Contents

1 Language Definition **4**

1.1 Terms . . . 4

1.1.1 Value Terms . . . 4

1.2 Type System . . . 4

1.2.1 Ground Effects . . . 4

1.2.2 Effect Po-Monoid Under a Effect Environment . . . 4

1.2.3 Types . . . 5

1.2.4 Type and Effect Environments . . . 5

1.2.5 Sub-typing . . . 6

1.2.6 Type Rules . . . 6

1.2.7 Ok Lemma . . . 7

2 Category Requirements **8**

2.1 CCC . . . 8

2.2 Graded Pre-Monad . . . 8

2.2.1 Left Unit . . . 8

2.2.2 Right Unit . . . 9

2.2.3 Associativity . . . 9

2.3 Tensor Strength . . . 9

2.3.1 Left Naturality . . . 9

2.3.2 Right Naturality . . . 9

2.3.3 Unitor Law . . . 9

2.3.4 Commutativity with Join . . . 10

2.4 Commutativity with Unit . . . 10

2.5 Commutativity with α . . . 10

2.6 Sub-effecting . . . 10

2.6.1 Sub-effecting and Tensor Strength . . . 10

2.6.2 Sub-effecting and Monadic Join . . . 11

2.7 Sub-typing . . . 11

3 Denotations **12**

3.1 Helper Morphisms . . . 12

3.1.1 Diagonal and Twist Morphisms . . . 12

3.2 Denotations of Types . . . 12

3.2.1 Denotation of Ground Types . . . 12

3.2.2 Denotation of Polymorphic Types . . . 12

3.2.3 Denotation of Computation Type . . . 12

3.2.4 Denotation of Function Types . . . 12

3.2.5 Denotation of Type Environments . . . 12

3.2.6 Denotation of Value Terms . . . 12

3.2.7 Denotation of Computation Terms . . . 12

4 Unique Denotations **13**
4.1 Reduced Type Derivation . . . 13
4.2 Reduced Type Derivations are Unique . . . 13
4.3 Each type derivation has a reduced equivalent with the same denotation. . . . 16
4.4 Denotations are Equivalent . . . 19

5 Weakening **20**
5.1 Effect Weakening Definition . . . 20
5.1.1 Relation . . . 20
5.1.2 Weakening Properties . . . 20
5.1.3 Effect Weakening Preserves `Ok` . . . 20
5.1.4 Domain Lemma . . . 21
5.1.5 Weakening Preserves Effect Well-Formed-Ness . . . 21
5.1.6 Weakening Preserves Type-Well-Formed-Ness . . . 21
5.1.7 Corollary . . . 22
5.1.8 Effect Weakening preserves Type Relations . . . 22
5.2 Type Environment Weakening . . . 23
5.2.1 Relation . . . 23
5.2.2 Domain Lemma . . . 23
5.2.3 Theorem 1 . . . 24
5.2.4 Theorem 2 . . . 24

6 Substitution **28**
6.1 Effect Substitutions . . . 28
6.1.1 Action of Effect Substitution on Effects . . . 28
6.1.2 Action of Effect Substitution on Types . . . 28
6.1.3 Action of Effect-Substitution on Type Environments . . . 29
6.1.4 Action of Effect Substitution on Terms . . . 29
6.1.5 Well-Formed-ness . . . 29
6.1.6 Property 1 . . . 29
6.1.7 Property 2 . . . 29
6.1.8 Property 3 . . . 30
6.2 Substitution Preserves the Well-formed-ness of Effects . . . 30
6.2.1 Effect Substitution preserves the sub-effect relation . . . 30
6.2.2 Substitution preserves well-formed-ness of Types . . . 30
6.2.3 Substitution of effects preserves Sub-Typing Relation . . . 31
6.2.4 Substitution preserves well-formed-ness of Type Environments . . . 31
6.2.5 Effect-Polymorphism Preserves the Typing Relation . . . 31
6.3 The Identity Substitution on Effect Environments . . . 33
6.3.1 Properties of the Identity Substitution . . . 33
6.4 Single Substitution on Effect Environments . . . 33
6.5 Term-Term Substitutions . . . 33
6.5.1 Substitutions as SNOC lists . . . 33
6.5.2 Trivial Properties of substitutions . . . 33
6.5.3 Action of substitutions . . . 34
6.5.4 Well-Formed-ness . . . 34
6.5.5 Simple Properties Of Substitution . . . 34
6.5.6 Substitution Preserves Typing . . . 35
6.5.7 Computation Terms . . . 36
6.6 The Identity Substitution on Type Environments . . . 38
6.6.1 Properties of the Identity Substitution . . . 38
6.7 Single Substitution on Type Environments . . . 38

7 Beta Eta Equivalence (Soundness) 39
7.1 Beta and Eta Equivalence . . . 39
7.1.1 Beta-Eta conversions . . . 39
7.1.2 Equivalence Relation . . . 39
7.1.3 Congruences . . . 40
7.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type . . . 40
7.2.1 Equivalence Relations . . . 40
7.2.2 Beta-Eta conversions . . . 40
7.2.3 Congruences . . . 42

Chapter 1

Language Definition

1.1 Terms

Making the language no-longer differentiate between values and computations.

1.1.1 Value Terms

$$\begin{aligned} v ::= &x \\ &\mid \lambda x : A.v \\ &\mid \mathtt{C}^A \\ &\mid \mathtt{()} \\ &\mid \mathtt{true} \mid \mathtt{false} \\ &\mid \Lambda\alpha.v \\ &\mid v\ \epsilon \\ &\mid \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \\ &\mid v_1\ v_2 \\ &\mid \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \\ &\mid \mathtt{return} v \end{aligned} \tag{1.1}$$

1.2 Type System

1.2.1 Ground Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, \mathtt{1}, \leq)$ with ground elements e.

1.2.2 Effect Po-Monoid Under a Effect Environment

Derive a new Po-Monoid for each Φ:

$$(E_\Phi, \cdot_\Phi, \mathtt{1}, \leq_\Phi) \tag{1.2}$$

Where meta-variables, ϵ, range over E_Φ Where

$$E_\Phi = E \cup \{\alpha \mid \alpha \in \Phi\} \tag{1.3}$$

And

$$()\frac{\epsilon_3 = \epsilon_1 \cdot \epsilon_2}{\epsilon_3 = \epsilon_1 \cdot_\Phi \epsilon_2} \tag{1.4}$$

Otherwise, $\cdot_\Phi$ is symbolic in nature.

$$\epsilon_1 \leq_\Phi \epsilon_2 \Leftrightarrow \forall \sigma \downarrow . \epsilon_1 [\sigma \downarrow] \leq \epsilon_2 [\sigma \downarrow] \tag{1.5}$$

Where $\sigma \downarrow$ denotes any ground-substitution of Φ. That is any substitution of all effect-variables in Φ to ground effects. Where it is obvious from the context, I shall use $\leq$ instead of $\leq_\Phi$.

1.2.3 Types

Ground Types There exists a set γ of ground types, including `Unit`, `Bool`

Term Types

$$A, B, C ::= \gamma \mid A \to B \mid \mathtt{M}_\epsilon A \mid \forall \alpha . A$$

1.2.4 Type and Effect Environments

A type environment is a snoc-list of tern-variable, type pairs, $G ::= \diamond \mid \Gamma, x : A$. An effect environment is a snoc-list of effect-variables.

$\Phi ::= \diamond \mid \Phi, \alpha$

Domain Function on Type Environments

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

Membership of Effect Environments Informally, $\alpha \in \Phi$ if α appears in the list represented by Φ.

`Ok` Predicate On Effect Environments

- (Atom) $\frac{}{\diamond \mathtt{Ok}}$
- (A) $\frac{\Phi \mathtt{Ok} \quad \alpha \notin \Phi}{\Phi, \alpha \mathtt{Ok}}$

Well-Formed-ness of effects We define a relation $\Phi \vdash \epsilon$.

- (Ground) $\frac{\Phi \mathtt{Ok}}{\Phi \vdash e}$
- (Var) $\frac{\Phi, \alpha \mathtt{Ok}}{\Phi, \alpha \vdash \alpha}$
- (Weaken) $\frac{\Phi \vdash \alpha}{\Phi, \beta \vdash \alpha}$ (if $\alpha \neq \beta$)
- (Monoid Op) $\frac{\Phi \vdash \epsilon_1 \quad \Phi \vdash \epsilon_2}{\Phi \vdash \epsilon_1 \cdot \epsilon_2}$

Well-Formed-ness of Types We define a relation $\Phi \vdash \tau$ on types.

- (Ground) $\frac{}{\Phi \vdash \gamma}$
- (Lambda) $\frac{\Phi \vdash A \quad \Phi \vdash B}{\Phi \vdash A \to B}$
- (Computation) $\frac{\Phi \vdash A \quad \Phi \vdash \epsilon}{\Phi \vdash \mathtt{M}_\epsilon A}$
- (For-All) $\frac{\Phi, \alpha \vdash A}{\Phi \vdash \forall \alpha . A}$

Ok Predicate on Type Environments We now define a predicate on type environments and effect environments: $\Phi \vdash \Gamma \texttt{Ok}$

- (Nil) $\frac{}{\Phi \vdash \diamond \texttt{Ok}}$
- (Var) $\frac{\Phi \vdash \Gamma \texttt{Ok} \quad x \notin \texttt{dom}(\Gamma) \quad \Phi \vdash A}{\Phi \vdash \Gamma, x:A \texttt{Ok}}$

1.2.5 Sub-typing

There exists a sub-typing pre-order relation $\leq:_\gamma$ over ground types that is:

- (Reflexive) $\frac{}{A \leq:_\gamma A}$
- (Transitive) $\frac{A \leq:_\gamma B \quad B \leq:_\gamma C}{A \leq:_\gamma C}$

We extend this relation with the function and effect-lambda sub-typing rules to yield the full sub-typing relation under an effect environment, Φ, $\leq:_\Phi$

- (ground) $\frac{A \leq:_\gamma B}{A \leq:_\Phi B}$
- (Fn) $\frac{A \leq:_\Phi A' \quad B' \leq:_\Phi B}{A' \to B' \leq:_\Phi A \to B}$
- (All) $\frac{A \leq:_\Phi A'}{\forall \alpha. A \leq:_\Phi \forall a. A'}$
- (Effect) $\frac{A \leq:_\Phi B \quad \epsilon_1 \leq_\Phi \epsilon_2}{\texttt{M}_{\epsilon_1} A \leq:_\Phi \texttt{M}_{\epsilon_2} B}$

1.2.6 Type Rules

- (Const) $\frac{\Phi \vdash \Gamma \texttt{Ok} \quad \Phi \vdash A}{\Phi | \Gamma \vdash \texttt{C}^A : A}$
- (Unit) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{()} : \texttt{Unit}}$
- (True) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{true} : \texttt{Bool}}$
- (False) $\frac{\Phi \vdash \Gamma \texttt{Ok}}{\Phi | \Gamma \vdash \texttt{false} : \texttt{Bool}}$
- (Var) $\frac{\Phi \vdash \Gamma, x:A \texttt{Ok}}{\Phi | \Gamma, x:A \vdash x:A}$
- (Weaken) $\frac{\Phi | \Gamma \vdash x:A \quad \Phi \vdash B}{\Phi | \Gamma, y:B \vdash x:A}$ (if $x \neq y$)
- (Fn) $\frac{\Phi | \Gamma, x:A \vdash v:\beta}{\Phi | \Gamma \vdash \lambda x:A.v : A \to B}$
- (Sub) $\frac{\Phi | \Gamma \vdash v:A \quad A \leq:_\Phi B}{\Phi | \Gamma \vdash v:B}$
- (Effect-Abs) $\frac{\Phi, \alpha | \Gamma \vdash v:A}{\Phi | \Gamma \vdash \Lambda \alpha.v : \forall \alpha.A}$
- (Effect-apply) $\frac{\Phi | \Gamma \vdash v : \forall \alpha.A \quad \Phi \vdash \epsilon}{\Phi | \Gamma \vdash v \ \epsilon : A[\epsilon/\alpha]}$
- (Return) $\frac{\Phi | \Gamma \vdash v:A}{\Phi | \Gamma \vdash \texttt{return} v : \texttt{M}_{\mathbf{1}} A}$
- (Apply) $\frac{\Phi | \Gamma \vdash v_1 : A \to \texttt{M}_\epsilon B \quad \Phi | \Gamma \vdash v_2 : A}{\Phi | \Gamma \vdash v_1 \ v_2 : \texttt{M}_\epsilon B}$
- (If) $\frac{\Phi | \Gamma \vdash v : \texttt{Bool} \quad \Phi | \Gamma \vdash v_1 : A \quad \Phi | \Gamma \vdash v_2 : A}{\Phi | \Gamma \vdash \texttt{if}_A \ V \ \texttt{then} \ v_1 \ \texttt{else} \ v_2 : A}$
- (Do) $\frac{\Phi | \Gamma \vdash v_1 : \texttt{M}_{\epsilon_1} A \quad \Phi | \Gamma, x:A \vdash v_2 : \texttt{M}_{\epsilon_2} B}{\Phi | \Gamma \vdash \texttt{do} \ x \leftarrow v_1 \ \texttt{in} \ v_2 : \texttt{M}_{\epsilon_1 \cdot \epsilon_2} B}$

1.2.7 Ok Lemma

If $\Phi \mid \Gamma \vdash t\colon \tau$ then $\Phi \vdash \Gamma\mathtt{Ok}$.

Proof If $\Gamma, x : A\mathtt{Ok}$ then by inversion $\Gamma\mathtt{Ok}$ Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Phi \vdash \Gamma\mathtt{Ok}$. And all non-axiom derivations preserve the `Ok` property.

Chapter 2

Category Requirements

2.1 CCC

The category at each index should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

Further more, it should have a co-product of the terminal object $\mathtt{1}$. This is required for the beta-eta equivalence of `if-then-else` terms.

$\mathtt{1} \xrightarrow{inl} A \xleftarrow{inr} \mathtt{1}$

For each:

$\mathtt{1} \xrightarrow{f} A \xleftarrow{g} \mathtt{1}$

There exists unique $[f, g] : \mathtt{1} + \mathtt{1} \to A$ such that:

$$\begin{tikzcd} & A & \\ \mathtt{1} \arrow[ur, "f"] \arrow[r, "\mathtt{inl}"'] & \mathtt{1}+\mathtt{1} \arrow[u, "{[f,g]}"] & \mathtt{1} \arrow[ul, "g"'] \arrow[l, "\mathtt{inr}"] \end{tikzcd}$$

2.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endo-functor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot\, \mathtt{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

2.2.1 Left Unit

$$\begin{tikzcd} T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"'] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon, \mathtt{1}, A}"] \\ & T_\epsilon A \end{tikzcd}$$

2.2.2 Right Unit

$$\begin{tikzcd} T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1},\epsilon,A}"] \\ & T_\epsilon A \end{tikzcd}$$

2.2.3 Associativity

$$\begin{tikzcd} T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3} A}"] \arrow[d, "T_{\epsilon_1} \mu_{\epsilon_2,\epsilon_3,A}"] & T_{\epsilon_1 \cdot \epsilon_2} T_{\epsilon_3} A \arrow[d, "\mu_{\epsilon_1 \cdot \epsilon_2,\epsilon_3,A}"] \\ T_{\epsilon_1} T_{\epsilon_2 \cdot \epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2 \cdot \epsilon_3,A}"] & T_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} A \end{tikzcd}$$

2.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$$

Satisfying the following rules:

2.3.1 Left Naturality

$$\begin{tikzcd} A \times T_\epsilon B \arrow[r, "\mathtt{Id}_A \times T_\epsilon f"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A \times T_\epsilon B' \arrow[d, "\mathtt{t}_{\epsilon,A,B'}"] \\ T_\epsilon(A \times B) \arrow[r, "T_\epsilon(\mathtt{Id}_A \times f)"] & T_\epsilon(A \times B') \end{tikzcd}$$

2.3.2 Right Naturality

$$\begin{tikzcd} A \times T_\epsilon B \arrow[r, "f \times \mathtt{Id}_{T_\epsilon B}"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A' \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A',B}"] \\ T_\epsilon(A \times B) \arrow[r, "T_\epsilon(f \times \mathtt{Id}_B)"] & T_\epsilon(A' \times B) \end{tikzcd}$$

2.3.3 Unitor Law

$$\begin{tikzcd} \mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\ & T_\epsilon A \end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

Tensor Strength and Projection Due to the left-unitor law, we can develop a new law for the commutativity of π_2 with $\mathtt{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B)$$

And $\pi_{2,\mathtt{1}}$ is the left unitor, so by tensorial strength:

$$\begin{aligned}
T_\epsilon \pi_2 \circ \mathtt{t}_{\epsilon,A,B} &= T_\epsilon \pi_{2,\mathbf{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon,A,B} \\
&= T_\epsilon \pi_{2,1,B} \circ \mathtt{t}_{\epsilon,\mathbf{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_2
\end{aligned} \tag{2.1}$$

So the following commutes:

$$\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{t}_{\epsilon,A,B}"] \arrow[dr, "\pi_2"] & T_\epsilon(A \times B) \arrow[d, "T_\epsilon \pi_2"] \\
& T_\epsilon B
\end{tikzcd}$$

2.3.4 Commutativity with Join

$$\begin{tikzcd}
A \times T_{\epsilon_1} T_{\epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2} B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1} T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A\times B}"] \\
& A \times T_{\epsilon_1 \cdot \epsilon_2} B \arrow[r, "\mathtt{t}_{\epsilon_1 \cdot \epsilon_2,A,B}"] & T_{\epsilon_1 \cdot \epsilon_2}(A \times B)
\end{tikzcd}$$

2.4 Commutativity with Unit

$$\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] \\
& T_\epsilon(A \times B)
\end{tikzcd}$$

2.5 Commutativity with α

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$\begin{tikzcd}
(A \times B) \times T_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,T_\epsilon C}"] & & T_\epsilon((A \times B) \times C) \arrow[d, "T_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times T_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times T_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & T_\epsilon(A \times (B \times C))
\end{tikzcd}$$

TODO: Needed?

2.6 Sub-effecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

2.6.1 Sub-effecting and Tensor Strength

$$\begin{tikzcd}
A \times T_{\epsilon_1} B \arrow[r, "\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2} B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "[\![\epsilon_1 \leq \epsilon_2]\!]_{A\times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}$$

2.6.2 Sub-effecting and Monadic Join

Since the monoid operation on effects is monotone, we can introduce the following diagram.

$$\begin{tikzcd}
T_{\epsilon_1} T_{\epsilon_2} \arrow[r, "T_{\epsilon_1} \llbracket \epsilon_2 \leq \epsilon_2' \rrbracket_M"] \arrow[d, "\mu_{\epsilon_1, \epsilon_2},"] & T_{\epsilon_1} T_{\epsilon_2'} \arrow[r, "\llbracket \epsilon_1 \leq \epsilon_1' \rrbracket_{M, T_{\epsilon_2'}}"] & T_{\epsilon_1'} T_{\epsilon_2'} \arrow[d, "\mu_{\epsilon_1', \epsilon_2'},"] \\
T_{\epsilon_1 \cdot \epsilon_2} \arrow[rr, "\llbracket \epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \epsilon_2' \rrbracket_M"] & & T_{\epsilon_1' \cdot \epsilon_2'}
\end{tikzcd}$$

2.7 Sub-typing

The denotation of ground types $\llbracket \_ \rrbracket_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function sub-typing to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)}\frac{f = \llbracket A' \leq: A \rrbracket_M \quad g = \llbracket B \leq: B' \rrbracket_M \quad h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket}{rhs = \llbracket A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B' \rrbracket_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned} rhs =& (h_{B'} \circ T_{\epsilon_1} g)^{A'} \circ (T_{\epsilon_1} B)^f \\ =& \mathtt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_{\epsilon_1} B^{A'}} \times f)) \end{aligned} \tag{2.2}$$

Chapter 3

Denotations

3.1 Helper Morphisms

3.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\tau_{A,B} : (A \times B) \to (B \times A) = \langle \pi_2, \pi_1 \rangle \tag{3.1}$$
$$\delta_A : A \to (A \times A) = \langle \mathtt{Id}_A, \mathtt{Id}_A \rangle \tag{3.2}$$

3.2 Denotations of Types

3.2.1 Denotation of Ground Types

3.2.2 Denotation of Polymorphic Types

3.2.3 Denotation of Computation Type

3.2.4 Denotation of Function Types

3.2.5 Denotation of Type Environments

3.2.6 Denotation of Value Terms

3.2.7 Denotation of Computation Terms

Chapter 4

Unique Denotations

4.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

4.2 Reduced Type Derivations are Unique

For each instance of the relation $\Phi \mid \Gamma \vdash v\colon A$,there exists at most one reduced derivation of $\Phi \mid \Gamma \vdash v\colon A$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

Proof: We induct on the structure of terms.

Case Variables: To find the unique derivation of $\Phi \mid \Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$: Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is, if $A' \leq:_\Phi A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\frac{\Phi \vdash \Gamma', x : A' \, \mathtt{Ok}}{\Phi \mid \Gamma, x : A' \vdash x : A'} \quad A' \leq: A}{\Phi \mid \Gamma', x : A' \vdash x\colon A} \tag{4.1}$$

Case $\Gamma = \Gamma', y : B$: with $y \neq x$.

Hence, if $\Phi \mid \Gamma \vdash x\colon A$ holds, then so must $\Phi \mid \Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi \mid \Gamma' \vdash x : A'} \quad A' \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{4.2}$$

Be the unique reduced derivation of $\Phi \mid \Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Phi \mid \Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\frac{()\frac{\Delta}{\Phi \mid \Gamma, x : A' \vdash x : A'}}{\Phi \mid \Gamma \vdash x : A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x\colon A} \tag{4.3}$$

Case Constants: For each of the constants, ($\mathtt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Phi \mid \Gamma \vdash c \colon A$ for a given A. I shall give examples using the case $\mathtt{C}^A$

$$\text{(Subtype)}\frac{\text{(Const)}\frac{\Gamma\,\mathtt{Ok}}{\Gamma\vdash \mathtt{C}^A : A} \quad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash \mathtt{C}^A \colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq:_{\Phi} A$).

Case Lambda: The reduced derivation of $\Phi \mid \Gamma \vdash \lambda x : A.v \colon A' \to B'$ is:

$$\text{(Subtype)}\frac{\text{(Lambda)}\frac{()\frac{\Delta}{\Phi\mid\Gamma,x:A\vdash v:B}}{\Phi\mid\Gamma\vdash\lambda x:A.B:A\to B} \quad A \to B \leq:_{\Phi} A' \to B'}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A' \to B'}$$

Where

$$\text{(Sub-Type)}\frac{()\frac{\Delta}{\Phi\mid\Gamma,x:A\vdash v:B} \quad B \leq:_{\Phi} B'}{\Phi \mid \Gamma, x : A \vdash v \colon B'} \tag{4.4}$$

is the reduced derivation of $\Phi \mid \Gamma, x : A \vdash v \colon B'$ if it exists.

Case Return: The reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{return} v \colon \mathtt{M}_{\epsilon} B$ is

$$\text{(Subtype)}\frac{\text{(Return)}\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v:A}}{\Phi\mid\Gamma\vdash\mathtt{return} v:\mathtt{M}_{\mathbf{1}} A} \quad \text{(Computation)}\frac{A\leq:_{\Phi}B \quad \mathbf{1}\leq_{\Phi}\epsilon}{\mathtt{M}_{\mathbf{1}}A\leq_{\Phi}\mathtt{M}_{\epsilon}B}}{\Phi \mid \Gamma \vdash \mathtt{return} v \colon B}$$

Where

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v:A} \quad A \leq: B}{\Phi \mid \Gamma \vdash v \colon B}$$

is the reduced derivation of $\Phi \mid \Gamma \vdash v \colon B$

Case Apply: If

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v_1:A\to B} \quad A \to B \leq: A' \to B'}{\Phi \mid \Gamma \vdash v_1 \colon A' \to B'}$$

and

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Phi\mid\Gamma\vdash v_2:A''} \quad A'' \leq: A'}{\Phi \mid \Gamma \vdash v_2 \colon A'}$$

Are the reduced type derivations of $\Phi \mid \Gamma \vdash v_1 \colon A' \to B'$ and $\Phi \mid \Gamma \vdash v_2 \colon A'$
Then we can construct the reduced derivation of $\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subtype)}\frac{\text{(Apply)}\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v_1:A\to B} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Phi\mid\Gamma\vdash v:A''} \quad A''\leq:_{\Phi}A}{\Phi\mid\Gamma\vdash v:A}}{\Phi\mid\Gamma\vdash v_1\ v_2:B} \quad \text{(Computation)}\frac{B\leq:_{\Phi}B' \quad \epsilon\leq_{\Phi}\epsilon'}{\mathtt{M}_{\epsilon}B\leq:_{\Phi}\mathtt{M}_{\epsilon'}B}}{\Phi \mid \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'}$$

Case If: Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi\mid\Gamma\vdash v:B} \quad B \leq: \mathtt{Bool}}{\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}} \tag{4.5}$$

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Phi\mid\Gamma\vdash v_1:A'} \quad A' \leq: A}{\Phi \mid \Gamma \vdash v_1 \colon A} \tag{4.6}$$

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 \colon A''} \quad A'' \leq: A}{\Phi \mid \Gamma \vdash v_2 \colon A} \tag{4.7}$$

Be the unique reduced reduced derivations of $\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$.
Then the only reduced derivation of $\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A$ is:
TODO: Scale this properly

$$\text{(Subtype)}\frac{\text{(If)}\frac{\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:B} \quad B\leq:\mathtt{Bool}}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_1:A'} \quad A'\leq:A}{\Phi|\Gamma\vdash v_1:A} \quad \text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash v_2:A''} \quad A''\leq:A}{\Phi|\Gamma\vdash v_2:A}}{\Phi|\Gamma\vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2:A \quad \epsilon\leq_\Phi\epsilon \quad A\leq:_\Phi A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{4.8}$$

Case Bind: Let

$$\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad \text{(Computation)}\frac{A\leq:_\Phi A' \quad \epsilon_1\leq_\Phi\epsilon_1'}{\mathtt{M}_{\epsilon_1} A\leq:_\Phi\mathtt{M}_{\epsilon_1'} A'}}{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{4.9}$$

$$\text{(Subtype)}\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B\leq:_\Phi B' \quad \epsilon_2\leq_\Phi\epsilon_2'}{\mathtt{M}_{\epsilon_2} B\leq:_\Phi\mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{4.10}$$

Be the respective unique reduced type derivations of the sub-terms.
By weakening, $\Phi \vdash \iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Phi \mid \Gamma, x : A' \vdash v_2 \colon B$, there's also one of $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.

$$\text{(Subtype)}\frac{()\frac{\Delta''}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B\leq:_\Phi B' \quad \epsilon_2\leq_\Phi\epsilon_2'}{\mathtt{M}_{\epsilon_2} B\leq:_\Phi\mathtt{M}_{\epsilon_2'} B'}}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{4.11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$
Hence the reduced type derivation of $\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:
TODO: Make this and the other smaller

$$\text{(Type)}\frac{\text{(Bind)}\frac{\text{(Subtype)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A} \quad \text{(Computation)}\frac{A\leq:_\Phi A' \quad \epsilon_1\leq_\Phi\epsilon_1'}{\mathtt{M}_{\epsilon_1} A\leq:_\Phi\mathtt{M}_{\epsilon_1'} A'}}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1'} A'} \quad \text{(Subtype)}\frac{()\frac{\Delta''}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2} B} \quad \text{(Computation)}\frac{B\leq:_\Phi B' \quad \epsilon_2\leq_\Phi\epsilon_2'}{\mathtt{M}_{\epsilon_2} B\leq:_\Phi\mathtt{M}_{\epsilon_2'} B'}}{\Phi|\Gamma,x:A'\vdash v_2:\mathtt{M}_{\epsilon_2'} B'}}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2} B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v - 2 \colon \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'} \tag{4.12}$$

Case Effect-Fn: The unique reduced derivation of $\Phi \mid \Gamma \vdash \Lambda\alpha.A \colon \forall\alpha.B$
is

$$\text{(Sub-type)}\frac{\text{(Effect-Fn)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi|\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A} \quad \forall\alpha.A \leq_\Phi \forall\alpha.B}{\Phi \mid \Gamma \vdash \Lambda\alpha.B \colon \forall\alpha.B} \tag{4.13}$$

Where

$$\text{(Sub-type)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A} \quad A \leq:_{\Phi,\alpha} B}{\Phi, \alpha \mid \Gamma \vdash v \colon B} \tag{4.14}$$

Is the unique reduced derivation of $\Phi, \alpha \mid \Gamma \vdash v \colon B$

Case Effect-App: The unique reduced derivation of $\Phi \mid \Gamma \vdash v\ \alpha \colon B'$ is

$$(\text{Subtype})\frac{(\text{Effect-App})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A\,[\epsilon/\alpha] \leq:_\Phi B'}{\Phi \mid \Gamma \vdash v\ \alpha \colon B'} \tag{4.15}$$

Where $B\,[\epsilon/\alpha] \leq:_\Phi B'$ and

$$(\text{Subtype})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.B} \quad (\text{Quantification})\frac{A\leq:_{\Phi,\alpha}B}{\forall\alpha.A\leq:_\Phi\forall\alpha.B}}{\Phi \mid \Gamma \vdash v\colon \forall\alpha.B} \tag{4.16}$$

4.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Phi \mid \Gamma \vdash v \colon A$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

Case Constants: For the constants $\texttt{true}, \texttt{false}, \texttt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced.

$\mathit{reduce}((\text{Const})\frac{\Phi\vdash\Gamma\texttt{Ok}}{\Phi|\Gamma\vdash\texttt{C}^A:A}) = (\text{Const})\frac{\Phi\vdash\Gamma\texttt{Ok}}{\Phi|\Gamma\vdash\texttt{C}^A:A}$

Case Var:

$$\mathit{reduce}((\text{Var})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A}) = (\text{Var})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \mid \Gamma, x : A \vdash x \colon A} \tag{4.17}$$

Case Weaken:

reduce **definition** To find:

$$\mathit{reduce}((\text{Weaken})\frac{()\frac{\Delta}{\Phi|\Gamma\vdash x:A}}{\Phi \mid \Gamma, y : B \vdash x \colon A}) \tag{4.18}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x \colon A} = \mathit{reduce}(\Delta) \tag{4.19}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\frac{()\frac{\Delta'}{\Phi|\Gamma\vdash x:A'}}{\Phi|\Gamma,y:B\vdash x:A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma, y : B \vdash x \colon A} \tag{4.20}$$

Case Lambda:

reduce **definition** To find:

$$\mathit{reduce}((\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v \colon A \to \epsilon_2 B}) \tag{4.21}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi|\Gamma,x:A\vdash v:B'} \quad B' \leq:_\Phi B}{\Phi \mid \Gamma, x : A \vdash v \colon B} = \mathit{reduce}(\Delta) \tag{4.22}$$

In

$$(\text{Sub-type})\frac{(\text{Fn})\frac{\Delta'}{\Phi \mid \Gamma, x{:}A \vdash v{:}\mathtt{M}_{\epsilon_1} B'} \quad A \to \epsilon_1 B' \leq:_\Phi A \to \epsilon_2 B}{\Phi \mid \Gamma \vdash \lambda x : A.v{:}\, A \to \epsilon_2 B} \tag{4.23}$$

Case Subtype:

reduce **definition** To find:

$$reduce((\text{Subtype})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v{:}A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v{:}\, B}) \tag{4.24}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash x{:}A} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash x{:}\, A} = reduce(\Delta) \tag{4.25}$$

In

$$(\text{Subtype})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v{:}A'} \quad A' \leq:_\Phi A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v{:}\, B} \tag{4.26}$$

Case Return:

reduce **definition** To find:

$$reduce((\text{Return})\frac{()\frac{\Delta}{\Phi \mid \Gamma \vdash v{:}A}}{\Phi \mid \Gamma \vdash \mathtt{return} v{:}\, \mathtt{M}_{\mathtt{1}} A}) \tag{4.27}$$

Let

$$(\text{Sub-type})\frac{()\frac{\Delta'}{\Phi \mid \Gamma \vdash v{:}A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v{:}\, A} = reduce(\Delta) \tag{4.28}$$

In

$$(\text{Sub-type})\frac{(\text{Return})\frac{\Delta'}{\Phi \mid \Gamma \vdash v{:}A} \quad (\text{Computation})\frac{\mathtt{1} \leq_\Phi \mathtt{1} \quad A' \leq:_\Phi A}{\mathtt{M}_{\mathtt{1}} A' \leq:_\Phi \mathtt{M}_{\mathtt{1}} A}}{\Phi \mid \Gamma \vdash \mathtt{return} v{:}\, \mathtt{M}_{\mathtt{1}} A} \tag{4.29}$$

Case Apply:

reduce **definition** To find:

$$reduce((\text{Apply})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A \to B} \quad ()\frac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1\ v_2{:}\, B}) \tag{4.30}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 : A' \to B'} \quad A' \to B' \leq:_\Phi A \to \epsilon B}{\Phi \mid \Gamma \vdash v_1{:}\, A \to B} = reduce(\Delta_1) \tag{4.31}$$

$$(\text{Subtype})\frac{()\frac{\Delta_2'}{\Phi \mid \Gamma \vdash v{:}A'} \quad A' \leq:_\Phi A}{\Phi \mid \Gamma \vdash v_1{:}\, A} = reduce(\Delta_2) \tag{4.32}$$

In

$$(\text{Subtype})\frac{(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi \mid \Gamma \vdash v_1 : A' \to B'} \quad (\text{Sub-type})\frac{()\frac{\Delta_2'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq:_\Phi A \leq:_\Phi A'}{\Phi \mid \Gamma \vdash v_2 : A'}}{\Phi \mid \Gamma \vdash v_1\ v_2 : \mathtt{M}_{\epsilon'} B'} \quad (\text{Computation})\frac{\epsilon' \leq_\Phi \epsilon \quad B' \leq:_\Phi B}{\mathtt{M}_{\epsilon'} B' \leq:_\Phi \mathtt{M}_\epsilon B}}{\Phi \mid \Gamma \vdash v_1\ v_2{:}\, B} \tag{4.33}$$

Case If:

reduce **definition**

$$reduce((\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A}) = (\text{If})\frac{()\frac{reduce(\Delta_1)}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{reduce(\Delta_2)}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{reduce(\Delta_3)}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{4.34}$$

Case Bind:

reduce **definition** To find

$$reduce((\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\texttt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\texttt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2}B}) \tag{4.35}$$

Let

$$(\text{Sub-Type})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\texttt{M}_{\epsilon_1'}A'} \quad (\text{Computation})\frac{\epsilon_1'\leq_\Phi\epsilon_1 \quad A'\leq:_\Phi A}{\texttt{M}_{\epsilon_1'}A'\leq:_\Phi\texttt{M}_{\epsilon_1}A}}{\Phi \mid \Gamma \vdash v_1 \colon \texttt{M}_{\epsilon_1}A} = reduce(\Delta_1) \tag{4.36}$$

Since $\Phi \vdash i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq:_\Phi A$, and by Δ_2, $\Phi \mid (\Gamma, x : A) \vdash v_2 \colon \texttt{M}_{\epsilon_2}B$, there also exists a derivation Δ_3 of $\Phi \mid (\Gamma, x : A') \vdash v_2 \colon \texttt{M}_{\epsilon_2}B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\texttt{M}_{\epsilon_2'}B'} \quad (\text{Computation})\frac{\epsilon_2'\leq_\Phi\epsilon_2 \quad B'\leq:_\Phi B}{\texttt{M}_{\epsilon_2'}B'\leq:_\Phi\texttt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma, x : A' \vdash v_2 \colon \texttt{M}_{\epsilon_2}B} = reduce(\Delta_3) \tag{4.37}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq_\Phi \epsilon_1'$ and $\epsilon_2 \leq_\Phi \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq_\Phi \epsilon_1' \cdot \epsilon_2'$
Then the result of reduction of the whole bind expression is:

$$(\text{Sub-Type})\frac{(\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v_1:\texttt{M}_{\epsilon_1'}A'} \quad ()\frac{\Delta_3'}{\Phi|\Gamma,x:A'\vdash v_2:\texttt{M}_{\epsilon_2'}B'}}{\Phi|\Gamma\vdash\texttt{do}\ x\leftarrow v_1\ \texttt{in}\ v_2:\texttt{M}_{\epsilon_1'\cdot\epsilon_2'}B} \quad (\text{Computation})\frac{\epsilon_1'\cdot\epsilon_2'\leq_\Phi\epsilon_1\cdot\epsilon_2 \quad B'\leq:_\Phi B}{\texttt{M}_{\epsilon_1'\cdot\epsilon_2'}B'\leq:_\Phi\texttt{M}_{\epsilon_1\cdot\epsilon_2}B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{4.38}$$

Case Effect-Fn:

reduce **definition** To find

$$reduce((\text{Effect-Lambda})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A}) \tag{4.39}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi,\alpha|\Gamma\vdash v:A'} \quad A' \leq:_\Phi A}{\Phi, \alpha \mid \Gamma \vdash v \colon A} = reduce(\Delta_1) \tag{4.40}$$

in

$$(\text{Subtype})\frac{(\text{Effect-Fn})\frac{()\frac{\Delta_1'}{\Phi,\alpha|\Gamma\vdash v:A'}}{\Phi|\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}}{\forall\alpha.A'\leq:_\Phi\forall\alpha.A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{4.41}$$

Case Effect-Application:

reduce **definition** To find

$$reduce((\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]}) \tag{4.42}$$

Let

$$(\text{Subtype})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v:\forall\alpha.A'} \quad (\text{Quantification})\frac{A'\leq:_{\Phi,\alpha}A}{\forall\alpha.A'\leq:_{\Phi}\forall\alpha.A}}{\Phi \mid \Gamma\vdash v\colon\forall\alpha.A} = reduce(\Delta_1) \tag{4.43}$$

In

$$(\text{Subtype})\frac{(\text{E-app})\frac{()\frac{\Delta_1'}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]} \quad A'\,[\epsilon/\alpha]\leq:_{\Phi} A\,[\epsilon/\alpha]}{\Phi \mid \Gamma\vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{4.44}$$

4.4 Denotations are Equivalent

For each type relation instance $\Phi \mid \Gamma \vdash v\colon A$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!]_M = [\![reduce\Delta]\!]_M = [\![reduce\Delta']\!]_M = [\![\Delta']\!]_M$, hence the denotation $[\![\Phi \mid \Gamma \vdash v\colon A]\!]_M$ is unique.

Chapter 5

Weakening

5.1 Effect Weakening Definition

Introduce a relation $\omega : \Phi' \triangleright \Phi$ relating effect-environments.

5.1.1 Relation

- (Id) $\frac{\Phi\texttt{Ok}}{\iota:\Phi\triangleright\Phi}$
- (Project) $\frac{\omega:\Phi'\triangleright\Phi}{\omega\pi:(\Phi',\alpha)\triangleright\Phi}$
- (Extend) $\frac{\omega:\Phi'\triangleright\Phi}{\omega\times:(\Phi',\alpha)\triangleright(\Phi,\alpha)}$

5.1.2 Weakening Properties

5.1.3 Effect Weakening Preserves Ok

$$\omega : \Phi' \triangleright \Phi \wedge \Phi\texttt{Ok} \Leftarrow \Phi'\texttt{Ok} \tag{5.1}$$

Proof

Case: ι

$$\Phi\texttt{Ok} \wedge \iota : \Phi \triangleright \Phi \Leftarrow \Phi\texttt{Ok}$$

Case: $\omega\pi$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{5.2}$$

So, by induction, $\Phi'\texttt{Ok}$ and hence $(\Phi', \alpha)\texttt{Ok}$

Case: $\omega\times$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{5.3}$$

So

$$\begin{aligned}
(\Phi, \alpha)\texttt{Ok} &\Rightarrow \Phi\texttt{Ok} && (5.4)\\
&\Rightarrow \Phi'\texttt{Ok} && (5.5)\\
&\Rightarrow (\Phi', \alpha)\texttt{Ok} && (5.6)\\
& && (5.7)
\end{aligned}$$

5.1.4 Domain Lemma

$$\omega : \Phi' \triangleright \Phi \Rightarrow (\alpha \notin \Phi \Rightarrow \alpha \notin \Phi')$$

Proof By trivial Induction.

5.1.5 Weakening Preserves Effect Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ then $\Phi \vdash \epsilon \implies \Phi' \vdash \epsilon$

Proof By induction over the well-formed-ness of effects

Case Ground By inversion, $\Phi \texttt{Ok} \wedge \epsilon \in E$. Hence by the ok-property, $\Phi' \texttt{Ok}$ So $\Phi' \vdash \epsilon$

Case Var $\Phi = \Phi'', \alpha$
So either:

Case: $\Phi' = \Phi''', \alpha$ So $\omega = \omega' \times$ So $\omega' : \Phi''' \triangleright \Phi''$, and hence:

$$(\text{Var})\frac{\Phi''', \alpha \texttt{Ok}}{\Phi''', \alpha \vdash \alpha} \tag{5.8}$$

Case: $\Phi' = \Phi''', \beta$ and $\beta \neq \alpha$
So $\omega = \omega' \pi$
By induction, $\omega' : \Phi''' \triangleright \Phi$ so

$$(\text{Weaken})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{5.9}$$

Case Weaken By inversion, $\Phi = \Phi'', \beta$.
So $\omega = \omega' \times$
And, $\Phi' = \Phi''', \beta$ So By inversion$\omega' : \Phi''' \triangleright \pi_1''$
So by induction

$$(\text{weak})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{5.10}$$

Case Monoid By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$. So by induction, $\Phi' \vdash \epsilon_1$ and $\Phi' \vdash \epsilon_2$, and so:

$$\Phi' \vdash \epsilon_1 \cdot \epsilon_2 \tag{5.11}$$

5.1.6 Weakening Preserves Type-Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ and $\Phi \vdash A$ then $\Phi' \vdash A$.

Proof:

Case Ground: By inversion, $\Phi \texttt{Ok}$, hence by property 1 of weakening, $\Phi' \texttt{Ok}$. Hence $\Phi' \vdash \gamma$.

Case Function: By inversion, $\Phi \vdash A$, $\Phi \vdash B$. So by induction $\Phi' \vdash A$, $\Phi' \vdash B$, hence,

$$\Phi' \vdash A \to B$$

Case Computation: By inversion$\Phi \vdash A$, and $\Phi \vdash \epsilon$.
So by induction and the effect-well-formed-ness theorem,
$\Phi' \vdash A$ and $\Phi' \vdash \epsilon$
So

$$\Phi' \vdash \mathtt{M}_{\epsilon} A$$

Case For All: By inversion, $\Phi, \alpha \vdash A$ Picking $\alpha \notin \Phi'$ using α-conversion.
So $\omega\times : (\Phi', \alpha) \triangleright (\Phi, \alpha)$
So $(\Phi', \alpha) \vdash A$
So $\Phi \vdash \forall \alpha.A$

5.1.7 Corollary

$$\omega : \Phi' \triangleright \Phi \wedge \Phi \vdash \Gamma\mathtt{Ok} \implies \Phi' \vdash \Gamma\mathtt{Ok}$$

Case Nil: By inversion$\Phi\mathtt{Ok}$ so $\Phi \vdash \diamond\mathtt{Ok}$

Case Var: By inversion$\Phi \vdash \Gamma\mathtt{Ok}$, $x \in \mathtt{dom}(\Gamma)$, $\Phi \vdash A$
So by induction $\Phi' \vdash \Gamma\mathtt{Ok}$, and $\pi_1' \vdash \Gamma\mathtt{Ok}$
So $\Phi' \vdash (\Gamma, x : A)\mathtt{Ok}$

5.1.8 Effect Weakening preserves Type Relations

$$\Phi \mid \Gamma \vdash v \colon A \wedge \omega : \Phi' \triangleright \Phi \implies \Phi' \mid \Gamma \vdash v \colon A \tag{5.12}$$

Proof:

Case Constants: If $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi' \vdash \Gamma\mathtt{Ok}$ so:

$$(\text{Const})\frac{\Phi' \vdash \Gamma\mathtt{Ok}}{\Phi' \mid \Gamma \vdash \mathtt{C}^A \colon A} \tag{5.13}$$

Case Variables: If $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi' \vdash \Gamma\mathtt{Ok}$ so: So, $\Phi' \mid G \vdash x \colon A$, if $\Phi \mid G \vdash x \colon A$

Case Lambda: By inversion, $\Phi \mid \Gamma, x : A \vdash v \colon B$, so by induction $\Phi' \mid \Gamma, x : A \vdash v \colon B$.
So,

$$\Phi' \mid \Gamma \vdash \lambda x : A.v \colon A \to B \tag{5.14}$$

Case Apply: By inversion$\Phi \mid \Gamma \vdash v_1 \colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 \colon A$.
Hence by induction, $\Phi' \mid \Gamma \vdash v_1 \colon A \to B$ and $\Phi' \mid \Gamma \vdash v_2 \colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{app} v_1 v_2 \colon B$$

Case Return: By inversion$\Phi \mid \Gamma \vdash v \colon A$
So by induction $\Phi' \mid \Gamma \vdash v \colon A$
Hence $\Phi' \mid \Gamma \vdash \mathtt{return} v \colon \mathtt{M}_{\mathbf{1}} A$

Case Bind: By inversion$\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash \epsilon_2 \colon \mathtt{M}_{\epsilon_2} A$.
Hence by induction $\Phi' \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi' \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B \tag{5.15}$$

Case If: By inversion$\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, and $\Phi \mid \Gamma \vdash v_2 \colon A$.
Hence by induction $\Phi' \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi' \mid \Gamma \vdash v_1 \colon A$, and $\Phi' \mid \Gamma \vdash v_2 \colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \tag{5.16}$$

Case Subtype: By inversion$\Phi \mid \Gamma \vdash v \colon A$, and $A \leq: B$.
So by induction: $\Phi' \mid \Gamma \vdash v \colon A$, and $A \leq: B$.
So

$$\Phi' \mid \Gamma \vdash v \colon B \tag{5.17}$$

Case Effect-Lambda: By inversion$\Phi, \alpha \mid \Gamma \vdash v \colon A$
By picking $\alpha \notin \Phi'$ using α-conversion.

$$\omega \times : \Phi', \alpha \triangleright \Phi, \alpha \tag{5.18}$$

So by induction, $\Phi', \alpha \mid \Gamma \vdash v \colon A$
Hence,

$$\Phi' \mid \Gamma \vdash \Lambda\alpha.v \colon \forall a.A \tag{5.19}$$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v \colon \forall\alpha.A$, and $\Phi \vdash \epsilon$.
So by induction, $\Phi' \mid \Gamma \vdash v \colon \forall\alpha.A$
And by the well-formed-ness-theorem $\Phi' \vdash \epsilon$
Hence,

$$\Phi' \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha] \tag{5.20}$$

5.2 Type Environment Weakening

5.2.1 Relation

We define the ternary weakening relation $\Phi \vdash w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Phi \vdash \Gamma \mathtt{Ok}}{\Phi \vdash \iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma') \quad A \leq: B}{\Phi \vdash w \times : \Gamma', x : A \triangleright \Gamma, x : B}$

5.2.2 Domain Lemma

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, then $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$.

Proof:

Case Id: Then $\Gamma' = \Gamma$ and so $\mathtt{dom}(\Gamma') = \mathtt{dom}(\Gamma)$.

Case Project: By inversion and induction, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma') \subseteq \texttt{dom}(\Gamma' \cup \{x\})$

Case Extend: By inversion and induction, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$ so

$$\texttt{dom}(\Gamma, x : A) = \texttt{dom}(\Gamma) \cup \{x\} \subseteq \texttt{dom}(\Gamma') \cup \{x\} = \texttt{dom}(\Gamma', x : A)$$

5.2.3 Theorem 1

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ and $\Phi \vdash \Gamma\texttt{Ok}$ then $\Phi \vdash \Gamma'\texttt{Ok}$

Proof:

Case Id:

$$(\text{Id})\frac{\Phi \vdash \Gamma\texttt{Ok}}{\Phi \vdash \iota : \Gamma \triangleright \Gamma}$$

By inversion, $\Phi \vdash \Gamma\texttt{Ok}$.

Case Project:

$$(\text{Project})\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ and $x \notin \texttt{dom}(\Gamma')$.
Hence by induction $\Phi \vdash \Gamma'\texttt{Ok}$, $\Phi \vdash \Gamma\texttt{Ok}$. Since $x \notin \texttt{dom}(\Gamma')$, we have $\Phi \vdash \Gamma', x : A\texttt{Ok}$.

Case Extend: $(\text{Extend})\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \texttt{dom}(\Gamma') \quad A \leq: B}{\Phi \vdash w\times : \Gamma', x : A \triangleright \Gamma, x : B}$,
By inversion, we have
$\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, $x \notin \texttt{dom}(\Gamma')$.
Hence we have $\Phi \vdash \Gamma\texttt{Ok}$, $\Phi \vdash \Gamma'\texttt{Ok}$, and by the domain Lemma, $\texttt{dom}(\Gamma) \subseteq \texttt{dom}(\Gamma')$, hence $x \notin \texttt{dom}(\Gamma)$. Hence, we have $\Phi \vdash \Gamma, x : A\texttt{Ok}$ and $\Phi \vdash \Gamma', x : A\texttt{Ok}$

5.2.4 Theorem 2

If $\Phi \mid \Gamma \vdash t\colon \tau$ and $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Phi \mid \Gamma' \vdash t\colon \tau$

Proof: We induct over the structure of typing derivations of $\Phi \mid \Gamma \vdash t\colon \tau$, assuming $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ holds.

Case Var and Weaken: We case split on the weakening ω.

Case: $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Phi \mid \Gamma' \vdash x\colon A$ holds and the derivation Δ' is the same as Δ

Case: $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Phi \mid \Gamma'' \vdash x\colon A$, such that:

$$(\text{Weaken})\frac{()\frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x : A}}{\Phi \mid \Gamma'', x' : A' \vdash x\colon A} \tag{5.21}$$

Case: $\omega = \omega'\times$ Then

$$\Gamma' = \Gamma''', x' : B \tag{5.22}$$
$$\Gamma = \Gamma'', x' : A' \tag{5.23}$$
$$B \leq: A \tag{5.24}$$

Case: $x = x'$ Then $A = A'$.
Then we derive the new derivation, Δ' as so:

$$\text{(Sub-type)}\frac{\text{(var)}\frac{}{\Phi|\Gamma''',x:B\vdash x:B} \quad B \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{5.25}$$

Case: $x \neq x'$ Then

$$\Delta = \text{(Weaken)}\frac{()\frac{\Delta_1}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma \vdash x\colon A} \tag{5.26}$$

By induction with $\Phi \vdash \omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Phi \mid \Gamma''' \vdash x\colon A$
We have the weakened derivation:

$$\Delta' = \text{(Weaken)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'''\vdash x:A}}{\Phi \mid \Gamma' \vdash x\colon A} \tag{5.27}$$

Case Constant: The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$\text{(Const)}\frac{\Gamma\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A\colon A} \tag{5.28}$$

By inversion, we have $\Phi \vdash \Gamma\mathtt{Ok}$, so we have $\Phi \vdash \Gamma'\mathtt{Ok}$.
Hence

$$\text{(Const)}\frac{\Phi \vdash \Gamma'\mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A\colon A} \tag{5.29}$$

Holds.

Case Lambda: By inversion, we have a derivation Δ_1 giving

$$\Delta = \text{(Fn)}\frac{()\frac{\Delta_1}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{5.30}$$

Since $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we have:

$$\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{5.31}$$

Hence, by induction, using $\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ_1':

$$\Delta' = \text{(Fn)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma',x:A\vdash v:B}}{\Phi \mid \Gamma', x : A \vdash \lambda x : A.v\colon A \to B} \tag{5.32}$$

Case Sub-typing:

$$\text{(Sub-type)}\frac{\Phi \mid \Gamma \vdash v\colon A \quad A \leq: B}{\Phi \mid \Gamma \vdash v\colon B} \tag{5.33}$$

by inversion, we have a derivation Δ_1

$$()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v\colon A} \tag{5.34}$$

So by induction, we have a derivation Δ_1' such that:

$$\text{(Sub-type)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:a} \quad A \leq: B}{\Phi \mid \Gamma' \vdash v\colon B} \tag{5.35}$$

Case Return: We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return}v\colon \mathtt{M_1}A} \tag{5.36}$$

Hence, by induction, with $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ_1' such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \mathtt{return}v\colon \mathtt{M_1}A} \tag{5.37}$$

Case Apply: By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{5.38}$$

By induction, this gives us the respective derivations: Δ_1', Δ_2' such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash v_1\ v_2\colon B} \tag{5.39}$$

Case If: By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A} \tag{5.40}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\mathtt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1:A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2\colon A} \tag{5.41}$$

Case Bind: By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\mathbb{E}_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{5.42}$$

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then $\Phi \vdash \omega\times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ_1', Δ_2' such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:\mathtt{M}_{\mathbb{E}_1}A} \quad ()\frac{\Delta_2'}{\Phi|\Gamma',x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma' \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2\colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{5.43}$$

Case Effect-Abstraction: By inversion,we have derivation Δ_1 deriving

$$(\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{5.44}$$

By α-conversion, we have $\iota\pi : \Phi, \alpha \triangleright \Phi$, So we have $\Phi, \alpha \vdash \omega : \Gamma' \triangleright \Gamma$ so by induction, there exists Δ_1 deriving:

$$\Delta' = (\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{5.45}$$

Case Effect-Application: By inversionwe have derivation Δ_1 deriving

$$\text{(Effect-App)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{5.46}$$

So by induction, we have Δ_1' deriving

$$\text{(Effect-App)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma' \vdash v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{5.47}$$

Chapter 6

Substitution

We need to define substitutions of effects on effects, effects on types, effects on terms, terms on terms.

6.1 Effect Substitutions

Define a substitution, σ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \tag{6.1}$$

Define the free-effect Variables of σ:

$$\begin{aligned} fev(\diamond) &= \emptyset \\ fev(\sigma, \alpha := \epsilon) &= fev(\sigma) \cup fev(\epsilon) \end{aligned}$$

We define the property:

$$\alpha \# \sigma \Leftrightarrow \alpha \notin (\mathtt{dom}(\sigma) \cup fev(\sigma)) \tag{6.2}$$

6.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \tag{6.3}$$

$$\sigma(e) = e \tag{6.4}$$

$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \tag{6.5}$$

$$\diamond(\alpha) = \alpha \tag{6.6}$$

$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \tag{6.7}$$

$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \tag{6.8}$$

6.1.2 Action of Effect Substitution on Types

Define the action of applying an effect substitution, σ to a type τ as:

$$\tau\,[\sigma]$$

Defined as so

$$\gamma\,[\sigma] = \gamma \tag{6.9}$$
$$(A \to \mathtt{M}_\epsilon B)\,[\sigma] = (A\,[\sigma]) \to \mathtt{M}_{\sigma(\epsilon)}(B\,[\sigma]) \tag{6.10}$$
$$(\mathtt{M}_\epsilon A)\,[\sigma] = \mathtt{M}_{\sigma(\epsilon)}(A\,[\sigma]) \tag{6.11}$$
$$(\forall\alpha.A)\,[\sigma] = \forall\alpha.(A\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{6.12}$$

6.1.3 Action of Effect-Substitution on Type Environments

Define the action of effect substitution on type environments:

$$\Gamma\,[\sigma]$$

Defined as so:

$$\diamond\,[\sigma] = \diamond$$
$$(\Gamma, x : A)\,[\sigma] = (\Gamma\,[\sigma]\,, x : (A\,[\sigma]))$$

6.1.4 Action of Effect Substitution on Terms

Define the action of effect-substitution on terms:

$$x\,[\sigma] = x \tag{6.13}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^{(A[\sigma])} \tag{6.14}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : (A\,[\sigma]).(C\,[\sigma]) \tag{6.15}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\sigma(\epsilon),(A[\sigma])}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{6.16}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{6.17}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \tag{6.18}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{6.19}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \sigma(\epsilon) \tag{6.20}$$
$$\tag{6.21}$$

6.1.5 Well-Formed-ness

For any two effect-environments, and a substitution, define the well-formed-ness relation:

$$\Phi' \vdash \sigma \colon \Phi \tag{6.22}$$

- (Nil) $\frac{\Phi'\mathtt{Ok}}{\Phi' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi' \vdash \sigma:\Phi \quad \Phi' \vdash \epsilon \quad \alpha \notin \Phi}{\Phi' \vdash \sigma, \alpha := \epsilon : (\Phi, \alpha)}$

6.1.6 Property 1

If $\Phi' \vdash \sigma\colon \Phi$ then $\Phi'\mathtt{Ok}$ (By the Nil case) and $\Phi\mathtt{Ok}$ Since each use of the extend case preserves $\mathtt{Ok}$.

6.1.7 Property 2

If $\Phi' \vdash \sigma\colon \Phi$ then $\omega : \Phi' \triangleright \Phi' \implies \Phi'' \vdash \sigma\colon \Phi$ since $\Phi' \vdash \epsilon \implies \Phi'' \vdash \epsilon$ and $\Phi'\mathtt{Ok} \implies \Phi''\mathtt{Ok}$

6.1.8 Property 3

If $\Phi' \vdash \sigma \colon \Phi$ then

$$\alpha \notin \Phi \wedge \alpha \notin \Phi' \implies (\Phi', \alpha) \vdash (\sigma, \alpha := \alpha) \colon (\Phi, \alpha) \tag{6.23}$$

Since $\iota\pi : \Phi', \alpha \triangleright \Phi'$ so $\Phi', \alpha \vdash \sigma \colon \Phi$ and $\Phi', \alpha \vdash \alpha$

6.2 Substitution Preserves the Well-formed-ness of Effects

I.e.

$$\Phi \vdash \epsilon \wedge \Phi' \vdash \iota \colon \Phi \implies \Phi' \vdash \sigma(\epsilon) \tag{6.24}$$

Proof:

Case Ground: $\sigma(e) = e$, so $\Phi' \vdash \sigma(\epsilon)$ holds.

Case Multiply: By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$ so $\Phi' \vdash \sigma(\epsilon_1)$ and $\Phi' \vdash \sigma(\epsilon_2)$ by induction and hence $\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)$

Case Var: By inversion, $\Phi = \Phi'', \alpha$ and $\Phi'', \alpha\mathtt{Ok}$. Hence byu case splitting on ι, we see that $\sigma = \sigma', \alpha := \epsilon$.

So by inversion, $\sigma \vdash \epsilon$ so $\Phi' \vdash \sigma(\alpha) = \epsilon$

Case Weaken: By inversion$\Phi = \Phi'', \beta$ and $\Phi'' \vdash \alpha$, so $\sigma = \sigma'\beta := \epsilon$.

So $\Phi' \vdash \sigma' \colon \Phi''$.

hence by induction, $\Phi' \vdash \sigma'(a)$, so $\Phi' \vdash \sigma(\alpha)$ since $\alpha \neq \beta$)

6.2.1 Effect Substitution preserves the sub-effect relation

If $\Phi' \vdash \sigma \colon \Phi$ and $\epsilon_1 \leq_\Phi \epsilon_2$, then $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

Proof: For any ground substitution σ' of Φ', then $\sigma\sigma'$ (the substitution σ' applied after σ) is also a ground substitution.

So $\epsilon_1 [\sigma] [\sigma'] \leq \epsilon_2 [\sigma] [\sigma']$.

So $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

6.2.2 Substitution preserves well-formed-ness of Types

$$\Phi' \vdash \sigma \colon \Phi \wedge \Phi \vdash A \implies \Phi' \vdash A [\sigma] \tag{6.25}$$

Proof:

Case Ground: $\Phi'\mathtt{Ok}$ so $\Phi' \vdash \gamma$ and $\gamma [\sigma] = \gamma$.

Hence $\Phi' \vdash \gamma [\sigma]$.

Case Lambda: By inversion$\Phi \vdash A$ and $\Phi \vdash B$.

So by induction, $\Phi' \vdash A [\sigma]$ and $\Phi' \vdash B [\sigma]$.

So

$$\Phi' \vdash (A [\sigma]) \to (B [\sigma]) \tag{6.26}$$

So

$$\Phi' \vdash (A \to B) [\sigma] \tag{6.27}$$

Case Computation: By inversion, $\Phi \vdash \epsilon$ and $\Phi \vdash A$ so by induction and substitution of effect preserving effect-well-formed-ness,

$\Phi' \vdash \sigma(\epsilon)$ and $\Phi' \vdash A\,[\sigma]$ so $\Phi \vdash \mathtt{M}_{\sigma(\epsilon)} A\,[\sigma]$ so $\Phi' \vdash (\mathtt{M}_{\epsilon} A)\,[\sigma]$

Case For All: By inversion, $\Phi, \alpha \vdash A$. So by picking $\alpha \notin \Phi \wedge \alpha \notin \Phi'$ using α-equivalence, we have $(\Phi', \alpha) \vdash (\sigma\alpha := \alpha) \colon (\Phi, \alpha)$.

So by induction $(\Phi, \alpha) \vdash A\,[\sigma, \alpha := \alpha]$

So $(\Phi', \alpha) \vdash A\,[\sigma]$

So $\Phi' \vdash (\forall \alpha. A)\,[\sigma]$

6.2.3 Substitution of effects preserves Sub-Typing Relation

If $\Phi' \vdash \sigma \colon \Phi$ and $A \leq:_{\Phi} B$ then $A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$

Proof: By induction on the sub-typing relation

Case Ground: By inversion, $A \leq:_{\gamma} B$, so A, B are ground types. Hence $A\,[\sigma] = A$ and $B\,[\sigma] = B$. So $A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$

Case Fn: By inversion, $A' \leq:_{\Phi} A$ and $B \leq:_{\Phi} B'$.

So by induction, $A'\,[\sigma] \leq:_{\Phi'} A\,[\sigma]$ and $B\,[\sigma] \leq:_{\Phi'} B'\,[\sigma]$.

So $(A\,[\sigma]) \rightarrow (B\,[\sigma]) \leq:_{\Phi'} (A'\,[\sigma]) \rightarrow (B'\,[\sigma])$

So $(A \rightarrow B)\,[\sigma] \leq:_{\Phi'} (A' \rightarrow B')\,[\sigma]$

Case Computation: By inversion, $A \leq:_{\Phi} B$, $\epsilon_1 \leq_{\Phi} \epsilon_2$.

So by induction and substitution preserving the sub-effect relation,

$A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$ and $\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)$

So $\mathtt{M}_{\sigma(\epsilon_1)}(A\,[\sigma]) \leq:_{\Phi'} \mathtt{M}_{\sigma(\epsilon_2)}(B\,[\sigma])$

So $(\mathtt{M}_{\epsilon_1} A)\,[\sigma] \leq:_{\Phi'} (\mathtt{M}_{\epsilon_2} B)\,[\sigma]$

6.2.4 Substitution preserves well-formed-ness of Type Environments

If $\Phi \vdash \Gamma \mathtt{Ok}$ and $\Phi' \vdash \sigma \colon \Phi$ then $\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}$

Proof:

Case Nil: $\Phi\mathtt{Ok} \implies \Phi'\mathtt{Ok}$ so $\Phi' \vdash \diamond\mathtt{Ok}$ and $\diamond\,[\sigma] = \diamond$

Case Var: By inversion, $\Phi \vdash \Gamma\mathtt{Ok}$ and $\Phi \vdash A$.

By induction and substitution preserving well-formed-ness of types, $\Phi' \vdash \Gamma'\,[\sigma]\,\mathtt{Ok}$ and $\Phi' \vdash A\,[\sigma]$.

So $\Phi' \vdash (\Gamma'\,[\sigma], x : A\,[\sigma])\mathtt{Ok}$.

Hence $\Phi' \vdash \Gamma, x : A\,[\sigma]\,\mathtt{Ok}$.

6.2.5 Effect-Polymorphism Preserves the Typing Relation

If $\Phi' \vdash \sigma \colon \Phi$ and $\Phi \mid \Gamma \vdash v \colon A$, then $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]$

Proof:

Case Const: By inversion, $\Phi \vdash \Gamma\mathtt{Ok}$.

So $\Phi' \vdash \Gamma\mathtt{Ok}$

So $\Phi' \mid \Gamma\,[\sigma] \vdash \mathtt{C}^{A[\sigma]} \colon A\,[\sigma]$

Case True, False, Unit: The logic is the same for each of these cases, so we look at the case true only.

By inversion, $\Phi \vdash \Gamma\mathtt{Ok}$.

So $\Phi' \vdash \Gamma\mathtt{Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{true}\colon \mathtt{Bool}$

Since $\mathtt{true}[\sigma] = \mathtt{true}$ and $\mathtt{Bool}[\sigma] = \mathtt{Bool}$.

Case Var: By inversion$\Gamma = \Gamma', x : A$ and $\Phi \vdash \Gamma', x : A\mathtt{Ok}$.

So since substitution preserves well-formed-ness of type environments, $\Phi' \vdash \Gamma'[\sigma], x : A[\sigma]\,\mathtt{Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash x\colon A[\sigma]$

Since $x[\sigma] = x$

Case Weaken: By inversion$\Gamma = \Gamma', y : B$, $\Phi \vdash B$, and $\Phi \mid \Gamma' \vdash x\colon A$. $x \neq y$

By induction and the theorem that effect-substitution preserves type well-formed-ness, we have: $\Phi' \mid \Gamma'[\sigma] \vdash x\colon A[\sigma]$ and $\Phi' \vdash B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash x[\sigma] : A[\sigma]$

Since $x[\sigma] = x$, $\Gamma[\sigma] = (\Gamma'[\sigma], y : B[\sigma])$

Case Lambda: By inversion$\Phi \mid \Gamma, x : A \vdash v\colon B$.

So, by induction $\Phi' \mid (\Gamma, x : A)[\sigma] \vdash v[\sigma] : B[\sigma]$.

So, $\Phi \mid \Gamma[\sigma], x : A[\sigma] \vdash v[\sigma] : B[\sigma]$.

Hence by the lambda type rule,

$\Phi' \mid \Gamma[\sigma] \vdash \lambda x : A[\sigma].v[\sigma] : (A[\sigma]) \to (B[\sigma])$

So

$\Phi' \mid \Gamma[\sigma] \vdash (\lambda x : A.v)[\sigma] : (A \to B)[\sigma])$

Case Apply: By inversion, $\Phi \mid \Gamma \vdash v_1\colon A \to B$, $\Phi \mid \Gamma \vdash V_2\colon A$.

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : (A[\sigma]) \to (B[\sigma])$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1[\sigma])\ (v_2[\sigma])\colon B[\sigma]$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1\ v_2)[\sigma] : (A \to B)[\sigma]$

Case Subtype: By inversion, $\Phi \mid \Gamma \vdash v\colon A$ and $\Phi \vdash A \leq: B$

So by induction and effect-substitution preserving sub-typing, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ and $\Phi' \vdash A[\sigma] \leq: B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : B[\sigma]$

Case Return: By inversion, $\Phi \mid \Gamma \vdash v\colon A$

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{return}(v[\sigma])\colon \mathtt{M}_{\mathbf{1}}(A[\sigma])$

Hence $\Phi' \mid \Gamma[\sigma] \vdash (\mathtt{return} v)[\sigma] : (\mathtt{M}_{\mathbf{1}} A)[\sigma]$

Case Bind: By inversion, $\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B$.

So by induction: $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : \mathtt{M}_{\sigma(\epsilon_1)}(A[\sigma])$, and $\Phi' \mid \Gamma[\sigma], x : A[\sigma] \vdash v_2\colon \mathtt{M}_{\sigma(\epsilon_2)}(B[\sigma])$.

And so $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{do}\ x \leftarrow (v_1[\sigma])\ \mathtt{in}\ (v_2[\sigma])\colon \mathtt{M}_{\sigma(\epsilon_1)\cdot(\epsilon_2[\sigma])} B[\sigma]$

Case If: By inversion, $\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1\colon A$, and $\Phi \mid \Gamma \vdash v_2\colon A$

So by induction $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \mathtt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_1\colon A[\sigma]$, and $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \mathtt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_2\colon A[\sigma]$. (Since $\mathtt{Bool}[\sigma] = \mathtt{Bool}$)

Hence:

$\Phi' \mid \Gamma[\sigma] \vdash \mathtt{if}_{A[\sigma]}\ v[\sigma]\ \mathtt{then}\ v_1[\sigma]\ \mathtt{else}\ v_2[\sigma] : A[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash (\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)[\sigma] : A[\sigma]$

Case Effect-lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v \colon A$.

So by the substitution property 3 (**TODO: Is this correct/reference correctly**), pick $\alpha \notin \Phi' \wedge \alpha \notin \Phi$ so we have:

$$(\Phi', \alpha) \vdash (\sigma, \alpha := \alpha) \colon (\Phi, \alpha)$$

So by induction, $\Phi', \alpha \mid \Gamma\,[\sigma, \alpha := \alpha] \vdash v\,[\sigma, \alpha := \alpha] : A\,[\sigma, \alpha := \alpha]$
So $\Phi', \alpha \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]$ since $\alpha \notin \Phi' \wedge \alpha \notin \Phi$.
So $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : (\forall \alpha.A)\,[\sigma]$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v \colon \forall \alpha.A$, $\Phi \vdash \epsilon$.

So by induction and effect-substitution preserving well-formed-ness of effects: $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : (\forall \alpha.A)\,[\sigma]$ and $\Phi' \vdash \sigma(\epsilon)$
So $\Phi' \mid \Gamma\,[\sigma] \vdash (v\,[\sigma])\ (\sigma(\epsilon)) \colon A\,[\sigma]\,[\sigma(\epsilon)/\alpha]$.
Since $\alpha \# \sigma$, we can commute the applications of substitution. **TODO: Do I need to prove this?**
So, $\Phi' \mid \Gamma\,[\sigma] \vdash (v\ \epsilon)\,[\sigma] : A\,[\epsilon/\alpha]\,[\sigma]$

6.3 The Identity Substitution on Effect Environments

For each type environment Φ, define the identity substitution I_Φ as so:

- $I_\diamond = \diamond$
- $I_{(\Phi,\alpha} = (I_\Phi, \alpha := \alpha)$

6.3.1 Properties of the Identity Substitution

Property 1 If $\Phi\mathtt{Ok}$ then $\Phi \vdash I_\Phi \colon \Phi$, proved trivially by induction over the `Ok` relation.

Property 2 TODO: The denotational property of id-substitution

6.4 Single Substitution on Effect Environments

If $\Phi \vdash \epsilon$, let the single substitution $\Phi \vdash [\epsilon/\alpha] : \Phi, \alpha$, be defined as:

$$[x/\alpha] = (I_\Phi, \alpha := \epsilon) \tag{6.28}$$

6.5 Term-Term Substitutions

6.5.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{6.29}$$

6.5.2 Trivial Properties of substitutions

$\mathtt{fv}(\sigma)$

$$\mathtt{fv}(\diamond) = \emptyset \tag{6.30}$$

$$\mathtt{fv}(\sigma, x := v) = \mathtt{fv}(\sigma) \cup \mathtt{fv}(v) \tag{6.31}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{6.32}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{6.33}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6.34}$$

6.5.3 Action of substitutions

We define the action of applying a substitution σ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{6.35}$$
$$x\,[\sigma, x := v] = v \tag{6.36}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{6.37}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.38}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.39}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{6.40}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{6.41}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.42}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \tag{6.43}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \epsilon \tag{6.44}$$
$$\tag{6.45}$$

6.5.4 Well-Formed-ness

Define the relation

$$\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil) $\frac{\Phi \vdash \Gamma' \mathtt{Ok}}{\Phi \mid \Gamma' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi \mid \Gamma' \vdash \sigma : \Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Phi \mid \Gamma' \vdash v : A}{\Phi \mid \Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

6.5.5 Simple Properties Of Substitution

If $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$ then: **TODO: Number these**

Property 1: $\Phi \vdash \Gamma\mathtt{Ok}$ and $\Phi \vdash \Gamma'\mathtt{Ok}$ Since $\Phi \vdash \Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Phi \vdash \Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Phi \mid \Gamma'' \vdash \sigma \colon \Gamma$. By induction over well-formed-ness relation. For each $x := v$ in σ, $\Phi \mid \Gamma'' \vdash v \colon A$ holds if $\Phi \mid \Gamma' \vdash v \colon A$ holds.

Property 3: $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ **implies** $\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Phi \mid \Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Phi \mid \Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{6.46}$$

6.5.6 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$(\Phi \mid \Gamma \vdash v \colon A) \wedge (\Phi \mid \Gamma' \vdash \sigma \colon \Gamma) \Rightarrow (\Phi \mid \Gamma' \vdash v\,[\sigma] : A) \tag{6.47}$$

Assuming $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Phi \mid \Gamma \vdash v \colon A \implies \Phi \mid \Gamma' \vdash v \colon A$

Proof:

Case Var: By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Phi \mid \Gamma'', x : A \vdash x \colon A \tag{6.48}$$

So by inversion, since $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma'', x : A$,

$$\sigma = (\sigma', x := v) \wedge \Phi \mid \Gamma' \vdash v \colon A \tag{6.49}$$

By the definition of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{6.50}$$

holds.

Case Weaken: By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists Δ such that

$$(\text{Weaken}) \frac{() \frac{\Delta}{\Phi \mid \Gamma'' \vdash x : A}}{\Phi \mid \Gamma'', y : B \vdash x \colon A} \tag{6.51}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Phi \mid \Gamma' \vdash \sigma' \colon \Gamma'' \tag{6.52}$$

So by induction,

$$\Phi \mid \Gamma' \vdash x\,[\sigma'] : A \tag{6.53}$$

And so by definition of the effect of σ, $x\,[\sigma] = x\,[\sigma']$

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{6.54}$$

Case Lambda: By inversion, there exists Δ such that:

$$(\text{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{6.55}$$

Using alpha equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$ Hence, by property 3, we have

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x)\colon \Gamma, x : A \tag{6.56}$$

So by induction using $\sigma, x := x$, we have Δ' such that:

$$(\text{Fn})\frac{()\frac{\Delta'}{\Phi|\Gamma',x:A\vdash v[\sigma,x:=v]:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\,[\sigma, x := x] : A \to B} \tag{6.57}$$

Since $\lambda x : A.(v\,[\sigma, x := x]) = \lambda x : A.(v\,[\sigma]) = (\lambda x : A.v)\,[\sigma]$, we have a typing derivation for $\Phi \mid \Gamma' \vdash (\lambda x : A.v)\,[\sigma] : A \to B$.

Case Constants: We use the same logic for all constants, $\texttt{()}, \texttt{true}, \texttt{false}, \texttt{C}^A$:
$\Phi \mid \Gamma \vdash \sigma\colon \Gamma \Rightarrow \Phi \vdash \Gamma'\texttt{Ok}$ and:

$$\texttt{C}^A\,[\sigma] = \texttt{C}^A \tag{6.58}$$

So

$$(\text{Const})\frac{\Phi \vdash \Gamma'\texttt{Ok}}{\Phi \mid \Gamma' \vdash \texttt{C}^A\colon A} \tag{6.59}$$

6.5.7 Computation Terms

Case Return: By inversion, we have Δ_1 such that:

$$(\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{6.60}$$

By induction, we have Δ_1' such that

$$(\text{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \texttt{return}(v\,[\sigma])\colon \texttt{M}_1 A} \tag{6.61}$$

Since $(\texttt{return}v)\,[\sigma] = \texttt{return}(v\,[\sigma])$, the type derivation above holds for $\Phi \mid \Gamma' \vdash (\texttt{return}v)\,[\sigma] : \texttt{M}_1 A$.

Case Apply: By inversion, we have Δ_1, Δ_2 such that:

$$(\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{6.62}$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that

$$(\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (v_1\,[\sigma])\ (v_2\,[\sigma])\colon B} \tag{6.63}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Phi \mid \Gamma' \vdash (v_1\ v_2)\,[\sigma] : B$

Case If: By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{6.64}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta_1', \Delta_2', \Delta_3'$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v[\sigma]:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash \texttt{if}_A\ (v\,[\sigma])\ \texttt{then}\ (v_1\,[\sigma])\ \texttt{else}\ (v_2\,[\sigma]) \colon A} \tag{6.65}$$

Since $(\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2)\,[\sigma] = \texttt{if}_A\ (v\,[\sigma])\ \texttt{then}\ (v_1\,[\sigma])\ \texttt{else}\ (v_2\,[\sigma])$ The derivation above holds for $\Phi \mid \Gamma' \vdash (\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2)\,[\sigma] : A$

Case Bind: By inversion, there exist Δ_1, Δ_2 such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\texttt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\texttt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{6.66}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$\Phi \mid (\Gamma, x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$$

By induction on Δ_1, Δ_2, we have Δ_1', Δ_2' such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:\texttt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma',x:A\vdash v_2[\sigma,x:=x]:\texttt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma' \vdash \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma, x := x]) \colon \texttt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{6.67}$$

Since $(\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] = \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma]) = \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma, x := x])$, the above derivation holds for $\Phi \mid \Gamma' \vdash (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] : \texttt{M}_{\epsilon_1\cdot\epsilon_2}B$

Case Sub-type: By inversion, there exists Δ such that

$$\text{(sub-type)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v \colon B} \tag{6.68}$$

By induction on Δ we derive Δ' such that:

$$\text{(sub-type)}\frac{()\frac{\Delta'}{\Phi|\Gamma'\vdash v[\sigma]:A} \quad A \leq:_\Phi B}{\Phi \mid \Gamma \vdash v\,[\sigma] : B} \tag{6.69}$$

Case Effect-Lambda: By inversion, there exists Δ such that

$$\text{(Effect-abs)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{6.70}$$

It is also the case that $\iota\pi : \Phi, \alpha \triangleright \Phi$.
So $\Phi, \alpha \mid \Gamma' \vdash \sigma \colon \Gamma$
So by induction there exists Δ',

$$\text{(Effect-abs)}\frac{()\frac{\Delta'}{\Phi,\alpha|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.(v\,[\sigma]) \colon \forall\alpha.A} \tag{6.71}$$

Where $\Lambda\alpha.(v\,[\sigma]) = (\Lambda\alpha.v)\,[\sigma]$

Case Effect Application: By inversion$\Phi \vdash \epsilon$ and there exists Δ such that

$$\text{(Effect-App)} \frac{() \frac{\Delta}{\Phi | \Gamma \vdash v : \forall \alpha . A}}{\Phi \mid \Gamma \vdash v \ \epsilon : A \left[\epsilon / \alpha\right]} \tag{6.72}$$

So by induction there exists Δ' such that:

$$\text{(Effect-App)} \frac{() \frac{\Delta'}{\Phi | \Gamma' \vdash v[\sigma] : \forall \alpha . A}}{\Phi \mid \Gamma' \vdash (v\left[\sigma\right]) \ \epsilon : A \left[\epsilon / \alpha\right]} \tag{6.73}$$

Where $(v\left[\sigma\right]) \ \epsilon = (v \ \epsilon) \left[\sigma\right]$

6.6 The Identity Substitution on Type Environments

For each type environment Γ, define the identity substitution I_Γ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma, x:A} = (I_\Gamma, x := x)$

6.6.1 Properties of the Identity Substitution

Property 1 If $\Phi \vdash \Gamma \texttt{Ok}$ then $\Phi \mid \Gamma \vdash I_\Gamma : \Gamma$, proved trivially by induction over the well-formed-ness relation.

Property 2 TODO: The denotational property of id-substitution

6.7 Single Substitution on Type Environments

If $\Phi \mid \Gamma \vdash v : A$, let the single substitution $\Phi \mid \Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{6.74}$$

Chapter 7

Beta Eta Equivalence (Soundness)

7.1 Beta and Eta Equivalence

7.1.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Phi|\Gamma,x{:}A\vdash v_2{:}B \quad \Phi|\Gamma\vdash v_1{:}A}{\Phi|\Gamma\vdash(\lambda x{:}A.v_1)\ v_2 =_{\beta\eta} v_1[v_2/x]{:}B}$
- (Lambda-Eta) $\frac{\Phi|\Gamma\vdash v{:}A\to B}{\Phi|\Gamma\vdash\lambda x{:}A.(v\ x) =_{\beta\eta} v{:}A\to B}$
- (Left Unit) $\frac{\Phi|\Gamma\vdash v_1{:}A \quad \Phi|\Gamma,x{:}A\vdash v_2{:}\mathtt{M}_\epsilon B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow\mathtt{return} v_1\ \mathtt{in}\ v_2 =_{\beta\eta} v_2[v_1/x]{:}\mathtt{M}_\epsilon B}$
- (Right Unit) $\frac{\Phi|\Gamma\vdash v{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v\ \mathtt{in}\ \mathtt{return} x =_{\beta\eta} v{:}\mathtt{M}_\epsilon A}$
- (Associativity) $\frac{\Phi|\Gamma\vdash v_1{:}\mathtt{M}_{\epsilon_1} A \quad \Phi|\Gamma,x{:}A\vdash v_2{:}\mathtt{M}_{\epsilon_2} B \quad \Phi|\Gamma,y{:}B\vdash v_3{:}\mathtt{M}_{\epsilon_3} C}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y\leftarrow v_2\ \mathtt{in}\ v_3) =_{\beta\eta} \mathtt{do}\ y\leftarrow(\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2)\ \mathtt{in}\ v_3{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C}$
- (Unit) $\frac{\Phi|\Gamma\vdash v{:}\mathtt{Unit}}{\Phi|\Gamma\vdash v =_{\beta\eta} \mathtt{()}{:}\mathtt{Unit}}$
- (if-true) $\frac{\Phi|\Gamma\vdash v_1{:}A \quad \Phi|\Gamma\vdash v_2{:}A}{\Phi|\Gamma\vdash\mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 =_{\beta\eta} v_1{:}A}$
- (if-false) $\frac{\Phi|\Gamma\vdash v_2{:}A \quad \Phi|\Gamma\vdash v_1{:}A}{\Phi|\Gamma\vdash\mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 =_{\beta\eta} v_2{:}A}$
- (If-Eta) $\frac{\Phi|\Gamma,x{:}\mathtt{Bool}\vdash v_2{:}A \quad \Phi|\Gamma\vdash v_1{:}\mathtt{Bool}}{\Phi|\Gamma\vdash\mathtt{if}_A\ v_1\ \mathtt{then}\ v_2[\mathtt{true}/x]\ \mathtt{else}\ v_2[\mathtt{false}/x] =_{\beta\eta} v_2[v_1/x]{:}A}$
- (Effect-beta) $\frac{\Phi\vdash\epsilon \quad \Phi,\alpha|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash(\Lambda\alpha.v\ \epsilon) =_{\beta\eta} v[\epsilon/\alpha]{:}A[\epsilon/\alpha]}$
- (Effect-eta) $\frac{\Phi|\Gamma\vdash v{:}\forall\alpha.A}{\Phi|\Gamma\vdash\Lambda\alpha.(v\ \alpha) =_{\beta\eta} v{:}\forall\alpha.A}$

7.1.2 Equivalence Relation

- (Reflexive) $\frac{\Phi|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash v =_{\beta\eta} v{:}A}$
- (Symmetric) $\frac{\Phi|\Gamma\vdash v_1 =_{\beta\eta} v_2{:}A}{\Phi|\Gamma\vdash v_2 =_{\beta\eta} v_1{:}A}$
- (Transitive) $\frac{\Phi|\Gamma\vdash v_1 =_{\beta\eta} v_2{:}A \quad \Phi|\Gamma\vdash v_2 =_{\beta\eta} v_3{:}A}{\Phi|\Gamma\vdash v_1 =_{\beta\eta} v_3{:}A}$

7.1.3 Congruences

- (Effect-Abs) $\frac{\Phi,\alpha|\Gamma\vdash v_1=_{\beta\eta}v_2:A}{\Phi|\Gamma\vdash\Lambda\alpha.v_1=_{\beta\eta}\Lambda\alpha.v_2:\forall\alpha.A}$
- (Effect-Apply) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2:\forall\alpha.A \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v_1\ \epsilon=_{\beta\eta}v_2\ \epsilon:A[\epsilon/\alpha]}$
- (Lambda) $\frac{\Phi|\Gamma,x:A\vdash v_1=_{\beta\eta}v_2:B}{\Phi|\Gamma\vdash\lambda x:A.v_1=_{\beta\eta}\lambda x:A.v_2:A\to B}$
- (Return) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2:A}{\Phi|\Gamma\vdash\mathtt{return}v_1=_{\beta\eta}\mathtt{return}v_2:\mathtt{M}_{\mathtt{1}}A}$
- (Apply) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':A\to B \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_2':A}{\Phi|\Gamma\vdash v_1\ v_2=_{\beta\eta}v_1'\ v_2':B}$
- (Bind) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':\mathtt{M}_{\epsilon_1}A \quad \Phi|\Gamma,x:A\vdash v_2=_{\beta\eta}v_2':\mathtt{M}_{\epsilon_2}B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2=_{\beta\eta}\mathtt{do}\ c\leftarrow v_1'\ \mathtt{in}\ v_2':\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$
- (If) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v':\mathtt{Bool} \quad \Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':A \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_2':A}{\Phi|\Gamma\vdash\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2=_{\beta\eta}\mathtt{if}_A\ v\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2':A}$
- (Subtype) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v':A \quad A\leq:_\Phi B}{\Phi|\Gamma\vdash v=_{\beta\eta}v':B}$

7.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

If $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$ then each derivation of $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$ can be converted to a derivation of $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$ by induction over the beta-eta equivalence relation derivation.

7.2.1 Equivalence Relations

Case Reflexive: By inversion we have a derivation of $\Phi \mid \Gamma \vdash v \colon A$.

Case Symmetric: By inversion $\Phi \mid \Gamma \vdash v' =_{\beta\eta} v \colon A$. Hence by induction, derivations of $\Phi \mid \Gamma \vdash v' \colon A$ and $\Phi \mid \Gamma \vdash v \colon A$ are given.

Case Transitive: By inversion, there exists v_2 such that $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_3 \colon A$. Hence by induction, we have derivations of $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma \vdash v_3 \colon A$

7.2.2 Beta-Eta conversions

Case Lambda: By inversion, we have $\Phi \mid \Gamma, x : A \vdash v_1 \colon B$ and $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing rules, we have:

$$\text{(Apply)}\frac{\text{(Lambda)}\frac{\Phi|\Gamma,x:A\vdash v_1:B}{\Phi|\Gamma\vdash\lambda x:A.v_1:A\to B} \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash (\lambda x : A.v_1)\ v_2 \colon A}$$

By the substitution rule **TODO: which?**, we have

$$\text{(Substitution)}\frac{\Phi \mid \Gamma, x : A \vdash v_1 \colon B \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash v_1\,[v_2/x] : B}$$

Case Left Unit: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B$

Hence we have:

$$\text{(Bind)}\frac{\text{(Return)}\frac{\Phi|\Gamma\vdash v_1:A}{\Phi|\Gamma\vdash \mathtt{return} v_1:\mathtt{M}_{\mathtt{1}} A} \quad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return} v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\mathtt{1}\cdot\epsilon} B = \mathtt{M}_\epsilon B} \tag{7.1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Phi \mid \Gamma \vdash v_2\left[v_1/x\right] \colon \mathtt{M}_\epsilon B \tag{7.2}$$

Case Right Unit: By inversion, we have $\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A$.

Hence we have:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A \quad \text{(Return)}\frac{\text{(var)}\frac{}{\Phi|\Gamma,x:A\vdash x:A}}{\Phi|\Gamma,x:A\vdash \mathtt{return} v:\mathtt{M}_{\mathtt{1}} A}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return} x \colon \mathtt{M}_{\epsilon\cdot\mathtt{1}} A = \mathtt{M}_\epsilon A} \tag{7.3}$$

Case Associativity: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$\Phi \vdash (\iota\pi\times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Phi \mid \Gamma, x : A, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \quad \text{(Bind)}\frac{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B \quad \Phi|\Gamma,x:A,y:B\vdash v_3:\mathtt{M}_{\epsilon_3} C}{\Phi|\Gamma,x:A\vdash x v_2 v_3:\mathtt{M}_{\epsilon_2\cdot\epsilon_3} C}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3) \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C} \tag{7.4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\frac{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1} A \quad \Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2} B}{\Phi|\Gamma\vdash \mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2} B} \quad \Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Phi \mid \Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2)\ \mathtt{in}\ v_3 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} C} \tag{7.5}$$

Case Eta: By inversion, we have $\Phi \mid \Gamma \vdash v \colon A \to B$

By weakening, we have $\Phi \vdash \iota\pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\frac{\text{(App)}\frac{\Phi|(\Gamma,x:A)\vdash x:A \quad \text{(weakening)}\frac{\Phi|\Gamma\vdash v:A\to B \quad \Phi\vdash\iota\pi:\Gamma,x:A\triangleright\Gamma}{\Phi|\Gamma,x:A\vdash v:A\to B}}{\Phi|\Gamma,x:A\vdash v\ x:B}}{\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) \colon A \to B} \tag{7.6}$$

Case If-True: By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing lemma **TODO: Which?**, we have $\Phi \vdash \Gamma\mathtt{Ok}$ so $\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\frac{\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \quad \Phi \mid \Gamma \vdash v_1 \colon A \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{7.7}$$

Case If-False: As above,

Hence

$$\text{(If)}\frac{\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \quad \Phi \mid \Gamma \vdash v_1 \colon A \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{7.8}$$

Case If-Eta: By inversion, we have:

$$\Phi \mid \Gamma \vdash v_1 \colon \texttt{Bool} \tag{7.9}$$

and

$$\Phi \mid \Gamma, x : \texttt{Bool} \vdash v_2 \colon A \tag{7.10}$$

Hence we also have $\Phi \vdash \Gamma \texttt{Ok}$. Hence, the following also hold:
$\Phi \mid \Gamma \vdash \texttt{true} \colon \texttt{Bool}$, and $\Phi \mid \Gamma \vdash \texttt{false} \colon \texttt{Bool}$.
Hence by the substitution theorem, we have:

$$\text{(If)}\frac{\Phi \mid \Gamma \vdash v_1 \colon \texttt{Bool} \quad \Phi \mid \Gamma \vdash v_2\,[\texttt{true}/x] : A \quad \Phi \mid \Gamma \vdash v_2\,[\texttt{false}/x] : A}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v_1\ \texttt{then}\ v_2\,[\texttt{true}/x]\ \texttt{else}\ v_2\,[\texttt{false}/x] : A} \tag{7.11}$$

and

$$\Phi \mid \Gamma \vdash v_2\,[v_1/x] : A \tag{7.12}$$

Case Effect-Beta: By inversion, $\Phi, \alpha \mid \Gamma \vdash v \colon A$ and $\Phi \vdash \epsilon$.
Then we have the following type derivation:

$$\text{(Effect-App)}\frac{\text{(Effect-Fn)}\frac{\Phi,\alpha \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v : \forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{7.13}$$

And we can construct the single-effect-substitution:

$$\text{(Single Substitution)}\frac{\Phi \vdash \epsilon}{\Phi \vdash [\epsilon/\alpha] : (\Phi, \alpha)} \tag{7.14}$$

Hence by the substitution theorem,

$$\Phi \mid \Gamma \vdash v\,[\epsilon/\alpha] : A\,[\epsilon/\alpha] \tag{7.15}$$

Case Effect-Eta: By inversion$\Phi \mid \Gamma \vdash v \colon \forall\alpha.A$
So the following derivation holds:

$$\text{(Effect-Fn)}\frac{\text{(Effect-App)}\frac{\text{(Effect-weakening)}\frac{\Phi \mid \Gamma \vdash v : \forall\alpha.A}{\Phi,\alpha \mid \Gamma \vdash v : \forall\alpha.A} \quad \Phi,\alpha \vdash \alpha}{\Phi,\alpha \mid \Gamma \vdash v\ \alpha : A[\alpha/\alpha] = A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha) \colon \forall\alpha.A} \tag{7.16}$$

And

$$\Phi \mid \Gamma \vdash v \colon \forall\alpha.A \tag{7.17}$$

7.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

Case Lambda: By inversion, $\Phi \mid \Gamma, x : A \vdash v_1 =_{\beta\eta} v_2 \colon B$. Hence by induction $\Phi \mid \Gamma, x : A \vdash v_1 \colon B$, and $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.
So

$$\Phi \mid \Gamma \vdash \lambda x : A.v_1 \colon A \to B \tag{7.18}$$

and

$$\Phi \mid \Gamma \vdash \lambda x : A.v_2 \colon A \to B \tag{7.19}$$

Hold.

Case Return: By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, so by induction

$$\Phi \mid \Gamma \vdash v_1 \colon A$$

and

$$\Phi \mid \Gamma \vdash v_2 \colon A$$

Hence we have

$$\Phi \mid \Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_{\mathbf{1}} A$$

and

$$\Phi \mid \Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_{\mathbf{1}} A$$

Case Apply: By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$. Hence we have by induction $\Phi \mid \Gamma \vdash v_1 \colon A \to B$, $\Phi \mid \Gamma \vdash v_2 \colon A$, $\Phi \mid \Gamma \vdash v_1' \colon A \to B$, and $\Phi \mid \Gamma \vdash v_2' \colon A$.

So we have:

$$\Phi \mid \Gamma \vdash v_1\ v_2 \colon B \tag{7.20}$$

and

$$\Phi \mid \Gamma \vdash v_1'\ v_2' \colon B \tag{7.21}$$

Case Bind: By inversion, we have: $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2 =_{\beta\eta} v_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, x : A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.22}$$

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1'\ \mathtt{in}\ v_2' \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.23}$$

Case If: By inversion, we have: $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A$, and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$.

Hence by induction, we have:

$\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v' \colon \mathtt{Bool}$,

$\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_1' \colon A$,

$\Phi \mid \Gamma \vdash v_2 \colon A$, and $\Phi \mid \Gamma \vdash v_2' \colon A$.

So

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \tag{7.24}$$

and

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v'\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2' \colon A \tag{7.25}$$

hold.

Case Subtype: By inversion, we have $A \leq :_\Phi B$ and $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$. By induction, we therefore have $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$.

Hence we have

$$\Phi \mid \Gamma \vdash v \colon B \tag{7.26}$$

$$\Phi \mid \Gamma \vdash v' \colon B \tag{7.27}$$

Case Effect-Lambda: By inversion, $\Phi, \alpha \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$. So

$$(\text{Effect-Lambda})\frac{\Phi, \alpha \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{7.28}$$

and

$$(\text{Effect-Lambda})\frac{\Phi, \alpha \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{7.29}$$

Case Effect-Apply: By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon \forall\alpha.A$ and $\Phi \vdash \epsilon$. So

$$(\text{Effect-App})\frac{\Phi \mid \Gamma \vdash v_1 \colon \forall\alpha.A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_1\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{7.30}$$

and

$$(\text{Effect-App})\frac{\Phi \mid \Gamma \vdash v_2 \colon \forall\alpha.A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_2\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{7.31}$$