# Contents

# Chapter 1

# Language Definition

## 1.1 Terms

Making the language no-longer differentiate between values and computations.

### 1.1.1 Value Terms

$$\begin{aligned} v ::=& x \\ &\mid \lambda x : A.v \\ &\mid \mathtt{C}^A \\ &\mid \mathtt{()} \\ &\mid \mathtt{true} \mid \mathtt{false} \\ &\mid \Lambda\alpha.v \\ &\mid v\ \epsilon \\ &\mid \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \\ &\mid v_1\ v_2 \\ &\mid \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \\ &\mid \mathtt{return} v \end{aligned} \tag{1.1}$$

## 1.2 Type System

### 1.2.1 Ground Effects

The effects should form a monotonous, pre-ordered monoid $(E, \cdot, 1, \leq)$ with ground elements $e$.

### 1.2.2 Effect Po-Monoid Under a Effect Environment

Derive a new Po-Monoid for each $\Phi$:

$$(E_\Phi, \cdot_\Phi, 1, \leq_\Phi) \tag{1.2}$$

Where meta-variables, $\epsilon$, range over $E_\Phi$ Where

$$E_\Phi = E \cup \{\alpha \mid \alpha \in \Phi\} \tag{1.3}$$

And

$$()\frac{\epsilon_3 = \epsilon_1 \cdot \epsilon_2}{\epsilon_3 = \epsilon_1 \cdot_\Phi \epsilon_2} \tag{1.4}$$

Otherwise, $\cdot_\Phi$ is symbolic in nature.

$$\epsilon_1 \leq_\Phi \epsilon_2 \Leftrightarrow \forall \sigma \downarrow . \epsilon_1 [\sigma \downarrow] \leq \epsilon_2 [\sigma \downarrow] \tag{1.5}$$

Where $\sigma \downarrow$ denotes any ground-substitution of $\Phi$. That is any substitution of all effect-variables in $\Phi$ to ground effects. Where it is obvious from the context, I shall use $\leq$ instead of $\leq_\Phi$.

### 1.2.3 Types

**Ground Types** There exists a set $\gamma$ of ground types, including `Unit`, `Bool`

**Term Types**

$$A, B, C ::= \gamma \mid A \rightarrow B \mid \mathtt{M}_\epsilon A \mid \forall \alpha . A$$

### 1.2.4 Type and Effect Environments

A type environment is a snoc-list of tern-variable, type pairs, $G ::= \diamond \mid \Gamma, x : A$ . An effect environment is a snoc-list of effect-variables.

$\Phi ::= \diamond \mid \Phi, \alpha$

**Domain Function on Type Environments**

- $\mathtt{dom}(\diamond) = \emptyset$
- $\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\}$

**Membership of Effect Environments** Informally, $\alpha \in \Phi$ if $\alpha$ appears in the list represented by $\Phi$.

**`Ok` Predicate On Effect Environments**

- (Atom) $\frac{}{\diamond \mathtt{Ok}}$
- (A) $\frac{\Phi \mathtt{Ok} \quad \alpha \notin \Phi}{\Phi, \alpha \mathtt{Ok}}$

**Well-Formed-ness of effects** We define a relation $\Phi \vdash \epsilon$.

- (Ground) $\frac{\Phi \mathtt{Ok}}{\Phi \vdash e}$
- (Var) $\frac{\Phi, \alpha \mathtt{Ok}}{\Phi, \alpha \vdash \alpha}$
- (Weaken) $\frac{\Phi \vdash \alpha}{\Phi, \beta \vdash \alpha}$ (if $\alpha \neq \beta$)
- (Monoid Op) $\frac{\Phi \vdash \epsilon_1 \quad \Phi \vdash \epsilon_2}{\Phi \vdash \epsilon_1 \cdot \epsilon_2}$

**Well-Formed-ness of Types** We define a relation $\Phi \vdash \tau$ on types.

- (Ground) $\frac{}{\Phi \vdash \gamma}$
- (Lambda) $\frac{\Phi \vdash A \quad \Phi \vdash B}{\Phi \vdash A \rightarrow B}$
- (Computation) $\frac{\Phi \vdash A \quad \Phi \vdash \epsilon}{\Phi \vdash \mathtt{M}_\epsilon A}$
- (For-All) $\frac{\Phi, \alpha \vdash A}{\Phi \vdash \forall \alpha . A}$

**Ok Predicate on Type Environments** We now define a predicate on type environments and effect environments: $\Phi \vdash \Gamma\mathtt{Ok}$

- (Nil) $\frac{}{\Phi\vdash\diamond\mathtt{Ok}}$
- (Var) $\frac{\Phi\vdash\Gamma\mathtt{Ok}\quad x\notin\mathtt{dom}(\Gamma)\quad \Phi\vdash A}{\Phi\vdash\Gamma,x:A\mathtt{Ok}}$

### 1.2.5 Sub-typing

There exists a sub-typing pre-order relation $\leq:_\gamma$ over ground types that is:

- (Reflexive) $\frac{}{A\leq:_\gamma A}$
- (Transitive) $\frac{A\leq:_\gamma B\quad B\leq:_\gamma C}{A\leq:_\gamma C}$

We extend this relation with the function and effect-lambda sub-typing rules to yield the full sub-typing relation under an effect environment, $\Phi$, $\leq:_\Phi$

- (ground) $\frac{A\leq:_\gamma B}{A\leq:_\Phi B}$
- (Fn) $\frac{A\leq:_\Phi A'\quad B'\leq:_\Phi B}{A'\to B'\leq:_\Phi A\to B}$
- (All) $\frac{A\leq:_\Phi A'}{\forall\alpha.A\leq:_\Phi\forall a.A'}$
- (Effect) $\frac{A\leq:_\Phi B\quad \epsilon_1\leq_\Phi\epsilon_2}{\mathtt{M}_{\epsilon_1}A\leq:_\Phi\mathtt{M}_{\epsilon_2}B}$

### 1.2.6 Type Rules

- (Const) $\frac{\Phi\vdash\Gamma\mathtt{Ok}\quad\Phi\vdash A}{\Phi|\Gamma\vdash\mathtt{C}^A:A}$
- (Unit) $\frac{\Phi\vdash\Gamma\mathtt{Ok}}{\Phi|\Gamma\vdash\mathtt{()}:\mathtt{Unit}}$
- (True) $\frac{\Phi\vdash\Gamma\mathtt{Ok}}{\Phi|\Gamma\vdash\mathtt{true}:\mathtt{Bool}}$
- (False) $\frac{\Phi\vdash\Gamma\mathtt{Ok}}{\Phi|\Gamma\vdash\mathtt{false}:\mathtt{Bool}}$
- (Var) $\frac{\Phi\vdash\Gamma,x:A\mathtt{Ok}}{\Phi|\Gamma,x:A\vdash x:A}$
- (Weaken) $\frac{\Phi|\Gamma\vdash x:A\quad\Phi\vdash B}{\Phi|\Gamma,y:B\vdash x:A}$ (if $x\neq y$)
- (Fn) $\frac{\Phi|\Gamma,x:A\vdash v:\beta}{\Phi|\Gamma\vdash\lambda x:A.v:A\to B}$
- (Sub) $\frac{\Phi|\Gamma\vdash v:A\quad A\leq:_\Phi B}{\Phi|\Gamma\vdash v:B}$
- (Effect-Abs) $\frac{\Phi,\alpha|\Gamma\vdash v:A}{\Phi|\Gamma\vdash\Lambda\alpha.v:\forall\alpha.A}$
- (Effect-apply) $\frac{\Phi|\Gamma\vdash v:\forall\alpha.A\quad\Phi\vdash\epsilon}{\Phi|\Gamma\vdash v\ \epsilon:A[\epsilon/\alpha]}$
- (Return) $\frac{\Phi|\Gamma\vdash v:A}{\Phi|\Gamma\vdash\mathtt{return}v:\mathtt{M}_{\mathbf{1}}A}$
- (Apply) $\frac{\Phi|\Gamma\vdash v_1:A\to\mathtt{M}_\epsilon B\quad\Phi|\Gamma\vdash v_2:A}{\Phi|\Gamma\vdash v_1\ v_2:\mathtt{M}_\epsilon B}$
- (If) $\frac{\Phi|\Gamma\vdash v:\mathtt{Bool}\quad\Phi|\Gamma\vdash v_1:A\quad\Phi|\Gamma\vdash v_2:A}{\Phi|\Gamma\vdash\mathtt{if}_A\ V\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2:A}$
- (Do) $\frac{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A\quad\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2:\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$

### 1.2.7 Ok Lemma

If $\Phi \mid \Gamma \vdash t\colon \tau$ then $\Phi \vdash \Gamma\texttt{Ok}$.

**Proof** If $\Gamma, x : A\texttt{Ok}$ then by inversion $\Gamma\texttt{Ok}$ Only the type rule `Weaken` adds terms to the environment from its preconditions to its post-condition and it does so in an `Ok` preserving way. Any type derivation tree has at least one leaf. All leaves are axioms which require $\Phi \vdash \Gamma\texttt{Ok}$. And all non-axiom derivations preserve the `Ok` property.

# Chapter 2

# Category Requirements

## 2.1 CCC

The category at each index should be a cartesian closed category. That is it should have:

- A Terminal object $\mathtt{1}$
- Binary products
- Exponentials

Further more, it should have a co-product of the terminal object $\mathtt{1}$. This is required for the beta-eta equivalence of `if-then-else` terms.

$\mathtt{1} \xrightarrow{inl} A \xleftarrow{inr} \mathtt{1}$

For each:

$\mathtt{1} \xrightarrow{f} A \xleftarrow{g} \mathtt{1}$

There exists unique $[f, g] : \mathtt{1} + \mathtt{1} \to A$ such that:

$$\begin{tikzcd}
 & A & \\
\mathtt{1} \arrow[ur, "f"] \arrow[r, "\mathtt{inl}"'] & \mathtt{1} + \mathtt{1} \arrow[u, "{[f,g]}"] & \mathtt{1} \arrow[ul, "g"'] \arrow[l, "\mathtt{inr}"]
\end{tikzcd}$$

## 2.2 Graded Pre-Monad

The category should have a graded pre-monad. That is:

- An endo-functor indexed by the po-monad on effects: $T : (\mathbb{E}, \cdot\, \mathtt{1}, \leq) \to \mathtt{Cat}(\mathbb{C}, \mathbb{C})$
- A unit natural transformation: $\eta : \mathtt{Id} \to T_{\mathtt{1}}$
- A join natural transformation: $\mu_{\epsilon_1, \epsilon_2,} : T_{\epsilon_1} T_{\epsilon_2} \to T_{\epsilon_1 \cdot \epsilon_2}$

Subject to the following commutative diagrams:

### 2.2.1 Left Unit

$$\begin{tikzcd}
T_\epsilon A \arrow[r, "T_\epsilon \eta_A"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_\epsilon T_{\mathtt{1}} A \arrow[d, "\mu_{\epsilon, \mathtt{1}, A}"] \\
 & T_\epsilon A
\end{tikzcd}$$

### 2.2.2 Right Unit

$$\begin{tikzcd} T_\epsilon A \arrow[r, "\eta_{T_\epsilon A}"] \arrow[dr, "\mathtt{Id}_{T_\epsilon A}"] & T_{\mathtt{1}} T_{\mathtt{1}} A \arrow[d, "\mu_{\mathtt{1},\epsilon,A}"] \\ & T_\epsilon A \end{tikzcd}$$

### 2.2.3 Associativity

$$\begin{tikzcd} T_{\epsilon_1} T_{\epsilon_2} T_{\epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2,T_{\epsilon_3} A}"] \arrow[d, "T_{\epsilon_1}\mu_{\epsilon_2,\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2} T_{\epsilon_3} A \arrow[d, "\mu_{\epsilon_1\cdot\epsilon_2,\epsilon_3,A}"] \\ T_{\epsilon_1} T_{\epsilon_2\cdot\epsilon_3} A \arrow[r, "\mu_{\epsilon_1,\epsilon_2\cdot\epsilon_3,A}"] & T_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3} A \end{tikzcd}$$

## 2.3 Tensor Strength

The category should also have tensorial strength over its products and monads. That is, it should have a natural transformation

$$\mathtt{t}_{\epsilon,A,B} : A \times T_\epsilon B \to T_\epsilon(A \times B)$$

Satisfying the following rules:

### 2.3.1 Left Naturality

$$\begin{tikzcd} A \times T_\epsilon B \arrow[r, "\mathtt{Id}_A \times T_\epsilon f"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A \times T_\epsilon B' \arrow[d, "\mathtt{t}_{\epsilon,A,B'}"] \\ T_\epsilon(A \times B) \arrow[r, "T_\epsilon(\mathtt{Id}_A \times f)"] & T_\epsilon(A \times B') \end{tikzcd}$$

### 2.3.2 Right Naturality

$$\begin{tikzcd} A \times T_\epsilon B \arrow[r, "f \times \mathtt{Id}_{T_\epsilon B}"] \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] & A' \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A',B}"] \\ T_\epsilon(A \times B) \arrow[r, "T_\epsilon(f \times \mathtt{Id}_B)"] & T_\epsilon(A' \times B) \end{tikzcd}$$

### 2.3.3 Unitor Law

$$\begin{tikzcd} \mathtt{1} \times T_\epsilon A \arrow[r, "\mathtt{t}_{\epsilon,\mathtt{1},A}"] \arrow[dr, "\lambda_{T_\epsilon A}"] & T_\epsilon(\mathtt{1} \times A) \arrow[d, "T_\epsilon(\lambda_A)"] \\ & T_\epsilon A \end{tikzcd}$$

Where $\lambda : \mathtt{1} \times \mathtt{Id} \to \mathtt{Id}$ is the left-unitor. ($\lambda = \pi_2$)

**Tensor Strength and Projection** Due to the left-unitor law, we can develop a new law for the commutativity of $\pi_2$ with $\mathtt{t}_{,,}$

$$\pi_{2,A,B} = \pi_{2,\mathtt{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B)$$

And $\pi_{2,\mathtt{1}}$ is the left unitor, so by tensorial strength:

$$
\begin{aligned}
T_\epsilon\pi_2 \circ \mathtt{t}_{\epsilon,A,B} &= T_\epsilon\pi_{2,\mathbf{1},B} \circ T_\epsilon(\langle\rangle_A \times \mathtt{Id}_B) \circ \mathtt{t}_{\epsilon,A,B} \\
&= T_\epsilon\pi_{2,1,B} \circ \mathtt{t}_{\epsilon,\mathbf{1},B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_{2,1,B} \circ (\langle\rangle_A \times \mathtt{Id}_B) \\
&= \pi_2
\end{aligned} \tag{2.1}
$$

So the following commutes:

$$
\begin{tikzcd}
A \times T_\epsilon B \arrow[r, "\mathtt{t}_{\epsilon,A,B}"] \arrow[dr, "\pi_2"'] & T_\epsilon(A \times B) \arrow[d, "T_\epsilon\pi_2"] \\
& T_\epsilon B
\end{tikzcd}
$$

### 2.3.4 Commutativity with Join

$$
\begin{tikzcd}
A \times T_{\epsilon_1}T_{\epsilon_2}B \arrow[r, "\mathtt{t}_{\epsilon_1,A,T_{\epsilon_2}B}"] \arrow[dr, "\mathtt{Id}_A \times \mu_{\epsilon_1,\epsilon_2,B}"] & T_{\epsilon_1}(A \times T_{\epsilon_2}B) \arrow[r, "T_{\epsilon_1}\mathtt{t}_{\epsilon_2,A,B}"] & T_{\epsilon_1}T_{\epsilon_2}(A \times B) \arrow[d, "\mu_{\epsilon_1,\epsilon_2,A\times B}"] \\
& A \times T_{\epsilon_1\cdot\epsilon_2}B \arrow[r, "\mathtt{t}_{\epsilon_1\cdot\epsilon_2,A,B}"] & T_{\epsilon_1\cdot\epsilon_2}(A \times B)
\end{tikzcd}
$$

## 2.4 Commutativity with Unit

$$
\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_\epsilon B \arrow[d, "\mathtt{t}_{\epsilon,A,B}"] \\
& T_\epsilon(A \times B)
\end{tikzcd}
$$

## 2.5 Commutativity with $\alpha$

Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$
\begin{tikzcd}
(A \times B) \times T_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,T_\epsilon C}"] & & T_\epsilon((A \times B) \times C) \arrow[d, "T_\epsilon\alpha_{A,B,C}"] \\
A \times (B \times T_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times T_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & T_\epsilon(A \times (B \times C))
\end{tikzcd}
$$

**TODO: Needed?**

## 2.6 Sub-effecting

For each instance of the pre-order $(\mathbb{E}, \leq)$, $\epsilon_1 \leq \epsilon_2$, there exists a natural transformation $[\![\epsilon_1 \leq \epsilon_2]\!] : T_{\epsilon_1} \to T_{\epsilon_2}$ that commutes with $\mathtt{t}_{,,}$:

### 2.6.1 Sub-effecting and Tensor Strength

$$
\begin{tikzcd}
A \times T_{\epsilon_1}B \arrow[r, "\mathtt{Id}_A \times [\![\epsilon_1 \leq \epsilon_2]\!]_B"] \arrow[d, "\mathtt{t}_{\epsilon_1,A,B}"] & A \times T_{\epsilon_2}B \arrow[d, "\mathtt{t}_{\epsilon_2,A,B}"] \\
T_{\epsilon_1}(A \times B) \arrow[r, "[\![\epsilon_1 \leq \epsilon_2]\!]_{A\times B}"] & T_{\epsilon_2}(A \times B)
\end{tikzcd}
$$

### 2.6.2 Sub-effecting and Monadic Join

Since the monoid operation on effects is monotone, we can introduce the following diagram.

$$\begin{tikzcd}
T_{\epsilon_1} T_{\epsilon_2} \arrow[r, "T_{\epsilon_1} \llbracket \epsilon_2 \leq \epsilon_2' \rrbracket_M"] \arrow[d, "\mu_{\epsilon_1, \epsilon_2},"] & T_{\epsilon_1} T_{\epsilon_2'} \arrow[r, "\llbracket \epsilon_1 \leq \epsilon_1' \rrbracket_{M, T_{\epsilon_2'}}"] & T_{\epsilon_1'} T_{\epsilon_2'} \arrow[d, "\mu_{\epsilon_1', \epsilon_2'},"] \\
T_{\epsilon_1 \cdot \epsilon_2} \arrow[rr, "\llbracket \epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \epsilon_2' \rrbracket_M"] & & T_{\epsilon_1' \cdot \epsilon_2'}
\end{tikzcd}$$

## 2.7 Sub-typing

The denotation of ground types $\llbracket \_ \rrbracket_M$ is a functor from the pre-order category of ground types $(\gamma, \leq:_\gamma)$ to $\mathbb{C}$. This pre-ordered sub-category of $\mathbb{C}$ is extended with the rule for function sub-typing to form a larger pre-ordered sub-category of $\mathbb{C}$.

$$\text{(Function Subtyping)} \frac{f = \llbracket A' \leq: A \rrbracket_M \quad g = \llbracket B \leq: B' \rrbracket_M \quad h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket}{rhs = \llbracket A \to \mathtt{M}_{\epsilon_1} B \leq: A' \to \mathtt{M}_{\epsilon_2} B' \rrbracket_M : (T_{\epsilon_1} B)^A \to (T_{\epsilon_2} B')^{A'}}$$

$$\begin{aligned}
rhs =& (h_{B'} \circ T_{\epsilon_1} g)^{A'} \circ (T_{\epsilon_1} B)^f \\
=& \mathtt{cur}(h_{B'} \circ T_{\epsilon_1} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id}_{T_{\epsilon_1} B^{A'}} \times f))
\end{aligned} \tag{2.2}$$

# Chapter 3

# Denotations

## 3.1 Helper Morphisms

### 3.1.1 Diagonal and Twist Morphisms

In the definition and proofs (Especially of the the If cases), I make use of the morphisms twist and diagonal.

$$\tau_{A,B} : (A \times B) \to (B \times A) = \langle \pi_2, \pi_1 \rangle \tag{3.1}$$
$$\delta_A : A \to (A \times A) = \langle \mathtt{Id}_A, \mathtt{Id}_A \rangle \tag{3.2}$$

## 3.2 Denotations of Types

### 3.2.1 Denotation of Ground Types

### 3.2.2 Denotation of Polymorphic Types

### 3.2.3 Denotation of Computation Type

### 3.2.4 Denotation of Function Types

### 3.2.5 Denotation of Type Environments

### 3.2.6 Denotation of Value Terms

### 3.2.7 Denotation of Computation Terms

# Chapter 4

# Unique Denotations

## 4.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t\colon \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

## 4.2 Reduced Type Derivations are Unique

### 4.2.1 Variables

### 4.2.2 Constants

### 4.2.3 Value Terms

### 4.2.4 Computation Terms

## 4.3 Each type derivation has a reduced equivalent with the same denotation.

### 4.3.1 Constants

### 4.3.2 Value Types

### 4.3.3 Computation Types

## 4.4 Denotations are Equivalent

# Chapter 5

# Weakening

## 5.1 Effect Weakening Definition

Introduce a relation $\omega : \Phi' \triangleright \Phi$ relating effect-environments.

### 5.1.1 Relation

- (Id) $\frac{\Phi \texttt{Ok}}{\iota : \Phi \triangleright \Phi}$
- (Project) $\frac{\omega : \Phi' \triangleright \Phi}{\omega\pi : (\Phi', \alpha) \triangleright \Phi}$
- (Extend) $\frac{\omega : \Phi' \triangleright \Phi}{\omega\times : (\Phi', \alpha) \triangleright (\Phi, \alpha)}$

### 5.1.2 Weakening Properties

### 5.1.3 Effect Weakening Preserves Ok

$$\omega : \Phi' \triangleright \Phi \wedge \Phi\texttt{Ok} \Leftarrow \Phi'\texttt{Ok} \tag{5.1}$$

**Proof**

**Case:** $\iota$

$$\Phi\texttt{Ok} \wedge \iota : \Phi \triangleright \Phi \Leftarrow \Phi\texttt{Ok}$$

**Case:** $\omega\pi$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{5.2}$$

So, by induction, $\Phi'\texttt{Ok}$ and hence $(\Phi', \alpha)\texttt{Ok}$

**Case:** $\omega\times$ By inversion,

$$\omega : \Phi' \triangleright \Phi \wedge \alpha \notin \Phi' \tag{5.3}$$

So

$$\begin{aligned}
(\Phi, \alpha)\texttt{Ok} &\Rightarrow \Phi\texttt{Ok} && (5.4)\\
&\Rightarrow \Phi'\texttt{Ok} && (5.5)\\
&\Rightarrow (\Phi', \alpha)\texttt{Ok} && (5.6)\\
& && (5.7)
\end{aligned}$$

### 5.1.4 Domain Lemma

$$\omega : \Phi' \triangleright \Phi \Rightarrow (\alpha \notin \Phi \Rightarrow \alpha \notin \Phi')$$

**Proof** By trivial Induction.

### 5.1.5 Weakening Preserves Effect Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ then $\Phi \vdash \epsilon \implies \Phi' \vdash \epsilon$

**Proof** By induction over the well-formed-ness of effects

**Case Ground** By inversion, $\Phi\mathtt{Ok} \wedge \epsilon \in E$. Hence by the ok-property, $\Phi'\mathtt{Ok}$ So $\Phi' \vdash \epsilon$

**Case Var** $\Phi = \Phi'', \alpha$
So either:

**Case:** $\Phi' = \Phi''', \alpha$ So $\omega = \omega' \times$ So $\omega' : \Phi''' \triangleright \Phi''$, and hence:

$$(\text{Var})\frac{\Phi''', \alpha\mathtt{Ok}}{\Phi''', \alpha \vdash \alpha} \tag{5.8}$$

**Case:** $\Phi' = \Phi''', \beta$ and $\beta \neq \alpha$
So $\omega = \omega'\pi$
By induction, $\omega' : \Phi''' \triangleright \Phi$ so

$$(\text{Weaken})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{5.9}$$

**Case Weaken** By inversion, $\Phi = \Phi'', \beta$.
So $\omega = \omega' \times$
And, $\Phi' = \Phi''', \beta$ So By inversion$\omega' : \Phi''' \triangleright \pi_1''$
So by induction

$$(\text{weak})\frac{\Phi''' \vdash \alpha}{\Phi' \vdash \alpha} \tag{5.10}$$

**Case Monoid** By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$. So by induction, $\Phi' \vdash \epsilon_1$ and $\Phi' \vdash \epsilon_2$, and so:

$$\Phi' \vdash \epsilon_1 \cdot \epsilon_2 \tag{5.11}$$

### 5.1.6 Weakening Preserves Type-Well-Formed-Ness

If $\omega : \Phi' \triangleright \Phi$ and $\Phi \vdash A$ then $\Phi' \vdash A$.

**Proof:**

**Case Ground:** By inversion, $\Phi\mathtt{Ok}$, hence by property 1 of weakening, $\Phi'\mathtt{Ok}$. Hence $\Phi' \vdash \gamma$.

**Case Function:** By inversion, $\Phi \vdash A$, $\Phi \vdash B$. So by induction $\Phi' \vdash A$, $\Phi' \vdash B$, hence,

$$\Phi' \vdash A \to B$$

**Case Computation:** By inversion$\Phi \vdash A$, and $\Phi \vdash \epsilon$.
So by induction and the effect-well-formed-ness theorem,
$\Phi' \vdash A$ and $\Phi' \vdash \epsilon$
So

$$\Phi' \vdash \mathtt{M}_\epsilon A$$

**Case For All:** By inversion, $\Phi, \alpha \vdash A$ Picking $\alpha \notin \Phi'$ using $\alpha$-conversion.
So $\omega\times : (\Phi', \alpha) \triangleright (\Phi, \alpha)$
So $(\Phi', \alpha) \vdash A$
So $\Phi \vdash \forall \alpha . A$

### 5.1.7 Corollary

$$\omega : \Phi' \triangleright \Phi \wedge \Phi \vdash \Gamma\mathtt{Ok} \implies \Phi' \vdash \Gamma\mathtt{Ok}$$

**Case Nil:** By inversion$\Phi\mathtt{Ok}$ so $\Phi \vdash \diamond\mathtt{Ok}$

**Case Var:** By inversion$\Phi \vdash \Gamma\mathtt{Ok}$, $x \in \mathtt{dom}(\Gamma)$, $\Phi \vdash A$
So by induction $\Phi' \vdash \Gamma\mathtt{Ok}$, and $\pi_1' \vdash \Gamma\mathtt{Ok}$
So $\Phi' \vdash (\Gamma, x : A)\mathtt{Ok}$

### 5.1.8 Effect Weakening preserves Type Relations

$$\Phi \mid \Gamma \vdash v \colon A \wedge \omega : \Phi' \triangleright \Phi \implies \Phi' \mid \Gamma \vdash v \colon A \tag{5.12}$$

**Proof:**

**Case Constants:** If $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi' \vdash \Gamma\mathtt{Ok}$ so:

$$(\text{Const})\frac{\Phi' \vdash \Gamma\mathtt{Ok}}{\Phi' \mid \Gamma \vdash \mathtt{C}^A \colon A} \tag{5.13}$$

**Case Variables:** If $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi' \vdash \Gamma\mathtt{Ok}$ so: So, $\Phi' \mid G \vdash x \colon A$, if $\Phi \mid G \vdash x \colon A$

**Case Lambda:** By inversion, $\Phi \mid \Gamma, x : A \vdash v \colon B$, so by induction $\Phi' \mid \Gamma, x : A \vdash v \colon B$.
So,

$$\Phi' \mid \Gamma \vdash \lambda x : A.v \colon A \to B \tag{5.14}$$

**Case Apply:** By inversion$\Phi \mid \Gamma \vdash v_1 \colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 \colon A$.
Hence by induction, $\Phi' \mid \Gamma \vdash v_1 \colon A \to B$ and $\Phi' \mid \Gamma \vdash v_2 \colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{app} v_1 v_2 \colon B$$

**Case Return:** By inversion$\Phi \mid \Gamma \vdash v \colon A$
So by induction $\Phi' \mid \Gamma \vdash v \colon A$
Hence $\Phi' \mid \Gamma \vdash \mathtt{return} v \colon \mathtt{M}_{\mathbf{1}} A$

**Case Bind:** By inversion$\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash \epsilon_2 \colon \mathtt{M}_{\epsilon_2} A$.
Hence by induction $\Phi' \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi' \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B \tag{5.15}$$

**Case If:** By inversion$\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 \colon A$, and $\Phi \mid \Gamma \vdash v_2 \colon A$.
Hence by induction $\Phi' \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi' \mid \Gamma \vdash v_1 \colon A$, and $\Phi' \mid \Gamma \vdash v_2 \colon A$.
So

$$\Phi' \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \tag{5.16}$$

**Case Subtype:** By inversion$\Phi \mid \Gamma \vdash v \colon A$, and $A \leq: B$.
So by induction: $\Phi' \mid \Gamma \vdash v \colon A$, and $A \leq: B$.
So

$$\Phi' \mid \Gamma \vdash v \colon B \tag{5.17}$$

**Case Effect-Lambda:** By inversion$\Phi, \alpha \mid \Gamma \vdash v \colon A$
By picking $\alpha \notin \Phi'$ using $\alpha$-conversion.

$$\omega\times : \Phi', \alpha \triangleright \Phi, \alpha \tag{5.18}$$

So by induction, $\Phi', \alpha \mid \Gamma \vdash v \colon A$
Hence,

$$\Phi' \mid \Gamma \vdash \Lambda\alpha.v \colon \forall a.A \tag{5.19}$$

**Case Effect-Apply:** By inversion, $\Phi \mid \Gamma \vdash v \colon \forall\alpha.A$, and $\Phi \vdash \epsilon$.
So by induction, $\Phi' \mid \Gamma \vdash v \colon \forall\alpha.A$
And by the well-formed-ness-theorem $\Phi' \vdash \epsilon$
Hence,

$$\Phi' \mid \Gamma \vdash v\ \epsilon \colon A\,[\epsilon/\alpha] \tag{5.20}$$

## 5.2 Type Environment Weakening

### 5.2.1 Relation

We define the ternary weakening relation $\Phi \vdash w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Phi \vdash \Gamma\, \mathtt{Ok}}{\Phi \vdash \iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma') \quad A \leq: B}{\Phi \vdash w\times : \Gamma', x : A \triangleright \Gamma, x : B}$

### 5.2.2 Domain Lemma

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, then $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$.

**Proof:**

**Case Id:** Then $\Gamma' = \Gamma$ and so $\mathtt{dom}(\Gamma') = \mathtt{dom}(\Gamma)$.

**Case Project:** By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma') \subseteq \mathtt{dom}(\Gamma' \cup \{x\})$

**Case Extend:** By inversion and induction, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$ so

$$\mathtt{dom}(\Gamma, x : A) = \mathtt{dom}(\Gamma) \cup \{x\} \subseteq \mathtt{dom}(\Gamma') \cup \{x\} = \mathtt{dom}(\Gamma', x : A)$$

### 5.2.3 Theorem 1

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ and $\Phi \vdash \Gamma \mathtt{Ok}$ then $\Phi \vdash \Gamma' \mathtt{Ok}$

**Proof:**

**Case Id:**

$$(\mathrm{Id}) \frac{\Phi \vdash \Gamma \mathtt{Ok}}{\Phi \vdash \iota : \Gamma \triangleright \Gamma}$$

By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$.

**Case Project:**

$$(\mathrm{Project}) \frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma')}{\Phi \vdash \omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ and $x \notin \mathtt{dom}(\Gamma')$.
Hence by induction $\Phi \vdash \Gamma' \mathtt{Ok}$, $\Phi \vdash \Gamma \mathtt{Ok}$. Since $x \notin \mathtt{dom}(\Gamma')$, we have $\Phi \vdash \Gamma', x : A \mathtt{Ok}$.

**Case Extend:** $(\mathrm{Extend}) \frac{\Phi \vdash \omega : \Gamma' \triangleright \Gamma \quad x \notin \mathtt{dom}(\Gamma') \quad A \leq: B}{\Phi \vdash \omega \times : \Gamma', x : A \triangleright \Gamma, x : B}$,
By inversion, we have
$\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, $x \notin \mathtt{dom}(\Gamma')$.
Hence we have $\Phi \vdash \Gamma \mathtt{Ok}$, $\Phi \vdash \Gamma' \mathtt{Ok}$, and by the domain Lemma, $\mathtt{dom}(\Gamma) \subseteq \mathtt{dom}(\Gamma')$, hence $x \notin \mathtt{dom}(\Gamma)$. Hence, we have $\Phi \vdash \Gamma, x : A \mathtt{Ok}$ and $\Phi \vdash \Gamma', x : A \mathtt{Ok}$

### 5.2.4 Theorem 2

If $\Phi \mid \Gamma \vdash t : \tau$ and $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Phi \mid \Gamma' \vdash t : \tau$

**Proof:** We induct over the structure of typing derivations of $\Phi \mid \Gamma \vdash t : \tau$, assuming $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ holds.

**Case Var and Weaken:** We case split on the weakening $\omega$.

**Case:** $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Phi \mid \Gamma' \vdash x : A$ holds and the derivation $\Delta'$ is the same as $\Delta$

**Case:** $\omega = \omega'\pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, $\Delta_1$ deriving $\Phi \mid \Gamma'' \vdash x : A$, such that:

$$(\mathrm{Weaken}) \frac{() \frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x : A}}{\Phi \mid \Gamma'', x' : A' \vdash x : A} \tag{5.21}$$

**Case:** $\omega = \omega' \times$ Then

$$\Gamma' = \Gamma''', x' : B \tag{5.22}$$
$$\Gamma = \Gamma'', x' : A' \tag{5.23}$$
$$B \leq: A \tag{5.24}$$

**Case:** $x = x'$ Then $A = A'$.
Then we derive the new derivation, $\Delta'$ as so:

$$\text{(Sub-type)}\frac{\text{(var)}\frac{}{\Phi|\Gamma''',x:B\vdash x:B} \quad B \leq: A}{\Phi \mid \Gamma' \vdash x\colon A} \tag{5.25}$$

**Case:** $x \neq x'$ Then

$$\Delta = \text{(Weaken)}\frac{()\frac{\Delta_1}{\Phi|\Gamma''\vdash x:A}}{\Phi \mid \Gamma \vdash x\colon A} \tag{5.26}$$

By induction with $\Phi \vdash \omega : \Gamma''' \triangleright \Gamma''$, we have a derivation $\Delta_1$ of $\Phi \mid \Gamma''' \vdash x\colon A$
We have the weakened derivation:

$$\Delta' = \text{(Weaken)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'''\vdash x:A}}{\Phi \mid \Gamma' \vdash x\colon A} \tag{5.27}$$

**Case Constant:** The constant typing rules, `()`, `true`, `false`, $\mathtt{C}^A$, all proceed by the same logic. Hence I shall only prove the theorems for the case $\mathtt{C}^A$.

$$\text{(Const)}\frac{\Gamma\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A\colon A} \tag{5.28}$$

By inversion, we have $\Phi \vdash \Gamma\mathtt{Ok}$, so we have $\Phi \vdash \Gamma'\mathtt{Ok}$.
Hence

$$\text{(Const)}\frac{\Phi \vdash \Gamma'\mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A\colon A} \tag{5.29}$$

Holds.

**Case Lambda:** By inversion, we have a derivation $\Delta_1$ giving

$$\Delta = \text{(Fn)}\frac{()\frac{\Delta_1}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{5.30}$$

Since $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we have:

$$\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \tag{5.31}$$

Hence, by induction, using $\Phi \vdash \omega\times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive $\Delta_1'$:

$$\Delta' = \text{(Fn)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma',x:A\vdash v:B}}{\Phi \mid \Gamma', x : A \vdash \lambda x : A.v\colon A \to B} \tag{5.32}$$

**Case Sub-typing:**

$$\text{(Sub-type)}\frac{\Phi \mid \Gamma \vdash v\colon A \quad A \leq: B}{\Phi \mid \Gamma \vdash v\colon B} \tag{5.33}$$

by inversion, we have a derivation $\Delta_1$

$$()\frac{\Delta_1}{\Phi \mid \Gamma \vdash v\colon A} \tag{5.34}$$

So by induction, we have a derivation $\Delta_1'$ such that:

$$\text{(Sub-type)}\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:a} \quad A \leq: B}{\Phi \mid \Gamma' \vdash v\colon B} \tag{5.35}$$

**Case Return:** We have the sub-derivation $\Delta_1$ such that

$$\Delta = (\text{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{5.36}$$

Hence, by induction, with $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$, we find the derivation $\Delta_1'$ such that:

$$\Delta' = (\text{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \texttt{return}v\colon \texttt{M}_1 A} \tag{5.37}$$

**Case Apply:** By inversion, we have derivations $\Delta_1$, $\Delta_2$ such that

$$\Delta = (\text{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{5.38}$$

By induction, this gives us the respective derivations: $\Delta_1', \Delta_2'$ such that

$$\Delta' = (\text{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash v_1\ v_2\colon B} \tag{5.39}$$

**Case If:** By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\colon A} \tag{5.40}$$

By induction, this gives us the sub-derivations $\Delta_1', \Delta_2', \Delta_3'$ such that

$$\Delta' = (\text{If})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_1:A} \quad ()\frac{\Delta_3'}{\Phi|\Gamma'\vdash v_2:A}}{\Phi \mid \Gamma' \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2\colon A} \tag{5.41}$$

**Case Bind:** By inversion, we have derivations $\Delta_1, \Delta_2$ such that:

$$\Delta = (\text{Bind})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\texttt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2\colon \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{5.42}$$

If $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ then $\Phi \vdash \omega\times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive $\Delta_1'$, $\Delta_2'$ such that:

$$\Delta' = (\text{Bind})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1:\texttt{M}_{\mathbb{E}_1} A} \quad ()\frac{\Delta_2'}{\Phi|\Gamma',x:A\vdash v_2:\texttt{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2\colon \texttt{M}_{\epsilon_1\cdot\epsilon_2} B} \tag{5.43}$$

**Case Effect-Abstraction:** By inversion,we have derivation $\Delta_1$ deriving

$$(\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{5.44}$$

By $\alpha$-conversion, we have $\iota\pi : \Phi, \alpha \triangleright \Phi$, So we have $\Phi, \alpha \vdash \omega : \Gamma' \triangleright \Gamma$ so by induction, there exists $\Delta_1$ deriving:

$$\Delta' = (\text{Effect-Abs})\frac{()\frac{\Delta_1}{\Phi,\alpha|\Gamma'\vdash v:A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.v\colon \forall\alpha.A} \tag{5.45}$$

**Case Effect-Application:** By inversionwe have derivation $\Delta_1$ deriving

$$(\text{Effect-App})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{5.46}$$

So by induction, we have $\Delta_1'$ deriving

$$(\text{Effect-App})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v:\forall\alpha.A} \quad \Phi\vdash\epsilon}{\Phi \mid \Gamma' \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{5.47}$$

# Chapter 6

# Substitution

We need to define substitutions of effects on effects, effects on types, effects on terms, terms on terms.

## 6.1 Effect Substitutions

Define a substitution, $\sigma$ as

$$\sigma ::= \diamond \mid \sigma, \alpha := \epsilon \tag{6.1}$$

Define the free-effect Variables of $\sigma$:

$$\begin{aligned} fev(\diamond) &= \emptyset \\ fev(\sigma, \alpha := \epsilon) &= fev(\sigma) \cup fev(\epsilon) \end{aligned}$$

We define the property:

$$\alpha \# \sigma \Leftrightarrow \alpha \notin (\texttt{dom}(\sigma) \cup fev(\sigma)) \tag{6.2}$$

### 6.1.1 Action of Effect Substitution on Effects

Define the action of applying an effect substitution to an effect symbol:

$$\sigma(\epsilon) \tag{6.3}$$

$$\sigma(e) = e \tag{6.4}$$

$$\sigma(\epsilon_1 \cdot \epsilon_2) = (\sigma(\epsilon_1)) \cdot (\sigma(\epsilon_2)) \tag{6.5}$$

$$\diamond(\alpha) = \alpha \tag{6.6}$$

$$(\sigma, \beta := \epsilon)(\alpha) = \sigma(\alpha) \tag{6.7}$$

$$(\sigma, \alpha := \epsilon)(\alpha) = \epsilon \tag{6.8}$$

### 6.1.2 Action of Effect Substitution on Types

Define the action of applying an effect substitution, $\sigma$ to a type $\tau$ as:

$$\tau\,[\sigma]$$

Defined as so

$$\gamma\,[\sigma] = \gamma \tag{6.9}$$
$$(A \to \mathtt{M}_\epsilon B)\,[\sigma] = (A\,[\sigma]) \to \mathtt{M}_{\sigma(\epsilon)}(B\,[\sigma]) \tag{6.10}$$
$$(\mathtt{M}_\epsilon A)\,[\sigma] = \mathtt{M}_{\sigma(\epsilon)}(A\,[\sigma]) \tag{6.11}$$
$$(\forall\alpha.A)\,[\sigma] = \forall\alpha.(A\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{6.12}$$

### 6.1.3 Action of Effect-Substitution on Type Environments

Define the action of effect substitution on type environments:

$$\Gamma\,[\sigma]$$

Defined as so:

$$\diamond\,[\sigma] = \diamond$$
$$(\Gamma, x : A)\,[\sigma] = (\Gamma\,[\sigma]\,, x : (A\,[\sigma]))$$

### 6.1.4 Action of Effect Substitution on Terms

Define the action of effect-substitution on terms:

$$x\,[\sigma] = x \tag{6.13}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^{(A[\sigma])} \tag{6.14}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : (A\,[\sigma]).(C\,[\sigma]) \tag{6.15}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\sigma(\epsilon),(A[\sigma])}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{6.16}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{6.17}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \tag{6.18}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \quad \text{If } \alpha\#\sigma \tag{6.19}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \sigma(\epsilon) \tag{6.20}$$
$$\tag{6.21}$$

### 6.1.5 Well-Formed-ness

For any two effect-environments, and a substitution, define the well-formed-ness relation:

$$\Phi' \vdash \sigma : \Phi \tag{6.22}$$

- (Nil) $\frac{\Phi' \mathtt{Ok}}{\Phi' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi' \vdash \sigma : \Phi \quad \Phi' \vdash \epsilon \quad \alpha \notin \Phi}{\Phi' \vdash \sigma, \alpha := \epsilon : (\Phi, \alpha)}$

### 6.1.6 Property 1

If $\Phi' \vdash \sigma : \Phi$ then $\Phi' \mathtt{Ok}$ (By the Nil case) and $\Phi \mathtt{Ok}$ Since each use of the extend case preserves $\mathtt{Ok}$.

### 6.1.7 Property 2

If $\Phi' \vdash \sigma : \Phi$ then $\omega : \Phi' \triangleright \Phi' \implies \Phi'' \vdash \sigma : \Phi$ since $\Phi' \vdash \epsilon \implies \Phi'' \vdash \epsilon$ and $\Phi' \mathtt{Ok} \implies \Phi'' \mathtt{Ok}$

### 6.1.8 Property 3

If $\Phi' \vdash \sigma \colon \Phi$ then

$$\alpha \notin \Phi \wedge \alpha \notin \Phi' \implies (\Phi', \alpha) \vdash (\sigma, \alpha := \alpha) \colon (\Phi, \alpha) \tag{6.23}$$

Since $\iota\pi : \Phi', \alpha \triangleright \Phi'$ so $\Phi', \alpha \vdash \sigma \colon \Phi$ and $\Phi', \alpha \vdash \alpha$

## 6.2 Substitution Preserves the Well-formed-ness of Effects

I.e.

$$\Phi \vdash \epsilon \wedge \Phi' \vdash \iota \colon \Phi \implies \Phi' \vdash \sigma(\epsilon) \tag{6.24}$$

**Proof:**

**Case Ground:** $\sigma(e) = e$, so $\Phi' \vdash \sigma(\epsilon)$ holds.

**Case Multiply:** By inversion, $\Phi \vdash \epsilon_1$ and $\Phi \vdash \epsilon_2$ so $\Phi' \vdash \sigma(\epsilon_1)$ and $\Phi' \vdash \sigma(\epsilon_2)$ by induction and hence $\Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2)$

**Case Var:** By inversion, $\Phi = \Phi'', \alpha$ and $\Phi'', \alpha\mathtt{Ok}$. Hence byu case splitting on $\iota$, we see that $\sigma = \sigma', \alpha := \epsilon$.

So by inversion, $\sigma \vdash \epsilon$ so $\Phi' \vdash \sigma(\alpha) = \epsilon$

**Case Weaken:** By inversion$\Phi = \Phi'', \beta$ and $\Phi'' \vdash \alpha$, so $\sigma = \sigma'\beta := \epsilon$.

So $\Phi' \vdash \sigma' \colon \Phi''$.

hence by induction, $\Phi' \vdash \sigma'(a)$, so $\Phi' \vdash \sigma(\alpha)$ since $\alpha \neq \beta$)

### 6.2.1 Effect Substitution preserves the sub-effect relation

If $\Phi' \vdash \sigma \colon \Phi$ and $\epsilon_1 \leq_\Phi \epsilon_2$, then $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

**Proof:** For any ground substitution $\sigma'$ of $\Phi'$, then $\sigma\sigma'$ (the substitution $\sigma'$ applied after $\sigma$) is also a ground substitution.

So $\epsilon_1 [\sigma] [\sigma'] \leq \epsilon_2 [\sigma] [\sigma']$.

So $\epsilon_1 [\sigma] \leq_{\Phi'} \epsilon_2 [\sigma]$.

### 6.2.2 Substitution preserves well-formed-ness of Types

$$\Phi' \vdash \sigma \colon \Phi \wedge \Phi \vdash A \implies \Phi' \vdash A [\sigma] \tag{6.25}$$

**Proof:**

**Case Ground:** $\Phi'\mathtt{Ok}$ so $\Phi' \vdash \gamma$ and $\gamma [\sigma] = \gamma$.

Hence $\Phi' \vdash \gamma [\sigma]$.

**Case Lambda:** By inversion$\Phi \vdash A$ and $\Phi \vdash B$.

So by induction, $\Phi' \vdash A [\sigma]$ and $\Phi' \vdash B [\sigma]$.

So

$$\Phi' \vdash (A [\sigma]) \to (B [\sigma]) \tag{6.26}$$

So

$$\Phi' \vdash (A \to B) [\sigma] \tag{6.27}$$

**Case Computation:** By inversion, $\Phi \vdash \epsilon$ and $\Phi \vdash A$ so by induction and substitution of effect preserving effect-well-formed-ness,

$\Phi' \vdash \sigma(\epsilon)$ and $\Phi' \vdash A\,[\sigma]$ so $\Phi \vdash \mathtt{M}_{\sigma(\epsilon)} A\,[\sigma]$ so $\Phi' \vdash (\mathtt{M}_{\epsilon} A)\,[\sigma]$

**Case For All:** By inversion, $\Phi, \alpha \vdash A$. So by picking $\alpha \notin \Phi \wedge \alpha \notin \Phi'$ using $\alpha$-equivalence, we have $(\Phi', \alpha) \vdash (\sigma\alpha := \alpha)\colon (\Phi, \alpha)$.

So by induction $(\Phi, \alpha) \vdash A\,[\sigma, \alpha := \alpha]$
So $(\Phi', \alpha) \vdash A\,[\sigma]$
So $\Phi' \vdash (\forall \alpha . A)\,[\sigma]$

### 6.2.3 Substitution of effects preserves Sub-Typing Relation

If $\Phi' \vdash \sigma\colon \Phi$ and $A \leq:_{\Phi} B$ then $A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$

**Proof:** By induction on the sub-typing relation

**Case Ground:** By inversion, $A \leq:_{\gamma} B$, so $A, B$ are ground types. Hence $A\,[\sigma] = A$ and $B\,[\sigma] = B$. So $A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$

**Case Fn:** By inversion, $A' \leq:_{\Phi} A$ and $B \leq:_{\Phi} B'$.
So by induction, $A'\,[\sigma] \leq:_{\Phi'} A\,[\sigma]$ and $B\,[\sigma] \leq:_{\Phi'} B'\,[\sigma]$.
So $(A\,[\sigma]) \to (B\,[\sigma]) \leq:_{\Phi'} (A'\,[\sigma]) \to (B'\,[\sigma])$
So $(A \to B)\,[\sigma] \leq:_{\Phi'} (A' \to B')\,[\sigma]$

**Case Computation:** By inversion, $A \leq:_{\Phi} B$, $\epsilon_1 \leq_{\Phi} \epsilon_2$.
So by induction and substitution preserving the sub-effect relation,
$A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]$ and $\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)$
So $\mathtt{M}_{\sigma(\epsilon_1)}(A\,[\sigma]) \leq:_{\Phi'} \mathtt{M}_{\sigma(\epsilon_2)}(B\,[\sigma])$
So $(\mathtt{M}_{\epsilon_1} A)\,[\sigma] \leq:_{\Phi'} (\mathtt{M}_{\epsilon_2} B)\,[\sigma]$

### 6.2.4 Substitution preserves well-formed-ness of Type Environments

If $\Phi \vdash \Gamma \mathtt{Ok}$ and $\Phi' \vdash \sigma\colon \Phi$ then $\Phi' \vdash \Gamma\,[\sigma]\,\mathtt{Ok}$

**Proof:**

**Case Nil:** $\Phi \mathtt{Ok} \implies \Phi' \mathtt{Ok}$ so $\Phi' \vdash \diamond \mathtt{Ok}$ and $\diamond\,[\sigma] = \diamond$

**Case Var:** By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$ and $\Phi \vdash A$.
By induction and substitution preserving well-formed-ness of types, $\Phi' \vdash \Gamma'\,[\sigma]\,\mathtt{Ok}$ and $\Phi' \vdash A\,[\sigma]$.
So $\Phi' \vdash (\Gamma'\,[\sigma], x : A\,[\sigma])\mathtt{Ok}$.
Hence $\Phi' \vdash \Gamma, x : A\,[\sigma]\,\mathtt{Ok}$.

### 6.2.5 Effect-Polymorphism Preserves the Typing Relation

If $\Phi' \vdash \sigma\colon \Phi$ and $\Phi \mid \Gamma \vdash v\colon A$, then $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]$

**Proof:**

**Case Const:** By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$.
So $\Phi' \vdash \Gamma \mathtt{Ok}$
So $\Phi' \mid \Gamma\,[\sigma] \vdash \mathtt{C}^{A[\sigma]}\colon A\,[\sigma]$

**Case True, False, Unit:** The logic is the same for each of these cases, so we look at the case `true` only.

By inversion, $\Phi \vdash \Gamma \mathtt{Ok}$.

So $\Phi' \vdash \Gamma \mathtt{Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{true}\colon \mathtt{Bool}$

Since $\mathtt{true}[\sigma] = \mathtt{true}$ and $\mathtt{Bool}[\sigma] = \mathtt{Bool}$.

**Case Var:** By inversion$\Gamma = \Gamma', x : A$ and $\Phi \vdash \Gamma', x : A \mathtt{Ok}$.

So since substitution preserves well-formed-ness of type environments, $\Phi' \vdash \Gamma'[\sigma], x : A[\sigma]\, \mathtt{Ok}$

So $\Phi' \mid \Gamma[\sigma] \vdash x\colon A[\sigma]$

Since $x[\sigma] = x$

**Case Weaken:** By inversion$\Gamma = \Gamma', y : B$, $\Phi \vdash B$, and $\Phi \mid \Gamma' \vdash x\colon A$. $x \neq y$

By induction and the theorem that effect-substitution preserves type well-formed-ness, we have: $\Phi' \mid \Gamma'[\sigma] \vdash x\colon A[\sigma]$ and $\Phi' \vdash B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash x[\sigma] : A[\sigma]$

Since $x[\sigma] = x$, $\Gamma[\sigma] = (\Gamma'[\sigma], y : B[\sigma])$

**Case Lambda:** By inversion$\Phi \mid \Gamma, x : A \vdash v\colon B$.

So, by induction $\Phi' \mid (\Gamma, x : A)[\sigma] \vdash v[\sigma] : B[\sigma]$.

So, $\Phi \mid \Gamma[\sigma], x : A[\sigma] \vdash v[\sigma] : B[\sigma]$.

Hence by the lambda type rule,

$\Phi' \mid \Gamma[\sigma] \vdash \lambda x : A[\sigma] .v[\sigma] : (A[\sigma]) \to (B[\sigma])$

So

$\Phi' \mid \Gamma[\sigma] \vdash (\lambda x : A.v)[\sigma] : (A \to B)[\sigma])$

**Case Apply:** By inversion, $\Phi \mid \Gamma \vdash v_1\colon A \to B$, $\Phi \mid \Gamma \vdash V_2\colon A$.

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : (A[\sigma]) \to (B[\sigma])$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1[\sigma])\ (v_2[\sigma])\colon B[\sigma]$.

So $\Phi' \mid \Gamma[\sigma] \vdash (v_1\ v_2)[\sigma] : (A \to B)[\sigma]$

**Case Subtype:** By inversion, $\Phi \mid \Gamma \vdash v\colon A$ and $\Phi \vdash A \leq: B$

So by induction and effect-substitution preserving sub-typing, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$ and $\Phi' \vdash A[\sigma] \leq: B[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : B[\sigma]$

**Case Return:** By inversion, $\Phi \mid \Gamma \vdash v\colon A$

So by induction, $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{return}(v[\sigma])\colon \mathtt{M}_{\mathbf{1}}(A[\sigma])$

Hence $\Phi' \mid \Gamma[\sigma] \vdash (\mathtt{return} v)[\sigma] : (\mathtt{M}_{\mathbf{1}} A)[\sigma]$

**Case Bind:** By inversion, $\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B$.

So by induction: $\Phi' \mid \Gamma[\sigma] \vdash v_1[\sigma] : \mathtt{M}_{\sigma(\epsilon_1)}(A[\sigma])$, and $\Phi' \mid \Gamma[\sigma], x : A[\sigma] \vdash v_2\colon \mathtt{M}_{\sigma(\epsilon_2)}(B[\sigma])$.

And so $\Phi' \mid \Gamma[\sigma] \vdash \mathtt{do}\ x \leftarrow (v_1[\sigma])\ \mathtt{in}\ (v_2[\sigma])\colon \mathtt{M}_{\sigma(\epsilon_1)\cdot(\epsilon_2[\sigma])} B[\sigma]$

**Case If:** By inversion, $\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1\colon A$, and $\Phi \mid \Gamma \vdash v_2\colon A$

So by induction $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \mathtt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_1\colon A[\sigma]$, and $\Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : \mathtt{Bool}$, $\Phi' \mid \Gamma[\sigma] \vdash v_2\colon A[\sigma]$. (Since $\mathtt{Bool}[\sigma] = \mathtt{Bool}$)

Hence:

$\Phi' \mid \Gamma[\sigma] \vdash \mathtt{if}_{A[\sigma]}\ v[\sigma]\ \mathtt{then}\ v_1[\sigma]\ \mathtt{else}\ v_2[\sigma] : A[\sigma]$

So $\Phi' \mid \Gamma[\sigma] \vdash (\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2)[\sigma] : A[\sigma]$

**Case Effect-lambda:** By inversion, $\Phi, \alpha \mid \Gamma \vdash v \colon A$.

So by the substitution property 3 (**TODO: Is this correct/reference correctly**), pick $\alpha \notin \Phi' \wedge \alpha \notin \Phi$ so we have:

$$(\Phi', \alpha) \vdash (\sigma, \alpha := \alpha) \colon (\Phi, \alpha)$$

So by induction, $\Phi', \alpha \mid \Gamma\,[\sigma, \alpha := \alpha] \vdash v\,[\sigma, \alpha := \alpha] : A\,[\sigma, \alpha := \alpha]$

So $\Phi', \alpha \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]$ since $\alpha \notin \Phi' \wedge \alpha \notin \Phi$.

So $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : (\forall \alpha.A)\,[\sigma]$

**Case Effect-Apply:** By inversion, $\Phi \mid \Gamma \vdash v \colon \forall \alpha.A$, $\Phi \vdash \epsilon$.

So by induction and effect-substitution preserving well-formed-ness of effects: $\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : (\forall \alpha.A)\,[\sigma]$ and $\Phi' \vdash \sigma(\epsilon)$

So $\Phi' \mid \Gamma\,[\sigma] \vdash (v\,[\sigma])\ (\sigma(\epsilon)) \colon A\,[\sigma]\,[\sigma(\epsilon)/\alpha]$.

Since $\alpha \# \sigma$, we can commute the applications of substitution. **TODO: Do I need to prove this?**

So, $\Phi' \mid \Gamma\,[\sigma] \vdash (v\ \epsilon)\,[\sigma] : A\,[\epsilon/\alpha]\,[\sigma]$

## 6.3 The Identity Substitution on Effect Environments

For each type environment $\Phi$, define the identity substitution $I_\Phi$ as so:

- $I_\diamond = \diamond$
- $I_{(\Phi,\alpha} = (I_\Phi, \alpha := \alpha)$

### 6.3.1 Properties of the Identity Substitution

**Property 1** If $\Phi\texttt{Ok}$ then $\Phi \vdash I_\Phi \colon \Phi$, proved trivially by induction over the `Ok` relation.

**Property 2** **TODO: The denotational property of id-substitution**

## 6.4 Single Substitution on Effect Environments

If $\Phi \vdash \epsilon$, let the single substitution $\Phi \vdash [\epsilon/\alpha] : \Phi, \alpha$, be defined as:

$$[x/\alpha] = (I_\Phi, \alpha := \epsilon) \tag{6.28}$$

## 6.5 Term-Term Substitutions

### 6.5.1 Substitutions as SNOC lists

$$\sigma ::= \diamond \mid \sigma, x := v \tag{6.29}$$

### 6.5.2 Trivial Properties of substitutions

$\texttt{fv}(\sigma)$

$$\texttt{fv}(\diamond) = \emptyset \tag{6.30}$$

$$\texttt{fv}(\sigma, x := v) = \texttt{fv}(\sigma) \cup \texttt{fv}(v) \tag{6.31}$$

$\mathtt{dom}(\sigma)$

$$\mathtt{dom}(\diamond) = \emptyset \tag{6.32}$$
$$\mathtt{dom}(\sigma, x := v) = \mathtt{dom}(\sigma) \cup \{x\} \tag{6.33}$$

$x\#\sigma$

$$x\#\sigma \Leftrightarrow x \notin (\mathtt{fv}(\sigma) \cup \mathtt{dom}(\sigma`)) \tag{6.34}$$

### 6.5.3 Action of substitutions

We define the action of applying a substitution $\sigma$ as

$$t\,[\sigma]$$

$$x\,[\diamond] = x \tag{6.35}$$
$$x\,[\sigma, x := v] = v \tag{6.36}$$
$$x\,[\sigma, x' := v'] = x\,[\sigma] \quad \text{If } x \neq x' \tag{6.37}$$
$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.38}$$
$$(\lambda x : A.C)\,[\sigma] = \lambda x : A.(C\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.39}$$
$$(\mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2)\,[\sigma] = \mathtt{if}_{\epsilon,A}\ v\,[\sigma]\ \mathtt{then}\ C_1\,[\sigma]\ \mathtt{else}\ C_2\,[\sigma] \tag{6.40}$$
$$(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ v_2\,[\sigma] \tag{6.41}$$
$$(\mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2) = \mathtt{do}\ x \leftarrow (C_1\,[\sigma])\ \mathtt{in}\ (C_2\,[\sigma]) \quad \text{If } x\#\sigma \tag{6.42}$$
$$(\Lambda\alpha.v)\,[\sigma] = \Lambda\alpha.(v\,[\sigma]) \tag{6.43}$$
$$(v\ \epsilon)\,[\sigma] = (v\,[\sigma])\ \epsilon \tag{6.44}$$
$$\tag{6.45}$$

### 6.5.4 Well-Formed-ness

Define the relation

$$\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$$

by:

- (Nil) $\frac{\Phi \vdash \Gamma' \mathtt{Ok}}{\Phi \mid \Gamma' \vdash \diamond : \diamond}$
- (Extend) $\frac{\Phi \mid \Gamma' \vdash \sigma : \Gamma \quad x \notin \mathtt{dom}(\Gamma) \quad \Phi \mid \Gamma' \vdash v : A}{\Phi \mid \Gamma' \vdash (\sigma, x := v) : (\Gamma, x : A)}$

### 6.5.5 Simple Properties Of Substitution

If $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$ then: **TODO: Number these**

**Property 1: $\Phi \vdash \Gamma\mathtt{Ok}$ and $\Phi \vdash \Gamma'\mathtt{Ok}$** Since $\Phi \vdash \Gamma'\mathtt{Ok}$ holds by the Nil-axiom. $\Phi \vdash \Gamma\mathtt{Ok}$ holds by induction on the well-formed-ness relation.

**Property 2: $\omega : \Gamma'' \triangleright \Gamma'$ implies $\Phi \mid \Gamma'' \vdash \sigma \colon \Gamma$** . By induction over well-formed-ness relation. For each $x := v$ in $\sigma$, $\Phi \mid \Gamma'' \vdash v \colon A$ holds if $\Phi \mid \Gamma' \vdash v \colon A$ holds.

**Property 3:** $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma''))$ **implies** $\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$ Since $\iota\pi : \Gamma', x : A \triangleright \Gamma'$, so by (Property 2) **TODO: Better referencing here**,

$$\Phi \mid \Gamma', x : A \vdash \sigma \colon \Gamma$$

In addition, $\Phi \mid \Gamma', x : A \vdash x \colon A$ trivially, so by the rule **Extend**, well-formed-ness holds for

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := v) \colon (\Gamma, x : A) \tag{6.46}$$

### 6.5.6 Substitution Preserves Typing

We have the following non-trivial property of substitution:

$$(\Phi \mid \Gamma \vdash v \colon A) \wedge (\Phi \mid \Gamma' \vdash \sigma \colon \Gamma) \Rightarrow (\Phi \mid \Gamma' \vdash v\,[\sigma] : A) \tag{6.47}$$

Assuming $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma$, we induct over the typing relation, proving $\Phi \mid \Gamma \vdash v \colon A \implies \Phi \mid \Gamma' \vdash v \colon A$

**Proof:**

**Case Var:** By inversion $\Gamma = (\Gamma'', x : A)$ So

$$\Phi \mid \Gamma'', x : A \vdash x \colon A \tag{6.48}$$

So by inversion, since $\Phi \mid \Gamma' \vdash \sigma \colon \Gamma'', x : A$,

$$\sigma = (\sigma', x := v) \wedge \Phi \mid \Gamma' \vdash v \colon A \tag{6.49}$$

By the definition of the effect of substitutions, $x\,[\sigma] = v$, So

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{6.50}$$

holds.

**Case Weaken:** By inversion, $\Gamma = \Gamma'', y : B, x \neq y$, and there exists $\Delta$ such that

$$(\text{Weaken})\frac{()\frac{\Delta}{\Phi \mid \Gamma'' \vdash x : A}}{\Phi \mid \Gamma'', y : B \vdash x \colon A} \tag{6.51}$$

By inversion, $\sigma = \sigma', y := v$ and:

$$\Phi \mid \Gamma' \vdash \sigma' \colon \Gamma'' \tag{6.52}$$

So by induction,

$$\Phi \mid \Gamma' \vdash x\,[\sigma'] : A \tag{6.53}$$

And so by definition of the effect of $\sigma$, $x\,[\sigma] = x\,[\sigma']$

$$\Phi \mid \Gamma' \vdash x\,[\sigma] : A \tag{6.54}$$

**Case Lambda:** By inversion, there exists $\Delta$ such that:

$$(\mathrm{Fn})\frac{()\frac{\Delta}{\Phi|\Gamma,x:A\vdash v:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\colon A \to B} \tag{6.55}$$

Using alpha equivalence, we pick $x \notin (\mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Gamma'))$ Hence, by property 3, we have

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x)\colon \Gamma, x : A \tag{6.56}$$

So by induction using $\sigma, x := x$, we have $\Delta'$ such that:

$$(\mathrm{Fn})\frac{()\frac{\Delta'}{\Phi|\Gamma',x:A\vdash v[\sigma,x:=v]:B}}{\Phi \mid \Gamma \vdash \lambda x : A.v\,[\sigma, x := x] : A \to B} \tag{6.57}$$

Since $\lambda x : A.(v\,[\sigma, x := x]) = \lambda x : A.(v\,[\sigma]) = (\lambda x : A.v)\,[\sigma]$, we have a typing derivation for $\Phi \mid \Gamma' \vdash (\lambda x : A.v)\,[\sigma] : A \to B$.

**Case Constants:** We use the same logic for all constants, $\mathtt{()}, \mathtt{true}, \mathtt{false}, \mathtt{C}^A$:
$\Phi \mid \Gamma \vdash \sigma\colon \Gamma \Rightarrow \Phi \vdash \Gamma'\mathtt{Ok}$ and:

$$\mathtt{C}^A\,[\sigma] = \mathtt{C}^A \tag{6.58}$$

So

$$(\mathrm{Const})\frac{\Phi \vdash \Gamma'\mathtt{Ok}}{\Phi \mid \Gamma' \vdash \mathtt{C}^A\colon A} \tag{6.59}$$

### 6.5.7 Computation Terms

**Case Return:** By inversion, we have $\Delta_1$ such that:

$$(\mathrm{Return})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \mathtt{return}v\colon \mathtt{M}_1 A} \tag{6.60}$$

By induction, we have $\Delta_1'$ such that

$$(\mathrm{Return})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \mathtt{return}(v\,[\sigma])\colon \mathtt{M}_1 A} \tag{6.61}$$

Since $(\mathtt{return}v)\,[\sigma] = \mathtt{return}(v\,[\sigma])$, the type derivation above holds for $\Phi \mid \Gamma' \vdash (\mathtt{return}v)\,[\sigma] : \mathtt{M}_1 A$.

**Case Apply:** By inversion, we have $\Delta_1$, $\Delta_2$ such that:

$$(\mathrm{Apply})\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:A\to B} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash v_1\ v_2\colon B} \tag{6.62}$$

By induction on $\Delta_1, \Delta_2$, we have $\Delta_1', \Delta_2'$ such that

$$(\mathrm{Apply})\frac{()\frac{\Delta_1'}{\Phi|\Gamma'\vdash v_1[\sigma]:A\to B} \quad ()\frac{\Delta_2'}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash (v_1\,[\sigma])\ (v_2\,[\sigma])\colon B} \tag{6.63}$$

Since $(v_1\ v_2)\,[\sigma] = (v_1\,[\sigma])\ (v_2\,[\sigma])$, we the above derivation holds for $\Phi \mid \Gamma' \vdash (v_1\ v_2)\,[\sigma] : B$

**Case If:** By inversion, we have $\Delta_1, \Delta_2, \Delta_3$ such that:

$$\text{(If)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v:\texttt{Bool}} \quad ()\frac{\Delta_2}{\Phi|\Gamma\vdash v_1:A} \quad ()\frac{\Delta_3}{\Phi|\Gamma\vdash v_2:A}}{\Phi \mid \Gamma \vdash \texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2 \colon A} \tag{6.64}$$

By induction on $\Delta_1, \Delta_2, \Delta_3$, we derive $\Delta'_1, \Delta'_2, \Delta'_3$ such that:

$$\text{(If)}\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v[\sigma]:\texttt{Bool}} \quad ()\frac{\Delta'_2}{\Phi|\Gamma'\vdash v_1[\sigma]:A} \quad ()\frac{\Delta'_3}{\Phi|\Gamma'\vdash v_2[\sigma]:A}}{\Phi \mid \Gamma' \vdash \texttt{if}_A\ (v\,[\sigma])\ \texttt{then}\ (v_1\,[\sigma])\ \texttt{else}\ (v_2\,[\sigma]) \colon A} \tag{6.65}$$

Since $(\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2)\,[\sigma] = \texttt{if}_A\ (v\,[\sigma])\ \texttt{then}\ (v_1\,[\sigma])\ \texttt{else}\ (v_2\,[\sigma])$ The derivation above holds for $\Phi \mid \Gamma' \vdash (\texttt{if}_A\ v\ \texttt{then}\ v_1\ \texttt{else}\ v_2)\,[\sigma] : A$

**Case Bind:** By inversion, there exist $\Delta_1, \Delta_2$ such that:

$$\text{(Bind)}\frac{()\frac{\Delta_1}{\Phi|\Gamma\vdash v_1:\mathtt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma,x:A\vdash v_2:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma \vdash \texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{6.66}$$

Using alpha-equivalence, we pick $x \notin (\texttt{dom}(\Gamma) \cup \texttt{dom}(\Gamma'))$. Hence by property 3,

$$\Phi \mid (\Gamma, x : A) \vdash (\sigma, x := x) \colon (\Gamma, x : A)$$

By induction on $\Delta_1, \Delta_2$, we have $\Delta'_1, \Delta'_2$ such that:

$$\text{(Bind)}\frac{()\frac{\Delta'_1}{\Phi|\Gamma'\vdash v_1[\sigma]:\mathtt{M}_{\epsilon_1}A} \quad ()\frac{\Delta_2}{\Phi|\Gamma',x:A\vdash v_2[\sigma,x:=x]:\mathtt{M}_{\epsilon_2}B}}{\Phi \mid \Gamma' \vdash \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma, x := x]) \colon \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B} \tag{6.67}$$

Since $(\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] = \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma]) = \texttt{do}\ x \leftarrow (v_1\,[\sigma])\ \texttt{in}\ (v_2\,[\sigma, x := x])$, the above derivation holds for $\Phi \mid \Gamma' \vdash (\texttt{do}\ x \leftarrow v_1\ \texttt{in}\ v_2)\,[\sigma] : \mathtt{M}_{\epsilon_1\cdot\epsilon_2}B$

**Case Sub-type:** By inversion, there exists $\Delta$ such that

$$\text{(sub-type)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:A} \quad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v \colon B} \tag{6.68}$$

By induction on $\Delta$ we derive $\Delta'$ such that:

$$\text{(sub-type)}\frac{()\frac{\Delta'}{\Phi|\Gamma'\vdash v[\sigma]:A} \quad A \leq:_{\Phi} B}{\Phi \mid \Gamma \vdash v\,[\sigma] : B} \tag{6.69}$$

**Case Effect-Lambda:** By inversion, there exists $\Delta$ such that

$$\text{(Effect-abs)}\frac{()\frac{\Delta}{\Phi,\alpha|\Gamma\vdash v:A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.v \colon \forall\alpha.A} \tag{6.70}$$

It is also the case that $\iota\pi : \Phi, \alpha \triangleright \Phi$.
So $\Phi, \alpha \mid \Gamma' \vdash \sigma \colon \Gamma$
So by induction there exists $\Delta'$,

$$\text{(Effect-abs)}\frac{()\frac{\Delta'}{\Phi,\alpha|\Gamma'\vdash v[\sigma]:A}}{\Phi \mid \Gamma' \vdash \Lambda\alpha.(v\,[\sigma]) \colon \forall\alpha.A} \tag{6.71}$$

Where $\Lambda\alpha.(v\,[\sigma]) = (\Lambda\alpha.v)\,[\sigma]$

**Case Effect Application:** By inversion$\Phi \vdash \epsilon$ and there exists $\Delta$ such that

$$\text{(Effect-App)}\frac{()\frac{\Delta}{\Phi|\Gamma\vdash v:\forall\alpha.A}}{\Phi \mid \Gamma \vdash v\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{6.72}$$

So by induction there exists $\Delta'$ such that:

$$\text{(Effect-App)}\frac{()\frac{\Delta'}{\Phi|\Gamma'\vdash v[\sigma]:\forall\alpha.A}}{\Phi \mid \Gamma' \vdash (v\,[\sigma])\ \epsilon\colon A\,[\epsilon/\alpha]} \tag{6.73}$$

Where $(v\,[\sigma])\ \epsilon = (v\ \epsilon)\,[\sigma]$

## 6.6 The Identity Substitution on Type Environments

For each type environment $\Gamma$, define the identity substitution $I_\Gamma$ as so:

- $I_\diamond = \diamond$
- $I_{(\Gamma,x:A} = (I_\Gamma, x := x)$

### 6.6.1 Properties of the Identity Substitution

**Property 1** If $\Phi \vdash \Gamma\mathtt{Ok}$ then $\Phi \mid \Gamma \vdash I_\Gamma\colon\Gamma$, proved trivially by induction over the well-formed-ness relation.

**Property 2 TODO: The denotational property of id-substitution**

## 6.7 Single Substitution on Type Environments

If $\Phi \mid \Gamma \vdash v\colon A$, let the single substitution $\Phi \mid \Gamma \vdash [v/x] : \Gamma, x : A$, be defined as:

$$[v/x] = (I_\Gamma, x := v) \tag{6.74}$$

# Chapter 7

# Beta Eta Equivalence (Soundness)

## 7.1 Beta and Eta Equivalence

### 7.1.1 Beta-Eta conversions

- (Lambda-Beta) $\frac{\Phi|\Gamma,x{:}A\vdash v_2{:}B \quad \Phi|\Gamma\vdash v_1{:}A}{\Phi|\Gamma\vdash(\lambda x{:}A.v_1)\ v_2 =_{\beta\eta} v_1[v_2/x]{:}B}$
- (Lambda-Eta) $\frac{\Phi|\Gamma\vdash v{:}A\to B}{\Phi|\Gamma\vdash\lambda x{:}A.(v\ x)=_{\beta\eta}v{:}A\to B}$
- (Left Unit) $\frac{\Phi|\Gamma\vdash v_1{:}A \quad \Phi|\Gamma,x{:}A\vdash v_2{:}\mathtt{M}_\epsilon B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow\mathtt{return}v_1\ \mathtt{in}\ v_2=_{\beta\eta}v_2[v_1/x]{:}\mathtt{M}_\epsilon B}$
- (Right Unit) $\frac{\Phi|\Gamma\vdash v{:}\mathtt{M}_\epsilon A}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v\ \mathtt{in}\ \mathtt{return}x=_{\beta\eta}v{:}\mathtt{M}_\epsilon A}$
- (Associativity) $\frac{\Phi|\Gamma\vdash v_1{:}\mathtt{M}_{\epsilon_1}A \quad \Phi|\Gamma,x{:}A\vdash v_2{:}\mathtt{M}_{\epsilon_2}B \quad \Phi|\Gamma,y{:}B\vdash v_3{:}\mathtt{M}_{\epsilon_3}C}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y\leftarrow v_2\ \mathtt{in}\ v_3)=_{\beta\eta}\mathtt{do}\ y\leftarrow(\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2)\ \mathtt{in}\ v_3{:}\mathtt{M}_{\epsilon_1\cdot\epsilon_2\cdot\epsilon_3}C}$
- (Unit) $\frac{\Phi|\Gamma\vdash v{:}\mathtt{Unit}}{\Phi|\Gamma\vdash v=_{\beta\eta}\mathtt{()}{:}\mathtt{Unit}}$
- (if-true) $\frac{\Phi|\Gamma\vdash v_1{:}A \quad \Phi|\Gamma\vdash v_2{:}A}{\Phi|\Gamma\vdash\mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2=_{\beta\eta}v_1{:}A}$
- (if-false) $\frac{\Phi|\Gamma\vdash v_2{:}A \quad \Phi|\Gamma\vdash v_1{:}A}{\Phi|\Gamma\vdash\mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2=_{\beta\eta}v_2{:}A}$
- (If-Eta) $\frac{\Phi|\Gamma,x{:}\mathtt{Bool}\vdash v_2{:}A \quad \Phi|\Gamma\vdash v_1{:}\mathtt{Bool}}{\Phi|\Gamma\vdash\mathtt{if}_A\ v_1\ \mathtt{then}\ v_2[\mathtt{true}/x]\ \mathtt{else}\ v_2[\mathtt{false}/x]=_{\beta\eta}v_2[v_1/x]{:}A}$
- (Effect-beta) $\frac{\Phi\vdash\epsilon \quad \Phi,\alpha|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash(\Lambda\alpha.v\ \epsilon)=_{\beta\eta}v[\epsilon/\alpha]{:}A[\epsilon/\alpha]}$
- (Effect-eta) $\frac{\Phi|\Gamma\vdash v{:}\forall\alpha.A}{\Phi|\Gamma\vdash\Lambda\alpha.(v\ \alpha)=_{\beta\eta}v{:}\forall\alpha.A}$

### 7.1.2 Equivalence Relation

- (Reflexive) $\frac{\Phi|\Gamma\vdash v{:}A}{\Phi|\Gamma\vdash v=_{\beta\eta}v{:}A}$
- (Symmetric) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2{:}A}{\Phi|\Gamma\vdash v_2=_{\beta\eta}v_1{:}A}$
- (Transitive) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2{:}A \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_3{:}A}{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_3{:}A}$

### 7.1.3 Congruences

- (Effect-Abs) $\frac{\Phi,\alpha|\Gamma\vdash v_1=_{\beta\eta}v_2:A}{\Phi|\Gamma\vdash\Lambda\alpha.v_1=_{\beta\eta}\Lambda\alpha.v_2:\forall\alpha.A}$
- (Effect-Apply) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2:\forall\alpha.A \quad \Phi\vdash\epsilon}{\Phi|\Gamma\vdash v_1\ \epsilon=_{\beta\eta}v_2\ \epsilon:A[\epsilon/\alpha]}$
- (Lambda) $\frac{\Phi|\Gamma,x:A\vdash v_1=_{\beta\eta}v_2:B}{\Phi|\Gamma\vdash\lambda x:A.v_1=_{\beta\eta}\lambda x:A.v_2:A\to B}$
- (Return) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_2:A}{\Phi|\Gamma\vdash\mathtt{return}v_1=_{\beta\eta}\mathtt{return}v_2:\mathtt{M}_{\mathbf{1}}A}$
- (Apply) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':A\to B \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_2':A}{\Phi|\Gamma\vdash v_1\ v_2=_{\beta\eta}v_1'\ v_2':B}$
- (Bind) $\frac{\Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':\mathtt{M}_{\epsilon_1}A \quad \Phi|\Gamma,x:A\vdash v_2=_{\beta\eta}v_2':\mathtt{M}_{\epsilon_2}B}{\Phi|\Gamma\vdash\mathtt{do}\ x\leftarrow v_1\ \mathtt{in}\ v_2=_{\beta\eta}\mathtt{do}\ c\leftarrow v_1'\ \mathtt{in}\ v_2':\mathtt{M}_{\epsilon_1\cdot\epsilon_2}B}$
- (If) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v':\mathtt{Bool} \quad \Phi|\Gamma\vdash v_1=_{\beta\eta}v_1':A \quad \Phi|\Gamma\vdash v_2=_{\beta\eta}v_2':A}{\Phi|\Gamma\vdash\mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2=_{\beta\eta}\mathtt{if}_A\ v\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2':A}$
- (Subtype) $\frac{\Phi|\Gamma\vdash v=_{\beta\eta}v':A \quad A\leq:_\Phi B}{\Phi|\Gamma\vdash v=_{\beta\eta}v':B}$

## 7.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

If $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$ then each derivation of $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$ can be converted to a derivation of $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$ by induction over the beta-eta equivalence relation derivation.

### 7.2.1 Equivalence Relations

**Case Reflexive:** By inversion we have a derivation of $\Phi \mid \Gamma \vdash v \colon A$.

**Case Symmetric:** By inversion $\Phi \mid \Gamma \vdash v' =_{\beta\eta} v \colon A$. Hence by induction, derivations of $\Phi \mid \Gamma \vdash v' \colon A$ and $\Phi \mid \Gamma \vdash v \colon A$ are given.

**Case Transitive:** By inversion, there exists $v_2$ such that $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_3 \colon A$. Hence by induction, we have derivations of $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma \vdash v_3 \colon A$

### 7.2.2 Beta-Eta conversions

**Case Lambda:** By inversion, we have $\Phi \mid \Gamma, x : A \vdash v_1 \colon B$ and $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing rules, we have:

$$(\text{Apply})\frac{(\text{Lambda})\frac{\Phi|\Gamma,x:A\vdash v_1:B}{\Phi|\Gamma\vdash\lambda x:A.v_1:A\to B} \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash (\lambda x : A.v_1)\ v_2 \colon A}$$

By the substitution rule **TODO: which?**, we have

$$(\text{Substitution})\frac{\Phi \mid \Gamma, x : A \vdash v_1 \colon B \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash v_1\,[v_2/x] : B}$$

**Case Left Unit:** By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$ and $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B$

Hence we have:

$$\text{(Bind)}\frac{\text{(Return)}\frac{\Phi|\Gamma \vdash v_1 : A}{\Phi|\Gamma \vdash \mathtt{return} v_1 : \mathtt{M}_{\mathbf{1}} A} \quad \Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_\epsilon B}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow \mathtt{return} v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\mathbf{1} \cdot \epsilon} B = \mathtt{M}_\epsilon B} \tag{7.1}$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Phi \mid \Gamma \vdash v_2 \left[ v_1 / x \right] \colon \mathtt{M}_\epsilon B \tag{7.2}$$

**Case Right Unit:** By inversion, we have $\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A$.

Hence we have:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v \colon \mathtt{M}_\epsilon A \quad \text{(Return)}\frac{\text{(var)}\frac{}{\Phi|\Gamma, x : A \vdash x : A}}{\Phi|\Gamma, x : A \vdash \mathtt{return} v : \mathtt{M}_{\mathbf{1}} A}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v\ \mathtt{in}\ \mathtt{return} x \colon \mathtt{M}_{\epsilon \cdot \mathbf{1}} A = \mathtt{M}_\epsilon A} \tag{7.3}$$

**Case Associativity:** By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$.

$$\Phi \vdash (\iota \pi \times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**, $\Phi \mid \Gamma, x : A, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)}\frac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A \quad \text{(Bind)}\frac{\Phi|\Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B \quad \Phi|\Gamma, x : A, y : B \vdash v_3 : \mathtt{M}_{\epsilon_3} C}{\Phi|\Gamma, x : A \vdash x v_2 v_3 : \mathtt{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ (\mathtt{do}\ y \leftarrow v_2\ \mathtt{in}\ v_3) \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.4}$$

and

$$\text{(Bind)}\frac{\text{(Bind)}\frac{\Phi|\Gamma \vdash v_1 : \mathtt{M}_{\epsilon_1} A \quad \Phi|\Gamma, x : A \vdash v_2 : \mathtt{M}_{\epsilon_2} B}{\Phi|\Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \Phi \mid \Gamma, y : B \vdash v_3 \colon \mathtt{M}_{\epsilon_3} C}{\Phi \mid \Gamma \vdash \mathtt{do}\ y \leftarrow (\mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2)\ \mathtt{in}\ v_3 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \tag{7.5}$$

**Case Eta:** By inversion, we have $\Phi \mid \Gamma \vdash v \colon A \to B$

By weakening, we have $\Phi \vdash \iota \pi : (\Gamma, x : A) \triangleright \Gamma$ Hence, we have

$$\text{(Fn)}\frac{\text{(App)}\frac{\Phi|(\Gamma, x : A) \vdash x : A \quad \text{(weakening)}\frac{\Phi|\Gamma \vdash v : A \to B \quad \Phi \vdash \iota \pi : \Gamma, x : A \triangleright \Gamma}{\Phi|\Gamma, x : A \vdash v : A \to B}}{\Phi|\Gamma, x : A \vdash v\ x : B}}{\Phi \mid \Gamma \vdash \lambda x : A.(v\ x) \colon A \to B} \tag{7.6}$$

**Case If-True:** By inversion, we have $\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_2 \colon A$. Hence by the typing lemma **TODO: Which?**, we have $\Phi \vdash \Gamma \mathtt{Ok}$ so $\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$ by the axiom typing rule.

Hence

$$\text{(If)}\frac{\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool} \quad \Phi \mid \Gamma \vdash v_1 \colon A \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{true}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{7.7}$$

**Case If-False:** As above,

Hence

$$\text{(If)}\frac{\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool} \quad \Phi \mid \Gamma \vdash v_1 \colon A \quad \Phi \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ \mathtt{false}\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A} \tag{7.8}$$

**Case If-Eta:** By inversion, we have:

$$\Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool} \tag{7.9}$$

and

$$\Phi \mid \Gamma, x : \mathtt{Bool} \vdash v_2 \colon A \tag{7.10}$$

Hence we also have $\Phi \vdash \Gamma\mathtt{Ok}$. Hence, the following also hold:
$\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}$, and $\Phi \mid \Gamma \vdash \mathtt{false} \colon \mathtt{Bool}$.
Hence by the substitution theorem, we have:

$$(\text{If})\frac{\Phi \mid \Gamma \vdash v_1 \colon \mathtt{Bool} \quad \Phi \mid \Gamma \vdash v_2\,[\mathtt{true}/x] : A \quad \Phi \mid \Gamma \vdash v_2\,[\mathtt{false}/x] : A}{\Phi \mid \Gamma \vdash \mathtt{if}_A\ v_1\ \mathtt{then}\ v_2\,[\mathtt{true}/x]\ \mathtt{else}\ v_2\,[\mathtt{false}/x] : A} \tag{7.11}$$

and

$$\Phi \mid \Gamma \vdash v_2\,[v_1/x] : A \tag{7.12}$$

**Case Effect-Beta:** By inversion, $\Phi, \alpha \mid \Gamma \vdash v \colon A$ and $\Phi \vdash \epsilon$.
Then we have the following type derivation:

$$(\text{Effect-App})\frac{(\text{Effect-Fn})\frac{\Phi,\alpha \mid \Gamma \vdash v:A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v:\forall\alpha.A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash \Lambda\alpha.v\ \epsilon \colon A\,[\epsilon/\alpha]} \tag{7.13}$$

And we can construct the single-effect-substitution:

$$(\text{Single Substitution})\frac{\Phi \vdash \epsilon}{\Phi \vdash [\epsilon/\alpha] : (\Phi, \alpha)} \tag{7.14}$$

Hence by the substitution theorem,

$$\Phi \mid \Gamma \vdash v\,[\epsilon/\alpha] : A\,[\epsilon/\alpha] \tag{7.15}$$

**Case Effect-Eta:** By inversion$\Phi \mid \Gamma \vdash v \colon \forall\alpha.A$
So the following derivation holds:

$$(\text{Effect-Fn})\frac{(\text{Effect-App})\frac{(\text{Effect-weakening})\frac{\Phi \mid \Gamma \vdash v:\forall\alpha.A}{\Phi,\alpha \mid \Gamma \vdash v:\forall\alpha.A} \quad \Phi,\alpha \vdash \alpha}{\Phi,\alpha \mid \Gamma \vdash v\ \alpha:A[\alpha/\alpha]=A}}{\Phi \mid \Gamma \vdash \Lambda\alpha.(v\ \alpha) \colon \forall\alpha.A} \tag{7.16}$$

And

$$\Phi \mid \Gamma \vdash v \colon \forall\alpha.A \tag{7.17}$$

### 7.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

**Case Lambda:** By inversion, $\Phi \mid \Gamma, x : A \vdash v_1 =_{\beta\eta} v_2 \colon B$. Hence by induction $\Phi \mid \Gamma, x : A \vdash v_1 \colon B$, and $\Phi \mid \Gamma, x : A \vdash v_2 \colon B$.
So

$$\Phi \mid \Gamma \vdash \lambda x : A.v_1 \colon A \to B \tag{7.18}$$

and

$$\Phi \mid \Gamma \vdash \lambda x : A.v_2 \colon A \to B \tag{7.19}$$

Hold.

**Case Return:** By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$, so by induction

$$\Phi \mid \Gamma \vdash v_1 \colon A$$

and

$$\Phi \mid \Gamma \vdash v_2 \colon A$$

Hence we have

$$\Phi \mid \Gamma \vdash \mathtt{return} v_1 \colon \mathtt{M}_{\mathbf{1}} A$$

and

$$\Phi \mid \Gamma \vdash \mathtt{return} v_2 \colon \mathtt{M}_{\mathbf{1}} A$$

**Case Apply:** By inversion, we have $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A \to B$ and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$. Hence we have by induction $\Phi \mid \Gamma \vdash v_1 \colon A \to B$, $\Phi \mid \Gamma \vdash v_2 \colon A$, $\Phi \mid \Gamma \vdash v_1' \colon A \to B$, and $\Phi \mid \Gamma \vdash v_2' \colon A$.

So we have:

$$\Phi \mid \Gamma \vdash v_1\ v_2 \colon B \tag{7.20}$$

and

$$\Phi \mid \Gamma \vdash v_1'\ v_2' \colon B \tag{7.21}$$

**Case Bind:** By inversion, we have: $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon \mathtt{M}_{\epsilon_1} A$ and $\Phi \mid \Gamma, x : A \vdash v_2 =_{\beta\eta} v_2' \colon \mathtt{M}_{\epsilon_2} B$. Hence by induction, we have $\Phi \mid \Gamma \vdash v_1 \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma \vdash v_1' \colon \mathtt{M}_{\epsilon_1} A$, $\Phi \mid \Gamma, x : A \vdash v_2 \colon \mathtt{M}_{\epsilon_2} B$, and $\Phi \mid \Gamma, x : A \vdash v_2' \colon \mathtt{M}_{\epsilon_2} B$

Hence we have

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1\ \mathtt{in}\ v_2 \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.22}$$

$$\Phi \mid \Gamma \vdash \mathtt{do}\ x \leftarrow v_1'\ \mathtt{in}\ v_2' \colon \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} A \tag{7.23}$$

**Case If:** By inversion, we have: $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_1' \colon A$, and $\Phi \mid \Gamma \vdash v_2 =_{\beta\eta} v_2' \colon A$.

Hence by induction, we have:
$\Phi \mid \Gamma \vdash v \colon \mathtt{Bool}$, $\Phi \mid \Gamma \vdash v' \colon \mathtt{Bool}$,
$\Phi \mid \Gamma \vdash v_1 \colon A$, $\Phi \mid \Gamma \vdash v_1' \colon A$,
$\Phi \mid \Gamma \vdash v_2 \colon A$, and $\Phi \mid \Gamma \vdash v_2' \colon A$.
So

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v\ \mathtt{then}\ v_1\ \mathtt{else}\ v_2 \colon A \tag{7.24}$$

and

$$\Phi \mid \Gamma \vdash \mathtt{if}_A\ v'\ \mathtt{then}\ v_1'\ \mathtt{else}\ v_2' \colon A \tag{7.25}$$

hold.

**Case Subtype:** By inversion, we have $A \leq:_\Phi B$ and $\Phi \mid \Gamma \vdash v =_{\beta\eta} v' \colon A$. By induction, we therefore have $\Phi \mid \Gamma \vdash v \colon A$ and $\Phi \mid \Gamma \vdash v' \colon A$.

Hence we have

$$\Phi \mid \Gamma \vdash v \colon B \tag{7.26}$$

$$\Phi \mid \Gamma \vdash v' \colon B \tag{7.27}$$

**Case Effect-Lambda:** By inversion, $\Phi, \alpha \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon A$. So

$$(\text{Effect-Lambda})\frac{\Phi, \alpha \mid \Gamma \vdash v_1 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{7.28}$$

and

$$(\text{Effect-Lambda})\frac{\Phi, \alpha \mid \Gamma \vdash v_2 \colon A}{\Phi \mid \Gamma \vdash \Lambda\alpha.v_2 \colon \forall\alpha.A} \tag{7.29}$$

**Case Effect-Apply:** By inversion, $\Phi \mid \Gamma \vdash v_1 =_{\beta\eta} v_2 \colon \forall\alpha.A$ and $\Phi \vdash \epsilon$.
So

$$(\text{Effect-App})\frac{\Phi \mid \Gamma \vdash v_1 \colon \forall\alpha.A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_1\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{7.30}$$

and

$$(\text{Effect-App})\frac{\Phi \mid \Gamma \vdash v_2 \colon \forall\alpha.A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v_2\ \epsilon \colon A\,[\alpha/\epsilon]} \tag{7.31}$$