Algebra I
Winter 2020

WAYNE STATE
UNIVERSITY

1.1 Divisors

1. Let $m, n.r.s \in \mathbb{Z}$. If $m^2 + n^2 = r^2 + s^2 = mr + ns$, prove that $m = r$ and $n = s$.

Joe Starr

We select $m, n.r.s \in \mathbb{Z}$, given $m^2 + n^2 = r^2 + s^2 = mr + ns$ which can write as $m^2 + n^2 - mr - ns = r^2 + s^2 - mr - ns$. From here we can simplify:

$$
\begin{aligned}
m^2 + n^2 - mr - ns = r^2 + s^2 - mr - ns &\Rightarrow m\left(m - r\right) + n\left(n - s\right) = r\left(r - m\right) + s\left(s - n\right) \\
&\Rightarrow m\left(m - r\right) + n\left(n - s\right) - r\left(r - m\right) - s\left(s - n\right) = 0 \\
&\Rightarrow m\left(m - r\right) + r\left(m - r\right) + n\left(n - s\right) + s\left(n - s\right) = 0 \\
&\Rightarrow \left(m - r\right)\left(m + r\right) + \left(n - s\right)\left(n + s\right) = 0
\end{aligned}
$$

from here we can see that in order for $\left(m - r\right)\left(m + r\right) + \left(n - s\right)\left(n + s\right) = 0$ to be true $m = r$ and $n = s$.

3. Find the quotient and reminder when a id divided by b.

a $a = 99$, $b = 17$

b $a = -99$, $b = 17$

c $a = 17$, $b = 99$

d $a = -1017$, $b = 99$

Joe Starr

a $99 = 17q + r \Rightarrow q = 5, r = 14$

b $-99 = 17q + r \Rightarrow q = -6, r = 3$

c $17 = 99q + r \Rightarrow q = 0, r = 17$

d $-1017 = 99q + r \Rightarrow q = -11, r = 72$

5. Use the Euclidean algorithm to find the following greatest common divisors

a $(6643, 2873)$

b $(7684, 4148)$

c $(26460, 12600)$

d $(6540, 1206)$

e $(12091, 8439)$

Joe Starr

(a) $(6643, 2873)$

$$6643 = 2873 * 2 + 897$$
$$2873 = 897 * 3 + 182$$
$$897 = 182 * 4 + 169$$
$$182 = 169 * 1 + 13$$
$$169 = 13 * 13$$

(b) $(7684, 4148)$

$$7684 = 4148 * 1 + 3536$$
$$4148 = 3536 * 1 + 612$$
$$3536 = 612 * 5 + 476$$
$$612 = 476 * 1 + 136$$
$$476 = 136 * 3 + 68$$
$$136 = 68 * 68$$

(c) $(26460, 12600)$

$$26460 = 12600 * 2 + 1260$$
$$12600 = 1260 * 10$$

(d) $(6540, 1206)$

$$
\begin{aligned}
6540 &= 1206 * 5 + 510 \\
1206 &= 510 * 2 + 186 \\
510 &= 186 * 2 + 138 \\
186 &= 138 * 1 + 48 \\
138 &= 48 * 2 + 42 \\
48 &= 42 * 1 + 6 \\
42 &= 6 * 7
\end{aligned}
$$

(e) $(12091, 8439)$

$$
\begin{aligned}
12091 &= 8439 * 1 + 3652 \\
8439 &= 3652 * 2 + 1135 \\
3652 &= 1135 * 3 + 247 \\
1135 &= 247 * 4 + 147 \\
247 &= 147 * 1 + 100 \\
147 &= 100 * 1 + 47 \\
100 &= 47 * 2 + 6 \\
47 &= 6 * 7 + 5 \\
6 &= 5 * 1 + 1 \\
5 &= 1 * 5
\end{aligned}
$$

7. For each part of Exercise 5, find integers m and n such that (a, b) is expressed in the form $ma + nb$.

Joe Starr

(a) $(6643, 2873)$

$(6643) - 16 + (2873)\,37 = 13$

(b) $(7684, 4148)$

$(7684)\,27 + (4148) - 50 = 68$

(c) $(26460, 12600)$ $(26460)\,1 + (12600) - 2 = 1260$

(d) $(6540, 1206)$ $(6540) - 26 + (1206)\,141 = 6$

(e) $(12091, 8439)$ $(12091)\,1435 + (8439) - 2056 = 1$

9. let a, b, c be integers such that $a + b + c = 0$. Show that if n is an integer which is a divisor of two of the three integers, then it is also a divisor of the third.

Joe Starr

Select $a, b, c \in \mathbb{Z}$ to satisfy $a + b + c = 0$, WLOG let $n \in \mathbb{Z}$ such that $n|a$ and $n|b$. Since $(a + b) + c = 0$ it must be that $(a + b) = -c$. From here we must show $n|(a + b)$, or $a + b = nq$. Since $n|a$ and $n|b$ we may write $a = nq_1$ and $b = nq_2$, yielding, $nq_1 + nq_2 = n(q_1 + q_2) = nq$ thus $n|c$, as desired.$\square$

13. Show that if n is any integer, then $(10n_3, 5n+2) = 1$

Joe Starr

We begin with the Euclidean algorithm,

$$10n + 3 = (5n + 2)\,1 + (5n + 1)$$

$$5n + 2 = (5n + 1)\,1 + 1$$

from here we have $(10n + 3, 5n + 2) = (5n + 2, 5n + 1) = 1$, as desired.

15. For what positive integers n is it true that $(n, n+2) = 2$? Prove your claim.

Joe Starr

The conjecture is that the statement is true for even values of n. We begin with rewriting n in terms of k, $n = 2k$the Euclidean algorithm,

$$(2k) + 2 = (2k)\,1 + (2)$$

$$2k = (2)\,k$$

from here we have $(n+2, n) = (2k+2, 2k) = 2$, as desired.

17. Show that the positive integer k is the difference of two odd squares if and only if k is divisible by 8.

Joe Starr

We begin by writing $k = a^2 - b^2$, since a and b are odd we can write,

$$a = 2r + 1$$

$$b = 2s + 1$$

from here we have $q^2 - b^2 = 4(r + s + 1)(r - s)$. Since $k > 0$ we must consider two cases $r - s = 2m + 1$ and $r - s = 2m$.

$r - s = 2m$:

In this case we have $q^2 - b^2 = 4(r + s + 1)\,2m = 8(r + s + 1)\,m$ and we are done.

$r - s = 2m + 1$:

In this case we have $r - s = 2m + 1$ and $r + s = r - s + 2s = 2m + 1 + 2s$

$$\begin{aligned} q^2 - b^2 &= 4\left(r+s+1\right)\left(2m+1\right) \\ &= 4\left(2m\left(r+s+1\right)+\left(r+s+1\right)\right) \\ &= 4\left(\left(2mr+2ms+2m\right)+\left(r+s+1\right)\right) \\ &= 4\left(2mr+2ms+2m+r+s+1\right) \\ &= 4\left(2mr+2ms+2m+2m+1+2s+1\right) \\ &= 4\left(2mr+2ms+2m+2m+2s+2\right) \\ &= 8\left(mr+ms+m+m+s+1\right) \end{aligned}$$

as desired.

1.2 Primes

1. Find the prime factorizations of each of the following numbers, and use the them to compute the greatest common divisor and least common multiple of the given pairs of numbers.

(a) $(35, 14)$

(b) $(15, 11)$

(c) $(252, 11)$

(d) $(7684, 4148)$

(e) $(6643, 2873)$

Joe Starr

(a) $(35, 14)$

$35 : 5, 7$

$14 : 2, 7$

gcd: 7

lcm: 70

(b) $(15, 11)$

$15 : 3, 5$

$11 : 11$

gcd: 1

lcm: 165

(c) $(252, 180)$

$252 : 2, 2, 3, 3, 7$

$180 : 2, 2, 3, 3, 5$

gcd: 36

lcm: 1260

(d) $(7684, 4148)$

$7684 : 2, 2, 17, 113$

$4148 : 2, 2, 17, 61$

gcd: 68

lcm: 468724

(e) $(6643, 2873)$

$6643 : 7, 13, 73$

$2873 : 13, 13, 17$

gcd: 13

lcm: 1468103

2. US the sieve of Eratosthenes to find all prime numbers less than 200.

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	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

3. For each composite number a. with $4 \leq a \leq 20$, find all positive numbers less than a that are relatively prime to a.

Joe Starr

$4: 2, 3$

$6: 2, 3, 5$

$8: 2, 3, 5, 7$

$9: 2, 3, 4, 5, 7, 8$

$10: 2, 3, 5, 7, 9$

$12: 2, 3, 5, 7, 11$

$14: 2, 3, 5, 7, 9, 11, 13$

$15: 2, 3, 4, 5, 7, 8, 11, 13, 14$

$16: 2, 3, 5, 7, 9, 11, 13, 15$

$18: 2, 3, 5, 7, 11, 13, 17$

$20: 2, 3, 5, 7, 9, 11, 13, 17, 19$

4. Find all positive integers less than 60 and relatively prime to 60.

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$60: \; 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59$

9. (a) For which $n \in \mathbb{Z}^+$ is $n^3 - 1$ a prime number?

(b) For which $n \in \mathbb{Z}^+$ is $n^3 + 1$ a prime number?

(c) For which $n \in \mathbb{Z}^+$ is $n^2 - 1$ a prime number?

(d) For which $n \in \mathbb{Z}^+$ is $n^2 + 1$ a prime number?

Joe Starr

(a) We can factor $n^3 - 1$ into $(n-1)(n^2+n+1)$. We have then $n-1|n^3-1$, for n^3-1 to be prime $n-1$ must be 1. This happens only for $n=2$.

(b) We can factor n^3+1 into $(n+1)(n^2-n+1)$. We have then $(n^2-n+1)|n^3+1$, for n^3+1 to be prime (n^2-n+1) must be 1. This happens only for $n=1$.

(c) We can factor n^2-1 into $(n-1)(n+1)$. We have then $(n11)|n^2-1$, for n^2-1 to be prime $(n-1)$ must be 1. This happens only for $n=2$. For which $n \in \mathbb{Z}^+$ is n^2-1 a prime number?

(d) ????

11. Prove that $n^4 + 4^n$ is composite if $n > 1$.

Joe Starr

We are presented with two potability's, n is even or n is odd.

n even

It's obvious that $n^4 + 4^n$ is an even not 2 and can't be prime.

n odd

We begin by completing the square

$$\begin{aligned} n^4 + 4^n &= n^4 + 4^n \\ &= \left(n^2\right)^2 + (2^n)^2 \\ &= \left(n^2 + 2^n\right)^2 - 2n^2 2^n \end{aligned}$$

We from here we observe that $2^n 2 = 2^{n+1}$, since n is odd $n+1$ is even yielding $2^{n+1} = 2^{2k}$. We can see we have a difference of squares

$$\begin{aligned} \left(n^2 + 2^n\right)^2 - 2n^2 2^n &= \left(n^2 + 2^n\right)^2 - (2^n n)^2 \\ &= \left(n^2 + 2^n + 2^n n\right)\left(n^2 + 2^n - 2^n n\right) \end{aligned}$$

since we are restricted to $n > 1$ we can see that both $(n^2 + 2^n + 2^n n) > 1$ and $(n^2 + 2^n - 2^n n) > 1$ for all n. Making $n^4 + 4^n$ composite as desired.

13. Let a, b, c be positive integers, and let $d = (a, b)$. Since $d|a$, there exists an integer h with $a = dh$. Show that $a|bc$, then $h|c$.

Joe Starr

14. Show that $a\mathbb{Z} \cap b\mathbb{Z} = [a, b]$.

Joe Starr

Let $x \in (a\mathbb{Z} \cap b\mathbb{Z})$, since $x \in a\mathbb{Z}$ we have $x = aq_1$, similarly since $x \in b\mathbb{Z}$ we have$x = bq_2$. We can see that $x = abq$, this means x is a multiple of $[a, b]$ putting $x \in [a, b]$. Next, we let $x \in [a, b]\,\mathbb{Z}$, this means x is of the form $x = [a, b]\,q$. We can see that $a|x$ and $b|x$ since $a|[a, b]$, This makes $x \in a\mathbb{Z}$ and $x \in b\mathbb{Z}$, as desired.

17. Let a, b be nonzero integers. Prove $(a, b) = 1$ if and only if $(a + b, ab) = 1$.

Joe Starr

$\Rightarrow$ We let $(a, b) = 1$, then consider the $(a + b, ab)$. We assume $(a + b, ab) = d$, with $d > 1$. Since $d > 1$ there must exist p a prime such that $p|d$. This means that $p|a + b$ and $p|ab$. Consequently, either $p|a$ or $p|b$. WOLG we have $p|a$, and since $p|a + b$ it must be that $p|b$. Finally, since $p|a$ and $p|b$, $p|(a, b)$ a contradiction. So $(a + b, ab) = 1$.

$\Leftarrow$ We let $(a + b, ab) = 1$, then consider the (a, b). We assume $(a, b) = d$, with $d > 1$. Since $d > 1$ there must exist p a prime such that $p|d$. This means that $p|a$ and $p|b$, further $p|ab$. Since p divides a and b, we have $p|a + b$. Finally, since $p|ab$, and $p|a + b$, $p|(a + b, ab)$, a contradiction so $(a, b) = 1$.

18. Let a, b be nonzero integers with $(a, b) = 1$. Compute $(a + b, a - b)$.

Joe Starr

19. Let a and b be positive integers, and let m be an integer such that $ab = m\,(a,b)$. Without using the prime factorization theorem, prove that $(a,b)\,[a,b] = ab$ by verifying that m satisfies the necessary properties of $[a,b]$.

Joe Starr

20. A positive integer a is called a square if $a = n^2$ for some $n \in \mathbb{Z}$. Show that the integer $a > 1$ is a square if and only if every exponent in its prime factorization is even.

Joe Starr

Let $a \in \mathbb{Z}$ be a square. Since a is a square by definiton there exists a n such that $nn = a$. Now by the fundamental theorem of arythmic we know n has a prime factorization,written $p_1^{n_1} \cdots p_k^{n_k}$. If we consider nn, we have $nn = (p_1^{n_1} \cdots p_k^{n_k})(p_1^{n_1} \cdots p_k^{n_k})$, by combining terms we can see that $nn = \left(p_1^{2n_1} \cdots p_k^{2n_k}\right)$, as desired.

23. Let p and q be prime numbers. Prove that $pq + 1$ is a square if and only if p and q are twin primes.

Joe Starr

26. Prove that if $a > 1$, then there is a prime p with $a < p \leq a! + 1$.

Joe Starr

29. Show that $\log 2/\log 3$ is not a rational number.

Joe Starr

CHAPTER 2 SECTION

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2.1 A Subsection

CHAPTER 3 SECTION

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3.1 A Subsection

CHAPTER 4 SECTION

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4.1 A Subsection

CHAPTER 5 SECTION

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5.1 A Subsection

CHAPTER 6 SECTION

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6.1 A Subsection

CHAPTER 7 SECTION

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CHAPTER 8 SECTION

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8.1 A Subsection

CHAPTER 9 SECTION

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9.1 A Subsection

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10.1 A Subsection