Given a set based S-Category $\mathbb{C}$ which is a model of the non-polymorphic effect calculus, we generate an indexed category capable of modelling the polymorphic effect calculus.

0.1 The Non-Polymorphic Model

Since $\mathbb{C}$ is a model of the non-polymorphic calculus,

- $\mathbb{C}$ is cartesian closed.
- $\mathbb{C}$ has a strong graded monad: $\mathtt{T}^0 : (E, \cdot, \leq_0, \mathtt{1}) \to [\mathbb{C}, \mathbb{C}]$
- $\mathbb{C}$ has a co-product on the terminal object $\mathtt{1}$.

In addition, we require that

- $\mathbb{C}$ should be complete (e.g a sub-category of `Set`)
- E should be small.

0.2 Base Category

We construct the base category, `Eff` as follows:

- $U = E$, the set of ground effects in the non-polymorphic language.
- $\mathtt{1}$ is a singleton set.
- $U^n = E^n$, set of n-wide tuples of effects, $\vec{\epsilon}$

Hence when we treat effects that are well formed in Φ as morphisms, $E^n \to E$ in `Eff`, we should treat them as functions $f : E^n \to E$. Ground effects become point functions: $e : \mathtt{1} \to E$, so the denotation of a ground effect is the constant value function: $[\![\Phi \vdash e \colon \mathtt{Effect}]\!]_M = \vec{\epsilon} \mapsto e$

We extend the multiplication of ground effects to multiplication on effect functions, giving us our `Mul` operation

$$\mathtt{Mul}(f, g) = f \cdot g \tag{1}$$
$$(f \cdot g)(\vec{\epsilon}) = (f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \tag{2}$$
$$\tag{3}$$

This satisfies naturality of `Mul`.

$$((f \cdot g) \circ \theta)\vec{\epsilon} = (f(\theta\vec{\epsilon})) \cdot (g(\theta\vec{\epsilon})) = ((f \circ \theta) \cdot (g \circ \theta))\vec{\epsilon} \tag{4}$$

0.3 S-Categories

The semantic category, $\mathbb{C}_0$ of the effect-environment $\diamond$ is $\mathbb{C}$.

Since each effect-environment is alpha equivalent to a natural number, the semantic category for Φ shall be represented as $\mathbb{C}_\Phi = \mathbb{C}_n = [E^n, \mathbb{C}]$, the category of functions $E^n \to \mathbb{C}$.

Objects in $[E^n, \mathbb{C}]$ are functions and we describe them by their actions on a generic vector of ground effects, $\vec{\epsilon}$.

Morphisms in $[E^n, \mathbb{C}]$ are natural transformations between the functions. So:

$$
\begin{aligned}
m: \quad & A \to B \quad \text{In } [E^n, \mathbb{C}] & (5)\\
m\vec{\epsilon}: \quad & A\vec{\epsilon} \to B\vec{\epsilon} \quad \text{In } \mathbb{C} & (6)\\
(f \circ g)\vec{\epsilon} = \quad & (f\vec{\epsilon}) \circ (g\vec{\epsilon}) & (7)\\
\mathtt{1}(\vec{\epsilon}) = \quad & \mathtt{1} & (8)
\end{aligned}
$$

So morphisms are dependently typed functions from a vector of ground effects to morphisms in $\mathbb{C}$.

0.3.1 Each S-Category is a CCC

Since $\mathbb{C}$ is complete and a CCC, and E^n is small, since E is small, $[E^n, \mathbb{C}]$ is a CCC.

$$
\begin{aligned}
(A \times B)\vec{\epsilon} &= (A\vec{\epsilon}) \times (B\vec{\epsilon}) & (9)\\
\mathtt{1}\vec{\epsilon} &= \mathtt{1} & (10)\\
(B^A)\vec{\epsilon} &= (B\vec{\epsilon})^{(A\vec{\epsilon})} & (11)\\
\pi_1\vec{\epsilon} &= \pi_1 & (12)\\
\pi_2\vec{\epsilon} &= \pi_2 & (13)\\
\mathtt{app}\vec{\epsilon} &= \mathtt{app} & (14)\\
\mathtt{cur}(f)\vec{\epsilon} &= \mathtt{cur}(f\vec{\epsilon}) & (15)\\
\langle f, g\rangle\, \vec{\epsilon} &= \langle f\vec{\epsilon}, g\vec{\epsilon}\rangle & (16)\\
& & (17)
\end{aligned}
$$

0.3.2 The Terminal Co-Product

We can define the co-product point-wise.

$$
\begin{aligned}
(\mathtt{1} + \mathtt{1})\vec{\epsilon} &= (\mathtt{1}\vec{\epsilon} + \mathtt{1}\vec{\epsilon}) & (18)\\
&= (\mathtt{1} + \mathtt{1}) & (19)\\
\mathtt{inl}\vec{\epsilon} &= \mathtt{inl} & (20)\\
\mathtt{inr}\vec{\epsilon} &= \mathtt{inr} & (21)\\
[f, g]\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}] & (22)\\
& & (23)
\end{aligned}
$$

This preserves the co-product diagram.

$$\begin{aligned}
([f,g]\circ \mathtt{inl})\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}]\circ \mathtt{inl} && (24)\\
&= f\vec{\epsilon} && (25)\\
&\square && (26)\\
([f,g]\circ \mathtt{inr})\vec{\epsilon} &= [f\vec{\epsilon}, g\vec{\epsilon}]\circ \mathtt{inr} && (27)\\
&= f\vec{\epsilon} && (28)\\
&\square && (29)\\
& && (30)
\end{aligned}$$

Suppose $l\circ \mathtt{inl} = f$ and $l \circ \mathtt{inr} = g$ then $l\vec{\epsilon}\circ \mathtt{inl} = f\vec{\epsilon}$ and $l\vec{\epsilon}\circ \mathtt{inr} = g\vec{\epsilon}$ then by in co-product in $\mathbb{C}$, $l = [f\vec{\epsilon}, g\vec{\epsilon}]$ so $l = [f,g]$.

0.3.3 Ground Types and Terms

Each ground type in the non-polymorphic calculus has a fixed denotation $[\![\gamma]\!]_M \in \mathtt{obj}\ \mathbb{C}$. The ground type in the polymorphic calculus hence has a denotation represented by the constant function.

$$\begin{aligned}
[\![\gamma]\!]_M : \quad & E^n \to \mathtt{obj}\ \mathbb{C} && (31)\\
\vec{\epsilon} \mapsto \quad & [\![\gamma]\!]_M && (32)\\
& && (33)
\end{aligned}$$

Each constant term $\mathtt{C}^A$ in the non-polymorphic calculus has a fixed denotation $[\![\mathtt{C}^A]\!]_M \in \mathbb{C}(\mathtt{1}, A)$.

So the morphism $[\![\mathtt{C}^A]\!]_M$ in $[E^n, \mathbb{C}]$ is the corresponding constant dependently typed morphism.

$$\begin{aligned}
[\![\mathtt{C}^A]\!]_M : \quad & [E^n, \mathbb{C}](\mathtt{1}, A) && (34)\\
\vec{\epsilon} \mapsto \quad & [\![\mathtt{C}^A]\!]_M && (35)
\end{aligned}$$

0.3.4 Graded Monad

Given the strong graded monad $(\mathtt{T}^0, \eta^0, \mu^0_{,,}, \mathtt{t}^0_{,,})$ on $\mathbb{C}$ we can construct an appropriate graded monad on $[E^n, \mathbb{C}]$.

$$\begin{aligned}
\mathtt{T}^n : \quad & (E^n, \cdot, \leq_n, \mathtt{1}_n) \to [[E^n, \mathbb{C}], [E^n, \mathbb{C}]] && (36)\\
(\mathtt{T}^n_f A)\vec{\epsilon} = \quad & \mathtt{T}^0_{(f\vec{\epsilon})} A\vec{\epsilon} && (37)\\
(\eta^n_A)\vec{\epsilon} = \quad & \eta^0_{A\vec{\epsilon}} && (38)\\
(\mu^n_{f,g,A})\vec{\epsilon} = \quad & \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} && (39)\\
(\mathtt{t}^n_{f,A,B})\vec{\epsilon} = \quad & \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} && (40)
\end{aligned}$$

Through some mechanical proof and the naturality of the $\mathbb{C}$ strong graded monad, these morphisms are natural in their type parameters and form a strong graded monad in $[E^n, \mathbb{C}]$

Naturality

$$\begin{array}{ccc} A\vec{\epsilon} & \xrightarrow{\eta^0_{(A\vec{\epsilon})}} & \mathtt{T}^0_{\mathtt{1}}(A\vec{\epsilon}) \\ \Big\downarrow f\vec{\epsilon} & & \Big\downarrow \mathtt{T}^0_{\mathtt{1}}(f\vec{\epsilon}) \\ B\vec{\epsilon} & \xrightarrow{\eta^0_{(B\vec{\epsilon})}} & \mathtt{T}^0_{\mathtt{1}}(B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} \mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(A\vec{\epsilon}) & \xrightarrow{\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}} & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon}) \\ \Big\downarrow \mathtt{T}^0_{f\vec{\epsilon}}\mathtt{T}^0_{g\vec{\epsilon}}m\vec{\epsilon} & & \Big\downarrow \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}m\vec{\epsilon} \\ \mathtt{T}^0_{(f\vec{\epsilon})}\mathtt{T}^0_{(g\vec{\epsilon})}(B\vec{\epsilon}) & \xrightarrow{\mu^0_{f\vec{\epsilon},g\vec{\epsilon},(B\vec{\epsilon})}} & \mathtt{T}^0_{(f\vec{\epsilon})\cdot(g\vec{\epsilon})}(A\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) & \xrightarrow{\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}} & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B\vec{\epsilon}) \\ \Big\downarrow (m\vec{\epsilon}\times\mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}B}) & & \Big\downarrow \mathtt{T}^0_{(f\vec{\epsilon})}(m\vec{\epsilon}\times\mathtt{Id}_{B\vec{\epsilon}}) \\ A'\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) & \xrightarrow{\mathtt{t}^0_{f\vec{\epsilon},(A'\vec{\epsilon}),(B\vec{\epsilon})}} & \mathtt{T}^0_{f\vec{\epsilon}}(A'\vec{\epsilon}\times B\vec{\epsilon}) \end{array}$$

$$\begin{array}{ccc} A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B\vec{\epsilon}) & \xrightarrow{\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B\vec{\epsilon})}} & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B\vec{\epsilon}) \\ \Big\downarrow (\mathtt{Id}_{A\vec{\epsilon}}\times\mathtt{T}^0_{f\vec{\epsilon}}(m\vec{\epsilon})) & & \Big\downarrow \mathtt{T}^0_{(f\vec{\epsilon})}(\mathtt{Id}_{A\vec{\epsilon}}\times m\vec{\epsilon}) \\ A\vec{\epsilon}\times\mathtt{T}^0_{f\vec{\epsilon}}(B'\vec{\epsilon}) & \xrightarrow{\mathtt{t}^0_{f\vec{\epsilon},(A\vec{\epsilon}),(B'\vec{\epsilon})}} & \mathtt{T}^0_{f\vec{\epsilon}}(A\vec{\epsilon}\times B'\vec{\epsilon}) \end{array}$$

Monad Laws

Left Unit

$$\begin{aligned} (\mu^n_{f,\mathtt{1},A}\circ\mathtt{T}^n_f\eta^n_A)\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),\mathtt{1},(A\vec{\epsilon})}\circ\mathtt{T}^0_{f\vec{\epsilon}}(\eta^0_{A\vec{\epsilon}}) && (41)\\ &= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} && (42)\\ &= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon} && (43) \end{aligned}$$

Right Unit

$$\begin{aligned} (\mu^n_{\mathtt{1},g,A}\circ\eta^n_{\mathtt{T}^n_f A})\vec{\epsilon} &= \mu^0_{\mathtt{1},(f\vec{\epsilon}),(A\vec{\epsilon})}\circ(\eta^0_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}}) && (44)\\ &= \mathtt{Id}_{\mathtt{T}^0_{f\vec{\epsilon}}A\vec{\epsilon}} && (45)\\ &= (\mathtt{Id}_{\mathtt{T}^n_f A})\vec{\epsilon} && (46) \end{aligned}$$

Monad Associativity

$$\begin{aligned} ((\mu^n_{f,(g\cdot h),A})\circ\mathtt{T}^n_f(\mu^n_{g,h,A}))\vec{\epsilon} &= \mu^0_{(f\vec{\epsilon}),((g\vec{\epsilon})\cdot(h\vec{\epsilon})),(A\vec{\epsilon})}\circ\mathtt{T}^0_{f\vec{\epsilon}}\mu^0_{(h\vec{\epsilon}),(g\vec{\epsilon}),A\vec{\epsilon}} && (47)\\ &= \mu^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(h\vec{\epsilon}),(A\vec{\epsilon})}\circ\mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(\mathtt{T}^0_{h\vec{\epsilon}}(A\vec{\epsilon}))} && (48)\\ &= (\mu^n_{f\cdot g,h,A}\circ\mu^n_{f,g,\mathtt{T}^0_h A})\vec{\epsilon} && (49) \end{aligned}$$

Tensorial Strength

Unitor Law

$$
\begin{aligned}
(\mathtt{T}^n_f \pi_2)\vec{\epsilon} &= \mathtt{T}^0_{(f\vec{\epsilon})}(\pi_2\vec{\epsilon}) && (50)\\
&= \mathtt{T}^0_{(f\vec{\epsilon})}(\pi_2) && (51)\\
&= \pi_2 && (52)\\
&= \pi_2\vec{\epsilon} && (53)
\end{aligned}
$$

Bind Law

$$
\begin{tikzcd}
A \times \mathtt{T}^n_f \mathtt{T}^n_g B \arrow[r, "\mathtt{t}_{f,A,\mathtt{T}^n_g B}"] \arrow[dr, "\mathtt{Id}_A \times \mu^n_{f,g,B}"] & \mathtt{T}^n_f(A \times \mathtt{T}^n_g B) \arrow[r, "\mathtt{T}^n_f \mathtt{t}_{g,A,B}"] & \mathtt{T}^n_f \mathtt{T}^n_g (A \times B) \arrow[d, "\mu^n_{f,g,A\times B}"] \\
& A \times \mathtt{T}^n_{f\cdot g} B \arrow[r, "\mathtt{t}_{f\cdot g,A,B}"] & \mathtt{T}^n_{f\cdot g}(A \times B)
\end{tikzcd}
$$

$$
\begin{aligned}
(\mathtt{t}^n_{(f\cdot g),A,B} \circ (\mathtt{Id}_A \times \mu^n_{f,g,B}))\vec{\epsilon} &= (\mathtt{t}^0_{((f\vec{\epsilon})\cdot(g\vec{\epsilon})),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mu^n_{(f\vec{\epsilon}),(g\vec{\epsilon}),(B\vec{\epsilon})})) && (54)\\
&= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\times B)\vec{\epsilon}} \circ \mathtt{T}^0_{f\vec{\epsilon}}(\mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})}) \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),\mathtt{T}^0_{g\vec{\epsilon}}(B\vec{\epsilon})} && (55)\\
&= (\mu^n_{f,g,(A\times B)} \circ \mathtt{T}^n_f(\mathtt{t}^n_{g,A,B}) \circ \mathtt{t}^n_{f,A,\mathtt{T}^n_g(B)})\vec{\epsilon} && (56)
\end{aligned}
$$

Commutativity with Unit

$$
\begin{tikzcd}
A \times B \arrow[r, "\mathtt{Id}_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"] & A \times T_1 B \arrow[d, "\mathtt{t}_{1,A,B}"] \\
& \mathtt{T}^n_1(A \times B)
\end{tikzcd}
$$

$$
\begin{aligned}
(\mathtt{t}^n_{1,A,B} \circ (\mathtt{Id}_A \times \eta^n_A))\vec{\epsilon} &= \mathtt{t}^0_{1,(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \eta^0_{A\vec{\epsilon}}) && (57)\\
&= \eta^0_{A\vec{\epsilon}\times B\vec{\epsilon}} && (58)\\
&= (\eta^n_{A\times B})\vec{\epsilon} && (59)
\end{aligned}
$$

Commutativity with α Let $\alpha_{A,B,C} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle : ((A \times B) \times C) \to (A \times (B \times C))$

$$
\begin{tikzcd}
(A \times B) \times \mathtt{T}^n_\epsilon C \arrow[rr, "\mathtt{t}_{\epsilon,(A\times B),C}"] \arrow[d, "\alpha_{A,B,\mathtt{T}^n_\epsilon C}"] & & \mathtt{T}^n_\epsilon((A \times B) \times C) \arrow[d, "\mathtt{T}^n_\epsilon \alpha_{A,B,C}"] \\
A \times (B \times \mathtt{T}^n_\epsilon C) \arrow[r, "\mathtt{Id}_A \times \mathtt{t}_{\epsilon,B,C}"] & A \times \mathtt{T}^n_\epsilon(B \times C) \arrow[r, "\mathtt{t}_{\epsilon,A,(B\times C)}"] & \mathtt{T}^n_\epsilon(A \times (B \times C))
\end{tikzcd}
$$

$$
\begin{aligned}
(\mathtt{T}^n_f \alpha_{A,B,C} \circ \mathtt{t}^n_{f,A\times B,C})\vec{\epsilon} &= \mathtt{T}^0_{f\vec{\epsilon}} \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\times B)\vec{\epsilon},(C\vec{\epsilon})} && (60)\\
&= \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon}\times C\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \mathtt{t}^0_{(f\vec{\epsilon}),(B\vec{\epsilon}),(C\vec{\epsilon})}) \circ \alpha_{A\vec{\epsilon},B\vec{\epsilon},C\vec{\epsilon}} && (61)\\
&= (\mathtt{t}^n_{f,A,(B\times C)} \circ (\mathtt{Id}_A \times \mathtt{t}^n_{f,B,C}) \circ \alpha_{A,B,C})\vec{\epsilon} && (62)\\
& && (63)
\end{aligned}
$$

0.3.5 Sub-Effecting

Given a collection of sub-effecting natural transformation in $\mathbb{C}$,

$$\llbracket \epsilon_1 \leq_0 \epsilon_2 \rrbracket_M : \quad \mathtt{T}^0_{\epsilon_1} \to \mathtt{T}^0_{\epsilon_2} \tag{64}$$

We can form sub-effect natural transformations in $[E^n, \mathbb{C}]$:

$$\begin{aligned}
\llbracket f \leq_n g \rrbracket_M : &\quad \mathtt{T}^n_f \to \mathtt{T}^n_g && (65)\\
\llbracket f \leq_n g \rrbracket_M A\vec{\epsilon} : &\quad \mathtt{T}^n_{f\vec{\epsilon}}(A\vec{\epsilon}) \to \mathtt{T}^n_{g\vec{\epsilon}}(B\vec{\epsilon}) && (66)\\
= &\quad \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M A\vec{\epsilon} && (67)
\end{aligned}$$

Naturality

$$\begin{tikzcd}
\mathtt{T}^0_{f\vec{\epsilon}} A\vec{\epsilon} \arrow[r, "\llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M A\vec{\epsilon}"] \arrow[d, "\mathtt{T}^0_{f\vec{\epsilon}} m\vec{\epsilon}"] & \mathtt{T}^0_{g\vec{\epsilon}} A\vec{\epsilon} \arrow[d, "\mathtt{T}^0_{g\vec{\epsilon}} m\vec{\epsilon}"] \\
\mathtt{T}^0_{f\vec{\epsilon}} B\vec{\epsilon} \arrow[r, "\llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_M B\vec{\epsilon}"] & \mathtt{T}^0_{g\vec{\epsilon}} B\vec{\epsilon}
\end{tikzcd}$$

Commutes With Tensor Strength

$$\begin{tikzcd}
A \times \mathtt{T}^n_f B \arrow[r, "\mathtt{Id}_A \times \llbracket f \leq_n g \rrbracket_B"] \arrow[d, "\mathtt{t}^n_{f,A,B}"] & A \times \mathtt{T}^n_g B \arrow[d, "\mathtt{t}^n_{g,A,B}"] \\
\mathtt{T}^n_f (A \times B) \arrow[r, "\llbracket f \leq_n g \rrbracket_{A \times B}"] & \mathtt{T}^n_g (A \times B)
\end{tikzcd}$$

$$\begin{aligned}
(\mathtt{t}^n_{g,A,B} \circ (\mathtt{Id}_A \times \llbracket f \leq_n g \rrbracket_B))\vec{\epsilon} &= \mathtt{t}^0_{(g\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} \circ (\mathtt{Id}_{A\vec{\epsilon}} \times \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_{B\vec{\epsilon}}) && (68)\\
&= \llbracket f\vec{\epsilon} \leq_0 g\vec{\epsilon} \rrbracket_{(A \times B)\vec{\epsilon}} \circ \mathtt{t}^0_{(f\vec{\epsilon}),(A\vec{\epsilon}),(B\vec{\epsilon})} && (69)\\
&= (\llbracket f \leq_n g \rrbracket_{(A \times B)} \circ \mathtt{t}^n_{f,A,B})\vec{\epsilon} && (70)\\
& && (71)
\end{aligned}$$

Commutes with Join

$$\begin{tikzcd}
\mathtt{T}^n_f \mathtt{T}^n_g \arrow[r, "\mathtt{T}^n_f \llbracket g \leq_n g' \rrbracket_M"] \arrow[d, "\mu^n_{f,g,}"] & \mathtt{T}^n_f \mathtt{T}^n_{g'} \arrow[r, "\llbracket f \leq_n f' \rrbracket_{M, \mathtt{T}^n_{g'}}"] & \mathtt{T}^n_{f'} \mathtt{T}^n_{g'} \arrow[d, "\mu^n_{f',g',}"] \\
\mathtt{T}^n_{f \cdot g} \arrow[rr, "\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_M"] & & \mathtt{T}^n_{f' \cdot g'}
\end{tikzcd}$$

$$\begin{aligned}
(\llbracket f \cdot g \leq_n f' \cdot g' \rrbracket_A \circ \mu^n_{f,g,A})\vec{\epsilon} &= \llbracket (f\vec{\epsilon}) \cdot (g\vec{\epsilon}) \leq_0 (f'\vec{\epsilon}) \cdot (g\vec{\epsilon}) \rrbracket_{A\vec{\epsilon}} \circ \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} && (72)\\
&= \mu^0_{(f\vec{\epsilon}),(g\vec{\epsilon}),(A\vec{\epsilon})} \circ \llbracket f\vec{\epsilon} \leq_0 f'\vec{\epsilon} \rrbracket_{\mathtt{T}^0_{g'\vec{\epsilon}}(A\vec{\epsilon})} \circ \mathtt{T}^0_{f\vec{\epsilon}} \llbracket g\vec{\epsilon} \leq_0 g'\vec{\epsilon} \rrbracket_{(A\vec{\epsilon})} && (73)\\
&= \mu^n_{f,g,A} \circ \llbracket f \leq_n f' \rrbracket_{\mathtt{T}^n_{g'} A} \circ \mathtt{T}^n_f \llbracket g \leq_n g' \rrbracket_A && (74)
\end{aligned}$$

0.3.6 Sub-Typing

Sub-typing in $[E^n, \mathbb{C}]$ holds via sub-typing in $\mathbb{C}$

$$\begin{aligned} [\![A \leq:_n B]\!]_M : &\quad A \to B && (75)\\ [\![A \leq:_n B]\!]_M \vec{\epsilon} = &\quad [\![A\vec{\epsilon} \leq:_0 B\vec{\epsilon}]\!]_M && (76)\end{aligned}$$

So the subtyping relation $A \leq: B$ forms a morphism in $[E^n, \mathbb{C}]$

0.4 Functors Between S-Categories

For a function $\theta : E^m \to E^n$, the re-indexing functor θ^* is defined as follows:

$$\begin{aligned} \theta^* : &\quad [E^n, \mathbb{C}] \to [E^m, \mathbb{C}] && (77)\\ \theta^*(A)\vec{\epsilon_m} = &\quad A(\theta(\vec{\epsilon_m})) && (78)\\ f : &\quad A \to B \in [E^n, \mathbb{C}] && (79)\\ \theta^*(f)\vec{\epsilon_m} = &\quad f(\theta(\vec{\epsilon_m})) : A(\theta(\vec{\epsilon_m}) \to B(\theta(\vec{\epsilon_m}))) && (80)\end{aligned}$$

0.4.1 θ^* is S-closed.

CCC

$$\begin{aligned} (\theta^*(A \times B))\vec{\epsilon} &= (A \times B)(\theta\vec{\epsilon}) && (81)\\ &= (A(\theta\vec{\epsilon}) \times B(\theta\vec{\epsilon})) && (82)\\ &= (\theta^* A \times \theta^* B)\vec{\epsilon} && (83)\\ & && (84)\end{aligned}$$

$$\begin{aligned} (\theta^*\pi_1)\vec{\epsilon} &= \pi_1(\theta\vec{\epsilon}) && (85)\\ &= \pi_1 \quad \text{Constant function} && (86)\\ &= \pi_1\vec{\epsilon} && (87)\end{aligned}$$

$$\begin{aligned} (\theta^*\pi_2)\vec{\epsilon} &= \pi_2(\theta\vec{\epsilon}) && (88)\\ &= \pi_2 \quad \text{Constant function} \qquad = \pi_2\vec{\epsilon} && (89)\end{aligned}$$

$$\begin{aligned} (\theta^* \langle f, g\rangle)\vec{\epsilon} &= (\langle f, g\rangle)(\theta\vec{\epsilon}) && (90)\\ &= \langle f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})\rangle && (91)\\ &= \langle \theta^* f, \theta^* g\rangle\, \vec{\epsilon} && (92)\end{aligned}$$

$$\begin{aligned} (\theta^*(A^B))\vec{\epsilon} &= (A^B)(\theta\vec{\epsilon}) && (93)\\ &= (A(\theta\vec{\epsilon}))^{(B(\theta\vec{\epsilon}))} && (94)\\ &= (\theta^* A)^{(\theta^* B)}\vec{\epsilon} && (95)\\ & && (96)\end{aligned}$$

$$
\begin{aligned}
(\theta^* \mathtt{app})\vec{\epsilon} &= \mathtt{app}(\theta\vec{\epsilon}) && (97)\\
&= \mathtt{app} \quad \text{Constant fn} && (98)\\
&= \mathtt{app}\vec{\epsilon} && (99)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* \mathtt{cur}(f))\vec{\epsilon} &= \mathtt{cur}(f)(\theta\vec{\epsilon}) && (100)\\
&= \mathtt{cur}(f(\theta\vec{\epsilon})) && (101)\\
&= \mathtt{cur}(\theta^* f) && (102)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* \mathtt{1})\vec{\epsilon} &= \mathtt{1}(\theta\vec{\epsilon}) && (103)\\
&= \mathtt{1} && (104)\\
&= \mathtt{1}\vec{\epsilon} && (105)\\
& && (106)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* \langle\rangle_A)\vec{\epsilon} &= \langle\rangle_A (\theta\vec{\epsilon}) && (107)\\
&= \langle\rangle_{A(\theta\vec{\epsilon})} && (108)\\
&= \langle\rangle_{\theta^* A}\, \vec{\epsilon} && (109)
\end{aligned}
$$

Co-Product

$$
\begin{aligned}
(\theta^* (\mathtt{1}+\mathtt{1}))\vec{\epsilon} &= (\mathtt{1}+\mathtt{1})(\theta\vec{\epsilon}) && (110)\\
&= (\mathtt{1}+\mathtt{1}) \quad \text{Constant function} && (111)\\
&= (\mathtt{1}+\mathtt{1})\vec{\epsilon} && (112)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* \mathtt{inl})\vec{\epsilon} &= \mathtt{inl}(\theta\vec{\epsilon}) && (113)\\
&= \mathtt{inl} \quad \text{Constant Fn} && (114)\\
&= \mathtt{inl}\vec{\epsilon} && (115)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* \mathtt{inr})\vec{\epsilon} &= \mathtt{inr}(\theta\vec{\epsilon}) && (116)\\
&= \mathtt{inr} \quad \text{Constant Fn} && (117)\\
&= \mathtt{inr}\vec{\epsilon} && (118)
\end{aligned}
$$

$$
\begin{aligned}
(\theta^* [f,g])\vec{\epsilon} &= [f,g](\theta\vec{\epsilon}) && (119)\\
&= [f(\theta\vec{\epsilon}), g(\theta\vec{\epsilon})] && (120)\\
&= [\theta^* f, \theta^* g]\vec{\epsilon} && (121)\\
& && (122)
\end{aligned}
$$

Strong Graded Monad

$$\begin{aligned}
(\theta^* \mathtt{T}^n_f A)\vec{\epsilon} &= \mathtt{T}^n_f A(\theta\vec{\epsilon}) && (123)\\
&= \mathtt{T}^0_{(f(\theta\vec{\epsilon}))}(A(\theta\vec{\epsilon})) && (124)\\
&= (\mathtt{T}^m_{(f\circ\theta)}\theta^* A)\vec{\epsilon} && (125)\\
&&& (126)
\end{aligned}$$

$$\begin{aligned}
(\theta^* \eta^n_A)\vec{\epsilon} &= \eta^n_A(\theta\vec{\epsilon}) && (127)\\
&= \eta^0_{A(\theta\vec{\epsilon})} && (128)\\
&= \eta^m_{\theta^* A}\vec{\epsilon} && (129)\\
&&& (130)
\end{aligned}$$

$$\begin{aligned}
(\theta^* \mathtt{t}^n_{f,A,B})\vec{\epsilon} &= \mathtt{t}^n_{f,A,B}(\theta\vec{\epsilon}) && (131)\\
&= \mathtt{t}^0_{(f(\theta\vec{\epsilon})),(A(\theta\vec{\epsilon})),(B(\theta\vec{\epsilon}))} && (132)\\
&= \mathtt{t}_{f\circ\theta,\theta^* A,\theta^* B}\vec{\epsilon} && (133)\\
&&& (134)
\end{aligned}$$

Sub-Effecting

$$\begin{aligned}
(\theta^*([\![f \leq_n g]\!]A))\vec{\epsilon} &= ([\![f \leq_n g]\!]A)(\theta\vec{\epsilon}) && (135)\\
&= ([\![f(\theta\vec{\epsilon}) \leq_n g(\theta\vec{\epsilon})]\!](A(\theta\vec{\epsilon}))) && (136)\\
&= ([\![\theta^* f \leq_m \theta^* g]\!](\theta^* A))\vec{\epsilon} && (137)\\
&&& (138)
\end{aligned}$$

Ground Sub-Typing

$$\begin{aligned}
\theta^*([\![A \leq:_\gamma B]\!]_M)\vec{\epsilon} &= [\![A \leq:_\gamma B]\!]_M(\theta\vec{\epsilon}) && (139)\\
&= [\![A \leq: B]\!]_M \quad \text{Constant Function} && (140)\\
&= [\![A \leq: B]\!]_M\vec{\epsilon} && (141)\\
&&& (142)
\end{aligned}$$

0.4.2 Quantification

We need to define $\forall_{E^n} : [E^{n+1}, \mathbb{C}] \to [E^n, \mathbb{C}]$

So

$$\begin{aligned}
(\forall_{E^n} A)\vec{\epsilon_n} =& \quad \Pi_{\epsilon\in E} A(\vec{\epsilon_n}, \epsilon) && (143)\\
m : & \quad A \to B && (144)\\
(\forall_{E^n} m) : & \quad \forall_{E^n} A \to \forall_{E^n} B && (145)\\
(\forall_{E^n} m)\vec{\epsilon_n} =& \Pi_{\epsilon\in E} m(\vec{\epsilon_n}, \epsilon) && (146)\\
&&& (147)
\end{aligned}$$

0.4.3 Adjunction

It is the case that:

$$\pi_1^* \dashv \forall_{E^n}$$

With unit:

$$\begin{aligned} \boldsymbol{\eta}_A : &\quad A \to \forall_{E^n} \pi_1^* A && (148)\\ \boldsymbol{\eta}_A(\vec{\epsilon_n}) = &\quad \left\langle \mathtt{Id}_{A(\vec{\epsilon_n},e)} \right\rangle_{e \in E} && (149) \end{aligned}$$

And co-unit

$$\begin{aligned} \boldsymbol{\epsilon}_B : &\quad \pi_1^* \forall_{E^n} B \to B && (150)\\ \boldsymbol{\epsilon}_B(\vec{\epsilon_n}, \epsilon) = &\quad \pi_\epsilon : \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) \to \Pi_{e \in E} B(\vec{\epsilon_n}, \epsilon) && (151) \end{aligned}$$

We then define the natural bi-jection as so:

$$\overline{(-)} : [E^{n_1}, \mathbb{C}](\pi_1^* A, B) \leftrightarrow [E^n, \mathbb{C}](A, \forall_{E^n} B) : \widehat{(-)} \quad (152)$$

$$\begin{aligned} m : &\quad \pi_1^* A \to B && (153)\\ \overline{m} : &\quad A \to \forall_{E^n} B && (154)\\ \overline{m}(\vec{\epsilon_n}) = &\quad \langle m(\vec{\epsilon_n}, \epsilon) \rangle_{e \in E} && (155) \end{aligned}$$

$$\begin{aligned} n : &\quad A \to \forall_{E^n} B && (156)\\ \widehat{n} : &\quad \pi_1^* A \to B && (157)\\ \widehat{n}(\vec{\epsilon_n}, \epsilon_{n+1}) = &\quad \pi_\epsilon \circ g(\vec{\epsilon_n}) && (158) \end{aligned}$$

This is an Adjunction

For any $g : \pi_1^* A \to B$,

$$\begin{aligned} (\boldsymbol{\epsilon}_B \circ \pi_1^*(\overline{g}))(\vec{\epsilon_n}, \epsilon_{n+1}) &= \pi_{\epsilon_{n+1}} \circ \langle g(\vec{\epsilon_n}, \epsilon') \rangle_{\epsilon' \in E} && (159)\\ &= g(\vec{\epsilon_n}, \epsilon_{n+1}) && (160) \end{aligned}$$

0.4.4 Beck-Chevalley Condition

For $\theta : E^m \to E^n$:

$$\begin{aligned} ((\theta^* \circ \forall_{E^n}) A)\vec{\epsilon_n} &= \theta^*(\forall_{E^n} A)\vec{\epsilon_n} && (161)\\ &= (\forall_{E^n} A)(\theta(\vec{\epsilon_n})) && (162)\\ &= \Pi_{\epsilon \in E}(A(\theta(\vec{\epsilon_n}), \epsilon)) && (163)\\ &= \Pi_{\epsilon \in E}(((\theta \times \mathtt{Id}_U)^* A)(\vec{\epsilon_n}, \epsilon)) && (164)\\ &= \forall_{E^m}((\theta \times \mathtt{Id}_E)^* A)\vec{\epsilon_n} && (165)\\ &= ((\forall_{E^m} \circ (\theta \times \mathtt{Id}_E)^*) A)\vec{\epsilon_n} && (166) \end{aligned}$$

And $\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}} = \mathtt{Id}_{\theta^* \circ \forall_I}$.

$$\begin{aligned}
\overline{(\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A} \vec{\epsilon} &= \langle (\theta \times \mathtt{Id}_U)^* \boldsymbol{\epsilon}_A(\vec{\epsilon}, \epsilon) \rangle_{\epsilon \in E} && (167)\\
&= \langle \boldsymbol{\epsilon}_A(\theta\vec{\epsilon}, \epsilon) \rangle_{e \in E} && (168)\\
&= \langle \pi_\epsilon \rangle_{\epsilon \in E} : \Pi_{\epsilon \in E} A(\theta\vec{\epsilon}, \epsilon) \to \Pi_{\epsilon \in E} A(\theta\vec{\epsilon}, \epsilon) && (169)\\
&= \mathtt{Id}_{\Pi_{\epsilon \in E} A(\theta\vec{\epsilon}, \epsilon)} && (170)\\
&= \mathtt{Id}_{\forall_{I'} \circ (\theta \times \mathtt{Id}_U)^* A} \vec{\epsilon} && (171)\\
&= \mathtt{Id}_{\theta^* \circ \forall_I} && (172)
\end{aligned}$$