Phi' \vdash \sigma(\alpha)\colon \mathtt{Type}]\!]_M && (3.11)\\
&= [\![\Phi' \vdash \sigma'(\alpha)\colon \mathtt{Type}]\!]_M && (3.12)\\
& && (3.13)
\end{aligned}
$$

Case Multiply:

$$
\begin{aligned}
[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2\colon \mathtt{Type}]\!]_M \circ \sigma &= \mathtt{Mul}([\![\Phi \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M) \circ \sigma && (3.14)\\
&= \mathtt{Mul}([\![\Phi \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M \circ \sigma) \quad \text{By Naturality} && (3.15)\\
&= \mathtt{Mul}([\![\Phi' \vdash \sigma(\epsilon_1)\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \sigma(\epsilon_2)\colon \mathtt{Effect}]\!]_M) && (3.16)\\
&= [\![\Phi' \vdash \sigma(\epsilon_1) \cdot \sigma(\epsilon_2)\colon \mathtt{Effect}]\!]_M && (3.17)\\
&= [\![\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)\colon \mathtt{Effect}]\!]_M && (3.18)\\
& && (3.19)
\end{aligned}
$$

3.2 Types

If $\sigma = [\![\Phi' \vdash \sigma\colon \Phi]\!]_M$ then $[\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M = \sigma^*[\![\Phi \vdash A\colon \mathtt{Type}]\!]_M = [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash A\colon \mathtt{Type}]\!]_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$
\begin{aligned}
[\![\Phi \vdash \gamma\colon \mathtt{Type}]\!]_M \circ \sigma &= [\![\gamma]\!]_M \circ \langle\rangle_I \circ \sigma && (3.20)\\
&= [\![\gamma]\!]_M \circ \langle\rangle_{I'} && (3.21)\\
&= [\![\Phi' \vdash \gamma\colon \mathtt{Type}]\!]_M && (3.22)\\
&= [\![\Phi' \vdash \gamma\,[\sigma] : \mathtt{Type}]\!]_M && (3.23)
\end{aligned}
$$

Case Monad:

$$
\begin{aligned}
[\![\Phi \vdash \mathtt{M}_\epsilon A\colon \mathtt{Type}]\!]_M \circ \sigma &= \mathtt{Eff}([\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M) \circ \sigma && (3.24)\\
&= \mathtt{Eff}([\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \sigma) \quad \text{By naturality} && (3.25)\\
&= \mathtt{Eff}([\![\Phi' \vdash \sigma(\epsilon)\colon \mathtt{Effect}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.26)\\
&= [\![\Phi' \vdash \mathtt{M}_{\sigma(\epsilon)} A\,[\sigma]\colon \mathtt{Type}]\!]_M && (3.27)\\
&= [\![\Phi' \vdash (\mathtt{M}_\epsilon A)\,[\sigma] : \mathtt{Type}]\!]_M && (3.28)
\end{aligned}
$$

Case Quantification:

$$\begin{aligned}
[\![\Phi \vdash \forall\alpha.A\colon \mathtt{Type}]\!]_M \circ \sigma &= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M) \circ \sigma && (3.29)\\
&= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M \circ (\sigma \times \mathtt{Id}_U)) && (3.30)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma, \alpha := \epsilon] : \mathtt{Type}]\!]_M) && (3.31)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.32)\\
&= [\![\Phi' \vdash \forall\alpha.A\,[\sigma] : \mathtt{Type}]\!]_M && (3.33)\\
&= [\![\Phi' \vdash (\forall\alpha.A)\,[\sigma] : \mathtt{Type}]\!]_M && (3.34)\\
& && (3.35)
\end{aligned}$$

Case Function:

$$\begin{aligned}
[\![\Phi \vdash A \to B\colon \mathtt{Type}]\!]_M \circ \sigma &= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M) \circ \sigma && (3.36)\\
&= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \sigma, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M \circ \sigma) \quad \text{By Naturality} && (3.37)\\
&= \diamond([\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M, [\![\Phi' \vdash B\,[\sigma] : \mathtt{Type}]\!]_M) && (3.38)\\
&= [\![\Phi' \vdash (A\,[\sigma]) \to (B\,[\sigma])\colon \mathtt{Type}]\!]_M && (3.39)\\
&= [\![\Phi' \vdash (A \to B)\,[\sigma] : \mathtt{Type}]\!]_M && (3.40)\\
& && (3.41)
\end{aligned}$$

3.3 Sub-typing

If $\sigma = [\![\Phi' \vdash \sigma\colon \Phi]\!]_M$ then $[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M = \sigma^*[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $[\![A \leq:_\Phi B]\!]_M$. Using S-closure of σ^*

Case Ground:

$$\sigma^*(\gamma_1 \leq:_\gamma \gamma_2) = (\gamma_1 \leq:_\gamma \gamma_2) \tag{3.42}$$

Since σ^* is s-closed.

Case Monad:

$$\begin{aligned}
\sigma^*[\![\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M &= \sigma^*([\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M) \circ \sigma^*(T_{\epsilon_1}([\![A \leq:_\Phi B]\!]_M)) && (3.43)\\
&= [\![\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)]\!]_M \circ T_{\sigma(\epsilon_1)}[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M \quad \text{By S-Closure} && (3.44)\\
&= [\![\mathtt{M}_{\sigma(\epsilon_1)} A\,[\sigma] \leq:_{\Phi'} \mathtt{M}_{\sigma(\epsilon_2)} B\,[\sigma]]\!]_M && (3.45)\\
&= [\![(\mathtt{M}_{\epsilon_1} A)\,[\sigma] \leq:_{\Phi'} \mathtt{M}_{\epsilon_2} B\,[\sigma]\,]\!]_M && (3.46)\\
& && (3.47)
\end{aligned}$$

Case For All:

$$\begin{aligned}
\sigma^*[\![\forall\alpha.A \leq:_\Phi \forall\alpha.B]\!]_M &= \sigma^*(\forall_I([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.48)\\
&= \forall_{I'}((\sigma \times \mathtt{Id}_U)^*([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.49)\\
&= \forall_{I'}([\![A\,[\sigma, \alpha := \alpha] \leq:_{\Phi',\alpha} B\,[\sigma, \alpha := \alpha]\,]\!]_M) && (3.50)\\
&= [\![(\forall\alpha.A)\,[\sigma] \leq:_{\Phi'} (\forall\alpha.B)\,[\sigma]\,]\!]_M && (3.51)\\
& && (3.52)
\end{aligned}$$

Case Fn:

$$\begin{aligned}
\sigma^*[\![(A \to B) \leq:_\Phi A' \to B']\!]_M &= \sigma^*([\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}) && (3.53)\\
&= \sigma^*(\mathtt{cur}([\![B \leq:_\Phi B']\!]_M \circ \mathtt{app})) \circ \sigma^*(\mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq:_\Phi A]\!]_M))) && (3.54)\\
&= \mathtt{cur}(\sigma^*([\![B \leq:_\Phi B']\!]_M) \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times \sigma^*([\![A' \leq:_\Phi A]\!]_M))) && (3.55)\\
&= \mathtt{cur}([\![B\,[\sigma] \leq:_{\Phi'} B'\,[\sigma]\,]\!]_M \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{B[\sigma]} \times [\![A'\,[\sigma] \leq:_{\Phi'} A\,[\sigma]\,]\!]_M)) && (3.56)\\
&= [\![(A\,[\sigma]) \to (B\,[\sigma]) \leq:_{\Phi'} (A'\,[\sigma]) \to (B'\,[\sigma])]\!]_M && (3.57)\\
&= [\![(A \to B)\,[\sigma] \leq:_{\Phi'} (A' \to B')\,[\sigma]\,]\!]_M && (3.58)
\end{aligned}$$

3.4 Type Environments

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}]\!]_M = \sigma^*[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M = [\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M \circ \sigma : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$\begin{aligned}
\sigma^*[\![\Phi \vdash \diamond\mathtt{Ok}]\!]_M &= \langle\rangle_I \circ \sigma && (3.59)\\
&= \langle\rangle_{I'} && (3.60)\\
&= [\![\Phi' \vdash \diamond\mathtt{Ok}]\!]_M && (3.61)\\
[\![\Phi' \vdash \diamond\,[\sigma]\,\mathtt{Ok}]\!]_M & && (3.62)\\
& && (3.63)
\end{aligned}$$

Case Var:

$$\begin{aligned}
\sigma^*[\![\Phi \vdash \Gamma, x : A\mathtt{Ok}]\!]_M &= \sigma^*(\Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M)) && (3.64)\\
&= \Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M) \circ \sigma && (3.65)\\
&= \Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M \circ \sigma, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M \circ \sigma) && (3.66)\\
&= \Box([\![\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.67)\\
&= [\![\Phi' \vdash \Gamma\,[\sigma]\,, x : A\,[\sigma]\,\mathtt{Ok}]\!]_M && (3.68)\\
&= [\![\Phi' \vdash (\Gamma, x : A)\,[\sigma]\,\mathtt{Ok}]\!]_M && (3.69)\\
& && (3.70)
\end{aligned}$$

3.5 Terms

If

$$\begin{aligned}
\sigma &= [\![\Phi' \vdash \sigma \colon \Phi]\!]_M && (3.71)\\
\Delta &= [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M && (3.72)\\
\Delta' &= [\![\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]\,]\!]_M && (3.73)\\
& && (3.74)
\end{aligned}$$

Then

$$\Delta' = \sigma^*(\Delta) \tag{3.75}$$

Proof: By induction over the derivation of Δ. Using the S-Closure of σ^*. We use Γ_I to indicate $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$, an A_I to indicate $[\![\Phi \vdash A \colon \mathtt{Effect}]\!]_M$

Case Unit:

$$\Delta = \langle\rangle_{\Gamma_I} \tag{3.76}$$

So

$$\sigma^*(\Delta) = \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.77}$$

Case True, False: Giving the case for true as false is the same but using `inr`

$$\Delta = \mathtt{inl} \circ \langle\rangle_{\Gamma_I} \tag{3.78}$$

So

$$\sigma^*(\Delta) = \mathtt{inl} \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.79}$$

Since σ^* is S-closed.

Case Constant:

$$\Delta = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I} \tag{3.80}$$

So

$$\sigma^*(\Delta) = \sigma^*[\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = [\![\mathtt{C}^{A[\sigma]}]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.81}$$

Since σ^* is S-closed.

Case Subtype: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon A]\!]_M \tag{3.82}$$

Then

$$\Delta = [\![A \leq :_\Phi B]\!]_M \circ \Delta_1 \tag{3.83}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*[\![A \leq :_\Phi B]\!]_M \circ \sigma^*\Delta_1 && (3.84)\\
&= [\![A\,[\sigma] \leq :_{\Phi'} B\,[\sigma]\,]\!]_M \circ \Delta_1' \quad \text{By induction} && (3.85)\\
&= D' && (3.86)
\end{aligned}$$

Case Lambda: Let

$$\Delta_1 = [\![\Phi \mid \Gamma, x : A \vdash v\colon B]\!]_M \tag{3.87}$$

Then

$$\Delta = \texttt{cur}(()\Delta_1) \tag{3.88}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\texttt{cur}(\Delta_1)) && (3.89)\\
&= \texttt{cur}(\sigma^*(\Delta_1)) \quad \text{By S-closure} && (3.90)\\
&= \texttt{cur}(\Delta_1') \quad \text{By induction} && (3.91)\\
&= \Delta' && (3.92)
\end{aligned}$$

Case Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v_1\colon A \to B]\!]_M && (3.93)\\
\Delta_2 &= [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]_M && (3.94)
\end{aligned}$$

Then

$$\Delta = \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \tag{3.95}$$

So

$$\begin{aligned}
\sigma^*\Delta &= \sigma^*(\texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle) && (3.96)\\
&= \texttt{app} \circ \langle \sigma^*(\Delta_1), \sigma^*(\Delta_2) \rangle \quad \text{By S-closure} && (3.97)\\
&= \texttt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Induction} && (3.98)\\
&= \Delta' && (3.99)
\end{aligned}$$

Case Return: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M \tag{3.100}$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \tag{3.101}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\eta_{A_I} \circ \Delta_1) && (3.102)\\
&= \eta_{A_{I'}} \circ \sigma^*(\Delta_1) \quad \text{By S-closure} && (3.103)\\
&= \eta_{A_{I'}} \circ \Delta_1' && (3.104)\\
&= \Delta' && (3.105)
\end{aligned}$$

Case Bind: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \rrbracket_M \tag{3.106}$$
$$\Delta_2 = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B \rrbracket_M \tag{3.107}$$

Then

$$\Delta = \mathtt{M}_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma_I, A_I} \circ \langle \mathtt{Id}_{\Gamma_I}, \Delta_1 \rangle \tag{3.108}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A} \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle) && (3.109)\\
&= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A}) \circ \sigma^*(T_{\epsilon_1} \Delta_2) \circ \sigma^*(\mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (3.110)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \sigma^*(\Delta_2) \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \right\rangle \quad \text{By S-Closure} && (3.111)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \Delta_2' \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \Delta_1' \right\rangle \quad \text{By Induction} && (3.112)\\
&= \Delta' && (3.113)\\
& && (3.114)
\end{aligned}$$

Case If: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v \colon \mathtt{Bool} \rrbracket_M \tag{3.115}$$
$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_1 \colon A \rrbracket_M \tag{3.116}$$
$$\Delta_3 = \llbracket \Phi \mid \Gamma \vdash v_2 \colon A \rrbracket_M \tag{3.117}$$
$$\tag{3.118}$$

Then

$$\Delta = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{3.119}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma) && (3.120)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\sigma^*(\Delta_2) \circ \pi_2), \mathtt{cur}(\sigma^*(\Delta_3) \circ \pi_2)] \circ \sigma^*(\Delta_1)) \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By S-Closure} && (3.121)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By Induction} && (3.122)\\
&= \Delta' && (3.123)\\
& && (3.124)
\end{aligned}$$

Case Effect-Lambda: Let

$$\Delta_1 = \llbracket \Phi, \alpha \mid \Gamma \vdash v \colon A \rrbracket_M \tag{3.125}$$

Then

$$\Delta = \widehat{\Delta_1} \tag{3.126}$$

And also

$$\sigma \times \mathtt{Id} = [\![(\Phi', \alpha) \vdash (\sigma, \alpha := \epsilon) \colon (\Phi, \alpha)]\!]_M \tag{3.127}$$

So

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\widehat{\Delta_1}) && (3.128)\\
&= \widehat{(\sigma \times \mathtt{Id}_U)^* \Delta_1} \quad \text{By naturality} && (3.129)\\
&= \widehat{\Delta_1'} \quad \text{By induction} && (3.130)\\
&= \Delta' && (3.131)
\end{aligned}$$

Case Effect-Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v \colon \forall \alpha. A]\!]_M && (3.132)\\
h &= [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M && (3.133)\\
& && (3.134)
\end{aligned}$$

Then

$$\Delta = \langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1 \tag{3.135}$$

So Due to the substitution theorem on effects

$$h \circ \sigma = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \sigma = [\![\Phi' \vdash \sigma(\epsilon) \colon \mathtt{Effect}]\!]_M = h' \tag{3.136}$$

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1) && (3.137)\\
&= (\langle \mathtt{Id}_\Gamma, h \rangle \circ \sigma)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \sigma^*(\Delta_1) && (3.138)\\
&= ((\sigma \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_\Gamma, h \circ \sigma \rangle)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' && (3.139)\\
&= (\langle \mathtt{Id}_\Gamma, h' \rangle)^* ((\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' && (3.140)\\
& && (3.141)
\end{aligned}$$

Looking at the inner part of the functor application: Let

$$\begin{aligned}
A &= [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M && (3.142)\\
& && (3.143)
\end{aligned}$$

$$\begin{aligned}
(\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M} &= (\sigma \times \mathtt{Id}_U)^* \epsilon_A && (3.144)\\
&= (\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}}) && (3.145)\\
&= \overline{\overline{(\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}})}} \quad \text{By bijection} && (3.146)\\
&= \overline{\sigma^* (\overline{\overline{\widehat{\mathtt{Id}_{\forall_I(A)}}}})} \quad \text{By naturality} && (3.147)\\
&= \widehat{\sigma^* (\mathtt{Id}_{\forall_I(A)})} \quad \text{By bijection} && (3.148)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A \circ (\sigma \times \mathtt{Id}_U))}} \quad \text{By S-Closure, naturality} && (3.149)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A[\sigma, \alpha := \alpha])}} \quad \text{By Substitution theorem} && (3.150)\\
&= \epsilon_{A[\sigma]} && (3.151)
\end{aligned}$$

Going back to the original expression:

$$\sigma^*\Delta = ((\langle \mathtt{Id}_\Gamma, h' \rangle)^*(\epsilon_{A[\sigma]}) \circ \Delta_1)' \tag{3.152}$$
$$= \Delta' \tag{3.153}$$
$$\tag{3.154}$$

Chapter 4

Effect Weakening Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-weakening upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the weakened relation, $\Delta' = \omega^*(\Delta)$.

4.1 Effects

If $\omega = [\![\omega : \Phi' \rhd \Phi]\!]_M$ then $\Phi' \vdash \epsilon\colon \mathtt{Effect} = \omega^*[\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M = [\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M \circ \omega$

Proof: By induction on the derivation on $[\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M$

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash e\colon \mathtt{Effect}]\!]_M \circ \omega &= [\![e]\!]_M \circ \langle\rangle_I \circ \omega &(4.1)\\
&= [\![e]\!]_M \circ \langle\rangle_{I'} &(4.2)\\
&= [\![\Phi' \vdash e\colon \mathtt{Type}]\!]_M &(4.3)\\
& &(4.4)
\end{aligned}$$

Case Var: Case split on ω.

Case: $\omega = \iota$ Then $\Phi' = \Phi$ and $\omega = \mathtt{Id}_I$. So the theorem holds trivially.

Case: $\omega = \omega'\times$ Then

$$\begin{aligned}
[\![\Phi, \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M \circ \omega &= \pi_2 \circ (\omega' \times \mathtt{Id}_U) &(4.5)\\
&= \pi_2 &(4.6)\\
&= [\![\Phi', \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M &(4.7)
\end{aligned}$$

Case: $\omega = \omega'\pi_1$ Then

$$[\![\Phi, \alpha \vdash \alpha\colon \mathtt{Effect}]\!]_M = \pi_2 \circ \omega' \circ \pi_1 \qquad (4.8)$$

Where $\Phi' = \Phi, \beta$ and $\omega' : \Phi'' \rhd \Phi$.

So

$$\pi_2 \circ \omega' = [\![\Phi'' \vdash \alpha \colon \texttt{Effect}]\!]_M \tag{4.9}$$
$$\pi_2 \circ \omega' \circ \pi_1 = [\![\Phi'', \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \qquad = [\![\Phi' \vdash \alpha \colon \texttt{Effect}]\!]_M \tag{4.10}$$

Case Weaken:

$$[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega = [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 \circ \omega \tag{4.11}$$

Case split of structure of w

Case: $\omega = \iota$ Then $\Phi' = \Phi, \beta$ so $\omega = \texttt{Id}_I$ So $[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega = [\![\Phi' \vdash \alpha \colon \texttt{Effect}]\!]_M$

Case: $\omega = \omega'\pi_1$ Then $\Phi' = \Phi'', \gamma$ and $\omega = \omega' \circ \pi_1$ Where $\omega' : \Phi'' \triangleright \Phi, \beta$. So

$$\begin{aligned}
[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega &= [\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega' \circ \pi_1 && (4.12)\\
&= \Phi'' \vdash \alpha \colon \texttt{Effect} \circ \pi_1 && (4.13)\\
&= \Phi'', \gamma \vdash \alpha \colon \texttt{Effect} && (4.14)\\
&= \Phi' \vdash \alpha \colon \texttt{Effect} && (4.15)\\
& && (4.16)
\end{aligned}$$

Case: $\omega = \omega' \times$ Then $\Phi' = \Phi'', \beta$ and $\omega' : \Phi' \triangleright \Phi$

So

$$\begin{aligned}
[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega &= [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 \circ (\omega' \times \texttt{Id}_U) && (4.17)\\
&= [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \omega' \circ \pi_1 && (4.18)\\
&= [\![\Phi'' \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 && (4.19)\\
&= [\![\Phi' \vdash \alpha \colon \texttt{Effect}]\!]_M && (4.20)\\
& && (4.21)
\end{aligned}$$

Case Multiply:

$$\begin{aligned}
[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Type}]\!]_M \circ \omega &= \texttt{Mul}([\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M) \circ \omega && (4.22)\\
&= \texttt{Mul}([\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M \circ \omega, [\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M \circ \omega) \quad \text{By Naturality} && (4.23)\\
&= \texttt{Mul}([\![\Phi' \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M) && (4.24)\\
&= [\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Effect}]\!]_M && (4.25)
\end{aligned}$$

4.2 Types

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![\Phi' \vdash A \colon \texttt{Type}]\!]_M = \omega^* [\![\Phi \vdash A \colon \texttt{Type}]\!]_M = [\![\Phi \vdash A \colon \texttt{Type}]\!]_M \circ \omega$.

Proof: By induction on the derivation on $[\![\Phi \vdash A \colon \texttt{Type}]\!]_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$
\begin{aligned}
[\![\Phi \vdash \gamma\colon \mathtt{Type}]\!]_M \circ \omega &= [\![\gamma]\!]_M \circ \langle\rangle_I \circ \omega && (4.26)\\
&= [\![\gamma]\!]_M \circ \langle\rangle_{I'} && (4.27)\\
&= [\![\Phi' \vdash \gamma\colon \mathtt{Type}]\!]_M && (4.28)\\
&= [\![\Phi' \vdash \gamma\colon \mathtt{Type}]\!]_M && (4.29)
\end{aligned}
$$

Case Monad:

$$
\begin{aligned}
[\![\Phi \vdash \mathtt{M}_\epsilon A\colon \mathtt{Type}]\!]_M \circ \omega &= \mathtt{Eff}([\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M) \circ \omega && (4.30)\\
&= \mathtt{Eff}([\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M \circ \omega, [\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \omega) \quad \text{By naturality} && (4.31)\\
&= \mathtt{Eff}([\![\Phi' \vdash \epsilon\colon \mathtt{Effect}]\!]_M, [\![\Phi' \vdash A\colon \mathtt{Type}]\!]_M) && (4.32)\\
&= [\![\Phi' \vdash (\mathtt{M}_\epsilon A)\colon \mathtt{Type}]\!]_M && (4.33)
\end{aligned}
$$

Case Quantification: Note $[\![\omega\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = \omega \times \mathtt{Id}_U$

$$
\begin{aligned}
[\![\Phi \vdash \forall\alpha.A\colon \mathtt{Type}]\!]_M \circ \omega &= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M) \circ \omega && (4.34)\\
&= \forall_I([\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M \circ (\omega \times \mathtt{Id}_U)) \quad \text{By naturality} && (4.35)\\
&= \forall_I([\![\Phi', \alpha \vdash A\colon \mathtt{Type}]\!]_M) \quad \text{By induction} && (4.36)\\
&= [\![\Phi' \vdash \forall\alpha.A\colon \mathtt{Type}]\!]_M && (4.37)\\
&= [\![\Phi' \vdash (\forall\alpha.A)\colon \mathtt{Type}]\!]_M && (4.38)\\
& && (4.39)
\end{aligned}
$$

Case Function:

$$
\begin{aligned}
[\![\Phi \vdash A \to B\colon \mathtt{Type}]\!]_M \circ \omega &= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M) \circ \omega && (4.40)\\
&= \diamond([\![\Phi \vdash A\colon \mathtt{Type}]\!]_M \circ \omega, [\![\Phi \vdash B\colon \mathtt{Type}]\!]_M \circ \omega) \quad \text{By Naturality} && (4.41)\\
&= \diamond([\![\Phi' \vdash A\colon \mathtt{Type}]\!]_M, [\![\Phi' \vdash B\colon \mathtt{Type}]\!]_M) && (4.42)\\
&= [\![\Phi' \vdash (A \to B)\colon \mathtt{Type}]\!]_M && (4.43)\\
& && (4.44)
\end{aligned}
$$

4.3 Sub-typing

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![A \leq:_{\Phi'} B]\!]_M = \omega^*[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $[\![A \leq:_\Phi B]\!]_M$. Using S-closure of ω^*

Case Ground:

$$
\omega^*(\gamma_1 \leq:_\gamma \gamma_2) = (\gamma_1 \leq:_\gamma \gamma_2) \qquad (4.45)
$$

Since ω^* is s-closed.

Case Monad:

$$\begin{aligned}
\omega^*[\![\mathtt{M}_{\epsilon_1} A \leq :_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M &= \omega^*([\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M) \circ \omega^*(T_{\epsilon_1}([\![A \leq :_\Phi B]\!]_M)) && (4.46)\\
&= [\![\epsilon_1 \leq_{\Phi'} \epsilon_2]\!]_M \circ T_{\epsilon_1}[\![A \leq :_{\Phi'} B]\!]_M \quad \text{By S-Closure} && (4.47)\\
&= [\![\mathtt{M}_{\epsilon_1} A \leq :_{\Phi'} \mathtt{M}_{\epsilon_2} B]\!]_M && (4.48)\\
&= [\![(\mathtt{M}_{\epsilon_1} A) \leq :_{\Phi'} \mathtt{M}_{\epsilon_2} B]\!]_M && (4.49)\\
& && (4.50)
\end{aligned}$$

Case For All: Note $[\![\omega\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = (\omega \times \mathtt{Id}_U)$

$$\begin{aligned}
\omega^*[\![\forall\alpha.A \leq :_\Phi \forall\alpha.B]\!]_M &= \omega^*(\forall_I([\![A \leq :_{\Phi,\alpha} B]\!]_M)) && (4.51)\\
&= \forall_{I'}((\omega \times \mathtt{Id}_U)^*([\![A \leq :_{\Phi,\alpha} B]\!]_M)) && (4.52)\\
&= \forall_{I'}([\![A \leq :_{\Phi',\alpha} B]\!]_M) && (4.53)\\
&= [\![(\forall\alpha.A) \leq :_{\Phi'} (\forall\alpha.B)]\!]_M && (4.54)\\
& && (4.55)
\end{aligned}$$

Case Fn:

$$\begin{aligned}
\omega^*[\![(A \to B) \leq :_\Phi A' \to B']\!]_M &= \omega^*([\![B \leq :_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq :_\Phi A]\!]_M}) && (4.56)\\
&= \omega^*(\mathtt{cur}([\![B \leq :_\Phi B']\!]_M \circ \mathtt{app})) \circ \omega^*(\mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq :_\Phi A]\!]_M))) && (4.57)\\
&= \mathtt{cur}(\omega^*([\![B \leq :_\Phi B']\!]_M) \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times \omega^*([\![A' \leq :_\Phi A]\!]_M))) && (4.58)\\
&= \mathtt{cur}([\![B \leq :_{\Phi'} B']\!]_M \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_B \times [\![A' \leq :_{\Phi'} A]\!]_M)) && (4.59)\\
&= [\![(A \to B) \leq :_{\Phi'} (A' \to B')]\!]_M && (4.60)
\end{aligned}$$

4.4 Type Environments

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma\mathtt{Ok}]\!]_M = \omega^*[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M = [\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M \circ \omega : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$\begin{aligned}
\omega^*[\![\Phi \vdash \diamond\mathtt{Ok}]\!]_M &= \langle\rangle_I \circ \omega && (4.61)\\
&= \langle\rangle_{I'} && (4.62)\\
&= [\![\Phi' \vdash \diamond\mathtt{Ok}]\!]_M && (4.63)\\
& && (4.64)
\end{aligned}$$

Case Var:

$$
\begin{aligned}
\omega^*[\![\Phi \vdash \Gamma, x : A\texttt{Ok}]\!]_M &= \omega^*(\Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M)) && (4.65)\\
&= \Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M) \circ \omega && (4.66)\\
&= \Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M \circ \omega, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M \circ \omega) && (4.67)\\
&= \Box([\![\Phi' \vdash \Gamma\texttt{Ok}]\!]_M, [\![\Phi' \vdash A\colon \texttt{Type}]\!]_M) && (4.68)\\
&= [\![\Phi' \vdash (\Gamma, x : A)\texttt{Ok}]\!]_M && (4.69)\\
& && (4.70)
\end{aligned}
$$

4.5 Terms

4.6 Terms

If

$$
\begin{aligned}
\omega &= [\![\omega : \Phi' \triangleright \Phi]\!]_M && (4.71)\\
\Delta &= [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M && (4.72)\\
\Delta' &= [\![\Phi' \mid \Gamma \vdash v\colon A]\!]_M && (4.73)\\
& && (4.74)
\end{aligned}
$$

Then

$$\Delta' = \omega^*(\Delta) \tag{4.75}$$

Proof: By induction over the derivation of Δ. Using the S-Closure of ω^*. We use Γ_I to indicate $[\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M$, an A_I to indicate $[\![\Phi \vdash A\colon \texttt{Effect}]\!]_M$

Case Unit:

$$\Delta = \langle\rangle_{\Gamma_I} \tag{4.76}$$

So

$$\omega^*(\Delta) = \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.77}$$

Case True, False: Giving the case for true as false is the same but using `inr`

$$\Delta = \texttt{inl} \circ \langle\rangle_{\Gamma_I} \tag{4.78}$$

So

$$\omega^*(\Delta) = \texttt{inl} \circ \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.79}$$

Since ω^* is S-closed.

Case Constant:

$$\Delta = [\![\mathtt{C}^{A}]\!]_{M} \circ \langle\rangle_{\Gamma_{I}} \tag{4.80}$$

So

$$\omega^{*}(\Delta) = \omega^{*}[\![\mathtt{C}^{A}]\!]_{M} \circ \langle\rangle_{\Gamma_{I'}} = [\![\mathtt{C}^{A_{I'}}]\!]_{M} \circ \langle\rangle_{\Gamma_{I'}} = \Delta' \tag{4.81}$$

Since ω^{*} is S-closed.

Case Subtype: Let

$$\Delta_{1} = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_{M} \tag{4.82}$$

Then

$$\Delta = [\![A \leq:_{\Phi} B]\!]_{M} \circ \Delta_{1} \tag{4.83}$$

So

$$\begin{aligned} \omega^{*}(\Delta) &= \omega^{*}[\![A \leq:_{\Phi} B]\!]_{M} \circ \omega^{*}\Delta_{1} && (4.84)\\ &= [\![A_{I'} \leq:_{\Phi'} B_{I'}]\!]_{M} \circ \Delta'_{1} \quad \text{By induction} && (4.85)\\ &= D' && (4.86) \end{aligned}$$

Case Lambda: Let

$$\Delta_{1} = [\![\Phi \mid \Gamma, x : A \vdash v\colon B]\!]_{M} \tag{4.87}$$

Then

$$\Delta = \mathtt{cur}(()\Delta_{1}) \tag{4.88}$$

So

$$\begin{aligned} \omega^{*}(\Delta) &= \omega^{*}(\mathtt{cur}(\Delta_{1})) && (4.89)\\ &= \mathtt{cur}(\omega^{*}(\Delta_{1})) \quad \text{By S-closure} && (4.90)\\ &= \mathtt{cur}(\Delta'_{1}) \quad \text{By induction} && (4.91)\\ &= \Delta' && (4.92) \end{aligned}$$

Case Application: Let

$$\begin{aligned} \Delta_{1} &= [\![\Phi \mid \Gamma \vdash v_{1}\colon A \to B]\!]_{M} && (4.93)\\ \Delta_{2} &= [\![\Phi \mid \Gamma \vdash v_{2}\colon A]\!]_{M} && (4.94) \end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ \langle \Delta_{1}, \Delta_{2} \rangle \tag{4.95}$$

So

$$\begin{aligned} \omega^{*}\Delta &= \omega^{*}(\mathtt{app} \circ \langle \Delta_{1}, \Delta_{2} \rangle) && (4.96)\\ &= \mathtt{app} \circ \langle \omega^{*}(\Delta_{1}), \omega^{*}(\Delta_{2}) \rangle \quad \text{By S-closure} && (4.97)\\ &= \mathtt{app} \circ \langle \Delta'_{1}, \Delta'_{2} \rangle \quad \text{By Induction} && (4.98)\\ &= \Delta' && (4.99) \end{aligned}$$

Case Return: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M \tag{4.100}$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \tag{4.101}$$

So

$$\begin{aligned}
\omega^*(\Delta) &= \omega^*(\eta_{A_I} \circ \Delta_1) && (4.102)\\
&= \eta_{A_{I'}} \circ \omega^*(\Delta_1) \quad \text{By S-closure} && (4.103)\\
&= \eta_{A_{I'}} \circ \Delta_1' && (4.104)\\
&= \Delta' && (4.105)
\end{aligned}$$

Case Bind: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A]\!]_M && (4.106)\\
\Delta_2 &= [\![\Phi \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B]\!]_M && (4.107)
\end{aligned}$$

Then

$$\Delta = \mathtt{M}_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathtt{t}_{\epsilon_1, \Gamma_I, A_I} \circ \langle \mathtt{Id}_{\Gamma_I}, \Delta_1 \rangle \tag{4.108}$$

So

$$\begin{aligned}
\omega^*(\Delta) &= \omega^*(\mu_{\epsilon_1,\epsilon_2,A} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1\rangle) && (4.109)\\
&= \omega^*(\mu_{\epsilon_1,\epsilon_2,A}) \circ \omega^*(T_{\epsilon_1}\Delta_2) \circ \omega^*(\mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \omega^*(\Delta_1)\right\rangle \quad \text{By S-Closure} && (4.110)\\
&= \mu_{\epsilon_1,\epsilon_2,A_{I'}} \circ T_{\epsilon_1}\omega^*(\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma_{I'},A_{I'}} \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \omega^*(\Delta_1)\right\rangle \quad \text{By S-Closure} && (4.111)\\
&= \mu_{\epsilon_1,\epsilon_2,A_{I'}} \circ T_{\epsilon_1}\Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma_{I'},A_{I'}} \circ \left\langle \omega^*(\mathtt{Id}_{\Gamma_I}), \Delta_1'\right\rangle \quad \text{By Induction} && (4.112)\\
&= \Delta' && (4.113)\\
& && (4.114)
\end{aligned}$$

Case If: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}]\!]_M && (4.115)\\
\Delta_2 &= [\![\Phi \mid \Gamma \vdash v_1\colon A]\!]_M && (4.116)\\
\Delta_3 &= [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]_M && (4.117)\\
& && (4.118)
\end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{4.119}$$

So

$$\omega^*(\Delta) = \omega^*(\mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma) \tag{4.120}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(\omega^*(\Delta_2) \circ \pi_2), \mathtt{cur}(\omega^*(\Delta_3) \circ \pi_2)] \circ \omega^*(\Delta_1)) \times \mathtt{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By S-Closure} \tag{4.121}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By Induction} \tag{4.122}$$
$$= \Delta' \tag{4.123}$$
$$\tag{4.124}$$

Case Effect-Lambda: Let

$$\Delta_1 = [\![\Phi, \alpha \mid \Gamma \vdash v \colon A]\!]_M \tag{4.125}$$

Then

$$\Delta = \widehat{\Delta_1} \tag{4.126}$$

And also

$$\omega \times \mathtt{Id} = [\![\omega \times : (\Phi', \alpha) \triangleright (\Phi, \alpha)]\!]_M \tag{4.127}$$

So

$$\omega^* \Delta = \omega^*(\widehat{\Delta_1}) \tag{4.128}$$
$$= \widehat{(\omega \times \mathtt{Id}_U)^* \Delta_1} \quad \text{By naturality} \tag{4.129}$$
$$= \widehat{\Delta_1'} \quad \text{By induction} \tag{4.130}$$
$$= \Delta' \tag{4.131}$$

Case Effect-Application: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v \colon \forall \alpha. A]\!]_M \tag{4.132}$$
$$h = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \tag{4.133}$$
$$\tag{4.134}$$

Then

$$\Delta = \langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1 \tag{4.135}$$

So due to the substitution theorem on effects

$$h \circ \omega = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \omega = [\![\Phi' \vdash \epsilon \colon \mathtt{Effect}]\!]_M = h' \tag{4.136}$$

Also note $(\omega \times \mathtt{Id}_U) = [\![\omega \times : \Phi', \alpha \triangleright \Phi\alpha]\!]_M$

$$\omega^* \Delta = \omega^*(\langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1) \tag{4.137}$$
$$= (\langle \mathtt{Id}_\Gamma, h \rangle \circ \omega)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \omega^*(\Delta_1) \tag{4.138}$$
$$= ((\omega \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_\Gamma, h \circ \omega \rangle)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' \tag{4.139}$$
$$= (\langle \mathtt{Id}_\Gamma, h' \rangle)^* ((\omega \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' \tag{4.140}$$
$$\tag{4.141}$$

Looking at the inner part of the functor application: Let

$$A = [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M \tag{4.142}$$

$$\tag{4.143}$$

$$\begin{aligned}
(\omega \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi,\beta \vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M} &= (\omega \times \mathtt{Id}_U)^* \epsilon_A && (4.144)\\
&= (\omega \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}}) && (4.145)\\
&= \overline{\widehat{(\omega \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}})}} \quad \text{By bijection} && (4.146)\\
&= \widehat{\omega^* (\overline{\widehat{\mathtt{Id}_{\forall_I(A)}}})} \quad \text{By naturality} && (4.147)\\
&= \widehat{\omega^* (\mathtt{Id}_{\forall_I(A)})} \quad \text{By bijection} && (4.148)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A \circ (\omega \times \mathtt{Id}_U))}} \quad \text{By S-Closure, naturality} && (4.149)\\
&= \widehat{\mathtt{Id}_{\forall_{I'}(A)}} \quad \text{By Substitution theorem} && (4.150)\\
&= \epsilon_{A_{I'}} && (4.151)
\end{aligned}$$

Going back to the original expression:

$$\begin{aligned}
\omega^* \Delta &= (\langle \mathtt{Id}_\Gamma, h' \rangle)^* (\epsilon_{A_{I'}}) \circ \Delta_1)' && (4.152)\\
&= \Delta' && (4.153)\\
& && (4.154)
\end{aligned}$$

4.7 Term-Substitution

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$, then $[\![\Phi' \mid \Gamma' \vdash \sigma : \Gamma]\!]_M = \omega^* [\![\Phi \mid \Gamma' \vdash \sigma : \Gamma]\!]_M$.

Proof: By induction on the structure of σ, making use of the weakening of term denotations above.

Case Nil: Then $\sigma = \langle\rangle_{\Gamma'_I}$, so $\omega^*(\sigma) = \langle\rangle_{\Gamma'_{I'}} = [\![\Phi' \mid \Gamma' \vdash \sigma : \Gamma]\!]_M$

Case Var: Then $\sigma = (\sigma', x := v)$

$$\begin{aligned}
\omega^* \sigma &= \omega * \langle \sigma', [\![\Phi \mid \Gamma \vdash v : A]\!]_M \rangle && (4.155)\\
&= \langle \omega^* \sigma', \omega^* [\![\Phi \mid \Gamma \vdash v : A]\!]_M \rangle && (4.156)\\
&= \langle [\![\Phi' \mid \Gamma' \vdash \sigma' : \Gamma]\!]_M, [\![\Gamma' \mid \Phi' \vdash v : A]\!]_M \rangle && (4.157)\\
&= [\![\Phi' \mid \Gamma' \vdash \sigma : \Gamma, x : A]\!]_M && (4.158)
\end{aligned}$$

4.8 Term-Weakening

If $\omega = [\![\omega : \Phi' \triangleright \Phi]\!]_M$, then $[\![\Phi' \vdash \omega_1 : \Gamma' \triangleright \Gamma]\!]_M = \omega^* [\![\Phi \vdash \omega_1 : \Gamma' \triangleright \Gamma]\!]_M$.

Proof: By induction on the structure of ω_1.

Case Id: Then $\omega_1 = \iota$, so its denotation is $\omega_1 = \mathtt{Id}_{\Gamma_I}$

So

$$\omega^*(\mathtt{Id}_{\Gamma_I}) = \mathtt{Id}_{\Gamma_{I'}} = [\![\Phi' \vdash \iota : \Gamma \triangleright \Gamma]\!]_M \tag{4.159}$$

Case Project: Then $\omega_1 = \omega_1' \pi$

$$(\text{Project})\frac{\Phi \vdash \omega_1' : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1 \pi : \Gamma', x : A \triangleright \Gamma} \tag{4.160}$$

So $\omega_1 = \omega_1' \circ \pi_1$

Hence

$$\begin{aligned}
\omega^*(\omega_1) &= \omega^*(\omega_1') \circ \omega^*(\pi_1) && (4.161)\\
&= [\![\Phi' \vdash \omega_1' : \Gamma' \triangleright \Gamma]\!]_M \circ \pi_1 && (4.162)\\
&= [\![\Phi' \vdash \omega_1' \pi : \Gamma', x : A \triangleright \Gamma]\!]_M && (4.163)\\
&= [\![\Phi' \vdash \omega_1 : \Gamma', x : A \triangleright \Gamma]\!]_M && (4.164)
\end{aligned}$$

Case Extend: Then $\omega_1 = \omega_1' \times$

$$(\text{Extend})\frac{\Phi \vdash \omega_1' : \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1 \times : \Gamma', x : A \triangleright \Gamma, x : A} \tag{4.165}$$

So $\omega_1 = \omega_1' \times \mathtt{Id}_{A_I}$

Hence

$$\begin{aligned}
\omega^*(\omega_1) &= (\omega^*(\omega_1') \times \omega^*(\mathtt{Id}_{A_I}) && (4.166)\\
&= ([\![\Phi' \vdash \omega_1' : \Gamma' \triangleright \Gamma]\!]_M \times \mathtt{Id}_{A_I}) && (4.167)\\
&= [\![\Phi' : \omega_1 \triangleright \Gamma', x : A\Gamma, x : A]\!]_M && (4.168)
\end{aligned}$$

Chapter 5

Value Substitution Theorem

If Δ derives $\Phi \mid \Gamma \vdash v\colon A$ and $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$ then the derivation Δ' deriving $\Phi \mid \Gamma' \vdash v\,[\sigma]\colon A$ satisfies:

$$\Delta' = \Delta \circ [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M \tag{5.1}$$

This is proved by induction over the derivation of $\Phi \mid \Gamma \vdash v\colon A$. We shall use σ to denote $[\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M$ where it is clear from the context.

Case Var: By inversion $\Gamma = \Gamma'', x : A$

$$(\text{Var})\frac{\Phi \vdash \Gamma\,\mathtt{Ok}}{\Phi \mid \Gamma'', x : A \vdash x\colon A} \tag{5.2}$$

By inversion, $\sigma = \sigma', x := v$ and $\Phi \mid \Gamma' \vdash v\colon A$.

Let

$$\sigma = [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle \sigma', \Delta' \rangle \tag{5.3}$$

$$\Delta = [\![\Phi \mid \Gamma'', x : A \vdash x\colon A]\!]_M = \pi_2 \tag{5.4}$$

$$\tag{5.5}$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \tag{5.6}$$

$$= \Delta' \quad \text{By product property} \tag{5.7}$$

Case Weaken: By inversion, $\Gamma = \Gamma', y : B$ and $\sigma = \sigma', y := v$ and we have Δ_1 deriving:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x:A}}{\Phi \mid \Gamma'', y : B \vdash x\colon A} \tag{5.8}$$

Also by inversion of the well-formed-ness of $\Phi \mid \Gamma' \vdash \sigma\colon \Gamma$, we have $\Phi \mid \Gamma' \vdash \sigma'\colon \Gamma''$ and

$$[\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma]\!]_M = \langle [\![\Phi \mid \Gamma' \vdash \sigma\colon \Gamma'']\!]_M, [\![\Phi \mid \Gamma' \vdash v\colon B]\!]_M \rangle \tag{5.9}$$

Hence by induction on Δ_1 we have Δ_1' such that

$$()\frac{\Delta_1'}{\Phi \mid \Gamma' \vdash x\,[\sigma] : A} \tag{5.10}$$

Hence

$$\begin{aligned}
\Delta' &= \Delta_1' \quad \text{By definition} && (5.11)\\
&= \Delta_1 \circ \sigma' \quad \text{By induction} && (5.12)\\
&= \Delta_1 \circ \pi_1 \circ \langle \sigma', [\![\Phi \mid \Gamma' \vdash v \colon B]\!]_M \rangle \quad \text{By product property} && (5.13)\\
&= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By defintion of the denotation of } \sigma && (5.14)\\
&= \Delta \circ \sigma \quad \text{By defintion.} && (5.15)
\end{aligned}$$

Case Constants: The logic for all constant terms ($\texttt{true}, \texttt{false}, \texttt{()}, \texttt{C}^A$) is the same. Let

$$c = [\![\texttt{C}^A]\!]_M \tag{5.16}$$

$$\begin{aligned}
\Delta' &= c \circ \langle\rangle_{\Gamma'} \quad \text{By Definition} && (5.17)\\
&= c \circ \langle\rangle_G \circ \sigma \quad \text{Terminal property} && (5.18)\\
&= \Delta \circ \sigma \quad \text{By definition} && (5.19)
\end{aligned}$$

Case Lambda: By inversion, we have Δ_1 such that

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Phi|\Gamma, x:A \vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A \to B} \tag{5.20}$$

By induction of Δ_1 we have Δ_1' such that

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Phi|\Gamma', x:A \vdash (v[\sigma]):B}}{\Phi \mid \Gamma \vdash (\lambda x : A.v)\,[\sigma] : A \to B} \tag{5.21}$$

By induction and the extension lemma, we have:

$$\Delta_1' = \Delta_1 \circ (\sigma \times \texttt{Id}_A) \tag{5.22}$$

Hence:

$$\begin{aligned}
\Delta' &= \texttt{cur}(\Delta_1') \quad \text{By definition} && (5.23)\\
&= \texttt{cur}(\Delta_1 \circ (\sigma \times \texttt{Id}_A)) \quad \text{By induction and extension lemma.} && (5.24)\\
&= \texttt{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} && (5.25)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.26)\\
& && (5.27)
\end{aligned}$$

Case Sub-type: By inversion, there exists derivation Δ_1 such that:

$$\Delta = (\text{Sub-type})\frac{()\frac{\Delta_1}{\Phi|\Gamma \vdash v:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v \colon B} \tag{5.28}$$

By induction on Δ_1, we find Δ_1' such that $\Delta_1' = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Sub-type})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma' \vdash v\,[\sigma] : B} \tag{5.29}$$

Hence,

$$\begin{aligned}
\Delta' &= [\![A \leq:_\Phi B]\!]_M \circ \Delta'_1 \quad \text{By definition} && (5.30)\\
&= [\![A \leq:_\Phi B]\!]_M \circ \Delta_1 \circ \sigma \quad \text{By induction} && (5.31)\\
&= \Delta \circ \sigma \quad \text{By definition} && (5.32)\\
& && (5.33)
\end{aligned}$$

Case Return: By inversion, we have Δ_1 such that:

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return}v : \mathtt{M_1}A} \tag{5.34}$$

By induction on Δ_1, we find Δ'_1 such that $\Delta'_1 = \Delta_1 \circ \sigma$ and:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash (\mathtt{return}v)\,[\sigma] : \mathtt{M_1}A} \tag{5.35}$$

Hence,

$$\begin{aligned}
\Delta' &= \eta_A \circ \Delta'_1 \quad \text{By Definition} && (5.36)\\
&= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} && (5.37)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.38)\\
& && (5.39)
\end{aligned}$$

Case Apply: By inversion, we find Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2 : B} \tag{5.40}$$

By induction we find Δ'_1, Δ'_2 such that

$$\begin{aligned}
\Delta'_1 &= \Delta_1 \circ \sigma && (5.41)\\
\Delta'_2 &= \Delta_2 \circ \sigma && (5.42)\\
& && (5.43)
\end{aligned}$$

And

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v_1[\sigma]:A\to B} \quad ()\frac{\Delta'_2}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (v_1\ v_2)\,[\sigma] : B} \tag{5.44}$$

Hence

$$\begin{aligned}
\Delta' &= \mathtt{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} && (5.45)\\
&= \mathtt{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} && (5.46)\\
&= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} && (5.47)\\
&= \Delta \circ \sigma \quad \text{By Definition} && (5.48)\\
& && (5.49)
\end{aligned}$$

Case If: By inversion, we find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{5.50}$$

By induction we find $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\begin{aligned}
\Delta'_1 &= \Delta_1 \circ \sigma && (5.51)\\
\Delta'_2 &= \Delta_2 \circ \sigma && (5.52)\\
\Delta'_3 &= \Delta_3 \circ \sigma && (5.53)\\
& && (5.54)
\end{aligned}$$

And

$$\Delta' = (\text{If})\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:\mathtt{Bool}} \quad ()\frac{\Delta'_2}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta'_3}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)\,[\sigma] : A} \tag{5.55}$$

Since $\sigma : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\sigma = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{5.56}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \mathtt{cur}(f) \tag{5.57}$$

And so:

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

$$
\begin{aligned}
\Delta' &= \texttt{app} \circ (([\texttt{cur}(\Delta_2' \circ \pi_2), \texttt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (5.58)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \sigma \circ \pi_2), \texttt{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta_1') \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (5.59)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma)), \texttt{cur}(\Delta_3 \circ \pi_2 \circ (\texttt{Id}_1 \times \sigma))] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (5.60)\\
&= \texttt{app} \circ (([(T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} && (5.61)\\
&= \texttt{app} \circ (((T_\epsilon A)^\sigma \circ [\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (5.62)\\
&= \texttt{app} \circ ((T_\epsilon A)^\sigma \times \texttt{Id}_{\Gamma'}) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (5.63)\\
&= \texttt{app} \circ (\texttt{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma'}) \circ (\sigma \times \texttt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \texttt{app}, (T_\epsilon A)^\sigma && (5.64)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (5.65)\\
&= \texttt{app} \circ (([\texttt{cur}(\Delta_2 \circ \pi_2), \texttt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \texttt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \sigma \quad \text{By Definition of the diagonal morphism.} && (5.66)\\
&= \Delta \circ \sigma && (5.67)
\end{aligned}
$$

Case Bind: By inversion, we have Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A} \quad ()\frac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_1 : B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.68}$$

By property 3,

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A) \tag{5.69}$$

With denotation (extension lemma)

$$[\![\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A)]\!]_M = \sigma \times \texttt{Id}_A \tag{5.70}$$

By induction, we derive Δ_1', Δ_2' such that:

$$
\begin{aligned}
\Delta_1' &= \Delta_1 \circ \sigma && (5.71)\\
\Delta_2' &= \Delta_2 \circ (\sigma \times \texttt{Id}_A) \quad \text{By Extension Lemma} && (5.72)
\end{aligned}
$$

And:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma' \vdash v_1[\sigma] : A} \quad ()\frac{\Delta_2'}{\Phi \mid \Gamma', x : A \vdash v_1[\sigma] : B}}{\Phi \mid \Gamma' \vdash (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{5.73}$$

Hence:

$$
\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} && (5.74)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} (\Delta_2 \circ (\sigma \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} && (5.75)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\sigma \times \mathtt{Id}_{T_{\epsilon_1} A}) \circ \langle \mathtt{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} && (5.76)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} && (5.77)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} && (5.78)\\
&= \Delta \circ \sigma \quad \text{By Defintion} && (5.79)\\
& && (5.80)
\end{aligned}
$$

Case Effect-Lambda: By inversion, we have Δ_1 such that

$$\Delta = (\text{Effect-Fn}) \frac{() \frac{\Delta_1}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda \alpha . v : \forall \epsilon . A} \tag{5.81}$$

By induction, we derive Δ'_1 such that

$$\Delta' = (\text{Effect-Fn}) \frac{() \frac{\Delta'_1}{\Phi, \alpha \mid \Gamma' \vdash v[\sigma] : A}}{\Phi \mid \Gamma' \vdash (\Lambda \alpha . v)\, [\sigma] : \forall \epsilon . A} \tag{5.82}$$

Where

$$
\begin{aligned}
\Delta'_1 &= \Delta_1 \circ [\![\Phi, \alpha \mid \Gamma' \vdash \sigma : \Gamma]\!]_M && (5.83)\\
&= \Delta_1 \circ [\![\iota\pi : \Phi, a \triangleright \Phi]\!]^*_M (\sigma) && (5.84)\\
&= \Delta_1 \circ \pi^*_1 (\sigma) && (5.85)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Delta \circ \sigma &= \overline{\Delta_1} \circ \sigma && (5.86)\\
&= \overline{\Delta_1 \circ \pi^*_1 (\sigma)} && (5.87)\\
&= \overline{\Delta'_1} && (5.88)\\
&= \Delta' && (5.89)
\end{aligned}
$$

Case Effect-Application: By inversion, we derive Δ_1 such that

$$\Delta = (\text{Effect-App}) \frac{() \frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \forall \alpha . A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon : A\, [\epsilon / \alpha]} \tag{5.90}$$

By induction, we derive Δ'_1 such that

$$\Delta' = (\text{Effect-App}) \frac{() \frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v[\sigma] : \forall \alpha . A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash (v\ \epsilon)\, [\sigma] : A\, [\epsilon / \alpha]} \tag{5.91}$$

Where

$$\Delta'_1 = \Delta \circ \sigma \tag{5.92}$$

Hence, if $h = \llbracket \Phi \vdash \epsilon \colon \mathtt{Effect} \rrbracket_M$

$$\begin{aligned}
\Delta \circ \sigma &= \langle \mathtt{Id}_I, h \rangle^* \left(\epsilon \llbracket \Phi, \beta \vdash A\,[\alpha/\beta] : \mathtt{Effect} \rrbracket_M \right) \circ \Delta_1 \circ \sigma && (5.93)\\
&= \langle \mathtt{Id}_I, h \rangle^* \left(\epsilon \llbracket \Phi, \beta \vdash A\,[\alpha/\beta] : \mathtt{Effect} \rrbracket_M \right) \circ \Delta_1' && (5.94)\\
&= \Delta' && (5.95)
\end{aligned}$$

Chapter 6

Type-Environment Weakening Theorem

If $w = [\![\Phi \vdash \omega : \Gamma' \triangleright G]\!]_M$ and $\Delta = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M$ then there exists $\Delta' = [\![\Phi \mid \Gamma' \vdash v\colon A]\!]_M$ such that $\Delta' = \Delta \circ \omega$

Proof: We induct over the structure of typing derivations of $\Phi \mid \Gamma \vdash v\colon A$, assuming $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Phi \mid \Gamma \vdash v\colon A$ and show that $\Delta \circ [\![\Phi \vdash \omega : \Gamma' \triangleright \Gamma]\!]_M = \Delta'$

Case Var and Weaken: We case split on the weakening ω.

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Phi \mid \Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \mathtt{Id}_\Gamma = \Delta \circ [\![\Phi \vdash \iota : \Gamma \triangleright \Gamma]\!]_M \tag{6.1}$$

If $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Phi \mid \Gamma'' \vdash x\colon A$, such that

$$\Delta_1 = \Delta \circ [\![\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma]\!]_M \quad \text{By Induction} \tag{6.2}$$

, and hence by the weaken rule, we have

$$(\text{Weaken})\frac{\Phi \mid \Gamma'' \vdash x\colon A}{\Phi \mid \Gamma'', x' : A' \vdash x\colon A} \tag{6.3}$$

This preserves denotations:

$$\begin{aligned} \Delta' &= \Delta_1 \circ \pi_1 \quad \text{By Definition} && (6.4)\\ &= \Delta \circ [\![\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma]\!]_M \circ \pi_1 \quad \text{By induction} && (6.5)\\ &= \Delta \circ [\![\Phi \vdash \omega'\pi_1 : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By denotation of weakening} && (6.6) \end{aligned}$$

If $\omega = \omega'\times$ Then

$$\begin{aligned} \Gamma' &= \Gamma''', x' : B && (6.7)\\ \Gamma &= \Gamma'', x' : A' && (6.8)\\ B &\leq:_\Phi A && (6.9) \end{aligned}$$

If $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$(\text{Sub-type})\frac{(\text{var})\frac{}{\Phi|\Gamma''',x:B\vdash x:B} \quad B \leq:_\Phi A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{6.10}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![B \leq:_\Phi A]\!]_M \circ \pi_2 \quad \text{By Definition} && (6.11)\\
&= \pi_2 \circ ([\![\Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![B \leq:_\Phi A]\!]_M) \quad \text{By the properties of binary products} && (6.12)\\
&= \Delta \circ [\![\Phi \vdash \omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By Definition} && (6.13)
\end{aligned}$$

Case $x \neq x'$ Then

$$\Delta = (\text{Weaken})\frac{()\frac{\Delta_1}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma \vdash x\colon A} \tag{6.14}$$

By induction with $\Phi \vdash \omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Phi \mid \Gamma''' \vdash x\colon A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'''\vdash x:A}}{\Phi \mid \Gamma' \vdash x\colon A} \tag{6.15}$$

This preserves denotations:

By induction, we have

$$\Delta_1' = \Delta_1 \circ [\![\Phi \vdash \omega : \Gamma''' \triangleright \Gamma'']\!]_M \tag{6.16}$$

So we have:

$$\begin{aligned}
\Delta' &= \Delta_1' \circ \pi_1 \quad \text{By denotation definition} && (6.17)\\
&= \Delta_1 \circ [\![\Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'']\!]_M \quad \text{By induction} \circ \pi_1 && (6.18)\\
&= \Delta_1 \circ \pi_1 \circ ([\![\Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'']\!]_M \times [\![A' \leq:_\Phi B]\!]_M) \quad \text{By product properties} && (6.19)\\
&= \Delta \circ [\![\Phi \vdash \omega : \Gamma' \triangleright \Gamma]\!]_M \quad \text{By definition} && (6.20)
\end{aligned}$$

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $[\![\Phi \vdash \omega : \Gamma' \triangleright \Gamma']\!]_M$, simply as ω.

Case Constant: The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$(\text{Const})\frac{\Phi \vdash \Gamma\mathtt{Ok}}{\Phi \mid \Gamma \vdash \mathtt{C}^A\colon A} \tag{6.21}$$

By inversion, we have $\Phi \vdash \Gamma\mathtt{Ok}$, so we have $\Phi \vdash \Gamma'\mathtt{Ok}$.

Hence

$$(\text{Const})\frac{\Phi \vdash \Gamma'\mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A\colon A} \tag{6.22}$$

Holds.

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma'} \quad \text{By definition} && (6.23)\\
&= [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} && (6.24)\\
&= \Delta \quad \text{By Definition} && (6.25)\\
& && (6.26)
\end{aligned}$$

Case Lambda: By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn})\frac{()\frac{\Delta_1}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{6.27}$$

Since $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we have:

$$\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{6.28}$$

Hence, by induction, using $\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = (\text{Fn})\frac{()\frac{\Delta_1'}{\Phi|\Gamma',x:A\vdash v:B}}{\Phi \mid \Gamma', x : A \vdash \lambda x : A.v\colon A \to B} \tag{6.29}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mathtt{cur}(\Delta_1') \quad \text{By Definition} && (6.30)\\
&= \mathtt{cur}(\Delta_1 \circ (\omega \times \mathtt{Id}_\Gamma)) \quad \text{By the denotation of } \omega\times && (6.31)\\
&= \mathtt{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} && (6.32)\\
&= \Delta \circ \omega \quad \text{By Definition} && (6.33)
\end{aligned}$$

Case Sub-typing:

$$(\text{Sub-type})\frac{\Phi \mid \Gamma \vdash v\colon A \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B} \tag{6.34}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v\colon A} \tag{6.35}$$

So by induction, we have a derivation Δ_1' such that:

$$(\text{Sub-type})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:a} \quad A \leq:_\Phi B}{\Phi \mid \Gamma' \vdash v\colon B} \tag{6.36}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= [\![A \leq:_\Phi B]\!]_M \circ \Delta_1' \quad \text{By Definition} && (6.37)\\
&= [\![A \leq:_\Phi B]\!]_M \circ \Delta_1 \circ \omega \quad \text{By induction} && (6.38)\\
&= \Delta \circ \omega \quad \text{By Definition} && (6.39)\\
& && (6.40)
\end{aligned}$$

Case Return: We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{6.41}$$

Hence, by induction, with $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ_1' such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{6.42}$$

This preserves denotations:

$$\Delta' = \eta_A \circ \Delta_1' \quad \text{By definition} \tag{6.43}$$
$$= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta_1' \tag{6.44}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{6.45}$$

Case Apply: By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{6.46}$$

By induction, this gives us the respective derivations: Δ_1', Δ_2' such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash v_1\ v_2\colon B} \tag{6.47}$$

This preserves denotations:

$$\Delta' = \texttt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Definition} \tag{6.48}$$
$$= \texttt{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \tag{6.49}$$
$$= \texttt{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \tag{6.50}$$
$$= \Delta \circ \omega \quad \text{By Definition} \tag{6.51}$$

Case If: By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\colon A} \tag{6.52}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1:A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\colon A} \tag{6.53}$$

And

$$\Delta_1' = \Delta_1 \circ \omega \tag{6.54}$$
$$\Delta_3' = \Delta_2 \circ \omega \tag{6.55}$$
$$\Delta_3' = \Delta_3 \circ \omega \tag{6.56}$$

This preserves denotations. Since $\omega : \Gamma' \to \Gamma$,
Let $(T_\epsilon A)^\omega : T_\epsilon A^\Gamma \to T_\epsilon A^{\Gamma'}$ be as defined in ExSh 3 ([1]) That is:

$$(T_\epsilon A)^\omega = \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_\epsilon A} \times w)) \tag{6.57}$$

. And hence, we have:

$$\mathtt{cur}(f \circ (\mathtt{Id} \times \omega)) = (T_\epsilon A)^\omega \circ \mathtt{cur}(f) \tag{6.58}$$

$$\begin{aligned}
\Delta' &= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} && (6.59)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \omega \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} && (6.60)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2 \circ (\mathtt{Id}_{\mathtt{1}} \times \omega)), \mathtt{cur}(\Delta_3 \circ \pi_2 \circ (\mathtt{Id}_{\mathtt{1}} \times \omega))] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} && (6.61)\\
&= \mathtt{app} \circ (([(T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\omega \circ \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\omega \text{ property} && (6.62)\\
&= \mathtt{app} \circ (((T_\epsilon A)^\omega \circ [\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} && (6.63)\\
&= \mathtt{app} \circ ((T_\epsilon A)^\omega \times \mathtt{Id}_{\Gamma'}) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} && (6.64)\\
&= \mathtt{app} \circ (\mathtt{Id}_{(T_\epsilon A)} \times \omega) \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma'}) \circ (\omega \times \mathtt{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By defintion of } \mathtt{app}, (T_\epsilon A)^\omega && (6.65)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} && (6.66)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_{\Gamma}) \circ \delta_{\Gamma} \circ \omega \quad \text{By Definition of the diagonal morphism.} && (6.67)\\
&= \Delta \circ \omega && (6.68)
\end{aligned}$$

Case Bind: By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{6.69}$$

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then $\Phi \vdash \omega \times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ_1', Δ_2' such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2'}{\Phi|\Gamma',x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{6.70}$$

This preserves denotations:

$$\begin{aligned}
\Delta' &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2' \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{G'}, \Delta_1' \rangle \quad \text{By definition} && (6.71)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}(\Delta_2 \circ (\omega \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma',A} \circ \langle \mathtt{Id}_{G'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta_1', \Delta_2' && (6.72)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} && (6.73)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} && (6.74)\\
&= \Delta \quad \text{By definition} && (6.75)
\end{aligned}$$

[1] https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf

Case Effect-Lambda: By inversion, we have Δ_1 such that

$$\Delta = (\text{Effect-Fn})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\epsilon.A} \tag{6.76}$$

By induction, we derive Δ_1' such that

$$\Delta' = (\text{Effect-Fn})\frac{()\frac{\Delta_1'}{\Phi,\alpha|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash (\Lambda\alpha.v)\colon \forall\epsilon.A} \tag{6.77}$$

Where

$$\Delta_1' = \Delta_1 \circ [\![\Phi, \alpha \vdash \omega : \Gamma' \triangleright \Gamma]\!]_M \tag{6.78}$$
$$= \Delta_1 \circ [\![\iota\pi : \Phi, a \triangleright \Phi]\!]^*_M(\omega) \tag{6.79}$$
$$= \Delta_1 \circ \pi_1^*(\omega) \tag{6.80}$$

Hence

$$\Delta \circ \omega = \overline{\Delta_1} \circ \omega \tag{6.81}$$
$$= \overline{\Delta_1 \circ \pi_1^*(\omega)} \tag{6.82}$$
$$= \overline{\Delta_1'} \tag{6.83}$$
$$= \Delta' \tag{6.84}$$

Case Effect-Application: By inversion, we derive Δ_1 such that

$$\Delta = (\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{6.85}$$

By induction, we derive Δ_1' such that

$$\Delta' = (\text{Effect-App})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{6.86}$$

Where

$$\Delta_1' = \Delta \circ \omega \tag{6.87}$$

Hence, if $h = [\![\Phi \vdash \epsilon\colon \texttt{Effect}]\!]_M$

$$\Delta \circ \omega = \langle \texttt{Id}_I, h\rangle^* \left(\epsilon[\![\Phi, \beta \vdash A\,[\alpha/\beta] : \texttt{Effect}]\!]_M\right) \circ \Delta_1 \circ \omega \tag{6.88}$$
$$= \langle \texttt{Id}_I, h\rangle^* \left(\epsilon[\![\Phi, \beta \vdash A\,[\alpha/\beta] : \texttt{Effect}]\!]_M\right) \circ \Delta_1' \tag{6.89}$$
$$= \Delta' \tag{6.90}$$

Chapter 7

Unique Denotation Theorem

7.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

7.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v\colon A$,there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Case Variables: To find the unique derivation of $\Phi \mid \Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case: $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is, if $A' \leq:_\Phi A$, as below:

$$(\text{Subtype})\frac{(\text{Var})\frac{\Phi \vdash \Gamma', x: A' \texttt{Ok}}{\Phi \mid \Gamma, x: A' \vdash x: A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma', x : A' \vdash x\colon A} \tag{7.1}$$

Case: $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x\colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x\colon A$.

Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma' \vdash x: A'} \quad A' \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{7.2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is:

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta}{\Phi\mid\Gamma,x:A'\vdash x:A'}}{\Phi\mid\Gamma\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} \tag{7.3}$$

Case Constants: For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$(\text{Subtype})\frac{(\text{Const})\frac{\Gamma\,\mathtt{Ok}}{\Gamma\vdash\mathtt{C}^A:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_\Phi A$).

Case Lambda: The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v\colon A' \to B'$ is:

$$(\text{Subtype})\frac{(\text{Lambda})\frac{()\frac{\Delta}{\Phi\mid\Gamma,x:A\vdash v:B}}{\Phi\mid\Gamma\vdash\lambda x:A.B:A\to B} \quad A \to B \leq:_\Phi A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A' \to B'}$$

Where

$$(\text{Sub-Type})\frac{()\frac{\Delta}{\Phi\mid\Gamma,x:A\vdash v:B} \quad B \leq:_\Phi B'}{\Phi \mid \Gamma, x : A \vdash v\colon B'} \tag{7.4}$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v\colon B$ if it exists.

Case Return: The reduced denotation of $\Phi \mid \Gamma \vdash \mathtt{return}v\colon B$ is

$$(\text{Subtype})\frac{(\text{Return})\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v:A}}{\Phi\mid\Gamma\vdash\mathtt{return}v:\mathtt{M_1}A} \quad (\text{Computation})\frac{\mathtt{1}\leq_\Phi\epsilon \quad A\leq:_\Phi B}{\mathtt{M_1}A\leq:_\Phi\mathtt{M}_\epsilon B}}{\Phi \mid \Gamma \vdash \mathtt{return}v\colon \mathtt{M}_\epsilon B}$$

Where

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v:A} \quad A \leq: B}{\Phi \mid \Gamma \vdash v\colon B}$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v\colon B$

Case Apply: If

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v_1:A\to B} \quad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1\colon A' \to B'}$$

and

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi\mid\Gamma\vdash v_2:A''} \quad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1\colon A' \to B'$ and $\Phi \mid \Gamma \vdash v_2\colon A'$

Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2\colon B$ as

$$(\text{Sub-Type})\frac{(\text{Apply})\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v_1:A\to B} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi\mid\Gamma\vdash v:A''} \quad A''\leq:_\Phi A}{\Phi\mid\Gamma\vdash v:A}}{\Phi\mid\Gamma\vdash v_1\ v_2:B} \quad B \leq:_\Phi B'}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B'}$$

Case If: Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v \colon B'} \quad B' \leq : \mathtt{Bool}}{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}} \tag{7.5}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_1 \colon A'} \quad A' \leq : A}{\Phi \mid \Gamma \vdash v_1 \colon A} \tag{7.6}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 \colon A''} \quad A'' \leq : A}{\Phi \mid \Gamma \vdash v_2 \colon A} \tag{7.7}$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$.

Then the only reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon B$ is:

TODO: Scale this properly

$$(\text{Subtype})\frac{(\text{If})\frac{(\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v : B'} \quad B' \leq : \mathtt{Bool}}{\Phi \mid \Gamma \vdash v : \mathtt{Bool}} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \quad A' \leq : A}{\Phi \mid \Gamma \vdash v_1 : A} \quad (\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq : A}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 : A \quad A \leq :_\Phi B}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon B} \tag{7.8}$$

Case Bind: Let

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A} \quad (\text{Computation})\frac{A \leq :_\Phi A' \quad \epsilon_1 \leq_\Phi \epsilon_1'}{\mathtt{M}_{\epsilon_1} A \leq :_\Phi \mathtt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{7.9}$$

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B} \quad (\text{Computation})\frac{B \leq :_\Phi B' \quad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq :_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{7.10}$$

Be the respective unique reduced type derivations of the sub-terms]

By weakening, $\Phi \vdash \iota \times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2 \colon B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.

$$(\text{Subtype})\frac{()\frac{\Delta''}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2} B} \quad (\text{Computation})\frac{B \leq :_\Phi B' \quad \epsilon_2 \leq_\Phi \epsilon_2'}{\mathtt{M}_{\epsilon_2} B \leq :_\Phi \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{7.11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$(\text{Sub-type})\frac{(\text{Bind})\frac{(\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A} \quad (\text{Computation})\frac{A \leq :_\Phi A' \quad \epsilon_1 \leq_\Phi \epsilon_1'}{\mathtt{M}_{\epsilon_1} A \leq :_\Phi \mathtt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1'} A'} \quad (\text{Subtype})\frac{()\frac{\Delta''}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2} B} \quad (\text{Computation})\frac{B \leq :_\Phi B' \quad \epsilon_2 \leq}{\mathtt{M}_{\epsilon_2} B \leq :_\Phi \mathtt{M}_{\epsilon_2'}}}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{7.12}$$

Case Effect-Fn: The unique reduced derivation of $\Phi \mid \Gamma \vdash \Lambda\alpha.A\colon \forall\alpha.B$

is

$$\text{(Sub-type)}\frac{\text{(Effect-Fn)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi|\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A} \quad \forall\alpha.A \leq_{\Phi} \forall\alpha.B}{\Phi \mid \Gamma \vdash \Lambda\alpha.B\colon \forall\alpha.B} \tag{7.13}$$

Where

$$\text{(Sub-type)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A} \quad A \leq:_{\Phi,\alpha} B}{\Phi, \alpha \mid \Gamma \vdash v\colon B} \tag{7.14}$$

Is the unique reduced derivation of $\Phi, \alpha \mid \Gamma \vdash v\colon B$

Case Effect-App: The unique reduced derivation of $\Phi \mid \Gamma \vdash v\ \alpha\colon B'$

is

$$\text{(Subtype)}\frac{\text{(Effect-App)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A\,[\epsilon/\alpha] \leq:_{\Phi} B'}{\Phi \mid \Gamma \vdash v\ \alpha\colon B'} \tag{7.15}$$

Where $B\,[\epsilon/\alpha] \leq:_{\Phi} B'$ and

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.B} \quad \text{(Quantification)}\frac{A\leq:_{\Phi,\alpha}B}{\forall\alpha.A\leq:_{\Phi}\forall\alpha.B}}{\Phi \mid \Gamma \vdash v\colon \forall\alpha.B} \tag{7.16}$$

7.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v\colon A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

Case Constants: For the constants $\texttt{true}, \texttt{false}, \texttt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$reduce(\text{(Const)}\frac{\Gamma\texttt{Ok}}{\Gamma\vdash\texttt{C}^A:A}) = \text{(Const)}\frac{\Gamma\texttt{Ok}}{\Gamma\vdash\texttt{C}^A:A}$

Case Var:

$$reduce(\text{(Var)}\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A}) = \text{(Var)}\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x\colon A} \tag{7.17}$$

Preserves denotation trivially.

Case Weaken:

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash x:A}}{\Phi \mid \Gamma, y : B \vdash x\colon A}) \tag{7.18}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{7.19}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A'}}{\Phi|\Gamma,y:B\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma, y : B \vdash x\colon A} \tag{7.20}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$[\![(\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash x:A}}{\Phi \mid \Gamma, y : B \vdash x\colon A}]\!]_M = \Delta \circ \pi_1 \tag{7.21}$$

Similarly, the denotation of the reduced denotation is:

$$[\![(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A'}}{\Phi|\Gamma,y:B\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma, y : B \vdash x\colon A}]\!]_M = [\![A' \leq:_\Phi A]\!]_M \circ \Delta' \circ \pi_1 \tag{7.22}$$

By induction on *reduce* preserving denotations and the reduction of Δ (7.19), we have:

$$\Delta = [\![A' \leq:_\Phi A]\!]_M \circ \Delta' \tag{7.23}$$

So the denotations of the un-reduced and reduced derivations are equal.

Case Lambda:

reduce **definition** To find:

$$reduce((\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B}) \tag{7.24}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:B'} B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A \vdash v\colon \mathtt{M}_{\epsilon_2} B} = reduce(\Delta) \tag{7.25}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:B'}}{\Phi|\Gamma\vdash\lambda x:A.v:A\to B'} \quad A \to B' \leq:_\Phi A \to B}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{7.26}$$

Preserves Denotation Let

$$f = [\![B' \leq:_\Phi B']\!]_M \tag{7.27}$$
$$[\![A \to B' \leq:_\Phi A \to B]\!]_M = f^A = \mathtt{cur}(f \circ \mathtt{app}) \tag{7.28}$$

Then

$$
\begin{aligned}
before &= \mathtt{cur}(\Delta) \quad \text{By definition} && (7.29)\\
&= \mathtt{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta && (7.30)\\
&= f^A \circ \mathtt{cur}(\Delta') \quad \text{By the property of } f^X \circ \mathtt{cur}(g) = \mathtt{cur}(f \circ g) && (7.31)\\
&= after \quad \text{By definition} && (7.32)\\
& && (7.33)
\end{aligned}
$$

Case Subtype:

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B}) \tag{7.34}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} = reduce(\Delta) \tag{7.35}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\colon B} \tag{7.36}$$

Preserves Denotation

$$
\begin{aligned}
before &= [\![A \leq:_\Phi B]\!]_M \circ \Delta && (7.37)\\
&= [\![A \leq:_\Phi B]\!]_M \circ ([\![A' \leq:_\Phi A]\!]_M \circ \Delta') \quad \text{by Denotation of reduction of } \Delta. && (7.38)\\
&= [\![A' \leq:_\Phi B]\!]_M \circ \Delta' \quad \text{Subtyping relations are unique} && (7.39)\\
&= after && (7.40)\\
& && (7.41)
\end{aligned}
$$

Case Return:

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return} v\colon \mathtt{M_1} A}) \tag{7.42}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v\colon A} = reduce(\Delta) \tag{7.43}$$

In

$$(\text{Sub-type})\frac{(\text{Return})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \texttt{return} v \colon \texttt{M}_{\mathtt{1}} A'} \quad (\text{Computation})\frac{\mathtt{1} \leq_{\Phi} \mathtt{1} \quad A' \leq :_{\Phi} A}{\texttt{M}_{\mathtt{1}} A' \leq :_{\Phi} \texttt{M}_{\mathtt{1}} A}}{\Phi \mid \Gamma \vdash \texttt{return} v \colon \texttt{M}_{\mathtt{1}} A} \tag{7.44}$$

Then

$$\begin{aligned}
before &= \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} && (7.45)\\
&= \eta_A \circ [\![A' \leq :_{\Phi} A]\!]_M \circ \Delta' \quad \text{By reduction of } \Delta && (7.46)\\
&= T_{\mathtt{1}} [\![A' \leq :_{\Phi} A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta && (7.47)\\
&= [\![\mathtt{1} \leq_{\Phi} \mathtt{1}]\!]_{M,A} \circ T_{\mathtt{1}} [\![A' \leq :_{\Phi} A]\!]_M \circ \eta_{A'} \circ \Delta' \quad \text{Since } [\![\mathtt{1} \leq_{\Phi} \mathtt{1}]\!]_M \text{ is the identity Nat-Trans} && (7.48)\\
&= after \quad \text{By definition} && (7.49)\\
& && (7.50)
\end{aligned}$$

Case Apply:

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A \to B} \quad ()\frac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B}) \tag{7.51}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 : A' \to B'} \quad A' \to B' \leq :_{\Phi} A \to B}{\Phi \mid \Gamma \vdash v_1 \colon A \to B} = reduce(\Delta_1) \tag{7.52}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Phi \mid \Gamma \vdash v : A'} \quad A' \leq :_{\Phi} A}{\Phi \mid \Gamma \vdash v_1 \colon A} = reduce(\Delta_2) \tag{7.53}$$

In

$$(\text{Sub-type})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 : A' \to B'} \quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq :_{\Phi} A \leq :_{\Phi} A'}{\Phi \mid \Gamma \vdash v_2 : A'}}{\Phi \mid \Gamma \vdash v_1\ v_2 : B'} \quad B' \leq :_{\Phi} B}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon B} \tag{7.54}$$

Preserves Denotation Let

$$\begin{aligned}
f &= [\![A \leq :_{\Phi} A']\!]_M : A \to A' && (7.55)\\
f' &= [\![A'' \leq :_{\Phi} A]\!]_M : A'' \to A && (7.56)\\
g &= [\![B' \leq :_{\Phi} B]\!]_M : B' \to B && (7.57)\\
& && (7.58)
\end{aligned}$$

Hence

$$\begin{aligned}
[\![A' \to B' \leq :_{\Phi} A \to B]\!]_M &= (g)^A \circ (B')^f && (7.59)\\
&= \texttt{cur}(app \circ \texttt{app}) \circ \texttt{cur}(\texttt{app} \circ (\texttt{Id} \times f)) && (7.60)\\
&= \texttt{cur}(g \circ \texttt{app} \circ (\texttt{Id} \times f)) && (7.61)
\end{aligned}$$

Then

$$\begin{aligned}
before &= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} && (7.62)\\
&= \mathtt{app} \circ \langle \mathtt{cur}(g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \circ \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 && (7.63)\\
&= \mathtt{app} \circ (\mathtt{cur}(g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \times \mathtt{Id}_A) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{Factoring out} && (7.64)\\
&= g \circ \mathtt{app} \circ (\mathtt{Id} \times f) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By the exponential property} && (7.65)\\
&= g \circ \mathtt{app} \circ \langle \Delta_1', f \circ f' \circ \Delta_2' \rangle && (7.66)\\
&= after \quad \text{By defintion} && (7.67)
\end{aligned}$$

Case If:

reduce **definition**

$$reduce\left((\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A}\right) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{reduce(\Delta_3)}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{7.68}$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Case Bind:

reduce **definition** To find

$$reduce\left((\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}\right) \tag{7.69}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'} A'} \quad (\text{Computation})\frac{\epsilon_1' \leq_\Phi \epsilon_1 \quad A' \leq:_\Phi A}{\mathtt{M}_{\epsilon_1'} A' \leq:_\Phi \mathtt{M}_{\epsilon_1} A}}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A} = reduce(\Delta_1) \tag{7.70}$$

Since $\Phi \vdash (i, \times) : (\Gamma, x : A') \triangleright (\Gamma, x : A)$ if $A' \leq:_\Phi A$, and by $\Delta_2 = \Phi \mid (\Gamma, x : A) \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-type})\frac{()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'} B'} \quad (\text{Computation})\frac{\epsilon_1' \leq_\Phi \epsilon_2 \quad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon_1'} B' \leq:_\Phi \mathtt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B} = reduce(\Delta_3) \tag{7.71}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-type})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'} A'} \quad ()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'} B'}}{\Phi|\Gamma\vdash \mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B} \quad (\text{Computation})\frac{\epsilon_1'\cdot\epsilon_2' \leq_\Phi \epsilon_1\cdot\epsilon_2 \quad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon_1'\cdot\epsilon_2'} B' \leq:_\Phi \mathtt{M}_{\epsilon_1\cdot\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{7.72}$$

Preserves Denotation Let

$$f = [\![A' \leq:_\Phi A]\!]_M : A' \to A \tag{7.73}$$
$$g = [\![B' \leq:_\Phi B]\!]_M : B' \to B \tag{7.74}$$
$$h_1 = [\![\epsilon_1' \leq_\Phi \epsilon_1]\!]_M : T_{\epsilon_1'} \to T_{\epsilon_1} \tag{7.75}$$
$$h_2 = [\![\epsilon_2' \leq_\Phi \epsilon_2]\!]_M : T_{\epsilon_2'} \to T_{\epsilon_2} \tag{7.76}$$
$$h = [\![\epsilon_1' \cdot \epsilon_2' \leq_\Phi \epsilon_1 \cdot \epsilon_2]\!]_M : T_{\epsilon_1' \cdot \epsilon_2'} \to T_{\epsilon_1 \cdot \epsilon_2} \tag{7.77}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3 = \Delta_2 \circ (\mathtt{Id}_\Gamma \times f) \tag{7.78}$$

And due to the reduction of Δ_3, we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3' \tag{7.79}$$

So:

$$\begin{aligned}
\mathit{before} &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \quad \text{By definition.} && (7.80)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{By reduction of } \Delta_1. && (7.81)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Factor out } h_1 && (7.82)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma\times A)} \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Tensor strength and sub-effecting } h_1 && (7.83)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Naturality of } h_1 && (7.84)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1'} f) \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out pairing again} && (7.85)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Tensorstrength} && (7.86)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_3) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the definition of } \Delta_3 && (7.87)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3') \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the reduction of } \Delta_3 && (7.88)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'} h_{2,B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out the functor} && (7.89)\\
&= h_B \circ \mu_{\epsilon_1',\epsilon_2',B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the } \mu \text{ and Sub-type rule} && (7.90)\\
&= h_B \circ T_{\epsilon_1'\cdot\epsilon_2'} g \circ \mu_{\epsilon_1',\epsilon_2',B'} \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By naturality of } \mu_{,,} && (7.91)\\
&= \mathit{after} \quad \text{By definition} && (7.92)
\end{aligned}$$

Case Effect-Fn:

reduce **definition** To find

$$\mathit{reduce}((\text{Effect-Lambda})\frac{()\frac{\Delta_1}{\Phi,\alpha \mid \Gamma\vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\alpha.A}) \tag{7.93}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi,\alpha\mid\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi,\alpha \mid \Gamma \vdash v : A} = \mathit{reduce}(\Delta_1) \tag{7.94}$$

in

$$(\text{Subtype})\frac{(\text{Effect-Fn})\frac{()\frac{\Delta_1'}{\Phi,\alpha\mid\Gamma\vdash v:A'}}{\Phi\mid\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}}{\forall\alpha.A'\leq:_{\Phi}\forall\alpha.A}}{\Phi\mid\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A} \tag{7.95}$$

Preserves Denotation

$$before = \overline{\Delta_1} \tag{7.96}$$
$$= \overline{[\![A' \leq:_{\Phi,\alpha} A]\!]_M \circ \Delta_1'} \quad \text{By induction} \tag{7.97}$$
$$= \forall_I([\![A' \leq:_{\Phi,\alpha} A]\!]_M) \circ \overline{\Delta_1'} \tag{7.98}$$
$$= [\![\forall\alpha.A' \leq:_{\Phi} \forall\alpha.A]\!]_M \circ \overline{\Delta_1'} \quad \text{By definition} \tag{7.99}$$
$$= after \quad \text{By definition} \tag{7.100}$$

Case Effect-Application:

reduce **definition** To find

$$reduce((\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi\mid\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi\mid\Gamma\vdash v\ \epsilon\colon A\,[\epsilon/\alpha]}) \tag{7.101}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi\mid\Gamma\vdash v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}A}{\forall\alpha.A'\leq:_{\Phi}\forall\alpha.A}}{\Phi\mid\Gamma\vdash v\colon\forall\alpha.A} = reduce(\Delta_1) \tag{7.102}$$

In

$$(\text{Subtype})\frac{(\text{E-app})\frac{()\frac{\Delta_1'}{\Phi\mid\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi\mid\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A'\,[\epsilon/\alpha] \leq:_{\Phi} A\,[\epsilon/\alpha]}{\Phi\mid\Gamma\vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{7.103}$$

Preserves Denotation Let

$$h = [\![\Phi\vdash\epsilon\colon\texttt{Effect}]\!]_M \tag{7.104}$$
$$A = [\![\Phi,\beta\vdash A\,[\beta/\alpha]:\texttt{Effect}]\!]_M \tag{7.105}$$
$$A' = [\![\Phi,\beta\vdash A'\,[\beta/\alpha]:\texttt{Effect}]\!]_M \tag{7.106}$$

Note that

$$\langle\texttt{Id}_I, h\rangle^*\,(\pi_1^*(f)) = (\pi_1\circ\langle\texttt{Id}_I, h\rangle)^*(f) = \texttt{Id}_I^*(f) = f \tag{7.107}$$

And that

$$\langle\texttt{Id}_I, h\rangle = [\![\Phi\vdash[\epsilon/\alpha]:\Phi,\alpha]\!]_M \tag{7.108}$$

With lemma:

$$[\![\forall\alpha.A' \leq:_{\Phi} \forall\alpha.A]\!]_M = \forall_I([\![A' \leq:_{\Phi,\alpha} A]\!]_M) \tag{7.109}$$
$$= \langle\texttt{Id}_I, h\rangle^*\,(\pi_1^*(\forall_I([\![A' \leq:_{\Phi,\alpha} A]\!]_M))) \tag{7.110}$$

In

$$
\begin{aligned}
before &= \langle \mathtt{Id}_I, h\rangle^* (\epsilon_A) \circ \Delta_1 && (7.111)\\
&= \langle \mathtt{Id}_I, h\rangle^* (\epsilon_A) \circ [\![\forall\alpha.A' \leq:_{\Phi} \forall\alpha.A]\!]_M \circ \Delta_1' \quad \text{By induction} && (7.112)\\
&= \langle \mathtt{Id}_I, h\rangle^* (\epsilon_A) \circ \langle \mathtt{Id}_I, h\rangle^* (\pi_1^*(\forall_I([\![A' \leq:_{\Phi,\alpha} A]\!]_M))) \circ \Delta_1' \quad \text{By lemma} && (7.113)\\
&= \langle \mathtt{Id}_I, h\rangle^* (\epsilon_A \circ \pi_1^*(\forall_I([\![A' \leq:_{\Phi,\alpha} A]\!]_M))) \circ \Delta_1' \quad \text{By functorality} && (7.114)\\
&= \langle \mathtt{Id}_I, h\rangle^* ([\![A' \leq:_{\Phi,\alpha} A]\!]_M \circ \epsilon_{A'}) \circ \Delta_1' \quad \text{By Naturality} && (7.115)\\
&= \langle \mathtt{Id}_I, h\rangle^* ([\![A' \leq:_{\Phi,\alpha} A]\!]_M) \circ \langle \mathtt{Id}_I, h\rangle^* (\epsilon_{A'}) \circ \Delta_1' && (7.116)\\
&= [\![A' [\epsilon/\alpha] \leq:_{\Phi,\alpha} A [\epsilon/\alpha]\,]\!]_M \circ \langle \mathtt{Id}_I, h\rangle^* (\epsilon_{A'}) \circ \Delta_1' \quad \text{By substitution of sub-typing} && (7.117)\\
&= after && (7.118)
\end{aligned}
$$

□

7.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v \colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\Delta]\!]_M = [\![reduce\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Phi \mid \Gamma \vdash v \colon A]\!]_M$ is unique.

Chapter 8

Beta-Eta-Equivalence Theorem (Soundness)

If
$eberelation \Phi v v' A$ then $[\![\Gamma \vdash v\colon A]\!]_M = [\![\Gamma \vdash v'\colon A]\!]_M$

By induction over Beta-eta equivalence relation.

8.0.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

Case Reflexive: Equality is reflexive, so if $\Phi \mid \Gamma \vdash v\colon A$ then $[\![\Phi \mid \Gamma \vdash v\colon A]\!]_M$ is equal to itself.

Case Symmetric: By inversion, if $\Phi \mid \Gamma \vdash v =_{\beta\eta} v'\colon A$ then $\Phi \mid \Gamma \vdash v' =_{\beta\eta} v\colon A$, so by induction $[\![\Gamma \vdash v'\colon A]\!]_M = [\![\Gamma \vdash v\colon A]\!]_M$ and hence $[\![\Gamma \vdash v\colon A]\!]_M = [\![\Gamma \vdash v'\colon A]\!]_M$

Case Transitive: There must exist v_2 such that $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2\colon A$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_3\colon A$, so by induction, $[\![\Gamma \vdash v_1\colon A]\!]_M = [\![\Gamma \vdash v_2\colon A]\!]_M$ and $[\![\Gamma \vdash v_2\colon A]\!]_M = [\![\Gamma \vdash v_3\colon A]\!]_M$. Hence by transitivity of equality, $[\![\Gamma \vdash v_1\colon A]\!]_M = [\![\Gamma \vdash v_3\colon A]\!]_M$

8.0.2 Beta-Eta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

Case Lambda: Let $f = [\![\Phi \mid \Gamma, x : A \vdash v_1\colon B]\!]_M : (\Gamma \times A) \to B$

Let $g = [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]_M : \Gamma \to A$

By the substitution denotation,

$$[\![\Phi \mid \Gamma \vdash [v_2/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Phi \mid \Gamma \vdash v_1\, [{}_2v/x] : B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned}
[\![\Phi \mid \Gamma \vdash (\lambda x : A.v)\ v \colon B]\!]_M &= \mathtt{app} \circ \langle \mathtt{cur}(f), g \rangle \\
&= \mathtt{app} \circ (\mathtt{cur}(f) \times \mathtt{Id}_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
&= f \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
&= [\![\Phi \mid \Gamma \vdash v_1\,[v_2/x] : B]\!]_M
\end{aligned} \tag{8.1}$$

Case Left Unit: Let $f = [\![\Phi \mid \Gamma, x : A \vdash v_1 \colon \mathtt{M}_\epsilon B]\!]_M$

Let $g = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M : \Gamma \to A$

By the substitution denotation,

$$[\![\Phi \mid \Gamma \vdash [v_2/x] : \Gamma, x : A]\!]_M : \Gamma \to (\Gamma \times A) = \langle \mathtt{Id}_\Gamma, g \rangle$$

We have

$$[\![\Phi \mid \Gamma \vdash v_1\,[v_2/x] : \mathtt{M}_\epsilon B]\!]_M = f \circ \langle \mathtt{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned}
[\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return} v_2\ \mathtt{in}\ v_1 \colon \mathtt{M}_\epsilon B]\!]_M =& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \eta_A \circ g \rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \mathtt{t}_{1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times \eta_A) \circ \langle \mathtt{Id}_\Gamma, g \rangle \\
=& \mu_{\mathbf{1},\epsilon,B} \circ T_{\mathbf{1}} f \circ \eta_{(\Gamma \times A)} \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\
=& \mu_{\mathbf{1},\epsilon,B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\
=& f \circ \langle \mathtt{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\
=& [\![\Phi \mid \Gamma \vdash v_1\,[v_2/x] : \mathtt{M}_\epsilon B]\!]_M
\end{aligned} \tag{8.2}$$

Case Right Unit: Let $f = [\![\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A]\!]_M$

$$\begin{aligned}
[\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return} x \colon \mathtt{M}_\epsilon A]\!]_M &= \mu_{\epsilon,\mathbf{1},A} \circ T_\epsilon(\eta_A \circ \pi_2) \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= \pi_2 \circ \langle \mathtt{Id}_\Gamma, f \rangle \\
&= f
\end{aligned} \tag{8.3}$$

Case Associative: Let

$$f = [\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{8.4}$$
$$g = [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{8.5}$$
$$h = [\![\Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C]\!]_M \tag{8.6}$$
$$\tag{8.7}$$

We also have the weakening:

$$\Phi \vdash \iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \tag{8.8}$$

With denotation:

$$[\![\Phi \vdash \iota\pi\times : \Gamma, x : A, y : B \triangleright \Gamma, y : B]\!]_M = (\pi_1 \times \mathtt{Id}_B) \tag{8.9}$$

We need to prove that the following are equal

$$
\begin{aligned}
&lhs = [\![\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ (\texttt{do}\ y \leftarrow v_2\ \texttt{in}\ v_3) \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2} C]\!]_M && (8.10)\\
&\quad = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \texttt{Id}_B) \circ \texttt{t}_{\epsilon_2, (\Gamma \times A), B} \circ \left\langle \texttt{Id}_{(\Gamma \times A)}, g \right\rangle) \circ \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle && (8.11)\\
&rhs = [\![\Phi \mid \Gamma \vdash \texttt{do}\ y \leftarrow (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\ \texttt{in}\ v_3 \colon \texttt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_2} C]\!]_M && (8.12)\\
&\quad = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \texttt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \texttt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle) \rangle && (8.13)\\
& && (8.14)
\end{aligned}
$$

Let's look at fragment F of rhs.

$$F = \texttt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \texttt{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle) \rangle \tag{8.15}$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \tag{8.16}$$

$$
\begin{aligned}
F &= \texttt{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\texttt{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\texttt{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle \\
&= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \texttt{t}_{\epsilon_2, \Gamma, B} \circ \texttt{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\texttt{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle && \text{By } \textbf{TODO: ref: mu+tstrength} \\
&= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\texttt{t}_{\epsilon_2, \Gamma, B} \circ (\texttt{Id}_\Gamma \times g)) \circ \texttt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle && \text{By naturality of v-strength}
\end{aligned} \tag{8.17}
$$

Since $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F$,

$$
\begin{aligned}
rhs =& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B))} \circ T_{\epsilon_1}(\texttt{t}_{\epsilon_2, \Gamma, B} \circ (\texttt{Id}_\Gamma \times g)) \circ \texttt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle \\
=& \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \texttt{t}_{\epsilon_2, \Gamma, B} \circ (\texttt{Id}_\Gamma \times g)) \circ \texttt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\
=& \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \texttt{t}_{\epsilon_2, \Gamma, B} \circ (\texttt{Id}_\Gamma \times g)) \circ \texttt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle
\end{aligned} \tag{8.18}
$$

Let's now look at the fragment G of rhs

$$G = T_{\epsilon_1}(\texttt{Id}_\Gamma \times g) \circ \texttt{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \texttt{Id}_\Gamma, \texttt{t}_{\epsilon_1, \Gamma, A} \circ \langle \texttt{Id}_\Gamma, f \rangle \rangle \tag{8.19}$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \texttt{t}_{\epsilon_2, \Gamma, B}) \circ G \tag{8.20}$$

By folding out the $\langle ..., ... \rangle$, we have

$$G = T_{\epsilon_1}(\texttt{Id}_\Gamma \times g) \circ \texttt{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ (\texttt{Id}_\Gamma \times \texttt{t}_{\epsilon_1, \Gamma, A}) \circ \langle \texttt{Id}_\Gamma, \langle \texttt{Id}_\Gamma, f \rangle \rangle \tag{8.21}$$

From the rule **TODO: Ref** showing the commutativity of tensor strength with α, the following commutes

$$
\begin{tikzcd}
\Gamma \arrow[r, "{\langle \texttt{Id}_\Gamma, \langle \texttt{Id}_\Gamma, f \rangle \rangle}"] & \Gamma \times (\Gamma \times T_{\epsilon_1} A) \arrow[d, "\texttt{Id}_\Gamma \times \texttt{t}_{\epsilon_1, \Gamma, A}"] & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \arrow[l, "{\alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}}"'] \arrow[d, "\texttt{t}_{\epsilon_1, (\Gamma \times \Gamma), A}"] \\
& \Gamma \times T_{\epsilon_1}(\Gamma \times A) \arrow[d, "\texttt{t}_{\epsilon_1, \Gamma, \Gamma \times A}"] & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \arrow[dl, "T_{\epsilon_1} \alpha_{\Gamma, \Gamma, A}"] \\
& T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) &
\end{tikzcd}
$$

Where $\alpha : ((\_ \times \_) \times \_) \to (\_ \times (\_ \times \_))$ is a natural isomorphism.

$$\begin{aligned} \alpha &= \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle && (8.22) \\ \alpha^{-1} &= \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle && (8.23) \end{aligned}$$

So:

$$\begin{aligned} G =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ \alpha^{-1}_{\Gamma,\Gamma,(T_{\epsilon_1}A)} \circ \langle \mathtt{Id}_\Gamma, \langle \mathtt{Id}_\Gamma, f \rangle \rangle \\ =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A}) \circ \mathtt{t}_{\epsilon_1,(\Gamma\times\Gamma),A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \times \mathtt{Id}_{T_{\epsilon_1}A}) \circ \langle \mathtt{Id}_\Gamma, f \rangle \quad \text{By definition of } \alpha \text{ and products} \\ =& T_{\epsilon_1}((\mathtt{Id}_\Gamma \times g) \circ \alpha_{\Gamma,\Gamma,A} \circ (\langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \times \mathtt{Id}_A)) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \quad \text{By tensor strength's left-naturality} \\ =& T_{\epsilon_1}((\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g \right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \end{aligned} \tag{8.24}$$

Since

$$rhs = \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B}) \circ G \tag{8.25}$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h) \circ \mathtt{t}_{\epsilon_2,\Gamma,B} \circ (\pi_1 \times \mathtt{Id}_{T_{\epsilon_2}B}) \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g \right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \\ &= \mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,C} \circ T_{\epsilon_1}(\mu_{\epsilon_2,\epsilon_3,C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \mathtt{Id}_B)) \circ \mathtt{t}_{\epsilon_2,(\Gamma\times A),B} \circ \left\langle \mathtt{Id}_{(\Gamma\times A)}, g \right\rangle) \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, f \rangle \quad \text{By Left-Tensor Stre} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \tag{8.26}$$

Case Eta: Let

$$f = [\![\Phi \mid \Gamma \vdash v \colon A \to B]\!]_M : \Gamma \to (B)^A \tag{8.27}$$

By weakening, we have

$$\begin{aligned} [\![\Phi \mid \Gamma, x : A \vdash v \colon A \to B]\!]_M &= f \circ \pi_1 : \Gamma \times A \to (B)^A && (8.28) \\ [\![\Phi \mid \Gamma, x : A \vdash v\ x \colon B]\!]_M &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle && (8.29) \\ & && (8.30) \end{aligned}$$

Hence, we have

$$\begin{aligned} [\![\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) \colon A \to B]\!]_M &= \mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \\ \mathtt{app} \circ ([\![\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) \colon A \to B]\!]_M \times \mathtt{Id}_A) &= \mathtt{app} \circ (\mathtt{cur}(\mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \times \mathtt{Id}_A) \\ &= \mathtt{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \\ &= \mathtt{app} \circ (f \times \mathtt{Id}_A) \end{aligned} \tag{8.31}$$

Hence, by the fact that $\mathtt{cur}(f)$ is unique in a cartesian closed category,

$$[\![\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) \colon A \to B]\!]_M = f = [\![\Phi \mid \Gamma \vdash v \colon A \to B]\!]_M \tag{8.32}$$

Case If-True: Let

$$
\begin{aligned}
f =& [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M && (8.33)\\
g =& [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M && (8.34)\\
& && (8.35)
\end{aligned}
$$

Then

$$
\begin{aligned}
[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A]\!]_M &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inl} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathtt{app} \circ ((\mathtt{cur}(f \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathtt{app} \circ (\mathtt{cur}(f \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= f \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\
&= f \\
&= [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M
\end{aligned} \tag{8.36}
$$

Case If-False: Let

$$
\begin{aligned}
f =& [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M && (8.37)\\
g =& [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M && (8.38)\\
& && (8.39)
\end{aligned}
$$

Then

$$
\begin{aligned}
[\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A]\!]_M &= \mathtt{app} \circ (([\mathtt{cur}(f \circ \pi_2), \mathtt{cur}(g \circ \pi_2)] \circ \mathtt{inr} \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathtt{app} \circ ((\mathtt{cur}(g \circ \pi_2) \circ \langle\rangle_\Gamma) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathtt{app} \circ (\mathtt{cur}(g \circ \pi_2) \times \mathtt{Id}_\Gamma) \circ (\langle\rangle_\Gamma \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\
&= g \circ \pi_2 \circ \langle \langle\rangle_\Gamma, \mathtt{Id}_\Gamma \rangle \\
&= g \\
&= [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M
\end{aligned} \tag{8.40}
$$

Case If-Eta: Let

$$
\begin{aligned}
f &= [\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool}]\!]_M && (8.41)\\
g &= [\![\Phi \mid \Gamma, x : \mathtt{Bool} \vdash v_2 \colon A]\!]_M && (8.42)\\
& && (8.43)
\end{aligned}
$$

Then by the substitution theorem,

$$
\begin{aligned}
[\![\Phi \mid \Gamma \vdash v_2\,[\mathtt{true}/x] : A]\!]_M &= g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \rangle && (8.44)\\
[\![\Phi \mid \Gamma \vdash v_2\,[\mathtt{false}/x] : A]\!]_M &= g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \rangle && (8.45)\\
[\![\Phi \mid \Gamma \vdash v_2\,[v_1/x] : A]\!]_M &= g \circ \langle \mathtt{Id}_\Gamma, f \rangle && (8.46)
\end{aligned}
$$

Hence we have (Using the diagonal and twist morphisms):

$$\llbracket \Phi \mid \Gamma \vdash \mathtt{if}_A\ v_1\ \mathtt{then}\ v_2\,[\mathtt{true}/x]\ \mathtt{else}\ v_2\,[\mathtt{false}/x] : A \rrbracket_M \tag{8.47}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \rangle \circ \pi_2), \mathtt{cur}(g \circ \langle \mathtt{Id}_\Gamma, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \rangle \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{8.48}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_2 \rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_2 \rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Pairing property} \tag{8.49}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \langle \pi_2, \mathtt{inl}_1 \circ \langle\rangle_\Gamma \circ \pi_1 \rangle), \mathtt{cur}(g \circ \langle \pi_2, \mathtt{inr}_1 \circ \langle\rangle_\Gamma \circ \pi_1 \rangle)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Terminal is unique} \tag{8.50}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inl}_1 \circ \langle\rangle_{\mathbf{1}})) \circ \tau_{\mathbf{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times (\mathtt{inr}_1 \circ \langle\rangle_{\mathbf{1}})) \circ \tau_{\mathbf{1},\Gamma})] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition} \tag{8.51}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inl}_1) \circ \tau_{\mathbf{1},\Gamma}), \mathtt{cur}(g \circ (\mathtt{Id}_\Gamma \times \mathtt{inr}_1) \circ \tau_{\mathbf{1},\Gamma})] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Identity} = \mathtt{Id}_{\mathbf{1}} \tag{8.52}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma} \circ (\mathtt{inl}_1 \times \mathtt{Id}_\Gamma)), \mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma} \circ (\mathtt{inr}_1 \times \mathtt{Id}_\Gamma))] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Twist commutivity} \tag{8.53}$$
$$= \mathtt{app} \circ (([\mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma}) \circ \mathtt{inl}_1, \mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma}) \circ \mathtt{inr}_1] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Exponential property} \tag{8.54}$$
$$= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma}) \circ [\mathtt{inl}_1, \mathtt{inr}_1] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring out } \mathtt{cur}(..) \tag{8.55}$$
$$= \mathtt{app} \circ ((\mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma}) \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Since } [\mathtt{inl}, \mathtt{inr}]\, istheidentity \tag{8.56}$$
$$= \mathtt{app} \circ (\mathtt{cur}(g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma}) \times \mathtt{Id}_\Gamma) \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring} \tag{8.57}$$
$$= g \circ \tau_{\mathbf{1}+\mathbf{1},\Gamma} \circ (f \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of } \mathtt{app}, \mathtt{cur}(..) \tag{8.58}$$
$$= g \circ (\mathtt{Id}_\Gamma \times f) \circ \tau_{\Gamma,\Gamma} \circ \delta_\Gamma \quad \text{Twist commutivity} \tag{8.59}$$
$$= g \circ (\mathtt{Id}_\Gamma \times f) \circ \langle \mathtt{Id}_\Gamma, \mathtt{Id}_\Gamma \rangle \quad \text{Twist, diagonal defintions} \tag{8.60}$$
$$= g \circ \langle \mathtt{Id}_\Gamma, f \rangle \tag{8.61}$$
$$= \llbracket \Phi \mid \Gamma \vdash v_2\,[v_1/x] : A \rrbracket_M \tag{8.62}$$
$$\tag{8.63}$$

Case Effect-Beta: let

$$h = \llbracket \Phi \vdash \epsilon : \mathtt{Effect} \rrbracket_M \tag{8.64}$$
$$f = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket_M \tag{8.65}$$
$$A = \llbracket \Phi, \alpha \vdash A\,[\alpha/\alpha] : \mathtt{Type} \rrbracket_M \tag{8.66}$$

Then

$$\llbracket \Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\alpha.A \rrbracket_M = \overline{f} \tag{8.67}$$

So

$$\llbracket \Phi \mid \Gamma \vdash (\Lambda\alpha.v)\ \epsilon : \forall\alpha.A \rrbracket_M = \langle \mathtt{Id}_I, h \rangle^* (\epsilon_A) \circ \overline{f} \tag{8.68}$$
$$= \langle \mathtt{Id}_I, h \rangle^* (\epsilon_A) \circ \langle \mathtt{Id}_I, h \rangle^* (\pi_1^*(\overline{f})) \quad \text{Identity functor} \tag{8.69}$$
$$= \langle \mathtt{Id}_I, h \rangle^* (\epsilon_A \circ \pi_1^*(\overline{f})) \tag{8.70}$$
$$= \langle \mathtt{Id}_I, h \rangle^* (f) \quad \text{By adjunction} \tag{8.71}$$
$$= \llbracket \Phi \mid \Gamma \vdash v\,[\epsilon/\alpha] : A\,[e/\alpha]\, \rrbracket_M \quad \text{By substitution theorem} \tag{8.72}$$
$$\tag{8.73}$$

Case Effect-Eta: Let

$$f = [\![\Phi \mid \Gamma \vdash v\colon \forall\alpha.A]\!]_M \tag{8.74}$$
$$A = [\![\Phi, \alpha \vdash A\colon \mathtt{Type}]\!]_M \tag{8.75}$$

so

$$[\![\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha)\colon \forall\alpha.A]\!]_M = \overline{[\![\Phi, \alpha \mid \Gamma \vdash \epsilon\ \alpha\colon \forall\alpha.A]\!]_M} \tag{8.76}$$
$$= \overline{\langle \mathtt{Id}_{I\times U}, \pi_2\rangle^* \left(\epsilon_{[\![\Phi,\alpha,\beta\vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M}\right) \circ \pi_1^*(f)} \tag{8.77}$$

Let's look at $[\![\Phi, \alpha, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M$.

We have the weakening:

$$\iota\pi\times : \Phi, \alpha, \beta \rhd \Phi, \beta \tag{8.78}$$

So by the weakening theorem on type denotations:

$$[\![\Phi, \alpha, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M = [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M \circ (\pi_1 \times \mathtt{Id}_U) \tag{8.79}$$
$$\forall_{I\times U}([\![\Phi, \alpha, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M) = \forall_I([\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M) \circ \pi_1 \tag{8.80}$$
$$= \pi_1^*\forall_I([\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M) \tag{8.81}$$
$$\epsilon_{[\![\Phi,\alpha,\beta\vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M} = \overline{\mathtt{Id}_{\pi_1^*\forall_I([\![\Phi,\beta\vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M)}} \tag{8.82}$$
$$= \overline{\mathtt{Id}_{\pi_1^*\forall_I A}} \tag{8.83}$$
$$= \overline{\pi_1^*(\mathtt{Id}_{\forall_I A})} \tag{8.84}$$
$$= \overline{\pi_1^*(\overline{\epsilon_A})} \tag{8.85}$$
$$= \overline{\overline{(\pi_1 \times \mathtt{Id}_U)^*(\epsilon_A)}} \tag{8.86}$$
$$= (\pi_1 \times \mathtt{Id}_U)^*(\epsilon_A) \tag{8.87}$$

So

$$[\![\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha)\colon \forall\alpha.A]\!]_M = \overline{\langle \mathtt{Id}_{I\times U}, \pi_2\rangle^* \left(\epsilon_{[\![\Phi,\alpha,\beta\vdash A[\beta/\alpha]:\mathtt{Type}]\!]_M}\right) \circ \pi_1^*(f)} \tag{8.88}$$
$$= \overline{\langle \mathtt{Id}_{I\times U}, \pi_2\rangle^* \left((\pi_1 \times \mathtt{Id}_U)^*(\epsilon_A)\right) \circ \pi_1^*(f)} \tag{8.89}$$
$$= \overline{\langle \pi_1, \pi_2\rangle^* (\epsilon_A) \circ \pi_1^*(f)} \tag{8.90}$$
$$= \overline{\mathtt{Id}_{I\times U}^*(\epsilon_A) \circ \pi_1^*(f)} \tag{8.91}$$
$$= \overline{\epsilon_A \circ \pi_1^*(f)} \quad \text{By adjunction} \tag{8.92}$$
$$= f \tag{8.93}$$

8.0.3 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of sub-expressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

Case Lambda: By inversion, we have
eberelation$\Phi\Gamma, x : Av_1v_2B$ By induction, we therefore have $[\![\Phi \mid \Gamma, x : A \vdash v_1\colon B]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2\colon B]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma, x : A \vdash v_1 \colon B]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon B]\!]_M \tag{8.94}$$

And so

$$[\![\Phi \mid \Gamma \vdash \lambda x : A.v_1 \colon A \to B]\!]_M = \mathtt{cur}(f) = [\![\Phi \mid \Gamma \vdash \lambda x : A.v_2 \colon A \to B]\!]_M \tag{8.95}$$

Case Return: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ By induction, we therefore have $[\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M \tag{8.96}$$

And so

$$[\![\Phi \mid \Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_1 A]\!]_M = \eta_A \circ f = [\![\Phi \mid \Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_1 A]\!]_M \tag{8.97}$$

Case Apply: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$ By induction, we therefore have $[\![\Phi \mid \Gamma \vdash v_1 \colon A \to B]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon A \to B]\!]_M$ and $[\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2' \colon A]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma \vdash v_1 \colon A \to B]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon A \to B]\!]_M \tag{8.98}$$
$$g = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2' \colon A]\!]_M \tag{8.99}$$

And so

$$[\![\Phi \mid \Gamma \vdash v_1\ v_2 \colon B]\!]_M = \mathtt{app} \circ \langle f, g\rangle = [\![\Phi \mid \Gamma \vdash v_1'\ v_2' \colon B]\!]_M \tag{8.100}$$

Case Bind: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon \mathtt{M}_{e_1} A$ and $eberelation \Phi\Gamma, x : A v_2 v_2' \mathtt{M}_{\epsilon_2} B$ By induction, we therefore have $[\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{e_1} A]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon \mathtt{M}_{e_1} A]\!]_M$ and $[\![\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{8.101}$$
$$g = [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{8.102}$$

And so

$$\begin{aligned} [\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\epsilon_1, \Gamma, A} \circ \langle \mathtt{Id}_\Gamma, f\rangle \\ &= [\![\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A]\!]_M \end{aligned} \tag{8.103}$$

Case If: By inversion, we have $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$ By induction, we therefore have $[\![\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Phi \mid \Gamma \vdash v' \colon \mathtt{Bool}]\!]_M$, $[\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon A]\!]_M$ and $[\![\Phi \mid \Gamma, x : A \vdash v_2 \colon A]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2' \colon A]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}]\!]_M = [\![\Phi \mid \Gamma \vdash v' \colon \mathtt{Bool}]\!]_M \tag{8.104}$$
$$g = [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_1' \colon A]\!]_M \tag{8.105}$$
$$h = [\![\Phi \mid \Gamma, x : A \vdash v_2 \colon A]\!]_M = [\![\Phi \mid \Gamma, x : A \vdash v_2' \colon A]\!]_M \tag{8.106}$$

And so

$$\begin{aligned} [\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A]\!]_M &= \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \\ &= [\![\Phi \mid \Gamma \vdash \mathtt{if}_A\ v'\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2' \colon A]\!]_M \end{aligned} \tag{8.107}$$

Case Subtype: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, and $A \leq :_\Phi B$ By induction, we therefore have $[\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2 \colon A]\!]_M$

Then let

$$f = [\![\Phi \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2 \colon B]\!]_M \tag{8.108}$$
$$g = [\![A \leq :_\Phi B]\!]_M \tag{8.109}$$

And so

$$[\![\Phi \mid \Gamma \vdash v_1 \colon B]\!]_M = g \circ f = [\![\Phi \mid \Gamma \vdash v_1 \colon B]\!]_M \tag{8.110}$$

Case Effect-Lambda: By inversion, we have $\Phi, \alpha \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$. So by induction, $[\![\Phi, \alpha \mid \Gamma \vdash v_1 \colon A]\!]_M = [\![\Phi, \alpha \mid \Gamma \vdash v_2 \colon A]\!]_M$

So

$$[\![\Phi \mid \Gamma \vdash \Lambda\alpha.v_1 \colon \forall\alpha.A]\!]_M = \overline{[\![\Phi, \alpha \mid \Gamma \vdash v_1 \colon A]\!]_M} \tag{8.111}$$
$$= \overline{[\![\Phi, \alpha \mid \Gamma \vdash v_2 \colon A]\!]_M} \tag{8.112}$$
$$= [\![\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A]\!]_M \tag{8.113}$$

Case Effect-Apply: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon \forall\alpha.A$ and $\Phi \vdash \epsilon \colon \texttt{Effect}$.

So by induction, we have $[\![\Phi \mid \Gamma \vdash v_1 \colon \forall\alpha.A]\!]_M = [\![\Phi \mid \Gamma \vdash v_2 \colon \forall\alpha.A]\!]_M$

So

$$[\![\Phi \mid \Gamma \vdash v_1\ \epsilon \colon A\,[\epsilon/\alpha]\,]\!]_M = \langle \texttt{Id}_I, [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle^* (\epsilon_A) \circ [\![\Phi \mid \Gamma \vdash v_1 \colon \forall\alpha.A]\!]_M \tag{8.114}$$
$$= \langle \texttt{Id}_I, [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle^* (\epsilon_A) \circ [\![\Phi \mid \Gamma \vdash v_2 \colon \forall\alpha.A]\!]_M \tag{8.115}$$
$$= [\![\Phi \mid \Gamma \vdash v_2\ \epsilon \colon A\,[\epsilon/\alpha]\,]\!]_M \tag{8.116}$$

□