Problem 2

(a) $f(y|\lambda) = \lambda \cdot e^{-\lambda y} = \exp(\log(\lambda \cdot e^{-\lambda y}))$

$= \exp(\log(\lambda) + \log(e^{-\lambda y}))$

$= \exp(\log(\lambda) - \lambda y \cdot \log(e))$

$= \exp(\log(\lambda) - \lambda y) = \exp(\log(\lambda)) \cdot \exp(-\lambda y)$

$= \lambda e^{-\lambda y}$

$\Rightarrow \theta = -\lambda,\ b(\theta) = -\log(\lambda),\ \phi = 1,\ c(y, \phi) = 0$

(b) $f(y|\pi) = \binom{n}{y} \pi^y (1-\pi)^{n-y} = \frac{n!}{y!(n-y)!} \pi^y (1-\pi)^{n-y}$

$= \exp\left(\log\left(\frac{n!}{y!(n-y)!} \pi^y (1-\pi)^{n-y}\right)\right)$

$= \exp\left(\log\left(\frac{n!}{y!(n-y)!} \pi^y\right) + \log\left((1-\pi)^{n-y}\right)\right)$

$= \exp\left(\log\left(\frac{n!}{y!(n-y)!}\right)\right) + \exp\left(\log(\pi^y) + \log\left((1-\pi)^{n-y}\right)\right)$

$= \binom{n}{y} + \exp(y \log(\pi) + (n-y)\log(1-\pi))$

$= \binom{n}{y} + \exp(y \log(\pi) - y \log(1-\pi) + n \log(1-\pi))$

$= \binom{n}{y} + \exp\left(y \log\left(\frac{\pi}{1-\pi}\right) + n \log(1-\pi)\right)$

$\Rightarrow \theta = \log\left(\frac{\pi}{1-\pi}\right)$

$\Rightarrow b(\theta) = -n \log(1-\pi) = n \log(1 + e^{\theta})$

$\Rightarrow \phi = 1,\ c(y, \phi) = \binom{n}{y}$

(c) $f(y|\theta) = \frac{1}{\theta} = \exp\left(\log\left(\frac{1}{\theta}\right)\right)$

$\Rightarrow$ Not part of the exponential family, since we cannot form the function in the form of $\exp\left(\frac{\theta y - b(\theta)}{\phi} + c(y, \phi)\right)$.

(d) ~~$f(y, u) = \frac{1}{\sqrt{2\pi \cdot 2}}$~~

$$f(y, u) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(y-u)^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} + \exp\left(\log\left(\exp\left(-\frac{(y-u)^2}{2\sigma^2}\right)\right)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} + \exp\left(\log\left(\exp(-(y-u)^2) - \exp(2\sigma^2)\right)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{y^2 - 2yu + u^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{y^2}{2\sigma^2} + \frac{yu}{\sigma^2} - \frac{u^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{y^2}{2\sigma^2} + \frac{yu}{\sigma^2} - \frac{u^2}{2\sigma^2} + \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\right)$$

$\Rightarrow \theta = \frac{u}{\sigma^2} \Leftrightarrow u = \theta\sigma^2$

$\Rightarrow b(\theta) = \frac{u^2}{2\sigma^2} = \frac{(\theta\sigma^2)^2}{2\sigma^2} = \frac{\theta^2\sigma^2}{2}$

$\Rightarrow \phi = 1, \; c(y, \phi) = -\frac{y^2}{2\sigma^2} + \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)$