Day8 exercise solutions

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Oct. 4th, 2024

```
# Set global code chunk options
knitr::opts_chunk$set(warning = FALSE)
```

```
# load required libraries
library("skimr")
library("dplyr")
library("magrittr")
library("ggplot2")

# define functions
`%notin%` <- Negate(`%in%`)
```

Problem 1

```
data(bliss, package = "faraway")
bliss %<>% mutate(ratio = dead/(alive + dead))
```

1.A)

```
skim(bliss)
```

Table 1: Data summary

Name	bliss
Number of rows	5
Number of columns	4
Column type frequency:	
numeric	4
Group variables	None

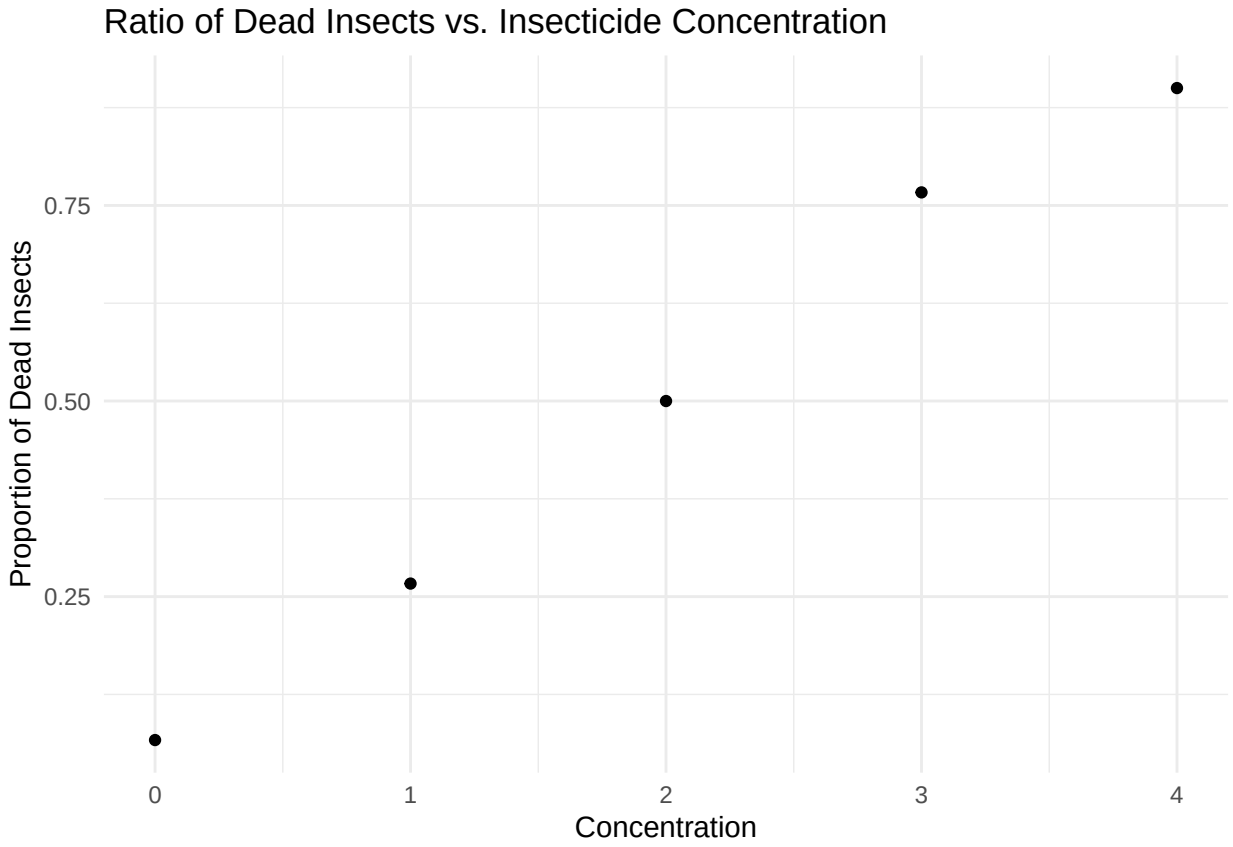
Variable type: numeric

skim_variable	n_missing	complete_rate	mean	sd	p0	p25	p50	p75	p100	hist
dead	0	1	15.0	10.32	2.00	8.00	15.0	23.00	27.0	
alive	0	1	15.0	10.32	3.00	7.00	15.0	22.00	28.0	
conc	0	1	2.0	1.58	0.00	1.00	2.0	3.00	4.0	
ratio	0	1	0.5	0.34	0.07	0.27	0.5	0.77	0.9	

```r
head(bliss)
```

```
##   dead alive conc      ratio
## 1    2    28    0 0.06666667
## 2    8    22    1 0.26666667
## 3   15    15    2 0.50000000
## 4   23     7    3 0.76666667
## 5   27     3    4 0.90000000
```

```r
# Load the data
data(bliss, package = "faraway")

# Calculate total insects and ratio
bliss$ratio <- bliss$dead / (bliss$dead + bliss$alive)

# Plot the ratio vs concentration
ggplot(bliss, aes(x = conc, y = ratio)) +
  geom_point() +
  labs(title = "Ratio of Dead Insects vs. Insecticide Concentration",
       x = "Concentration",
       y = "Proportion of Dead Insects") +
  theme_minimal()
```



1.B)

```r
logit_model <- glm(cbind(dead,alive)~conc, family = binomial(link=logit), data = bliss)
```

```r
summary(logit_model)
```

```
##
## Call:
## glm(formula = cbind(dead, alive) ~ conc, family = binomial(link = logit),
##     data = bliss)
##
## Deviance Residuals:
##       1        2        3        4        5
## -0.4510   0.3597   0.0000   0.0643  -0.2045
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -2.3238     0.4179  -5.561 2.69e-08 ***
## conc          1.1619     0.1814   6.405 1.51e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 64.76327  on 4  degrees of freedom
## Residual deviance:  0.37875  on 3  degrees of freedom
## AIC: 20.854
##
## Number of Fisher Scoring iterations: 4
```

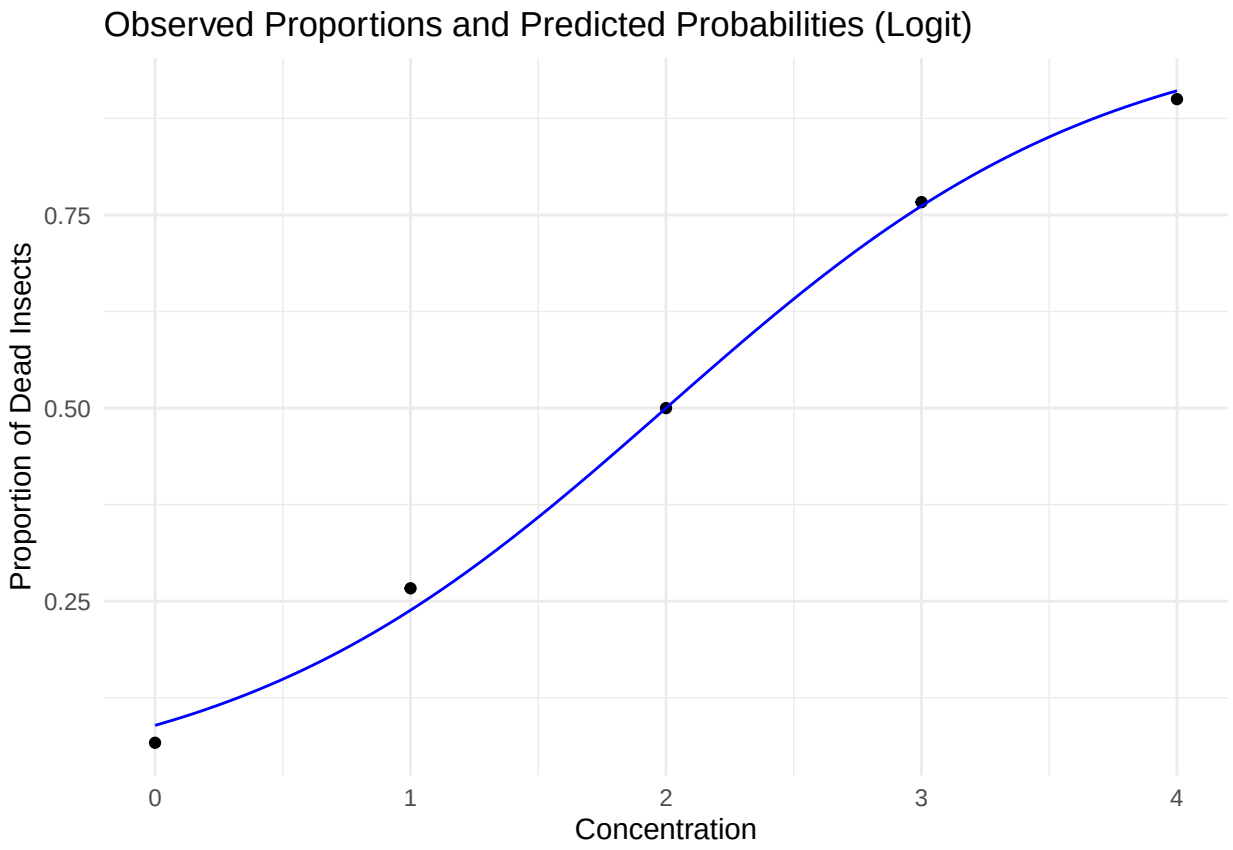
1.C)

```r
# Calculate predicted values manually
coeffs <- coef(logit_model)
linear_pred <- coeffs[1] + coeffs[2] * bliss$conc
manual_pred <- exp(linear_pred) / (1 + exp(linear_pred))

# Compare with fitted values
fitted_pred <- fitted(logit_model)

# Compare results
comparison_df <- data.frame(
  concentration = bliss$conc,
  manual = manual_pred,
  fitted = fitted_pred,
  difference = abs(manual_pred - fitted_pred)
)
print(comparison_df)
```

```
##   concentration     manual     fitted difference
## 1             0 0.08917177 0.08917177          0
## 2             1 0.23832314 0.23832314          0
## 3             2 0.50000000 0.50000000          0
## 4             3 0.76167686 0.76167686          0
## 5             4 0.91082823 0.91082823          0
```

1.D)

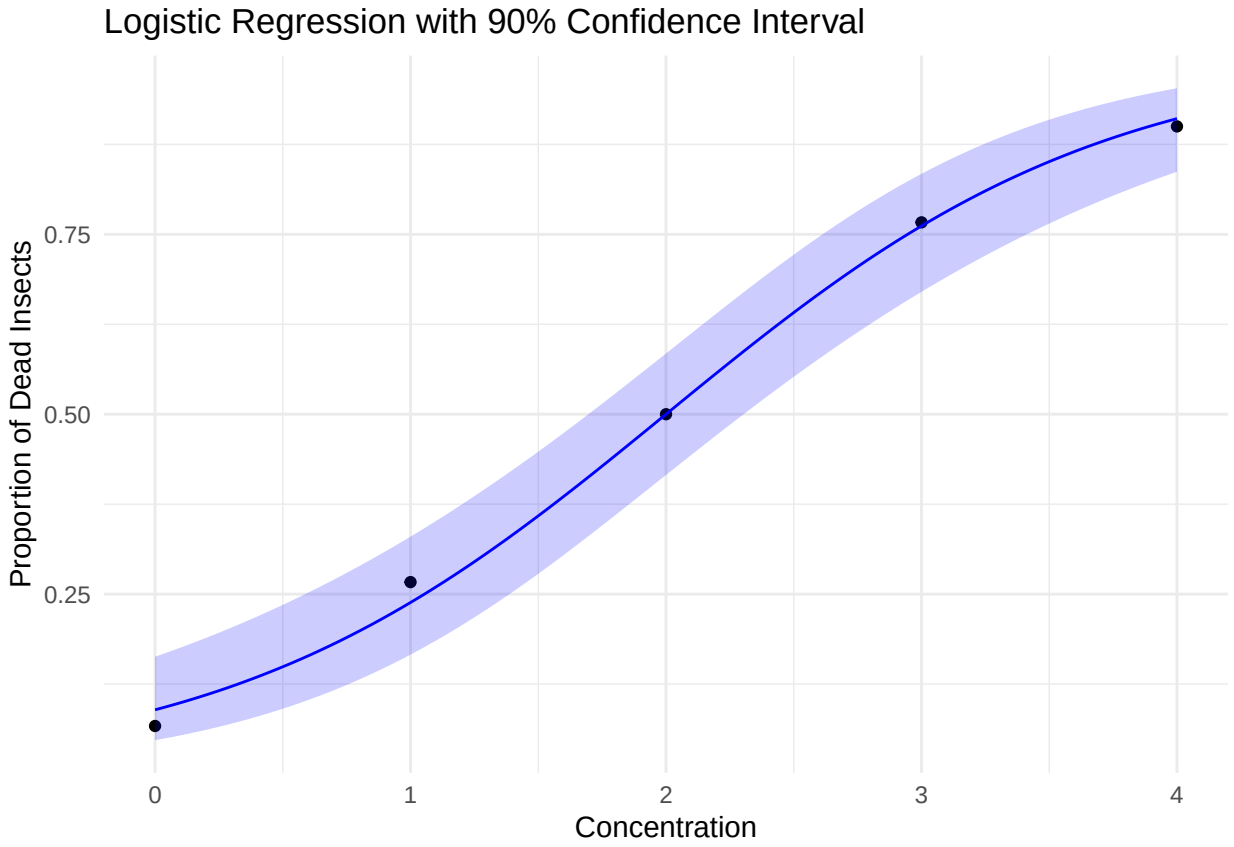
```r
# Create prediction grid
pred_grid <- data.frame(conc = seq(min(bliss$conc), max(bliss$conc), length.out = 100))
pred_grid$pred <- predict(logit_model, newdata = pred_grid, type = "response")

ggplot() +
  geom_point(data = bliss, aes(x = conc, y = ratio)) +
  geom_line(data = pred_grid, aes(x = conc, y = pred), color = "blue") +
  labs(title = "Observed Proportions and Predicted Probabilities (Logit)",
       x = "Concentration",
       y = "Proportion of Dead Insects") +
  theme_minimal()
```



1.E)

```r
# Calculate confidence intervals
pred_ci <- predict(logit_model,
                   newdata = pred_grid,
                   type = "link",
                   se.fit = TRUE)

# Transform to probability scale
ci_lower <- plogis(pred_ci$fit - 1.645 * pred_ci$se.fit)  # 90% CI
ci_upper <- plogis(pred_ci$fit + 1.645 * pred_ci$se.fit)

# Add to plot
```

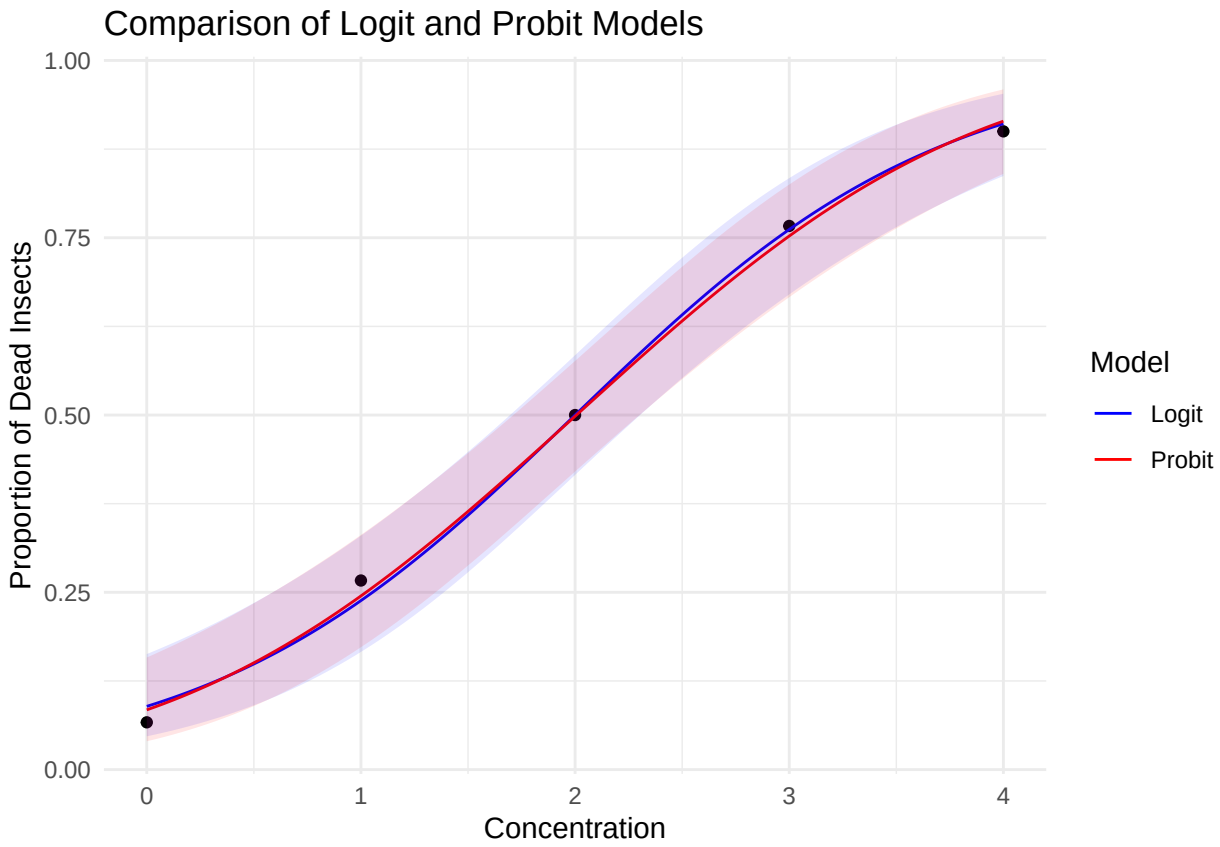
```r
pred_grid$lower <- ci_lower
pred_grid$upper <- ci_upper

ggplot() +
  geom_point(data = bliss, aes(x = conc, y = ratio)) +
  geom_line(data = pred_grid, aes(x = conc, y = pred), color = "blue") +
  geom_ribbon(data = pred_grid,
              aes(x = conc, ymin = lower, ymax = upper),
              alpha = 0.2,
              fill = "blue") +
  labs(title = "Logistic Regression with 90% Confidence Interval",
       x = "Concentration",
       y = "Proportion of Dead Insects") +
  theme_minimal()
```




```r
# Fit probit model
probit_model <- glm(cbind(dead, alive) ~ conc,
                    family = binomial(link = "probit"),
                    data = bliss)

# Get predictions
pred_grid$probit_pred <- predict(probit_model,
                                 newdata = pred_grid,
                                 type = "response")

# Calculate confidence intervals
probit_ci <- predict(probit_model,
```

```
                    newdata = pred_grid,
                    type = "link",
                    se.fit = TRUE)

pred_grid$probit_lower <- pnorm(probit_ci$fit - 1.645 * probit_ci$se.fit)
pred_grid$probit_upper <- pnorm(probit_ci$fit + 1.645 * probit_ci$se.fit)
```

```
# Plot both models
ggplot() +
  geom_point(data = bliss, aes(x = conc, y = ratio)) +
  geom_line(data = pred_grid, aes(x = conc, y = pred, color = "Logit")) +
  geom_line(data = pred_grid, aes(x = conc, y = probit_pred, color = "Probit")) +
  geom_ribbon(data = pred_grid,
              aes(x = conc, ymin = lower, ymax = upper),
              alpha = 0.1, fill = "blue",
              fill = "blue") +
  geom_ribbon(data = pred_grid,
              aes(x = conc, ymin = probit_lower, ymax = probit_upper),
              alpha = 0.1, fill = "red") +
  scale_color_manual(values = c("blue", "red")) +
  labs(title = "Comparison of Logit and Probit Models",
       x = "Concentration",
       y = "Proportion of Dead Insects",
       color = "Model") +
  theme_minimal()
```



Problem 2: Exponential Family

The exponential family has the form:

$f(y;\theta,\phi) = exp((y\theta - b(\theta))/\phi + c(y,\phi))$

(a) Exponential Distribution

The probability density function is: $f(y;\lambda) = \lambda e^{-\lambda y}$

We can rewrite this as: $f(y;\lambda) = exp(log(\lambda) - \lambda y) = exp(-\lambda y + log(\lambda))$

This belongs to the exponential family with:

- $\theta = -\lambda$
- $\phi = 1$
- $b(\theta) = -log(-\theta)$
- $c(y,\phi) = 0$

(b) Binomial Distribution

The probability density function is: $f(y;\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$

We can rewrite this as: $f(y;\pi) = exp(ylog(\pi/(1-\pi)) + nlog(1-\pi) + log(\binom{n}{y}))$ $= exp(y\theta - nlog(1+e^\theta) + log(\binom{n}{y}))$

where $\theta = log(\pi/(1-\pi))$

This belongs to the exponential family with:

- $\theta = log(\pi/(1-\pi))$
- $\phi = 1$
- $b(\theta) = nlog(1+e^\theta)$
- $c(y,\phi) = log(\binom{n}{y})$

(c) Uniform Distribution

– version 1

The probability density function is: $f(y;\theta) = 1/\theta, \quad 0 < y < \theta$

This cannot be written in exponential family form because the support of y depends on θ.

– version 2

The probability density function is: The continuous uniform distribution Uniform(a,b) has the following probability density function:

$$p(x \mid a,b) = \frac{1}{b-a}, \quad x \in [a,b].$$

where:

1. **Sufficient statistic** ϕ: Since the distribution is constant over $[a,b]$, it does not depend on x. Thus, we set $\phi = 0$.
2. **Natural parameter** θ: The natural parameter, θ, is also zero, as there's no variation in x in the density.
3. **Base measure** $c(y,\phi)$: This is the constant $c(y,\phi) = \frac{1}{b-a}$, representing the flat density within the interval $[a,b]$.

4. **Log-partition function** $b(\theta)$: Since there's no dependence on x or θ, we have $b(\theta) = 0$.

(d) Normal Distribution (known variance)

The probability density function is: $f(y; \mu, \sigma^2) = (1/\sqrt{2\pi\sigma^2})exp(-(y-\mu)^2/(2\sigma^2))$

We can rewrite this as: $f(y; \mu, \sigma^2) = exp(y\mu/\sigma^2 - \mu^2/(2\sigma^2) - y^2/(2\sigma^2) - (1/2)log(2\pi\sigma^2))$

– Version 1

This belongs to the exponential family with:

- $\theta = \mu$
- $\phi = \sigma^2$
- $b(\theta) = \theta^2/2$
- $c(y, \phi) = -y^2/(2\phi) - (1/2)log(2\pi\phi)$

– Version 2

- **Natural parameter**: $\theta = \frac{\mu}{\sigma^2}$
- **Dispersion parameter**: $\phi = \sigma^2$
- **Function** $b(\theta)$: $b(\theta) = \frac{\mu^2}{2\sigma^2}$
- **Function** $c(y, \phi)$: $c(y, \phi) = -\frac{y^2}{2\phi^2} - \frac{1}{2}\ln(2\pi\phi^2)$