

## 0.1 Beta and Eta Equivalence

### 0.1.1 Beta-Eta conversions

- (Lambda-Beta) 
$$\frac{\Gamma, x:A \vdash C:\mathbb{M}_\epsilon B \quad \Gamma \vdash v:A}{\Gamma \vdash (\lambda x:A. C) v =_{\beta_\eta} C[v/x]:\mathbb{M}_\epsilon B}$$
- (Lambda-Eta) 
$$\frac{\Gamma \vdash v:A \rightarrow \mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x:A. (v \ x) =_{\beta_\eta} v:A \rightarrow \mathbb{M}_\epsilon B}$$
- (Left Unit) 
$$\frac{\Gamma \vdash v:A \quad \Gamma, x:A \vdash C:\mathbb{M}_\epsilon B}{\Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C =_{\beta_\eta} C[v/x]:\mathbb{M}_\epsilon B}$$
- (Right Unit) 
$$\frac{\Gamma \vdash C:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x =_{\beta_\eta} C:\mathbb{M}_\epsilon A}$$
- (Associativity) 
$$\frac{\Gamma \vdash C_1:\mathbb{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2:\mathbb{M}_{\epsilon_2} B \quad \Gamma, y:B \vdash C_3:\mathbb{M}_{\epsilon_3} C}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) =_{\beta_\eta} \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C}$$
- (Unit) 
$$\frac{\Gamma \vdash v:\mathbf{Unit}}{\Gamma \vdash v =_{\beta_\eta} ():\mathbf{Unit}}$$
- (if-true) 
$$\frac{\Gamma \vdash C_1:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_2:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ true then } C_1 \text{ else } C_2 =_{\beta_\eta} C_1:\mathbb{M}_\epsilon A}$$
- (if-false) 
$$\frac{\Gamma \vdash C_2:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_1:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ false then } C_1 \text{ else } C_2 =_{\beta_\eta} C_2:\mathbb{M}_\epsilon A}$$
- (If-Eta) 
$$\frac{\Gamma, x:\mathbf{Bool} \vdash C:\mathbb{M}_\epsilon A \quad \Gamma \vdash v:\mathbf{Bool}}{\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C[\text{true}/x] \text{ else } C[\text{false}/x] =_{\beta_\eta} C[v/x]:\mathbb{M}_\epsilon A}$$

### 0.1.2 Equivalence Relation

- (Reflexive) 
$$\frac{\Gamma \vdash t:\tau}{\Gamma \vdash t =_{\beta_\eta} t:\tau}$$
- (Symmetric) 
$$\frac{\Gamma \vdash t_1 =_{\beta_\eta} t_2:\tau}{\Gamma \vdash t_2 =_{\beta_\eta} t_1:\tau}$$
- (Transitive) 
$$\frac{\Gamma \vdash t_1 =_{\beta_\eta} t_2:\tau \quad \Gamma \vdash t_2 =_{\beta_\eta} t_3:\tau}{\Gamma \vdash t_1 =_{\beta_\eta} t_3:\tau}$$

### 0.1.3 Congruences

- (Lambda) 
$$\frac{\Gamma, x:A \vdash C_1 =_{\beta_\eta} C_2:\mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x:A. C_1 =_{\beta_\eta} \lambda x:A. C_2:A \rightarrow \mathbb{M}_\epsilon B}$$
- (Return) 
$$\frac{\Gamma \vdash v_1 =_{\beta_\eta} v_2:A}{\Gamma \vdash \text{return } v_1 =_{\beta_\eta} \text{return } v_2:\mathbb{M}_1 A}$$
- (Apply) 
$$\frac{\Gamma \vdash v_1 =_{\beta_\eta} v'_1:A \rightarrow \mathbb{M}_\epsilon B \quad \Gamma \vdash v_2 =_{\beta_\eta} v'_2:A}{\Gamma \vdash v_1 \ v_2 =_{\beta_\eta} v'_1 \ v'_2:\mathbb{M}_\epsilon B}$$
- (Bind) 
$$\frac{\Gamma \vdash C_1 =_{\beta_\eta} C'_1:\mathbb{M}_{\epsilon_1} A \quad \Gamma, x:A \vdash C_2 =_{\beta_\eta} C'_2:\mathbb{M}_{\epsilon_2} B}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 =_{\beta_\eta} \text{do } x \leftarrow C'_1 \text{ in } C'_2:\mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B}$$
- (If) 
$$\frac{\Gamma \vdash v =_{\beta_\eta} v':\mathbf{Bool} \quad \Gamma \vdash C_1 =_{\beta_\eta} C'_1:\mathbb{M}_\epsilon A \quad \Gamma \vdash C_2 =_{\beta_\eta} C'_2:\mathbb{M}_\epsilon A}{\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 =_{\beta_\eta} \text{if}_{\epsilon, A} v \text{ then } C'_1 \text{ else } C'_2:\mathbb{M}_\epsilon A}$$
- (Subtype) 
$$\frac{\Gamma \vdash v =_{\beta_\eta} v':A \quad A \leq B}{\Gamma \vdash v =_{\beta_\eta} v':B}$$
- (Subeffect) 
$$\frac{\Gamma \vdash C =_{\beta_\eta} C':\mathbb{M}_{\epsilon_1} A \quad A \leq B \quad \epsilon_1 \leq \epsilon_2}{\Gamma \vdash C =_{\beta_\eta} C':\mathbb{M}_{\epsilon_2} B}$$

## 0.2 Beta-Eta Equivalence Implies Both Sides Have the Same Type

Each derivation of  $\Gamma \vdash t =_{\beta\eta} t' : \tau$  can be converted to a derivation of  $\Gamma \vdash t : \tau$  and  $\Gamma \vdash t' : \tau$  by induction over the beta-eta equivalence relation derivation.

### 0.2.1 Equivalence Relations

**Case Reflexive** By inversion we have a derivation of  $\Gamma \vdash t : \tau$ .

**Case Symmetric** By inversion  $\Gamma \vdash t' =_{\beta\eta} t : \tau$ . Hence by induction, derivations of  $\Gamma \vdash t' : \tau$  and  $\Gamma \vdash t : \tau$  are given.

**Case Transitive** By inversion, there exists  $t_2$  such that  $\Gamma \vdash t_1 =_{\beta\eta} t_2 : \tau$  and  $\Gamma \vdash t_2 =_{\beta\eta} t_3 : \tau$ . Hence by induction, we have derivations of  $\Gamma \vdash t_1 : \tau$  and  $\Gamma \vdash t_3 : \tau$

### 0.2.2 Beta-Eta Conversions

**Case Lambda** By inversion, we have  $\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B$  and  $\Gamma \vdash v : A$ . Hence by the typing rules, we have:

$$\text{(Apply)} \frac{\text{(Lambda)} \frac{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B}{\Gamma \vdash \lambda x : A. C : A \rightarrow \mathbb{M}_\epsilon B} \quad \Gamma \vdash v : A}{\Gamma \vdash (\lambda x : A. C) v : \mathbb{M}_\epsilon A}$$

By the substitution rule **TODO: which?**, we have

$$\text{(Substitution)} \frac{\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B \quad \Gamma \vdash v : A}{\Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B}$$

**Case Left Unit** By inversion, we have  $\Gamma \vdash v : A$  and  $\Gamma, x : A \vdash C : \mathbb{M}_\epsilon B$

Hence we have:

$$\text{(Bind)} \frac{\text{(Return)} \frac{\Gamma \vdash v : A}{\Gamma \vdash \text{return } v : \mathbb{M}_1 A} \quad \Gamma, x : A \vdash C : \mathbb{M}_\epsilon B}{\Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C : \mathbb{M}_{1.\epsilon} B = \mathbb{M}_\epsilon B} \quad (1)$$

And by the substitution typing rule we have: **TODO: Which Rule?**

$$\Gamma \vdash C[v/x] : \mathbb{M}_\epsilon B \quad (2)$$

**Case Right Unit** By inversion, we have  $\Gamma \vdash C : \mathbb{M}_\epsilon A$ .

Hence we have:

$$\text{(Bind)} \frac{\Gamma \vdash C : \mathbb{M}_\epsilon A \quad \text{(Return)} \frac{\text{(var)} \frac{\Gamma, x : A \vdash x : A}{\Gamma, x : A \vdash \text{return } x : \mathbb{M}_1 A}}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_{\epsilon.1} A = \mathbb{M}_\epsilon A}}{\Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_{\epsilon.1} A = \mathbb{M}_\epsilon A} \quad (3)$$

**Case Associativity** By inversion, we have  $\Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A$ ,  $\Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B$ , and  $\Gamma, y : B \vdash C_3 : \mathbf{M}_{\epsilon_3} C$ .

$$(\iota\pi \times) : (\Gamma, x : A, y : B) \triangleright (\Gamma, y : B)$$

So by the weakening property **TODO: which?**,  $\Gamma, x : A, y : B \vdash C_3 : \mathbf{M}_{\epsilon_3} C$

Hence we can construct the type derivations:

$$\text{(Bind)} \frac{\Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A \quad \text{(Bind)} \frac{\Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B \quad \Gamma, x : A, y : B \vdash C_3 : \mathbf{M}_{\epsilon_3} C}{\Gamma, x : A \vdash x C_2 C_3 : \mathbf{M}_{\epsilon_2 \cdot \epsilon_3} C}}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \quad (4)$$

and

$$\text{(Bind)} \frac{\text{(Bind)} \frac{\Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A \quad \Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B}{\Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \Gamma, y : B \vdash C_3 : \mathbf{M}_{\epsilon_3} C}{\Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} C} \quad (5)$$

**Case Eta** By inversion, we have  $\Gamma \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B$

By weakening, we have  $\iota\pi : (\Gamma, x : A) \triangleright \Gamma$  Hence, we have

$$\text{(Fn)} \frac{\text{(App)} \frac{(\Gamma, x : A) \vdash x : A \quad \text{(weakening)} \frac{\Gamma \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B \quad \iota\pi : \Gamma, x : A \triangleright \Gamma}{\Gamma, x : A \vdash v : A \rightarrow \mathbf{M}_{\epsilon} B}}{\Gamma, x : A \vdash v \ x : \mathbf{M}_{\epsilon} B}}{\Gamma \vdash \lambda x : A. (v \ x) : A \rightarrow \mathbf{M}_{\epsilon} B} \quad (6)$$

**Case If-True** By inversion, we have  $\Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A$ ,  $\Gamma \vdash C_2 : \mathbf{M}_{\epsilon} A$ . Hence by the typing lemma **TODO: Which?**, we have  $\Gamma \vdash \text{true} : \text{Bool}$  by the axiom typing rule.

Hence

$$\text{(If)} \frac{\Gamma \vdash \text{true} : \text{Bool} \quad \Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A \quad \Gamma \vdash C_2 : \mathbf{M}_{\epsilon} A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ true then } C_1 \text{ else } C_2 : \mathbf{M}_{\epsilon} A} \quad (7)$$

**Case If-False** As above,

Hence

$$\text{(If)} \frac{\Gamma \vdash \text{false} : \text{Bool} \quad \Gamma \vdash C_1 : \mathbf{M}_{\epsilon} A \quad \Gamma \vdash C_2 : \mathbf{M}_{\epsilon} A}{\Gamma \vdash \text{if}_{\epsilon, A} \text{ false then } C_1 \text{ else } C_2 : \mathbf{M}_{\epsilon} A} \quad (8)$$

**Case If-Eta** By inversion, we have:

$$\Gamma \vdash v : \text{Bool} \quad (9)$$

and

$$\Gamma, x : \text{Bool} \vdash C : \mathbf{M}_{\epsilon} A \quad (10)$$

Hence we also have  $\Gamma \vdash \text{true} : \text{Bool}$ . Hence, the following also hold:

$\Gamma \vdash \text{true} : \text{Bool}$ , and  $\Gamma \vdash \text{false} : \text{Bool}$ .

Hence by the substitution theorem, we have:

$$\text{(If)} \frac{\Gamma \vdash v : \text{Bool} \quad \Gamma \vdash C [\text{true}/x] : \mathbf{M}_{\epsilon} A \quad \Gamma \vdash C [\text{false}/x] : \mathbf{M}_{\epsilon} A}{\Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C [\text{true}/x] \text{ else } C [\text{false}/x] : \mathbf{M}_{\epsilon} A} \quad (11)$$

and

$$\Gamma \vdash C [v/x] : \mathbf{M}_{\epsilon} A \quad (12)$$

### 0.2.3 Congruences

Each congruence rule corresponds exactly to a type derivation rule. To convert to a type derivation, convert all preconditions, then use the equivalent type derivation rule.

**Case Lambda** By inversion,  $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 : \mathbb{M}_\epsilon B$ . Hence by induction  $\Gamma, x : A \vdash C_1 : \mathbb{M}_\epsilon B$ , and  $\Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B$ .

So

$$\Gamma \vdash \lambda x : A. C_1 : A \rightarrow \mathbb{M}_\epsilon B \quad (13)$$

and

$$\Gamma \vdash \lambda x : A. C_2 : A \rightarrow \mathbb{M}_\epsilon B \quad (14)$$

Hold.

**Case Return** By inversion,  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$ , so by induction

$$\Gamma \vdash v_1 : A$$

and

$$\Gamma \vdash v_2 : A$$

Hence we have

$$\Gamma \vdash \mathbf{return} v_1 : \mathbb{M}_1 A$$

and

$$\Gamma \vdash \mathbf{return} v_2 : \mathbb{M}_1 A$$

**Case Apply** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v'_1 : A \rightarrow \mathbb{M}_\epsilon B$  and  $\Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$ . Hence we have by induction  $\Gamma \vdash v_1 : A \rightarrow \mathbb{M}_\epsilon B$ ,  $\Gamma \vdash v_2 : A$ ,  $\Gamma \vdash v'_1 : A \rightarrow \mathbb{M}_\epsilon B$ , and  $\Gamma \vdash v'_2 : A$ .

So we have:

$$\Gamma \vdash v_1 \ v_2 : \mathbb{M}_\epsilon B \quad (15)$$

and

$$\Gamma \vdash v'_1 \ v'_2 : \mathbb{M}_\epsilon B \quad (16)$$

**Case Bind** By inversion, we have:  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbb{M}_{\epsilon_1} A$  and  $\Gamma, x : A \vdash C_2 =_{\beta\eta} C'_2 : \mathbb{M}_{\epsilon_2} B$ . Hence by induction, we have  $\Gamma \vdash C_1 : \mathbb{M}_{\epsilon_1} A$ ,  $\Gamma \vdash C'_1 : \mathbb{M}_{\epsilon_1} A$ ,  $\Gamma, x : A \vdash C_2 : \mathbb{M}_{\epsilon_2} B$ , and  $\Gamma, x : A \vdash C'_2 : \mathbb{M}_{\epsilon_2} B$

Hence we have

$$\Gamma \vdash \mathbf{do} \ x \leftarrow C_1 \ \mathbf{in} \ C_2 : \mathbb{M}_{\epsilon_1, \epsilon_2} A \quad (17)$$

$$\Gamma \vdash \mathbf{do} \ x \leftarrow C'_1 \ \mathbf{in} \ C'_2 : \mathbb{M}_{\epsilon_1, \epsilon_2} A \quad (18)$$

**Case If** By inversion, we have:  $\Gamma \vdash v =_{\beta\eta} v': \mathbf{Bool}$ ,  $\Gamma \vdash C_1 =_{\beta\eta} C'_1: \mathbf{M}_\epsilon A$ , and  $\Gamma \vdash C_2 =_{\beta\eta} C'_2: \mathbf{M}_\epsilon A$ .

Hence by induction, we have:

$$\Gamma \vdash v: \mathbf{Bool}, \Gamma \vdash v': \mathbf{Bool},$$

$$\Gamma \vdash C_1: \mathbf{M}_\epsilon A, \Gamma \vdash C'_1: \mathbf{M}_\epsilon A,$$

$$\Gamma \vdash C_2: \mathbf{M}_\epsilon A, \text{ and } \Gamma \vdash C'_2: \mathbf{M}_\epsilon A.$$

So

$$\Gamma \vdash \mathbf{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2: \mathbf{M}_\epsilon A \quad (19)$$

and

$$\Gamma \vdash \mathbf{if}_{\epsilon, A} v \text{ then } C'_1 \text{ else } C'_2: \mathbf{M}_\epsilon A \quad (20)$$

Hold.

**Case Subtype** By inversion, we have  $A \leq: B$  and  $\Gamma \vdash v =_{\beta\eta} v': A$ . By induction, we therefore have  $\Gamma \vdash v: A$  and  $\Gamma \vdash v': A$ .

Hence we have

$$\Gamma \vdash v: B \quad (21)$$

$$\Gamma \vdash v': B \quad (22)$$

**Case subeffect** By inversion we have:  $A \leq: B$ ,  $\epsilon_1 \leq \epsilon_2$ , and  $\Gamma \vdash C =_{\beta\eta} C': \mathbf{M}_{\epsilon_1} A$ .

Hence by inductive hypothesis, we have  $\Gamma \vdash C: \mathbf{M}_{\epsilon_1} A$  and  $\Gamma \vdash C': \mathbf{M}_{\epsilon_1} A$ .

Hence,

$$\Gamma \vdash C: \mathbf{M}_{\epsilon_2} B \quad (23)$$

and

$$\Gamma \vdash C': \mathbf{M}_{\epsilon_2} B \quad (24)$$

hold.

## 0.3 Beta-Eta equivalent terms have equal denotations

If  $t \vdash t' =_{\beta\eta} \tau$ : then  $\llbracket \Gamma \vdash t: \tau \rrbracket_M = \llbracket \Gamma \vdash t': \tau \rrbracket_M$

By induction over Beta-eta equivalence relation.

### 0.3.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

**Case Reflexive** Equality is reflexive, so if  $\Gamma \vdash t: \tau$  then  $\llbracket \Gamma \vdash t: \tau \rrbracket_M$  is equal to itself.

**Case Symmetric** By inversion, if  $\Gamma \vdash t =_{\beta\eta} t': \tau$  then  $\Gamma \vdash t' =_{\beta\eta} t: \tau$ , so by induction  $\llbracket \Gamma \vdash t': \tau \rrbracket_M = \llbracket \Gamma \vdash t: \tau \rrbracket_M$  and hence  $\llbracket \Gamma \vdash t: \tau \rrbracket_M = \llbracket \Gamma \vdash t': \tau \rrbracket_M$

**Case Transitive** There must exist  $t_2$  such that  $\Gamma \vdash t_1 =_{\beta\eta} t_2 : \tau$  and  $\Gamma \vdash t_2 =_{\beta\eta} t_3 : \tau$ , so by induction,  $\llbracket \Gamma \vdash t_1 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_2 : \tau \rrbracket_M$  and  $\llbracket \Gamma \vdash t_2 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_3 : \tau \rrbracket_M$ . Hence by transitivity of equality,  $\llbracket \Gamma \vdash t_1 : \tau \rrbracket_M = \llbracket \Gamma \vdash t_3 : \tau \rrbracket_M$

### 0.3.2 Beta-Eta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

**Case Lambda** Let  $f = \llbracket \Gamma, x : A \vdash C : \mathbb{M}_\epsilon B \rrbracket_M : (\Gamma \times A) \rightarrow T_\epsilon B$

Let  $g = \llbracket \Gamma \vdash v : A \rrbracket_M : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Gamma \vdash [v/x] : \Gamma, x : A \rrbracket_M : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon B \rrbracket_M = f \circ \langle \text{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned} \llbracket \Gamma \vdash (\lambda x : A. C) v : \mathbb{M}_\epsilon B \rrbracket_M &= \text{app} \circ \langle \text{cur}(f), g \rangle \\ &= \text{app} \circ (\text{cur}(f) \times \text{Id}_A) \circ \langle \text{Id}_\Gamma, g \rangle \\ &= f \circ \langle \text{Id}_\Gamma, g \rangle \\ &= \llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon B \rrbracket_M \end{aligned} \tag{25}$$

**Case Left Unit** Let  $f = \llbracket \Gamma, x : A \vdash C : \mathbb{M}_\epsilon B \rrbracket_M$

Let  $g = \llbracket \Gamma \vdash v : A \rrbracket_M : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Gamma \vdash [v/x] : \Gamma, x : A \rrbracket_M : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon B \rrbracket_M = f \circ \langle \text{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow \text{return } v \text{ in } C : \mathbb{M}_\epsilon B \rrbracket_M &= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathfrak{t}_{1, \Gamma, A} \circ \langle \text{Id}_\Gamma, \eta_A \circ g \rangle \\ &= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathfrak{t}_{1, \Gamma, A} \circ (\text{Id}_\Gamma \times \eta_A) \circ \langle \text{Id}_\Gamma, g \rangle \\ &= \mu_{1, \epsilon, B} \circ T_1 f \circ \eta_{(\Gamma \times A)} \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\ &= \mu_{1, \epsilon, B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\ &= f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\ &= \llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon B \rrbracket_M \end{aligned} \tag{26}$$

**Case Right Unit** Let  $f = \llbracket \Gamma \vdash C : \mathbb{M}_\epsilon A \rrbracket_M$

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow C \text{ in return } x : \mathbb{M}_\epsilon A \rrbracket_M &= \mu_{\epsilon, 1, A} \circ T_\epsilon (\eta_A \circ \pi_2) \circ \mathfrak{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= T_\epsilon \pi_2 \circ \mathfrak{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \pi_2 \circ \langle \text{Id}_\Gamma, f \rangle \\ &= f \end{aligned} \tag{27}$$

**Case Associative** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M \quad (28)$$

$$g = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M \quad (29)$$

$$h = \llbracket \Gamma, y : B \vdash C_3 : \mathbb{M}_\epsilon C \rrbracket_M \quad (30)$$

We also have the weakening:

$$\iota\pi \times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \quad (31)$$

With denotation:

$$\llbracket \iota\pi \times : \Gamma, x : A, y : B \triangleright \Gamma, y : B \rrbracket_M = (\pi_1 \times \text{Id}_B) \quad (32)$$

We need to prove that the following are equal

$$lhs = \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } (\text{do } y \leftarrow C_2 \text{ in } C_3) : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} \rrbracket_M \quad (33)$$

$$= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \text{Id}_B) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad (34)$$

$$rhs = \llbracket \Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow C_1 \text{ in } C_2) \text{ in } C_3 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3} \rrbracket_M \quad (35)$$

$$= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2} (h) \circ \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \quad (36)$$

$$(37)$$

Let's look at fragment  $F$  of  $rhs$ .

$$F = \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \quad (38)$$

So

$$rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2} (h) \circ F \quad (39)$$

$$\begin{aligned} F &= \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\text{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathbf{t}_{\epsilon_2, \Gamma, B} \circ \mathbf{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\text{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{By \textbf{TODO: ref: mu+tstrength}} \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} (\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{By naturality of t-strength} \end{aligned} \quad (40)$$

Since  $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2} (h) \circ F$ ,

$$\begin{aligned} rhs &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2} (h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} (\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1} (T_{\epsilon_2} (h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\ &= \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} (h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \end{aligned} \quad (41)$$

Let's now look at the fragment  $G$  of  $rhs$

$$G = T_{\epsilon_1} (\text{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad (42)$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2 \cdot \epsilon_3, C} \circ T_{\epsilon_1} (\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} (h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (43)$$

By folding out the  $\langle \dots, \dots \rangle$ , we have

$$G = T_{\epsilon_1}(\text{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ (\text{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A}) \circ \langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle \quad (44)$$

From the rule **TODO: Ref** showing the commutativity of tensor strength with  $\alpha$ , the following commutes

$$\begin{array}{ccc} \Gamma \xrightarrow{\langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle} \Gamma \times (\Gamma \times T_{\epsilon_1} A) & \xleftarrow{\alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}} & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \\ \downarrow \text{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A} & & \downarrow \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \\ \Gamma \times T_{\epsilon_1}(\Gamma \times A) & & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \\ \downarrow \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} & \swarrow T_{\epsilon_1} \alpha_{\Gamma, \Gamma, A} & \\ T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) & & \end{array}$$

Where  $\alpha : ((- \times -) \times -) \rightarrow (- \times (- \times -))$  is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle \quad (45)$$

$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle \quad (46)$$

So:

$$\begin{aligned} G &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}^{-1} \circ \langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \langle \langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \times \text{Id}_{T_{\epsilon_1} A} \rangle \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By definition of } \alpha \text{ and products} \\ &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A} \circ \langle \langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \times \text{Id}_A \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By tensor strength's left-naturality} \\ &= T_{\epsilon_1}((\pi_1 \times \text{Id}_{T_{\epsilon_2} B}) \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \end{aligned} \quad (47)$$

Since

$$rhs = \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (48)$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\pi_1 \times \text{Id}_{T_{\epsilon_2} B}) \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \text{Id}_B))) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By Left-Tensor Strength} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \quad (49)$$

**Case Eta** Let

$$f = \llbracket \Gamma \vdash v : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M : \Gamma \rightarrow (T_\epsilon B)^A \quad (50)$$

By weakening, we have

$$\llbracket \Gamma, x : A \vdash v : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M = f \circ \pi_1 : \Gamma \times A \rightarrow (T_\epsilon B)^A \quad (51)$$

$$\llbracket \Gamma, x : A \vdash v x : \mathbf{M}_\epsilon B \rrbracket_M = \mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \quad (52)$$

$$(53)$$



Hence, we have

$$\begin{aligned}
\llbracket \Gamma \vdash \lambda x : A.(v x) : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M &= \mathbf{cur}(\mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \\
\mathbf{app} \circ (\llbracket \Gamma \vdash \lambda x : A.(v x) : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M \times \mathbf{Id}_A) &= \mathbf{app} \circ (\mathbf{cur}(\mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \times \mathbf{Id}_A) \\
&= \mathbf{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \\
&= \mathbf{app} \circ (f \times \mathbf{Id}_A)
\end{aligned} \tag{54}$$

Hence, by the fact that  $\mathbf{cur}(f)$  is unique in a cartesian closed category,

$$\llbracket \Gamma \vdash \lambda x : A.(v x) : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M = f = \llbracket \Gamma \vdash v : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M \tag{55}$$

**Case If-True** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M \tag{56}$$

$$g = \llbracket \Gamma \vdash C_2 : \mathbb{M}_\epsilon A \rrbracket_M \tag{57}$$

$$\tag{58}$$

Then

$$\begin{aligned}
\llbracket \Gamma \vdash \mathbf{if}_{\mathbf{true}, A} v \text{ then } C_1 \text{ else } C_2 : \mathbb{M}_\epsilon A \rrbracket_M &= \mathbf{app} \circ (([\mathbf{cur}(f \circ \pi_2), \mathbf{cur}(g \circ \pi_2)] \circ \mathbf{inl} \circ \langle \rangle_\Gamma) \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathbf{app} \circ ((\mathbf{cur}(f \circ \pi_2) \circ \langle \rangle_\Gamma) \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathbf{app} \circ (\mathbf{cur}(f \circ \pi_2) \times \mathbf{Id}_\Gamma) \circ (\langle \rangle_\Gamma \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= f \circ \pi_2 \circ \langle \rangle_\Gamma, \mathbf{Id}_\Gamma \\
&= f \\
&= \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M
\end{aligned} \tag{59}$$

**Case If-False** Let

$$f = \llbracket \Gamma \vdash C_1 : \mathbb{M}_\epsilon A \rrbracket_M \tag{60}$$

$$g = \llbracket \Gamma \vdash C_2 : \mathbb{M}_\epsilon A \rrbracket_M \tag{61}$$

$$\tag{62}$$

Then

$$\begin{aligned}
\llbracket \Gamma \vdash \mathbf{if}_{\mathbf{true}, A} v \text{ then } C_1 \text{ else } C_2 : \mathbb{M}_\epsilon A \rrbracket_M &= \mathbf{app} \circ (([\mathbf{cur}(f \circ \pi_2), \mathbf{cur}(g \circ \pi_2)] \circ \mathbf{inr} \circ \langle \rangle_\Gamma) \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathbf{app} \circ ((\mathbf{cur}(g \circ \pi_2) \circ \langle \rangle_\Gamma) \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= \mathbf{app} \circ (\mathbf{cur}(g \circ \pi_2) \times \mathbf{Id}_\Gamma) \circ (\langle \rangle_\Gamma \times \mathbf{Id}_\Gamma) \circ \delta_\Gamma \\
&= g \circ \pi_2 \circ \langle \rangle_\Gamma, \mathbf{Id}_\Gamma \\
&= g \\
&= \llbracket \Gamma \vdash C_2 : \mathbb{M}_\epsilon A \rrbracket_M
\end{aligned} \tag{63}$$

### 0.3.3 Case If-Eta

Let

$$f = \llbracket \Gamma \vdash v : \mathbf{Bool} \rrbracket_M \tag{64}$$

$$g = \llbracket \Gamma, x : \mathbf{Bool} \vdash C : \mathbb{M}_\epsilon A \rrbracket_M \tag{65}$$

$$\tag{66}$$

Then by the substitution theorem,

$$\llbracket \Gamma \vdash C [\mathbf{true}/x] : \mathbb{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \quad (67)$$

$$\llbracket \Gamma \vdash C [\mathbf{false}/x] : \mathbb{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \quad (68)$$

$$\llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon A \rrbracket_M = g \circ \langle \text{Id}_\Gamma, f \rangle \quad (69)$$

Hence we have (Using the diagonal and twist morphisms):

$$\llbracket \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C [\mathbf{true}/x] \text{ else } C [\mathbf{false}/x] : \mathbb{M}_\epsilon A \rrbracket_M \quad (70)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \text{Id}_\Gamma, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2), \text{cur}(g \circ \langle \text{Id}_\Gamma, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad (71)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Pairing property} \quad (72)$$

$$= \text{app} \circ (((\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_1), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_1)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Terminal is unique} \quad (73)$$

$$= \text{app} \circ ((([\text{cur}(g \circ (\text{Id}_\Gamma \times (\text{inl}_1 \circ \langle \rangle_1)) \circ \tau_{1, \Gamma}), \text{cur}(g \circ (\text{Id}_\Gamma \times (\text{inr}_1 \circ \langle \rangle_1)) \circ \tau_{1, \Gamma})] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of } \tau \quad (74)$$

$$= \text{app} \circ ((([\text{cur}(g \circ (\text{Id}_\Gamma \times \text{inl}_1)) \circ \tau_{1, \Gamma}), \text{cur}(g \circ (\text{Id}_\Gamma \times \text{inr}_1)) \circ \tau_{1, \Gamma})] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Identity} = \text{Id}_1 \quad (75)$$

$$= \text{app} \circ ((([\text{cur}(g \circ \tau_{1+1, \Gamma} \circ (\text{inl}_1 \times \text{Id}_\Gamma)), \text{cur}(g \circ \tau_{1+1, \Gamma} \circ (\text{inr}_1 \times \text{Id}_\Gamma))] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Twist commutivity} \quad (76)$$

$$= \text{app} \circ ((([\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ \text{inl}_1, \text{cur}(g \circ \tau_{1+1, \Gamma}) \circ \text{inr}_1] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Exponential property} \quad (77)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ [\text{inl}_1, \text{inr}_1] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring out } \text{cur}(\cdot) \quad (78)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1, \Gamma}) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Since } [\text{inl}, \text{inr}] \text{ is the identity} \quad (79)$$

$$= \text{app} \circ (\text{cur}(g \circ \tau_{1+1, \Gamma}) \times \text{Id}_\Gamma) \circ (f \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring} \quad (80)$$

$$= g \circ \tau_{1+1, \Gamma} \circ (f \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of } \text{app}, \text{cur}(\cdot) \quad (81)$$

$$= g \circ (\text{Id}_\Gamma \times f) \circ \tau_{\Gamma, \Gamma} \circ \delta_\Gamma \quad \text{Twist commutivity} \quad (82)$$

$$= g \circ (\text{Id}_\Gamma \times f) \circ \langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \quad \text{Twist, diagonal definitions} \quad (83)$$

$$= g \circ \langle \text{Id}_\Gamma, f \rangle \quad (84)$$

$$= \llbracket \Gamma \vdash C [v/x] : \mathbb{M}_\epsilon A \rrbracket_M \quad (85)$$

$$(86)$$

### 0.3.4 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of sub-expressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

**Case Lambda** By inversion, we have  $\Gamma, x : A \vdash C_1 =_{\beta\eta} C_2 : \mathbb{M}_\epsilon B$  By induction, we therefore have  $\llbracket \Gamma, x : A \vdash C_1 : \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M$

Then let

$$f = \llbracket \Gamma, x : A \vdash C_1 : \mathbb{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C_2 : \mathbb{M}_\epsilon B \rrbracket_M \quad (87)$$

And so

$$\llbracket \Gamma \vdash \lambda x : A. C_1 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M = \text{cur}(f) = \llbracket \Gamma \vdash \lambda x : A. C_2 : A \rightarrow \mathbb{M}_\epsilon B \rrbracket_M \quad (88)$$

**Case Return** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$  By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M \quad (89)$$

And so

$$\llbracket \Gamma \vdash \text{return } v_1 : \mathbf{M}_1 A \rrbracket_M = \eta_A \circ f = \llbracket \Gamma \vdash \text{return } v_2 : \mathbf{M}_1 A \rrbracket_M \quad (90)$$

**Case Apply** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v'_1 : A \rightarrow \mathbf{M}_\epsilon B$  and  $\Gamma \vdash v_2 =_{\beta\eta} v'_2 : A$  By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M = \llbracket \Gamma \vdash v'_1 : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M$  and  $\llbracket \Gamma \vdash v_2 : A \rrbracket_M = \llbracket \Gamma \vdash v'_2 : A \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M = \llbracket \Gamma \vdash v'_1 : A \rightarrow \mathbf{M}_\epsilon B \rrbracket_M \quad (91)$$

$$g = \llbracket \Gamma \vdash v_2 : A \rrbracket_M = \llbracket \Gamma \vdash v'_2 : A \rrbracket_M \quad (92)$$

And so

$$\llbracket \Gamma \vdash v_1 v_2 : \mathbf{M}_\epsilon A \rrbracket_M = \text{app} \circ \langle f, g \rangle = \llbracket \Gamma \vdash v'_1 v'_2 : \mathbf{M}_\epsilon A \rrbracket_M \quad (93)$$

**Case Bind** By inversion, we have  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbf{M}_\epsilon A$  and  $\Gamma, x : A \vdash C_2 =_{\beta\eta} C'_2 : \mathbf{M}_\epsilon B$  By induction, we therefore have  $\llbracket \Gamma \vdash C_1 : \mathbf{M}_\epsilon A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbf{M}_\epsilon A \rrbracket_M$  and  $\llbracket \Gamma, x : A \vdash C_2 : \mathbf{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbf{M}_\epsilon B \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M \quad (94)$$

$$g = \llbracket \Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M \quad (95)$$

And so

$$\begin{aligned} \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M \end{aligned} \quad (96)$$

**Case If** By inversion, we have  $\Gamma \vdash v =_{\beta\eta} v' : \text{Bool}$ ,  $\Gamma \vdash C_1 =_{\beta\eta} C'_1 : \mathbf{M}_\epsilon A$  and  $\Gamma \vdash C_2 =_{\beta\eta} C'_2 : \mathbf{M}_\epsilon A$  By induction, we therefore have  $\llbracket \Gamma \vdash v : \text{Bool} \rrbracket_M = \llbracket \Gamma \vdash v' : \text{Bool} \rrbracket_M$ ,  $\llbracket \Gamma \vdash C_1 : \mathbf{M}_\epsilon A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbf{M}_\epsilon A \rrbracket_M$  and  $\llbracket \Gamma, x : A \vdash C_2 : \mathbf{M}_\epsilon B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbf{M}_\epsilon B \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v : \text{Bool} \rrbracket_M = \llbracket \Gamma \vdash v' : \text{Bool} \rrbracket_M \quad (97)$$

$$g = \llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C'_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M \quad (98)$$

$$h = \llbracket \Gamma, x : A \vdash C_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M = \llbracket \Gamma, x : A \vdash C'_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M \quad (99)$$

And so

$$\begin{aligned} \llbracket \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } C_1 \text{ else } C_2 : \rrbracket_M &= \text{app} \circ ((\text{cur}(g \circ \pi_2), \text{cur}(h \circ \pi_2)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \llbracket \Gamma \vdash \text{do } x \leftarrow C_1 \text{ in } C_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket_M \end{aligned} \quad (100)$$

**Case Subtype** By inversion, we have  $\Gamma \vdash v_1 =_{\beta\eta} v_2 : A$ , and  $A \leq B$  By induction, we therefore have  $\llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : A \rrbracket_M \quad (101)$$

$$g = \llbracket A \leq B \rrbracket_M \quad (102)$$

And so

$$\llbracket \Gamma \vdash v_1 : B \rrbracket_M = g \circ f = \llbracket \Gamma \vdash v_1 : B \rrbracket_M \quad (103)$$

**Case subeffect** By inversion, we have  $\Gamma \vdash C_1 =_{\beta\eta} C_2 : \mathbf{M}_{\epsilon_1} A$ , and  $A \leq B$  and  $\epsilon_1 \leq \epsilon_2$ . By induction, we therefore have  $\llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_1} A \rrbracket_M = \llbracket \Gamma \vdash C_2 : \mathbf{M}_{\epsilon_1} A \rrbracket_M$

Then let

$$f = \llbracket \Gamma \vdash v_1 : A \rrbracket_M = \llbracket \Gamma \vdash v_2 : B \rrbracket_M \quad (104)$$

$$g = \llbracket A \leq B \rrbracket_M \quad (105)$$

$$h = \llbracket \epsilon_1 \leq \epsilon_2 \rrbracket_M \quad (106)$$

And so

$$\llbracket \Gamma \vdash C_1 : \mathbf{M}_{\epsilon_2} B \rrbracket_M = h_B \circ T_{\epsilon_1} g \circ f = \llbracket \Gamma \vdash C_2 : \mathbf{M}_{\epsilon_2} B \rrbracket_M \quad (107)$$

□