Projectors

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Abstract

This note loosely follows [1] and explains basic properties of projectors in the context of linear algebra. Prerequisites include subspaces and unitary operators. The last section assumes an understanding of the eigenvalue problem of Hermitian operators.

A trick of constructing a projector onto a subspace without explicitly knowing its basis is given in section 3. This trick is commonly used in physics.

Contents

1 Definition and switching basis

Motivation 1.1. We will define projectors *in a basis-independent manner* and see that they are useful tools for dealing with subspaces. In fact a subspace and the projector onto it carry virtually the same amount of information, and for this reason we often use one of them to denote the other.

Definition 1.1. For a d-dimensional vector space $\mathbb{V}$, the *projector* $P_{\mathbb{W}}$ onto a subspace $\mathbb{W}$ of $\mathbb{V}$ is an operator such that for any vector $|v\rangle \in \mathbb{V}$,

- If $|v\rangle \in \mathbb{W}$ then $P_{\mathbb{W}} |v\rangle = |v\rangle$,
- If $|v\rangle \perp \mathbb{W}$ then $P_{\mathbb{W}} |v\rangle = 0$.

Lemma 1.1. *A vector $|w\rangle$ belongs to $\mathbb{W}$ if and only if $P_{\mathbb{W}} |w\rangle = |w\rangle$.*

Proof. By definition, $|w\rangle \in \mathbb{W}$ implies $P_{\mathbb{W}} = |w\rangle$. We need only establish implication in the other direction. We prove by its contrapositive.

If $|v\rangle \notin \mathbb{V}$ then we write $|v\rangle = |v'\rangle + |\theta\rangle$ with $|v'\rangle \in \mathbb{V}$, $|\theta\rangle \perp \mathbb{V}$ and $|\theta\rangle \neq |\text{null}\rangle$. It follows that $P_{\mathbb{W}} |v\rangle = |v'\rangle \neq |v\rangle$, hence the contrapositive is proven. □

Remark 1.1. From lemma 1.1 we see that a subspace and the projector onto it carry the same amount of information. They both specify the subspace.

Lemma 1.2. *Given an orthonormal basis $\{|w_i\rangle\}_{i=1}^{k}$ of $\mathbb{W}$, the projector onto $\mathbb{W}$ can be constructed as*

$$P_{\mathbb{W}} = \sum_{i=1}^{k} |w_i\rangle \langle w_i| \,. \tag{1.1}$$

Proof. We only need to check whether this explicit constrction fits definition 1.1. For $|v\rangle \in \mathbb{W}$, write $|v\rangle = \sum_{j=1}^{k} v_j |w_j\rangle$. Then

$$P_{\mathbb{W}} |v\rangle = \sum_{j=1}^{k}\sum_{i=1}^{k} |w_i\rangle \langle w_i| v_j |w_j\rangle = \sum_{i,j} \delta_{i,j} v_j |w_i\rangle = |v\rangle . \tag{1.2}$$

For $|v'\rangle \perp \mathbb{W}$, write $|v'\rangle = \sum_{j=1}^{d-k} v'_j |u_j\rangle$ with $|u_j\rangle \perp \mathbb{W}$,

$$P_{\mathbb{W}} |v'\rangle = \sum_{i,j} |w_i\rangle \langle w_i| v'_j |u_j\rangle = 0. \tag{1.3}$$

Now the lemma follows. □

Lemma 1.3. *Using eqn.* (1.1) *to define projectors is also acceptable, because the equation is invariant upon change of basis.*

Remark 1.2. This is natural, since our definition of projectors only involves the subspace. There is no reason to expect the projector to depend on any basis. On the other hand, had we defined the projector with eqn. (1.1), we would need to *explicitly verify* its invariance under change of basis.

Proof. First, we need to re-phrase this claim. For a projector $P = \sum_{i=1}^{k} |i\rangle \langle i|$ onto a subspace, supposed we choose another basis $\left\{|\tilde{j}\rangle\right\}_{j=1}^{k}$ of this subspace to construct the projector $P' = \sum_{j=1}^{k} |\tilde{j}\rangle \langle \tilde{j}|$.

Now, in order to prove that $P' = P$, we take an arbitrary vector $|\phi\rangle = \sum_{m=1}^{d} c_m |m\rangle$, then expand $P' |\phi\rangle$ and $P |\phi\rangle$ under the same basis. Here we take $\{|m\rangle\}_{m=1}^{d}$ so that this basis coincides with $\{|i\rangle\}_{i=1}^{k}$ within the subspace span$\{|1\rangle, \ldots, |k\rangle\}$.

Expanding as mentioned we have

$$P |\phi\rangle = \sum_{i=1}^{k}\sum_{m=1}^{k} c_m |i\rangle \langle i|m\rangle = \sum_{i=1}^{k} c_i |i\rangle , \tag{1.4}$$

$$\begin{aligned} P' |\phi\rangle &= \sum_{j=1}^{k}\sum_{m=1}^{d} c_m |\tilde{j}\rangle \langle \tilde{j}|m\rangle & (1.5)\\ &= \sum_{j=1}^{k}\left(\sum_{m=1}^{k} + \sum_{m=k+1}^{d}\right) c_m |\tilde{j}\rangle \langle \tilde{j}|m\rangle & (1.6)\\ &= \sum_{j=1}^{k}\sum_{m=1}^{k} c_m |\tilde{j}\rangle \langle \tilde{j}|m\rangle , & (1.7) \end{aligned}$$

where the blue term does not contribute because $\forall k+1 \leq m \leq d, \quad \langle \tilde{j}|m\rangle = 0$.

Suppose the two bases are related by a unitary T such that $|\tilde{j}\rangle = \sum_{\ell=1}^{k} T_{j\ell} |\ell\rangle$. Then $\langle \tilde{j}| = \sum_p T^*_{jp} \langle p|$, and inserting this into eqn. 1.7 gives

$$\begin{aligned} P' |\phi\rangle &= \sum_{j=1}^{k}\sum_{\ell,p=1}^{k} T_{j\ell} T^*_{jp} |\ell\rangle \langle p| \sum_{m=1}^{k} c_m |m\rangle & (1.8)\\ &= \sum_{\ell,p=1}^{k} \delta_{\ell,p} \sum_{m=1}^{k} c_m |m\rangle = \sum_{\ell=1}^{k} c_\ell |\ell\rangle . & (1.9) \end{aligned}$$

This completes our expansion and finishes the proof. □

2 Idempotence and orthogonal complement

Motivation 2.1. We now gain more intuition into projector by showing some of their properties. First, projectors are idempotent, *i.e.* applying them more than once is the same as applying them only once. This is natural because projectors make a binary decision concerning the vector they act on, and they only need to make this decision once. Secondly, given a projector onto subspace $\mathbb{V}$, we can quickly obtain the projector onto its orthogonal complement $\{|w\rangle : |w\rangle \perp \mathbb{V}\}$, because a projector is just *the identity within some subspace.*

Theorem 2.1. *If P is a projector then $P^2 = P$.*

Proof. We work under a basis $\{|i\rangle\}_{i=1}^{k}$. Expanding P^2 gives

$$P^2 = \sum_{j,\ell=1}^{k} |j\rangle \langle j|\ell\rangle \langle \ell| = \sum_{j,\ell=1}^{k} \delta_{j,\ell} |j\rangle \langle \ell| = P, \tag{2.1}$$

and our lemma follows. □

Remark 2.1. This proof exploits the orthonormality of basis vectors within the subspace P projects onto, but fails to capture the intuition mentioned in motivation 2.1.

Motivation 2.2. We defined projectors to be operators which preserve operators in a subspace and kills vectors orthogonal to this subspace. The opposite effect is killing vectors within this subspace and preserving vectors orthogonal to it. This “opposite effect” also sounds like a projector. Let us construct it.

Definition 2.1. Given a projector P onto a subspace $\mathbb{V}$, its *orthogonal complement* Q is the projector onto the subspace $\{|q\rangle : |q\rangle \perp \mathbb{V}\}$.

Theorem 2.2. *The orthogonal complement Q of a projector P is*

$$Q = \mathbb{1} - P. \tag{2.2}$$

Proof. We can check this explicitly.

- If $|v\rangle \in \mathbb{V}$ then $Q|v\rangle = |v\rangle - P|v\rangle = 0$,
- If $|w\rangle \perp \mathbb{V}$ then $Q|w\rangle = |w\rangle - P|w\rangle = |w\rangle - 0 = |w\rangle$,

so $\mathbb{1} - P$ fits the definition of orthogonal complement. □

3 A useful trick

Motivation 3.1. Our definition of projectors was basis independent, so in principle there should exist other methods to construct them than that given by eqn. (1.1). Here we give an example of such a method.

Lemma 3.1. *Given a Hermitian operator M with eigenvalues $\{+1, -1\}$, the projector P_{+1} onto the invariant subspace $\mathbb{V}_{+1}$ is*

$$P_{+1} = \frac{1}{2}(\mathbb{1} + M). \tag{3.1}$$

The validity of this projector can be checked easily. Note that this example works because M is Hermitian; had M been an arbitrary normal matrix (e.g. a unitary), then this method will not work.

Remark 3.1. This trick is very useful in physics, because the Pauli matrices are Hermitians with eigenvalues $\{\pm 1\}$. Various observables are built from these matrices, and they therefore satisfy the condition for using this trick.

References

[1] Michael A. Nielsen and Isaac L. Chuang. *Quantum computation and quantum information.* 10th anniversary ed. Cambridge ; New York: Cambridge University Press, 2010. 676 pp. ISBN: 978-1-107-00217-3.