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Chapter 1

Preliminaries

1.1 Base Category Requirements

There are 3 distinct objects in the base category, $\mathbb{C}$:

- U - The kind of `Effect`
- W - The kind of `Type`
- `1` - A terminal object

And we have finite products on U.

- $U^0 = \mathtt{1}$
- $U^{n+1} = U^n \times U$

We also have the following natural operations on morphisms in $\mathbb{C}$.
Let $I = U^n$.

- $\diamond : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates exponential types.
- $\Box : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - Generates products of types.
- $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I, W)$ - generates quantified types.
- $\mathtt{Eff} : \mathbb{C}(I, U) \times \mathbb{C}(I, W) \to \mathbb{C}(I, W)$ - generates monad types.
- $\mathtt{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \to \mathbb{C}(I, U)$ - Generates multiplication of effects.

With naturality conditions which mean, for $\theta : \mathtt{Unit}^m \to \mathtt{Unit}^n(I' \to I)$,

- $\diamond(\phi, \psi) \circ \theta = \diamond(\phi \circ \theta, \psi \circ \theta)$
- $\Box(\phi, \psi) \circ \theta = \Box(\phi \circ \theta, \psi \circ \theta)$
- $\forall_I(\phi) \circ \theta = \forall_{I'}(\phi \circ (\theta \times \mathtt{Id}_U))$
- $\mathtt{Eff}(\phi, \psi) \circ \theta = \mathtt{Eff}(\phi \circ \theta, \psi \circ \theta)$
- $\mathtt{Mul}(\phi, \psi) \circ \theta = \mathtt{Mul}(\phi \circ \theta, \psi \circ \theta)$

1.2 Well-Formed-ness

Each instance of the well-formed-ness relation on effects, $\Phi \vdash \epsilon$ has a denotation in $\mathbb{C}$:

$$[\![\Phi \vdash \epsilon\colon \mathtt{Effect}]\!]_M : I \to U \tag{1.1}$$

Each instance of the well-formed-ness relation on types, $\Phi \vdash A$ has a denotation in $\mathbb{C}$:

$$[\![P \vdash A\colon \mathtt{Type}]\!]_M : I \to W \tag{1.2}$$

It should also be the case that

$$\mathtt{Mul}([\![\Phi \vdash \epsilon_1\colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2\colon \mathtt{Effect}]\!]_M) = [\![\Phi \vdash \epsilon_1 \cdot \epsilon_2\colon \mathtt{Effect}]\!]_M \in \mathbb{C}(I, U) \tag{1.3}$$

That is, $\mathtt{Mul}$ should be have identity $[\![\Phi \vdash \mathtt{1}\colon \mathtt{Effect}]\!]_M$ and be associative.

1.3 Substitution and Weakening of the Effect Environment

For each instance of the well-formed-ness relation on substitution of effects $\Phi' \vdash \sigma\colon \Phi$, there exists a denotation in $\mathbb{C}$:

$$[\![\Phi' \vdash \sigma\colon \Phi]\!]_M : I' \to I \tag{1.4}$$

For each instance of the well-formed weakening relation on effect-environments, $\omega : \Phi' \triangleright \Phi$ there exists a denotation in $\mathbb{C}$:

$$[\![\omega : \Phi' \triangleright \Phi]\!]_M : I' \to I \tag{1.5}$$

.

1.4 Fibre Categories

Each set of morphisms $\mathbb{C}(I, W)$ forms the objects of a semantic-closed (S-closed) category. That is, a category satisfying all the properties needed for the non-polymorphic language:

- Cartesian Closed
- Co-product of the terminal object with itself ($\mathtt{1} + \mathtt{1}$)
- Ground morphisms for each ground constant ($\mathtt{C}^A : \mathtt{1} \to A$)
- Partial order morphisms on ground types ($[\![A \leq:_\gamma]\!]_M B$)
- A strong, monad, graded by the po-monoid $(E_\Phi, \cdot_\Phi, \leq_\Phi, \mathtt{1})$.

1.5 Re-indexing Functors

For each morphism $f : I' \to I$ in $\mathbb{C}$, there should be a co-variant, re-indexing functor $f^* : \mathbb{C}(I, W) \to \mathbb{C}(I', W)$, which is S-closed. That is, it preserves the S-closed properties of $\mathbb{C}(I, W)$. (See below).

$(-)^*$ should be a contra-variant functor in its $\mathbb{C}$ argument and co-variant in its right argument.

- $(g \circ f)^*(a) = f^*(\gamma^*(a))$
- $\mathtt{Id}_I^*(a) = a$
- $f^*(\mathtt{Id}_A) = \mathtt{Id}_{f^*(A)}$
- $f^*(a \circ b) = f^*(a) \circ f^*(b)$

1.5.1 f^* Preserves Products

If $\langle g, h\rangle : \mathbb{C}(I,W)(Z, X \times Y)$ Then

$$f^*(X \times Y) = f^*(X) \times f^*(Y) \tag{1.6}$$
$$f^*(\langle g, h\rangle) = \langle f^*(g), f^*h\rangle \qquad : \mathbb{C}(I',W)(f^*Z, f^*(X) \times f^*(Y)) \tag{1.7}$$
$$f^*(\pi_1) = \pi_1 \qquad : \mathbb{C}(I',W)(f^*(X) \times f^*(Y), f^*(X)) \tag{1.8}$$
$$f^*(\pi_2) = \pi_2 \qquad : \mathbb{C}(I',W)(f^*(X) \times f^*(Y), f^*(Y)) \tag{1.9}$$

1.5.2 f^* Preserves Terminal Object

If $\mathtt{Id}_A : \mathbb{C}(I,W)(A, \mathtt{1})$ Then

$$f^*(\mathtt{1}) = \mathtt{1} \tag{1.10}$$
$$f^*(\langle\rangle_A) = \langle\rangle_{f^*(A)} \qquad : \mathbb{C}(I',W)(f^*A, \mathtt{1}) \tag{1.11}$$
$$\tag{1.12}$$

1.5.3 f^* Preserves Exponentials

$$f^*(Z^X) = (f^*(Z))^{(f^*(X))} \tag{1.13}$$
$$f^*(\mathtt{app}) = \mathtt{app} \qquad : \mathbb{C}(I',W)(f^*(Z^X) \times f^*(X), f^*(Z)) \tag{1.14}$$
$$f^*(\mathtt{cur}(g)) = \mathtt{cur}(f^*(g)) \qquad : \mathbb{C}(I',W)(f^*(Y) \times f^*(X), f^*(Z)^{f^*(X)}) \tag{1.15}$$

1.5.4 f^* Preserves Co-product on Terminal

$$f^*(\mathtt{1}+\mathtt{1}) = \mathtt{1}+\mathtt{1} \tag{1.16}$$
$$f^*(\mathtt{inl}) = \mathtt{inl} \qquad : \mathbb{C}(I',W)(\mathtt{1}, \mathtt{1}+\mathtt{1}) \tag{1.17}$$
$$f^*(\mathtt{inr}) = \mathtt{inr} \qquad : \mathbb{C}(I',W)(\mathtt{1}, \mathtt{1}+\mathtt{1}) \tag{1.18}$$
$$f^*([g,h]) = [f^*(g), f^*(h)] \qquad : \mathbb{C}(I',W)(\mathtt{1}+\mathtt{1}, f^*(Z)) \tag{1.19}$$

1.5.5 f^* Preserves Graded Monad

$$f^*(T_\epsilon A) = T_{f^*(\epsilon)} f^*(A) \qquad : \mathbb{C}(I',W) \tag{1.20}$$
$$f^*(\mathtt{1}) = \mathtt{1} \quad \text{The unit effect} \tag{1.21}$$
$$f^*(\eta_A) = \eta_{f^*(A)} \qquad : \mathbb{C}(I',W)(f^*(A), f^*(T_{\mathtt{1}}A)) \tag{1.22}$$
$$f^*(\mu_{\epsilon_1,\epsilon_2,A}) = \mu_{f^*(\epsilon_1),f^*(\epsilon_2),f^*(A)} \qquad : \mathbb{C}(I',W)(f^*(T_{\epsilon_1}T_{\epsilon_2}A), f^*(T_{f^*(\epsilon_1)\cdot f^*(\epsilon_2)} f^*(A))) \tag{1.23}$$
$$f^*(\epsilon_1 \cdot \epsilon_2) = f^*(\epsilon_1) \cdot f^*(\epsilon_2) \tag{1.24}$$
$$\tag{1.25}$$

1.5.6 f^* Preserves Tensor Strength

$$f^*(\mathtt{t}_{\epsilon,A,B}) = \mathtt{t}_{f^*(\epsilon),f^*(A),f^*(B)} \qquad : \mathbb{C}(I', W)(f^*(A \times T_\epsilon B), f^*(T_\epsilon(A \times B))) \tag{1.26}$$

1.5.7 f^* Preserves Ground Constants

For each ground constant $[\![\mathtt{C}^A]\!]_M$ in $\mathbb{C}(I, W)$,

$$f^*([\![\mathtt{C}^A]\!]_M) = \mathtt{C}^{f^*(A)} : \mathbb{C}(I', W)(\mathtt{1}, f^*(A)) \tag{1.27}$$

1.5.8 f^* Preserves Ground Sub-effecting

For ground effects e_1, e_2 such that $e_1 \leq e_2$

$$f^*(e) = e : \mathbb{C}(I', U) \tag{1.28}$$
$$f^*[\![\epsilon_1 \leq e_2]\!]_A = [\![e_1 \leq e_2]\!]_{f^*(A)} : \mathbb{C}(I', W)f^*(T_{e_1}A), f^*(T_{e_2}A) \tag{1.29}$$
$$\tag{1.30}$$

1.5.9 f^* Preserves Ground Sub-typing

For ground types γ_1, γ_2 such that $\gamma_1 \leq:_\gamma \gamma_2$

$$f^*\gamma = \gamma : \mathbb{C}(I', W)(\mathtt{1}, \gamma) \tag{1.31}$$
$$f^*([\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M) = [\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M \qquad : \mathbb{C}(I', W)(\gamma_1, \gamma_2) \tag{1.32}$$
$$\tag{1.33}$$

1.5.10 Action on Objects

It follows that the action of f^* on an object A in $\mathbb{C}(I, W)$ (i.e. a morphism $I \to U$ in $\mathbb{C}$) is:

$$f^*(A) = A \circ f : I' \to I \to W \tag{1.34}$$

1.6 Naturality Properties

1.7 The $\forall_I$ functor

We expand $\forall_I : \mathbb{C}(I \times U, W) \to \mathbb{C}(I, W)$ to be a functor which is right adjoint to the re-indexing functor π_1^*.

$$\overline{(\_)} : \mathbb{C}(I \times U, W)(\pi_1^* A, B) \leftrightarrow \mathbb{C}(I, W)(A, \forall_I B) : \widehat{(\_)} \tag{1.35}$$

For $A : \mathbb{C}(I, W)$, $B : \mathbb{C}(I \times U, W)$.

Hence the action of $\forall_I$ on a morphism $l : A \to A'$ is as follows:

$$\forall_I(l) = \overline{l \circ \epsilon_A} \tag{1.36}$$

Where $\epsilon_A : \mathbb{C}(I \times U, W)(\pi_1^* \forall_I A \to A)$ is the co-unit of the adjunction.

1.8 Naturality Corollaries

Here are some simple corollaries of the adjunction between π_1^* and $\forall_I$.

1.8.1 Naturality

By the definition of an adjunction:

$$\overline{f \circ \pi_1^*(n)} = \overline{f} \circ n \tag{1.37}$$

1.8.2 $\overline{(-)}$ and Re-indexing Functors

TODO: Why does this occur? it comes from page 222 of Crole?

$$\begin{aligned}
\theta^*(\overline{f}) &= (\pi_1 \circ (\theta \times \mathtt{Id}_U))^*(\overline{f}) && (1.38)\\
&= (\theta \times \mathtt{Id}_U)^*(\pi_1^*(\overline{f})) && (1.39)\\
& && (1.40)\\
& && (1.41)\\
&= \overline{(\theta \times \mathtt{Id}_U)^* f} && (1.42)\\
& && (1.43)\\
& && (1.44)
\end{aligned}$$

1.8.3 $(\hat{-})$ and Re-Indexing Functors

$$\begin{aligned}
\theta^*(\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m})) &= (\langle \mathtt{Id}_I, \rho\rangle \circ \theta)^*(\widehat{m}) && (1.45)\\
&= ((\theta \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_I, \rho\rangle)^*(\widehat{m}) && (1.46)\\
&= \langle \mathtt{Id}_I, \rho \circ \theta\rangle^* (\theta \times \mathtt{Id}_U)^*(\widehat{m}) && (1.47)\\
&= \langle \mathtt{Id}_I, \theta^* \rho\rangle^* (\theta^*(\widehat{m})) && (1.48)
\end{aligned}$$

1.8.4 Pushing Morphisms into f^*

$$\begin{aligned}
\langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ n &= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m}) \circ \langle \mathtt{Id}_I, \rho\rangle^* \pi_1^*(n) && (1.49)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m} \circ \pi_1^*(n)) && (1.50)\\
&= \langle \mathtt{Id}_I, \rho\rangle^* (\widehat{m \circ n}) && (1.51)
\end{aligned}$$

Chapter 2

Denotations

2.1 Effects

For each instance of the well-formed-ness relation on effects, we define a morphism $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M : \mathbb{C}(I, U)$

- $[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M = [\![\epsilon]\!]_M \circ \langle\rangle_I :\to U$
- $[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M = \pi_2 : I \times U \to U$
- $[\![\Phi, \beta \vdash \alpha \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \pi_1 : I \times U \to U$
- $[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \texttt{Effect}]\!]_M = \texttt{Mul}([\![\Phi \vdash \epsilon_2 \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_1 \colon \texttt{Effect}]\!]_M) : I \to U$

2.2 Types

For each instance of the well-formed-ness relation on types, we define a morphism $[\![\Phi \vdash A \colon \texttt{Type}]\!]_M : \mathbb{C}(I, W)$.

$[\![\texttt{Unit}]\!]_M$ is the morphism generating the terminal object of $\mathbb{C}(I, W)$. `Bool` is the morphism generating the co-product of this terminal object, $\mathtt{1} + \mathtt{1}$.

- $[\![\Phi \vdash \texttt{Unit} \colon \texttt{Type}]\!]_M = [\![\texttt{Unit}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \texttt{Bool} \colon \texttt{Type}]\!]_M = [\![\texttt{Bool}]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash \gamma \colon \texttt{Type}]\!]_M = [\![\gamma]\!]_M \circ \langle\rangle_I : I \to W$
- $[\![\Phi \vdash A \to B \colon \texttt{Type}]\!]_M = \diamond([\![\Phi \vdash A \colon \texttt{Type}]\!]_M, [\![\Phi \vdash B \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \texttt{M}_\epsilon A \colon \texttt{Type}]\!]_M = \texttt{Eff}([\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M, [\![\Phi \vdash A \colon \texttt{Type}]\!]_M) : I \to W$
- $[\![\Phi \vdash \forall \alpha.A \colon \texttt{Type}]\!]_M = \forall_I([\![\Phi, \alpha \vdash A \colon \texttt{Type}]\!]_M) : I \to W$

2.3 Effect Substitution

For each effect-substitution well-formed-ness-relation, define a denotation morphism, $[\![\Phi' \vdash \sigma \colon \Phi]\!]_M : \mathbb{C}(I', I)$

- $[\![\Phi' \vdash \diamond \colon \diamond]\!]_M = \langle\rangle_I : \mathbb{C}(I', \mathtt{1})$
- $[\![\Phi' \vdash (\sigma, \alpha := \epsilon) \colon \Phi, \alpha]\!]_M = \langle [\![\Phi' \vdash \sigma \colon \Phi]\!]_M, [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \rangle : \mathbb{C}(I', I \times U)$

2.4 Effect Weakening

For each instance of the effect-environment weakening relation, define a denotation morphism: $[\![\omega : \Phi' \triangleright P]\!]_M : \mathbb{C}(I', I)$

- $[\![\iota : \Phi \triangleright \Phi]\!]_M = \mathtt{Id}_I : I \to I$
- $[\![w\pi : \Phi', \alpha \triangleright \Phi]\!]_M = [\![\omega : \Phi' \triangleright \Phi]\!]_M \circ \pi_1 : I' \times U \to I$
- $[\![w\times : \Phi', \alpha \triangleright \Phi, \alpha]\!]_M = ([\![\omega : \Phi' \triangleright \Phi]\!]_M \times \mathtt{Id}_U) : I' \times U \to I \times U$

2.5 Sub-Typing

For each instance of the sub-typing relation with respect to an effect environment, there exists a denotation, $[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I, W)(A, B)$.

- $[\![\gamma_1 \leq:_\Phi \gamma_2]\!]_M = [\![\gamma_1 \leq:_\gamma \gamma_2]\!]_M : \mathbb{C}(I, W)(\gamma_1, \gamma_2)$
- $[\![A \to B \leq:_\Phi A' \to B']\!]_M = [\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}$
- $[\![\mathtt{M}_{\epsilon_1} A \leq:_\Phi \mathtt{M}_{\epsilon_2} B]\!]_M = [\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M \circ T_{\epsilon_1}[\![A \leq:_\Phi B]\!]_M$
- $[\![\forall \alpha.A \leq:_\Phi \forall \alpha.B]\!]_M = \forall_I [\![A \leq:_{\Phi,\alpha} B]\!]_M$

2.6 Type-Environments

For each instance of the well-formed relation on type environments, define an object in $[\![I \vdash W\mathtt{Ok}]\!]_M \in \mathbb{C}(I, W)$.

- $[\![\Phi \vdash \diamond\mathtt{Ok}]\!]_M = \mathtt{1} : \mathbb{C}(I, W)$
- $[\![\Phi \vdash \Gamma, x : A\mathtt{Ok}]\!]_M = \Box([\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M)$

2.7 Terms

For each instance of the typing relation, define a denotation morphism: $[\![\Phi \mid \Gamma \vdash v \colon A]\!]_M : \mathbb{C}(I, W)(\Gamma_I, A_I)$. Writing Γ_I and A_I for $[\![\Phi \vdash \Gamma\mathtt{Ok}]\!]_M$ and $[\![\Phi \vdash A \colon \mathtt{Type}]\!]_M$.

For each ground constant, $\mathtt{C}^A$, there exists $c : \mathtt{1} \to A_I$ in $\mathbb{C}(I, W)$.

- (Unit) $\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{()} \colon \mathtt{Unit}]\!]_M = \langle\rangle_\Gamma : \Gamma_I \to \mathtt{1}}$
- (Const) $\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{C}^A \colon A]\!]_M = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_\Gamma : \Gamma \to [\![A]\!]_M}$
- (True) $\dfrac{\Phi \vdash \Gamma\mathtt{Ok}}{[\![\Phi \mid \Gamma \vdash \mathtt{true} \colon \mathtt{Bool}]\!]_M = \mathtt{inl} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = \mathtt{1} + \mathtt{1}}$

- (False) $\dfrac{\Phi \vdash \Gamma\,\mathtt{Ok}}{[\![\Phi|\Gamma \vdash \mathtt{false}{:}\mathtt{Bool}]\!]_M = \mathtt{inr} \circ \langle\rangle_\Gamma : \Gamma \to [\![\mathtt{Bool}]\!]_M = 1+1}$

- (Var) $\dfrac{\Phi \vdash \Gamma\,\mathtt{Ok}}{[\![\Phi|\Gamma, x{:}A \vdash x{:}A]\!]_M = \pi_2 : \Gamma \times A \to A}$

- (Weaken) $\dfrac{f = [\![\Phi|\Gamma \vdash x{:}A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma, y{:}B \vdash x{:}A]\!]_M = f \circ \pi_1 : \Gamma \times B \to A}$

- (Lambda) $\dfrac{f = [\![\Phi|\Gamma, x{:}A \vdash C{:}\mathtt{M}_\epsilon B]\!]_M : \Gamma \times A \to T_\epsilon B}{[\![\Phi|\Gamma \vdash \lambda x{:}A.C{:}A \to \mathtt{M}_\epsilon B]\!]_M = \mathtt{cur}(f) : \Gamma \to (T_\epsilon B)^A}$

- (Subtype) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}A]\!]_M : \Gamma \to A \quad g = [\![A \leq{:} B]\!]_M}{[\![\Phi|\Gamma \vdash v{:}B]\!]_M = g \circ f : \Gamma \to B}$

- (Return) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{return}\, v{:}\mathtt{M}_{\mathtt{1}} A]\!]_M = \eta_A \circ f}$

- (If) $\dfrac{f = [\![\Phi|\Gamma \vdash v{:}\mathtt{Bool}]\!]_M : \Gamma \to 1+1 \quad g = [\![\Phi|\Gamma \vdash C_1{:}\mathtt{M}_\epsilon A]\!]_M \quad h = [\![\Phi|\Gamma \vdash C_2{:}\mathtt{M}_\epsilon A]\!]_M}{[\![\Phi|\Gamma \vdash \mathtt{if}_{\epsilon,A}\ v\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2{:}\mathtt{M}_\epsilon A]\!]_M = \mathtt{app} \circ (([\mathtt{cur}(g \circ \pi_2), \mathtt{cur}(h \circ \pi_2)] \circ f) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma : \Gamma \to T_\epsilon A}$

- (Bind) $\dfrac{f = [\![\Phi|\Gamma \vdash C_1{:}\mathtt{M}_{\epsilon_1} A{:}\Gamma \to T_{\epsilon_1} A \quad g = [\![\Phi|\Gamma, x{:}A \vdash C_2{:}\mathtt{M}_{\epsilon_2} B]\!]_M]\!]_M : \Gamma \times A \to T_{\epsilon_2} B}{[\![\Phi|\Gamma \vdash \mathtt{do}\ x \leftarrow C_1\ \mathtt{in}\ C_2{:}\mathtt{M}_{\epsilon_1 \cdot \epsilon_2}]\!]_M = \mu_{\epsilon_1,\epsilon_2,B} \circ T_{\epsilon_1} g \circ \mathtt{t}_{\Gamma,A,\epsilon_1} \circ \left\langle \mathtt{Id}_\Gamma, f \right\rangle : \Gamma \to T_{\epsilon_1 \cdot \epsilon_2} B}$

- (Apply) $\dfrac{f = [\![\Phi|\Gamma \vdash v_1{:}A \to \mathtt{M}_\epsilon B]\!]_M : \Gamma \to (T_\epsilon B)^A \quad g = [\![\Phi|\Gamma \vdash v_2{:}A]\!]_M : \Gamma \to A}{[\![\Phi|\Gamma \vdash v_1\ v_2{:}\mathtt{M}_\epsilon B]\!]_M = \mathtt{app} \circ \langle f, g \rangle : \Gamma \to T_\epsilon B}$

- (Effect-Lambda) $\dfrac{f = [\![\Phi, \alpha|\Gamma \vdash v{:}A]\!]_M : \mathbb{C}(I \times U, W)(\Gamma, A)}{[\![\Phi|\Gamma \vdash \Lambda\alpha.A{:}\forall\epsilon.A]\!]_M = \overline{f} : \mathbb{C}(I, W)(\Gamma, \forall_I(A))}$

- (Effect-App) $\dfrac{g = [\![\Phi|\Gamma \vdash v{:}\forall\alpha.A]\!]_M : \mathbb{C}(I, W)(\Gamma, \forall_I(A)) \quad h = [\![\Phi \vdash \epsilon{:}\mathtt{Effect}]\!]_M : \mathbb{C}(I, U)}{[\![\Phi|\Gamma \vdash v\ \epsilon{:}A[\epsilon/\alpha]]\!]_M = \left\langle \mathtt{Id}_I, h \right\rangle^* (\epsilon_{[\![\Phi,\beta \vdash A[\beta/\alpha]{:}\mathtt{Type}]\!]_M}) \circ g : \mathbb{C}(I, W)(\Gamma, A[\epsilon/\alpha])}$

Chapter 3

Effect Substitution Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-variable substitution upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism Δ of some relation, the denotation of the substituted relation, $\Delta' = \sigma^*(\Delta)$.

3.1 Effects

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash \sigma(\epsilon) \colon \texttt{Effect}]\!]_M = \sigma^*[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M = [\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash \epsilon \colon \texttt{Effect}]\!]_M$

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash e \colon \texttt{Effect}]\!]_M \circ \sigma &= [\![e]\!]_M \circ \langle\rangle_I \circ \sigma && (3.1)\\
&= [\![e]\!]_M \circ \langle\rangle_{I'} && (3.2)\\
&= [\![\Phi' \vdash e \colon \texttt{Type}]\!]_M && (3.3)\\
& && (3.4)
\end{aligned}$$

Case Var:

$$\begin{aligned}
[\![\Phi, \alpha \vdash \alpha \colon \texttt{Effect}]\!]_M \circ \sigma' &= \pi_2 \circ \langle \sigma, [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M \rangle \quad \text{By inversion } \sigma' = (\sigma, \alpha := \epsilon) && (3.5)\\
&= [\![\Phi' \vdash \epsilon \colon \texttt{Effect}]\!]_M && (3.6)\\
&= [\![\Phi' \vdash \sigma'(\alpha) \colon \texttt{Effect}]\!]_M && (3.7)\\
& && (3.8)
\end{aligned}$$

Case Weaken:

$$\begin{aligned}
[\![\Phi, \beta \vdash \alpha \colon \mathtt{Type}]\!]_M \circ \sigma' &= [\![\Phi \vdash \alpha \colon \mathtt{Type}]\!]_M \circ \pi_1 \circ \langle \sigma, [\![\Phi' \vdash \epsilon \colon \mathtt{Effect}]\!]_M \rangle \quad \text{By inversion, } \sigma' = (\sigma, \beta := \epsilon) && (3.9)\\
&= [\![\Phi \vdash \alpha \colon \mathtt{Type}]\!]_M \circ \sigma && (3.10)\\
&= [\![\Phi' \vdash \sigma(\alpha) \colon \mathtt{Type}]\!]_M && (3.11)\\
&= [\![\Phi' \vdash \sigma'(\alpha) \colon \mathtt{Type}]\!]_M && (3.12)\\
& && (3.13)
\end{aligned}$$

Case Multiply:

$$\begin{aligned}
[\![\Phi \vdash \epsilon_1 \cdot \epsilon_2 \colon \mathtt{Type}]\!]_M \circ \sigma &= \mathtt{Mul}([\![\Phi \vdash \epsilon_1 \colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \epsilon_2 \colon \mathtt{Effect}]\!]_M) \circ \sigma && (3.14)\\
&= \mathtt{Mul}([\![\Phi \vdash \epsilon_1 \colon \mathtt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash \epsilon_2 \colon \mathtt{Effect}]\!]_M \circ \sigma) \quad \text{By Naturality} && (3.15)\\
&= \mathtt{Mul}([\![\Phi' \vdash \sigma(\epsilon_1) \colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash \sigma(\epsilon_2) \colon \mathtt{Effect}]\!]_M) && (3.16)\\
& && (3.17)
\end{aligned}$$

3.2 Types

If $\sigma = [\![\Phi' \vdash \sigma \colon \Phi]\!]_M$ then $[\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M = \sigma^* [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M = [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M \circ \sigma$.

Proof: By induction on the derivation on $[\![\Phi \vdash A \colon \mathtt{Type}]\!]_M$. Making use of naturality properties of the type constructors.

Case Ground:

$$\begin{aligned}
[\![\Phi \vdash \gamma \colon \mathtt{Type}]\!]_M \circ \sigma &= [\![\gamma]\!]_M \circ \langle\rangle_I \circ \sigma && (3.18)\\
&= [\![\gamma]\!]_M \circ \langle\rangle_{I'} && (3.19)\\
&= [\![\Phi' \vdash \gamma \colon \mathtt{Type}]\!]_M && (3.20)\\
&= [\![\Phi' \vdash \gamma\,[\sigma] : \mathtt{Type}]\!]_M && (3.21)
\end{aligned}$$

Case Monad:

$$\begin{aligned}
[\![\Phi \vdash \mathtt{M}_\epsilon A \colon \mathtt{Type}]\!]_M \circ \sigma &= \mathtt{Eff}([\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M) \circ \sigma && (3.22)\\
&= \mathtt{Eff}([\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \sigma, [\![\Phi \vdash A \colon \mathtt{Type}]\!]_M \circ \sigma) \quad \text{By naturality} && (3.23)\\
&= \mathtt{Eff}([\![\Phi' \vdash \sigma(\epsilon) \colon \mathtt{Effect}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \mathtt{Type}]\!]_M) && (3.24)\\
&= [\![\Phi' \vdash \mathtt{M}_{\sigma(\epsilon)} A\,[\sigma] \colon \mathtt{Type}]\!]_M && (3.25)\\
&= [\![\Phi' \vdash (\mathtt{M}_\epsilon A)\,[\sigma] : \mathtt{Type}]\!]_M && (3.26)
\end{aligned}$$

Case Quantification:

$$\begin{aligned}
[\![\Phi \vdash \forall\alpha.A\colon \texttt{Type}]\!]_M \circ \sigma &= \forall_I([\![\Phi, \alpha \vdash A\colon \texttt{Type}]\!]_M) \circ \sigma && (3.27)\\
&= \forall_I([\![\Phi, \alpha \vdash A\colon \texttt{Type}]\!]_M \circ (\sigma \times \texttt{Id}_U)) && (3.28)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma, \alpha := \epsilon] : \texttt{Type}]\!]_M) && (3.29)\\
&= \forall_I([\![\Phi', \alpha \vdash A\,[\sigma] : \texttt{Type}]\!]_M) && (3.30)\\
&= [\![\Phi' \vdash \forall\alpha.A\,[\sigma] : \texttt{Type}]\!]_M && (3.31)\\
&= [\![\Phi' \vdash (\forall\alpha.A)\,[\sigma] : \texttt{Type}]\!]_M && (3.32)\\
& && (3.33)
\end{aligned}$$

Case Function:

$$\begin{aligned}
[\![\Phi \vdash A \to B\colon \texttt{Type}]\!]_M \circ \sigma &= \diamond([\![\Phi \vdash A\colon \texttt{Type}]\!]_M, [\![\Phi \vdash B\colon \texttt{Type}]\!]_M) \circ \sigma && (3.34)\\
&= \diamond([\![\Phi \vdash A\colon \texttt{Type}]\!]_M \circ \sigma, [\![\Phi \vdash B\colon \texttt{Type}]\!]_M \circ \sigma) \quad \text{By Naturality} && (3.35)\\
&= \diamond([\![\Phi' \vdash A\,[\sigma] : \texttt{Type}]\!]_M, [\![\Phi' \vdash B\,[\sigma] : \texttt{Type}]\!]_M) && (3.36)\\
&= [\![\Phi' \vdash (A\,[\sigma]) \to (B\,[\sigma])\colon \texttt{Type}]\!]_M && (3.37)\\
&= [\![\Phi' \vdash (A \to B)\,[\sigma] : \texttt{Type}]\!]_M && (3.38)\\
& && (3.39)
\end{aligned}$$

3.3 Sub-typing

If $\sigma = [\![\Phi' \vdash \sigma\colon \Phi]\!]_M$ then $[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M = \sigma^*[\![A \leq:_\Phi B]\!]_M : \mathbb{C}(I', W)(A, B)$.

Proof: By induction on the derivation on $[\![A \leq:_\Phi B]\!]_M$. Using S-closure of σ^*

Case Ground:

$$\sigma^*(\gamma_1 \leq:_\gamma \gamma_2) = (\gamma_1 \leq:_\gamma \gamma_2) \tag{3.40}$$

Since σ^* is s-closed.

Case Monad:

$$\begin{aligned}
\sigma^*[\![\texttt{M}_{\epsilon_1} A \leq:_\Phi \texttt{M}_{\epsilon_2} B]\!]_M &= \sigma^*([\![\epsilon_1 \leq_\Phi \epsilon_2]\!]_M) \circ \sigma^*(T_{\epsilon_1}([\![A \leq:_\Phi B]\!]_M)) && (3.41)\\
&= [\![\sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2)]\!]_M \circ T_{\sigma(\epsilon_1)}[\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M \quad \text{By S-Closure} && (3.42)\\
&= [\![\texttt{M}_{\sigma(\epsilon_1)} A\,[\sigma] \leq:_{\Phi'} \texttt{M}_{\sigma(\epsilon_2)} B\,[\sigma]]\!]_M && (3.43)\\
&= [\![(\texttt{M}_{\epsilon_1} A)\,[\sigma] \leq:_{\Phi'} \texttt{M}_{\epsilon_2} B\,[\sigma]\,]\!]_M && (3.44)\\
& && (3.45)
\end{aligned}$$

Case For All:

$$\begin{aligned}
\sigma^*[\![\forall\alpha.A \leq:_\Phi \forall\alpha.B]\!]_M &= \sigma^*(\forall_I([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.46)\\
&= \forall_{I'}((\sigma \times \texttt{Id}_U)^*([\![A \leq:_{\Phi,\alpha} B]\!]_M)) && (3.47)\\
&= \forall_{I'}([\![A\,[\sigma, \alpha := \alpha] \leq:_{\Phi',\alpha} B\,[\sigma, \alpha := \alpha]\,]\!]_M) && (3.48)\\
&= [\![(\forall\alpha.A)\,[\sigma] \leq:_{\Phi'} (\forall\alpha.B)\,[\sigma]\,]\!]_M && (3.49)\\
& && (3.50)
\end{aligned}$$

Case Fn:

$$\begin{aligned}
\sigma^*[\![(A \to B) \leq:_\Phi A' \to B']\!]_M &= \sigma^*([\![B \leq:_\Phi B']\!]_M^{A'} \circ B^{[\![A' \leq:_\Phi A]\!]_M}) && (3.51)\\
&= \sigma^*(\texttt{cur}([\![B \leq:_\Phi B']\!]_M \circ \texttt{app})) \circ \sigma^*(\texttt{cur}(\texttt{app} \circ (\texttt{Id}_B \times [\![A' \leq:_\Phi A]\!]_M))) && (3.52)\\
&= \texttt{cur}(\sigma^*([\![B \leq:_\Phi B']\!]_M) \circ \texttt{app}) \circ \texttt{cur}(\texttt{app} \circ (\texttt{Id}_B \times \sigma^*([\![A' \leq:_\Phi A]\!]_M))) && (3.53)\\
&= \texttt{cur}([\![B\,[\sigma] \leq:_{\Phi'} B'\,[\sigma]\,]\!]_M \circ \texttt{app}) \circ \texttt{cur}(\texttt{app} \circ (\texttt{Id}_{B[\sigma]} \times [\![A'\,[\sigma] \leq:_{\Phi'} A\,[\sigma]\,]\!]_M)) && (3.54)\\
&= [\![(A\,[\sigma]) \to (B\,[\sigma]) \leq:_{\Phi'} (A'\,[\sigma]) \to (B'\,[\sigma])]\!]_M && (3.55)\\
&= [\![(A \to B)\,[\sigma] \leq:_{\Phi'} (A' \to B')\,[\sigma]\,]\!]_M && (3.56)
\end{aligned}$$

3.4 Type Environments

If $\sigma = [\![\Phi' \vdash \sigma\colon \Phi]\!]_M$ then $[\![\Phi' \vdash \Gamma\,[\sigma]\,\texttt{Ok}]\!]_M = \sigma^*[\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M = [\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M : \mathbb{C}(I', W)$.

Proof: By induction on the derivation on $[\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M$. Using Naturality.

Case Nil:

$$\begin{aligned}
\sigma^*[\![\Phi \vdash \diamond\texttt{Ok}]\!]_M &= \langle\rangle_I \circ \sigma && (3.57)\\
&= \langle\rangle_{I'} && (3.58)\\
&= [\![\Phi' \vdash \diamond\texttt{Ok}]\!]_M && (3.59)\\
[\![\Phi' \vdash \diamond\,[\sigma]\,\texttt{Ok}]\!]_M & && (3.60)\\
& && (3.61)
\end{aligned}$$

Case Var:

$$\begin{aligned}
\sigma^*[\![\Phi \vdash \Gamma, x : A\texttt{Ok}]\!]_M &= \sigma^*(\Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M)) && (3.62)\\
&= \Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M) \circ \sigma && (3.63)\\
&= \Box([\![\Phi \vdash \Gamma\texttt{Ok}]\!]_M \circ \sigma, [\![\Phi \vdash A\colon \texttt{Type}]\!]_M \circ \sigma) && (3.64)\\
&= \Box([\![\Phi' \vdash \Gamma\,[\sigma]\,\texttt{Ok}]\!]_M, [\![\Phi' \vdash A\,[\sigma] : \texttt{Type}]\!]_M) && (3.65)\\
&= [\![\Phi' \vdash \Gamma\,[\sigma]\,, x : A\,[\sigma]\,\texttt{Ok}]\!]_M && (3.66)\\
&= [\![\Phi' \vdash (\Gamma, x : A)\,[\sigma]\,\texttt{Ok}]\!]_M && (3.67)\\
& && (3.68)
\end{aligned}$$

3.5 Terms

If

$$\begin{aligned}
\sigma &= [\![\Phi' \vdash \sigma\colon \Phi]\!]_M && (3.69)\\
\Delta &= [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M && (3.70)\\
\Delta' &= [\![\Phi' \mid \Gamma\,[\sigma] \vdash v\,[\sigma] : A\,[\sigma]\,]\!]_M && (3.71)\\
& && (3.72)
\end{aligned}$$

Then

$$\Delta' = \sigma^*(\Delta) \tag{3.73}$$

Proof: By induction over the derivation of Δ. Using the S-Closure of σ^*. We use Γ_I to indicate $[\![\Phi \vdash \Gamma \mathtt{Ok}]\!]_M$, an A_I to indicate $[\![\Phi \vdash A\colon \mathtt{Effect}]\!]_M$

Case Unit:

$$\Delta = \langle\rangle_{\Gamma_I} \tag{3.74}$$

So

$$\sigma^*(\Delta) = \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.75}$$

Case True, False: Giving the case for true as false is the same but using `inr`

$$\Delta = \mathtt{inl} \circ \langle\rangle_{\Gamma_I} \tag{3.76}$$

So

$$\sigma^*(\Delta) = \mathtt{inl} \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.77}$$

Since σ^* is S-closed.

Case Constant:

$$\Delta = [\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I} \tag{3.78}$$

So

$$\sigma^*(\Delta) = \sigma^*[\![\mathtt{C}^A]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = [\![\mathtt{C}^{A[\sigma]}]\!]_M \circ \langle\rangle_{\Gamma_I[\sigma]} = \Delta' \tag{3.79}$$

Since σ^* is S-closed.

Case Subtype: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon A]\!]_M \tag{3.80}$$

Then

$$\Delta = [\![A \leq:_\Phi B]\!]_M \circ \Delta_1 \tag{3.81}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*[\![A \leq:_\Phi B]\!]_M \circ \sigma^*\Delta_1 && (3.82)\\
&= [\![A\,[\sigma] \leq:_{\Phi'} B\,[\sigma]\,]\!]_M \circ \Delta_1' \quad \text{By induction} && (3.83)\\
&= D' && (3.84)
\end{aligned}$$

Case Lambda: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma, x : A \vdash v \colon B \rrbracket_M \tag{3.85}$$

Then

$$\Delta = \mathtt{cur}(()\Delta_1) \tag{3.86}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mathtt{cur}(\Delta_1)) && (3.87)\\
&= \mathtt{cur}(\sigma^*(\Delta_1)) \quad \text{By S-closure} && (3.88)\\
&= \mathtt{cur}(\Delta_1') \quad \text{By induction} && (3.89)\\
&= \Delta' && (3.90)
\end{aligned}$$

Case Application: Let

$$\begin{aligned}
\Delta_1 &= \llbracket \Phi \mid \Gamma \vdash v_1 \colon A \to B \rrbracket_M && (3.91)\\
\Delta_2 &= \llbracket \Phi \mid \Gamma \vdash v_2 \colon A \rrbracket_M && (3.92)
\end{aligned}$$

Then

$$\Delta = \mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle \tag{3.93}$$

So

$$\begin{aligned}
\sigma^*\Delta &= \sigma^*(\mathtt{app} \circ \langle \Delta_1, \Delta_2 \rangle) && (3.94)\\
&= \mathtt{app} \circ \langle \sigma^*(\Delta_1), \sigma^*(\Delta_2) \rangle \quad \text{By S-closure} && (3.95)\\
&= \mathtt{app} \circ \langle \Delta_1', \Delta_2' \rangle \quad \text{By Induction} && (3.96)\\
&= \Delta' && (3.97)
\end{aligned}$$

Case Return: Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v \colon A \rrbracket_M \tag{3.98}$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \tag{3.99}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\eta_{A_I} \circ \Delta_1) && (3.100)\\
&= \eta_{A_{I'}} \circ \sigma^*(\Delta_1) \quad \text{By S-closure} && (3.101)\\
&= \eta_{A_{I'}} \circ \Delta_1' && (3.102)\\
&= \Delta' && (3.103)
\end{aligned}$$

Case Bind: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v_1\colon \mathtt{M}_{\epsilon_1} A]\!]_M \tag{3.104}$$
$$\Delta_2 = [\![\Phi \mid \Gamma, x : A \vdash v_2\colon \mathtt{M}_{\epsilon_2} B]\!]_M \tag{3.105}$$

Then

$$\Delta = \mathtt{M}_{\epsilon_1}\epsilon_2 A_I \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma_I,A_I} \circ \langle \mathtt{Id}_{\Gamma_I}, \Delta_1\rangle \tag{3.106}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A} \circ T_{\epsilon_1}\Delta_2 \circ \mathtt{t}_{\epsilon_1,\Gamma,A} \circ \langle \mathtt{Id}_\Gamma, \Delta_1\rangle) && (3.107)\\
&= \sigma^*(\mu_{\epsilon_1,\epsilon_2,A}) \circ \sigma^*(T_{\epsilon_1}\Delta_2) \circ \sigma^*(\mathtt{t}_{\epsilon_1,\Gamma,A}) \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1)\right\rangle \quad \text{By S-Closure} && (3.108)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)}\sigma^*(\Delta_2) \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \sigma^*(\Delta_1)\right\rangle \quad \text{By S-Closure} && (3.109)\\
&= \mu_{\sigma(\epsilon_1),\sigma(\epsilon_2),A[\sigma]'} \circ T_{\sigma(\epsilon_1)}\Delta_2' \circ \mathtt{t}_{\sigma(\epsilon_1),\Gamma[\sigma],A[\sigma]} \circ \left\langle \sigma^*(\mathtt{Id}_{\Gamma_I}), \Delta_1'\right\rangle \quad \text{By Induction} && (3.110)\\
&= \Delta' && (3.111)\\
& && (3.112)
\end{aligned}$$

Case If: Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v\colon \mathtt{Bool}]\!]_M \tag{3.113}$$
$$\Delta_2 = [\![\Phi \mid \Gamma \vdash v_1\colon A]\!]_M \tag{3.114}$$
$$\Delta_3 = [\![\Phi \mid \Gamma \vdash v_2\colon A]\!]_M \tag{3.115}$$
$$\tag{3.116}$$

Then

$$\Delta = \mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma \tag{3.117}$$

So

$$\begin{aligned}
\sigma^*(\Delta) &= \sigma^*(\mathtt{app} \circ (([\mathtt{cur}(\Delta_2 \circ \pi_2), \mathtt{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \mathtt{Id}_\Gamma) \circ \delta_\Gamma) && (3.118)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\sigma^*(\Delta_2) \circ \pi_2), \mathtt{cur}(\sigma^*(\Delta_3) \circ \pi_2)] \circ \sigma^*(\Delta_1)) \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By S-Closure} && (3.119)\\
&= \mathtt{app} \circ (([\mathtt{cur}(\Delta_2' \circ \pi_2), \mathtt{cur}(\Delta_3' \circ \pi_2)] \circ \Delta_1') \times \mathtt{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By Induction} && (3.120)\\
&= \Delta' && (3.121)\\
& && (3.122)
\end{aligned}$$

Case Effect-Lambda: Let

$$\Delta_1 = [\![\Phi, \alpha \mid \Gamma \vdash v\colon A]\!]_M \tag{3.123}$$

Then

$$\Delta = \hat{\Delta_1} \tag{3.124}$$

And also

$$\sigma \times \mathtt{Id} = [\![(\Phi', \alpha) \vdash (\sigma, \alpha := \epsilon) \colon (\Phi, \alpha)]\!]_M \tag{3.125}$$

So

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\hat{\Delta_1}) && \text{(3.126)}\\
&= (\sigma \times \hat{\mathtt{Id}_U})^* \Delta_1 \quad \text{By naturality} && \text{(3.127)}\\
&= \hat{\Delta_1'} \quad \text{By induction} && \text{(3.128)}\\
&= \Delta' && \text{(3.129)}
\end{aligned}$$

Case Effect-Application: Let

$$\begin{aligned}
\Delta_1 &= [\![\Phi \mid \Gamma \vdash v \colon \forall \alpha . A]\!]_M && \text{(3.130)}\\
h &= [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M && \text{(3.131)}\\
& && \text{(3.132)}
\end{aligned}$$

Then

$$\Delta = \langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1 \tag{3.133}$$

So Due to the substitution theorem on effects

$$h \circ \sigma = [\![\Phi \vdash \epsilon \colon \mathtt{Effect}]\!]_M \circ \sigma = [\![\Phi' \vdash \sigma(\epsilon) \colon \mathtt{Effect}]\!]_M = h' \tag{3.134}$$

$$\begin{aligned}
\sigma^* \Delta &= \sigma^*(\langle \mathtt{Id}_\Gamma, h \rangle^* \left(\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}\right) \circ \Delta_1) && \text{(3.135)}\\
&= (\langle \mathtt{Id}_\Gamma, h \rangle \circ \sigma)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \sigma^*(\Delta_1) && \text{(3.136)}\\
&= ((\sigma \times \mathtt{Id}_U) \circ \langle \mathtt{Id}_\Gamma, h \circ \sigma \rangle)^* (\epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' && \text{(3.137)}\\
&= (\langle \mathtt{Id}_\Gamma, h' \rangle)^* ((\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M}) \circ \Delta_1)' && \text{(3.138)}\\
& && \text{(3.139)}
\end{aligned}$$

Looking at the inner part of the functor application: Let

$$\begin{aligned}
A &= [\![\Phi, \beta \vdash A\,[\beta/\alpha] : \mathtt{Type}]\!]_M && \text{(3.140)}\\
& && \text{(3.141)}
\end{aligned}$$

$$\begin{aligned}
(\sigma \times \mathtt{Id}_U)^* \epsilon_{[\![\Phi, \beta \vdash A[\beta/\alpha] : \mathtt{Type}]\!]_M} &= (\sigma \times \mathtt{Id}_U)^* \epsilon_A && \text{(3.142)}\\
&= (\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}}) && \text{(3.143)}\\
&= \widehat{\overline{(\sigma \times \mathtt{Id}_U)^* (\widehat{\mathtt{Id}_{\forall_I(A)}})}} \quad \text{By bijection} && \text{(3.144)}\\
&= \widehat{\sigma^* (\overline{\widehat{\mathtt{Id}_{\forall_I(A)}}})} \quad \text{By naturality} && \text{(3.145)}\\
&= \widehat{\sigma^* (\mathtt{Id}_{\forall_I(A)})} \quad \text{By bijection} && \text{(3.146)}\\
&= \widehat{\mathtt{Id}_{\forall_I(A \circ (\sigma \times \mathtt{Id}_U))}} \quad \text{By S-Closure, naturality} && \text{(3.147)}\\
&= \widehat{\mathtt{Id}_{\forall_I(A[\sigma, \alpha := \alpha])}} \quad \text{By Substitution theorem} && \text{(3.148)}\\
&= \epsilon_{A[\sigma]} && \text{(3.149)}
\end{aligned}$$

Going back to the original expression:

$$\sigma^*\Delta = (\langle \mathtt{Id}_\Gamma, h' \rangle)^*((\sigma \times \mathtt{Id}_U)^*\epsilon_{A[\sigma]}) \circ \Delta_1)' \tag{3.150}$$

$$= \Delta' \tag{3.151}$$

$$\tag{3.152}$$

Chapter 4

Effect Weakening Theorem

4.1 Effects

4.2 Types

4.3 Type Environments

4.4 Sub-typing

4.5 Terms

Chapter 5

Value Substitution Theorem

Chapter 6

Type-Environment Weakening Theorem

Chapter 7

Unique Denotation Theorem

Chapter 8

Beta-Eta-Equivalence Theorem (Soundness)