random process assignment 1

Alireza Mousazadeh

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1 One

$X \sim Binom(n,k); n = 2000, k = 600$

$$\Pr(k = 600) = \binom{2000}{600} \frac{1}{2}^{600} \frac{1}{2}^{1400}$$

2 Two

$PDF: f_X(x) = A\exp(-x)u(x), X \sim f_X(x)$

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x')dx' &= 1 \\ \int_{-\infty}^{\infty} Ae^{-x'}u(x')dx' &= 1 \\ \int_{0}^{\infty} Ae^{-x'}dx' &= 1 \\ -Ae^{-x'}\Big|_0^{\infty} &= A = 1 \end{aligned} \tag{1}$$

$$F_X(x) = \int_0^x e^{-x'}dx' = -e^{-x'}\Big|_0^x = 1 - e^{-x} \tag{2}$$

$$\begin{aligned} &\Pr\{1 < x < 2\} = \Pr\{x > 1, x < 2\} \\ &= F_X(2) - F_X(1) = e^{-1} - e^{-2} \end{aligned} \tag{3}$$

3 Three

$f_{XY}(x,y) = A\exp(-(x+y))u(x)u(y)$

$$\iint_{\mathcal{X},\mathcal{Y}} f_{XY}(x,y)dxdy = 1$$

$$\begin{aligned} \iint_0^{\infty} Ae^{-(x+y)}dxdy = A\int_0^{\infty} e^{-x}\int_0^{\infty} e^{-y}dxdy &= 1 \\ = A &= 1 \end{aligned} \tag{4}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}}{f_Y} = f_X \\ f_{XY} &= f_Y . f_X \\ i \quad f_X &= \int_{\mathcal{Y}} f_{XY} = e^{-x} \\ ii \quad f_Y &= \int_{\mathcal{X}} f_{XY} = e^{-y} \\ i, ii \rightarrow f_{XY} &= e^{-x}.e^{-y} = e^{-(x+y)} \quad \diamond \end{aligned} \tag{5}$$

$\rightarrow X, Y$ are independent.

$$\begin{aligned} \rightarrow f_{X|Y} &= f_X = e^{-x} \\ f_{Y|X} &= f_Y = e^{-y} \end{aligned}$$

4 Four

$Y = g(x) = a\sin(X + \theta)$ let solutions be x_n :

$$g(x_n) = y \quad g'(x_n) = a\cos(x_n + \theta)$$

$$\cos(x_n + \theta) = \sqrt{1 - \sin^2(x_n + \theta)} = \sqrt{1 - (\frac{y}{a})^2}$$

$$\rightarrow g(x_n) = \frac{1}{a}\sqrt{a^2 - y^2} \rightarrow g'(x_n) = \sqrt{a^2 - y^2}$$

$$\rightarrow f_Y = \sum_n \left(\frac{f_X(x_n)}{\sqrt{a^2 - y^2}} \right)$$

and if $X \sim Uni(-\pi, \pi)$ then $g(x_n) = y$ has two solution over $(-\pi, \pi)$ for $-4 < y < 4$:

$$f_Y = \left(\frac{1}{2\pi\sqrt{16 - y^2}} \right) + \left(\frac{1}{2\pi\sqrt{16 - y^2}} \right) = \left(\frac{2}{2\pi\sqrt{16 - y^2}} \right)$$

5 Five

$X \sim P(a)$; $P(X = k) = \exp(-a)a^k/k! \rightarrow E_x, \sigma_X^2 = ?$

$$E[X] = \sum_{k=1}^{\infty} k \frac{e^{-a}a^k}{k!}$$

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} = \frac{\mathrm{d}}{\mathrm{da}} e^a = \frac{1}{a} \sum_{k=1}^{\infty} k \frac{a^k}{k!} \rightarrow E[X] = a \quad \diamond$$

$$\frac{\mathrm{d}}{\mathrm{da}}\left(\frac{\mathrm{d}}{\mathrm{da}}\right) = \frac{1}{a^2} \sum_{k=1}^{\infty} k(k-1)\frac{a^k}{k!} \rightarrow E[k^2] = a^2 + a$$

$$\rightarrow \sigma_X^2 = E[k^2] - E^2[k] = a^2 \quad \diamond$$

6 Six

$\Phi_X = e^{j\mu\omega - 1/_2\sigma^2\omega^2} = E[e^{j\mu\omega}]$ defining new r.v. Z:

$$X = \mu + \sigma Z \; ; \; Z \sim \mathcal{N}(0,1)$$

$$\Phi_X(\omega) = e^{j\mu\omega}.E[e^{j\sigma\omega z}] = e^{j\mu\omega}.\Phi_Z(\sigma\omega)$$

$$\begin{aligned}\Phi_Z(t) &= \frac{1}{\sqrt{2\pi}} \int e^{jtz}.e^{-z^2/2}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int e^{(jtz - z^2/2)}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}((z-jt)^2 + t^2)}\, dz \\ &= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(z-jt)^2}\, dz = e^{\frac{-t^2}{2}} \\ &\rightarrow \Phi_X(\omega) = e^{j\mu\omega}.e^{-\frac{(\sigma\omega)^2}{2}} \quad \diamond\end{aligned}$$

7 Seven

because X and Y are normal, Z, W and their joint distribution are normal too. and we have the following:

$$\mu_z = 10 + 0 = 10$$

$$\mu_w = 10 - 0 = 10$$

$$\sigma_z^2 = 4 + 1 - 2(.5)$$

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$$C_{zw} = C_{x+y,x-y} = Var(x) - Var(y) = 4 - 1 = 3$$

covariance matrix: $\rightarrow \Sigma_{zw} = \begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix}$, $\mu_z = 10$, $\mu_w = 10$

$$\det(\Sigma_{zw}) = 24 - 9 = 15 \rightarrow \Sigma_{zw}^{-1} = \frac{1}{15}\begin{pmatrix} 4 & -3 \\ -3 & 6 \end{pmatrix}$$

$$\rightarrow f_{zw} = \frac{1}{2\pi\sqrt{15}} \exp\Big((-1/(2.15)\left(z-10 \quad w-10\right) . \begin{pmatrix} 4 & -3 \\ -3 & 6 \end{pmatrix} . \Big($$

8 Eight

following code generates a gif describing the central limit theory:

```
Nsim = 50000;

n_max = 100;
n_values = 1:n_max;

gif_name = 'CLT_animation.gif';

for n = n_values

X = rand(Nsim, n);

S = sum(X, 2);

mu = n * 0.5;
sigma = sqrt(n * 1/12);

histogram(S, 'Normalization', 'pd

x = linspace(min(S), max(S), 300)
y = 1/(sigma*sqrt(2*pi)) * exp(-(
plot(x, y, 'r', 'LineWidth', 2);

title(['Central Limit Theorem
| n = ' num2str(n)]);
xlabel('S_n');
ylabel('Density');
xlim([mu - 5*sigma, mu + 5*sigma]
grid on;

drawnow;

frame = getframe(gcf);
im = frame2im(frame);
[A, map] = rgb2ind(im, 256);

if n == 1
```

```
imwrite(A, map, gif_name, 'gif', 'LoopCount', Inf, 'DelayTime', 0.1);
else
imwrite(A, map, gif_name, 'gif', 'WriteMode', 'append', 'DelayTime', 0.
end

hold off;
end
```