0.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of $\Gamma \vdash t\colon \tau$. Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

0.2 Reduced Type Derivations are Unique

For each instance of the relation $\Gamma \vdash t\colon \tau$,there exists at most one reduced derivation of $\Gamma \vdash t\colon \tau$. This is proved by induction over the typing rules on the bottom rule used in each derivation.

0.2.1 Variables

To find the unique derivation of $\Gamma \vdash x\colon A$, we case split on the type-environment, Γ.

Case $\Gamma = \Gamma', x : A'$ Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is, if $A' \leq: A$, as below:

$$\text{(Subtype)}\frac{\text{(Var)}\dfrac{\Gamma', x : A'\texttt{Ok}}{\Gamma, x : A' \vdash x\colon A'} \qquad A' \leq: A}{\Gamma', x : A' \vdash x\colon A} \tag{1}$$

Case $\Gamma = \Gamma', y : B$ with $y \neq x$.

Hence, if $\Gamma \vdash x\colon A$ holds, then so must $\Gamma' \vdash x\colon A$.

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta}{\Gamma' \vdash x\colon A'} \qquad A' \leq: A}{\Gamma' \vdash x\colon A} \tag{2}$$

Be the unique reduced derivation of $\Gamma' \vdash x\colon A$.

Then the unique reduced derivation of $\Gamma \vdash x\colon A$ is:

$$\text{(Subtype)}\frac{\text{(Weaken)}\dfrac{\dfrac{\Delta}{\Gamma, x : A' \vdash x\colon A'}}{\Gamma \vdash x\colon A'} \qquad A' \leq: A}{\Gamma \vdash x\colon A} \tag{3}$$

0.2.2 Constants

For each of the constants, ($\texttt{C}^A$, `true`, `false`, `()`), there is exactly one possible derivation for $\Gamma \vdash c\colon A$ for a given A. I shall give examples using the case $\texttt{C}^A$

$$\text{(Subtype)}\dfrac{\text{(Const)}\dfrac{\Gamma\,\mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A\colon A} \qquad A \leq: B}{\Gamma \vdash \mathtt{C}^A\colon B}$$

If $A = B$, then the subtype relation is the identity subtype ($A \leq: A$).

0.2.3 Value Terms

Case Lambda The reduced derivation of $\Gamma \vdash \lambda x : A.C : A' \to \mathtt{M}_{\epsilon'} B'$ is:

$$\text{(Subtype)}\dfrac{\text{(Lambda)}\dfrac{\dfrac{\Delta}{\Gamma, x : A \vdash C\colon \mathtt{M}_\epsilon B}}{\Gamma \vdash \lambda x : A.B : A \to \mathtt{M}_\epsilon B} \qquad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash \lambda x : A.C : A' \to \mathtt{M}_{\epsilon'} B'}$$

Where

$$\text{(Sub-Effect)}\dfrac{\dfrac{\Delta}{\Gamma, x : A \vdash C\colon \mathtt{M}_\epsilon B} \qquad B \leq: B'}{\epsilon \leq \epsilon'}\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon'} B' \tag{4}$$

is the reduced derivation of $\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon'} B$ if it exists.

Case Subtype **TODO: Do we need to write anything here? (Probably needs an explanation)**

0.2.4 Computation Terms

Case Return The reduced denotation of $\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_\epsilon B$ is

$$\text{(Subtype)}\dfrac{\text{(Return)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v\colon A}}{\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_{\mathtt{1}} A} \qquad A \leq: B}{\mathtt{1} \leq \epsilon}\Gamma \vdash \mathtt{return}\ v : \mathtt{M}_\epsilon B$$

Where

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v\colon A} \qquad A \leq: B}{\Gamma \vdash v\colon B}$$

is the reduced derivation of $\Gamma \vdash v\colon B$

Case Apply If

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v_1\colon A \to \mathtt{M}_\epsilon B} \qquad A \to \mathtt{M}_\epsilon B \leq: A' \to \mathtt{M}_{\epsilon'} B'}{\Gamma \vdash v_1\colon A' \to \mathtt{M}_{\epsilon'} B'}$$

and

$$\text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Gamma \vdash v_2\colon A''} \qquad A'' \leq: A'}{\Gamma \vdash v_2\colon A'}$$

Are the reduced type derivations of $\Gamma \vdash v_1 \colon A' \to \mathtt{M}_{\epsilon'} B'$ and $\Gamma \vdash v_2 \colon A'$

Then we can construct the reduced derivation of $\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'$ as

$$\text{(Subeffect)}\dfrac{\text{(Apply)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v_1 \colon A \to \mathtt{M}_\epsilon B} \qquad \text{(Subtype)}\dfrac{\dfrac{\Delta'}{\Gamma \vdash v \colon A''} \qquad A'' \leq: A}{\Gamma \vdash v \colon A}}{\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_\epsilon B} \qquad B \leq: B'}{\epsilon \leq \epsilon'} \Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'$$

Case If Let

$$\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v \colon B} \qquad B \leq: \mathtt{Bool}}{\Gamma \vdash v \colon \mathtt{Bool}} \tag{5}$$

$$\text{(Subeffect)}\dfrac{\dfrac{\Delta'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon'} A'} \qquad A' \leq: A \qquad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A} \tag{6}$$

$$\text{(Subeffect)}\dfrac{\dfrac{\Delta''}{\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon''} A''} \qquad A'' \leq: A \qquad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A} \tag{7}$$

Be the unique reduced reduced derivations of $\Gamma \vdash v \colon \mathtt{Bool}$, $\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A$, $\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A$.

Then the only reduced derivation of $\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A$ is:

TODO: Scale this properly

$$\text{(Subtype)}\dfrac{\text{(If)}\dfrac{\text{(Subtype)}\dfrac{\dfrac{\Delta}{\Gamma \vdash v \colon B} \qquad B \leq: \mathtt{Bool}}{\Gamma \vdash v \colon \mathtt{Bool}} \qquad \text{(Subeffect)}\dfrac{\dfrac{\Delta'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon'} A'} \qquad A' \leq: A \qquad \epsilon' \leq \epsilon}{\Gamma \vdash C_1 \colon \mathtt{M}_\epsilon A} \qquad \text{(Subeffect)}\dfrac{\dfrac{\Delta''}{\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon''} A''} \qquad A'' \leq: A \qquad \epsilon'' \leq \epsilon}{\Gamma \vdash C_2 \colon \mathtt{M}_\epsilon A}}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A} \qquad \mathtt{M}_\epsilon A \leq: \mathtt{M}_\epsilon A}{\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_\epsilon A} \tag{8}$$

Case Bind Let

$$\text{(Subeffect)}\dfrac{\dfrac{\Delta}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad A \leq: A' \qquad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1'} A'} \tag{9}$$

$$\text{(Subeffect)}\dfrac{\dfrac{\Delta'}{\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad B \leq: B' \qquad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{10}$$

Be the respective unique reduced type derivations of the sub-terms]

By weakening, $\iota\times : \Gamma, x : A \triangleright \Gamma, x : A'$ so if there's a derivation of $\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_\epsilon B$, there's also one of $\Gamma, x : A \vdash C_2 \colon \mathtt{M}_\epsilon B$.

$$\text{(Subeffect)}\frac{\dfrac{\Delta''}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad B \leq: B' \qquad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2'} B'} \tag{11}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Hence the reduced type derivation of $\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B'$ is the following:

TODO: Make this and the other smaller

$$\text{(Subeffect)}\frac{\text{(Bind)}\dfrac{\text{(Subeffect)}\dfrac{\dfrac{\Delta}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A} \qquad A \leq: A' \qquad \epsilon_1 \leq \epsilon_1'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1'} A'} \qquad \text{(Subeffect)}\dfrac{\dfrac{\Delta''}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B} \qquad B \leq: B' \qquad \epsilon_2 \leq \epsilon_2'}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \qquad B \leq: B'}{\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'} \Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C - 2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B' \tag{12}$$

0.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of $\Gamma \vdash t \colon \tau$ to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

0.3.1 Constants

For the constants $\mathtt{true}, \mathtt{false}, \mathtt{C}^A$, etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$$reduce(\text{(Const)}\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A}) = \text{(Const)}\frac{\Gamma \mathtt{Ok}}{\Gamma \vdash \mathtt{C}^A \colon A}$$

0.3.2 Value Types

Var

$$reduce(\text{(Var)}\frac{\Gamma \mathtt{Ok}}{\Gamma, x : A \vdash x \colon A}) = \text{(Var)}\frac{\Gamma \mathtt{Ok}}{\Gamma, x : A \vdash x \colon A} \tag{13}$$

Preserves denotation trivially.

Weaken

reduce **definition** To find:

$$reduce((\text{Weaken})\frac{\dfrac{\Delta}{\Gamma \vdash x\colon A}}{\Gamma, y : B \vdash x\colon A}) \tag{14}$$

Let

$$(\text{Subtype})\frac{\dfrac{\Delta'}{\Gamma \vdash x\colon A} \qquad A' \leq: A}{\Gamma \vdash x\colon A} = reduce(\Delta) \tag{15}$$

In

$$(\text{Subtype})\frac{(\text{Weaken})\dfrac{\dfrac{\Delta'}{\Gamma \vdash x\colon A'}}{\Gamma, y : B \vdash x\colon A'} \qquad A' \leq: A}{\Gamma, y : B \vdash x\colon A} \tag{16}$$

Preserves Denotation Using the construction of denotations, we can find the denotation of the original derivation to be:

$$[\![(\text{Weaken})\frac{\dfrac{\Delta}{\Gamma \vdash x\colon A}}{\Gamma, y : B \vdash x\colon A}]\!] = \Delta \circ \pi_1 \tag{17}$$

Similarly, the denotation of the reduced denotation is:

$$[\![(\text{Subtype})\frac{(\text{Weaken})\dfrac{\dfrac{\Delta'}{\Gamma \vdash x\colon A'}}{\Gamma, y : B \vdash x\colon A'} \qquad A' \leq: A}{\Gamma, y : B \vdash x\colon A}]\!] = [\![A' \leq: A]\!] \circ \Delta' \circ \pi_1 \tag{18}$$

By induction on *reduce* preserving denotations and the reduction of Δ (15), we have:

$$\Delta = [\![A' \leq: A]\!] \circ \Delta' \tag{19}$$

So the denotations of the un-reduced and reduced derivations are equal.

Lambda

reduce **definition** To find:

$$reduce((\text{Fn})\frac{\dfrac{\Delta}{\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_{\epsilon_2} B}) \tag{20}$$

Let

$$(\text{Sub-effect})\frac{\dfrac{\Delta'}{\Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon_1} B'} \qquad \epsilon_1 \leq \epsilon_2}{B' \leq: B} \Gamma, x : A \vdash C\colon \mathtt{M}_{\epsilon_2} B = reduce(\Delta) \tag{21}$$

In

$$\text{(Sub-type)}\frac{\text{(Fn)}\dfrac{\Delta'}{\Gamma, x : A \vdash C \colon \mathtt{M}_{\epsilon_1} B'} \qquad A \to \mathtt{M}_{\epsilon_1} B' \leq: A \to \mathtt{M}_{\epsilon_2} B}{\Gamma \vdash \lambda x : A.C : A \to \mathtt{M}_{\epsilon_2} B} \tag{22}$$

Preserves Denotation Let

$$f = [\![\mathtt{M}_{\epsilon_1} B' \leq: \mathtt{M}_{\epsilon_2} B]\!] = [\![\epsilon_1 \leq \epsilon_2]\!]_{M,B} \circ T_{\epsilon_1}([\![B' \leq: B]\!]) \tag{23}$$

$$[\![A \to \mathtt{M}_{\epsilon_1} B' \leq: A \to \mathtt{M}_{\epsilon_2} B]\!] = f^A = \mathtt{cur}(f \circ \mathtt{app}) \tag{24}$$

Then

$$\begin{aligned} before &= \mathtt{cur}(\Delta) \quad \text{By definition} && (25)\\ &= \mathtt{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta && (26)\\ &= f^A \circ \mathtt{cur}(\Delta') \quad \text{By the property of } f^X \circ \mathtt{cur}(g) = \mathtt{cur}(f \circ g) && (27)\\ &= after \quad \text{By definition} && (28)\\ & && (29) \end{aligned}$$

Subtype

reduce **definition** To find:

$$reduce(\text{(Subtype)}\frac{\dfrac{\Delta}{\Gamma \vdash v \colon A} \qquad A \leq: B}{\Gamma \vdash v \colon B}) \tag{30}$$

Let

$$\text{(Subtype)}\frac{\dfrac{\Delta'}{\Gamma \vdash x \colon A} \qquad A' \leq: A}{\Gamma \vdash x \colon A} = reduce(\Delta) \tag{31}$$

In

$$\text{(Subtype)}\frac{\dfrac{\Delta'}{\Gamma \vdash v \colon A'} \qquad A' \leq: A \leq: B}{\Gamma \vdash v \colon B} \tag{32}$$

Preserves Denotation

$$\begin{aligned} before &= [\![A \leq: B]\!] \circ \Delta && (33)\\ &= [\![A \leq: B]\!] \circ ([\![A' \leq: A]\!] \circ \Delta') \quad \text{byDenotation of reduction of } \Delta. && (34)\\ &= [\![A' \leq: B]\!] \circ \Delta' \quad \text{Subtyping relations are unique} && (35)\\ &= after && (36)\\ & && (37) \end{aligned}$$

0.3.3 Computation Types

Return

reduce **definition** To find:

$$reduce((\text{Return})\frac{\frac{\Delta}{\Gamma \vdash v\colon A}}{\Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}} A}) \tag{38}$$

Let

$$(\text{Sub-type})\frac{\frac{\Delta'}{\Gamma \vdash v\colon A'} \qquad A' \leq: A}{\Gamma \vdash v\colon A} = reduce(\Delta) \tag{39}$$

In

$$(\text{Sub-effect})\frac{(\text{Return})\frac{\Delta'}{\Gamma \vdash v\colon A} \qquad \texttt{1} \leq \texttt{1}}{A' \leq: A} \Gamma \vdash \texttt{return}\ v : \texttt{M}_{\texttt{1}} A \tag{40}$$

Then

$$\begin{aligned}
before &= \eta_A \circ \Delta \quad \text{By definition} \quad \text{By defintion} && (41)\\
&= \eta_A \circ [\![A' \leq: A]\!] \circ \Delta' \quad \text{BY reduction of } \Delta && (42)\\
&= T_{\texttt{1}} [\![A' \leq: A]\!] \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta && (43)\\
&= [\![\texttt{1} \leq \texttt{1}]\!]_{M,A} \circ T_{\texttt{1}} [\![A' \leq: A]\!] \circ \eta_{A'} \circ \Delta' \quad \text{Since } [\![\texttt{1} \leq \texttt{1}]\!] \text{ is the identity Nat-Trans} && (44)\\
&= after \quad \text{By definition} && (45)\\
& && (46)
\end{aligned}$$

Apply

reduce **definition** To find:

$$reduce((\text{Apply})\frac{\frac{\Delta_1}{\Gamma \vdash v_1\colon A \to \texttt{M}_\epsilon B} \qquad \frac{\Delta_2}{\Gamma \vdash v_2\colon A}}{\Gamma \vdash v_1\ v_2\colon \texttt{M}_\epsilon B}) \tag{47}$$

Let

$$(\text{Subtype})\frac{\frac{\Delta_1'}{\Gamma \vdash v_1\colon A' \to \texttt{M}_{\epsilon'} B'} \qquad A' \to \texttt{M}_{\epsilon'} B' \leq: A \to \texttt{M}_\epsilon B}{\Gamma \vdash v_1\colon A \to \texttt{M}_\epsilon B} = reduce(\Delta_1) \tag{48}$$

$$(\text{Subtype})\frac{\frac{\Delta_2'}{\Gamma \vdash v\colon A'} \qquad A' \leq: A}{\Gamma \vdash v_1\colon A} = reduce(\Delta_2) \tag{49}$$

In

$$
(\text{Sub-effect})\frac{(\text{Apply})\dfrac{\dfrac{\Delta_1'}{\Gamma \vdash v_1 \colon A' \to \mathtt{M}_{\epsilon'} B'} \qquad (\text{Sub-type})\dfrac{\dfrac{\Delta_2'}{\Gamma \vdash v_2 \colon A''} \quad A'' \leq: A \leq: A'}{\Gamma \vdash v_2 \colon A'}}{\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon'} B'} \qquad B' \leq: B \qquad \epsilon' \leq \epsilon}{\Gamma \vdash v_1\ v_2 \colon \mathtt{M}_{\epsilon} B} \tag{50}
$$

Preserves Denotation Let

$$
\begin{aligned}
f &= [\![A \leq: A']\!] : A \to A' &(51)\\
f' &= [\![A'' \leq: A]\!] : A'' \to A &(52)\\
g &= [\![B' \leq: B]\!] : B' \to B &(53)\\
h &= [\![\epsilon' \leq \epsilon]\!] : T_{\epsilon'} \to T_{\epsilon} &(54)
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\![A' \to \mathtt{M}_{\epsilon'} B' \leq: A \to \mathtt{M}_{\epsilon} B]\!] &= (h_B \circ T_{\epsilon'} g)^A \circ (T_{\epsilon'} B')^f &(55)\\
&= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app}) \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{Id} \times f)) &(56)\\
&= \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) &(57)
\end{aligned}
$$

Then

$$
\begin{aligned}
before &= \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} &(58)\\
&= \mathtt{app} \circ \langle \mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \circ \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 &(59)\\
&= \mathtt{app} \circ (\mathtt{cur}(h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f)) \times \mathtt{Id}_A) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{Factoring out} &(60)\\
&= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ (\mathtt{Id} \times f) \circ \langle \Delta_1', f' \circ \Delta_2' \rangle \quad \text{By the exponential property} &(61)\\
&= h_B \circ T_{\epsilon'} g \circ \mathtt{app} \circ \langle \Delta_1', f \circ f' \circ \Delta_2' \rangle &(62)\\
&= after \quad \text{By defintion} &(63)
\end{aligned}
$$

If

reduce **definition**

$$
reduce\left((\text{If})\frac{\dfrac{\Delta_1}{\Gamma \vdash v \colon \mathtt{Bool}} \qquad \dfrac{\Delta_2}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon} A} \qquad \dfrac{\Delta_3}{\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon} A}}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_{\epsilon} A}\right) = (\text{If})\frac{\dfrac{reduce(\Delta_1)}{\Gamma \vdash v \colon \mathtt{Bool}} \qquad \dfrac{reduce(\Delta_2)}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon} A} \qquad \dfrac{reduce(\Delta_3)}{\Gamma \vdash C_2 \colon \mathtt{M}_{\epsilon} A}}{\Gamma \vdash \mathtt{if}_{\epsilon, A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 : \mathtt{M}_{\epsilon} A} \tag{64}
$$

Preserves Denotation Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

Bind

reduce **definition** To find

$$reduce((\text{Bind})\frac{\dfrac{\Delta_1}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A} \quad \dfrac{\Delta_2}{\Gamma, x : A \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B}) \tag{65}$$

Let

$$(\text{Sub-effect})\frac{\dfrac{\Delta_1'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1'} A'} \quad \epsilon_1' \leq: \epsilon_1 \quad A' \leq: A}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1} A} = reduce(\Delta_1) \tag{66}$$

Since $i, \times : \Gamma, x : A' \triangleright \Gamma, x : A$ if $A' \leq: A$, and by Δ_2, $(\Gamma, x : A) \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$, there also exists a derivation Δ_3 of $(\Gamma, x : A') \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B$. Δ_3 is derived from Δ_2 simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-effect})\frac{\dfrac{\Delta_3'}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2'} B'} \quad \epsilon_2' \leq: \epsilon_2 \quad B' \leq: B}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2} B} = reduce(\Delta_3) \tag{67}$$

Since the effects monoid operation is monotone, if $\epsilon_1 \leq \epsilon_1'$ and $\epsilon_2 \leq \epsilon_2'$ then $\epsilon_1 \cdot \epsilon_2 \leq \epsilon_1' \cdot \epsilon_2'$

Then the result of reduction of the whole bind expression is:

$$(\text{Sub-effect})\frac{(\text{Bind})\dfrac{\dfrac{\Delta_1'}{\Gamma \vdash C_1 \colon \mathtt{M}_{\epsilon_1'} A'} \quad \dfrac{\Delta_3'}{\Gamma, x : A' \vdash C_2 \colon \mathtt{M}_{\epsilon_2'} B'}}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1' \cdot \epsilon_2'} B} \quad B' \leq: B \quad \epsilon_1' \cdot \epsilon_2' \leq \epsilon_1 \cdot \epsilon_2}{\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2 : \mathtt{M}_{\epsilon_1 \cdot \epsilon_2} B} \tag{68}$$

Preserves Denotation Let

$$f = [\![A' \leq: A]\!] : A' \to A \tag{69}$$

$$g = [\![B' \leq: B]\!] : B' \to B \tag{70}$$

$$h_1 = [\![\epsilon_1' \leq \epsilon_1]\!] : T_{\epsilon_1'} \to T_{\epsilon_1} \tag{71}$$

$$h_2 = [\![\epsilon_2' \leq \epsilon_2]\!] : T_{\epsilon_2'} \to T_{\epsilon_2} \tag{72}$$

$$h = [\![\epsilon_1' \cdot \epsilon_2' \leq \epsilon_1 \cdot \epsilon_2]\!] : T_{\epsilon_1' \cdot \epsilon_2'} \to T_{\epsilon_1 \cdot \epsilon_2} \tag{73}$$

Due to the denotation of the weakening used to derive Δ_3 from Δ_2, we have

$$\Delta_3 = \Delta_2 \circ (\mathtt{Id}_\Gamma \times f) \tag{74}$$

And due to the reduction of Δ_3, we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3' \tag{75}$$

So:

$$
\begin{aligned}
before &= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1 \rangle \quad \text{By definition.} && (76)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, h_{1,A} \circ T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{By reduction of } \Delta_1. && (77)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ (\mathtt{Id}_\Gamma \times h_{1,A}) \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Factor out } h_1 && (78)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1}\Delta_2 \circ h_{1,(\Gamma\times A)} \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Tensor strength and sub-effecting } h_1 && (79)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ \left\langle \mathtt{Id}_\Gamma, T_{\epsilon_1'} f \circ \Delta_1' \right\rangle \quad \text{Naturality of } h_1 && (80)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}\Delta_2 \circ \mathtt{t}_{\epsilon_1',\Gamma,A} \circ (\mathtt{Id}_\Gamma \times T_{\epsilon_1'} f) \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out pairing again} && (81)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_2 \circ (\mathtt{Id}_\Gamma \times f)) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Tensorstrength} && (82)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(\Delta_3) \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the definition of } \Delta_3 && (83)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'}(h_{2,B} \circ T_{\epsilon_2'} g \circ \Delta_3') \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the reduction of } \Delta_3 && (84)\\
&= \mu_{\epsilon_1,\epsilon_2,B} \circ h_{1,B} \circ T_{\epsilon_1'} h_{2,B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{Factor out the functor} && (85)\\
&= h_B \circ \mu_{\epsilon_1',\epsilon_2',B} \circ T_{\epsilon_1'} T_{\epsilon_2'} g \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By the } \mu \text{ and Sub-effect rule} && (86)\\
&= h_B \circ T_{\epsilon_1' \cdot \epsilon_2'} g \circ \mu_{\epsilon_1',\epsilon_2',B'} \circ T_{\epsilon_1'}\Delta_3' \circ \mathtt{t}_{\epsilon_1',\Gamma,A'} \circ \langle \mathtt{Id}_\Gamma, \Delta_1' \rangle \quad \text{By naturality of } \mu_{,,} && (87)\\
&= after \quad \text{By definition} && (88)
\end{aligned}
$$

Subeffect

reduce **definition** To find:

$$
reduce((\text{Subeffect})\frac{\dfrac{\Delta}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon'} B'}}{\epsilon' \le \epsilon} B' \le: B \Gamma \vdash C\colon \mathtt{M}_\epsilon B) \tag{89}
$$

Let

$$
(\text{Subeffect})\frac{\dfrac{\Delta'}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon''} B''}}{\epsilon'' \le \epsilon'} \mathtt{Bool}'' \le: B \Gamma \vdash C\colon \mathtt{M}_{\epsilon'} B = reduce(\Delta) \tag{90}
$$

in

$$
(\text{subeffect})\frac{\dfrac{\Delta'}{\Gamma \vdash C\colon \mathtt{M}_{\epsilon''} B''}}{\epsilon'' \le \epsilon} B'' \le: B \Gamma \vdash C\colon \mathtt{M}_\epsilon B \tag{91}
$$

Preserves Denotation Let

$$\begin{aligned}
f &= [\![B' \leq: B]\!] && (92)\\
g &= [\![B'' \leq: B']\!] && (93)\\
h_1 &= [\![\epsilon' \leq \epsilon]\!] && (94)\\
h_2 &= [\![\epsilon' \leq \epsilon']\!] && (95)\\
f \circ g &= [\![B'' \leq: B]\!] && (96)\\
h_1 \circ h_2 &= [\![\epsilon'' \leq \epsilon']\!] && (97)\\
& && (98)
\end{aligned}$$

Hence we can find the denotation of the derivation before reduction.

$$\begin{aligned}
before &= h_{1,B} \circ T_{\epsilon'} f \circ \Delta \quad \text{By definition} && (99)\\
&= (h_{1,B} \circ T_{\epsilon'} f) \circ (h_{2,B'} \circ T_{\epsilon''} g) \circ \Delta' \quad \text{By reduction of } \Delta && (100)\\
&= (h_{1,B} \circ h_{2,B}) \circ (T_{\epsilon''} f \circ g) \circ \Delta' \quad \text{By naturality of } h_2 \qquad = after \quad \text{By definition.} && (101)
\end{aligned}$$

0.4 Denotations are Equivalent

For each type relation instance $\Gamma \vdash t \colon \tau$ there exists a unique reduced derivation of the relation instance. For all derivations Δ, Δ' of the type relation instance, $[\![\Delta]\!] = [\![reduce \Delta]\!] = [\![reduce \Delta']\!] = [\![\Delta']\!]$, hence the denotation $[\![\Gamma \vdash t \colon \tau]\!]$ is unique.