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# Chapter 1

## Preliminaries

### 1.1 Base Category Requirements

There are 3 distinct objects in the base category,  $\mathbb{C}$ :

- $U$  - The kind of **Effect**
- $W$  - The kind of **Type**
- $1$  - A terminal object

And we have finite products on  $U$ .

- $U^0 = 1$
- $U^{n+1} = U^n \times U$

We also have the following natural operations on morphisms in  $\mathbb{C}$ .

Let  $I = U^n$ .

- $\diamond : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$  - Generates exponential types.
- $\square : \mathbb{C}(I, W) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$  - Generates products of types.
- $\forall_I : \mathbb{C}(I \times U, W) \rightarrow \mathbb{C}(I, W)$  - generates quantified types.
- $\text{Eff} : \mathbb{C}(I, U) \times \mathbb{C}(I, W) \rightarrow \mathbb{C}(I, W)$  - generates monad types.
- $\text{Mul} : \mathbb{C}(I, U) \times \mathbb{C}(I, U) \rightarrow \mathbb{C}(I, U)$  - Generates multiplication of effects.

With naturality conditions which mean, for  $\theta : U^m \rightarrow U^n(I' \rightarrow I)$ ,

- $\diamond(\phi, \psi) \circ \theta = \diamond(\phi \circ \theta, \psi \circ \theta)$
- $\square(\phi, \psi) \circ \theta = \square(\phi \circ \theta, \psi \circ \theta)$
- $\forall_I(\phi) \circ \theta = \forall_{I'}(\phi \circ (\theta \times \text{Id}_U))$
- $\text{Eff}(\phi, \psi) \circ \theta = \text{Eff}(\phi \circ \theta, \psi \circ \theta)$
- $\text{Mul}(\phi, \psi) \circ \theta = \text{Mul}(\phi \circ \theta, \psi \circ \theta)$

## 1.2 Well-Formed-ness

Each instance of the well-formed-ness relation on effects,  $\Phi \vdash \epsilon$  has a denotation in  $\mathbb{C}$ :

$$\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket : I \rightarrow U \quad (1.1)$$

Each instance of the well-formed-ness relation on types,  $\Phi \vdash A$  has a denotation in  $\mathbb{C}$ :

$$\llbracket P \vdash A : \text{Type} \rrbracket : I \rightarrow W \quad (1.2)$$

It should also be the case that

$$\text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket) = \llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Effect} \rrbracket \in \mathbb{C}(I, U) \quad (1.3)$$

That is,  $\text{Mul}$  should be have identity  $\llbracket \Phi \vdash 1 : \text{Effect} \rrbracket$  and be associative.

## 1.3 Substitution and Weakening of the Effect Environment

For each instance of the well-formed-ness relation on substitution of effects  $\Phi' \vdash \sigma : \Phi$ , there exists a denotation in  $\mathbb{C}$ :

$$\llbracket \Phi' \vdash \sigma : \Phi \rrbracket : I' \rightarrow I \quad (1.4)$$

For each instance of the well-formed weakening relation on effect-environments,  $\omega : \Phi' \triangleright \Phi$  there exists a denotation in  $\mathbb{C}$ :

$$\llbracket \omega : \Phi' \triangleright \Phi \rrbracket : I' \rightarrow I \quad (1.5)$$

## 1.4 Fibre Categories

Each set of morphisms  $\mathbb{C}(I, W)$  forms the objects of a semantic-closed (S-closed) category. That is, a category satisfying all the properties needed for the non-polymorphic language:

- Cartesian Closed
- Co-product of the terminal object with itself ( $1 + 1$ )
- Ground morphisms for each ground constant ( $\mathbb{C}^A : 1 \rightarrow A$ )
- Partial order morphisms on ground types ( $\llbracket A \leq_\gamma B \rrbracket$ )
- A strong, monad, graded by the po-monoid  $(E_\Phi, \cdot_\Phi, \leq_\Phi, 1)$ .

## 1.5 Re-indexing Functors

For each morphism  $f : I' \rightarrow I$  in  $\mathbb{C}$ , there should be a co-variant, re-indexing functor  $f^* : \mathbb{C}(I, W) \rightarrow \mathbb{C}(I', W)$ , which is S-closed. That is, it preserves the S-closed properties of  $\mathbb{C}(I, W)$ . (See below).

$(-)^*$  should be a contra-variant functor in its  $\mathbb{C}$  argument and co-variant in its right argument.

- $(g \circ f)^*(a) = f^*(\gamma^*(a))$
- $\text{Id}_I^*(a) = a$
- $f^*(\text{Id}_A) = \text{Id}_{f^*(A)}$
- $f^*(a \circ b) = f^*(a) \circ f^*(b)$

### 1.5.1 $f^*$ Preserves Products

If  $\langle g, h \rangle : \mathbb{C}(I, W)(Z, X \times Y)$  Then

$$f^*(X \times Y) = f^*(X) \times f^*(Y) \quad (1.6)$$

$$f^*(\langle g, h \rangle) = \langle f^*(g), f^*(h) \rangle \quad : \mathbb{C}(I', W)(f^*(Z), f^*(X) \times f^*(Y)) \quad (1.7)$$

$$f^*(\pi_1) = \pi_1 \quad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(X)) \quad (1.8)$$

$$f^*(\pi_2) = \pi_2 \quad : \mathbb{C}(I', W)(f^*(X) \times f^*(Y), f^*(Y)) \quad (1.9)$$

### 1.5.2 $f^*$ Preserves Terminal Object

If  $\text{Id}_A : \mathbb{C}(I, W)(A, 1)$  Then

$$f^*(1) = 1 \quad (1.10)$$

$$f^*(\langle \rangle_A) = \langle \rangle_{f^*(A)} \quad : \mathbb{C}(I', W)(f^*(A), 1) \quad (1.11)$$

$$(1.12)$$

### 1.5.3 $f^*$ Preserves Exponentials

$$f^*(Z^X) = (f^*(Z))^{f^*(X)} \quad (1.13)$$

$$f^*(\text{app}) = \text{app} \quad : \mathbb{C}(I', W)(f^*(Z^X) \times f^*(X), f^*(Z)) \quad (1.14)$$

$$f^*(\text{cur}(g)) = \text{cur}(f^*(g)) \quad : \mathbb{C}(I', W)(f^*(Y) \times f^*(X), f^*(Z)^{f^*(X)}) \quad (1.15)$$

### 1.5.4 $f^*$ Preserves Co-product on Terminal

$$f^*(1 + 1) = 1 + 1 \quad (1.16)$$

$$f^*(\text{inl}) = \text{inl} \quad : \mathbb{C}(I', W)(1, 1 + 1) \quad (1.17)$$

$$f^*(\text{inr}) = \text{inr} \quad : \mathbb{C}(I', W)(1, 1 + 1) \quad (1.18)$$

$$f^*([g, h]) = [f^*(g), f^*(h)] \quad : \mathbb{C}(I', W)(1 + 1, f^*(Z)) \quad (1.19)$$

### 1.5.5 $f^*$ Preserves Graded Monad

$$f^*(T_\epsilon A) = T_{f^*(\epsilon)} f^*(A) \quad : \mathbb{C}(I', W) \quad (1.20)$$

$$f^*(1) = 1 \quad \text{The unit effect} \quad (1.21)$$

$$f^*(\eta_A) = \eta_{f^*(A)} \quad : \mathbb{C}(I', W)(f^*(A), f^*(T_1 A)) \quad (1.22)$$

$$f^*(\mu_{\epsilon_1, \epsilon_2, A}) = \mu_{f^*(\epsilon_1), f^*(\epsilon_2), f^*(A)} \quad : \mathbb{C}(I', W)(f^*(T_{\epsilon_1} T_{\epsilon_2} A), f^*(T_{f^*(\epsilon_1) \cdot f^*(\epsilon_2)} f^*(A))) \quad (1.23)$$

$$f^*(\epsilon_1 \cdot \epsilon_2) = f^*(\epsilon_1) \cdot f^*(\epsilon_2) \quad (1.24)$$

$$(1.25)$$

### 1.5.6 $f^*$ Preserves Tensor Strength

$$f^*(\mathfrak{t}_{\epsilon, A, B}) = \mathfrak{t}_{f^*(\epsilon), f^*(A), f^*(B)} : \mathbb{C}(I', W)(f^*(A \times T_\epsilon B), f^*(T_\epsilon(A \times B))) \quad (1.26)$$

### 1.5.7 $f^*$ Preserves Ground Constants

For each ground constant  $\llbracket \mathbf{c}^A \rrbracket$  in  $\mathbb{C}(I, W)$ ,

$$f^*(\llbracket \mathbf{c}^A \rrbracket) = \mathbf{c}^{f^*(A)} : \mathbb{C}(I', W)(1, f^*(A)) \quad (1.27)$$

### 1.5.8 $f^*$ Preserves Ground Sub-effecting

For ground effects  $e_1, e_2$  such that  $e_1 \leq e_2$

$$f^*(e) = e : \mathbb{C}(I', U) \quad (1.28)$$

$$f^*(\llbracket e_1 \leq e_2 \rrbracket_A) = \llbracket e_1 \leq e_2 \rrbracket_{f^*(A)} : \mathbb{C}(I', W)(f^*(T_{e_1} A), f^*(T_{e_2} A)) \quad (1.29)$$

$$(1.30)$$

### 1.5.9 $f^*$ Preserves Ground Sub-typing

For ground types  $\gamma_1, \gamma_2$  such that  $\gamma_1 \leq_{\gamma} \gamma_2$

$$f^*\gamma = \gamma : \mathbb{C}(I', W)(1, \gamma) \quad (1.31)$$

$$f^*(\llbracket \gamma_1 \leq_{\gamma} \gamma_2 \rrbracket) = \llbracket \gamma_1 \leq_{\gamma} \gamma_2 \rrbracket : \mathbb{C}(I', W)(\gamma_1, \gamma_2) \quad (1.32)$$

$$(1.33)$$

### 1.5.10 Action on Objects

It follows that the action of  $f^*$  on an object  $A$  in  $\mathbb{C}(I, W)$  (i.e. a morphism  $I \rightarrow U$  in  $\mathbb{C}$ ) is:

$$f^*(A) = A \circ f : I' \rightarrow I \rightarrow W \quad (1.34)$$

## 1.6 The $\forall_I$ functor

We expand  $\forall_I : \mathbb{C}(I \times U, W) \rightarrow \mathbb{C}(I, W)$  to be a functor which is right adjoint to the re-indexing functor  $\pi_1^*$ .

$$\overline{(-)} : \mathbb{C}(I \times U, W)(\pi_1^* A, B) \leftrightarrow \mathbb{C}(I, W)(A, \forall_I B) : \widehat{(-)} \quad (1.35)$$

For  $A : \mathbb{C}(I, W)$ ,  $B : \mathbb{C}(I \times U, W)$ .

Hence the action of  $\forall_I$  on a morphism  $l : A \rightarrow A'$  is as follows:

$$\forall_I(l) = \overline{l \circ \epsilon_A} \quad (1.36)$$

Where  $\epsilon_A : \mathbb{C}(I \times U, W)(\pi_1^* \forall_I A \rightarrow A)$  is the co-unit of the adjunction.

## 1.7 Naturality Corollaries

Here are some simple corollaries of the adjunction between  $\pi_1^*$  and  $\forall_I$ .

### 1.7.1 Naturality

By the definition of an adjunction:

$$\overline{f \circ \pi_1^*(n)} = \bar{f} \circ n \quad (1.37)$$

### 1.7.2 $\overline{(-)}$ and Re-indexing Functors

By assuming the Beck-Chevalley condition that:

$$\overline{(\theta \times \text{Id}_U)^*(\epsilon)} = \text{Id} : \theta^* \circ \forall_I \rightarrow \forall_{I'} \circ (\theta \times \text{Id}_U)^* \quad (1.38)$$

We then have:

$$\theta^* \eta_A : \theta^* A \rightarrow \theta^* \circ \forall_I \circ \pi_1^* A \quad (1.39)$$

$$\theta^* \eta = \overline{(\theta \times \text{Id}_U)^*(\epsilon_{\pi_1^*})} \circ \theta^* \eta \quad (1.40)$$

$$= (\forall_{I'} \circ (\theta \times \text{Id}_U)^*)(\epsilon_{\pi_1^*}) \circ \eta_{(\forall_{I'} \circ (\theta \times \text{Id}_U)^*) \circ \pi_1^*} \circ \theta^* \eta \quad (1.41)$$

$$= (\forall_{I'} \circ (\theta \times \text{Id}_U)^*)(\epsilon_{\pi_1^*}) \circ \eta_{\theta^* \circ \forall_I \circ \pi_1^*} \circ \theta^* \eta \quad (1.42)$$

$$= (\forall_{I'} \circ (\theta \times \text{Id}_U)^*)(\epsilon_{\pi_1^*}) \circ (\theta^* \circ \forall_I \circ \pi_1^*) \eta \circ \eta_{(\theta \times \text{Id}_U)^*} \quad (1.43)$$

$$= (\theta^* \circ \forall_I)(\epsilon_{\pi_1^*} \circ \pi_1^* \eta) \circ \eta_{(\theta \times \text{Id}_U)^*} \quad (1.44)$$

$$= (\theta^* \circ \forall_I)(\text{Id}) \circ \eta_{(\theta \times \text{Id}_U)^*} \quad (1.45)$$

$$= \eta_{(\theta \times \text{Id}_U)^*} \quad (1.46)$$

$$\theta^*(\bar{f}) = \theta^*(\forall_I(f) \circ \eta_A) \quad (1.47)$$

$$= \theta^*(\forall_I(f)) \circ \theta^*(\eta_A) \quad (1.48)$$

$$= (\forall_{I'} \circ (\theta \times \text{Id}_U)^*) f \circ \eta_{(\theta \times \text{Id}_U)^* A} \quad (1.49)$$

$$= \overline{(\theta \times \text{Id}_U)^* f} \quad (1.50)$$

$$(1.51)$$

### 1.7.3 $\hat{(-)}$ and Re-Indexing Functors

$$\theta^*(\langle \text{Id}_I, \rho \rangle^*(\widehat{m})) = (\langle \text{Id}_I, \rho \rangle \circ \theta)^*(\widehat{m}) \quad (1.52)$$

$$= ((\theta \times \text{Id}_U) \circ \langle \text{Id}_I, \rho \rangle)^*(\widehat{m}) \quad (1.53)$$

$$= \langle \text{Id}_I, \rho \circ \theta \rangle^*(\theta \times \text{Id}_U)^*(\widehat{m}) \quad (1.54)$$

$$= \langle \text{Id}_I, \theta^* \rho \rangle^*(\theta^*(\widehat{m})) \quad (1.55)$$

#### 1.7.4 Pushing Morphisms into $f^*$

$$\langle \mathrm{Id}_I, \rho \rangle^* (\widehat{m}) \circ n = \langle \mathrm{Id}_I, \rho \rangle^* (\widehat{m}) \circ \langle \mathrm{Id}_I, \rho \rangle^* \pi_1^*(n) \quad (1.56)$$

$$= \langle \mathrm{Id}_I, \rho \rangle^* (\widehat{m} \circ \pi_1^*(n)) \quad (1.57)$$

$$= \langle \mathrm{Id}_I, \rho \rangle^* (\widehat{m \circ n}) \quad (1.58)$$



## Chapter 2

# Denotations

### 2.1 Effects

For each instance of the well-formed-ness relation on effects, we define a morphism  $\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket : \mathbb{C}(I, U)$

- $\llbracket \Phi \vdash e : \text{Effect} \rrbracket = \llbracket \epsilon \rrbracket \circ \langle \rangle_I : I \rightarrow U$
- $\llbracket \Phi, \alpha \vdash \alpha : \text{Effect} \rrbracket = \pi_2 : I \times U \rightarrow U$
- $\llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket = \llbracket \Phi \vdash \alpha : \text{Effect} \rrbracket \circ \pi_1 : I \times U \rightarrow U$
- $\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Effect} \rrbracket = \text{Mul}(\llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket, \llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket) : I \rightarrow U$

### 2.2 Types

For each instance of the well-formed-ness relation on types, we define a morphism  $\llbracket \Phi \vdash A : \text{Type} \rrbracket : \mathbb{C}(I, W)$ .

$\llbracket \text{Unit} \rrbracket$  is the morphism generating the terminal object of  $\mathbb{C}(I, W)$ .  $\llbracket \text{Bool} \rrbracket$  is the morphism generating the co-product of this terminal object,  $1 + 1$ .

- $\llbracket \Phi \vdash \text{Unit} : \text{Type} \rrbracket = \llbracket \text{Unit} \rrbracket \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash \text{Bool} : \text{Type} \rrbracket = \llbracket \text{Bool} \rrbracket \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash \gamma : \text{Type} \rrbracket = \llbracket \gamma \rrbracket \circ \langle \rangle_I : I \rightarrow W$
- $\llbracket \Phi \vdash A \rightarrow B : \text{Type} \rrbracket = \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket, \llbracket \Phi \vdash B : \text{Type} \rrbracket) : I \rightarrow W$
- $\llbracket \Phi \vdash M_\epsilon A : \text{Type} \rrbracket = \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket) : I \rightarrow W$
- $\llbracket \Phi \vdash \forall \alpha. A : \text{Type} \rrbracket = \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket) : I \rightarrow W$

### 2.3 Effect Substitution

For each effect-substitution well-formed-ness-relation, define a denotation morphism,  $\llbracket \Phi' \vdash \sigma : \Phi \rrbracket : \mathbb{C}(I', I)$

- $\llbracket \Phi' \vdash \diamond : \diamond \rrbracket = \langle \rangle_I : \mathbb{C}(I', 1)$
- $\llbracket \Phi' \vdash (\sigma, \alpha : = \epsilon) : \Phi, \alpha \rrbracket = \langle \llbracket \Phi' \vdash \sigma : \Phi \rrbracket, \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \rangle : \mathbb{C}(I', I \times U)$

## 2.4 Effect Weakening

For each instance of the effect-environment weakening relation, define a denotation morphism:  $\llbracket \omega : \Phi' \triangleright P \rrbracket : \mathbb{C}(I', I)$

- $\llbracket \iota : \Phi \triangleright \Phi \rrbracket = \text{Id}_I : I \rightarrow I$
- $\llbracket w\pi : \Phi', \alpha \triangleright \Phi \rrbracket = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket \circ \pi_1 : I' \times U \rightarrow I$
- $\llbracket w\times : \Phi', \alpha \triangleright \Phi, \alpha \rrbracket = (\llbracket \omega : \Phi' \triangleright \Phi \rrbracket \times \text{Id}_U) : I' \times U \rightarrow I \times U$

## 2.5 Sub-Typing

For each instance of the sub-typing relation with respect to an effect environment, there exists a denotation,  $\llbracket A \leq_{\Phi} B \rrbracket : \mathbb{C}(I, W)(A, B)$ .

- $\llbracket \gamma_1 \leq_{\Phi} \gamma_2 \rrbracket = \llbracket \gamma_1 \leq_{\gamma} \gamma_2 \rrbracket : \mathbb{C}(I, W)(\gamma_1, \gamma_2)$
- $\llbracket A \rightarrow B \leq_{\Phi} A' \rightarrow B' \rrbracket = \llbracket B \leq_{\Phi} B' \rrbracket^{A'} \circ B \llbracket A' \leq_{\Phi} A \rrbracket$
- $\llbracket M_{\epsilon_1} A \leq_{\Phi} M_{\epsilon_2} B \rrbracket = \llbracket \epsilon_1 \leq_{\Phi} \epsilon_2 \rrbracket \circ T_{\epsilon_1} \llbracket A \leq_{\Phi} B \rrbracket$
- $\llbracket \forall \alpha. A \leq_{\Phi} \forall \alpha. B \rrbracket = \forall_I \llbracket A \leq_{\Phi, \alpha} B \rrbracket$

## 2.6 Type-Environments

For each instance of the well-formed relation on type environments, define an object in  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket \in \mathbb{C}(I, W)$ .

- $\llbracket \Phi \vdash \diamond \text{ Ok} \rrbracket = 1 : \mathbb{C}(I, W)$
- $\llbracket \Phi \vdash \Gamma, x : A \text{ Ok} \rrbracket = \square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket)$

## 2.7 Terms

For each instance of the typing relation, define a denotation morphism:  $\llbracket \Phi \mid \Gamma \vdash v : A \rrbracket : \mathbb{C}(I, W)(\Gamma_I, A_I)$ . Writing  $\Gamma_I$  and  $A_I$  for  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket$  and  $\llbracket \Phi \vdash A : \text{Type} \rrbracket$ .

For each ground constant,  $\mathcal{C}^A$ , there exists  $c : 1 \rightarrow A_I$  in  $\mathbb{C}(I, W)$ .

- (Unit) 
$$\frac{\Phi \vdash \Gamma \text{ Ok}}{\llbracket \Phi \mid \Gamma \vdash () : \text{Unit} \rrbracket = \langle \rangle_{\Gamma} : \Gamma_I \rightarrow 1}$$
- (Const) 
$$\frac{\Phi \vdash \Gamma \text{ Ok}}{\llbracket \Phi \mid \Gamma \vdash \mathcal{C}^A : A \rrbracket = \llbracket \mathcal{C}^A \rrbracket \circ \langle \rangle_{\Gamma} : \Gamma_I \rightarrow A_I}$$

- (True)  $\frac{\Phi \vdash \Gamma \text{ Ok}}{\llbracket \Phi \mid \Gamma \vdash \text{true} : \text{Bool} \rrbracket = \text{inl} \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket \text{Bool} \rrbracket = 1 + 1}$
- (False)  $\frac{\Phi \vdash \Gamma \text{ Ok}}{\llbracket \Phi \mid \Gamma \vdash \text{false} : \text{Bool} \rrbracket = \text{inr} \circ \langle \rangle_{\Gamma} : \Gamma \rightarrow \llbracket \text{Bool} \rrbracket = 1 + 1}$
- (Var)  $\frac{\Phi \vdash \Gamma \text{ Ok}}{\llbracket \Phi \mid \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \Gamma \times A \rightarrow A}$
- (Weaken)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash x : A \rrbracket : \Gamma \rightarrow A}{\llbracket \Phi \mid \Gamma, y : B \vdash x : A \rrbracket = f \circ \pi_1 : \Gamma \times B \rightarrow A}$
- (Lambda)  $\frac{f = \llbracket \Phi \mid \Gamma, x : A \vdash v : B \rrbracket : \Gamma \times A \rightarrow B}{\llbracket \Phi \mid \Gamma \vdash \lambda x : A. v : A \rightarrow B \rrbracket = \text{cur}(f) : \Gamma \rightarrow (B)^A}$
- (Subtype)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket : \Gamma \rightarrow A \quad g = \llbracket A \leq_{\Phi} B \rrbracket}{\llbracket \Phi \mid \Gamma \vdash v : B \rrbracket = g \circ f : \Gamma \rightarrow B}$
- (Return)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket}{\llbracket \Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A \rrbracket = \eta_A \circ f}$
- (If)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash v : \text{Bool} \rrbracket : \Gamma \rightarrow 1 + 1 \quad g = \llbracket \Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon} A \rrbracket \quad h = \llbracket \Phi \mid \Gamma \vdash v_2 : \mathbf{M}_{\epsilon} A \rrbracket}{\llbracket \Phi \mid \Gamma \vdash \text{if}_{\epsilon, A} v \text{ then } v_1 \text{ else } v_2 : \mathbf{M}_{\epsilon} A \rrbracket = \text{app} \circ ((\text{cur}(g \circ \pi_2), \text{cur}(h \circ \pi_2)) \circ f) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} : \Gamma \rightarrow T_{\epsilon} A}$
- (Bind)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A : \Gamma \rightarrow T_{\epsilon_1} A \rrbracket \quad g = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \rrbracket : \Gamma \times A \rightarrow T_{\epsilon_2} B}{\llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B \rrbracket = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\Gamma, A, \epsilon_1} \circ \langle \text{Id}_{\Gamma}, f \rangle : \Gamma \rightarrow T_{\epsilon_1 \cdot \epsilon_2} B}$
- (Apply)  $\frac{f = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rightarrow B \rrbracket : \Gamma \rightarrow (B)^A \quad g = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket : \Gamma \rightarrow A}{\llbracket \Phi \mid \Gamma \vdash v_1 v_2 : \beta \rrbracket = \text{app} \circ \langle f, g \rangle : \Gamma \rightarrow B}$
- (Effect-Lambda)  $\frac{f = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket : \mathbb{C}(I \times U, W)(\Gamma, A)}{\llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. A : \forall \epsilon. A \rrbracket = \bar{f} : \mathbb{C}(I, W)(\Gamma, \forall_I(A))}$
- (Effect-App)  $\frac{g = \llbracket \Phi \mid \Gamma \vdash v : \forall \alpha. A \rrbracket : \mathbb{C}(I, W)(\Gamma, \forall_I(A)) \quad h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket : \mathbb{C}(I, U)}{\llbracket \Phi \mid \Gamma \vdash v \epsilon : A[\epsilon/\alpha] \rrbracket = \langle \text{Id}_I, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ g : \mathbb{C}(I, W)(\Gamma, A[\epsilon/\alpha])}$

## 2.8 Term Weakening

For each instance of the type-environment weakening relation, define a morphism  $\llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket : \Gamma' \rightarrow \Gamma \in \mathbb{C}(I, W)$

- $\llbracket \Phi \vdash \iota : \Gamma \triangleright \Gamma \rrbracket = \text{Id}_{\Gamma} : \Gamma \rightarrow \Gamma \in \mathbb{C}(I)$
- $\llbracket \Phi \vdash \omega \pi : \Gamma', ax \triangleright \Gamma \rrbracket = \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \circ \pi_1 : \Gamma' \times A \rightarrow \Gamma$
- $\llbracket \Phi \vdash \omega \times : \Gamma', x : A \triangleright \Gamma, x : B \rrbracket = \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \times \llbracket A \leq_{\Phi} B \rrbracket : \Gamma' \times A \rightarrow \Gamma \times B$

## 2.9 Term Substitutions

For each instance of the type-environment weakening relation, define a morphism  $\llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket : \Gamma' \rightarrow \Gamma \in \mathbb{C}(I, W)$

- $\llbracket \Phi \vdash \iota : \Gamma \triangleright \Gamma \rrbracket = \text{Id}_\Gamma : \Gamma \rightarrow \Gamma \in \mathbb{C}(I)$
- $\llbracket \Phi \vdash \omega\pi : \Gamma', ax \triangleright \Gamma \rrbracket = \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \circ \pi_1 : \Gamma' \times A \rightarrow \Gamma$
- $\llbracket \Phi \vdash \omega \times : \Gamma', x : A \triangleright \Gamma, x : B \rrbracket = \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \times \llbracket A \leq_\Phi B \rrbracket : \Gamma' \times A \rightarrow \Gamma \times B$

## Chapter 3

# Effect Substitution Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-variable substitution upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism  $\Delta$  of some relation, the denotation of the substituted relation,  $\Delta' = \sigma^*(\Delta)$ .

### 3.1 Effects

If  $\sigma = \llbracket \Phi' \vdash \sigma: \Phi \rrbracket$  then  $\llbracket \Phi' \vdash \sigma(\epsilon): \text{Effect} \rrbracket = \sigma^* \llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket = \llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket \circ \sigma$ .

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket$

**Case Ground:**

$$\llbracket \Phi \vdash e: \text{Effect} \rrbracket \circ \sigma = \llbracket e \rrbracket \circ \langle \rangle_I \circ \sigma \quad (3.1)$$

$$= \llbracket e \rrbracket \circ \langle \rangle_{I'} \quad (3.2)$$

$$= \llbracket \Phi' \vdash e: \text{Type} \rrbracket \quad (3.3)$$

$$(3.4)$$

**Case Var:**

$$\llbracket \Phi, \alpha \vdash \alpha: \text{Effect} \rrbracket \circ \sigma' = \pi_2 \circ \langle \sigma, \llbracket \Phi' \vdash \epsilon: \text{Effect} \rrbracket \rangle \quad \text{By inversion } \sigma' = (\sigma, \alpha: \epsilon) \quad (3.5)$$

$$= \llbracket \Phi' \vdash \epsilon: \text{Effect} \rrbracket \quad (3.6)$$

$$= \llbracket \Phi' \vdash \sigma'(\alpha): \text{Effect} \rrbracket \quad (3.7)$$

$$(3.8)$$

**Case Weaken:**

$$\llbracket \Phi, \beta \vdash \alpha: \text{Type} \rrbracket \circ \sigma' = \llbracket \Phi \vdash \alpha: \text{Type} \rrbracket \circ \pi_1 \circ \langle \sigma, \llbracket \Phi' \vdash \epsilon: \text{Effect} \rrbracket \rangle \quad \text{By inversion, } \sigma' = (\sigma, \beta: \epsilon) \quad (3.9)$$

$$= \llbracket \Phi \vdash \alpha: \text{Type} \rrbracket \circ \sigma \quad (3.10)$$

$$= \llbracket \Phi' \vdash \sigma(\alpha): \text{Type} \rrbracket \quad (3.11)$$

$$= \llbracket \Phi' \vdash \sigma'(\alpha): \text{Type} \rrbracket \quad (3.12)$$

$$(3.13)$$

**Case Multiply:**

$$\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Type} \rrbracket \circ \sigma = \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket) \circ \sigma \quad (3.14)$$

$$= \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket \circ \sigma, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket \circ \sigma) \quad \text{By Naturality} \quad (3.15)$$

$$= \text{Mul}(\llbracket \Phi' \vdash \sigma(\epsilon_1) : \text{Effect} \rrbracket, \llbracket \Phi \vdash \sigma(\epsilon_2) : \text{Effect} \rrbracket) \quad (3.16)$$

$$= \llbracket \Phi' \vdash \sigma(\epsilon_1) \cdot \sigma(\epsilon_2) : \text{Effect} \rrbracket \quad (3.17)$$

$$= \llbracket \Phi' \vdash \sigma(\epsilon_1 \cdot \epsilon_2) : \text{Effect} \rrbracket \quad (3.18)$$

$$(3.19)$$

## 3.2 Types

If  $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket$  then  $\llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket = \sigma^* \llbracket \Phi \vdash A : \text{Type} \rrbracket = \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \sigma$ .

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash A : \text{Type} \rrbracket$ . Making use of naturality properties of the type constructors.

**Case Ground:**

$$\llbracket \Phi \vdash \gamma : \text{Type} \rrbracket \circ \sigma = \llbracket \gamma \rrbracket \circ \langle \rangle_I \circ \sigma \quad (3.20)$$

$$= \llbracket \gamma \rrbracket \circ \langle \rangle_{I'} \quad (3.21)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket \quad (3.22)$$

$$= \llbracket \Phi' \vdash \gamma[\sigma] : \text{Type} \rrbracket \quad (3.23)$$

**Case Monad:**

$$\llbracket \Phi \vdash M_\epsilon A : \text{Type} \rrbracket \circ \sigma = \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket) \circ \sigma \quad (3.24)$$

$$= \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \circ \sigma, \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \sigma) \quad \text{By naturality} \quad (3.25)$$

$$= \text{Eff}(\llbracket \Phi' \vdash \sigma(\epsilon) : \text{Effect} \rrbracket, \llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket) \quad (3.26)$$

$$= \llbracket \Phi' \vdash M_{\sigma(\epsilon)} A[\sigma] : \text{Type} \rrbracket \quad (3.27)$$

$$= \llbracket \Phi' \vdash (M_\epsilon A)[\sigma] : \text{Type} \rrbracket \quad (3.28)$$

**Case Quantification:**

$$\llbracket \Phi \vdash \forall \alpha. A : \text{Type} \rrbracket \circ \sigma = \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket) \circ \sigma \quad (3.29)$$

$$= \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket \circ (\sigma \times \text{Id}_U)) \quad (3.30)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A[\sigma, \alpha = \epsilon] : \text{Type} \rrbracket) \quad (3.31)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A[\sigma] : \text{Type} \rrbracket) \quad (3.32)$$

$$= \llbracket \Phi' \vdash \forall \alpha. A[\sigma] : \text{Type} \rrbracket \quad (3.33)$$

$$= \llbracket \Phi' \vdash (\forall \alpha. A)[\sigma] : \text{Type} \rrbracket \quad (3.34)$$

$$(3.35)$$

**Case Function:**

$$\llbracket \Phi \vdash A \rightarrow B : \text{Type} \rrbracket \circ \sigma = \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket, \llbracket \Phi \vdash B : \text{Type} \rrbracket) \circ \sigma \quad (3.36)$$

$$= \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \sigma, \llbracket \Phi \vdash B : \text{Type} \rrbracket \circ \sigma) \quad \text{By Naturality} \quad (3.37)$$

$$= \diamond(\llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket, \llbracket \Phi' \vdash B[\sigma] : \text{Type} \rrbracket) \quad (3.38)$$

$$= \llbracket \Phi' \vdash (A[\sigma]) \rightarrow (B[\sigma]) : \text{Type} \rrbracket \quad (3.39)$$

$$= \llbracket \Phi' \vdash (A \rightarrow B)[\sigma] : \text{Type} \rrbracket \quad (3.40)$$

$$(3.41)$$

### 3.3 Sub-typing

If  $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket$  then  $\llbracket A[\sigma] \leq_{\Phi'} B[\sigma] \rrbracket = \sigma^* \llbracket A \leq_{\Phi} B \rrbracket : \mathbb{C}(I', W)(A, B)$ .

**Proof:** By induction on the derivation on  $\llbracket A \leq_{\Phi} B \rrbracket$ . Using S-closure of  $\sigma^*$

**Case Ground:**

$$\sigma^*(\gamma_1 \leq_{\gamma} \gamma_2) = (\gamma_1 \leq_{\gamma} \gamma_2) \quad (3.42)$$

Since  $\sigma^*$  is s-closed.

**Case Monad:**

$$\sigma^* \llbracket M_{\epsilon_1} A \leq_{\Phi} M_{\epsilon_2} B \rrbracket = \sigma^* (\llbracket \epsilon_1 \leq_{\Phi} \epsilon_2 \rrbracket) \circ \sigma^* (T_{\epsilon_1} (\llbracket A \leq_{\Phi} B \rrbracket)) \quad (3.43)$$

$$= \llbracket \sigma(\epsilon_1) \leq_{\Phi'} \sigma(\epsilon_2) \rrbracket \circ T_{\sigma(\epsilon_1)} \llbracket A[\sigma] \leq_{\Phi'} B[\sigma] \rrbracket \quad \text{By S-Closure} \quad (3.44)$$

$$= \llbracket M_{\sigma(\epsilon_1)} A[\sigma] \leq_{\Phi'} M_{\sigma(\epsilon_2)} B[\sigma] \rrbracket \quad (3.45)$$

$$= \llbracket (M_{\epsilon_1} A)[\sigma] \leq_{\Phi'} M_{\epsilon_2} B[\sigma] \rrbracket \quad (3.46)$$

$$(3.47)$$

**Case For All:**

$$\sigma^* \llbracket \forall \alpha. A \leq_{\Phi} \forall \alpha. B \rrbracket = \sigma^* (\forall_I (\llbracket A \leq_{\Phi, \alpha} B \rrbracket)) \quad (3.48)$$

$$= \forall_{I'} ((\sigma \times \text{Id}_U)^* (\llbracket A \leq_{\Phi, \alpha} B \rrbracket)) \quad (3.49)$$

$$= \forall_{I'} (\llbracket A[\sigma, \alpha := \alpha] \leq_{\Phi', \alpha} B[\sigma, \alpha := \alpha] \rrbracket) \quad (3.50)$$

$$= \llbracket (\forall \alpha. A)[\sigma] \leq_{\Phi'} (\forall \alpha. B)[\sigma] \rrbracket \quad (3.51)$$

$$(3.52)$$

**Case Fn:**

$$\sigma^* \llbracket (A \rightarrow B) \leq_{\Phi} A' \rightarrow B' \rrbracket = \sigma^* (\llbracket B \leq_{\Phi} B' \rrbracket^{A'} \circ B^{[A' \leq_{\Phi} A]}) \quad (3.53)$$

$$= \sigma^* (\text{cur}(\llbracket B \leq_{\Phi} B' \rrbracket \circ \text{app})) \circ \sigma^* (\text{cur}(\text{app} \circ (\text{Id}_B \times \llbracket A' \leq_{\Phi} A \rrbracket))) \quad (3.54)$$

$$= \text{cur}(\sigma^* (\llbracket B \leq_{\Phi} B' \rrbracket) \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_B \times \sigma^* (\llbracket A' \leq_{\Phi} A \rrbracket))) \quad (3.55)$$

$$= \text{cur}(\llbracket B[\sigma] \leq_{\Phi'} B'[\sigma] \rrbracket \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_{B[\sigma]} \times \llbracket A'[\sigma] \leq_{\Phi'} A[\sigma] \rrbracket)) \quad (3.56)$$

$$= \llbracket (A[\sigma]) \rightarrow (B[\sigma]) \leq_{\Phi'} (A'[\sigma]) \rightarrow (B'[\sigma]) \rrbracket \quad (3.57)$$

$$= \llbracket (A \rightarrow B)[\sigma] \leq_{\Phi'} (A' \rightarrow B')[\sigma] \rrbracket \quad (3.58)$$

### 3.4 Type Environments

If  $\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket$  then  $\llbracket \Phi' \vdash \Gamma[\sigma] \text{ Ok} \rrbracket = \sigma^* \llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket = \llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket \circ \sigma : \mathbb{C}(I', W)$ .

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket$ . Using Naturality.

**Case Nil:**

$$\sigma^* \llbracket \Phi \vdash \diamond \text{ Ok} \rrbracket = \langle \rangle_I \circ \sigma \quad (3.59)$$

$$= \langle \rangle_{I'} \quad (3.60)$$

$$= \llbracket \Phi' \vdash \diamond \text{ Ok} \rrbracket \quad (3.61)$$

$$\llbracket \Phi' \vdash \diamond [\sigma] \text{ Ok} \rrbracket \quad (3.62)$$

$$(3.63)$$

**Case Var:**

$$\sigma^* \llbracket \Phi \vdash \Gamma, x : A \text{ Ok} \rrbracket = \sigma^* (\square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket)) \quad (3.64)$$

$$= \square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket) \circ \sigma \quad (3.65)$$

$$= \square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket \circ \sigma, \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \sigma) \quad (3.66)$$

$$= \square(\llbracket \Phi' \vdash \Gamma[\sigma] \text{ Ok} \rrbracket, \llbracket \Phi' \vdash A[\sigma] : \text{Type} \rrbracket) \quad (3.67)$$

$$= \llbracket \Phi' \vdash \Gamma[\sigma], x : A[\sigma] \text{ Ok} \rrbracket \quad (3.68)$$

$$= \llbracket \Phi' \vdash (\Gamma, x : A) [\sigma] \text{ Ok} \rrbracket \quad (3.69)$$

$$(3.70)$$

### 3.5 Terms

If

$$\sigma = \llbracket \Phi' \vdash \sigma : \Phi \rrbracket \quad (3.71)$$

$$\Delta = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket \quad (3.72)$$

$$\Delta' = \llbracket \Phi' \mid \Gamma[\sigma] \vdash v[\sigma] : A[\sigma] \rrbracket \quad (3.73)$$

$$(3.74)$$

Then

$$\Delta' = \sigma^*(\Delta) \quad (3.75)$$

**Proof:** By induction over the derivation of  $\Delta$ . Using the S-Closure of  $\sigma^*$ . We use  $\Gamma_I$  to indicate  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket$ , an  $A_I$  to indicate  $\llbracket \Phi \vdash A : \text{Type} \rrbracket$

**Case Unit:**

$$\Delta = \langle \rangle_{\Gamma_I} \quad (3.76)$$

So

$$\sigma^*(\Delta) = \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.77)$$



**Case True, False:** Giving the case for true as false is the same but using **inr**

$$\Delta = \text{inl} \circ \langle \rangle_{\Gamma_I} \quad (3.78)$$

So

$$\sigma^*(\Delta) = \text{inl} \circ \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.79)$$

Since  $\sigma^*$  is S-closed.

**Case Constant:**

$$\Delta = \llbracket \mathbf{C}^A \rrbracket \circ \langle \rangle_{\Gamma_I} \quad (3.80)$$

So

$$\sigma^*(\Delta) = \sigma^* \llbracket \mathbf{C}^A \rrbracket \circ \langle \rangle_{\Gamma_I[\sigma]} = \llbracket \mathbf{C}^{A[\sigma]} \rrbracket \circ \langle \rangle_{\Gamma_I[\sigma]} = \Delta' \quad (3.81)$$

Since  $\sigma^*$  is S-closed.

**Case Subtype:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket \quad (3.82)$$

Then

$$\Delta = \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta_1 \quad (3.83)$$

So

$$\sigma^*(\Delta) = \sigma^* \llbracket A \leq_{\Phi} B \rrbracket \circ \sigma^* \Delta_1 \quad (3.84)$$

$$= \llbracket A[\sigma] \leq_{\Phi'} B[\sigma] \rrbracket \circ \Delta'_1 \quad \text{By induction} \quad (3.85)$$

$$= D' \quad (3.86)$$

**Case Lambda:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma, x : A \vdash v : B \rrbracket \quad (3.87)$$

Then

$$\Delta = \text{cur}((\Delta_1)) \quad (3.88)$$

So

$$\sigma^*(\Delta) = \sigma^*(\text{cur}(\Delta_1)) \quad (3.89)$$

$$= \text{cur}(\sigma^*(\Delta_1)) \quad \text{By S-closure} \quad (3.90)$$

$$= \text{cur}(\Delta'_1) \quad \text{By induction} \quad (3.91)$$

$$= \Delta' \quad (3.92)$$

**Case Application:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rightarrow B \rrbracket \quad (3.93)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket \quad (3.94)$$

Then

$$\Delta = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad (3.95)$$

So

$$\sigma^* \Delta = \sigma^*(\text{app} \circ \langle \Delta_1, \Delta_2 \rangle) \quad (3.96)$$

$$= \text{app} \circ \langle \sigma^*(\Delta_1), \sigma^*(\Delta_2) \rangle \quad \text{By S-closure} \quad (3.97)$$

$$= \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Induction} \quad (3.98)$$

$$= \Delta' \quad (3.99)$$

**Case Return:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket \quad (3.100)$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \quad (3.101)$$

So

$$\sigma^*(\Delta) = \sigma^*(\eta_{A_I} \circ \Delta_1) \quad (3.102)$$

$$= \eta_{A_{I'}} \circ \sigma^*(\Delta_1) \quad \text{By S-closure} \quad (3.103)$$

$$= \eta_{A_{I'}} \circ \Delta'_1 \quad (3.104)$$

$$= \Delta' \quad (3.105)$$

**Case Bind:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : M_{\epsilon_1} A \rrbracket \quad (3.106)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : M_{\epsilon_2} B \rrbracket \quad (3.107)$$

Then

$$\Delta = M_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma_I, A_I} \circ \langle \text{Id}_{\Gamma_I}, \Delta_1 \rangle \quad (3.108)$$

So

$$\sigma^*(\Delta) = \sigma^*(M_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma_I, A_I} \circ \langle \text{Id}_{\Gamma_I}, \Delta_1 \rangle) \quad (3.109)$$

$$= \sigma^*(M_{\epsilon_1} \epsilon_2 A_I) \circ \sigma^*(T_{\epsilon_1} \Delta_2) \circ \sigma^*(\mathbf{t}_{\epsilon_1, \Gamma_I, A_I}) \circ \langle \sigma^*(\text{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (3.110)$$

$$= \mu_{\sigma(\epsilon_1), \sigma(\epsilon_2), A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \sigma^*(\Delta_2) \circ \mathbf{t}_{\sigma(\epsilon_1), \Gamma[\sigma], A[\sigma]} \circ \langle \sigma^*(\text{Id}_{\Gamma_I}), \sigma^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (3.111)$$

$$= \mu_{\sigma(\epsilon_1), \sigma(\epsilon_2), A[\sigma]'} \circ T_{\sigma(\epsilon_1)} \Delta'_2 \circ \mathbf{t}_{\sigma(\epsilon_1), \Gamma[\sigma], A[\sigma]} \circ \langle \sigma^*(\text{Id}_{\Gamma_I}), \Delta'_1 \rangle \quad \text{By Induction} \quad (3.112)$$

$$= \Delta' \quad (3.113)$$

$$(3.114)$$

**Case If:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v: \text{Bool} \rrbracket \quad (3.115)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_1: A \rrbracket \quad (3.116)$$

$$\Delta_3 = \llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket \quad (3.117)$$

$$(3.118)$$

Then

$$\Delta = \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad (3.119)$$

So

$$\sigma^*(\Delta) = \sigma^*(\text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_\Gamma) \circ \delta_\Gamma) \quad (3.120)$$

$$= \text{app} \circ (([\text{cur}(\sigma^*(\Delta_2) \circ \pi_2), \text{cur}(\sigma^*(\Delta_3) \circ \pi_2)] \circ \sigma^*(\Delta_1)) \times \text{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By S-Closure} \quad (3.121)$$

$$= \text{app} \circ (([\text{cur}(\Delta'_2 \circ \pi_2), \text{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma[\sigma]}) \circ \delta_{\Gamma[\sigma]} \quad \text{By Induction} \quad (3.122)$$

$$= \Delta' \quad (3.123)$$

$$(3.124)$$

**Case Effect-Lambda:** Let

$$\Delta_1 = \llbracket \Phi, \alpha \mid \Gamma \vdash v: A \rrbracket \quad (3.125)$$

Then

$$\Delta = \widehat{\Delta_1} \quad (3.126)$$

And also

$$\sigma \times \text{Id} = \llbracket (\Phi', \alpha) \mid (\sigma, \alpha: = \epsilon): (\Phi, \alpha) \rrbracket \quad (3.127)$$

So

$$\sigma^* \Delta = \sigma^*(\widehat{\Delta_1}) \quad (3.128)$$

$$= \overline{(\sigma \times \text{Id}_U)^* \Delta_1} \quad \text{By naturality} \quad (3.129)$$

$$= \widehat{\Delta'_1} \quad \text{By induction} \quad (3.130)$$

$$= \Delta' \quad (3.131)$$

**Case Effect-Application:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v: \forall \alpha. A \rrbracket \quad (3.132)$$

$$h = \llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket \quad (3.133)$$

$$(3.134)$$

Then

$$\Delta = \langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha]: \text{Type} \rrbracket}) \circ \Delta_1 \quad (3.135)$$

So Due to the substitution theorem on effects

$$h \circ \sigma = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \circ \sigma = \llbracket \Phi' \vdash \sigma(\epsilon) : \text{Effect} \rrbracket = h' \quad (3.136)$$

$$\sigma^* \Delta = \sigma^* (\langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta_1) \quad (3.137)$$

$$= (\langle \text{Id}_\Gamma, h \rangle \circ \sigma)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \sigma^* (\Delta_1) \quad (3.138)$$

$$= ((\sigma \times \text{Id}_U) \circ \langle \text{Id}_\Gamma, h \circ \sigma \rangle)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta'_1 \quad (3.139)$$

$$= (\langle \text{Id}_\Gamma, h' \rangle)^* ((\sigma \times \text{Id}_U)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta'_1) \quad (3.140)$$

$$(3.141)$$

Looking at the inner part of the functor application: Let

$$A = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket \quad (3.142)$$

$$(3.143)$$

$$(\sigma \times \text{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket} = (\sigma \times \text{Id}_U)^* \epsilon_A \quad (3.144)$$

$$= (\sigma \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}}) \quad (3.145)$$

$$= \overline{(\sigma \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By bijection} \quad (3.146)$$

$$= \overline{\sigma^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By naturality} \quad (3.147)$$

$$= \overline{\sigma^* (\text{Id}_{\forall_I(A)})} \quad \text{By bijection} \quad (3.148)$$

$$= \overline{\text{Id}_{\forall_{I'}(A \circ (\sigma \times \text{Id}_U))}} \quad \text{By S-Closure, naturality} \quad (3.149)$$

$$= \overline{\text{Id}_{\forall_{I'}(A[\sigma, \alpha := \alpha])}} \quad \text{By Substitution theorem} \quad (3.150)$$

$$= \epsilon_{A[\sigma]} \quad (3.151)$$

Going back to the original expression:

$$\sigma^* \Delta = (\langle \text{Id}_\Gamma, h' \rangle)^* (\epsilon_{A[\sigma]}) \circ \Delta'_1 \quad (3.152)$$

$$= \Delta' \quad (3.153)$$

$$(3.154)$$

## Chapter 4

# Effect Weakening Theorem

In this section, we state and prove a theorem that the action of a simultaneous effect-weakening upon a structure in the language has a consistent effect upon the denotation of the language. More formally, for the denotation morphism  $\Delta$  of some relation, the denotation of the weakened relation,  $\Delta' = \omega^*(\Delta)$ .

### 4.1 Effects

If  $\omega = \llbracket \omega: \Phi' \triangleright \Phi \rrbracket$  then  $\llbracket \Phi' \vdash \epsilon: \text{Effect} \rrbracket = \omega^* \llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket = \llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket \circ \omega$

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash \epsilon: \text{Effect} \rrbracket$

**Case Ground:**

$$\llbracket \Phi \vdash e: \text{Effect} \rrbracket \circ \omega = \llbracket e \rrbracket \circ \langle \rangle_I \circ \omega \quad (4.1)$$

$$= \llbracket e \rrbracket \circ \langle \rangle_{I'} \quad (4.2)$$

$$= \llbracket \Phi' \vdash e: \text{Type} \rrbracket \quad (4.3)$$

$$(4.4)$$

**Case Var:** Case split on  $\omega$ .

**Case:**  $\omega = \iota$  Then  $\Phi' = \Phi$  and  $\omega = \text{Id}_I$ . So the theorem holds trivially.

**Case:**  $\omega = \omega' \times$  Then

$$\llbracket \Phi, \alpha \vdash \alpha: \text{Effect} \rrbracket \circ \omega = \pi_2 \circ (\omega' \times \text{Id}_{I'}) \quad (4.5)$$

$$= \pi_2 \quad (4.6)$$

$$= \llbracket \Phi', \alpha \vdash \alpha: \text{Effect} \rrbracket \quad (4.7)$$

**Case:**  $\omega = \omega' \pi_1$  Then

$$\llbracket \Phi, \alpha \vdash \alpha: \text{Effect} \rrbracket = \pi_2 \circ \omega' \circ \pi_1 \quad (4.8)$$

Where  $\Phi' = \Phi, \beta$  and  $\omega': \Phi'' \triangleright \Phi$ .

So

$$\pi_2 \circ \omega' = \llbracket \Phi'' \vdash \alpha : \text{Effect} \rrbracket \quad (4.9)$$

$$\pi_2 \circ \omega' \circ \pi_1 = \llbracket \Phi'', \beta \vdash \alpha : \text{Effect} \rrbracket = \llbracket \Phi' \vdash \alpha : \text{Effect} \rrbracket \quad (4.10)$$

**Case Weaken:**

$$\llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket \circ \omega = \llbracket \Phi \vdash \alpha : \text{Effect} \rrbracket \circ \pi_1 \circ \omega \quad (4.11)$$

Case split of structure of  $w$

**Case:**  $\omega = \iota$  Then  $\Phi' = \Phi, \beta$  so  $\omega = \text{Id}_I$  So  $\llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket \circ \omega = \llbracket \Phi' \vdash \alpha : \text{Effect} \rrbracket$

**Case:**  $\omega = \omega' \pi_1$  Then  $\Phi' = \Phi'', \gamma$  and  $\omega = \omega' \circ \pi_1$  Where  $\omega' : \Phi'' \triangleright \Phi, \beta$ . So

$$\llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket \circ \omega = \llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket \circ \omega' \circ \pi_1 \quad (4.12)$$

$$= \llbracket \Phi'' \vdash \alpha : \text{Effect} \rrbracket \circ \pi_1 \quad (4.13)$$

$$= \llbracket \Phi'', \gamma \vdash \alpha : \text{Effect} \rrbracket \quad (4.14)$$

$$= \llbracket \Phi' \vdash \alpha : \text{Effect} \rrbracket \quad (4.15)$$

$$(4.16)$$

**Case:**  $\omega = \omega' \times$  Then  $\Phi' = \Phi'', \beta$  and  $\omega' : \Phi' \triangleright \Phi$

So

$$\llbracket \Phi, \beta \vdash \alpha : \text{Effect} \rrbracket \circ \omega = \llbracket \Phi \vdash \alpha : \text{Effect} \rrbracket \circ \pi_1 \circ (\omega' \times \text{Id}_U) \quad (4.17)$$

$$= \llbracket \Phi \vdash \alpha : \text{Effect} \rrbracket \circ \omega' \circ \pi_1 \quad (4.18)$$

$$= \llbracket \Phi'' \vdash \alpha : \text{Effect} \rrbracket \circ \pi_1 \quad (4.19)$$

$$= \llbracket \Phi' \vdash \alpha : \text{Effect} \rrbracket \quad (4.20)$$

$$(4.21)$$

**Case Multiply:**

$$\llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Type} \rrbracket \circ \omega = \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket) \circ \omega \quad (4.22)$$

$$= \text{Mul}(\llbracket \Phi \vdash \epsilon_1 : \text{Effect} \rrbracket \circ \omega, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket \circ \omega) \quad \text{By Naturality} \quad (4.23)$$

$$= \text{Mul}(\llbracket \Phi' \vdash \epsilon_1 : \text{Effect} \rrbracket, \llbracket \Phi \vdash \epsilon_2 : \text{Effect} \rrbracket) \quad (4.24)$$

$$= \llbracket \Phi \vdash \epsilon_1 \cdot \epsilon_2 : \text{Effect} \rrbracket \quad (4.25)$$

## 4.2 Types

If  $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket$  then  $\llbracket \Phi' \vdash A : \text{Type} \rrbracket = \omega^* \llbracket \Phi \vdash A : \text{Type} \rrbracket = \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \omega$ .

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash A : \text{Type} \rrbracket$ . Making use of naturality properties of the type constructors.

**Case Ground:**

$$\llbracket \Phi \vdash \gamma : \text{Type} \rrbracket \circ \omega = \llbracket \gamma \rrbracket \circ \langle \rangle_I \circ \omega \quad (4.26)$$

$$= \llbracket \gamma \rrbracket \circ \langle \rangle_{I'} \quad (4.27)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket \quad (4.28)$$

$$= \llbracket \Phi' \vdash \gamma : \text{Type} \rrbracket \quad (4.29)$$

**Case Monad:**

$$\llbracket \Phi \vdash M_\epsilon A : \text{Type} \rrbracket \circ \omega = \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket) \circ \omega \quad (4.30)$$

$$= \text{Eff}(\llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \circ \omega, \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \omega) \quad \text{By naturality} \quad (4.31)$$

$$= \text{Eff}(\llbracket \Phi' \vdash \epsilon : \text{Effect} \rrbracket, \llbracket \Phi' \vdash A : \text{Type} \rrbracket) \quad (4.32)$$

$$= \llbracket \Phi' \vdash (M_\epsilon A) : \text{Type} \rrbracket \quad (4.33)$$

**Case Quantification:** Note  $\llbracket \omega \times : \Phi', \alpha \triangleright \Phi, \alpha \rrbracket = \omega \times \text{Id}_U$

$$\llbracket \Phi \vdash \forall \alpha. A : \text{Type} \rrbracket \circ \omega = \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket) \circ \omega \quad (4.34)$$

$$= \forall_I(\llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket \circ (\omega \times \text{Id}_U)) \quad \text{By naturality} \quad (4.35)$$

$$= \forall_I(\llbracket \Phi', \alpha \vdash A : \text{Type} \rrbracket) \quad \text{By induction} \quad (4.36)$$

$$= \llbracket \Phi' \vdash \forall \alpha. A : \text{Type} \rrbracket \quad (4.37)$$

$$= \llbracket \Phi' \vdash (\forall \alpha. A) : \text{Type} \rrbracket \quad (4.38)$$

$$(4.39)$$

**Case Function:**

$$\llbracket \Phi \vdash A \rightarrow B : \text{Type} \rrbracket \circ \omega = \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket, \llbracket \Phi \vdash B : \text{Type} \rrbracket) \circ \omega \quad (4.40)$$

$$= \diamond(\llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \omega, \llbracket \Phi \vdash B : \text{Type} \rrbracket \circ \omega) \quad \text{By Naturality} \quad (4.41)$$

$$= \diamond(\llbracket \Phi' \vdash A : \text{Type} \rrbracket, \llbracket \Phi' \vdash B : \text{Type} \rrbracket) \quad (4.42)$$

$$= \llbracket \Phi' \vdash (A \rightarrow B) : \text{Type} \rrbracket \quad (4.43)$$

$$(4.44)$$

## 4.3 Sub-typing

If  $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket$  then  $\llbracket A \leq_{:\Phi'} B \rrbracket = \omega^* \llbracket A \leq_{:\Phi} B \rrbracket : \mathbb{C}(I', W)(A, B)$ .

**Proof:** By induction on the derivation on  $\llbracket A \leq_{:\Phi} B \rrbracket$ . Using S-closure of  $\omega^*$

**Case Ground:**

$$\omega^*(\gamma_1 \leq_{:\gamma} \gamma_2) = (\gamma_1 \leq_{:\gamma} \gamma_2) \quad (4.45)$$

Since  $\omega^*$  is s-closed.

**Case Monad:**

$$\omega^* \llbracket M_{\epsilon_1} A \leq_{:\Phi} M_{\epsilon_2} B \rrbracket = \omega^* (\llbracket \epsilon_1 \leq_{\Phi} \epsilon_2 \rrbracket) \circ \omega^* (T_{\epsilon_1} (\llbracket A \leq_{:\Phi} B \rrbracket)) \quad (4.46)$$

$$= \llbracket \epsilon_1 \leq_{\Phi'} \epsilon_2 \rrbracket \circ T_{\epsilon_1} \llbracket A \leq_{:\Phi'} B \rrbracket \quad \text{By S-Closure} \quad (4.47)$$

$$= \llbracket M_{\epsilon_1} A \leq_{:\Phi'} M_{\epsilon_2} B \rrbracket \quad (4.48)$$

$$= \llbracket (M_{\epsilon_1} A) \leq_{:\Phi'} (M_{\epsilon_2} B) \rrbracket \quad (4.49)$$

$$(4.50)$$

**Case For All:** Note  $\llbracket \omega \times : \Phi', \alpha \triangleright \Phi, \alpha \rrbracket = (\omega \times \text{Id}_U)$

$$\omega^* \llbracket \forall \alpha. A \leq_{:\Phi} \forall \alpha. B \rrbracket = \omega^* (\forall_I (\llbracket A \leq_{:\Phi, \alpha} B \rrbracket)) \quad (4.51)$$

$$= \forall_{I'} ((\omega \times \text{Id}_U)^* (\llbracket A \leq_{:\Phi, \alpha} B \rrbracket)) \quad (4.52)$$

$$= \forall_{I'} (\llbracket A \leq_{:\Phi', \alpha} B \rrbracket) \quad (4.53)$$

$$= \llbracket (\forall \alpha. A) \leq_{:\Phi'} (\forall \alpha. B) \rrbracket \quad (4.54)$$

$$(4.55)$$

**Case Fn:**

$$\omega^* \llbracket (A \rightarrow B) \leq_{:\Phi} A' \rightarrow B' \rrbracket = \omega^* (\llbracket B \leq_{:\Phi} B' \rrbracket^{A'} \circ B^{[A' \leq_{:\Phi} A]}) \quad (4.56)$$

$$= \omega^* (\text{cur}(\llbracket B \leq_{:\Phi} B' \rrbracket \circ \text{app})) \circ \omega^* (\text{cur}(\text{app} \circ (\text{Id}_B \times \llbracket A' \leq_{:\Phi} A \rrbracket))) \quad (4.57)$$

$$= \text{cur}(\omega^* (\llbracket B \leq_{:\Phi} B' \rrbracket) \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_B \times \omega^* (\llbracket A' \leq_{:\Phi} A \rrbracket))) \quad (4.58)$$

$$= \text{cur}(\llbracket B \leq_{:\Phi'} B' \rrbracket \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id}_B \times \llbracket A' \leq_{:\Phi'} A \rrbracket)) \quad (4.59)$$

$$= \llbracket (A \rightarrow B) \leq_{:\Phi'} (A' \rightarrow B') \rrbracket \quad (4.60)$$

## 4.4 Type Environments

If  $\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket$  then  $\llbracket \Phi' \vdash \Gamma \text{ Ok} \rrbracket = \omega^* \llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket = \llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket \circ \omega : \mathbb{C}(I', W)$ .

**Proof:** By induction on the derivation on  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket$ . Using Naturality.

**Case Nil:**

$$\omega^* \llbracket \Phi \vdash \diamond \text{ Ok} \rrbracket = \langle \rangle_I \circ \omega \quad (4.61)$$

$$= \langle \rangle_{I'} \quad (4.62)$$

$$= \llbracket \Phi' \vdash \diamond \text{ Ok} \rrbracket \quad (4.63)$$

$$(4.64)$$

**Case Var:**

$$\omega^* \llbracket \Phi \vdash \Gamma, x : A \text{ Ok} \rrbracket = \omega^* (\square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket)) \quad (4.65)$$

$$= \square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi \vdash A : \text{Type} \rrbracket) \circ \omega \quad (4.66)$$

$$= \square(\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket \circ \omega, \llbracket \Phi \vdash A : \text{Type} \rrbracket \circ \omega) \quad (4.67)$$

$$= \square(\llbracket \Phi' \vdash \Gamma \text{ Ok} \rrbracket, \llbracket \Phi' \vdash A : \text{Type} \rrbracket) \quad (4.68)$$

$$= \llbracket \Phi' \vdash (\Gamma, x : A) \text{ Ok} \rrbracket \quad (4.69)$$

$$(4.70)$$



## 4.5 Terms

If

$$\omega = \llbracket \omega : \Phi' \triangleright \Phi \rrbracket \quad (4.71)$$

$$\Delta = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket \quad (4.72)$$

$$\Delta' = \llbracket \Phi' \mid \Gamma \vdash v : A \rrbracket \quad (4.73)$$

$$(4.74)$$

Then

$$\Delta' = \omega^*(\Delta) \quad (4.75)$$

**Proof:** By induction over the derivation of  $\Delta$ . Using the S-Closure of  $\omega^*$ . We use  $\Gamma_I$  to indicate  $\llbracket \Phi \vdash \Gamma \text{ Ok} \rrbracket$ , an  $A_I$  to indicate  $\llbracket \Phi \vdash A : \text{Type} \rrbracket$

**Case Unit:**

$$\Delta = \langle \rangle_{\Gamma_I} \quad (4.76)$$

So

$$\omega^*(\Delta) = \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.77)$$

**Case True, False:** Giving the case for true as false is the same but using **inr**

$$\Delta = \text{inl} \circ \langle \rangle_{\Gamma_I} \quad (4.78)$$

So

$$\omega^*(\Delta) = \text{inl} \circ \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.79)$$

Since  $\omega^*$  is S-closed.

**Case Constant:**

$$\Delta = \llbracket C^A \rrbracket \circ \langle \rangle_{\Gamma_I} \quad (4.80)$$

So

$$\omega^*(\Delta) = \omega^* \llbracket C^A \rrbracket \circ \langle \rangle_{\Gamma_{I'}} = \llbracket C^{A_{I'}} \rrbracket \circ \langle \rangle_{\Gamma_{I'}} = \Delta' \quad (4.81)$$

Since  $\omega^*$  is S-closed.

**Case Subtype:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket \quad (4.82)$$

Then

$$\Delta = \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta_1 \quad (4.83)$$

So

$$\omega^*(\Delta) = \omega^*[\![A \leq_{:\Phi} B]\!] \circ \omega^* \Delta_1 \quad (4.84)$$

$$= [\![A_{I'} \leq_{:\Phi'} B_{I'}]\!] \circ \Delta'_1 \quad \text{By induction} \quad (4.85)$$

$$= D' \quad (4.86)$$

**Case Lambda:** Let

$$\Delta_1 = [\![\Phi \mid \Gamma, x:A \vdash v:B]\!] \quad (4.87)$$

Then

$$\Delta = \text{cur}(\Delta_1) \quad (4.88)$$

So

$$\omega^*(\Delta) = \omega^*(\text{cur}(\Delta_1)) \quad (4.89)$$

$$= \text{cur}(\omega^*(\Delta_1)) \quad \text{By S-closure} \quad (4.90)$$

$$= \text{cur}(\Delta'_1) \quad \text{By induction} \quad (4.91)$$

$$= \Delta' \quad (4.92)$$

**Case Application:** Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v_1:A \rightarrow B]\!] \quad (4.93)$$

$$\Delta_2 = [\![\Phi \mid \Gamma \vdash v_2:A]\!] \quad (4.94)$$

Then

$$\Delta = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad (4.95)$$

So

$$\omega^* \Delta = \omega^*(\text{app} \circ \langle \Delta_1, \Delta_2 \rangle) \quad (4.96)$$

$$= \text{app} \circ \langle \omega^*(\Delta_1), \omega^*(\Delta_2) \rangle \quad \text{By S-closure} \quad (4.97)$$

$$= \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Induction} \quad (4.98)$$

$$= \Delta' \quad (4.99)$$

**Case Return:** Let

$$\Delta_1 = [\![\Phi \mid \Gamma \vdash v:A]\!] \quad (4.100)$$

Then

$$\Delta = \eta_{A_I} \circ \Delta_1 \quad (4.101)$$

So

$$\omega^*(\Delta) = \omega^*(\eta_{A_I} \circ \Delta_1) \quad (4.102)$$

$$= \eta_{A_{I'}} \circ \omega^*(\Delta_1) \quad \text{By S-closure} \quad (4.103)$$

$$= \eta_{A_{I'}} \circ \Delta'_1 \quad (4.104)$$

$$= \Delta' \quad (4.105)$$

**Case Bind:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \rrbracket \quad (4.106)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \rrbracket \quad (4.107)$$

Then

$$\Delta = \mathbf{M}_{\epsilon_1} \epsilon_2 A_I \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma_I, A_I} \circ \langle \text{Id}_{\Gamma_I}, \Delta_1 \rangle \quad (4.108)$$

So

$$\omega^*(\Delta) = \omega^*(\mu_{\epsilon_1, \epsilon_2, A} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, \Delta_1 \rangle) \quad (4.109)$$

$$= \omega^*(\mu_{\epsilon_1, \epsilon_2, A}) \circ \omega^*(T_{\epsilon_1} \Delta_2) \circ \omega^*(\mathbf{t}_{\epsilon_1, \Gamma, A}) \circ \langle \omega^*(\text{Id}_{\Gamma_I}), \omega^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (4.110)$$

$$= \mu_{\epsilon_1, \epsilon_2, A_{I'}} \circ T_{\epsilon_1} \omega^*(\Delta_2) \circ \mathbf{t}_{\epsilon_1, \Gamma_{I'}, A_{I'}} \circ \langle \omega^*(\text{Id}_{\Gamma_I}), \omega^*(\Delta_1) \rangle \quad \text{By S-Closure} \quad (4.111)$$

$$= \mu_{\epsilon_1, \epsilon_2, A_{I'}} \circ T_{\epsilon_1} \Delta'_2 \circ \mathbf{t}_{\epsilon_1, \Gamma_{I'}, A_{I'}} \circ \langle \omega^*(\text{Id}_{\Gamma_I}), \Delta'_1 \rangle \quad \text{By Induction} \quad (4.112)$$

$$= \Delta' \quad (4.113)$$

$$(4.114)$$

**Case If:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \mathbf{Bool} \rrbracket \quad (4.115)$$

$$\Delta_2 = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket \quad (4.116)$$

$$\Delta_3 = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket \quad (4.117)$$

$$(4.118)$$

Then

$$\Delta = \mathbf{app} \circ (([\mathbf{cur}(\Delta_2 \circ \pi_2), \mathbf{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma} \quad (4.119)$$

So

$$\omega^*(\Delta) = \omega^*(\mathbf{app} \circ (([\mathbf{cur}(\Delta_2 \circ \pi_2), \mathbf{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma}) \circ \delta_{\Gamma}) \quad (4.120)$$

$$= \mathbf{app} \circ (([\mathbf{cur}(\omega^*(\Delta_2) \circ \pi_2), \mathbf{cur}(\omega^*(\Delta_3) \circ \pi_2)] \circ \omega^*(\Delta_1)) \times \text{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By S-Closure} \quad (4.121)$$

$$= \mathbf{app} \circ (([\mathbf{cur}(\Delta'_2 \circ \pi_2), \mathbf{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma_{I'}}) \circ \delta_{\Gamma_{I'}} \quad \text{By Induction} \quad (4.122)$$

$$= \Delta' \quad (4.123)$$

$$(4.124)$$

**Case Effect-Lambda:** Let

$$\Delta_1 = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket \quad (4.125)$$

Then

$$\Delta = \overline{\Delta_1} \quad (4.126)$$

And also

$$\omega \times \text{Id} = \llbracket \omega \times : (\Phi', \alpha) \triangleright (\Phi, \alpha) \rrbracket \quad (4.127)$$

So

$$\omega^* \Delta = \omega^* (\overline{\Delta_1}) \quad (4.128)$$

$$= \overline{(\omega \times \text{Id}_U)^* \Delta_1} \quad \text{By naturality} \quad (4.129)$$

$$= \overline{\Delta'_1} \quad \text{By induction} \quad (4.130)$$

$$= \Delta' \quad (4.131)$$

**Case Effect-Application:** Let

$$\Delta_1 = \llbracket \Phi \mid \Gamma \vdash v : \forall \alpha. A \rrbracket \quad (4.132)$$

$$h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \quad (4.133)$$

$$(4.134)$$

Then

$$\Delta = \langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta_1 \quad (4.135)$$

So due to the substitution theorem on effects

$$h \circ \omega = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \circ \omega = \llbracket \Phi' \vdash \epsilon : \text{Effect} \rrbracket = h' \quad (4.136)$$

Also note  $(\omega \times \text{Id}_U) = \llbracket \omega \times : \Phi', \alpha \triangleright \Phi \alpha \rrbracket$

$$\omega^* \Delta = \omega^* (\langle \text{Id}_\Gamma, h \rangle^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta_1) \quad (4.137)$$

$$= (\langle \text{Id}_\Gamma, h \rangle \circ \omega)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \omega^* (\Delta_1) \quad (4.138)$$

$$= ((\omega \times \text{Id}_U) \circ \langle \text{Id}_\Gamma, h \circ \omega \rangle)^* (\epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta'_1 \quad (4.139)$$

$$= (\langle \text{Id}_\Gamma, h' \rangle)^* ((\omega \times \text{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \Delta'_1 \quad (4.140)$$

$$(4.141)$$

Looking at the inner part of the functor application: Let

$$A = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket \quad (4.142)$$

$$(4.143)$$

$$(\omega \times \text{Id}_U)^* \epsilon_{\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket} = (\omega \times \text{Id}_U)^* \epsilon_A \quad (4.144)$$

$$= (\omega \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}}) \quad (4.145)$$

$$= \overline{(\omega \times \text{Id}_U)^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By bijection} \quad (4.146)$$

$$= \overline{\omega^* (\widehat{\text{Id}_{\forall_I(A)}})} \quad \text{By naturality} \quad (4.147)$$

$$= \overline{\omega^* (\text{Id}_{\forall_I(A)})} \quad \text{By bijection} \quad (4.148)$$

$$= \overline{\text{Id}_{\forall_{I'}(A \circ (\omega \times \text{Id}_U))}} \quad \text{By S-Closure, naturality} \quad (4.149)$$

$$= \overline{\text{Id}_{\forall_{I'}(A)}} \quad \text{By Substitution theorem} \quad (4.150)$$

$$= \epsilon_{A_{I'}} \quad (4.151)$$

Going back to the original expression:

$$\omega^* \Delta = (\langle \text{Id}_\Gamma, h' \rangle)^* (\epsilon_{A_{I'}}) \circ \Delta'_1 \quad (4.152)$$

$$= \Delta' \quad (4.153)$$

$$(4.154)$$

## 4.6 Term-Substitution

If  $\omega = \llbracket \omega: \Phi' \triangleright \Phi \rrbracket$ , then  $\llbracket \Phi' \mid \Gamma' \vdash \sigma: \Gamma \rrbracket = \omega^* \llbracket \Phi \mid \Gamma' \vdash \sigma: \Gamma \rrbracket$ .

**Proof:** By induction on the structure of  $\sigma$ , making use of the weakening of term denotations above.

**Case Nil:** Then  $\sigma = \langle \rangle_{\Gamma'_I}$ , so  $\omega^*(\sigma) = \langle \rangle_{\Gamma'_I} = \llbracket \Phi' \mid \Gamma' \vdash \sigma: \Gamma \rrbracket$

**Case Var:** Then  $\sigma = (\sigma', x: = v)$

$$\omega^* \sigma = \omega * \langle \sigma', \llbracket \Phi \mid \Gamma \vdash v: A \rrbracket \rangle \quad (4.155)$$

$$= \langle \omega^* \sigma', \omega^* \llbracket \Phi \mid \Gamma \vdash v: A \rrbracket \rangle \quad (4.156)$$

$$= \langle \llbracket \Phi' \mid \Gamma' \vdash \sigma': \Gamma \rrbracket, \llbracket \Gamma' \mid \Phi' \vdash v: A \rrbracket \rangle \quad (4.157)$$

$$= \llbracket \Phi' \mid \Gamma' \vdash \sigma: \Gamma, x: A \rrbracket \quad (4.158)$$

## 4.7 Term-Weakening

If  $\omega = \llbracket \omega: \Phi' \triangleright \Phi \rrbracket$ , then  $\llbracket \Phi' \vdash \omega_1: \Gamma' \triangleright \Gamma \rrbracket = \omega^* \llbracket \Phi \vdash \omega_1: \Gamma' \triangleright \Gamma \rrbracket$ .

**Proof:** By induction on the structure of  $\omega_1$ .

**Case Id:** Then  $\omega_1 = \iota$ , so its denotation is  $\omega_1 = \text{Id}_{\Gamma_I}$

So

$$\omega^*(\text{Id}_{\Gamma_I}) = \text{Id}_{\Gamma_{I'}} = \llbracket \Phi' \vdash \iota: \Gamma \triangleright \Gamma \rrbracket \quad (4.159)$$

**Case Project:** Then  $\omega_1 = \omega'_1 \pi$

$$(\text{Project}) \frac{\Phi \vdash \omega'_1: \Gamma' \triangleright \Gamma}{\Phi \vdash \omega_1 \pi: \Gamma', x: A \triangleright \Gamma} \quad (4.160)$$

So  $\omega_1 = \omega'_1 \circ \pi_1$

Hence

$$\omega^*(\omega_1) = \omega^*(\omega'_1) \circ \omega^*(\pi_1) \quad (4.161)$$

$$= \llbracket \Phi' \vdash \omega'_1: \Gamma' \triangleright \Gamma \rrbracket \circ \pi_1 \quad (4.162)$$

$$= \llbracket \Phi' \vdash \omega'_1 \pi: \Gamma', x: A \triangleright \Gamma \rrbracket \quad (4.163)$$

$$= \llbracket \Phi' \vdash \omega_1: \Gamma', x: A \triangleright \Gamma \rrbracket \quad (4.164)$$

**Case Extend:** Then  $\omega_1 = \omega'_1 \times$

$$(\text{Extend}) \frac{\textcolor{green}{\Phi} \vdash \omega'_1 : \textcolor{blue}{\Gamma'} \triangleright \textcolor{blue}{\Gamma} \quad A \leq_{:\Phi} B}{\textcolor{green}{\Phi} \vdash \omega_1 \times : \textcolor{blue}{\Gamma'}, x : A \triangleright \textcolor{blue}{\Gamma}, x : B} \quad (4.165)$$

So  $\omega_1 = \omega'_1 \times \llbracket A \leq_{:\Phi} B \rrbracket$

Hence

$$\omega^*(\omega_1) = (\omega^*(\omega'_1) \times \omega^*(\llbracket A \leq_{:\Phi} B \rrbracket)) \quad (4.166)$$

$$= (\llbracket \textcolor{green}{\Phi'} \vdash \omega'_1 : \textcolor{blue}{\Gamma'} \triangleright \textcolor{blue}{\Gamma} \rrbracket \times \llbracket A \leq_{:\Phi'} B \rrbracket) \quad (4.167)$$

$$= \llbracket \textcolor{green}{\Phi'} \vdash \omega_1 : \textcolor{blue}{\Gamma'}, x : A \triangleright \textcolor{blue}{\Gamma}, x : B \rrbracket \quad (4.168)$$

## Chapter 5

# Value Substitution Theorem

If  $\Delta$  derives  $\Phi \mid \Gamma \vdash v : A$  and  $\Phi \mid \Gamma' \vdash \sigma : \Gamma$  then the derivation  $\Delta'$  deriving  $\Phi \mid \Gamma' \vdash v[\sigma] : A$  satisfies:

$$\Delta' = \Delta \circ \llbracket \Phi \mid \Gamma' \vdash \sigma : \Gamma \rrbracket \quad (5.1)$$

This is proved by induction over the derivation of  $\Phi \mid \Gamma \vdash v : A$ . We shall use  $\sigma$  to denote  $\llbracket \Phi \mid \Gamma' \vdash \sigma : \Gamma \rrbracket$  where it is clear from the context.

**Case Var:** By inversion  $\Gamma = \Gamma'', x : A$

$$(\text{Var}) \frac{\Phi \vdash \Gamma \text{ Ok}}{\Phi \mid \Gamma'', x : A \vdash x : A} \quad (5.2)$$

By inversion,  $\sigma = \sigma', x := v$  and  $\Phi \mid \Gamma' \vdash v : A$ .

Let

$$\sigma = \llbracket \Phi \mid \Gamma' \vdash \sigma : \Gamma \rrbracket = \langle \sigma', \Delta' \rangle \quad (5.3)$$

$$\Delta = \llbracket \Phi \mid \Gamma'', x : A \vdash x : A \rrbracket = \pi_2 \quad (5.4)$$

$$(5.5)$$

$$\Delta \circ \sigma = \pi_2 \circ \langle \sigma', \Delta' \rangle \quad \text{By definition} \quad (5.6)$$

$$= \Delta' \quad \text{By product property} \quad (5.7)$$

**Case Weaken:** By inversion,  $\Gamma = \Gamma', y : B$  and  $\sigma = \sigma', y := v$  and we have  $\Delta_1$  deriving:

$$(\text{Weaken}) \frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x : A} \quad (5.8)$$

Also by inversion of the well-formed-ness of  $\Phi \mid \Gamma' \vdash \sigma : \Gamma$ , we have  $\Phi \mid \Gamma' \vdash \sigma' : \Gamma''$  and

$$\llbracket \Phi \mid \Gamma' \vdash \sigma : \Gamma \rrbracket = \langle \llbracket \Phi \mid \Gamma' \vdash \sigma' : \Gamma'' \rrbracket, \llbracket \Phi \mid \Gamma' \vdash v : B \rrbracket \rangle \quad (5.9)$$

Hence by induction on  $\Delta_1$  we have  $\Delta'_1$  such that

$$\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash x[\sigma] : A} \quad (5.10)$$

Hence

$$\Delta' = \Delta'_1 \quad \text{By definition} \quad (5.11)$$

$$= \Delta_1 \circ \sigma' \quad \text{By induction} \quad (5.12)$$

$$= \Delta_1 \circ \pi_1 \circ \langle \sigma', \llbracket \Phi \mid \Gamma' \vdash v : B \rrbracket \rangle \quad \text{By product property} \quad (5.13)$$

$$= \Delta_1 \circ \pi_1 \circ \sigma \quad \text{By definition of the denotation of } \sigma \quad (5.14)$$

$$= \Delta \circ \sigma \quad \text{By definition.} \quad (5.15)$$

**Case Constants:** The logic for all constant terms (**true**, **false**,  $()$ ,  $\mathsf{C}^A$ ) is the same. Let

$$c = \llbracket \mathsf{C}^A \rrbracket \quad (5.16)$$

$$\Delta' = c \circ \langle \rangle_{\Gamma'} \quad \text{By Definition} \quad (5.17)$$

$$= c \circ \langle \rangle_G \circ \sigma \quad \text{Terminal property} \quad (5.18)$$

$$= \Delta \circ \sigma \quad \text{By definition} \quad (5.19)$$

**Case Lambda:** By inversion, we have  $\Delta_1$  such that

$$\Delta = (\text{Fn}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A. v : A \rightarrow B} \quad (5.20)$$

By induction of  $\Delta_1$  we have  $\Delta'_1$  such that

$$\Delta' = (\text{Fn}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma', x : A \vdash (v[\sigma]) : B}}{\Phi \mid \Gamma \vdash (\lambda x : A. v) [\sigma] : A \rightarrow B} \quad (5.21)$$

By induction and the extension lemma, we have:

$$\Delta'_1 = \Delta_1 \circ (\sigma \times \text{Id}_A) \quad (5.22)$$

Hence:

$$\Delta' = \text{cur}(\Delta'_1) \quad \text{By definition} \quad (5.23)$$

$$= \text{cur}(\Delta_1 \circ (\sigma \times \text{Id}_A)) \quad \text{By induction and extension lemma.} \quad (5.24)$$

$$= \text{cur}(\Delta_1) \circ \sigma \quad \text{By the exponential property (Uniqueness)} \quad (5.25)$$

$$= \Delta \circ \sigma \quad \text{By Definition} \quad (5.26)$$

$$(5.27)$$



**Case Sub-type:** By inversion, there exists derivation  $\Delta_1$  such that:

$$\Delta = (\text{Sub-type}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : A} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v : B} \quad (5.28)$$

By induction on  $\Delta_1$ , we find  $\Delta'_1$  such that  $\Delta'_1 = \Delta_1 \circ \sigma$  and:

$$\Delta' = (\text{Sub-type}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v[\sigma] : A} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma' \vdash v[\sigma] : B} \quad (5.29)$$

Hence,

$$\Delta' = \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta'_1 \quad \text{By definition} \quad (5.30)$$

$$= \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta_1 \circ \sigma \quad \text{By induction} \quad (5.31)$$

$$= \Delta \circ \sigma \quad \text{By definition} \quad (5.32)$$

$$(5.33)$$

**Case Return:** By inversion, we have  $\Delta_1$  such that:

$$\Delta = (\text{Return}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A} \quad (5.34)$$

By induction on  $\Delta_1$ , we find  $\Delta'_1$  such that  $\Delta'_1 = \Delta_1 \circ \sigma$  and:

$$\Delta' = (\text{Return}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v[\sigma] : A}}{\Phi \mid \Gamma' \vdash (\text{return } v)[\sigma] : \mathbf{M}_1 A} \quad (5.35)$$

Hence,

$$\Delta' = \eta_A \circ \Delta'_1 \quad \text{By Definition} \quad (5.36)$$

$$= \eta_A \circ \Delta_1 \circ \sigma \quad \text{By induction} \quad (5.37)$$

$$= \Delta \circ \sigma \quad \text{By Definition} \quad (5.38)$$

$$(5.39)$$

**Case Apply:** By inversion, we find  $\Delta_1, \Delta_2$  such that

$$\Delta = (\text{Apply}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1 v_2 : B} \quad (5.40)$$

By induction we find  $\Delta'_1, \Delta'_2$  such that

$$\Delta'_1 = \Delta_1 \circ \sigma \quad (5.41)$$

$$\Delta'_2 = \Delta_2 \circ \sigma \quad (5.42)$$

$$(5.43)$$

And

$$\Delta' = (\text{Apply}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1 [\sigma] : A \rightarrow B} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_2 [\sigma] : A}}{\Phi \mid \Gamma' \vdash (v_1 v_2) [\sigma] : B} \quad (5.44)$$

Hence

$$\Delta' = \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \quad (5.45)$$

$$= \text{app} \circ \langle \Delta_1 \circ \sigma, \Delta_2 \circ \sigma \rangle \quad \text{By induction} \quad (5.46)$$

$$= \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \sigma \quad \text{By Product Property} \quad (5.47)$$

$$= \Delta \circ \sigma \quad \text{By Definition} \quad (5.48)$$

$$(5.49)$$

**Case If:** By inversion, we find  $\Delta_1, \Delta_2, \Delta_3$  such that

$$\Delta = (\text{If}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \text{Bool}} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_1 : A} \quad \frac{\Delta_3}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A} \quad (5.50)$$

By induction we find  $\Delta'_1, \Delta'_2, \Delta'_3$  such that

$$\Delta'_1 = \Delta_1 \circ \sigma \quad (5.51)$$

$$\Delta'_2 = \Delta_2 \circ \sigma \quad (5.52)$$

$$\Delta'_3 = \Delta_3 \circ \sigma \quad (5.53)$$

$$(5.54)$$

And

$$\Delta' = (\text{If}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v [\sigma] : \text{Bool}} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_1 [\sigma] : A} \quad \frac{\Delta'_3}{\Phi \mid \Gamma' \vdash v_2 [\sigma] : A}}{\Phi \mid \Gamma' \vdash (\text{if}_A v \text{ then } v_1 \text{ else } v_2) [\sigma] : A} \quad (5.55)$$

Since  $\sigma : \Gamma' \rightarrow \Gamma$ ,

Let  $(T_\epsilon A)^\sigma : T_\epsilon A^\Gamma \rightarrow T_\epsilon A^{\Gamma'}$  be as defined in ExSh 3 <sup>(1)</sup> That is:

$$(T_\epsilon A)^\sigma = \text{cur}(\text{app} \circ (\text{Id}_{T_\epsilon A} \times w)) \quad (5.56)$$

. And hence, we have:

$$\text{cur}(f \circ (\text{Id} \times \sigma)) = (T_\epsilon A)^\sigma \circ \text{cur}(f) \quad (5.57)$$

And so:

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<sup>1</sup><https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf>

$$\Delta' = \text{app} \circ (([\text{cur}(\Delta'_2 \circ \pi_2), \text{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} \quad (5.58)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \sigma \circ \pi_2), \text{cur}(\Delta_3 \circ \sigma \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} \quad (5.59)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2 \circ (\text{Id}_1 \times \sigma)), \text{cur}(\Delta_3 \circ \pi_2 \circ (\text{Id}_1 \times \sigma))] \circ \Delta_1 \circ \sigma) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} \quad (5.60)$$

$$= \text{app} \circ (((T_\epsilon A)^\sigma \circ \text{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\sigma \circ \text{cur}(\Delta_3 \circ \pi_2))] \circ \Delta_1 \circ \sigma) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\sigma \text{ property} \quad (5.61)$$

$$= \text{app} \circ (((T_\epsilon A)^\sigma \circ [\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \sigma) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} \quad (5.62)$$

$$= \text{app} \circ ((T_\epsilon A)^\sigma \times \text{Id}_{\Gamma'}) \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\sigma \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} \quad (5.63)$$

$$= \text{app} \circ (\text{Id}_{(T_\epsilon A)} \times \sigma) \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\sigma \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By definition of } \text{app}, (T_\epsilon A)^\sigma \quad (5.64)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\sigma \times \sigma) \circ \delta_{\Gamma'} \quad \text{Push through pairs} \quad (5.65)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \circ \sigma \quad \text{By Definition of the diagonal morphism.} \quad (5.66)$$

$$= \Delta \circ \sigma \quad (5.67)$$

**Case Bind:** By inversion, we have  $\Delta_1, \Delta_2$  such that:

$$\Delta = (\text{Bind}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A} \quad \frac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_1 : B}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1, \epsilon_2} B} \quad (5.68)$$

By property 3,

$$\Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A)) \quad (5.69)$$

With denotation (extension lemma)

$$\llbracket \Phi \mid (\Gamma', x : A) \vdash (\sigma, x := x : (\Gamma, x : A)) \rrbracket = \sigma \times \text{Id}_A \quad (5.70)$$

By induction, we derive  $\Delta'_1, \Delta'_2$  such that:

$$\Delta'_1 = \Delta_1 \circ \sigma \quad (5.71)$$

$$\Delta'_2 = \Delta_2 \circ (\sigma \times \text{Id}_A) \quad \text{By Extension Lemma} \quad (5.72)$$

And:

$$\Delta' = (\text{Bind}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1 [\sigma] : A} \quad \frac{\Delta'_2}{\Phi \mid \Gamma', x : A \vdash v_1 [\sigma] : B}}{\Phi \mid \Gamma' \vdash (\text{do } x \leftarrow v_1 \text{ in } v_2) [\sigma] : \mathbf{M}_{\epsilon_1, \epsilon_2} B} \quad (5.73)$$

Hence:

$$\Delta' = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathfrak{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By Definition} \quad (5.74)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2 \circ (\sigma \times \text{Id}_A)) \circ \mathfrak{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Induction using the extension lemma} \quad (5.75)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathfrak{t}_{\epsilon_1, \Gamma, A} \circ (\sigma \times \text{Id}_{T_{\epsilon_1} A}) \circ \langle \text{Id}_{\Gamma'}, \Delta_1 \circ \sigma \rangle \quad \text{By Tensor Strength} \quad (5.76)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathfrak{t}_{\epsilon_1, \Gamma, A} \circ \langle \sigma, \Delta_1 \circ \sigma \rangle \quad \text{By Product rule} \quad (5.77)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2) \circ \mathfrak{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, \Delta_1 \rangle \circ \sigma \quad \text{By Product rule} \quad (5.78)$$

$$= \Delta \circ \sigma \quad \text{By Definition} \quad (5.79)$$

$$(5.80)$$

**Case Effect-Lambda:** By inversion, we have  $\Delta_1$  such that

$$\Delta = (\text{Effect-Fn}) \frac{\frac{\Delta_1}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda \alpha.v : \forall \epsilon. A} \quad (5.81)$$

By induction, we derive  $\Delta'_1$  such that

$$\Delta' = (\text{Effect-Fn}) \frac{\frac{\Delta'_1}{\Phi, \alpha \mid \Gamma' \vdash v [\sigma] : A}}{\Phi \mid \Gamma' \vdash (\Lambda \alpha.v) [\sigma] : \forall \epsilon. A} \quad (5.82)$$

Where

$$\Delta'_1 = \Delta_1 \circ \llbracket \Phi, \alpha \mid \Gamma' \vdash \sigma : \Gamma \rrbracket \quad (5.83)$$

$$= \Delta_1 \circ \llbracket \iota \pi : \Phi, a \triangleright \Phi \rrbracket^* (\sigma) \quad (5.84)$$

$$= \Delta_1 \circ \pi_1^* (\sigma) \quad (5.85)$$

Hence

$$\Delta \circ \sigma = \overline{\Delta_1} \circ \sigma \quad (5.86)$$

$$= \overline{\Delta_1 \circ \pi_1^* (\sigma)} \quad (5.87)$$

$$= \overline{\Delta'_1} \quad (5.88)$$

$$= \Delta' \quad (5.89)$$

**Case Effect-Application:** By inversion, we derive  $\Delta_1$  such that

$$\Delta = (\text{Effect-App}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v \epsilon : A [\epsilon/\alpha]} \quad (5.90)$$

By induction, we derive  $\Delta'_1$  such that

$$\Delta' = (\text{Effect-App}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v [\sigma] : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash (v \epsilon) [\sigma] : A [\epsilon/\alpha]} \quad (5.91)$$

Where

$$\Delta'_1 = \Delta \circ \sigma \quad (5.92)$$

Hence, if  $h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket$

$$\Delta \circ \sigma = \langle \text{Id}_I, h \rangle^* (\epsilon \llbracket \Phi, \beta \vdash A[\alpha/\beta] : \text{Effect} \rrbracket) \circ \Delta_1 \circ \sigma \quad (5.93)$$

$$= \langle \text{Id}_I, h \rangle^* (\epsilon \llbracket \Phi, \beta \vdash A[\alpha/\beta] : \text{Effect} \rrbracket) \circ \Delta'_1 \quad (5.94)$$

$$= \Delta' \quad (5.95)$$

## Chapter 6

# Type-Environment Weakening Theorem

If  $w = \llbracket \Phi \vdash \omega : \Gamma' \triangleright G \rrbracket$  and  $\Delta = \llbracket \Phi \mid \Gamma \vdash v : A \rrbracket$  then there exists  $\Delta' = \llbracket \Phi \mid \Gamma' \vdash v : A \rrbracket$  such that  $\Delta' = \Delta \circ w$

**Proof:** We induct over the structure of typing derivations of  $\Phi \mid \Gamma \vdash v : A$ , assuming  $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$  holds. In each case, we construct the new derivation  $\Delta'$  from the derivation  $\Delta$  giving  $\Phi \mid \Gamma \vdash v : A$  and show that  $\Delta \circ \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket = \Delta'$

**Case Var and Weaken:** We case split on the weakening  $\omega$ .

**If  $\omega = \iota$**  Then  $\Gamma' = \Gamma$ , and so  $\Phi \mid \Gamma' \vdash x : A$  holds and the derivation  $\Delta'$  is the same as  $\Delta$

$$\Delta' = \Delta = \Delta \circ \text{Id}_\Gamma = \Delta \circ \llbracket \Phi \vdash \iota : \Gamma \triangleright \Gamma \rrbracket \quad (6.1)$$

**If  $\omega = \omega' \pi$**  Then  $\Gamma' = (\Gamma'', x' : A')$  and  $\Phi \vdash \omega' : \Gamma'' \triangleright \Gamma$ . So by induction, there is a tree,  $\Delta_1$  deriving  $\Phi \mid \Gamma'' \vdash x : A$ , such that

$$\Delta_1 = \Delta \circ \llbracket \Phi \vdash \omega' : \Gamma'' \triangleright \Gamma \rrbracket \quad \text{By Induction} \quad (6.2)$$

, and hence by the weaken rule, we have

$$\text{(Weaken)} \frac{\Phi \mid \Gamma'' \vdash x : A}{\Phi \mid \Gamma'', x' : A' \vdash x : A} \quad (6.3)$$

This preserves denotations:

$$\Delta' = \Delta_1 \circ \pi_1 \quad \text{By Definition} \quad (6.4)$$

$$= \Delta \circ \llbracket \Phi \vdash \omega' : \Gamma'' \triangleright \Gamma \rrbracket \circ \pi_1 \quad \text{By induction} \quad (6.5)$$

$$= \Delta \circ \llbracket \Phi \vdash \omega' \pi_1 : \Gamma' \triangleright \Gamma \rrbracket \quad \text{By denotation of weakening} \quad (6.6)$$

**If  $\omega = \omega' \times$**  Then

$$\Gamma' = \Gamma''', x' : B \quad (6.7)$$

$$\Gamma = \Gamma'', x' : A' \quad (6.8)$$

$$B \leq_\Phi A \quad (6.9)$$

**If**  $x = x'$  Then  $A = A'$ .

Then we derive the new derivation,  $\Delta'$  as so:

$$\text{(Sub-type)} \frac{(\text{var}) \frac{\Phi \vdash \Gamma \text{ Ok}}{\Phi \mid \Gamma''', x : B \vdash x : B} \quad B \leq_{\Phi} A}{\Phi \mid \Gamma' \vdash x : A} \quad (6.10)$$

This preserves denotations:

$$\Delta' = \llbracket B \leq_{\Phi} A \rrbracket \circ \pi_2 \quad \text{By Definition} \quad (6.11)$$

$$= \pi_2 \circ (\llbracket \Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket \times \llbracket B \leq_{\Phi} A \rrbracket) \quad \text{By the properties of binary products} \quad (6.12)$$

$$= \Delta \circ \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \quad \text{By Definition} \quad (6.13)$$

**Case**  $x \neq x'$  Then

$$\Delta = (\text{Weaken}) \frac{\Delta_1}{\Phi \mid \Gamma'' \vdash x : A} \quad (6.14)$$

By induction with  $\Phi \vdash \omega : \Gamma''' \triangleright \Gamma''$ , we have a derivation  $\Delta_1$  of  $\Phi \mid \Gamma''' \vdash x : A$

We have the weakened derivation:

$$\Delta' = (\text{Weaken}) \frac{\Delta'_1}{\Phi \mid \Gamma' \vdash x : A} \quad (6.15)$$

This preserves denotations:

By induction, we have

$$\Delta'_1 = \Delta_1 \circ \llbracket \Phi \vdash \omega : \Gamma''' \triangleright \Gamma'' \rrbracket \quad (6.16)$$

So we have:

$$\Delta' = \Delta'_1 \circ \pi_1 \quad \text{By denotation definition} \quad (6.17)$$

$$= \Delta_1 \circ \llbracket \Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket \quad \text{By induction} \circ \pi_1 \quad (6.18)$$

$$= \Delta_1 \circ \pi_1 \circ (\llbracket \Phi \vdash \omega' : \Gamma''' \triangleright \Gamma'' \rrbracket \times \llbracket A' \leq_{\Phi} B \rrbracket) \quad \text{By product properties} \quad (6.19)$$

$$= \Delta \circ \llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \quad \text{By definition} \quad (6.20)$$

From this point onwards, since we no-longer case split over the weakening relations, we write the denotation  $\llbracket \Phi \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket$ , simply as  $\omega$ .

**Case Constant:** The constant typing rules,  $()$ , **true**, **false**,  $\mathsf{C}^A$ , all proceed by the same logic. Hence I shall only prove the theorems for the case  $\mathsf{C}^A$ .

$$(\text{Const}) \frac{\Phi \vdash \Gamma \text{ Ok}}{\Phi \mid \Gamma \vdash \mathsf{C}^A : A} \quad (6.21)$$

By inversion, we have  $\Phi \vdash \Gamma \text{ Ok}$ , so we have  $\Phi \vdash \Gamma' \text{ Ok}$ .

Hence

$$(\text{Const}) \frac{\Phi \vdash \Gamma' \text{ Ok}}{\Phi \mid \Gamma' \vdash \mathbf{c}^A : A} \quad (6.22)$$

Holds.

This preserves denotations:

$$\Delta' = \llbracket \mathbf{c}^A \rrbracket \circ \langle \rangle_{\Gamma'} \quad \text{By definition} \quad (6.23)$$

$$= \llbracket \mathbf{c}^A \rrbracket \circ \langle \rangle_{\Gamma} \circ \omega \quad \text{By the terminal property} \quad (6.24)$$

$$= \Delta \quad \text{By Definition} \quad (6.25)$$

$$(6.26)$$

**Case Lambda:** By inversion, we have a derivation  $\Delta_1$  giving

$$\Delta = (\text{Fn}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma, x : A \vdash v : B}}{\Phi \mid \Gamma \vdash \lambda x : A. v : A \rightarrow B} \quad (6.27)$$

Since  $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ , we have:

$$\Phi \vdash \omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A) \quad (6.28)$$

Hence, by induction, using  $\Phi \vdash \omega \times : (\Gamma, x : A) \triangleright (\Gamma, x : A)$ , we derive  $\Delta'_1$ :

$$\Delta' = (\text{Fn}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma', x : A \vdash v : B}}{\Phi \mid \Gamma', x : A \vdash \lambda x : A. v : A \rightarrow B} \quad (6.29)$$

This preserves denotations:

$$\Delta' = \text{cur}(\Delta'_1) \quad \text{By Definition} \quad (6.30)$$

$$= \text{cur}(\Delta_1 \circ (\omega \times \text{Id}_{\Gamma})) \quad \text{By the denotation of } \omega \times \quad (6.31)$$

$$= \text{cur}(\Delta_1) \circ \omega \quad \text{By the exponential property} \quad (6.32)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (6.33)$$

**Case Sub-typing:**

$$(\text{Sub-type}) \frac{\Phi \mid \Gamma \vdash v : A \quad A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v : B} \quad (6.34)$$

by inversion, we have a derivation  $\Delta_1$

$$\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : A} \quad (6.35)$$

So by induction, we have a derivation  $\Delta'_1$  such that:

$$(\text{Sub-type}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v : a} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma' \vdash v : B} \quad (6.36)$$



This preserves denotations:

$$\Delta' = \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta'_1 \quad \text{By Definition} \quad (6.37)$$

$$= \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta_1 \circ \omega \quad \text{By induction} \quad (6.38)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (6.39)$$

$$(6.40)$$

**Case Return:** We have the sub-derivation  $\Delta_1$  such that

$$\Delta = (\text{Return}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A} \quad (6.41)$$

Hence, by induction, with  $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$ , we find the derivation  $\Delta'_1$  such that:

$$\Delta' = (\text{Return}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v : A}}{\Phi \mid \Gamma' \vdash \text{return } v : \mathbf{M}_1 A} \quad (6.42)$$

This preserves denotations:

$$\Delta' = \eta_A \circ \Delta'_1 \quad \text{By definition} \quad (6.43)$$

$$= \eta_A \circ \Delta_1 \circ \omega \quad \text{By induction of } \Delta_1, \Delta'_1 \quad (6.44)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (6.45)$$

**Case Apply:** By inversion, we have derivations  $\Delta_1, \Delta_2$  such that

$$\Delta = (\text{Apply}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1 v_2 : B} \quad (6.46)$$

By induction, this gives us the respective derivations:  $\Delta'_1, \Delta'_2$  such that

$$\Delta' = (\text{Apply}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1 : A \rightarrow B} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_2 : A}}{\Phi \mid \Gamma' \vdash v_1 v_2 : B} \quad (6.47)$$

This preserves denotations:

$$\Delta' = \text{app} \circ \langle \Delta'_1, \Delta'_2 \rangle \quad \text{By Definition} \quad (6.48)$$

$$= \text{app} \circ \langle \Delta_1 \circ \omega, \Delta_2 \circ \omega \rangle \quad \text{By induction on } \Delta_1, \Delta_2 \quad (6.49)$$

$$= \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \circ \omega \quad (6.50)$$

$$= \Delta \circ \omega \quad \text{By Definition} \quad (6.51)$$

**Case If:** By inversion, we have the sub-derivations  $\Delta_1, \Delta_2, \Delta_3$ , such that:

$$\Delta = (\text{If}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \text{Bool}} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_1 : A} \quad \frac{\Delta_3}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A} \quad (6.52)$$

By induction, this gives us the sub-derivations  $\Delta'_1, \Delta'_2, \Delta'_3$  such that

$$\Delta' = (\text{If}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v : \text{Bool}} \quad \frac{\Delta'_2}{\Phi \mid \Gamma' \vdash v_1 : A} \quad \frac{\Delta'_3}{\Phi \mid \Gamma' \vdash v_2 : A}}{\Phi \mid \Gamma' \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A} \quad (6.53)$$

And

$$\Delta'_1 = \Delta_1 \circ \omega \quad (6.54)$$

$$\Delta'_3 = \Delta_2 \circ \omega \quad (6.55)$$

$$\Delta'_3 = \Delta_3 \circ \omega \quad (6.56)$$

This preserves denotations. Since  $\omega : \Gamma' \rightarrow \Gamma$ ,  
Let  $(T_\epsilon A)^\omega : T_\epsilon A^\Gamma \rightarrow T_\epsilon A^{\Gamma'}$  be as defined in ExSh 3 <sup>(1)</sup> That is:

$$(T_\epsilon A)^\omega = \text{cur}(\text{app} \circ (\text{Id}_{T_\epsilon A} \times w)) \quad (6.57)$$

. And hence, we have:

$$\text{cur}(f \circ (\text{Id} \times \omega)) = (T_\epsilon A)^\omega \circ \text{cur}(f) \quad (6.58)$$

$$\Delta' = \text{app} \circ (([\text{cur}(\Delta'_2 \circ \pi_2), \text{cur}(\Delta'_3 \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Definition} \quad (6.59)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \omega \circ \pi_2), \text{cur}(\Delta_3 \circ \omega \circ \pi_2)] \circ \Delta'_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By Induction} \quad (6.60)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2 \circ (\text{Id}_1 \times \omega)), \text{cur}(\Delta_3 \circ \pi_2 \circ (\text{Id}_1 \times \omega))] \circ \Delta_1 \circ \omega) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By product property} \quad (6.61)$$

$$= \text{app} \circ (((T_\epsilon A)^\omega \circ \text{cur}(\Delta_2 \circ \pi_2), (T_\epsilon A)^\omega \circ \text{cur}(\Delta_3 \circ \pi_2)) \circ \Delta_1 \circ \omega) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By } (T_\epsilon A)^\omega \text{ property} \quad (6.62)$$

$$= \text{app} \circ (((T_\epsilon A)^\omega \circ [\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1 \circ \omega) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out transformation} \quad (6.63)$$

$$= \text{app} \circ ((T_\epsilon A)^\omega \times \text{Id}_{\Gamma'}) \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\omega \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{Factor out Identity pairs} \quad (6.64)$$

$$= \text{app} \circ (\text{Id}_{(T_\epsilon A)} \times \omega) \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\omega \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \quad \text{By definition of app, } (T_\epsilon A)^\omega \quad (6.65)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ (\omega \times \omega) \circ \delta_{\Gamma'} \quad \text{Push through pairs} \quad (6.66)$$

$$= \text{app} \circ (([\text{cur}(\Delta_2 \circ \pi_2), \text{cur}(\Delta_3 \circ \pi_2)] \circ \Delta_1) \times \text{Id}_{\Gamma'}) \circ \delta_{\Gamma'} \circ \omega \quad \text{By Definition of the diagonal morphism.} \quad (6.67)$$

$$= \Delta \circ \omega \quad (6.68)$$

<sup>1</sup><https://www.cl.cam.ac.uk/teaching/1819/L108/exercises/L108-exercise-sheet-3.pdf>

**Case Bind:** By inversion, we have derivations  $\Delta_1, \Delta_2$  such that:

$$\Delta = (\text{Bind}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : \mathbb{M}_{\epsilon_1} A} \quad \frac{\Delta_2}{\Phi \mid \Gamma, x : A \vdash v_2 : \mathbb{M}_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (6.69)$$

If  $\Phi \vdash \omega : \Gamma' \triangleright \Gamma$  then  $\Phi \vdash \omega \times : \Gamma', x : A \triangleright \Gamma, x : A$ , so by induction, we can derive  $\Delta'_1, \Delta'_2$  such that:

$$\Delta' = (\text{Bind}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v_1 : \mathbb{M}_{\epsilon_1} A} \quad \frac{\Delta'_2}{\Phi \mid \Gamma', x : A \vdash v_2 : \mathbb{M}_{\epsilon_2} B}}{\Phi \mid \Gamma' \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbb{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad (6.70)$$

This preserves denotations:

$$\Delta' = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta'_2 \circ \mathbf{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta'_1 \rangle \quad \text{By definition} \quad (6.71)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} (\Delta_2 \circ (\omega \times \text{Id}_A)) \circ \mathbf{t}_{\epsilon_1, \Gamma', A} \circ \langle \text{Id}_{\Gamma'}, \Delta_1 \circ \omega \rangle \quad \text{By induction on } \Delta'_1, \Delta'_2 \quad (6.72)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \omega, \Delta_1 \circ \omega \rangle \quad \text{By tensor strength} \quad (6.73)$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, \Delta_1 \rangle \circ \omega \quad \text{By product property} \quad (6.74)$$

$$= \Delta \quad \text{By definition} \quad (6.75)$$

**Case Effect-Lambda:** By inversion, we have  $\Delta_1$  such that

$$\Delta = (\text{Effect-Fn}) \frac{\frac{\Delta_1}{\Phi, \alpha \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \epsilon. A} \quad (6.76)$$

By induction, we derive  $\Delta'_1$  such that

$$\Delta' = (\text{Effect-Fn}) \frac{\frac{\Delta'_1}{\Phi, \alpha \mid \Gamma' \vdash v : A}}{\Phi \mid \Gamma' \vdash (\Lambda \alpha. v) : \forall \epsilon. A} \quad (6.77)$$

Where

$$\Delta'_1 = \Delta_1 \circ \llbracket \Phi, \alpha \vdash \omega : \Gamma' \triangleright \Gamma \rrbracket \quad (6.78)$$

$$= \Delta_1 \circ \llbracket \iota \pi : \Phi, a \triangleright \Phi \rrbracket^*(\omega) \quad (6.79)$$

$$= \Delta_1 \circ \pi_1^*(\omega) \quad (6.80)$$

Hence

$$\Delta \circ \omega = \overline{\Delta_1} \circ \omega \quad (6.81)$$

$$= \overline{\Delta_1 \circ \pi_1^*(\omega)} \quad (6.82)$$

$$= \overline{\Delta'_1} \quad (6.83)$$

$$= \Delta' \quad (6.84)$$

**Case Effect-Application:** By inversion, we derive  $\Delta_1$  such that

$$\Delta = (\text{Effect-App}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v : A[\epsilon/\alpha]} \quad (6.85)$$

By induction, we derive  $\Delta'_1$  such that

$$\Delta' = (\text{Effect-App}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma' \vdash v : \forall \alpha. A} \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma' \vdash v : A[\epsilon/\alpha]} \quad (6.86)$$

Where

$$\Delta'_1 = \Delta \circ \omega \quad (6.87)$$

Hence, if  $h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket$

$$\Delta \circ \omega = \langle \text{Id}_I, h \rangle^* (\epsilon \llbracket \Phi, \beta \vdash A[\alpha/\beta] : \text{Effect} \rrbracket) \circ \Delta_1 \circ \omega \quad (6.88)$$

$$= \langle \text{Id}_I, h \rangle^* (\epsilon \llbracket \Phi, \beta \vdash A[\alpha/\beta] : \text{Effect} \rrbracket) \circ \Delta'_1 \quad (6.89)$$

$$= \Delta' \quad (6.90)$$

## Chapter 7

# Unique Denotation Theorem

### 7.1 Reduced Type Derivation

A reduced type derivation is one where subtype and sub-effect rules must, and may only, occur at the root or directly above an **if**, or **apply** rule.

In this section, I shall prove that there is at most one reduced derivation of  $\Phi \mid \Gamma \vdash v : A$ . Secondly, I shall present a function for generating reduced derivations from arbitrary typing derivations, in a way that does not change the denotations. These imply that all typing derivations of a type-relation have the same denotation.

### 7.2 Reduced Type Derivations are Unique

For each instance of the relation  $\Phi \mid \Gamma \vdash v : A$ , there exists at most one reduced derivation of  $\Phi \mid \Gamma \vdash v : A$ . This is proved by induction over the typing rules on the bottom rule used in each derivation.

**Case Variables:** To find the unique derivation of  $\Phi \mid \Gamma \vdash x : A$ , we case split on the type-environment,  $\Gamma$ .

**Case:**  $\Gamma = \Gamma', x : A'$  Then the unique reduced derivation of  $\Phi \mid \Gamma \vdash x : A$  is, if  $A' \leq_{\Phi} A$ , as below:

$$\text{(Subtype)} \frac{\text{(Var)} \frac{\Phi \vdash \Gamma', x : A' \text{ Ok}}{\Phi \mid \Gamma, x : A' \vdash x : A'} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma', x : A' \vdash x : A} \quad (7.1)$$

**Case:**  $\Gamma = \Gamma', y : B$  with  $y \neq x$ .

Hence, if  $\Phi \mid \Gamma \vdash x : A$  holds, then so must  $\Phi \mid \Gamma' \vdash x : A$ .

Let

$$\text{(Subtype)} \frac{\Delta \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma' \vdash x : A'} \quad (7.2)$$

Be the unique reduced derivation of  $\Phi \mid \Gamma' \vdash x : A$ .

Then the unique reduced derivation of  $\Phi \mid \Gamma \vdash x : A$  is:

$$\begin{array}{c}
\Delta \\
\text{(Weaken)} \frac{\Phi \mid \Gamma, x : A' \vdash x : A'}{\Phi \mid \Gamma \vdash x : A'} \quad A' \leq_{\Phi} A \\
\text{(Subtype)} \frac{}{\Phi \mid \Gamma \vdash x : A}
\end{array} \tag{7.3}$$

**Case Constants:** For each of the constants, ( $\mathbf{C}^A$ , **true**, **false**,  $()$ ), there is exactly one possible derivation for  $\Phi \mid \Gamma \vdash c : A$  for a given  $A$ . I shall give examples using the case  $\mathbf{C}^A$

$$\begin{array}{c}
\text{(Const)} \frac{\Gamma \text{ Ok}}{\Gamma \vdash \mathbf{C}^A : A} \quad A \leq_{\Phi} B \\
\text{(Subtype)} \frac{}{\Phi \mid \Gamma \vdash \mathbf{C}^A : B}
\end{array}$$

If  $A = B$ , then the subtype relation is the identity subtype ( $A \leq_{\Phi} A$ ).

**Case Lambda:** The reduced derivation of  $\Phi \mid \Gamma \vdash \lambda x : A.v : A' \rightarrow B'$  is:

$$\begin{array}{c}
\Delta \\
\text{(Lambda)} \frac{\Phi \mid \Gamma, x : A \vdash v : B}{\Phi \mid \Gamma \vdash \lambda x : A.B : A \rightarrow B} \quad A \rightarrow B \leq_{\Phi} A' \rightarrow B' \\
\text{(Subtype)} \frac{}{\Phi \mid \Gamma \vdash \lambda x : A.v : A' \rightarrow B'}
\end{array}$$

Where

$$\begin{array}{c}
\Delta \\
\text{(Sub-Type)} \frac{\Phi \mid \Gamma, x : A \vdash v : B}{\Phi \mid \Gamma, x : A \vdash v : B'} \quad B \leq_{\Phi} B'
\end{array} \tag{7.4}$$

is the reduced derivation of  $\Phi \mid \Gamma, x : A \vdash v : B$  if it exists.

**Case Return:** The reduced denotation of  $\Phi \mid \Gamma \vdash \mathbf{return} v : B$  is

$$\begin{array}{c}
\Delta \\
\text{(Return)} \frac{\Phi \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash \mathbf{return} v : \mathbf{M}_1 A} \quad \text{(Effect)} \frac{1 \leq_{\Phi} \epsilon \quad A \leq_{\Phi} B}{\mathbf{M}_1 A \leq_{\Phi} \mathbf{M}_{\epsilon} B} \\
\text{(Subtype)} \frac{}{\Phi \mid \Gamma \vdash \mathbf{return} v : \mathbf{M}_{\epsilon} B}
\end{array}$$

Where

$$\begin{array}{c}
\Delta \\
\text{(Subtype)} \frac{\Phi \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash v : B} \quad A \leq_{\Phi} B
\end{array}$$

is the reduced derivation of  $\Phi \mid \Gamma \vdash v : B$

**Case Apply:** If

$$\begin{array}{c}
\Delta \\
\text{(Subtype)} \frac{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B \quad A \rightarrow B \leq_{\Phi} A' \rightarrow B'}{\Phi \mid \Gamma \vdash v_1 : A' \rightarrow B'}
\end{array}$$

and

$$\text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq A'}{\Phi \mid \Gamma \vdash v_2 : A'}$$

Are the reduced type derivations of  $\Phi \mid \Gamma \vdash v_1 : A' \rightarrow B'$  and  $\Phi \mid \Gamma \vdash v_2 : A'$

Then we can construct the reduced derivation of  $\Phi \mid \Gamma \vdash v_1 v_2 : B$  as

$$\text{(Sub-Type)} \frac{\text{(Apply)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B} \quad \text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v : A''} \quad A'' \leq_{\Phi} A}{\Phi \mid \Gamma \vdash v : A}}{\Phi \mid \Gamma \vdash v_1 v_2 : B}}{\Phi \mid \Gamma \vdash v_1 v_2 : B'} \quad B \leq_{\Phi} B'$$

**Case If:** Let

$$\text{(Subtype)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v : B'} \quad B' \leq \text{Bool}}{\Phi \mid \Gamma \vdash v : \text{Bool}} \quad (7.5)$$

$$\text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \quad A' \leq A}{\Phi \mid \Gamma \vdash v_1 : A} \quad (7.6)$$

$$\text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq A}{\Phi \mid \Gamma \vdash v_2 : A} \quad (7.7)$$

Be the unique reduced reduced derivations of  $\Phi \mid \Gamma \vdash v : \text{Bool}$ ,  $\Phi \mid \Gamma \vdash v_1 : A$ ,  $\Phi \mid \Gamma \vdash v_2 : A$ .

Then the only reduced derivation of  $\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : B$  is:

**TODO: Scale this properly**

$$\text{(Subtype)} \frac{\text{(If)} \frac{\text{(Subtype)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v : B'} \quad B' \leq \text{Bool}}{\Phi \mid \Gamma \vdash v : \text{Bool}}} \quad \frac{\text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v_1 : A'} \quad A' \leq A}{\Phi \mid \Gamma \vdash v_1 : A} \quad \text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v_2 : A''} \quad A'' \leq A}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : B} \quad A \leq_{\Phi} B \quad (7.8)$$

**Case Bind:** Let

$$\text{(Subtype)} \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v_1 : M_{\epsilon_1} A} \quad \text{(Computation)} \frac{A \leq_{\Phi} A' \quad \epsilon_1 \leq_{\Phi} \epsilon'_1}{M_{\epsilon_1} A \leq_{\Phi} M_{\epsilon'_1} A'}}{\Phi \mid \Gamma \vdash v_1 : M_{\epsilon'_1} A'} \quad (7.9)$$

$$\begin{array}{c}
\Delta' \\
\hline
\Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon_2} B \\
\text{(Subtype)} \quad \hline
\Phi \mid \Gamma, x : A \vdash v_2 : \mathbf{M}_{\epsilon'_2} B'
\end{array}
\quad
\begin{array}{c}
\text{(Computation)} \quad \frac{B \leq_{\Phi} B' \quad \epsilon_2 \leq_{\Phi} \epsilon'_2}{\mathbf{M}_{\epsilon_2} B \leq_{\Phi} \mathbf{M}_{\epsilon'_2} B'} \\
\hline
\end{array}
\quad (7.10)$$

Be the respective unique reduced type derivations of the sub-terms

By weakening,  $\Phi \vdash \iota \times : \Gamma, x : A \triangleright \Gamma, x : A'$  so if there's a derivation of  $\Phi \mid \Gamma, x : A' \vdash v_2 : B$ , there's also one of  $\Phi \mid \Gamma, x : A \vdash v_2 : B$ .

$$\begin{array}{c}
\Delta'' \\
\hline
\Phi \mid \Gamma, x : A' \vdash v_2 : \mathbf{M}_{\epsilon_2} B \\
\text{(Subtype)} \quad \hline
\Phi \mid \Gamma, x : A' \vdash v_2 : \mathbf{M}_{\epsilon'_2} B'
\end{array}
\quad
\begin{array}{c}
\text{(Computation)} \quad \frac{B \leq_{\Phi} B' \quad \epsilon_2 \leq_{\Phi} \epsilon'_2}{\mathbf{M}_{\epsilon_2} B \leq_{\Phi} \mathbf{M}_{\epsilon'_2} B'} \\
\hline
\end{array}
\quad (7.11)$$

Since the effects monoid operation is monotone, if  $\epsilon_1 \leq_{\Phi} \epsilon'_1$  and  $\epsilon_2 \leq_{\Phi} \epsilon'_2$  then  $\epsilon_1 \cdot \epsilon_2 \leq_{\Phi} \epsilon'_1 \cdot \epsilon'_2$

Hence the reduced type derivation of  $\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B'$  is the following:

**TODO: Make this and the other smaller**

$$\begin{array}{c}
\Delta \\
\hline
\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon_1} A \\
\text{(Subtype)} \quad \hline
\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon'_1} A'
\end{array}
\quad
\begin{array}{c}
\text{(Computation)} \quad \frac{A \leq_{\Phi} A' \quad \epsilon_1 \leq_{\Phi} \epsilon'_1}{\mathbf{M}_{\epsilon_1} A \leq_{\Phi} \mathbf{M}_{\epsilon'_1} A'} \\
\hline
\end{array}$$

$$\begin{array}{c}
\Delta'' \\
\hline
\Phi \mid \Gamma, x : A' \vdash v_2 : \mathbf{M}_{\epsilon_2} B \\
\text{(Subtype)} \quad \hline
\Phi \mid \Gamma, x : A' \vdash v_2 : \mathbf{M}_{\epsilon'_2} B'
\end{array}
\quad
\begin{array}{c}
\text{(Computation)} \quad \frac{B \leq_{\Phi} B' \quad \epsilon_2 \leq_{\Phi} \epsilon'_2}{\mathbf{M}_{\epsilon_2} B \leq_{\Phi} \mathbf{M}_{\epsilon'_2} B'} \\
\hline
\end{array}$$

$$\begin{array}{c}
\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B \\
\text{(Bind)} \quad \hline
\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B'
\end{array}$$

$$\begin{array}{c}
\epsilon_1 \cdot \epsilon_2 \leq_{\Phi} \epsilon'_1 \cdot \epsilon'_2 \quad B \leq_{\Phi} B' \\
\text{(Effect)} \quad \hline
\mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B \leq_{\Phi} \mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B'
\end{array}$$

$$\begin{array}{c}
\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B' \\
\text{(Sub-type)} \quad \hline
\end{array}
\quad (7.12)$$

**Case Effect-Fn:** The unique reduced derivation of  $\Phi \mid \Gamma \vdash \Lambda \alpha. A : \forall \alpha. B$

is

$$\begin{array}{c}
\Delta \\
\hline
\Phi, \alpha \mid \Gamma \vdash v : A \\
\text{(Effect-Fn)} \quad \hline
\Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \alpha. A
\end{array}
\quad
\forall \alpha. A \leq_{\Phi} \forall \alpha. B$$

$$\begin{array}{c}
\Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \alpha. A \\
\text{(Sub-type)} \quad \hline
\Phi \mid \Gamma \vdash \Lambda \alpha. B : \forall \alpha. B
\end{array}
\quad (7.13)$$

Where

$$\begin{array}{c}
\Delta \\
\hline
\Phi, \alpha \mid \Gamma \vdash v : A \\
\text{(Sub-type)} \quad \hline
\Phi, \alpha \mid \Gamma \vdash v : B
\end{array}
\quad
A \leq_{\Phi, \alpha} B
\quad (7.14)$$

Is the unique reduced derivation of  $\Phi, \alpha \mid \Gamma \vdash v : B$

**Case Effect-App:** The unique reduced derivation of  $\Phi \mid \Gamma \vdash v \alpha : B'$

is



$$\begin{array}{c}
\Delta \\
\text{(Effect-App)} \frac{\Phi \mid \Gamma \vdash v: \forall \alpha. A \quad \Phi \vdash \epsilon}{\Phi \mid \Gamma \vdash v \epsilon: A[\epsilon/\alpha]} \quad A[\epsilon/\alpha] \leq_{\Phi} B' \\
\text{(Subtype)} \frac{}{\Phi \mid \Gamma \vdash v \alpha: B'}
\end{array} \quad (7.15)$$

Where  $B[\epsilon/\alpha] \leq_{\Phi} B'$  and

$$\begin{array}{c}
\Delta \\
\text{(Subtype)} \frac{\Phi \mid \Gamma \vdash v: \forall \alpha. B}{\Phi \mid \Gamma \vdash v: \forall \alpha. B} \quad \text{(Quantification)} \frac{A \leq_{\Phi, \alpha} B}{\forall \alpha. A \leq_{\Phi} \forall \alpha. B}
\end{array} \quad (7.16)$$

### 7.3 Each type derivation has a reduced equivalent with the same denotation.

We introduce a function, *reduce* that maps each valid type derivation of  $\Phi \mid \Gamma \vdash v: A$  to a reduced equivalent with the same denotation. To do this, we do case analysis over the root type rule of a derivation and prove that the denotation is not changed.

**Case Constants:** For the constants `true`, `false`,  $\mathcal{C}^A$ , etc, *reduce* simply returns the derivation, as it is already reduced. This trivially preserves the denotation.

$$\text{reduce}((\text{Const}) \frac{\Gamma \text{ Ok}}{\Gamma \vdash \mathcal{C}^A: A}) = (\text{Const}) \frac{\Gamma \text{ Ok}}{\Gamma \vdash \mathcal{C}^A: A}$$

**Case Var:**

$$\text{reduce}((\text{Var}) \frac{\Phi \vdash \Gamma \text{ Ok}}{\Phi \mid \Gamma, x: A \vdash x: A}) = (\text{Var}) \frac{\Phi \vdash \Gamma \text{ Ok}}{\Phi \mid \Gamma, x: A \vdash x: A} \quad (7.17)$$

Preserves denotation trivially.

**Case Weaken:**

*reduce* **definition** To find:

$$\text{reduce}((\text{Weaken}) \frac{\Delta}{\Phi \mid \Gamma \vdash x: A} \frac{}{\Phi \mid \Gamma, y: B \vdash x: A}) \quad (7.18)$$

Let

$$\text{(Subtype)} \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash x: A} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma \vdash x: A} = \text{reduce}(\Delta) \quad (7.19)$$

In

$$\text{(Subtype)} \frac{\text{(Weaken)} \frac{\Delta'}{\Phi \mid \Gamma \vdash x: A'} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma, y: B \vdash x: A'} \quad (7.20)$$

**Preserves Denotation** Using the construction of denotations, we can find the denotation of the original derivation to be:

$$\llbracket (\text{Weaken}) \frac{\overline{\Delta} \quad \overline{\Phi \mid \Gamma \vdash x: A}}{\Phi \mid \Gamma, y: B \vdash x: A} \rrbracket = \Delta \circ \pi_1 \quad (7.21)$$

Similarly, the denotation of the reduced denotation is:

$$\llbracket (\text{Subtype}) \frac{(\text{Weaken}) \frac{\overline{\Delta'} \quad \overline{\Phi \mid \Gamma \vdash x: A'}}{\Phi \mid \Gamma, y: B \vdash x: A'} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma, y: B \vdash x: A} \rrbracket = \llbracket A' \leq_{\Phi} A \rrbracket \circ \Delta' \circ \pi_1 \quad (7.22)$$

By induction on *reduce* preserving denotations and the reduction of  $\Delta$  (7.19), we have:

$$\Delta = \llbracket A' \leq_{\Phi} A \rrbracket \circ \Delta' \quad (7.23)$$

So the denotations of the un-reduced and reduced derivations are equal.

**Case Lambda:**

*reduce* **definition** To find:

$$\text{reduce}((\text{Fn}) \frac{\overline{\Delta} \quad \overline{\Phi \mid \Gamma, x: A \vdash v: B}}{\Phi \mid \Gamma \vdash \lambda x: A.v : A \rightarrow B}) \quad (7.24)$$

Let

$$(\text{Sub-type}) \frac{\overline{\Delta'} \quad \overline{\Phi \mid \Gamma, x: A \vdash v: B'} \quad B' \leq_{\Phi} B \quad \Phi \mid \Gamma, x: A \vdash v: \mathbf{M}_{\epsilon_2} B}{=} \text{reduce}(\Delta) \quad (7.25)$$

In

$$(\text{Sub-type}) \frac{(\text{Fn}) \frac{\overline{\Delta'} \quad \overline{\Phi \mid \Gamma, x: A \vdash v: B'}}{\Phi \mid \Gamma \vdash \lambda x: A.v : A \rightarrow B'} \quad A \rightarrow B' \leq_{\Phi} A \rightarrow B}{\Phi \mid \Gamma \vdash \lambda x: A.v : A \rightarrow B} \quad (7.26)$$

**Preserves Denotation** Let

$$f = \llbracket B' \leq_{\Phi} B \rrbracket \quad (7.27)$$

$$\llbracket A \rightarrow B' \leq_{\Phi} A \rightarrow B \rrbracket = f^A = \text{cur}(f \circ \text{app}) \quad (7.28)$$

Then

$$before = \text{cur}(\Delta) \quad \text{By definition} \quad (7.29)$$

$$= \text{cur}(f \circ \Delta') \quad \text{By reduction of } \Delta \quad (7.30)$$

$$= f^A \circ \text{cur}(\Delta') \quad \text{By the property of } f^X \circ \text{cur}(g) = \text{cur}(f \circ g) \quad (7.31)$$

$$= after \quad \text{By definition} \quad (7.32)$$

$$(7.33)$$

### Case Subtype:

*reduce* **definition** To find:

$$reduce((\text{Subtype}) \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v: A} \quad A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v: B}) \quad (7.34)$$

Let

$$(\text{Subtype}) \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash x: A} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma \vdash x: A} = reduce(\Delta) \quad (7.35)$$

In

$$(\text{Subtype}) \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v: A'} \quad A' \leq_{\Phi} A \leq_{\Phi} B}{\Phi \mid \Gamma \vdash v: B} \quad (7.36)$$

### Preserves Denotation

$$before = \llbracket A \leq_{\Phi} B \rrbracket \circ \Delta \quad (7.37)$$

$$= \llbracket A \leq_{\Phi} B \rrbracket \circ (\llbracket A' \leq_{\Phi} A \rrbracket \circ \Delta') \quad \text{by Denotation of reduction of } \Delta. \quad (7.38)$$

$$= \llbracket A' \leq_{\Phi} B \rrbracket \circ \Delta' \quad \text{Subtyping relations are unique} \quad (7.39)$$

$$= after \quad (7.40)$$

$$(7.41)$$

### Case Return:

*reduce* **definition** To find:

$$reduce((\text{Return}) \frac{\frac{\Delta}{\Phi \mid \Gamma \vdash v: A}}{\Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A}) \quad (7.42)$$

Let

$$(\text{Sub-type}) \frac{\frac{\Delta'}{\Phi \mid \Gamma \vdash v: A'} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma \vdash v: A} = reduce(\Delta) \quad (7.43)$$

In

$$\begin{array}{c}
\Delta' \\
\hline
\text{(Return)} \frac{\Phi \mid \Gamma \vdash v : A}{\Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A'} \quad \text{(Effect)} \frac{1 \leq_{\Phi} 1 \quad A' \leq_{\Phi} A}{\mathbf{M}_1 A' \leq_{\Phi} \mathbf{M}_1 A} \\
\hline
\text{(Sub-type)} \frac{}{\Phi \mid \Gamma \vdash \text{return } v : \mathbf{M}_1 A}
\end{array} \quad (7.44)$$

Then

$$before = \eta_A \circ \Delta \quad \text{By definition} \quad \text{By definition} \quad (7.45)$$

$$= \eta_A \circ \llbracket A' \leq_{\Phi} A \rrbracket \circ \Delta' \quad \text{By reduction of } \Delta \quad (7.46)$$

$$= T_1 \llbracket A' \leq_{\Phi} A \rrbracket \circ \eta_{A'} \circ \Delta' \quad \text{By naturality of } \eta \quad (7.47)$$

$$= \llbracket 1 \leq_{\Phi} 1 \rrbracket_{M,A} \circ T_1 \llbracket A' \leq_{\Phi} A \rrbracket \circ \eta_{A'} \circ \Delta' \quad \text{Since } \llbracket 1 \leq_{\Phi} 1 \rrbracket \text{ is the identity Nat-Trans} \quad (7.48)$$

$$= after \quad \text{By definition} \quad (7.49)$$

$$(7.50)$$

**Case Apply:**

*reduce* **definition** To find:

$$\text{reduce}((\text{Apply}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_2 : A}}{\Phi \mid \Gamma \vdash v_1 v_2 : B}) \quad (7.51)$$

Let

$$\text{(Subtype)} \frac{\frac{\Delta'_1}{\Phi \mid \Gamma \vdash v_1 : A' \rightarrow B'} \quad A' \rightarrow B' \leq_{\Phi} A \rightarrow B}{\Phi \mid \Gamma \vdash v_1 : A \rightarrow B} = \text{reduce}(\Delta_1) \quad (7.52)$$

$$\text{(Subtype)} \frac{\frac{\Delta'_2}{\Phi \mid \Gamma \vdash v : A'} \quad A' \leq_{\Phi} A}{\Phi \mid \Gamma \vdash v_1 : A} = \text{reduce}(\Delta_2) \quad (7.53)$$

In

$$\begin{array}{c}
\Delta'_1 \quad \Delta'_2 \quad A'' \leq_{\Phi} A \leq_{\Phi} A' \\
\hline
\text{(Apply)} \frac{\frac{\Phi \mid \Gamma \vdash v_1 : A' \rightarrow B'}{\Phi \mid \Gamma \vdash v_1 v_2 : B'} \quad \text{(Sub-type)} \frac{\frac{\Phi \mid \Gamma \vdash v_2 : A''}{\Phi \mid \Gamma \vdash v_2 : A'}}{\Phi \mid \Gamma \vdash v_2 : A'}}{\Phi \mid \Gamma \vdash v_1 v_2 : B'} \quad B' \leq_{\Phi} B \\
\hline
\text{(Sub-type)} \frac{}{\Phi \mid \Gamma \vdash v_1 v_2 : B}
\end{array} \quad (7.54)$$

**Preserves Denotation** Let

$$f = \llbracket A \leq_{\Phi} A' \rrbracket : A \rightarrow A' \quad (7.55)$$

$$f' = \llbracket A'' \leq_{\Phi} A \rrbracket : A'' \rightarrow A \quad (7.56)$$

$$g = \llbracket B' \leq_{\Phi} B \rrbracket : B' \rightarrow B \quad (7.57)$$

$$(7.58)$$

Hence

$$\llbracket A' \rightarrow B' \leq_{\Phi} A \rightarrow B \rrbracket = (g)^A \circ (B')^f \quad (7.59)$$

$$= \text{cur}(\text{app} \circ \text{app}) \circ \text{cur}(\text{app} \circ (\text{Id} \times f)) \quad (7.60)$$

$$= \text{cur}(g \circ \text{app} \circ (\text{Id} \times f)) \quad (7.61)$$

Then

$$\text{before} = \text{app} \circ \langle \Delta_1, \Delta_2 \rangle \quad \text{By definition} \quad (7.62)$$

$$= \text{app} \circ \langle \text{cur}(g \circ \text{app} \circ (\text{Id} \times f)) \circ \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{By reductions of } \Delta_1, \Delta_2 \quad (7.63)$$

$$= \text{app} \circ (\text{cur}(g \circ \text{app} \circ (\text{Id} \times f)) \times \text{Id}_A) \circ \langle \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{Factoring out} \quad (7.64)$$

$$= g \circ \text{app} \circ (\text{Id} \times f) \circ \langle \Delta'_1, f' \circ \Delta'_2 \rangle \quad \text{By the exponential property} \quad (7.65)$$

$$= g \circ \text{app} \circ \langle \Delta'_1, f \circ f' \circ \Delta'_2 \rangle \quad (7.66)$$

$$= \text{after} \quad \text{By definition} \quad (7.67)$$

**Case If:**

*reduce* **definition**

$$\text{reduce}((\text{If}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v: \text{Bool}} \quad \frac{\Delta_2}{\Phi \mid \Gamma \vdash v_1: A} \quad \frac{\Delta_3}{\Phi \mid \Gamma \vdash v_2: A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2: A}) = (\text{If}) \frac{\frac{\text{reduce}(\Delta_1)}{\Phi \mid \Gamma \vdash v: \text{Bool}} \quad \frac{\text{reduce}(\Delta_2)}{\Phi \mid \Gamma \vdash v_1: A} \quad \frac{\text{reduce}(\Delta_3)}{\Phi \mid \Gamma \vdash v_2: A}}{\Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2: A} \quad (7.68)$$

**Preserves Denotation** Since calling *reduce* on the sub-derivations preserves their denotations, this definition trivially preserves the denotation of the derivation.

**Case Bind:**

*reduce* **definition** To find

$$\text{reduce}((\text{Bind}) \frac{\frac{\Delta_1}{\Phi \mid \Gamma \vdash v_1: M_{\epsilon_1} A} \quad \frac{\Delta_2}{\Phi \mid \Gamma, x: A \vdash v_2: M_{\epsilon_2} B}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2: M_{\epsilon_1, \epsilon_2} B}) \quad (7.69)$$

Let

$$(\text{Sub-type}) \frac{\frac{\Delta'_1}{\Phi \mid \Gamma \vdash v_1: M_{\epsilon'_1} A'} \quad (\text{Effect}) \frac{\epsilon'_1 \leq_{\Phi} \epsilon_1 \quad A' \leq_{\Phi} A}{M_{\epsilon'_1} A' \leq_{\Phi} M_{\epsilon_1} A}}{\Phi \mid \Gamma \vdash v_1: M_{\epsilon_1} A} = \text{reduce}(\Delta_1) \quad (7.70)$$

Since  $\Phi \vdash (i, \times): (\Gamma, x: A') \triangleright (\Gamma, x: A)$  if  $A' \leq_{\Phi} A$ , and by  $\Delta_2 = \Phi \mid (\Gamma, x: A) \vdash v_2: M_{\epsilon_2} B$ , there also exists a derivation  $\Delta_3$  of  $\Phi \mid (\Gamma, x: A') \vdash v_2: M_{\epsilon_2} B$ .  $\Delta_3$  is derived from  $\Delta_2$  simply by inserting a (Sub-type) rule below all instances of the (Var) rule.

Let

$$(\text{Sub-type}) \frac{\frac{\Delta'_3}{\Phi \mid \Gamma, x: A' \vdash v_2: M_{\epsilon'_2} B'} \quad (\text{Effect}) \frac{\epsilon'_1 \leq_{\Phi} \epsilon_2 \quad B' \leq_{\Phi} B}{M_{\epsilon'_1} B' \leq_{\Phi} M_{\epsilon_2} B}}{\Phi \mid \Gamma, x: A' \vdash v_2: M_{\epsilon_2} B} = \text{reduce}(\Delta_3) \quad (7.71)$$

Since the effects monoid operation is monotone, if  $\epsilon_1 \leq_{\Phi} \epsilon'_1$  and  $\epsilon_2 \leq_{\Phi} \epsilon'_2$  then  $\epsilon_1 \cdot \epsilon_2 \leq_{\Phi} \epsilon'_1 \cdot \epsilon'_2$ .  
Then the result of reduction of the whole bind expression is:

$$\begin{array}{c}
 \text{(Sub-type)} \frac{\text{(Bind)} \frac{\frac{\Delta'_1}{\Phi \mid \Gamma \vdash v_1 : \mathbf{M}_{\epsilon'_1} A'}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B} \quad \frac{\Delta'_3}{\Phi \mid \Gamma, x : A' \vdash v_2 : \mathbf{M}_{\epsilon'_2} B'}}{\Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \quad \text{(Effect)} \frac{\epsilon'_1 \cdot \epsilon'_2 \leq_{\Phi} \epsilon_1 \cdot \epsilon_2 \quad B' \leq_{\Phi} B}{\mathbf{M}_{\epsilon'_1 \cdot \epsilon'_2} B' \leq_{\Phi} \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B} \\
 \hline
 \Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2 : \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} B
 \end{array} \tag{7.72}$$

**Preserves Denotation** Let

$$f = \llbracket A' \leq_{\Phi} A \rrbracket : A' \rightarrow A \tag{7.73}$$

$$g = \llbracket B' \leq_{\Phi} B \rrbracket : B' \rightarrow B \tag{7.74}$$

$$h_1 = \llbracket \epsilon'_1 \leq_{\Phi} \epsilon_1 \rrbracket : T_{\epsilon'_1} \rightarrow T_{\epsilon_1} \tag{7.75}$$

$$h_2 = \llbracket \epsilon'_2 \leq_{\Phi} \epsilon_2 \rrbracket : T_{\epsilon'_2} \rightarrow T_{\epsilon_2} \tag{7.76}$$

$$h = \llbracket \epsilon'_1 \cdot \epsilon'_2 \leq_{\Phi} \epsilon_1 \cdot \epsilon_2 \rrbracket : T_{\epsilon'_1 \cdot \epsilon'_2} \rightarrow T_{\epsilon_1 \cdot \epsilon_2} \tag{7.77}$$

Due to the denotation of the weakening used to derive  $\Delta_3$  from  $\Delta_2$ , we have

$$\Delta_3 = \Delta_2 \circ (\text{Id}_{\Gamma} \times f) \tag{7.78}$$

And due to the reduction of  $\Delta_3$ , we have

$$\Delta_3 = h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3 \tag{7.79}$$

So:

$$\text{before} = \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, \Delta_1 \rangle \quad \text{By definition.} \tag{7.80}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, h_{1,A} \circ T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{By reduction of } \Delta_1. \tag{7.81}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ (\text{Id}_{\Gamma} \times h_{1,A}) \circ \langle \text{Id}_{\Gamma}, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Factor out } h_1 \tag{7.82}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} \Delta_2 \circ h_{1, (\Gamma \times A)} \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Tensor strength and sub-effecting } h_1 \tag{7.83}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} \Delta_2 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ \langle \text{Id}_{\Gamma}, T_{\epsilon'_1} f \circ \Delta'_1 \rangle \quad \text{Naturality of } h_1 \tag{7.84}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} \Delta_2 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A} \circ (\text{Id}_{\Gamma} \times T_{\epsilon'_1} f) \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{Factor out pairing again} \tag{7.85}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} (\Delta_2 \circ (\text{Id}_{\Gamma} \times f)) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{Tensor strength} \tag{7.86}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} (\Delta_3) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{By the definition of } \Delta_3 \tag{7.87}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} (h_{2,B} \circ T_{\epsilon'_2} g \circ \Delta'_3) \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{By the reduction of } \Delta_3 \tag{7.88}$$

$$= \mu_{\epsilon_1, \epsilon_2, B} \circ h_{1,B} \circ T_{\epsilon'_1} h_{2,B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{Factor out the functor} \tag{7.89}$$

$$= h_B \circ \mu_{\epsilon'_1, \epsilon'_2, B} \circ T_{\epsilon'_1} T_{\epsilon'_2} g \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{By the } \mu \text{ and Sub-type rule} \tag{7.90}$$

$$= h_B \circ T_{\epsilon'_1 \cdot \epsilon'_2} g \circ \mu_{\epsilon'_1, \epsilon'_2, B'} \circ T_{\epsilon'_1} \Delta'_3 \circ \mathbf{t}_{\epsilon'_1, \Gamma, A'} \circ \langle \text{Id}_{\Gamma}, \Delta'_1 \rangle \quad \text{By naturality of } \mu, \tag{7.91}$$

$$= \text{after} \quad \text{By definition} \tag{7.92}$$

**Case Effect-Fn:**

*reduce* **definition** To find

$$reduce((\text{Effect-Lambda}) \frac{\Delta_1}{\overline{\Phi, \alpha \mid \Gamma \vdash v: A}}) \quad (7.93)$$

Let

$$(\text{Subtype}) \frac{\frac{\Delta'_1}{\overline{\Phi, \alpha \mid \Gamma \vdash v: A'}} \quad A' \leq_{\Phi} A}{\overline{\Phi, \alpha \mid \Gamma \vdash v: A}} = reduce(\Delta_1) \quad (7.94)$$

in

$$(\text{Subtype}) \frac{(\text{Effect-Fn}) \frac{\Delta'_1}{\overline{\Phi, \alpha \mid \Gamma \vdash v: A'}} \quad (\text{Quantification}) \frac{A' \leq_{\Phi, \alpha}}{\forall \alpha. A' \leq_{\Phi} \forall \alpha. A}}{\overline{\Phi \mid \Gamma \vdash \Lambda \alpha. v: \forall \alpha. A}} \quad (7.95)$$

**Preserves Denotation**

$$before = \overline{\Delta_1} \quad (7.96)$$

$$= \overline{[A' \leq_{\Phi, \alpha} A] \circ \Delta'_1} \quad \text{By induction} \quad (7.97)$$

$$= \forall_I([A' \leq_{\Phi, \alpha} A]) \circ \overline{\Delta'_1} \quad (7.98)$$

$$= [\forall \alpha. A' \leq_{\Phi} \forall \alpha. A] \circ \overline{\Delta'_1} \quad \text{By definition} \quad (7.99)$$

$$= after \quad \text{By definition} \quad (7.100)$$

**Case Effect-Application:**

*reduce* **definition** To find

$$reduce((\text{Effect-App}) \frac{\Delta_1}{\overline{\Phi \mid \Gamma \vdash v: \forall \alpha. A}} \quad \Phi \vdash \epsilon) \quad (7.101)$$

Let

$$(\text{Subtype}) \frac{\frac{\Delta'_1}{\overline{\Phi \mid \Gamma \vdash v: \forall \alpha. A'}} \quad (\text{Quantification}) \frac{A' \leq_{\Phi, \alpha} A}{\forall \alpha. A' \leq_{\Phi} \forall \alpha. A}}{\overline{\Phi \mid \Gamma \vdash v: \forall \alpha. A}} = reduce(\Delta_1) \quad (7.102)$$

In

$$(\text{Subtype}) \frac{(\text{E-app}) \frac{\frac{\Delta'_1}{\overline{\Phi \mid \Gamma \vdash v: \forall \alpha. A}} \quad \Phi \vdash \epsilon}{\overline{\Phi \mid \Gamma \vdash v \in: A[\epsilon/\alpha]}} \quad A'[\epsilon/\alpha] \leq_{\Phi} A[\epsilon/\alpha]}{\overline{\Phi \mid \Gamma \vdash v \in: A[\epsilon/\alpha]}} \quad (7.103)$$

**Preserves Denotation** Let

$$h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \quad (7.104)$$

$$A = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Effect} \rrbracket \quad (7.105)$$

$$A' = \llbracket \Phi, \beta \vdash A'[\beta/\alpha] : \text{Effect} \rrbracket \quad (7.106)$$

Note that

$$\langle \text{Id}_I, h \rangle^*(\pi_1^*(f)) = (\pi_1 \circ \langle \text{Id}_I, h \rangle)^*(f) = \text{Id}_I^*(f) = f \quad (7.107)$$

And that

$$\langle \text{Id}_I, h \rangle = \llbracket \Phi \vdash [\epsilon/\alpha] : \Phi, \alpha \rrbracket \quad (7.108)$$

With lemma:

$$\llbracket \forall \alpha. A' \leq_{\Phi} \forall \alpha. A \rrbracket = \forall_I(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket) \quad (7.109)$$

$$= \langle \text{Id}_I, h \rangle^*(\pi_1^*(\forall_I(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket))) \quad (7.110)$$

In

$$before = \langle \text{Id}_I, h \rangle^*(\epsilon_A) \circ \Delta_1 \quad (7.111)$$

$$= \langle \text{Id}_I, h \rangle^*(\epsilon_A) \circ \llbracket \forall \alpha. A' \leq_{\Phi} \forall \alpha. A \rrbracket \circ \Delta'_1 \quad \text{By induction} \quad (7.112)$$

$$= \langle \text{Id}_I, h \rangle^*(\epsilon_A) \circ \langle \text{Id}_I, h \rangle^*(\pi_1^*(\forall_I(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket))) \circ \Delta'_1 \quad \text{By lemma} \quad (7.113)$$

$$= \langle \text{Id}_I, h \rangle^*(\epsilon_A \circ \pi_1^*(\forall_I(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket))) \circ \Delta'_1 \quad \text{By functoriality} \quad (7.114)$$

$$= \langle \text{Id}_I, h \rangle^*(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket \circ \epsilon_{A'}) \circ \Delta'_1 \quad \text{By Naturality} \quad (7.115)$$

$$= \langle \text{Id}_I, h \rangle^*(\llbracket A' \leq_{\Phi, \alpha} A \rrbracket) \circ \langle \text{Id}_I, h \rangle^*(\epsilon_{A'}) \circ \Delta'_1 \quad (7.116)$$

$$= \llbracket A'[\epsilon/\alpha] \leq_{\Phi, \alpha} A[\epsilon/\alpha] \rrbracket \circ \langle \text{Id}_I, h \rangle^*(\epsilon_{A'}) \circ \Delta'_1 \quad \text{By substitution of sub-typing} \quad (7.117)$$

$$= after \quad (7.118)$$

□

## 7.4 Denotations are Equivalent

For each type relation instance  $\Phi \mid \Gamma \vdash v : A$  there exists a unique reduced derivation of the relation instance. For all derivations  $\Delta, \Delta'$  of the type relation instance,  $\llbracket \Delta \rrbracket = \llbracket reduce \Delta \rrbracket = \llbracket reduce \Delta' \rrbracket = \llbracket \Delta' \rrbracket$ , hence the denotation  $\llbracket \Phi \mid \Gamma \vdash v : A \rrbracket$  is unique.



## Chapter 8

# Beta-Eta-Equivalence Theorem (Soundness)

If

*equivalence*  $\Phi v v' A$  then  $\llbracket \Gamma \vdash v : A \rrbracket = \llbracket \Gamma \vdash v' : A \rrbracket$

By induction over Beta-eta equivalence relation.

### 8.0.1 Equivalence Relation

The cases over the equivalence relation laws hold by the uniqueness of denotations and the fact that equality over morphisms is an equivalence relation.

**Case Reflexive:** Equality is reflexive, so if  $\Phi \mid \Gamma \vdash v : A$  then  $\llbracket \Phi \mid \Gamma \vdash v : A \rrbracket$  is equal to itself.

**Case Symmetric:** By inversion, if  $\Phi \mid \Gamma \vdash v \approx v' : A$  then  $\Phi \mid \Gamma \vdash v' \approx v : A$ , so by induction  $\llbracket \Gamma \vdash v' : A \rrbracket = \llbracket \Gamma \vdash v : A \rrbracket$  and hence  $\llbracket \Gamma \vdash v : A \rrbracket = \llbracket \Gamma \vdash v' : A \rrbracket$

**Case Transitive:** There must exist  $v_2$  such that  $\Phi \mid \Gamma \vdash v_1 \approx v_2 : A$  and  $\Phi \mid \Gamma \vdash v_2 \approx v_3 : A$ , so by induction,  $\llbracket \Gamma \vdash v_1 : A \rrbracket = \llbracket \Gamma \vdash v_2 : A \rrbracket$  and  $\llbracket \Gamma \vdash v_2 : A \rrbracket = \llbracket \Gamma \vdash v_3 : A \rrbracket$ . Hence by transitivity of equality,  $\llbracket \Gamma \vdash v_1 : A \rrbracket = \llbracket \Gamma \vdash v_3 : A \rrbracket$

### 8.0.2 Beta-Eta Conversions

These cases are typically proved using the properties of a cartesian closed category with a strong graded monad.

**Case Lambda:** Let  $f = \llbracket \Phi \mid \Gamma, x : A \vdash v_1 : B \rrbracket : (\Gamma \times A) \rightarrow B$

Let  $g = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Phi \mid \Gamma \vdash [v_2/x] : \Gamma, x : A \rrbracket : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Phi \mid \Gamma \vdash v_1 [v_2/x] : B \rrbracket = f \circ \langle \text{Id}_\Gamma, g \rangle$$

and hence

$$\begin{aligned}
\llbracket \Phi \mid \Gamma \vdash (\lambda x: A. v) v: B \rrbracket &= \text{app} \circ \langle \text{cur}(f), g \rangle \\
&= \text{app} \circ (\text{cur}(f) \times \text{Id}_A) \circ \langle \text{Id}_\Gamma, g \rangle \\
&= f \circ \langle \text{Id}_\Gamma, g \rangle \\
&= \llbracket \Phi \mid \Gamma \vdash v_1 [v_2/x]: B \rrbracket
\end{aligned} \tag{8.1}$$

**Case Left Unit:** Let  $f = \llbracket \Phi \mid \Gamma, x: A \vdash v_1: M_\epsilon B \rrbracket$

Let  $g = \llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket : \Gamma \rightarrow A$

By the substitution denotation,

$$\llbracket \Phi \mid \Gamma \vdash [v_2/x]: \Gamma, x: A \rrbracket : \Gamma \rightarrow (\Gamma \times A) = \langle \text{Id}_\Gamma, g \rangle$$

We have

$$\llbracket \Phi \mid \Gamma \vdash v_1 [v_2/x]: M_\epsilon B \rrbracket = f \circ \langle \text{Id}_\Gamma, g \rangle$$

And hence

$$\begin{aligned}
\llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow \text{return } v_2 \text{ in } v_1: M_\epsilon B \rrbracket &= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathbf{t}_{1, \Gamma, A} \circ \langle \text{Id}_\Gamma, \eta_A \circ g \rangle \\
&= \mu_{1, \epsilon, B} \circ T_1 f \circ \mathbf{t}_{1, \Gamma, A} \circ (\text{Id}_\Gamma \times \eta_A) \circ \langle \text{Id}_\Gamma, g \rangle \\
&= \mu_{1, \epsilon, B} \circ T_1 f \circ \eta_{(\Gamma \times A)} \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Tensor strength + unit} \\
&= \mu_{1, \epsilon, B} \circ \eta_{T_\epsilon B} \circ f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By Naturality of } \eta \\
&= f \circ \langle \text{Id}_\Gamma, g \rangle \quad \text{By left unit law} \\
&= \llbracket \Phi \mid \Gamma \vdash v_1 [v_2/x]: M_\epsilon B \rrbracket
\end{aligned} \tag{8.2}$$

**Case Right Unit:** Let  $f = \llbracket \Phi \mid \Gamma \vdash v: M_\epsilon A \rrbracket$

$$\begin{aligned}
\llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow v \text{ in return } x: M_\epsilon A \rrbracket &= \mu_{\epsilon, 1, A} \circ T_\epsilon (\eta_A \circ \pi_2) \circ \mathbf{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\
&= T_\epsilon \pi_2 \circ \mathbf{t}_{\epsilon, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\
&= \pi_2 \circ \langle \text{Id}_\Gamma, f \rangle \\
&= f
\end{aligned} \tag{8.3}$$

**Case Associative:** Let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1: M_{\epsilon_1} A \rrbracket \tag{8.4}$$

$$g = \llbracket \Phi \mid \Gamma, x: A \vdash v_2: M_{\epsilon_2} B \rrbracket \tag{8.5}$$

$$h = \llbracket \Phi \mid \Gamma, y: B \vdash v_3: M_{\epsilon_3} C \rrbracket \tag{8.6}$$

$$\tag{8.7}$$

We also have the weakening:

$$\Phi \vdash \iota\pi \times: \Gamma, x: A, y: B \triangleright \Gamma, y: B \tag{8.8}$$

With denotation:

$$\llbracket \Phi \vdash \iota\pi \times: \Gamma, x: A, y: B \triangleright \Gamma, y: B \rrbracket = (\pi_1 \times \text{Id}_B) \tag{8.9}$$

We need to prove that the following are equal

$$lhs = \llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } (\text{do } y \leftarrow v_2 \text{ in } v_3) : \mathbf{M}_{\epsilon_1, \epsilon_2, \epsilon_2} C \rrbracket \quad (8.10)$$

$$= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2} h \circ (\pi_1 \times \text{Id}_B) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad (8.11)$$

$$rhs = \llbracket \Phi \mid \Gamma \vdash \text{do } y \leftarrow (\text{do } x \leftarrow v_1 \text{ in } v_2) \text{ in } v_3 : \mathbf{M}_{\epsilon_1, \epsilon_2, \epsilon_2} C \rrbracket \quad (8.12)$$

$$= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \quad (8.13)$$

$$(8.14)$$

Let's look at fragment  $F$  of  $rhs$ .

$$F = \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ \langle \text{Id}_\Gamma, (\mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle) \rangle \quad (8.15)$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F \quad (8.16)$$

$$\begin{aligned} F &= \mathbf{t}_{\epsilon_1 \cdot \epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times \mu_{\epsilon_1, \epsilon_2, B}) \circ (\text{Id}_\Gamma \times T_{\epsilon_1} g) \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1} \mathbf{t}_{\epsilon_2, \Gamma, B} \circ \mathbf{t}_{\epsilon_1, \Gamma, (T_{\epsilon_2} B)} \circ (\text{Id}_\Gamma \circ T_{\epsilon_1} g) \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{By \textbf{TODO: ref: mu+tstrength}} \\ &= \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1}(\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{By naturality of v-strength} \end{aligned} \quad (8.17)$$

Since  $rhs = \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ F$ ,

$$\begin{aligned} rhs &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1 \cdot \epsilon_2}(h) \circ \mu_{\epsilon_1, \epsilon_2, (\Gamma \times B)} \circ T_{\epsilon_1}(\mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= \mu_{\epsilon_1 \cdot \epsilon_2, \epsilon_3, C} \circ \mu_{\epsilon_1, \epsilon_2, (T_{\epsilon_3} C)} \circ T_{\epsilon_1}(T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad \text{Naturality of } \mu \\ &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\text{Id}_\Gamma \times g)) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \end{aligned} \quad (8.18)$$

Let's now look at the fragment  $G$  of  $rhs$

$$G = T_{\epsilon_1}(\text{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, (\Gamma \times A)} \circ \langle \text{Id}_\Gamma, \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \rangle \quad (8.19)$$

So

$$rhs = \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (8.20)$$

By folding out the  $\langle \dots, \dots \rangle$ , we have

$$G = T_{\epsilon_1}(\text{Id}_\Gamma \times g) \circ \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} \circ (\text{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A}) \circ \langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle \quad (8.21)$$

From the rule **TODO: Ref** showing the commutativity of tensor strength with  $\alpha$ , the following commutes

$$\begin{array}{ccc} \Gamma \xrightarrow{\langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle} \Gamma \times (\Gamma \times T_{\epsilon_1} A) & \xleftarrow{\alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}} & (\Gamma \times \Gamma) \times T_{\epsilon_1} A \\ \downarrow \text{Id}_\Gamma \times \mathbf{t}_{\epsilon_1, \Gamma, A} & & \downarrow \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \\ \Gamma \times T_{\epsilon_1}(\Gamma \times A) & & T_{\epsilon_1}((\Gamma \times \Gamma) \times A) \\ \downarrow \mathbf{t}_{\epsilon_1, \Gamma, \Gamma \times A} & \swarrow T_{\epsilon_1} \alpha_{\Gamma, \Gamma, A} & \\ T_{\epsilon_1}(\Gamma \times (\Gamma \times A)) & & \end{array}$$

Where  $\alpha : ((- \times -) \times -) \rightarrow (- \times (- \times -))$  is a natural isomorphism.

$$\alpha = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle \quad (8.22)$$

$$\alpha^{-1} = \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle \quad (8.23)$$

So:

$$\begin{aligned} G &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ \alpha_{\Gamma, \Gamma, (T_{\epsilon_1} A)}^{-1} \circ \langle \text{Id}_\Gamma, \langle \text{Id}_\Gamma, f \rangle \rangle \\ &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A}) \circ \mathbf{t}_{\epsilon_1, (\Gamma \times \Gamma), A} \circ (\langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \times \text{Id}_{T_{\epsilon_1} A}) \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By definition of } \alpha \text{ and products} \\ &= T_{\epsilon_1}((\text{Id}_\Gamma \times g) \circ \alpha_{\Gamma, \Gamma, A} \circ (\langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \times \text{Id}_A)) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By tensor strength's left-naturality} \\ &= T_{\epsilon_1}((\pi_1 \times \text{Id}_{T_{\epsilon_2} B}) \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \end{aligned} \quad (8.24)$$

Since

$$rhs = \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B}) \circ G \quad (8.25)$$

We Have

$$\begin{aligned} rhs &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h) \circ \mathbf{t}_{\epsilon_2, \Gamma, B} \circ (\pi_1 \times \text{Id}_{T_{\epsilon_2} B}) \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle) \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \mu_{\epsilon_1, \epsilon_2, \epsilon_3, C} \circ T_{\epsilon_1}(\mu_{\epsilon_2, \epsilon_3, C} \circ T_{\epsilon_2}(h \circ (\pi_1 \times \text{Id}_B))) \circ \mathbf{t}_{\epsilon_2, (\Gamma \times A), B} \circ \langle \text{Id}_{(\Gamma \times A)}, g \rangle \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \quad \text{By Left-Tensor Strength} \\ &= lhs \quad \text{Woohoo!} \end{aligned} \quad (8.26)$$

**Case Eta:** Let

$$f = \llbracket \Phi \mid \Gamma \vdash v : A \rightarrow B \rrbracket : \Gamma \rightarrow (B)^A \quad (8.27)$$

By weakening, we have

$$\llbracket \Phi \mid \Gamma, x : A \vdash v : A \rightarrow B \rrbracket = f \circ \pi_1 : \Gamma \times A \rightarrow (B)^A \quad (8.28)$$

$$\llbracket \Phi \mid \Gamma, x : A \vdash v x : B \rrbracket = \text{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \quad (8.29)$$

$$(8.30)$$

Hence, we have

$$\begin{aligned} \llbracket \Phi \mid \Gamma \vdash \lambda x : A. (v x) : A \rightarrow B \rrbracket &= \text{cur}(\text{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \\ \text{app} \circ (\llbracket \Phi \mid \Gamma \vdash \lambda x : A. (v x) : A \rightarrow B \rrbracket \times \text{Id}_A) &= \text{app} \circ (\text{cur}(\text{app} \circ \langle f \circ \pi_1, \pi_2 \rangle) \times \text{Id}_A) \\ &= \text{app} \circ \langle f \circ \pi_1, \pi_2 \rangle \\ &= \text{app} \circ (f \times \text{Id}_A) \end{aligned} \quad (8.31)$$

Hence, by the fact that  $\text{cur}(f)$  is unique in a cartesian closed category,

$$\llbracket \Phi \mid \Gamma \vdash \lambda x : A. (v x) : A \rightarrow B \rrbracket = f = \llbracket \Phi \mid \Gamma \vdash v : A \rightarrow B \rrbracket \quad (8.32)$$

**Case If-True:** Let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket \quad (8.33)$$

$$g = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket \quad (8.34)$$

$$(8.35)$$

Then

$$\begin{aligned} \llbracket \Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A \rrbracket &= \text{app} \circ ((\text{cur}(f \circ \pi_2), \text{cur}(g \circ \pi_2)) \circ \text{inl} \circ \langle \rangle_\Gamma \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ ((\text{cur}(f \circ \pi_2) \circ \langle \rangle_\Gamma) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ (\text{cur}(f \circ \pi_2) \times \text{Id}_\Gamma) \circ (\langle \rangle_\Gamma \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= f \circ \pi_2 \circ \langle \rangle_\Gamma, \text{Id}_\Gamma \rangle \\ &= f \\ &= \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket \end{aligned} \quad (8.36)$$

**Case If-False:** Let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket \quad (8.37)$$

$$g = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket \quad (8.38)$$

$$(8.39)$$

Then

$$\begin{aligned} \llbracket \Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2 : A \rrbracket &= \text{app} \circ ((\text{cur}(f \circ \pi_2), \text{cur}(g \circ \pi_2)) \circ \text{inr} \circ \langle \rangle_\Gamma \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ ((\text{cur}(g \circ \pi_2) \circ \langle \rangle_\Gamma) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \text{app} \circ (\text{cur}(g \circ \pi_2) \times \text{Id}_\Gamma) \circ (\langle \rangle_\Gamma \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= g \circ \pi_2 \circ \langle \rangle_\Gamma, \text{Id}_\Gamma \rangle \\ &= g \\ &= \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket \end{aligned} \quad (8.40)$$

**Case If-Eta:** Let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1 : \text{Bool} \rrbracket \quad (8.41)$$

$$g = \llbracket \Phi \mid \Gamma, x : \text{Bool} \vdash v_2 : A \rrbracket \quad (8.42)$$

$$(8.43)$$

Then by the substitution theorem,

$$\llbracket \Phi \mid \Gamma \vdash v_2 [\text{true}/x] : A \rrbracket = g \circ \langle \text{Id}_\Gamma, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \quad (8.44)$$

$$\llbracket \Phi \mid \Gamma \vdash v_2 [\text{false}/x] : A \rrbracket = g \circ \langle \text{Id}_\Gamma, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \quad (8.45)$$

$$\llbracket \Phi \mid \Gamma \vdash v_2 [v_1/x] : A \rrbracket = g \circ \langle \text{Id}_\Gamma, f \rangle \quad (8.46)$$

Hence we have (Using the diagonal and twist morphisms):

$$\llbracket \Phi \mid \Gamma \vdash \text{if}_{\mathcal{A}} v_1 \text{ then } v_2 [\text{true}/x] \text{ else } v_2 [\text{false}/x] : A \rrbracket \quad (8.47)$$

$$= \text{app} \circ (([\text{cur}(g \circ \langle \text{Id}_\Gamma, \text{inl}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2), \text{cur}(g \circ \langle \text{Id}_\Gamma, \text{inr}_1 \circ \langle \rangle_\Gamma \rangle \circ \pi_2)] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad (8.48)$$

$$= \text{app} \circ (([\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_\Gamma \circ \pi_2 \rangle), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_\Gamma \circ \pi_2 \rangle)] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Pairing property} \quad (8.49)$$

$$= \text{app} \circ (([\text{cur}(g \circ \langle \pi_2, \text{inl}_1 \circ \langle \rangle_\Gamma \circ \pi_1 \rangle), \text{cur}(g \circ \langle \pi_2, \text{inr}_1 \circ \langle \rangle_\Gamma \circ \pi_1 \rangle)] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Terminal is unique} \quad (8.50)$$

$$= \text{app} \circ (([\text{cur}(g \circ (\text{Id}_\Gamma \times (\text{inl}_1 \circ \langle \rangle_1)) \circ \tau_{1,\Gamma}), \text{cur}(g \circ (\text{Id}_\Gamma \times (\text{inr}_1 \circ \langle \rangle_1)) \circ \tau_{1,\Gamma})] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition} \quad (8.51)$$

$$= \text{app} \circ (([\text{cur}(g \circ (\text{Id}_\Gamma \times \text{inl}_1)) \circ \tau_{1,\Gamma}), \text{cur}(g \circ (\text{Id}_\Gamma \times \text{inr}_1)) \circ \tau_{1,\Gamma})] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Identity} = \text{Id}_1 \quad (8.52)$$

$$= \text{app} \circ (([\text{cur}(g \circ \tau_{1+1,\Gamma} \circ (\text{inl}_1 \times \text{Id}_\Gamma)), \text{cur}(g \circ \tau_{1+1,\Gamma} \circ (\text{inr}_1 \times \text{Id}_\Gamma))] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Twist commutivity} \quad (8.53)$$

$$= \text{app} \circ (([\text{cur}(g \circ \tau_{1+1,\Gamma}) \circ \text{inl}_1, \text{cur}(g \circ \tau_{1+1,\Gamma}) \circ \text{inr}_1] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Exponential property} \quad (8.54)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1,\Gamma}) \circ [\text{inl}_1, \text{inr}_1] \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring out cur(..)} \quad (8.55)$$

$$= \text{app} \circ ((\text{cur}(g \circ \tau_{1+1,\Gamma}) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Since } [\text{inl}, \text{inr}] \text{ is the identity} \quad (8.56)$$

$$= \text{app} \circ (\text{cur}(g \circ \tau_{1+1,\Gamma}) \times \text{Id}_\Gamma) \circ (f \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Factoring} \quad (8.57)$$

$$= g \circ \tau_{1+1,\Gamma} \circ (f \times \text{Id}_\Gamma) \circ \delta_\Gamma \quad \text{Definition of app, cur(..)} \quad (8.58)$$

$$= g \circ (\text{Id}_\Gamma \times f) \circ \tau_{1,\Gamma} \circ \delta_\Gamma \quad \text{Twist commutivity} \quad (8.59)$$

$$= g \circ (\text{Id}_\Gamma \times f) \circ \langle \text{Id}_\Gamma, \text{Id}_\Gamma \rangle \quad \text{Twist, diagonal definitions} \quad (8.60)$$

$$= g \circ \langle \text{Id}_\Gamma, f \rangle \quad (8.61)$$

$$= \llbracket \Phi \mid \Gamma \vdash v_2 [v_1/x] : A \rrbracket \quad (8.62)$$

$$(8.63)$$

**Case Effect-Beta:** let

$$h = \llbracket \Phi \vdash \epsilon : \text{Effect} \rrbracket \quad (8.64)$$

$$f = \llbracket \Phi, \alpha \mid \Gamma \vdash v : A \rrbracket \quad (8.65)$$

$$A = \llbracket \Phi, \alpha \vdash A[\alpha/\alpha] : \text{Type} \rrbracket \quad (8.66)$$

Then

$$\llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. v : \forall \alpha. A \rrbracket = \bar{f} \quad (8.67)$$

So

$$\llbracket \Phi \mid \Gamma \vdash (\Lambda \alpha. v) : \epsilon : \forall \alpha. A \rrbracket = \langle \text{Id}_I, h \rangle^* (\epsilon_A) \circ \bar{f} \quad (8.68)$$

$$= \langle \text{Id}_I, h \rangle^* (\epsilon_A) \circ \langle \text{Id}_I, h \rangle^* (\pi_1^* (\bar{f})) \quad \text{Identity functor} \quad (8.69)$$

$$= \langle \text{Id}_I, h \rangle^* (\epsilon_A \circ \pi_1^* (\bar{f})) \quad (8.70)$$

$$= \langle \text{Id}_I, h \rangle^* (f) \quad \text{By adjunction} \quad (8.71)$$

$$= \llbracket \Phi \mid \Gamma \vdash v [\epsilon/\alpha] : A[\epsilon/\alpha] \rrbracket \quad \text{By substitution theorem} \quad (8.72)$$

$$(8.73)$$

**Case Effect-Eta:** **TODO:** Use re-indexing functors rather than post composition Let

$$f = \llbracket \Phi \mid \Gamma \vdash v : \forall \alpha. A \rrbracket \quad (8.74)$$

$$A = \llbracket \Phi, \alpha \vdash A : \text{Type} \rrbracket \quad (8.75)$$

so

$$\llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. (v \alpha) : \forall \alpha. A \rrbracket = \overline{\llbracket \Phi, \alpha \mid \Gamma \vdash \epsilon \alpha : \forall \alpha. A \rrbracket} \quad (8.76)$$

$$= \overline{\langle \text{Id}_{I \times U}, \pi_2 \rangle^* (\epsilon_{\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \pi_1^*(f)} \quad (8.77)$$

Let's look at  $\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket$ .

We have the weakening:

$$\iota \pi \times : \Phi, \alpha, \beta \triangleright \Phi, \beta \quad (8.78)$$

So by the weakening theorem on type denotations:

$$\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket = \llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket \circ (\pi_1 \times \text{Id}_U) \quad (8.79)$$

$$\forall_{I \times U} (\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket) = \forall_I (\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket) \circ \pi_1 \quad (8.80)$$

$$= \pi_1^* \forall_I (\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket) \quad (8.81)$$

$$\epsilon_{\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket} = \overline{\text{Id}_{\pi_1^* \forall_I (\llbracket \Phi, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket)}} \quad (8.82)$$

$$= \overline{\text{Id}_{\pi_1^* \forall_I A}} \quad (8.83)$$

$$= \overline{\pi_1^* (\text{Id}_{\forall_I A})} \quad (8.84)$$

$$= \overline{\pi_1^* (\overline{\epsilon_A})} \quad (8.85)$$

$$= \overline{(\pi_1 \times \text{Id}_U)^* (\epsilon_A)} \quad (8.86)$$

$$= (\pi_1 \times \text{Id}_U)^* (\epsilon_A) \quad (8.87)$$

So

$$\llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. (v \alpha) : \forall \alpha. A \rrbracket = \overline{\langle \text{Id}_{I \times U}, \pi_2 \rangle^* (\epsilon_{\llbracket \Phi, \alpha, \beta \vdash A[\beta/\alpha] : \text{Type} \rrbracket}) \circ \pi_1^*(f)} \quad (8.88)$$

$$= \overline{\langle \text{Id}_{I \times U}, \pi_2 \rangle^* ((\pi_1 \times \text{Id}_U)^* (\epsilon_A)) \circ \pi_1^*(f)} \quad (8.89)$$

$$= \overline{\langle \pi_1, \pi_2 \rangle^* (\epsilon_A) \circ \pi_1^*(f)} \quad (8.90)$$

$$= \overline{\text{Id}_{I \times U}^* (\epsilon_A) \circ \pi_1^*(f)} \quad (8.91)$$

$$= \overline{\epsilon_A \circ \pi_1^*(f)} \quad \text{By adjunction} \quad (8.92)$$

$$= f \quad (8.93)$$

### 8.0.3 Congruences

These cases can be proved fairly mechanically by assuming the preconditions, using induction to prove that the matching pairs of sub-expressions have equal denotations, then constructing the denotations of the expressions using the equal denotations which gives trivially equal denotations.

**Case Lambda:** By inversion, we have

*ebere*lation  $\Phi \Gamma, x : A v_1 v_2 B$  By induction, we therefore have  $\llbracket \Phi \mid \Gamma, x : A \vdash v_1 : B \rrbracket = \llbracket \Phi \mid \Gamma, x : A \vdash v_2 : B \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma, x: A \vdash v_1: B \rrbracket = \llbracket \Phi \mid \Gamma, x: A \vdash v_2: B \rrbracket \quad (8.94)$$

And so

$$\llbracket \Phi \mid \Gamma \vdash \lambda x: A. v_1: A \rightarrow B \rrbracket = \text{cur}(f) = \llbracket \Phi \mid \Gamma \vdash \lambda x: A. v_2: A \rightarrow B \rrbracket \quad (8.95)$$

**Case Return:** By inversion, we have  $\Phi \mid \Gamma \vdash v_1 \approx v_2: A$ . By induction, we therefore have  $\llbracket \Phi \mid \Gamma \vdash v_1: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket \quad (8.96)$$

And so

$$\llbracket \Phi \mid \Gamma \vdash \text{return } v_1: \mathbf{M}_1 A \rrbracket = \eta_A \circ f = \llbracket \Phi \mid \Gamma \vdash \text{return } v_2: \mathbf{M}_1 A \rrbracket \quad (8.97)$$

**Case Apply:** By inversion, we have  $\Phi \mid \Gamma \vdash v_1 \approx v'_1: A \rightarrow B$  and  $\Phi \mid \Gamma \vdash v_2 \approx v'_2: A$ . By induction, we therefore have  $\llbracket \Phi \mid \Gamma \vdash v_1: A \rightarrow B \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: A \rightarrow B \rrbracket$  and  $\llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_2: A \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1: A \rightarrow B \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: A \rightarrow B \rrbracket \quad (8.98)$$

$$g = \llbracket \Phi \mid \Gamma \vdash v_2: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_2: A \rrbracket \quad (8.99)$$

And so

$$\llbracket \Phi \mid \Gamma \vdash v_1 v_2: B \rrbracket = \text{app} \circ \langle f, g \rangle = \llbracket \Phi \mid \Gamma \vdash v'_1 v'_2: B \rrbracket \quad (8.100)$$

**Case Bind:** By inversion, we have  $\Phi \mid \Gamma \vdash v_1 \approx v'_1: \mathbf{M}_{\epsilon_1} A$  and  $\text{eberelation} \Phi \Gamma, x: A v_2 v'_2 \mathbf{M}_{\epsilon_2} B$ . By induction, we therefore have  $\llbracket \Phi \mid \Gamma \vdash v_1: \mathbf{M}_{\epsilon_1} A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: \mathbf{M}_{\epsilon_1} A \rrbracket$  and  $\llbracket \Phi \mid \Gamma, x: A \vdash v_2: \mathbf{M}_{\epsilon_2} B \rrbracket = \llbracket \Phi \mid \Gamma, x: A \vdash v'_2: \mathbf{M}_{\epsilon_2} B \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1: \mathbf{M}_{\epsilon_1} A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: \mathbf{M}_{\epsilon_1} A \rrbracket \quad (8.101)$$

$$g = \llbracket \Phi \mid \Gamma, x: A \vdash v_2: \mathbf{M}_{\epsilon_2} B \rrbracket = \llbracket \Phi \mid \Gamma, x: A \vdash v'_2: \mathbf{M}_{\epsilon_2} B \rrbracket \quad (8.102)$$

And so

$$\begin{aligned} \llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2: \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket &= \mu_{\epsilon_1, \epsilon_2, B} \circ T_{\epsilon_1} g \circ \mathbf{t}_{\epsilon_1, \Gamma, A} \circ \langle \text{Id}_\Gamma, f \rangle \\ &= \llbracket \Phi \mid \Gamma \vdash \text{do } x \leftarrow v_1 \text{ in } v_2: \mathbf{M}_{\epsilon_1 \cdot \epsilon_2} A \rrbracket \end{aligned} \quad (8.103)$$

**Case If:** By inversion, we have  $\Phi \mid \Gamma \vdash v \approx v': \text{Bool}$ ,  $\Phi \mid \Gamma \vdash v_1 \approx v'_1: A$  and  $\Phi \mid \Gamma \vdash v_2 \approx v'_2: A$ . By induction, we therefore have  $\llbracket \Phi \mid \Gamma \vdash v: \text{Bool} \rrbracket = \llbracket \Phi \mid \Gamma \vdash v': \text{Bool} \rrbracket$ ,  $\llbracket \Phi \mid \Gamma \vdash v_1: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: A \rrbracket$  and  $\llbracket \Phi \mid \Gamma, x: A \vdash v_2: A \rrbracket = \llbracket \Phi \mid \Gamma, x: A \vdash v'_2: A \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma \vdash v: \text{Bool} \rrbracket = \llbracket \Phi \mid \Gamma \vdash v': \text{Bool} \rrbracket \quad (8.104)$$

$$g = \llbracket \Phi \mid \Gamma \vdash v_1: A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v'_1: A \rrbracket \quad (8.105)$$

$$h = \llbracket \Phi \mid \Gamma, x: A \vdash v_2: A \rrbracket = \llbracket \Phi \mid \Gamma, x: A \vdash v'_2: A \rrbracket \quad (8.106)$$

And so

$$\begin{aligned} \llbracket \Phi \mid \Gamma \vdash \text{if}_A v \text{ then } v_1 \text{ else } v_2: A \rrbracket &= \text{app} \circ (((\text{cur}(g \circ \pi_2), \text{cur}(h \circ \pi_2)) \circ f) \times \text{Id}_\Gamma) \circ \delta_\Gamma \\ &= \llbracket \Phi \mid \Gamma \vdash \text{if}_A v' \text{ then } v'_1 \text{ else } v'_2: A \rrbracket \end{aligned} \quad (8.107)$$



**Case Subtype:** By inversion, we have  $\Phi \mid \Gamma \vdash v_1 \approx v_2 : A$ , and  $A \leq_{\Phi} B$ . By induction, we therefore have  $\llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v_2 : A \rrbracket$

Then let

$$f = \llbracket \Phi \mid \Gamma \vdash v_1 : A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v_2 : B \rrbracket \quad (8.108)$$

$$g = \llbracket A \leq_{\Phi} B \rrbracket \quad (8.109)$$

And so

$$\llbracket \Phi \mid \Gamma \vdash v_1 : B \rrbracket = g \circ f = \llbracket \Phi \mid \Gamma \vdash v_1 : B \rrbracket \quad (8.110)$$

**Case Effect-Lambda:** By inversion, we have  $\Phi, \alpha \mid \Gamma \vdash v_1 \approx v_2 : A$ . So by induction,  $\llbracket \Phi, \alpha \mid \Gamma \vdash v_1 : A \rrbracket = \llbracket \Phi, \alpha \mid \Gamma \vdash v_2 : A \rrbracket$

So

$$\llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. v_1 : \forall \alpha. A \rrbracket = \overline{\llbracket \Phi, \alpha \mid \Gamma \vdash v_1 : A \rrbracket} \quad (8.111)$$

$$= \overline{\llbracket \Phi, \alpha \mid \Gamma \vdash v_2 : A \rrbracket} \quad (8.112)$$

$$= \llbracket \Phi \mid \Gamma \vdash \Lambda \alpha. v_2 : \forall \alpha. A \rrbracket \quad (8.113)$$

**Case Effect-Apply:** By inversion, we have  $\Phi \mid \Gamma \vdash v_1 \approx v_2 : \forall \alpha. A$  and  $\Phi \vdash \epsilon : \mathbf{Effect}$ .

So by induction, we have  $\llbracket \Phi \mid \Gamma \vdash v_1 : \forall \alpha. A \rrbracket = \llbracket \Phi \mid \Gamma \vdash v_2 : \forall \alpha. A \rrbracket$

So

$$\llbracket \Phi \mid \Gamma \vdash v_1 \epsilon : A [\epsilon/\alpha] \rrbracket = \langle \text{Id}_I, \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket \rangle^* (\epsilon_A) \circ \llbracket \Phi \mid \Gamma \vdash v_1 : \forall \alpha. A \rrbracket \quad (8.114)$$

$$= \langle \text{Id}_I, \llbracket \Phi \vdash \epsilon : \mathbf{Effect} \rrbracket \rangle^* (\epsilon_A) \circ \llbracket \Phi \mid \Gamma \vdash v_2 : \forall \alpha. A \rrbracket \quad (8.115)$$

$$= \llbracket \Phi \mid \Gamma \vdash v_2 \epsilon : A [\epsilon/\alpha] \rrbracket \quad (8.116)$$

□