

Given a set based S-Category  $\mathbb{C}$  which is a model of the non-polymorphic effect calculus, we generate an indexed category capable of modelling the polymorphic effect calculus.

## 0.1 The Non-Polymorphic Model

Since  $\mathbb{C}$  is a model of the non-polymorphic calculus,

- $\mathbb{C}$  is cartesian closed.
- $\mathbb{C}$  has a strong graded monad:  $T^0 : (E, \cdot, \leq_0, 1) \rightarrow [\mathbb{C}, \mathbb{C}]$
- $\mathbb{C}$  has a co-product on the terminal object 1.

In addition, we require that

- $\mathbb{C}$  should be complete (e.g a sub-category of **Set**)
- $E$  should be small.
- The Non-polymorphic effect calculus has a least effect  $\perp \in E$  such that  $\forall \epsilon \in E. \perp \leq_0 \epsilon$ .

## 0.2 Base Category

We construct the base category, **Eff** as follows:

- $U = E$ , the set of ground effects in the non-polymorphic language.
- $1$  is a singleton set.
- $U^n = E^n$ , set of  $n$ -wide tuples of effects,  $\vec{\epsilon}$

Hence when we treat effects that are well formed in  $\Phi$  as morphisms,  $E^n \rightarrow E$  in **Eff**, we should treat them as functions  $f : E^n \rightarrow E$ . Ground effects become point functions:  $e : 1 \rightarrow E$ , so the denotation of a ground effect is the constant value function:  $\llbracket \Phi \vdash e : \mathbf{Effect} \rrbracket_M = \vec{\epsilon} \mapsto e$

We extend the multiplication of ground effects to multiplication on effect functions, giving us our **Mul** operation

$$\mathbf{Mul}(f, g) = f \cdot g \tag{1}$$

$$(f \cdot g)(\vec{\epsilon}) = (f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \tag{2}$$

$$\tag{3}$$

This satisfies naturality of **Mul**.

$$((f \cdot g) \circ \theta)\vec{\epsilon} = (f(\theta\vec{\epsilon})) \cdot (g(\theta\vec{\epsilon})) = ((f \circ \theta) \cdot (g \circ \theta))\vec{\epsilon} \tag{4}$$

### 0.3 S-Categories

The semantic category,  $\mathbb{C}_0$  of the effect-environment  $\diamond$  is  $\mathbb{C}$ .

Since each effect-environment is alpha equivalent to a natural number, the semantic category for  $\Phi$  shall be represented as  $\mathbb{C}_\Phi = \mathbb{C}_n = [E^n, \mathbb{C}]$ , the category of functions  $E^n \rightarrow \mathbb{C}$ .

Objects in  $[E^n, \mathbb{C}]$  are functions and we describe them by their actions on a generic vector of ground effects,  $\vec{\epsilon}$ .

Morphisms in  $[E^n, \mathbb{C}]$  are natural transformations between the functions. So:

$$m : A \rightarrow B \quad \text{In } [E^n, \mathbb{C}] \quad (5)$$

$$m\vec{\epsilon} : A\vec{\epsilon} \rightarrow B\vec{\epsilon} \quad \text{In } \mathbb{C} \quad (6)$$

$$(f \circ g)\vec{\epsilon} = (f\vec{\epsilon}) \circ (g\vec{\epsilon}) \quad (7)$$

$$1(\vec{\epsilon}) = 1 \quad (8)$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in  $\mathbb{C}$ .

#### 0.3.1 Each S-Category is a CCC

Since  $\mathbb{C}$  is complete and a CCC, and  $E^n$  is small, since  $E$  is small,  $[E^n, \mathbb{C}]$  is a CCC.

$$(A \times B)\vec{\epsilon} = (A\vec{\epsilon}) \times (B\vec{\epsilon}) \quad (9)$$

$$1\vec{\epsilon} = 1 \quad (10)$$

$$(B^A)\vec{\epsilon} = (B\vec{\epsilon})^{(A\vec{\epsilon})} \quad (11)$$

$$\pi_1\vec{\epsilon} = \pi_1 \quad (12)$$

$$\pi_2\vec{\epsilon} = \pi_2 \quad (13)$$

$$\mathbf{app}\vec{\epsilon} = \mathbf{app} \quad (14)$$

$$\mathbf{cur}(f)\vec{\epsilon} = \mathbf{cur}(f\vec{\epsilon}) \quad (15)$$

$$\langle f, g \rangle \vec{\epsilon} = \langle f\vec{\epsilon}, g\vec{\epsilon} \rangle \quad (16)$$

$$(17)$$

#### 0.3.2 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation  $\llbracket \gamma \rrbracket_M \in \mathbf{obj} \ \mathbb{C}$ . The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$\llbracket \gamma \rrbracket_M : E^n \rightarrow \mathbf{obj} \ \mathbb{C} \quad (18)$$

$$\vec{\epsilon} \mapsto \llbracket \gamma \rrbracket_M \quad (19)$$

$$(20)$$

Each constant term  $\mathbb{C}^A$  in the non-polymorphic calculus has a fixed denotation  $\llbracket \mathbb{C}^A \rrbracket_M \in \mathbb{C}(1, A)$ .

So the morphism  $\llbracket \mathbb{C}^A \rrbracket_M$  in  $[E^n, \mathbb{C}]$  is the corresponding constant dependently typed morphism.

$$[\mathbb{C}^A]_M : [E^n, \mathbb{C}](1, A) \quad (21)$$

$$\vec{\epsilon} \mapsto [\mathbb{C}^A]_M \quad (22)$$

### 0.3.3 Graded Monad

Given the strong graded monad  $(\mathbf{T}^0, \eta^0, \mu^0, \mathbf{t}^0)$  on  $\mathbb{C}$  we can construct an appropriate graded monad on  $[E^n, \mathbb{C}]$ .

$$\mathbf{T}^n : (E^n, \cdot, \leq_n, 1_n) \rightarrow [[E^n, \mathbb{C}], [E^n, \mathbb{C}]] \quad (23)$$

$$(\mathbf{T}_f^n A)\vec{\epsilon} = \mathbf{T}_{(f\vec{\epsilon})}^0 A\vec{\epsilon} \quad (24)$$

$$(\eta_A^n)\vec{\epsilon} = \eta_{A\vec{\epsilon}}^0 \quad (25)$$

$$(\mu_{f,g,A}^n)\vec{\epsilon} = \mu_{(f\vec{\epsilon}), (g\vec{\epsilon}), (A\vec{\epsilon})}^0 \quad (26)$$

$$(\mathbf{t}_{f,A,B}^n)\vec{\epsilon} = \mathbf{t}_{(f\vec{\epsilon}), (A\vec{\epsilon}), (B\vec{\epsilon})}^0 \quad (27)$$

Through some mechanical proof and the naturality of the  $\mathbb{C}$  strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in  $[E^n, \mathbb{C}]$

#### Naturality

$$\begin{array}{ccc} A\vec{\epsilon} & \xrightarrow{\eta_{(A\vec{\epsilon})}^0} & \mathbf{T}_1^0(A\vec{\epsilon}) \\ \downarrow f\vec{\epsilon} & & \downarrow \mathbf{T}_1^0(f\vec{\epsilon}) \\ B\vec{\epsilon} & \xrightarrow{\eta_{(B\vec{\epsilon})}^0} & \mathbf{T}_1^0(B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} \mathbf{T}_{(f\vec{\epsilon})}^0 \mathbf{T}_{(g\vec{\epsilon})}^0 (A\vec{\epsilon}) & \xrightarrow{\mu_{f\vec{\epsilon}, g\vec{\epsilon}, (B\vec{\epsilon})}^0} & \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0 (A\vec{\epsilon}) \\ \downarrow \mathbf{T}_{f\vec{\epsilon}}^0 \mathbf{T}_{g\vec{\epsilon}}^0 m\vec{\epsilon} & & \downarrow \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0 m\vec{\epsilon} \\ \mathbf{T}_{(f\vec{\epsilon})}^0 \mathbf{T}_{(g\vec{\epsilon})}^0 (B\vec{\epsilon}) & \xrightarrow{\mu_{f\vec{\epsilon}, g\vec{\epsilon}, (B\vec{\epsilon})}^0} & \mathbf{T}_{(f\vec{\epsilon}) \cdot (g\vec{\epsilon})}^0 (B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B\vec{\epsilon}) \\ \downarrow (m\vec{\epsilon} \times \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 B\vec{\epsilon}}) & & \downarrow \mathbf{T}_{f\vec{\epsilon}}^0(m\vec{\epsilon} \times \text{Id}_{B\vec{\epsilon}}) \\ A'\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A'\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A'\vec{\epsilon} \times B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B\vec{\epsilon}) \\ \downarrow (\text{Id}_{A\vec{\epsilon}} \times \mathbf{T}_{f\vec{\epsilon}}^0(m\vec{\epsilon})) & & \downarrow \mathbf{T}_{f\vec{\epsilon}}^0(\text{Id}_{A\vec{\epsilon}} \times m\vec{\epsilon}) \\ A\vec{\epsilon} \times \mathbf{T}_{f\vec{\epsilon}}^0(B'\vec{\epsilon}) & \xrightarrow{\mathbf{t}_{f\vec{\epsilon}, (A\vec{\epsilon}), (B'\vec{\epsilon})}^0} & \mathbf{T}_{f\vec{\epsilon}}^0(A\vec{\epsilon} \times B'\vec{\epsilon}) \end{array}$$

## Monad Laws

### Left Unit

$$(\mu_{f,1,A}^n \circ \mathbf{T}_f^n \eta_A^n) \vec{\epsilon} = \mu_{(f\vec{\epsilon}),1,(A\vec{\epsilon})}^0 \circ \mathbf{T}_{f\vec{\epsilon}}^0 (\eta_{A\vec{\epsilon}}^0) \quad (28)$$

$$= \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 A \vec{\epsilon}} \quad (29)$$

$$= (\text{Id}_{\mathbf{T}_f^n A}) \vec{\epsilon} \quad (30)$$

### Right Unit

$$(\mu_{1,g,A}^n \circ \eta_{\mathbf{T}_f^n A}^n) \vec{\epsilon} = \mu_{1,(f\vec{\epsilon}),(A\vec{\epsilon})}^0 \circ (\eta_{\mathbf{T}_{f\vec{\epsilon}}^0 A \vec{\epsilon}}^0) \quad (31)$$

$$= \text{Id}_{\mathbf{T}_{f\vec{\epsilon}}^0 A \vec{\epsilon}} \quad (32)$$

$$= (\text{Id}_{\mathbf{T}_f^n A}) \vec{\epsilon} \quad (33)$$

### Monad Associativity

$$((\mu_{f,(g \cdot h),A}^n \circ \mathbf{T}_f^n (\mu_{g,h,A}^n)) \vec{\epsilon} = \mu_{(f\vec{\epsilon}),((g\vec{\epsilon}) \cdot (h\vec{\epsilon})),(A\vec{\epsilon})}^0 \circ \mathbf{T}_{f\vec{\epsilon}}^0 \mu_{(h\vec{\epsilon}),(g\vec{\epsilon}),A\vec{\epsilon}}^0) \quad (34)$$

$$= \mu_{((f\vec{\epsilon}) \cdot (g\vec{\epsilon})),(h\vec{\epsilon}),(A\vec{\epsilon})}^0 \circ \mu_{(f\vec{\epsilon}),(g\vec{\epsilon}),(\mathbf{T}_{h\vec{\epsilon}}^0(A\vec{\epsilon}))}^0 \quad (35)$$

$$= (\mu_{f \cdot g,h,A}^n \circ \mu_{f,g,\mathbf{T}_h^n A}^n) \vec{\epsilon} \quad (36)$$

### Tensorial Strength

### Unitor Law

$$(\mathbf{T}_f^n \pi_2) \vec{\epsilon} = \mathbf{T}_{(f\vec{\epsilon})}^0 (\pi_2 \vec{\epsilon}) \quad (37)$$

$$= \mathbf{T}_{(f\vec{\epsilon})}^0 (\pi_2) \quad (38)$$

$$= \pi_2 \quad (39)$$

$$= \pi_2 \vec{\epsilon} \quad (40)$$

### Bind Law

$$\begin{array}{ccc} A \times \mathbf{T}_f^n \mathbf{T}_g^n B & \xrightarrow{\mathbf{t}_{f,A,\mathbf{T}_g^n B}} & \mathbf{T}_f^n (A \times \mathbf{T}_g^n B) \xrightarrow{\mathbf{T}_f^n \mathbf{t}_{g,A,B}} \mathbf{T}_f^n \mathbf{T}_g^n (A \times B) \\ & \searrow \text{Id}_A \times \mu_{f,g,B}^n & \downarrow \mu_{f,g,A \times B}^n \\ & A \times \mathbf{T}_{f \cdot g}^n B & \xrightarrow{\mathbf{t}_{f \cdot g,A,B}} \mathbf{T}_{f \cdot g}^n (A \times B) \end{array}$$

$$(\mathbf{t}_{(f \cdot g),A,B}^n \circ (\text{Id}_A \times \mu_{f,g,B}^n)) \vec{\epsilon} = (\mathbf{t}_{((f\vec{\epsilon}) \cdot (g\vec{\epsilon})),(A\vec{\epsilon}),(B\vec{\epsilon})}^0 \circ (\text{Id}_{A\vec{\epsilon}} \times \mu_{(f\vec{\epsilon}),(g\vec{\epsilon}),(B\vec{\epsilon})}^n)) \quad (41)$$

$$= \mu_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A \times B)\vec{\epsilon}}^0 \circ \mathbf{T}_{f\vec{\epsilon}}^0 (\mathbf{t}_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}^0) \circ \mathbf{t}_{(f\vec{\epsilon}),(A\vec{\epsilon}),\mathbf{T}_{g\vec{\epsilon}}^0(B\vec{\epsilon})}^0 \quad (42)$$

$$= (\mu_{f,g,(A \times B)}^n \circ \mathbf{T}_f^n (\mathbf{t}_{g,A,B}^n) \circ \mathbf{t}_{f,A,\mathbf{T}_g^n(B)}^n) \vec{\epsilon} \quad (43)$$

### Commutativity with Unit

$$\begin{array}{ccc} A \times B & \xrightarrow{\text{Id}_A \times \eta_B} & A \times \mathbf{T}_1 B \\ & \searrow \eta_{A \times B} & \downarrow \mathbf{t}_{1,A,B} \\ & & \mathbf{T}_1^n (A \times B) \end{array}$$

$$(\mathbf{t}_{1,A,B}^n \circ (\text{Id}_A \times \eta_A^n))\vec{\epsilon} = \mathbf{t}_{1,(A\vec{\epsilon}),(B\vec{\epsilon})}^0 \circ (\text{Id}_{A\vec{\epsilon}} \times \eta_{A\vec{\epsilon}}^0) \quad (44)$$

$$= \eta_{A\vec{\epsilon} \times B\vec{\epsilon}}^0 \quad (45)$$

$$= (\eta_{A \times B}^n)\vec{\epsilon} \quad (46)$$

**Commutativity with  $\alpha$**  Let  $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle : ((A \times B) \times C) \rightarrow (A \times (B \times C))$

$$\begin{array}{ccc} (A \times B) \times \mathbf{T}_\epsilon^n C & \xrightarrow{\mathbf{t}_{\epsilon,(A \times B),C}} & \mathbf{T}_\epsilon^n((A \times B) \times C) \\ \downarrow \alpha_{A,B,\mathbf{T}_\epsilon^n C} & & \downarrow \mathbf{T}_\epsilon^n \alpha_{A,B,C} \\ A \times (B \times \mathbf{T}_\epsilon^n C) & \xrightarrow{\text{Id}_A \times \mathbf{t}_{\epsilon,B,C}} & A \times \mathbf{T}_\epsilon^n(B \times C) \xrightarrow{\mathbf{t}_{\epsilon,A,(B \times C)}} \mathbf{T}_\epsilon^n(A \times (B \times C)) \end{array}$$

$$(\mathbf{T}_f^n \alpha_{A,B,C} \circ \mathbf{t}_{f,A \times B,C}^n)\vec{\epsilon} = \mathbf{T}_{f\vec{\epsilon}}^0 \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathbf{t}_{(f\vec{\epsilon}),(A \times B)\vec{\epsilon},(C\vec{\epsilon})}^0 \quad (47)$$

$$= \mathbf{t}_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon} \times C\vec{\epsilon})}^0 \circ (\text{Id}_{A\vec{\epsilon}} \times \mathbf{t}_{(f\vec{\epsilon}),(B\vec{\epsilon}),(C\vec{\epsilon})}^0) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \quad (48)$$

$$= (\mathbf{t}_{f,A,(B \times C)}^n \circ (\text{Id}_A \times \mathbf{t}_{f,B,C}^n) \circ \alpha_{A,B,C})\vec{\epsilon} \quad (49)$$

$$(50)$$

### 0.3.4 Sub-Effecting

Given a collection of sub-effecting natural transformation in  $\mathbb{C}$ ,

$$[\epsilon_1 \leq_0 \epsilon_2]_M : \mathbf{T}_{\epsilon_1}^0 \rightarrow \mathbf{T}_{\epsilon_2}^0 \quad (51)$$

We can form sub-effect natural transformations in  $[E^n, \mathbb{C}]$ :

$$[f \leq_n g]_M : \mathbf{T}_f^n \rightarrow \mathbf{T}_g^n \quad (52)$$

$$[f \leq_n g]_M A\vec{\epsilon} : \mathbf{T}_{f\vec{\epsilon}}^n(A\vec{\epsilon}) \rightarrow \mathbf{T}_{g\vec{\epsilon}}^n(B\vec{\epsilon}) \quad (53)$$

$$= [f\vec{\epsilon} \leq_0 g\vec{\epsilon}]_M A\vec{\epsilon} \quad (54)$$

### Naturality

$$\begin{array}{ccc} \mathbf{T}_{f\vec{\epsilon}}^0 [f\vec{\epsilon} \leq_0 g\vec{\epsilon}]_M A\vec{\epsilon} & \xrightarrow{\quad} & \mathbf{T}_{g\vec{\epsilon}}^0 A\vec{\epsilon} \\ \downarrow \mathbf{T}_{f\vec{\epsilon}}^0 m\vec{\epsilon} & & \downarrow \mathbf{T}_{g\vec{\epsilon}}^0 m\vec{\epsilon} \\ \mathbf{T}_{f\vec{\epsilon}}^0 [f\vec{\epsilon} \leq_0 g\vec{\epsilon}]_M B\vec{\epsilon} & \xrightarrow{\quad} & \mathbf{T}_{g\vec{\epsilon}}^0 B\vec{\epsilon} \end{array}$$

### Commutes With Tensor Strength

$$\begin{array}{ccc} A \times \mathbf{T}_f^n B & \xrightarrow{\text{Id}_A \times [f \leq_n g]_B} & A \times \mathbf{T}_g^n B \\ \downarrow \mathbf{t}_{f,A,B}^n & & \downarrow \mathbf{t}_{g,A,B}^n \\ \mathbf{T}_f^n(A \times B) & \xrightarrow{[f \leq_n g]_{A \times B}} & \mathbf{T}_g^n(A \times B) \end{array}$$

$$(\mathfrak{t}_{g,A,B}^n \circ (\text{Id}_A \times \llbracket f \leq_n g \rrbracket_B))\vec{e} = \mathfrak{t}_{(g\vec{e}), (A\vec{e}), (B\vec{e})}^0 \circ (\text{Id}_{A\vec{e}} \times \llbracket f\vec{e} \leq_0 g\vec{e} \rrbracket_{B\vec{e}}) \quad (55)$$

$$= \llbracket f\vec{e} \leq_0 g\vec{e} \rrbracket_{(A \times B)\vec{e}} \circ \mathfrak{t}_{(f\vec{e}), (A\vec{e}), (B\vec{e})}^0 \quad (56)$$

$$= (\llbracket f \leq_n g \rrbracket_{(A \times B)} \circ \mathfrak{t}_{f,A,B}^n)\vec{e} \quad (57)$$

$$(58)$$

### Commutates with Join

$$\begin{array}{ccc} \mathfrak{T}_f^n \mathfrak{T}_g^n & \xrightarrow{\mathfrak{T}_f^n \llbracket g \leq_n g' \rrbracket_M} \mathfrak{T}_f^n \mathfrak{T}_{g'}^n & \xrightarrow{\llbracket f \leq_n f' \rrbracket_{M, \mathfrak{T}_{g'}^n}} \mathfrak{T}_{f'}^n \mathfrak{T}_{g'}^n \\ \downarrow \mu_{f,g}^n & & \downarrow \mu_{f',g'}^n \\ \mathfrak{T}_{f \cdot g}^n & \xrightarrow{\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_M} & \mathfrak{T}_{f' \cdot g'}^n \end{array}$$

$$(\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_A \circ \mu_{f,g,A}^n)\vec{e} = \llbracket (f\vec{e}) \cdot (g\vec{e}) \leq_0 (f'\vec{e}) \cdot (g'\vec{e}) \rrbracket_{A\vec{e}} \circ \mu_{(f\vec{e}), (g\vec{e}), (A\vec{e})}^0 \quad (59)$$

$$= \mu_{(f\vec{e}), (g\vec{e}), (A\vec{e})}^0 \circ \llbracket f\vec{e} \leq_0 f'\vec{e} \rrbracket_{\mathfrak{T}_{g'\vec{e}}^0(A\vec{e})} \circ \mathfrak{T}_{f\vec{e}}^0 \llbracket g\vec{e} \leq_0 g'\vec{e} \rrbracket_{(A\vec{e})} \quad (60)$$

$$= \mu_{f,g,A}^n \circ \llbracket f \leq_n f' \rrbracket_{\mathfrak{T}_{g'}^n A} \circ \mathfrak{T}_f^n \llbracket g \leq_n g' \rrbracket_A \quad (61)$$

### 0.3.5 Sub-Typing

Sub-typing in  $[E^n, \mathbb{C}]$  holds via sub-typing in  $\mathbb{C}$

$$\llbracket A \leq_n B \rrbracket_M : A \rightarrow B \quad (62)$$

$$\llbracket A \leq_n B \rrbracket_M \vec{e} = \llbracket A\vec{e} \leq_0 B\vec{e} \rrbracket_M \quad (63)$$

So the subtyping relation  $A \leq B$  forms a morphism in  $[E^n, \mathbb{C}]$

## 0.4 Functors Between S-Categories

For a function  $\theta : E^m \rightarrow E^n$ , the re-indexing functor  $\theta^*$  is defined as follows:

$$\theta^* : [E^n, \mathbb{C}] \rightarrow [E^m, \mathbb{C}] \quad (64)$$

$$\theta^*(A)\epsilon_m^\rightarrow = A(\theta(\epsilon_m^\rightarrow)) \quad (65)$$

$$f : A \rightarrow B \in [E^n, \mathbb{C}] \quad (66)$$

$$\theta^*(f)\epsilon_m^\rightarrow = f(\theta(\epsilon_m^\rightarrow)) : A(\theta(\epsilon_m^\rightarrow)) \rightarrow B(\theta(\epsilon_m^\rightarrow)) \quad (67)$$

### 0.4.1 Quantification

We need to define  $\forall_{E^n} : [E^{n+1}, \mathbb{C}] \rightarrow [E^n, \mathbb{C}]$

So

$$(\forall_{E^n} A)\vec{\epsilon}_n = A(\vec{\epsilon}_n, \perp) \quad (68)$$

$$m : A \rightarrow B \quad (69)$$

$$(\forall_{E^n} m) : \forall_{E^n} A \rightarrow \forall_{E^n} B \quad (70)$$

$$(\forall_{E^n} m)\vec{\epsilon}_n = m(\vec{\epsilon}_n, \perp) \quad (71)$$

$$(72)$$

## 0.4.2 Adjunction

It is the case that:

$$\pi_1^* \dashv \forall_{E^n}$$

With unit:

$$\eta_A : A \rightarrow \forall_{E^n} \pi_1^* A \quad (73)$$

$$\eta_A(\vec{\epsilon}_n) = \text{Id}_{A\vec{\epsilon}_n} \quad (74)$$

Since  $(\forall_{E^n} \pi_1^* A)\vec{\epsilon}_n = (\pi_1^* A)(\vec{\epsilon}_n, \perp) = A\vec{\epsilon}_n$ , so  $A = \forall_{E^n} \pi_1^* A$

And co-unit

$$\epsilon_B : \pi_1^* \forall_{E^n} B \rightarrow B \quad (75)$$

$$\epsilon_B(\vec{\epsilon}_n, \epsilon_{n+1}) = \llbracket (\vec{\epsilon}_n, \perp) \leq_0 (\vec{\epsilon}_n, \epsilon_{n+1}) \rrbracket_B \quad (76)$$

Which uses the point-wise  $E^n$  sub-effect natural transformation.

We then define the natural bijection as so:

$$\overline{(-)} : [E^{n+1}, \mathbb{C}](\pi_1^* A, B) \leftrightarrow [E^n, \mathbb{C}](A, \forall_{E^n} B) : \widehat{(-)} \quad (77)$$

$$m : \pi_1^* A \rightarrow B \quad (78)$$

$$\overline{m} : A \rightarrow \forall_{E^n} B \quad (79)$$

$$\overline{m}(\vec{\epsilon}_n) = m(\vec{\epsilon}_n, \perp) \quad (80)$$

$$n : A \rightarrow \forall_{E^n} B \quad (81)$$

$$\widehat{n} : \pi_1^* A \rightarrow B \quad (82)$$

$$\widehat{n}(\vec{\epsilon}_n, \epsilon_{n+1}) = \llbracket (\vec{\epsilon}_n, \perp) \leq_0 (\vec{\epsilon}_n, \epsilon_{n+1}) \rrbracket_B \circ n(\vec{\epsilon}_n) \quad (83)$$

## This is an Adjunction

For any  $g : \pi_1^* A \rightarrow B$ ,

$$(\epsilon_B \circ \pi_1^*(\overline{g}))(\vec{\epsilon}_n, \epsilon_{n+1}) = \llbracket (\vec{\epsilon}_n, \perp) \leq_0 (\vec{\epsilon}_n, \epsilon_{n+1}) \rrbracket_B \circ g(\vec{\epsilon}_n, \perp) \quad (84)$$

$$= g(\vec{\epsilon}_n, \epsilon_{n+1}) \quad \text{By the limit.} \quad (85)$$

Since  $g(\vec{\epsilon}_n, \perp)$  is the limit of  $g(\vec{\epsilon}_n, -)$

### 0.4.3 Beck-Chevalley Condition

For  $\theta : E^m \rightarrow E^n$ :

$$((\theta^* \circ \forall_{E^n})A)\epsilon_n^\rightarrow = \theta^*(\forall_{E^n} A)\epsilon_n^\rightarrow \quad (86)$$

$$= (\forall_{E^n} A)(\theta(\epsilon_n^\rightarrow)) \quad (87)$$

$$= A(\theta(\epsilon_n^\rightarrow), \perp) \quad (88)$$

$$= A((\theta \times \text{Id}_E)(\epsilon_n^\rightarrow, \perp)) \quad (89)$$

$$= ((\theta \times \text{Id}_E)^* A)(\epsilon_n^\rightarrow, \perp) \quad (90)$$

$$= \forall_{E^m} ((\theta \times \text{Id}_E)^* A)\epsilon_n^\rightarrow \quad (91)$$

$$= ((\forall_{E^m} \circ (\theta \times \text{Id}_E)^*)A)\epsilon_n^\rightarrow \quad (92)$$