The Sochocki Formula

Wentao Li

September 16, 2025

Abstract

The Sochocki formula (proven by Polish mathematician Julian Karol Sochocki) is stated and proven. This formula is also known as the "Dirac relation" in physics literature.

We want to derive the distribution identity

$$\frac{1}{x - i0^+} = \mathcal{P}\frac{1}{x} + i\pi\delta(x). \tag{0.1}$$

This should be understood in the sense of distributions, i.e. after integrating against a smooth test function $f(x)$.

1 Real part: principal value

Consider

$$\operatorname{Re}\frac{1}{x - i\eta} = \frac{x}{x^2+\eta^2}, \qquad \eta > 0. \tag{1.1}$$

We want to show

$$\lim_{\eta\to 0^+}\int_{-\infty}^{\infty} f(x)\,\frac{x}{x^2+\eta^2}\,dx = \mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx. \tag{1.2}$$

Write

$$f(x) = f(0) + xg(x), \qquad g(x) = \frac{f(x)-f(0)}{x}, \quad (x \neq 0). \tag{1.3}$$

Then

$$\int f(x)\frac{x}{x^2+\eta^2}\,dx = f(0)\int\frac{x}{x^2+\eta^2}\,dx + \int g(x)\frac{x^2}{x^2+\eta^2}\,dx, \tag{1.4}$$

$$= 0 + \int g(x)\frac{x^2}{x^2+\eta^2}\,dx. \tag{1.5}$$

As $\eta \to 0$, $\frac{x^2}{x^2+\eta^2} \to 1$, so by dominated convergence

$$\lim_{\eta\to 0^+}\int g(x)\frac{x^2}{x^2+\eta^2}\,dx = \int g(x)\,dx. \tag{1.6}$$

Meanwhile,

$$\mathcal{P}\int\frac{f(x)}{x}\,dx = \int g(x)\,dx, \tag{1.7}$$

since the constant term vanishes in the principal value sense. Thus the real part gives the PV integral.

2 Imaginary part: delta function

Now consider

$$\operatorname{Im}\frac{1}{x-i\eta}=\frac{\eta}{x^2+\eta^2}. \tag{2.1}$$

We compute

$$\int_{-\infty}^{\infty} f(x)\,\frac{\eta}{x^2+\eta^2}\,dx. \tag{2.2}$$

Change variables $x=\eta t$:

$$\int_{-\infty}^{\infty}\frac{f(\eta t)}{1+t^2}\,dt. \tag{2.3}$$

As $\eta\to 0$, $f(\eta t)\to f(0)$, so

$$\lim_{\eta\to 0^+}\int f(x)\,\frac{\eta}{x^2+\eta^2}\,dx=f(0)\int_{-\infty}^{\infty}\frac{dt}{1+t^2}=\pi f(0) \tag{2.4}$$

where we have used

$$\int_{-\infty}^{+\infty}\mathrm{d}t\;\frac{1}{1+t^2}=\arctan(+\infty)-\arctan(-\infty)=\pi.$$

Thus

$$\frac{\eta}{x^2+\eta^2}\xrightarrow{\eta\to 0^+}\pi\delta(x). \tag{2.5}$$

3 Combined result (Sokhotski–Plemelj formula)

Putting the two limits together:

$$\frac{1}{x-i0^+}=\mathcal{P}\frac{1}{x}+i\pi\delta(x),\qquad \frac{1}{x+i0^+}=\mathcal{P}\frac{1}{x}-i\pi\delta(x). \tag{3.1}$$

The spelling of the Polish name "Sochocki" has some alternatives, such as "Sokhotski". In the physics literature this is the Dirac relation.