

0.1 Weakening Definition

0.1.1 Relation

We define the ternary weaking relation $w : \Gamma' \triangleright \Gamma$ using the following rules.

- (Id) $\frac{\Gamma 0k}{\iota : \Gamma \triangleright \Gamma}$
- (Project) $\frac{\omega : \Gamma' \triangleright \Gamma, x \notin \text{dom}(\Gamma')}{\omega \pi : \Gamma, x : A \triangleright \Gamma}$
- (Extend) $\frac{\omega : \Gamma' \triangleright \Gamma, x \notin \text{dom}(\Gamma') A \leq B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B}$

0.1.2 Weakening Denotations

The denotation of a weakening relation is defined as follows:

$$\llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M : \Gamma' \rightarrow \Gamma \quad (1)$$

- $\llbracket \iota : \Gamma \triangleright \Gamma \rrbracket_M = \text{Id}_\Gamma : \Gamma \rightarrow \Gamma$
- (Project) $\frac{f = \llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M : \Gamma' \rightarrow \Gamma}{\llbracket \omega \pi : \Gamma, x : A \triangleright \Gamma \rrbracket_M = f \circ \pi_1 : \Gamma' \times A \rightarrow \Gamma}$
- (Extend) $\frac{f = \llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M : \Gamma' \rightarrow \Gamma, g = \llbracket A \leq B \rrbracket_M : A \rightarrow B}{\llbracket w \times : \Gamma', x : A \triangleright \Gamma, x : B \rrbracket_M = (f \times g) : (\Gamma' \times A) \rightarrow (\Gamma \times B)}$

0.2 Weakening Theorems

0.2.1 Domain Lemma

If $\omega : \Gamma' \triangleright \Gamma$, then $\text{dom}(\Gamma) \subseteq \text{dom}(\Gamma')$.

Proof

Case Id Then $\Gamma' = \Gamma$ and so $\text{dom}(\Gamma') = \text{dom}(\Gamma)$.

Case Project By inversion and induction, $\text{dom}(\Gamma) \subseteq \text{dom}(\Gamma') \subseteq \text{dom}(\Gamma' \cup \{x\})$

Case Extend By inversion and induction, $\text{dom}(\Gamma) \subseteq \text{dom}(\Gamma')$ so

$$\text{dom}(\Gamma, x : A) = \text{dom}(\Gamma) \cup \{x\} \subseteq \text{dom}(\Gamma') \cup \{x\} = \text{dom}(\Gamma', x : A)$$

0.2.2 Theorem 1

If $\omega : \Gamma' \triangleright \Gamma$ and $\Gamma 0k$ then $\Gamma' 0k$

Proof

Case Id

$$(\text{Id}) \frac{\Gamma 0k}{\iota : \Gamma \triangleright \Gamma}$$

By inversion, $\Gamma 0k$.

Case Project

$$(\text{Project}) \frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \text{dom}(\Gamma')}{\omega\pi : \Gamma, x : A \triangleright \Gamma}$$

By inversion, $\omega : \Gamma' \triangleright \Gamma$ and $x \notin \text{dom}(\Gamma')$.

Hence by induction $\Gamma' \mathbf{Ok}$, $\Gamma \mathbf{Ok}$. Since $x \notin \text{dom}(\Gamma')$, we have $\Gamma', x : A \mathbf{Ok}$.

$$\text{Case Extend (Extend)} \frac{\omega : \Gamma' \triangleright \Gamma \quad x \notin \text{dom}(\Gamma') \quad A \leq B}{w \times : \Gamma', x : A \triangleright \Gamma, x : B},$$

By inversion, we have

$\omega : \Gamma' \triangleright \Gamma$, $x \notin \text{dom}(\Gamma')$.

Hence we have $\Gamma \mathbf{Ok}$, $\Gamma' \mathbf{Ok}$, and by the domain Lemma, $\text{dom}(\Gamma) \subseteq \text{dom}(\Gamma')$, hence $x \notin \text{dom}(\Gamma)$. Hence, we have $\Gamma, x : A \mathbf{Ok}$ and $\Gamma', x : A \mathbf{Ok}$.

0.2.3 Theorem 2

If $\Gamma \vdash t : \tau$ and $\omega : \Gamma' \triangleright \Gamma$ then there is a derivation of $\Gamma' \vdash t : \tau$

Proof Proved in parallel with theorem 3 below

0.2.4 Theorem 3

If $\omega : \Gamma' \triangleright \Gamma$ and $\Delta = \llbracket \Gamma \vdash t : \tau \rrbracket_M$ and $\Delta' = \llbracket \Gamma' \vdash t : \tau \rrbracket_M$, derived using Theorem 2, then

$$\Delta \circ \llbracket \omega \rrbracket_M = \Delta' : \Gamma' \rightarrow \llbracket \tau \rrbracket_M$$

Proof Below

0.3 Proof of Theorems 2 and 3

We induct over the structure of typing derivations of $\Gamma \vdash t : \tau$, assuming $\omega : \Gamma' \triangleright \Gamma$ holds. In each case, we construct the new derivation Δ' from the derivation Δ giving $\Gamma \vdash t : \tau$ and show that $\Delta \circ \llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M = \Delta'$

0.3.1 Variable Terms

Case Var and Weaken We case split on the weakening ω .

If $\omega = \iota$ Then $\Gamma' = \Gamma$, and so $\Gamma' \vdash x : A$ holds and the derivation Δ' is the same as Δ

$$\Delta' = \Delta = \Delta \circ \text{Id}_\Gamma = \Delta \circ \llbracket \iota : \Gamma \triangleright \Gamma \rrbracket_M \quad (2)$$

If $\omega = \omega' \pi$ Then $\Gamma' = (\Gamma'', x' : A')$ and $\omega' : \Gamma'' \triangleright \Gamma$. So by induction, there is a tree, Δ_1 deriving $\Gamma'' \vdash x : A$, such that

$$\Delta_1 = \Delta \circ \llbracket \omega' : \Gamma'' \triangleright \Gamma \rrbracket_M \quad \text{By Induction} \quad (3)$$

, and hence by the weaken rule, we have

$$(\text{Weaken}) \frac{\Gamma'' \vdash x : A}{\Gamma'', x' : A' \vdash x : A} \quad (4)$$

This preserves denotations:

$$\Delta' = \Delta_1 \circ \pi_1 \quad \text{By Definition} \quad (5)$$

$$= \Delta \circ \llbracket \omega' : \Gamma'' \triangleright \Gamma \rrbracket_M \circ \pi_1 \quad \text{By induction} \quad (6)$$

$$= \Delta \circ \llbracket \omega' \pi_1 : \Gamma' \triangleright \Gamma \rrbracket_M \quad \text{By denotation of weakening} \quad (7)$$

If $\omega = \omega' \times$ Then

$$\Gamma' = \Gamma''', x' : B \quad (8)$$

$$\Gamma = \Gamma'', x' : A' \quad (9)$$

$$B \leq: A \quad (10)$$

If $x = x'$ Then $A = A'$.

Then we derive the new derivation, Δ' as so:

$$\text{(Sub-type)} \frac{(\text{var}) \frac{}{\Gamma''', x: B \vdash x: B} \quad B \leq: A}{\Gamma' \vdash x: A} \quad (11)$$

This preserves denotations:

$$\Delta' = \llbracket B \leq: A \rrbracket_M \circ \pi_2 \quad \text{By Definition} \quad (12)$$

$$= \pi_2 \circ (\llbracket \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket_M \times \llbracket B \leq: A \rrbracket_M) \quad \text{By the properties of binary products} \quad (13)$$

$$= \Delta \circ \llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M \quad \text{By Definition} \quad (14)$$

Case $x \neq x'$ Then

$$\Delta = (\text{Weaken}) \frac{() \frac{\Delta_1}{\Gamma''' \vdash x: A}}{\Gamma' \vdash x: A} \quad (15)$$

By induction with $\omega : \Gamma''' \triangleright \Gamma''$, we have a derivation Δ_1 of $\Gamma''' \vdash x: A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken}) \frac{() \frac{\Delta'_1}{\Gamma''' \vdash x: A}}{\Gamma' \vdash x: A} \quad (16)$$

This preserves denotations:

By induction, we have

$$\Delta'_1 = \Delta_1 \circ \llbracket \omega : \Gamma''' \triangleright \Gamma'' \rrbracket_M \quad (17)$$

So we have:

$$\Delta' = \Delta'_1 \circ \pi_1 \quad \text{By denotation definition} \quad (18)$$

$$= \Delta_1 \circ \llbracket \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket_M \quad \text{By induction} \circ \pi_1 \quad (19)$$

$$= \Delta_1 \circ \pi_1 \circ (\llbracket \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket_M \times \llbracket A' \leq: B \rrbracket_M) \quad \text{By product properties} \quad (20)$$

$$= \Delta \circ \llbracket \omega : \Gamma' \triangleright \Gamma \rrbracket_M \quad \text{By definition} \quad (21)$$

0.3.2 Value Terms

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation $\llbracket \omega : \Gamma' \triangleright \Gamma' \rrbracket_M$, simply as ω .

Case Constant The constant typing rules, $()$, **true**, **false**, \mathbb{C}^A , all proceed by the same logic. Hence I shall only prove the theorems for the case \mathbb{C}^A .

$$(\text{Const}) \frac{\Gamma 0k}{\Gamma \vdash \mathbb{C}^A: A} \quad (22)$$

By inversion, we have $\Gamma 0k$, so we have $\Gamma' 0k$.

Hence

$$(\text{Const}) \frac{\Gamma' 0k}{\Gamma' \vdash \mathbb{C}^A: A} \quad (23)$$

Holds.

This preserves denotations:

$$\Delta' = \llbracket \mathbf{C}^A \rrbracket_M \circ \langle \rangle_{\Gamma'} \quad \text{By definition} \quad (24)$$

$$= \llbracket \mathbf{C}^A \rrbracket_M \circ \langle \rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} \quad (25)$$

$$= \Delta \quad \text{By Definition} \quad (26)$$

$$(27)$$

Case Lambda By inversion, we have a derivation Δ_1 giving

$$\Delta = (\text{Fn}) \frac{() \frac{\Delta_1}{\Gamma, x:A \vdash C: \mathbf{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.C: A \rightarrow \mathbf{M}_\epsilon B} \quad (28)$$

Since $\omega : \Gamma' \triangleright \Gamma$, we have:

$$\omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \quad (29)$$

Hence, by induction, using $\omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$, we derive Δ'_1 :

$$\Delta' = (\text{Fn}) \frac{() \frac{\Delta'_1}{\Gamma', x:A \vdash C: \mathbf{M}_\epsilon B}}{\Gamma', x : A \vdash \lambda x : A.C: A \rightarrow \mathbf{M}_\epsilon B} \quad (30)$$

This preserves denotations:

$$\Delta' = \text{cur}(\Delta'_1) \quad \text{By Definition} \quad (31)$$

$$= \text{cur}(\Delta_1 \circ (\omega \times \text{Id}_\Gamma)) \quad \text{By the denotation of } \omega \times \quad (32)$$

$$= \text{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} \quad (33)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (34)$$

Case Sub-typing

$$(\text{Sub-type}) \frac{\Gamma \vdash v: A \quad A \leq: B}{\Gamma \vdash v: B} \quad (35)$$

by inversion, we have a derivation Δ_1

$$() \frac{\Delta_1}{\Gamma \vdash v: A} \quad (36)$$

So by induction, we have a derivation Δ'_1 such that:

$$(\text{Sub-type}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v: A} \quad A \leq: B}{\Gamma' \vdash v: B} \quad (37)$$

This preserves denotations:

$$\Delta' = \llbracket A \leq: B \rrbracket_M \circ \Delta'_1 \quad \text{By Definition} \quad (38)$$

$$= \llbracket A \leq: B \rrbracket_M \circ \Delta_1 \circ \omega \quad \text{By induction} \quad (39)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (40)$$

$$(41)$$

0.3.3 Computation Terms

Case Return We have the sub-derivation Δ_1 such that

$$\Delta = (\text{Return}) \frac{() \frac{\Delta_1}{\Gamma \vdash v:A}}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A} \quad (42)$$

Hence, by induction, with $\omega : \Gamma' \triangleright \Gamma$, we find the derivation Δ'_1 such that:

$$\Delta' = (\text{Return}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v:A}}{\Gamma' \vdash \text{return } v : \mathbb{M}_1 A} \quad (43)$$

This preserves denotations:

$$\Delta' = \eta_A \circ \Delta'_1 \quad \text{By definition} \quad (44)$$

$$= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1 \quad (45)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (46)$$

Case Apply By inversion, we have derivations Δ_1, Δ_2 such that

$$\Delta = (\text{Apply}) \frac{() \frac{\Delta_1}{\Gamma \vdash v_1:A \rightarrow \mathbb{M}_\epsilon B} \quad () \frac{\Delta_2}{\Gamma \vdash v_2:A}}{\Gamma \vdash v_1 \quad v_2 : \mathbb{M}_\epsilon B} \quad (47)$$

By induction, this gives us the respective derivations: Δ'_1, Δ'_2 such that

$$\Delta' = (\text{Apply}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v_1:A \rightarrow \mathbb{M}_\epsilon B} \quad () \frac{\Delta'_2}{\Gamma' \vdash v_2:A}}{\Gamma' \vdash v_1 \quad v_2 : \mathbb{M}_\epsilon B} \quad (48)$$

This preserves denotations:

$$\Delta' = \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \quad (49)$$

$$= \text{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \quad (50)$$

$$= \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \quad (51)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (52)$$

Case If By inversion, we have the sub-derivations $\Delta_1, \Delta_2, \Delta_3$, such that:

$$\Delta = (\text{If}) \frac{() \frac{\Delta_1}{\Gamma \vdash v:\text{Bool}} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon,A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A} \quad (53)$$

By induction, this gives us the sub-derivations $\Delta'_1, \Delta'_2, \Delta'_3$ such that

$$\Delta' = (\text{If}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash v:\text{Bool}} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A}{\Gamma' \vdash \text{if}_{\epsilon,A} \quad v \quad \text{then} \quad C_1 \quad \text{else} \quad C_2 : \mathbb{M}_\epsilon A} \quad (54)$$

This preserves denotations

$$\Delta' = \text{If}_{T_\epsilon A} \circ \langle \Delta'_1, \langle \Delta'_2, \Delta'_3 \rangle \rangle \quad \text{By Definition} \quad (55)$$

$$= \text{If}_{T_\epsilon A} \circ \langle \Delta_1 \circ \omega, \langle \Delta_2 \circ \omega, \Delta_3 \circ \omega \rangle \rangle \quad \text{By induction} \quad (56)$$

$$= \text{If}_{T_\epsilon A} \circ \langle \Delta_1, \langle \Delta_2, \Delta_3 \rangle \rangle \circ \omega \quad \text{By Product property} \quad (57)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (58)$$

$$(59)$$

Case Bind By inversion, we have derivations Δ_1, Δ_2 such that:

$$\Delta = (\text{Bind}) \frac{() \frac{\Delta_1}{\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A} \quad () \frac{\Delta_2}{\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (60)$$

If $\omega : \Gamma' \triangleright \Gamma$ then $\omega \times : \Gamma', x : A \triangleright \Gamma, x : A$, so by induction, we can derive Δ'_1, Δ'_2 such that:

$$\Delta' = (\text{Bind}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash C_1 : \mathbb{M}_{\epsilon_1} A} \quad () \frac{\Delta'_2}{\Gamma', x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B}}{\Gamma' \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (61)$$

This preserves denotations:

$$\Delta' = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathbf{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By definition} \quad (62)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2 \circ (\omega \times \text{Id}_A)) \circ \mathbf{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta'_1, \Delta'_2 \quad (63)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} \quad (64)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} \quad (65)$$

$$= \Delta \quad \text{By definition} \quad (66)$$

Case Sub-effect

$$(\text{Sub-effect}) \frac{\Gamma \vdash C : \mathbb{M}_{\epsilon_1} A \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C : \mathbb{M}_{\epsilon_2} B} \quad (67)$$

by inversion, we have a derivation Δ_1

$$() \frac{\Delta_1}{\Gamma \vdash C : \mathbb{M}_{\epsilon_1} A} \quad (68)$$

So by induction, we have a derivation Δ'_1 such that:

$$(\text{Sub-effect}) \frac{() \frac{\Delta'_1}{\Gamma' \vdash C : \mathbb{M}_{\epsilon_1} A} \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Gamma' \vdash C : \mathbb{M}_{\epsilon_2} B} \quad (69)$$

This preserves denotations:

Let

$$g = \llbracket A \leq B \rrbracket_M : A \rightarrow B \quad (70)$$

$$h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_M : T_{\epsilon_1} \rightarrow T_{\epsilon_2} \quad (71)$$

Then

$$\Delta' = h_B \circ T_{\epsilon_1} g \circ \Delta'_1 \quad \text{By Definition} \quad (72)$$

$$= h_B \circ T_{\epsilon_1} g \circ \Delta_1 \circ \omega \quad \text{By Induction} \quad (73)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (74)$$