Algebra I

Winter 2020

WAYNE STATE
UNIVERSITY

1.1 Divisors

1. Let $m, n.r.s \in \mathbb{Z}$. If $m^2 + n^2 = r^2 + s^2 = mr + ns$, prove that $m = r$ and $n = s$.

Joe Starr

We select $m, n.r.s \in \mathbb{Z}$, given $m^2 + n^2 = r^2 + s^2 = mr + ns$ which can write as $m^2 + n^2 - mr - ns = r^2 + s^2 - mr - ns$. From here we can simplify:

$$\begin{aligned} m^2 + n^2 - mr - ns = r^2 + s^2 - mr - ns &\Rightarrow m(m-r) + n(n-s) = r(r-m) + s(s-n) \\ &\Rightarrow m(m-r) + n(n-s) - r(r-m) - s(s-n) = 0 \\ &\Rightarrow m(m-r) + r(m-r) + n(n-s) + s(n-s) = 0 \\ &\Rightarrow (m-r)(m+r) + (n-s)(n+s) = 0 \end{aligned}$$

from here we can see that in order for $(m-r)(m+r) + (n-s)(n+s) = 0$ to be true $m = r$ and $n = s$.

3. Find the quotient and reminder when a id divided by b.

a $a = 99$, $b = 17$

b $a = -99$, $b = 17$

c $a = 17$, $b = 99$

d $a = -1017$, $b = 99$

Joe Starr

a $99 = 17q + r \Rightarrow q = 5, r = 14$

b $-99 = 17q + r \Rightarrow q = -6, r = 3$

c $17 = 99q + r \Rightarrow q = 0, r = 17$

d $-1017 = 99q + r \Rightarrow q = -11, r = 72$

5. Use the Euclidean algorithm to find the following greatest common divisors

a $(6643, 2873)$

b $(7684, 4148)$

c $(26460, 12600)$

d $(6540, 1206)$

e $(12091, 8439)$

Joe Starr

(a) $(6643, 2873)$

$$6643 = 2873 * 2 + 897$$

$$2873 = 897 * 3 + 182$$

$$897 = 182 * 4 + 169$$

$$182 = 169 * 1 + 13$$

$$169 = 13 * 13$$

(b) $(7684, 4148)$

$$7684 = 4148 * 1 + 3536$$
$$4148 = 3536 * 1 + 612$$
$$3536 = 612 * 5 + 476$$
$$612 = 476 * 1 + 136$$
$$476 = 136 * 3 + 68$$
$$136 = 68 * 68$$

(c) $(26460, 12600)$

$$26460 = 12600 * 2 + 1260$$
$$12600 = 1260 * 10$$

(d) $(6540, 1206)$

$$\begin{aligned}
6540 &= 1206 * 5 + 510 \\
1206 &= 510 * 2 + 186 \\
510 &= 186 * 2 + 138 \\
186 &= 138 * 1 + 48 \\
138 &= 48 * 2 + 42 \\
48 &= 42 * 1 + 6 \\
42 &= 6 * 7
\end{aligned}$$

(e) $(12091, 8439)$

$$\begin{aligned}
12091 &= 8439 * 1 + 3652 \\
8439 &= 3652 * 2 + 1135 \\
3652 &= 1135 * 3 + 247 \\
1135 &= 247 * 4 + 147 \\
247 &= 147 * 1 + 100 \\
147 &= 100 * 1 + 47 \\
100 &= 47 * 2 + 6 \\
47 &= 6 * 7 + 5 \\
6 &= 5 * 1 + 1 \\
5 &= 1 * 5
\end{aligned}$$

7. For each part of Exercise 5, find integers m and n such that (a, b) is expressed in the form $ma + nb$.

Joe Starr

(a) $(6643, 2873)$

$(6643) - 16 + (2873)\,37 = 13$

(b) $(7684, 4148)$

$(7684)\,27 + (4148) - 50 = 68$

(c) $(26460, 12600)$ $(26460)\,1 + (12600) - 2 = 1260$

(d) $(6540, 1206)$ $(6540) - 26 + (1206)\,141 = 6$

(e) $(12091, 8439)$ $(12091)\,1435 + (8439) - 2056 = 1$

9. let a, b, c be integers such that $a + b + c = 0$. Show that if n is an integer which is a divisor of two of the three integers, then it is also a divisor of the third.

Joe Starr

Select $a, b, c \in \mathbb{Z}$ to satisfy $a + b + c = 0$, WLOG let $n \in \mathbb{Z}$ such that $n|a$ and $n|b$. Since $(a + b) + c = 0$ it must be that $(a + b) = -c$. From here we must show $n|(a + b)$, or $a + b = nq$. Since $n|a$ and $n|b$ we may write $a = nq_1$ and $b = nq_2$, yielding, $nq_1 + nq_2 = n\,(q_1 + q_2) = nq$ thus $n|c$, as desired.$\square$

13. Show that if n is any integer, then $(10n_3, 5n+2) = 1$

Joe Starr

We begin with the Euclidean algorithm,

$$10n + 3 = (5n + 2)\,1 + (5n + 1)$$

$$5n + 2 = (5n + 1)\,1 + 1$$

from here we have $(10n + 3, 5n + 2) = (5n + 2, 5n + 1) = 1$, as desired.

15. For what positive integers n is it true that $(n, n+2) = 2$? Prove your claim.

Joe Starr

The conjecture is that the statement is true for even values of n. We begin with rewriting n in terms of k, $n = 2k$the Euclidean algorithm,

$$(2k) + 2 = (2k)\,1 + (2)$$

$$2k = (2)\,k$$

from here we have $(n+2, n) = (2k+2, 2k) = 2$, as desired.

17. Show that the positive integer k is the difference of two odd squares if and only if k is divisible by 8.

Joe Starr

We begin by writing $k = a^2 - b^2$, since a and b are odd we can write,

$$a = 2r + 1$$

$$b = 2s + 1$$

from here we have $q^2 - b^2 = 4\left(r + s + 1\right)\left(r - s\right)$. Since $k > 0$ we must consider two cases $r - s = 2m + 1$ and $r - s = 2m$.

$r - s = 2m$:

In this case we have $q^2 - b^2 = 4\left(r + s + 1\right)2m = 8\left(r + s + 1\right)m$ and we are done.

$r - s = 2m + 1$:

In this case we have $r - s = 2m + 1$ and $r + s = r - s + 2s = 2m + 1 + 2s$

$$
\begin{aligned}
q^2 - b^2 &= 4\left(r + s + 1\right)\left(2m + 1\right) \\
&= 4\left(2m\left(r + s + 1\right) + \left(r + s + 1\right)\right) \\
&= 4\left(\left(2mr + 2ms + 2m\right) + \left(r + s + 1\right)\right) \\
&= 4\left(2mr + 2ms + 2m + r + s + 1\right) \\
&= 4\left(2mr + 2ms + 2m + 2m + 1 + 2s + 1\right) \\
&= 4\left(2mr + 2ms + 2m + 2m + 2s + 2\right) \\
&= 8\left(mr + ms + m + m + s + 1\right)
\end{aligned}
$$

as desired.

CHAPTER 2 SECTION

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CHAPTER 3 SECTION

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