Friedrich-Alexander Universität Erlangen-Nürnberg
Institute for Digital Communications

Tutorials on

Mobile Communications

A supplementary manuscript for the lecture
Mobile Communications
consistent with
Lecture-notes on Mobile Communications by *Ralf R. Müller & Wolfgang Koch*
Current Edition: Summer 2022

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Preface

This manuscript has been developed through several years of teaching tutorials on *Mobile Communications* at Friedrich-Alexander University of Erlangen-Nuremberg (FAU). Unlike many fundamental lectures, the contents of the lecture *Mobile Communications* vary from a textbook to another, and hence the students are often not provided with sufficient materials for further practice. This manuscript provides summary discussions and exercises with comprehensive solutions to fundamental topics in mobile communications. The key goal is to give a better understanding of the basic concepts and also to provide some further practice on the subject. The contents of this manuscript have been collected and prepared in consistency with the course *Mobile Communications* which is lectured by Prof. Dr.-Ing. Ralf Müller at FAU.

I would like to kindly thank my former colleague Dr.-Ing. Prasanth Karunakaran who gave some good thoughts on some of the exercises. Also, I should thank Hassan Nazim Bicer and Arda Buglagil who helped typesetting this manuscript.

I hope that this writing helps the readers to have a better and deeper understanding of fundamental concepts in mobile communications. As this is an initial version of the manuscript, it contains several typos and writing mistakes. I would hence be grateful to receive any comments or feedback on the manuscript. You can always reach me by sending me an email at ali.bereyhi@fau.de or contacting me in person.

Ali Bereyhi
Summer 2022, Erlangen, Germany

Contents

How to Use

To study the exercises, I would suggest to do the following:

1. Try to solve each exercise after the corresponding tutorial session without looking at the solutions.

2. If you have done your best with the exercises; then, go through the solutions. I would *strongly* suggest to read all solutions, *even those you have solved correctly.* This would help you understanding all details.

At the end of each chapter, you are given by homework. The homework contains new exercise out of the textbook and some important exercises form the textbook. I would suggest to examine yourself at the end of each chapter by going through the homework.

The last two chapters contain sample exams given in the previous semesters at FAU. In Chapter 10, recent exams are given with solutions. I would suggest that you go through respective problems in this chapter, after you finish with studying each topic. The final chapter then gives you some older exams without solutions. These exams could be used for self-test to check whether you are ready for the exam or not.

Chapter 1

Introduction

This chapter review the basic concepts of mobile communications such as cellular principles, impairments and fundamental limits. It then gives some exercises which cover the first chapter of the lecture-notes.

1.1 Brief Review on Main Concepts

The first chapter goes through the general concepts in mobile communications. Namely, it gives some brief answers to the following questions:

- What is the frequency reuse, and why do we need it?
- Why the hexagonal cells are optimal to be used and what are their properties?
- What are the sources of distortion in wireless communications and how we can model them?
- What are the general metrics by which we could characterize the performance of a cellular network?
- What is the up-to-down architecture of a cellular network?

Throughout the answers to these questions, we learn several definitions and derivations. In the next sections, we try to have an overview on the most important definitions and derivations.

1.2 Exercises

1.2.1 Principles of Cellular Networks

1. Consider a cellular network with hexagonal cells. The outer radius of the cells is $R = 1$ km. The network is grouped into clusters, such that in each cluster there are four cells.

   (a) What is the cluster size K and what it means in practice?

(b) Determine the frequency reuse distance in this network?

2. Consider a cellular network whose cells are *squares* with edge length R_0. For this network,

   (a) Draw the neighbouring cells and determine the inner radius R_{in} and outer radius R of a single cell.
   (b) Determine the cell area A_{C} in terms of the outer radius R.
   (c) Introduce a non-orthogonal (i, j) coordinate for this cell and determine the corresponding points on the Cartesian coordinate, i.e., (x, y)-axes; similar to the derivations in page 21 of the lecture notes.
   (d) Determine the distance of a cell center at point (i, j) from the origin.
   (e) Determine the cluster size K in terms of the frequency reuse distance D in this network when a repetitive pattern is used for frequency reuse. Represent the result in the (i, j) coordinate.
   (f) Can we have a cluster size of 1, 2, 3 or 5 in this network with a useful repetition pattern?

3. In a cellular network, base stations are allowed to transmit with power P. Assume that a user is placed in distance d meters from the nearest base station. The received power at the user place is

$$P_{\text{r}} = 10^3 \frac{P}{d^n}$$

where n is some real number more than two.

You need at least signal-to-noise ratio (SNR) to be

$$\log \text{SNR}_{\text{min}} = -10 \text{ dB}$$

to make sure that the user receives suitable services. The noise power in this network is $P_{\text{noise}} = 10$ mW.

   (a) How do you think that n changes from a village to a city?
   (b) Determine the minimum transmit power P_{min} in terms of d and n such that the user is properly served.
   (c) Now assume that $d = 10$ km and $n = 2$. What is the minimum required transmit power?
   (d) Repeat the last part for $n = 3$. What is your conclusion comparing these two parts?
   (e) Assume that you are ensured that for any chosen distance, the minimum required SNR is achieved. Is it a good idea to cover the whole region with a single cell?

1.2.2 Fundamental Impairments in Cellular Networks

1. Consider a receiver with noise figure $\log F = 9$ dB whose input filter $H_I(f)$ in baseband reads

$$H_I(f) = \begin{cases} 1 & |f| \leq 72 \text{ kHz} \\ \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sin\dfrac{\pi}{2}\left(\dfrac{90-f}{18}\right)} & 72 \text{ kHz} \leq |f| \leq 108 \text{ kHz} \\ 0 & |f| \geq 108 \text{ kHz} \end{cases}$$

Assume that the only impairment is the thermal noise. The noise power density is given by $N_0 = kT$ where the constant k reads $k = 1.38 \times 10^{-23}$ Ws/K, and the temperature is $T = 290$ K.

   (a) Calculate the equivalent noise bandwidth B_{noise}.

   (b) Calculate the noise power including the noise generated by amplification at the RF stage.

1.2.3 Network Capacity

1. A network provider intends to allocate 20 MHz for a duplex system in a hexagonal cellular network. Based on calculations, for each simplex sub-channel in this network 25 KHz bandwidth is required.

   (a) Determine the total number of *duplex* channels N_{sys} in this network.

   (b) Assume that your initial calculations indicate that the cluster size K in this network should be $K > 4.2$. Determine the spectral efficiency of this network in [channels/MHz/cell].

1.2.4 Design Challenges

1. Assume that a mobile user is moving in a cellular network while being on a call. We intend to guarantee that this displacement would not cause any drop in the call. The technical terminology for such a property is *mobility support*.

   (a) Name the procedure which you need to design in order to support mobility in the system.

   (b) State three major steps required for this procedure.

Assume that we implement the following algorithm to support mobility:

The mobile user determines the power of received signals from different base stations periodically. It switches then to the base station whose received signal is the strongest.

(c) Suppose that a user is moving along a cell edge where there are several buildings and trees. Using the above algorithm, what kind of behavior do you expect in this case?

2. Consider a cellular network with cluster size $K = 3$. The total available bandwidth in this network is divided into nine adjacent *sub-channels* whose bandwidth is B, and whose central frequencies are given by

$$f_n = f_0 + nB$$

for $n \in \{1, \dots, 9\}$ and some $f_0 \gg B$. You are asked to allocate same number of sub-channels to each cell. How do you distribute the carriers $f_1, \dots, f_9$ among the cells?

1.3 Solutions to Exercises

1.3.1 Principles of Cellular Networks

1. Consider a cellular network with hexagonal cells. The outer radius of the cells is $R = 1$ km. The network is grouped into clusters, such that in each cluster there are four cells.

   (a) What is the cluster size K and what it means in practice?

   (b) Determine the frequency reuse distance in this network?

♠ **Solution:**

   (a) Since we have four cells in each cluster, the cluster size is $K = 4$. This means that the total frequency band is divided into 4 sub-channels and each is given to one cell. This architecture of 4 cells is then repeated; see page 19 of the lecture notes.

   (b) Using equation (1.3) in page 25 of the lecture notes, the frequency reuse distance D reads

$$D = \sqrt{3K}R = \sqrt{12} \approx 3.46 \text{ km}$$

2. Consider a cellular network whose cells are *squares* with edge length R_0. For this network,

   (a) Draw the neighboring cells and determine the inner radius R_{in} and outer radius R of a single cell.

   (b) Determine the cell area A_{C} in terms of the outer radius R.

   (c) Introduce a non-orthogonal (i, j) coordinate for this cell and determine the corresponding points on the Cartesian coordinate, i.e., (x, y)-axes; similar to the derivations in page 21 of the lecture notes.

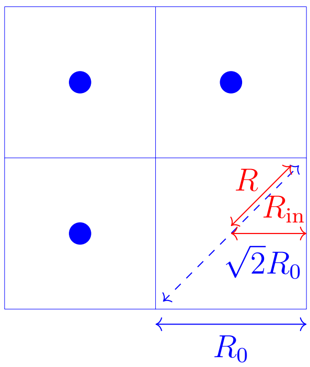
   (d) Determine the distance of a cell center at point (i, j) from the origin.

(e) Determine the cluster size K in terms of the frequency reuse distance D in this network when a repetitive pattern is used for frequency reuse. Represent the result in the (i, j) coordinate.

(f) Can we have a cluster size of 1, 2, 3 or 5 in this network with a useful repetition pattern?

♠ **Solution:**

(a) A group of 4 neighboring cells in this cellular network is shown in the following figure. In this figure, the center of the cells are denoted by bullets.



As shown in the figure, for a single cell we have

$$R = \frac{\sqrt{2}R_0}{2} = \frac{R_0}{\sqrt{2}},$$

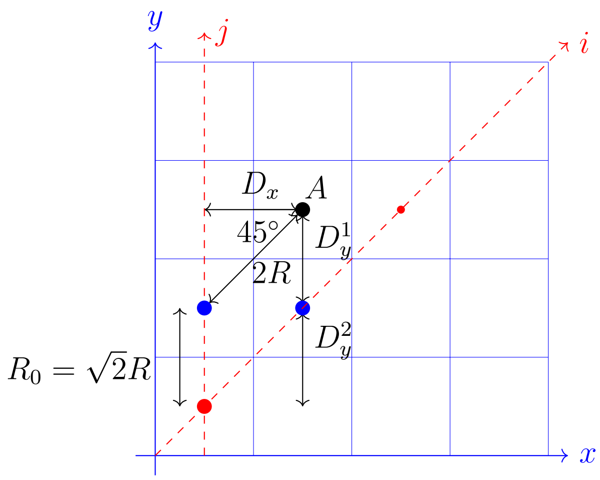
and

$$R_{\text{in}} = \frac{R_0}{2} = \frac{R}{\sqrt{2}}.$$

(b) Noting that $R_0 = \sqrt{2}R$, the area of a cell reads

$$A_{\text{C}} = R_0^2 = 2R^2.$$

(c) Following the approach in page 21 of the lecture notes, the non-orthogonal (i, j) coordinate is given as in the following figure



In this figure the red point shows the origin and the i- and j-axes are denoted by dashed lines. To calculate the corresponding (x, y) coordinates, let us consider a simple example: Consider point A in the figure which is located at $(i, j) = (1, 1)$. The distance of point A form the origin on the x-axis is denoted by D_x which reads

$$D_x = 2R \times \cos(45°) = \sqrt{2}R.$$

In general, when the center of a cell is located at point (i, j), it is straightforward to find out that

$$x = 2Ri \times \cos(45°) \Rightarrow \boxed{x = \sqrt{2}Ri}$$

The distance of point A form the origin on the y-axis is given by

$$D_y = D_y^1 + D_y^2.$$

For D_y^1, we have

$$D_y^1 = 2R \times \sin(45°) = \sqrt{2}R.$$

Moreover,

$$D_y^2 = R_0 = \sqrt{2}R.$$

Thus $D_y = 2\sqrt{2}R$. In general, for a center located at point (i, j), we can write

$$\begin{aligned} D_y^1 &= 2Ri \times \sin(45°) = \sqrt{2}Ri \\ D_y^2 &= R_0 j = \sqrt{2}Rj. \end{aligned}$$

Hence, we have

$$\boxed{y = \sqrt{2}R(i + j)}$$

(d) A cell center at (i, j) is in fact at

$$(x, y) = \left(\sqrt{2}Ri, \sqrt{2}R(i + j)\right)$$

in Cartesian coordinate whose distance from the origin reads

$$d = \sqrt{x^2 + y^2} = R\sqrt{2(2i^2 + 2ij + j^2)}.$$

(e) To determine the cluster size K in terms of the reuse distance D, we construct a *hyper-cluster* by connecting the four nearest co-channels around a given cell; see Figure 1.10 in page 25 of the lecture notes. The hyper-cluster consists of

- The cluster to which the given cell belongs.
- Fractions of the 4 neighboring clusters.

To determine the area of the fraction of these neighboring clusters which is in the hyper-cluster, we note that each of these neighboring clusters is also a neighboring cluster to 3 other hyper-clusters. Noting the the geometry of the network is symmetric, we conclude that each neighboring cluster is equally shared among 4 hyper-clusters. This means that, $1/4 = 0.25$ of its area is in the hyper-cluster. As the result, the area of the hyper-cluster $A_{\rm H}$ satisfies

$$A_{\rm H} = \underbrace{A_{\rm K}}_{\text{main cluster}} + 4 \times \underbrace{0.25 A_{\rm K}}_{\text{each neighboring cluster}} = 2A_{\rm K}$$

where $A_{\rm K}$ is the cluster area. On the other hand, we know that the hyper-cluster is a square with outer radius D, and hence we can write

$$A_{\rm H} = 2D^2.$$

Noting that, $A_{\rm K} = KA_{\rm C}$, we have

$$K = \frac{A_{\rm H}}{2A_{\rm C}} = \frac{D^2}{2R^2}.$$

Thus, we have

$$\boxed{K = 2i^2 + 2ij + j^2} \tag{EQ:K}$$

(f) From (EQ:K), we can see that $K = 1, 2, 5$ are possible in this architecture; however, $K = 3$ is not possible, since there exist *no integers* i and j for which $2i^2 + 2ij + j^2 = 3$.

3. In a cellular network, base stations are allowed to transmit with power P. Assume that a user is placed in distance d meters from the nearest base station. The received power at the user place is

$$P_{\rm r} = 10^3 \frac{P}{d^n}$$

where n is some real number more than two.

You need at least signal-to-noise ratio (SNR) to be

$$\log \mathrm{SNR}_{\min} = -10 \text{ dB}$$

to make sure that the user receives suitable services. The noise power in this network is $P_{\rm noise} = 10$ mW.

(a) How do you think that n changes from a village to a city?

(b) Determine the minimum transmit power $P_{\min}$ in terms of d and n such that the user is properly served.

(c) Now assume that $d = 10$ km and $n = 2$. What is the minimum required transmit power?

(d) Repeat the last part for $n = 3$. What is your conclusion comparing these two parts?

(e) Assume that you are ensured that for any chosen distance, the minimum required SNR is achieved. Is it a good idea to cover the whole region with a single cell?

♠ **Solution:**

(a) In a city n would be larger, as there are further obstacles which can boost the channel loss.

(b) The SNR is defined as

$$\mathrm{SNR} = \frac{P_\mathrm{r}}{P_\mathrm{noise}}.$$

Since $\log \mathrm{SNR}_\mathrm{min} = -10$ dB, or equivalently $\mathrm{SNR}_\mathrm{min} = 10^{-1}$, we can write

$$0.1 = \mathrm{SNR}_\mathrm{min} = \frac{P_\mathrm{r,min}}{P_\mathrm{noise}} = \frac{10^3 P_\mathrm{min}}{d^n P_\mathrm{noise}} \Rightarrow P_\mathrm{min} = 10^{-4} d^n P_\mathrm{noise}.$$

Noting that $P_\mathrm{noise} = 10$ mW $= 10^{-2}$ W, we have

$$P_\mathrm{min} = 10^{-6} d^n \ \mathrm{W}.$$

(c) For $d = 10$ km and $n = 2$,

$$P_\mathrm{min} = 10^{-6} \times \left(10^4\right)^2 = 100 \ \mathrm{W}$$

which is rather large.

(d) For $d = 10$ km and $n = 3$,

$$P_\mathrm{min} = 10^{-6} \times \left(10^4\right)^3 = 1 \ \mathrm{MW}$$

which is huge!

> This result indicates that one of the key reasons for partitioning a wireless networks into cells is the limited transmit power which we can afford.

(e) Even if there exist no power limitation, we still need to partition into cells. In fact, since we are limited in bandwidth, we cannot share the available bandwidth among all the users.

> ⇝ REMARK:
>
> Remember that there are two major restrictions which lead us to a cellular design:
>
> (A) Limited Power, and
>
> (B) Restricted Bandwidth.

1.3.2 Fundamental Impairments in Cellular Networks

1. Consider a receiver with noise figure $\log F = 9$ dB whose input filter $H_I(f)$ in baseband reads

$$H_I(f) = \begin{cases} 1 & |f| \leq 72 \text{ kHz} \\ \sqrt{\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{2}\left(\frac{90-f}{18}\right)} & 72 \text{ kHz} \leq |f| \leq 108 \text{ kHz} \\ 0 & |f| \geq 108 \text{ kHz} \end{cases}$$

Assume that the only impairment is the thermal noise. The noise power density is given by $N_0 = kT$ where the constant k reads $k = 1.38 \times 10^{-23}$ Ws/K, and the temperature is $T = 290$ K.

(a) Calculate the equivalent noise bandwidth B_{noise}.

(b) Calculate the noise power including the noise generated by amplification at the RF stage.

♠ **Solution:**

(a) The equivalent noise bandwidth B_{noise} is given by equation (1.8) in page 32 of the lecture notes.

$$B_{\text{noise}} = \frac{1}{|H_I(0)|^2} \int_{-\infty}^{\infty} |H_I(f)|^2 \mathrm{d}f = 180 \text{ kHz}.$$

(b) The noise power after amplification in the RF stage is given by equation (1.10) in page 32 which reads

$$\begin{aligned} P_{\text{noise}} &= N_0 B_{\text{noise}} F \\ &= kTB_{\text{noise}} F. \end{aligned}$$

Noting that $\log F = 9$ dB, or equivalently $F = 10^{0.9}$, we finally have

$$\log P_{\text{noise}} = -142.43 \text{ dB} = -112.43 \text{ dBm}$$

1.3.3 Network Capacity

1. A network provider intends to allocate 20 MHz for a duplex system in a hexagonal cellular network. Based on calculations, for each simplex sub-channel in this network 25 KHz bandwidth is required.

(a) Determine the total number of *duplex* channels N_{sys} in this network.

(b) Assume that your initial calculations indicate that the cluster size K in this network should be $K > 4.2$. Determine the spectral efficiency of this network in [channels/MHz/cell].

♠ **Solution:**

(a) The total bandwidth is $B_{\text{sys}} = 20$ MHz. Since we need $B_{\text{simplex}} = 25$ kHz for each simplex channel, we have in total

$$N_{\text{simplex}} = \frac{20 \times 10^3}{25} = 800$$

simplex channels. For each duplex channel, we need to employ two simplex channels, i.e., one uplink and one downlink. Thus the total number of duplex channels reads

$$N_{\text{sys}} = \frac{N_{\text{simplex}}}{2} = 400.$$

(b) As the network has hexagonal cells, the cluster size K can only take values given in Table 1.1 in page 26. Since $K > 4.2$, we choose K to be the first possible value for K which is larger than 4.2, i.e., $K = 7$. The spectral efficiency η is then determined from equation (1.14) in page 37 as

$$\eta = \frac{N_{\text{sys}}}{KB_{\text{sys}}} = \frac{400}{7 \times 20} \approx 2.86 \text{ channels/MHz/cell.}$$

1.3.4 Design Challenges

1. Assume that a mobile user is moving in a cellular network while being on a call. We intend to guarantee that this displacement would not cause any drop in the call. The technical terminology for such a property is *mobility support*.

   (a) Name the procedure which you need to design in order to support mobility in the system.

   (b) State three major steps required for this procedure.

   Assume that we implement the following algorithm to support mobility:

The mobile user determines the power of received signals from different base stations periodically. It switches then to the base station whose received signal is the strongest.

   (c) Suppose that a user is moving along a cell edge where there are several buildings and trees. Using the above algorithm, what kind of behavior do you expect in this case?

♠ **Solution:**

(a) This procedure is called *handover* which allows user to switch from a base station to another without experiencing a call drop.

(b) The basic steps of a handover algorithm are as follows:

- The base station and the mobile station should determine the signal quality periodically. There should be a measure with which the handover procedure is decided to be done or not in each period.
- When the handover procedure is decided to be done, the mobile station should broadcast a signal to base stations. The base stations should measure their channel qualities and communicate over the network backbone. The new serving base station is then chosen based on the these information.
- The information of the mobile terminal should be transmitted to the new base station and a channel resource should be allocated to the user.

The handover procedure needs a lot of signaling and coordination and thus designing a good algorithm could save a lot of resources in the network.

(c) When the algorithm in Part (c) is employed, the user will be handed over periodically between the base stations. This is due to the fact that the channel quality fluctuates a lot. Therefore, in the edges of cells the signal quality from the base station in the cell would periodically be better and worse than the signal quality from the base station of the neighboring cell. This will cause a periodic series of handovers. Thus, such an algorithm is not a good algorithm.

2. Consider a cellular network with cluster size $K = 3$. The total available bandwidth in this network is divided into nine adjacent *sub-channels* whose bandwidth is B, and whose central frequencies are given by

$$f_n = f_0 + nB$$

for $n \in \{1, \ldots, 9\}$ and some $f_0 \gg B$. You are asked to allocate same number of sub-channels to each cell. How do you distribute the carriers $f_1, \ldots, f_9$ among the cells?

♠ **Solution:** There are nine sub-channels and three cells, so each cell gets three sub-channels. The best allocation would be

- **Cell 1:** f_1, f_4, f_7
- **Cell 2:** f_2, f_5, f_8
- **Cell 3:** f_3, f_6, f_9

In this case, each cell has non-adjacent sub-channels. Therefore, the out-band radiations of the users within each cell does not interfere another user in the cell who is using another sub-channel of the cell. This allocation can reduce the amount of the *co-channel interference*.

1.4 Homework

1. In a cellular network, we have two transmission modes:

   - Uplink
   - Downlink

   In the first mode, transmission is done from the mobile stations to the base station. The latter mode refers to transmission from the base station to mobile stations.

   Which of these modes suffer from near-far effects? Justify your answer.

♠ **Solution:** The near-far effect occurs in *uplink*. The reason can be found in the first chapter.

2. Discuss near-far effects from two perspectives

♠ **Solution:** Check Section 1.5 of the lecture notes.

3. Consider a cellular network in which two users are transmitting signals to a base station. The first user is placed in $d_1 = 50$ *meters* away from the base station while the second user is $d_2 = 30$ *kilometers* far away from the base station. Both the users transmit with the same power P.

   For a pair of transmitter and receiver, the received power $P_{\rm T}$ in this network is determined as

$$P_{\rm R} = K_0 \frac{P_{\rm T}}{d^4}$$

   where $P_{\rm T}$ is the transmit power, and K_0 is a constant.

   For this setting,

   (a) Determine ratio

$$\beta = \frac{P_{\rm R}^{(1)}}{P_{\rm R}^{(2)}} \tag{1.4.1}$$

   where $P_{\rm R}^{(1)}$ and $P_{\rm R}^{(2)}$ denote the powers received from the first and second user at base station, respectively.

   (b) Determine the delays in the signal arrival time at each user.

♠ **Solution:** The ratio of the received powers would be around 111.13 dB. Based on discussions in page 40 of the lecture notes, the delay in the arrival time is twice the propagation delay. Noting that the signal propagates with the speed of light $c = 3 \times 10^8$ m/sec, the delays are derived for the first and second users as $\tau_1 = 0.33$ μsec and $\tau_2 = 200$ μsec, respectively.

1.5 A Side Note on Logarithmic Scale

The unit *bel* is defined to quantify the logarithm of a *ratio*. The ratio itself has no unit. An example of a ratio is the SNR which is defined as

$$\mathrm{SNR} = \frac{\text{Power of the useful signal}}{\text{Power of the noise}}.$$

When you determine the logarithm of a given ratio R in the decimal base, you are determining the logarithm of this ratio in bell shown by [B]. As a result, this value in decibels, denoted by [dB], is 10 times more. The correct notation is therefore to write

$$\log R \text{ [in dB]} = 10 \log R \text{ [in B]} = 10 \log_{10} R.$$

For example, when we say that $\log \mathrm{SNR} = 10$ dB, it means that

$$10 \log_{10} \mathrm{SNR} = 10 \Rightarrow \mathrm{SNR} = 10^1 = 10.$$

Note that the notation $10 \log \mathrm{SNR} = 10$ dB for this example is **wrong**!

Chapter 2

Antennas

This chapter discusses characteristics of antennas. The discussions include basic formulations and definitions and cover the second chapter of the lecture-notes. These definitions and formulations are important to be known, as antennas are fundamental components in wireless communications.

2.1 Brief Review of Main Concepts

Antennas are often described by their pattern. The pattern describes how and antenna radiates in the three-dimensional space. For almost all antennas, the pattern for radiation and also reception is the same. This means that in the direction where an antenna radiates more power, the power received by the antenna, when the antenna is used for signal reception, is also more.

There are several parameters for an antenna which can describe its characteristics in a brief form. These parameters are all derived from the radiation pattern of the antenna. Some of them that are given in the lecture are: directivity, gain and effective area.

In general, we use antennas in two different ways:

1. Single components which means that a single antenna is used for communication.

2. In the form of an array antenna, which means that multiple single antenna elements (often same antennas) are located next to each other in the form of an array and the whole array is now seen as a new big antenna.

The key reason that we go for array antennas is that by doing so, we could design our desired radiation pattern. This can help us making more efficient antennas (or in fact arrays). There are several forms of array antennas which have been introduced in the lecture-notes. The main derivations are however given for the fundamental case of linear arrays in which the antenna elements are located on a line next to each other. We now review the important aspects through some exercises.

2.2 Exercises

2.2.1 Basic Definitions

1. Consider an imaginary omnidirectional antenna whose vertical radiation pattern is given by

$$g_{\mathrm{v}}(\vartheta) = |\cos \vartheta|.$$

The power efficiency of this antenna is $\eta = 0.5$.

   (a) Write the 3-dimensional radiation pattern of this antenna.

   (b) Sketch the radiation pattern in both the horizontal and vertical planes.

   (c) Determine the directivity D and gain G of this antenna in dB$_{\mathrm{i}}$ and dB$_{\mathrm{d}}$.

2. A receive antenna with efficiency $\eta = 1$ is employed at a mobile station whose directivity is D. Assume that the antenna receives a constant signal stream such that the receive power flux is fixed for a long period of time.

   (a) Write the receive power as a function of the carrier frequency.

   (b) Determine the change in the receive power, when we move from the GSM band, i.e. 900 MHz, to the beginning of the millimeter-wave (mmWave) band, i.e. 30 GHz.

3. To avoid high correlation among the antennas in an array, the distance between two adjacent antennas should be in the order of the wave length. Let us consider an $N \times N$ planner antenna array in which the adjacent antennas in horizontal and vertical directions are distanced $\alpha\lambda$ and $\beta\lambda$, respectively, for $\alpha, \beta \geq 0.5$.

   (a) Determine the area of the antenna array in terms of α, β and N when we operate in the GSM band, i.e. 900 MHz, and when we are in the mmWave band, i.e. 30 GHz.

   (b) For these two frequency bands, calculate the maximum possible number of antennas which can be packed within a square-meter.

2.2.2 Antenna Arrays

1. Consider an electromagnetic wave with wave-length λ. At each point in the space, the power of this wave is

$$P = K|E|^2$$

for some constant K, where E is the magnitude of the electric field.

   (a) Determine the phase shift in the electric field of a wave when it travels distance d.

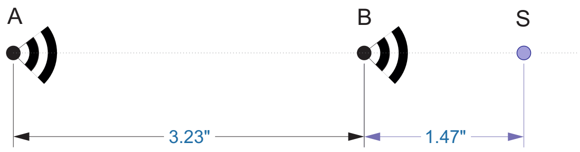


Figure 2.2.1: Two isotropic antennas transmuting the same signal.

(b) Assume that a signal has been fed into two antennas which are located next to each other with distance a on a horizontal line; see Figure 2.1. What is the superposed field at distance d on the horizontal line assuming that the magnitude of the field is E_0, and there is no power loss.

(c) Determine the power of the superposed signal on the horizontal line for $a = \lambda/4$, $a = \lambda/2$ and $a = \lambda$ in terms of $P_0 = K|E_0|^2$ assuming no power loss.

2. Consider the antenna array in Figure 2.2. The array consists of two $\lambda/2$-dipoles with horizontal spacing a. The antennas are fed by the same signal. The feed into the first antenna is delayed with a path-delay d. This means that the wave at the input of the first antenna has travelled an extra path of length d compared to the wave at the input of the second antenna.

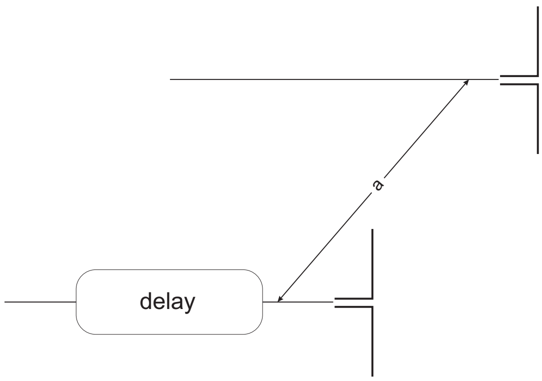


Figure 2.2.2: An antenna array of size two.

(a) Assume that $d = a = \lambda/4$. Determine the main beam of the array, as well as its null direction, i.e. the direction in which no power is radiated, on the horizontal plane.

(b) Determine the *array gain* $g_{\mathrm{A}}(\alpha)$, and the radiation pattern of the antenna array, i.e. $g(\vartheta, \alpha)$. Check your answers to the previous part using $g(\vartheta, \alpha)$.

(c) Now set $a = \lambda/4$. Determine d such that the main beam of the antenna array in the horizontal plane points to angle θ.

2.3 Solutions to Exercises

2.3.1 Basic Definitions

1. Consider an imaginary omnidirectional antenna whose vertical radiation pattern is given by

$$g_{\mathrm{v}}(\vartheta) = |\cos\vartheta|.$$

The power efficiency of this antenna is $\eta = 0.5$.

(a) Write the 3-dimensional radiation pattern of this antenna.

(b) Sketch the radiation pattern in both the horizontal and vertical planes.

(c) Determine the directivity D and gain G of this antenna in dB$_{\mathrm{i}}$ and dB$_{\mathrm{d}}$.

♠ **Solution:**

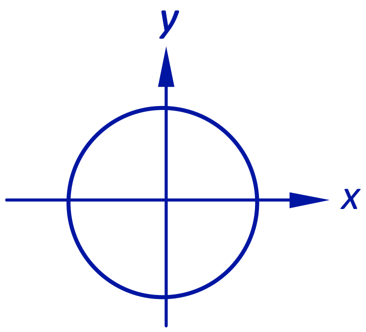
(a) Since the antenna is omnidirectional we have,

$$g_{\mathrm{h}}(\alpha) = 1.$$

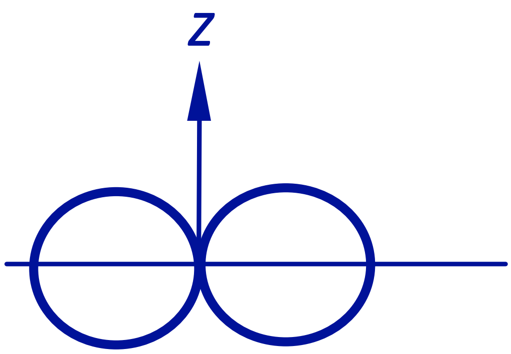
Consequently, the 3-dimensional radiation pattern reads

$$g(\vartheta,\alpha) = g_{\mathrm{v}}(\vartheta)\, g_{\mathrm{h}}(\alpha) = |\cos\vartheta|.$$

(b) The horizontal pattern is



The vertical pattern looks like



(c) Invoking equation (2.9) in page 52 of the lecture notes, we have

$$D = \frac{4\pi}{\displaystyle\int_0^{2\pi}\int_0^{\pi} |\cos\vartheta| \sin\vartheta \, \mathrm{d}\vartheta \mathrm{d}\alpha} = \frac{2}{\displaystyle\int_0^{\pi/2} \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta - \int_{\pi/2}^{\pi} \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta}$$
$$\stackrel{\star}{=} \frac{2}{\displaystyle\frac{1}{2}\int_0^{\pi/2} \sin 2\vartheta \, \mathrm{d}\vartheta - \frac{1}{2}\int_{\pi/2}^{\pi} \sin 2\vartheta \, \mathrm{d}\vartheta} = 2$$

where $\star$ comes from the fact that $\sin 2\vartheta = 2\sin\vartheta\cos\vartheta$. Consequently, $D = 3$ dB$_\mathrm{i}$ and

$$\log D \; [\mathrm{dB_d}] = 10 \log \frac{D}{D_{\mathrm{dipole}}} = \log D \; [\mathrm{dB_i}] - \log D_{\mathrm{dipole}} \; [\mathrm{dB_i}]$$
$$\stackrel{\dagger}{=} 3 - 2.15 = 0.85.$$

where $\dagger$ comes from the fact that $\log D_{\mathrm{dipole}} = 2.15$ dB$_\mathrm{i}$; see page 49. From page 48, we know that $G = \eta D$, hence

$$\log G = \log \frac{1}{2} + \log D = \log D - 3[\mathrm{dB}]$$

which means that $\log G = 0$ dB$_\mathrm{i}$ $= -2.15$ dB$_\mathrm{d}$.

2. A receive antenna with efficiency $\eta = 1$ is employed at a mobile station whose directivity is D. Assume that the antenna receives a constant signal stream such that the receive power flux is fixed for a long period of time.

   (a) Write the receive power as a function of the carrier frequency.

   (b) Determine the change in the receive power, when we move from the GSM band, i.e. 900 MHz, to the beginning of the millimeter-wave (mmWave) band, i.e. 30 GHz.

♠ **Solution:**

(a) From page 48 of the lecture notes, we know that the receive power P_r is written in terms of the power flux S and the effective antenna area A_w as

$$P_\mathrm{r} = S A_\mathrm{w}.$$

Using equation (2.3) in page 48, we have

$$P_\mathrm{r} = S A_\mathrm{w}$$
$$= S D \frac{\lambda^2}{4\pi}$$
$$\stackrel{\star}{=} S D \frac{c^2}{4\pi f_\mathrm{C}^2}$$

where D is the directivity of the receive antenna and $\star$ comes from $\boxed{\lambda f_\mathrm{C} = c}$ with f_C being the carrier frequency and $c = 3 \times 10^8$ m/sec being the speed of light.

(b) Denote the receive power in the GSM and mmWave bands with P_{GSM} and P_{mmW}, respectively. Therefore, we have

$$\frac{P_{\mathrm{GSM}}}{P_{\mathrm{mmW}}} = \left(\frac{f_{\mathrm{C,mmW}}}{f_{\mathrm{C,GSM}}}\right)^2 = \left(\frac{3\times 10^{10}}{9\times 10^{8}}\right)^2 \approx 1.11\times 10^3 \equiv 30.04 \text{ dB}.$$

Therefore, the power received in the mmWave band is of the order of 30 dB less than the power received in the standard GSM band. This means that if the receive power in the GSM bound in 1 mW, then it would be 1 μW in the mmWave band. This is a well-known effect which indicates that

⇝ REMARK:

By increasing the carrier frequency, the receive power decreases proportional to $1/f_{\mathrm{C}}^2$.

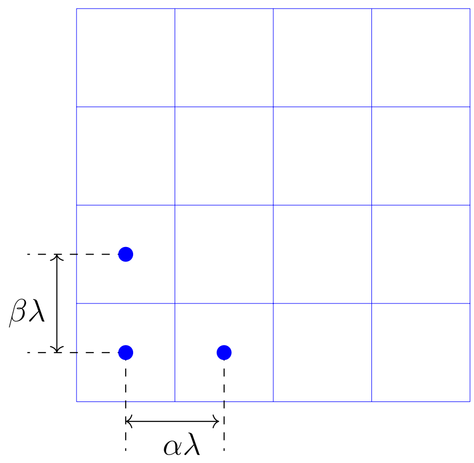
3. To avoid high correlation among the antennas in an array, the distance between two adjacent antennas should be in the order of the wave length. Let us consider an $N \times N$ planner antenna array in which the adjacent antennas in horizontal and vertical directions are distanced $\alpha\lambda$ and $\beta\lambda$, respectively, for $\alpha, \beta \geq 0.5$.

   (a) Determine the area of the antenna array in terms of α, β and N when we operate in the GSM band, i.e. 900 MHz, and when we are in the mmWave band, i.e. 30 GHz.

   (b) For these two frequency bands, calculate the maximum possible number of antennas which can be packed within a square-meter.

♠ **Solution:**

(a) The diagram for this planner array is shown in the following figure:



From the diagram, it is observed that the planner antenna array is of the size $N\alpha\lambda \times N\beta\lambda$. Therefore, the area reads

$$A_{\rm array} = N^2\lambda^2\alpha\beta.$$

In the GSM band

$$\lambda_{\rm GSM} = \frac{c}{f_{\rm C,GSM}} = \frac{1}{3} \text{ m}.$$

Thus, the array reads

$$A_{\rm array,GSM} = \frac{N^2\alpha\beta}{9} \text{ m}^2.$$

In the mmWave band, we have

$$\lambda_{\rm mmW} = \frac{c}{f_{\rm C,mmW}} = 10^{-2} \text{ m}.$$

Hence, the area reads

$$A_{\rm array,mmW} = 10^{-4}N^2\alpha\beta \text{ m}^2.$$

(b) To find the number of antennas within a unit of surface, we set $A_{\rm array} = 1$. This leads to

$$A_{\rm array} = 1 \Rightarrow \boxed{N = \frac{1}{\sqrt{\lambda\alpha\beta}}.} \tag{2.3.1}$$

Noting that $\alpha, \beta \geq 0.5$, we set α and β to their minimum values, i.e., $\alpha = \beta = 0.5$, in order to determine the maximum number of antennas. Hence, the maximum number of antennas within a square-meter is

$$N_{\max} = \frac{2}{\lambda}. \tag{2.3.2}$$

This means that in the GSM band $N_{\max} = 6$, and in the mmWave band $N_{\max} = 200$ antennas can be packed within 1 m^2.

As it is observed, by increasing the frequency, the array size reduces meaning that base stations become smaller, ore equivalently, the number of antennas which can be used in the limited space of a base station increases. This is again a well-known feature. Combining with the result of the previous exercise, we can conclude that

$\rightsquigarrow$ REMARK:

$$\text{Carrier frequency} \uparrow \Rightarrow \begin{cases} \text{Receive power at the mobile station } \downarrow \\ \text{Number of antennas at the base station } \uparrow \end{cases}$$

2.3.2 Antenna Arrays

1. Consider an electromagnetic wave with wave-length λ. At each point in the space, the power of this wave is

$$P = K|E|^2$$

for some constant K, where E is the magnitude of the electric field.

(a) Determine the phase shift in the electric field of a wave when it travels distance d.

(b) Assume that a signal has been fed into two antennas which are located next to each other with distance a on a horizontal line; see Figure 2.1. What is the superposed field at distance d on the horizontal line assuming that the magnitude of the field is E_0, and there is no power loss.

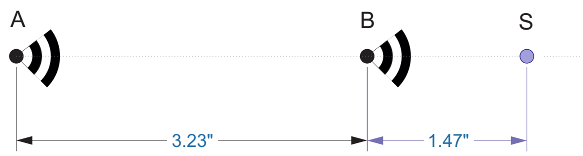


Figure 2.3.1: Two isotropic antennas transmuting the same signal.

(c) Determine the power of the superposed signal on the horizontal line for $a = \lambda/4$, $a = \lambda/2$ and $a = \lambda$ in terms of $P_0 = K|E_0|^2$ assuming no power loss.

♠ **Solution:**

(a) An electromagnetic wave rotates while it travels. When an electromagnetic wave travels a distance equivalent to its wave-length λ the vector of the electric field becomes exactly the same as it was at the starting point. This means that for traveling distance λ, there is 2π phase shift. As a result, when the wave travels distance d, the phase shift is

$$\boxed{\Delta\varphi = \frac{2\pi}{\lambda} d}$$

⇝ REMINDER:

From the theory of electromagnetic waves, we know that the electric field in this case is shown by the so-called *phasor*. The phasor of the electric field for a wave which has traveled distance d is defined as

$$E = E_0 \exp\{-j\Delta\varphi\} = E_0 \exp\left\{-j\frac{2\pi}{\lambda}d\right\}.$$

The effective magnitude of the field at this point is then calculated as

$$E_X = \Re(E) = E_0 \cos\left(\frac{2\pi}{\lambda}d\right).$$

The power of the wave is moreover given in terms of its electric field phasor as

$$P = K|E|^2$$

for some constant K.

(b) Feeding signal into two antennas as in Figure 2.1, the electric field at point S is written as the superposition of the electric fields transmitted from the antennas at points A and B. To calculate the electric field at this point, assume that the phase of the signal which is fed to A and B is zero. Let E_0 be the electric field of this signal. The phasor of the electric field at point S is then given by

$$E_\mathsf{S} = E_\mathsf{A} + E_\mathsf{B}$$

where E_A is the phasor of the electric field of the wave traveled from A to S, and E_B is the phasor of the electric field of the wave traveled from B to S. Noting that the magnitude of the electric field is E_0 and that there is no power loss, E_A and E_B are written as

$$E_\mathsf{A} = E_0 \exp\left\{-j\frac{2\pi}{\lambda}(a+d)\right\}$$
$$E_\mathsf{B} = E_0 \exp\left\{-j\frac{2\pi}{\lambda}d\right\}.$$

Hence, E_S is calculated as

$$\begin{aligned} E_\mathsf{S} &= E_0 \exp\left\{-j\frac{2\pi}{\lambda}(a+d)\right\} + E_0 \exp\left\{-j\frac{2\pi}{\lambda}d\right\} \\ &= E_0 \exp\left\{-j\frac{2\pi}{\lambda}d\right\}\left(\exp\left\{-j\frac{2\pi}{\lambda}a\right\} + 1\right). \end{aligned}$$

(c) The power of the signal received at point S is calculated as

$$
\begin{aligned}
P_{\mathrm{S}} &= K|E_{\mathrm{S}}|^2 \\
&= K|E_0 \exp\left\{-j\frac{2\pi}{\lambda}d\right\}\left(\exp\left\{-j\frac{2\pi}{\lambda}a\right\}+1\right)|^2 \\
&= \underbrace{K|E_0|^2}_{P_0}\ \underbrace{|\exp\left\{-j\frac{2\pi}{\lambda}d\right\}|^2}_{1}\ |\exp\left\{-j\frac{2\pi}{\lambda}a\right\}+1|^2 \\
&= P_0\left[\left(1+\cos\left(\frac{2\pi}{\lambda}a\right)\right)^2+\sin\left(\frac{2\pi}{\lambda}a\right)^2\right]
\end{aligned}
$$

This leads to

$$
\boxed{P_{\mathrm{S}} = 2P_0\left[1+\cos\left(\frac{2\pi}{\lambda}a\right)\right]}
$$

Consequently, we can write

- When $a = \lambda/4$, we have $P_{\mathrm{S}} = 2P_0$. In this case, the waves are superposed *orthogonally*.
- When $a = \lambda/2$, we have $P_{\mathrm{S}} = 0$. In this case, the waves are superposed *destructively*.
- When $a = \lambda$, we have $P_{\mathrm{S}} = 4P_0$. In this case, the waves are superposed *constructively*.

2. Consider the antenna array in Figure 2.2. The array consists of two $\lambda/2$-dipoles with horizontal spacing a. The antennas are fed by the same signal. The feed into the first antenna is delayed with a path-delay d. This means that the wave at the input of the first antenna has traveled an extra path of length d compared to the wave at the input of the second antenna.

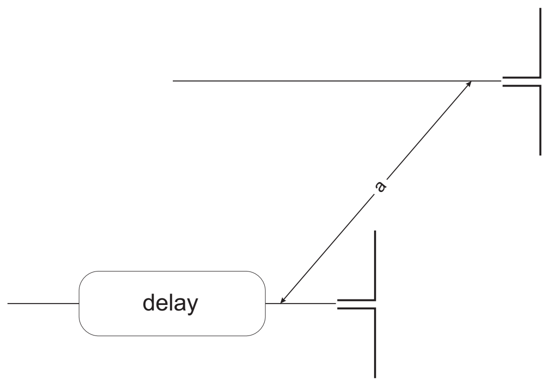


Figure 2.3.2: An antenna array of size two.

(a) Assume that $d = a = \lambda/4$. Determine the main beam of the array, as well as its null direction, i.e. the direction in which no power is radiated, on the horizontal plane.

(b) Determine the *array gain* $g_{\mathrm{A}}(\alpha)$, and the radiation pattern of the antenna array, i.e. $g(\vartheta, \alpha)$. Check your answers to the previous part using $g(\vartheta, \alpha)$.

(c) Now set $a = \lambda/4$. Determine d such that the main beam of the antenna array in the horizontal plane points to angle θ.

♠ **Solution:**

(a) The spacing is horizontal, i.e., over the angle α. The horizontal view of this array is shown in Figure 2.3. Note that in the horizontal plane, $\lambda/2$-dipole antennas are omnidirectional, i.e.,

$$g_{\mathrm{h,dipole}}(\alpha) = 1.$$

When $a = d = \lambda/4$, one can follow the same approach as in the previous exercise to find the main beam and the null:

- At $\alpha = 90^\circ$, the wave traveling from the the lower antenna travels in total

$$d_{\mathrm{tot,low}} = d + a = \frac{\lambda}{2}.$$

  The wave from the upper antenna however travels

$$d_{\mathrm{tot,up}} = 0.$$

  Hence, it has a phase shift $\Delta\varphi = \pi$ from the signal radiated from the upper antenna at this angle. In other words, the waves have same phasors with different signs. Consequently, the waves are superposed at this angle *destructively*, and at any point on the line $\alpha = 90^\circ$ the superposed electric field is zero. Hence, $\alpha = 90^\circ$ is a null point in the horizontal pattern.

- On the line $\alpha = -90^\circ$, the wave from the lower antenna travels

$$d_{\mathrm{tot,low}} = d = \frac{\lambda}{4}.$$

  For the upper antenna, we also have

$$d_{\mathrm{tot,up}} = a = \frac{\lambda}{4}.$$

  Therefore, the electric fields from the both antennas have a same phase and superpose *constructively*. As a result, the maximum power is radiated on the line $\alpha = -90^\circ$ which concludes that the main beam of the array is on $\alpha = -90^\circ$.

(b) To determine the array gain, we use equations (2.14) and (2.15) in page 56. Here, we have two antennas, i.e., $M = 2$. Noting that the signal at the lower antenna has phase shift

$$\Delta\varphi = \frac{2\pi}{\lambda} d$$

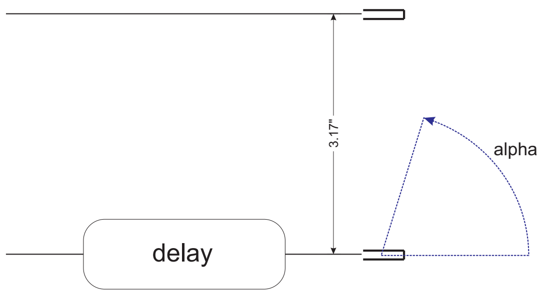


Figure 2.3.3: The horizontal view of the array in Figure 2.2.

for this array, we have[1]

$$\begin{aligned} w_0 &= 1 && \text{(upper antenna)} \\ w_1 &= \exp\{-j\Delta\varphi\} && \text{(lower antenna)} \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathrm{AF}(\alpha) &= 1 + \exp\{-j\Delta\varphi\}\exp\left\{-j2\pi\frac{a}{\lambda}\sin\alpha\right\} \\ &= 1 + \exp\left\{-j\frac{2\pi}{\lambda}(d + a\sin\alpha)\right\}. \end{aligned}$$

Note that

$$\max_{\alpha} |\mathrm{AF}(\alpha)|^2 = 2^2 = 4.$$

Hence, the array gain reads

$$\begin{aligned} g_{\mathrm{A}}(\alpha) &= \frac{|\mathrm{AF}(\alpha)|^2}{4} \\ &= \frac{1}{4}\left[2 + 2\cos\left(\frac{2\pi}{\lambda}(d + a\sin\alpha)\right)\right] \\ &= \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{\lambda}(d + a\sin\alpha)\right). \end{aligned}$$

The radiation pattern of this array is then written as

$$g(\vartheta, \alpha) = g_{\mathrm{dipole}}(\vartheta, \alpha)\, g_{\mathrm{A}}(\alpha)$$

where $g_{\mathrm{dipole}}(\vartheta, \alpha)$ is the radiation pattern of a $\lambda/2$-dipole antenna. In this case,

$$\begin{aligned} g(\vartheta, \alpha) &= \underbrace{g_{\mathrm{v,dipole}}(\vartheta)}_{\text{Eq. (2.5) - page 49}} \underbrace{g_{\mathrm{h,dipole}}(\alpha)}_{1}\, g_{\mathrm{A}}(\alpha) \\ &= \frac{1}{2}\frac{\cos^2(\pi\cos\vartheta/2)}{\sin^2\vartheta}\left[1 + \cos\left(\frac{2\pi}{\lambda}d + \frac{2\pi}{\lambda}a\sin\alpha\right)\right]. \end{aligned}$$

[1]See Figure 2.4 in page 55 of the lecture notes.

Note that the pattern in the horizontal plane is completely characterized by $g_{\mathrm{A}}(\alpha)$, since in this plane $\lambda/2$-dipole antennas are omnidirectional.

By setting $\alpha = -90^\circ$ and $\alpha = 90^\circ$ in $g_{\mathrm{A}}(\alpha)$, one can double check the initial conclusions in Part (a).

(c) To fix the main beam in the horizontal plane toward the angle θ, we should find d such that the maximum of the array gain occurs at $\alpha = \theta$, i.e.,

$$\max_{\alpha} g_{\mathrm{A}}(\alpha) = g_{\mathrm{A}}(\theta).$$

$g_{\mathrm{A}}(\alpha)$ is maximized, when the argument of $\cos(\cdot)$ equals to $2k\pi$ for some integer k. Thus, we have

$$\frac{2\pi}{\lambda}d + \frac{2\pi}{\lambda}a\sin\theta = 2k\pi \;\Rightarrow\; \boxed{d = k\lambda - a\sin\theta}.$$

By setting $a = \lambda/4$, the main beam is at the horizontal angle θ, when

$$d = \left(k - \frac{1}{4}\sin\theta\right)\lambda$$

for some integer k. For example, for $\theta = -90^0$, we need to set

$$d = \frac{\lambda}{4} \text{ or } d = \frac{5\lambda}{4} \text{ or } \ldots \qquad \text{(in general } d = k\lambda + \frac{\lambda}{4}\text{)}$$

which agrees with our initial conclusion.

2.4 Homework

Smart antennas are widely utilized in the current technologies, e.g. radars and cellular networks. These setups are often referred to as *adaptive antenna arrays*. The basic feature of these arrays is that they adopt their pattern. In other words, the radiation patterns of these antenna arrays is not fixed and can be tuned. In this section, we discuss a simple example of adaptive arrays which illustrates how the these setups change the radiation pattern.

1. Consider the following array of four $\lambda/2$-dipole antennas. These antennas are horizontally spaced $\lambda/4$. The feed into antennas B and D is delayed with path delays $\lambda/4$ and $3\lambda/4$, respectively. This means that the signals at the inputs of antennas B and D have traveled extra distances $\lambda/4$ and $3\lambda/4$, respectively, compared to the inputs of antennas A and C. Determine the main beam when

   (a) Only antennas A and C are switched on.

   (b) Only antennas A and B are switched on.

   (c) Only antennas C and D are switched on.

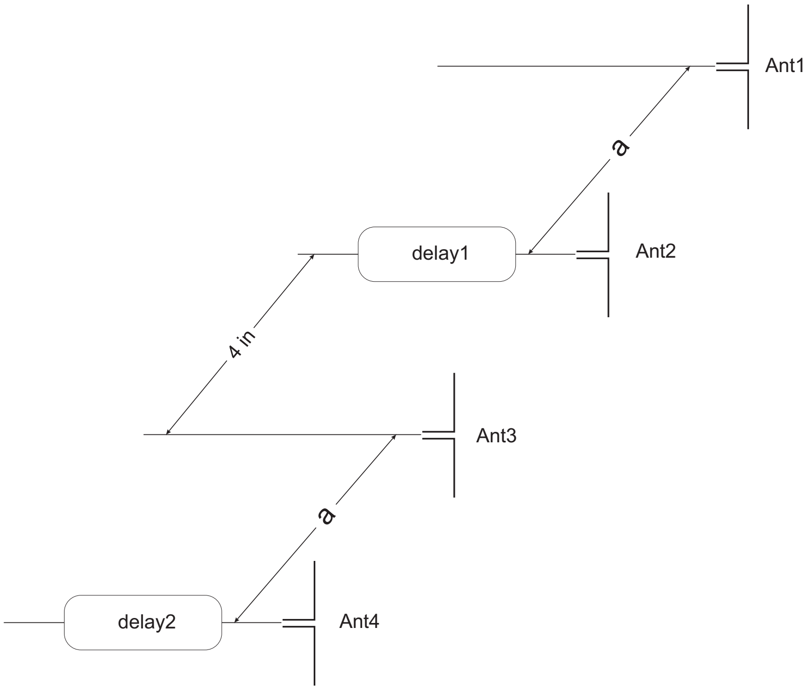


Figure 2.4.1: A horizontally configured linear array of antennas.

♠ **Solution:** Similar to the last exercise of Tutorial 2, it is concluded that in Cases (a), (b) and (c), the main beam points to $\alpha = 0°$(and $180°$), $\alpha = -90°$ and $\alpha = 90°$, respectively.

2. Discuss how we can employ the structure in Question 1, in order to cover three different sectors of the space within three different time slots.

2.5 A Side Note: Difference between dBW and dBm

When we say that $\log P = 1$ dBW, we mean that

$$\log \frac{P}{P_0} = 1 \text{ dB}$$

where P_0 is the reference power $P_0 = 1$ W. If you change this reference to $P_0 = 1$ mW, then the result is given in dBm. Hence, when $\log P = Q$ dBW we can say

$$Q = 10 \log \frac{P}{1\text{W}} = 10 \log \frac{P}{10^3 \text{mW}} = 10 \log \frac{P}{1\text{mW}} - 30.$$

Therefore, we always have

$$\log P \text{ in [dBm]} = \log P \text{ in [dBW]} + 30.$$

For example, $\log P = 0$ dBW is equivalent to say $\log P = 30$ dBm. In some texts, dBW is simply written dB.

Chapter 3

Large-Scale Fading

In the lecture-notes, the characteristics of a wireless channel have been illustrated in a long chapter; namely, Chapter 3. This is the most important part of the lecture, which gives the most fundamental definitions we need to know about wireless communication networks. We hence have this chapter as well as the next two chapters dedicated to this part of the lecture.

3.1 Brief Review of Main Concepts

The wireless channel suffer from three key distortion impacts:

1. Path-loss which indicates the average received power drops at least quadratic with respect to the distance to the transmitter.

2. Shadowing which model significant drop in the received signal (in addition to the path-loss) due to existence of obstacles between the transmitter and the receiver (for instance buildings and trees).

3. Small-scale fading which describe the changes in the received signal due to the *movement of the receiver* and the *availability of multiple paths* between the transmitter and receiver, due to reflection from the area surrounding the receiver.

Sometimes, we formulate the first two impacts as *large-scale* fading which in contrast to the small-scale fading results in much smoother changes in the curve of received power against the distance.

The exact modeling of each of above impacts is very hard and this is why a large literature has tried to give a good model. These models include experimental analysis as well as stochastic analyses. In this chapter, we start with the first two impacts. We learn the most important models for them through the exercises.

3.2 Exercises

3.2.1 Path Loss

1. Consider a cellular network in which the antennas at base stations and mobile stations have unit gains, i.e., $\log G_{\mathrm{BS}} = \log G_{\mathrm{MS}} = 0\ \mathrm{dB_i}$. Assume that we use the free space model to determine the path loss.

   (a) The operating carrier frequencies for GSM-800, GSM-1900 and millimeter wave (mmWave) are given in the following table.

System	Carrier frequency
GSM-900	900 MHz
GSM-1800	1.8 GHz
mmWave	30 GHz

   Sort the network in these frequencies in terms of the *area of each cell*.

   (b) Assume a mobile station is located in distance d of a base station. Write the path loss in terms of the distance, and calculate its limit when d tends to zero. Is this result correct? Justify your observation.

   In practice, the path loss does not follow the free space model. Hence, we want to modify this model, such that it calculates the path loss more accurately. For this aim, we let the loss exponent be a positive real number n. This means that the power flux at distance d from an isotropic antenna is assumed to be

   $$S = \frac{P_{\mathrm{s}}}{4\pi d^n}.$$

   (c) Find the expression for the path loss in this case.

   (d) Give a lower bound on n.

   (e) Assume that n_1 and n_2 are efficient exponents which accurately determine the path loss in Erlangen and Tokyo, respectively. How do you sort n_1 and n_2?

2. A base station is located at the height of 100 m from the roof of a living tower in Büchenbach with 20 floors. It is used to serve a mobile station which is located in an apartment on the ground floor of a building with 15 floors in Erlangen Zentrum. The network operates in GSM-900 band, and both the base and mobile stations use dipole antennas.

   (a) Using the Okumura-Hata model, calculate the path loss in decibel when the the distance between the apartment in Erlangen Zentrum and the living tower in Büchenbach is $d = 8$ km.

   (b) Now, assume that you intend to use the modified free space model proposed in the previous exercise for the path loss. Determine the exponent n, such that this model gives same path loss as the one derived via the Okumura-Hata model in the previous part. Does it satisfy the lower bound on n?

   (c) Assume that the mobile station has been moved to the roof of this building. Do you still use the Okumura-Hata model to calculate the path-loss? Justify your answer by giving a reason.

3.2.2 Shadowing

1. Consider the transmission scenario in Exercise 2 of Section 3.2.1. Assume that the antenna at the base station is replaced with a directive antenna whose gain is $g_{\mathrm{BS}} = 4$. The base station transmits signals with power

$$P_{\mathrm{transmit}} = 10 \text{ mW}$$

At the receive side, the mobile station is located on the roof of a car at height $h_{\mathrm{MS}} = 2$ m from the ground. The car moves around the base station, such that its distance to the base station is kept approximately fixed. The noise power at the mobile station is further

$$P_{\mathrm{noise}} = 10^{-10} \text{ mW}$$

(a) Name the stochastic model which describes the receive power in this case.

(b) Determine the mean value of this process.

For a successful transmission, the minimum signal-to-noise ratio (SNR) $\mathrm{SNR}_{\min} = -15$ dB is required. This means that the mobile station can decode the transmitted data only if the SNR of its receive signal is more than this minimum value.

(c) Assuming that the standard deviation of the shadowing process is $\sigma_{\Lambda} = 4$ dB, determine the percentage of the times in which the mobile station is not able to decode the transmitted data. This quantity is often called the *percentage of outage*.

3.3 Solutions to Exercises

3.3.1 Path Loss

1. Consider a cellular network in which the antennas at base stations and mobile stations have unit gains, i.e., $\log G_{\mathrm{BS}} = \log G_{\mathrm{MS}} = 0 \text{ dB}_{\mathrm{i}}$. Assume that we use the free space model to determine the path loss.

(a) The operating carrier frequencies for GSM-800, GSM-1900 and millimeter wave (mmWave) are given in the following table.

System	Carrier frequency
GSM-900	900 MHz
GSM-1800	1.8 GHz
mmWave	30 GHz

Sort the network in these frequencies in terms of the *area of each cell*.

(b) Assume a mobile station is located in distance d of a base station. Write the path loss in terms of the distance, and calculate its limit when d tends to zero. Is this result correct? Justify your observation.

In practice, the path loss does not follow the free space model. Hence, we want to modify this model, such that it calculates the path loss more accurately. For this aim, we let the loss exponent be a positive real number n. This means that the power flux at distance d from an isotropic antenna is assumed to be

$$S = \frac{P_\text{s}}{4\pi d^n}.$$

(c) Find the expression for the path loss in this case.

(d) Give a lower bound on n.

(e) Assume that n_1 and n_2 are efficient exponents which accurately determine the path loss in Erlangen and Tokyo, respectively. How do you sort n_1 and n_2?

♠ **Solution:**

(a) Let PL_1, PL_2 and PL_3 represent the path-loss for GSM-900, GSM-1800 and mm-wave band, respectively. From equation (3.10) in page 79, one observes that the path-loss grows proportional to the carrier frequency. Therefore, we have

$$\mathrm{PL}_1 < \mathrm{PL}_2 < \mathrm{PL}_3.$$

This indicates that for a fixed transmit power the coverage area for GSM-800 band is larger than the area for GSM-1900 band, and the area for mm-wave band is smaller than the two other bands.

(b) Considering either equation (3.9) or equation (3.10) in page 79, as d tends to zero, PL tends to zero, as well. In other words, the receive power tends to *infinity (!). This is not impossible.* The reason of such incorrect observation here is that the models for the path loss consider *far-fields*, meaning that $d \gg \lambda$. As a result, d in these models cannot be sent to zero.

> ⇝ REMARK:
>
> Remember that all the derivations for antennas consider *far-fields* meaning that they are true for distances which read $d \gg \lambda$.

(c) For the modified model, we have

$$S = \frac{P_\text{s}}{4\pi d^n}.$$

The receive power reads

$$P_\text{r} = S A_\text{w}$$

where A_w is the effective area of the antenna and is given in page 79 as

$$A_\text{w} = \frac{\lambda^2}{4\pi}.$$

Substituting $\lambda = \frac{c}{f}$ into it, we have

$$A_{\mathrm{w}} = \frac{c^2}{4\pi f^2}.$$

As a result the receive power reads

$$\begin{aligned} P_{\mathrm{r}} &= G_{\mathrm{BS}} G_{\mathrm{MS}} \frac{P_{\mathrm{s}}}{4\pi d^n} A_{\mathrm{w}} \\ &= G_{\mathrm{BS}} G_{\mathrm{MS}} P_{\mathrm{s}} \left(\frac{c}{4\pi}\right)^2 \frac{1}{f^2 d^n} \end{aligned}$$

Consequently, the path loss as a function of n is given by

$$a(n) = \frac{P_{\mathrm{s}}}{P_{\mathrm{r}}} = \frac{1}{G_{\mathrm{BS}} G_{\mathrm{MS}}} \left(\frac{4\pi}{c}\right)^2 d^n f^2.$$

Hence the path loss in [dB] reads

$$\begin{aligned} \mathrm{PL}(n) &= \log a(n) && \text{(3.3.1)} \\ &= 10 \log \left(\frac{4\pi}{c}\right)^2 + 10n \log d + 20 \log f - \log G_{\mathrm{BS}} - \log G_{\mathrm{MS}}. && \text{(EQ:A)} \end{aligned}$$

Here, d is in *meters* and the frequency is in *hertz*. We note that,

- For the first, we have

$$10 \log \left(\frac{4\pi}{c}\right)^2 = -147.56$$

- For the second term, we have

$$\begin{aligned} 10n \log (d\,[\mathrm{m}]) &= 10n \log \left(10^3 d\,[\mathrm{km}]\right) \\ &= 30n + 10n \log (d\,[\mathrm{km}]). \end{aligned}$$

- For the third term, we have

$$\begin{aligned} 20 \log (f\,[\mathrm{Hz}]) &= 20 \log \left(10^6 f\,[\mathrm{MHz}]\right) \\ &= 120 + 20 \log (f\,[\mathrm{MHz}]). \end{aligned}$$

Thus, we have replacing into the (EQ:A)

$$\begin{aligned} \mathrm{PL}(n) &= \log a(n) \\ &= -27.56 + 10n \log d\,[\mathrm{km}] + 20 \log f\,[\mathrm{MHz}] \\ &\quad + 30n - \log G_{\mathrm{BS}} - \log G_{\mathrm{MS}} \\ &= -27.56 + \underbrace{(60-60)}_{0} + 10n \log d\,[\mathrm{km}] + 20 \log f\,[\mathrm{MHz}] \\ &\quad + 30n - \log G_{\mathrm{BS}} - \log G_{\mathrm{MS}} \\ &= 32.44 + 10n \log d\,[\mathrm{km}] + 20 \log f\,[\mathrm{MHz}] \\ &\quad + 30n - 60 - \log G_{\mathrm{BS}} - \log G_{\mathrm{MS}} \\ &= 32.44 + 10n \log d\,[\mathrm{km}] + 20 \log f\,[\mathrm{MHz}] \\ &\quad + 30(n-2) - \log G_{\mathrm{BS}} - \log G_{\mathrm{MS}}. \end{aligned}$$

Setting $n = 2$, the equation reduces to equation (3.10) in page 79 of the lecture notes.

(d) In general, the exponent n increases in this model as the interference increases in the medium. The smallest value for n is $n = 2$ which occurs in the interference-free medium: the so-called *free space*. Thus, we have the lower bound

$$\boxed{n \geq 2}$$

(e) Since Erlangen is less crowded and has buildings with lower heights compared to Tokyo, the amount of interference is less in Erlangen. Hence $n_1 < n_2$.

2. A base station is located at the height of 100 m from the roof of a living tower in Büchenbach with 20 floors. It is used to serve a mobile station which is located in an apartment on the ground floor of a building with 15 floors in Erlangen Zentrum. The network operates in GSM-900 band, and both the base and mobile stations use dipole antennas.

(a) Using the Okumura-Hata model, calculate the path loss in decibel when the the distance between the apartment in Erlangen Zentrum and the living tower in Büchenbach is $d = 8$ km.

(b) Now, assume that you intend to use the modified free space model proposed in the previous exercise for the path loss. Determine the exponent n, such that this model gives same path loss as the one derived via the Okumura-Hata model in the previous part. Does it satisfy the lower bound on n?

(c) Assume that the mobile station has been moved to the roof of this building. Do you still use the Okumura-Hata model to calculate the path-loss? Justify your answer by giving a reason.

♠ **Solution:**

(a) To evaluate the path loss, we employ the Okumura-Hata model in pages 89 and 90. It is important to note that Erlangen is categorized as a *suburban* area in this model. Since the carrier frequency reads

$$150 \text{ MHz} < f_0 < 1.5 \text{ GHz},$$

the path loss is therefore determined in this case from equation (3.29) in page 90. For this problem, we assume each floor is approximately 3 meters[1]. Hence,

- $h_{\text{MS}} = 2$ m
- $h_{\text{BS}} = 100 + 20 \times 3 = 160$ m

Substituting into equation (3.27) in page 89, it is concluded that for isotropic antennas

$$\text{PL}_{\text{urban,isotropic}} = 142.51 \text{ dB}.$$

Using equation (3.29) in page 90, we have

$$\text{PL}_{\text{suburban,isotropic}} = 142.51 - 9.94 = 132.57 \text{ dB}.$$

[1] It is just an assumption, you could assume something else.

⇝ REMARK:

Remember that the formulas in pages 89 and 90 are given for *isotropic antennas*.

To calculate the path loss for dipole antennas, we note that

$$a_{\text{isotropic}} = \frac{P_{\text{s}}}{P_{\text{r,isotropic}}}$$

where P_{s} is the transmit power, and $P_{\text{r,isotropic}}$ is the receive power, when we use isotropic antennas at the both ends. When we use antennas with gains G_{BS} and G_{MS} at the base station and the mobile station, respectively, the receive power reads

$$P_{\text{r}} = P_{\text{r,isotropic}} G_{\text{BS}} G_{\text{MS}}$$

Thus, we have

$$a = \frac{P_{\text{s}}}{G_{\text{BS}} G_{\text{MS}} P_{\text{r}}} = \frac{a_{\text{isotropic}}}{G_{\text{BS}} G_{\text{MS}}}$$

where a simply indicates the attenuation, when we use the directive antennas. As a result, we have

$$\text{PL} = \text{PL}_{\text{isotropic}} - \log G_{\text{BS}} \text{ [dB}_{\text{i}}\text{]} - \log G_{\text{MS}} \text{ [dB}_{\text{i}}\text{]}$$

For dipole antennas, we have $\log G = 2.15 \text{ dB}_{\text{i}}$. the path loss is

$$\begin{aligned} \text{PL}_{\text{suburban}} &= \text{PL}_{\text{suburban,isotropic}} - \log G_{\text{BS}} \text{ [dB}_{\text{i}}\text{]} - \log G_{\text{MS}} \text{ [dB}_{\text{i}}\text{]} \\ &= 132.57 - 2.15 - 2.15 = 128.27 \text{ dB}. \end{aligned}$$

(b) In the previous exercise, we found the path loss for the modified free space model as

$$\text{PL}(n) = 32.44 + 10n \log d + 20 \log f + 30\,(n-2) - \log G_{\text{BS}} - \log G_{\text{MS}}.$$

To find the corresponding n for this given scenario, we should find n such that $\text{PL}(n)$ is equal to $\text{PL}_{\text{suburban}} = 128.27$. Thus, we have

$$\begin{aligned} 128.27 &= \text{PL}(n) \\ &= 32.44 + 10n \log 8 + 20 \log 900 + 30\,(n-2) - 4.3 \Rightarrow \boxed{n \approx 2.6} \end{aligned}$$

The result satisfy the lower bound $n > 2$ derived in the previous exercise.

(c) When the mobile station moves to the roof of a building with 15 floors, the height of the mobile station is calculated as the height of the mobile station itself plus the building. Assuming that each floor has 3 meters height, the height of the mobile station is

$$h_{\text{MS}} = 2 + 15 \times 3 = 47$$

From page 90 of the lecture notes we know that the Okumura-Hata model is valid for

$$1 \text{ m} < h_{\text{MS}} < 10 \text{ m}.$$

Since $47 \gg 10$, we conclude that in this case, the model is not valid.

3.3.2 Shadowing

1. Consider the transmission scenario in Exercise 2 of Section 3.3.1. Assume that the antenna at the base station is replaced with a directive antenna whose gain is $G_{\text{BS}} = 4$. The base station transmits signals with power

$$P_{\text{transmit}} = 10 \text{ mW}$$

At the receive side, the mobile station is located on the roof of a car at height $h_{\text{MS}} = 2$ m from the ground. The car moves around the base station, such that its distance to the base station is kept approximately fixed. The noise power at the mobile station is further

$$P_{\text{noise}} = 10^{-10} \text{ mW}$$

(a) Name the stochastic model which describes the receive power in this case.

(b) Determine the mean value of this process.

For a successful transmission, the minimum signal-to-noise ratio (SNR) $\text{SNR}_{\text{min}} = -15$ dB is required. This means that the mobile station can decode the transmitted data only if the SNR of its receive signal is more than this minimum value.

(c) Assuming that the standard deviation of the shadowing process is $\sigma_\Lambda = 4$ dB, determine the percentage of the times in which the mobile station is not able to decode the transmitted data. This quantity is often called the *percentage of outage*.

♠ **Solution:**

(a) Noting that the distance of the car to the base station is fixed always, the path loss is fixed. The receive power has however some fluctuations, due to the obstacles in the area. This fluctuations are called *shadowing* and is modeled as a *log-normal* process. This means that the logarithm of the receive power reads $\log P_{\text{r}} = \Lambda$, where

$$\Lambda \sim \mathcal{N}\left(\mu_\Lambda, \sigma_\Lambda^2\right).$$

(b) The mean value of the process is the average receive power in logarithmic scale. As the heights and the distance are the same, the path loss with isotropic antennas are the same in both the scenarios. The only difference here is that the antenna at the base station is changed. The path loss hence is calculated as

$$\begin{aligned}\mathrm{PL} &= \mathrm{PL}_{\text{suburban,isotropic}} - \log G_{\mathrm{BS}} \text{ [dB}_\text{i}\text{]} - \log G_{\mathrm{MS}} \text{ [dB}_\text{i}\text{]} \\ &= 132.57 - 6 - 2.15 = 124.42 \text{ dB}.\end{aligned}$$

The average receive power is hence given by

$$\log \bar{P}_{\mathrm{r}} = \log P_{\text{transmit}} - \mathrm{PL}$$

Noting that $\log P_{\text{transmit}} = 10$ dBm, we have

$$\log \bar{P}_{\mathrm{r}} = 10 - 124.42 = -114.42 \text{ [dBm]}.$$

Therefore, $\boxed{\mu_\Lambda = -114.42}$.

(c) SNR in the logarithmic scale reads

$$\mathrm{SNR} = \log P_{\mathrm{r}} - \log P_{\text{noise}}.$$

To have at least $\mathrm{SNR}_{\min} = -15$, we need to have

$$\log P_{\mathrm{r}} \geq \mathrm{SNR}_{\min} + \log P_{\text{noise}} = \log P_{\text{noise}} - 15$$

Since $\log P_{\text{noise}} = -100$ dBm, we need to have

$$\log P_{\mathrm{r}} \geq -115 \text{ dBm},$$

to make sure that the mobile station can decode the transmitted data. In other words, when

$$\log P_{\mathrm{r}} < -115 \text{ dBm}$$

the mobile station cannot decode the transmitted data.

In this case, we say that the mobile station is in *outage*.

Noting that $\log P_{\mathrm{r}} = \Lambda$ is a random process in the presence of shadowing, the probability that the mobile station be in outage is

$$\Pr_{\text{outage}} = \Pr\{\log P_{\mathrm{r}} < -115\} = \Pr\{\Lambda < -115\}$$

Since $\sigma_\Lambda = 4$, we have

$$\Lambda \sim \mathcal{N}\left(-114.42, 16\right).$$

As the result, the outage probability reads

$$\begin{aligned}\mathrm{Pr}_{\mathrm{outage}} &= \mathrm{Pr}\{\Lambda < -115\} \\ &= 1 - \mathrm{Pr}\{\Lambda \geq -115\} \\ &= 1 - \mathrm{Q}\left(\frac{-115 - \mu_\Lambda}{\sigma_\Lambda}\right) \\ &= 1 - \mathrm{Q}\left(\frac{-115 + 114.42}{4}\right) \\ &= 1 - \mathrm{Q}\left(0.145\right) = 1 - 0.56 = 0.44.\end{aligned}$$

This means that the mobile station has in 44% of the times unsuccessful receptions.

3.4 Homework

1. A base station of length 10 meters is placed on the roof of a tower with 30 floors in the center of New York city. Each floor is approximately 3 meters long. The network operates at carrier frequency 1900 MHz, and is employed to serve a mobile station of the height 1 meter locate on the roof of an van moving in the city. The gain of the antenna at the base station is $G_{\mathrm{BS}} = 6\ \mathrm{dB_i}$. The mobile station uses a simple $\lambda/2$-dipole antenna. The van moves in the city, such that its distance from the base station is approximately constant.

   (a) Assume that the distance between the van and the base station is $d = 5$ km. Find the path-loss using Okumura-Hata's model.

   (b) Calculate the path-loss for the previous part using the free-space propagation model.

   (c) Compare the results in Parts (a) and (b) and justify the difference by giving a reason.

   (d) Assume that the variance of shadowing process is $\sigma_\Lambda^2 = 9$ dB. Find the minimum required transmit power, such that the receive power in 95% of the times is more than -110 dBm.

♠ **Solution:** We have

- an *urban area*
- $h_{\mathrm{BS}} \approx 3 \times 30 + 10 = 100$ meters
- $h_{\mathrm{MS}} \approx 1 + 1 = 2$ meters, assuming that the van is 1 meter long

Using Okumura-Hata model, $\mathrm{PL}_{\mathrm{urban}} = 148.41$ dB. From the free-space model, we get $\mathrm{PL}_{\mathrm{free}} = 112$ dB when we do not consider the antenna gains. The impact of antennas are included by subtracting $G_{\mathrm{BS}} = 6\ \mathrm{dB_i}$ and $G_{\mathrm{MS}} = 2.15\ \mathrm{dB_i}$ from the calculated values.

The average receive power μ_Λ reads

$$\mu_\Lambda = P_\mathrm{T} - 148.41 + 6 + 2.15 = P_\mathrm{T} - 140.26 \text{ dBm}$$

where P_T is the transmit power in dBm. To guarantee less than 5% outage at -110 dBm, we need

$$\Pr\{\Lambda > -110\} = \mathrm{Q}\left(\frac{-110 - \mu_\Lambda}{\sigma_\Lambda}\right) > 0.95$$

which concludes that $P_\mathrm{T} \geq 36.84$ dBm.

Chapter 4

Basics of Small-Scale Fading

In this chapter, we go through the second part of Chapter 3 in the lecture-notes. Here, we learn the concept of small-scale fading which is also often referred to as *multipath* fading. This chapter only includes two important exercises through which the key reasons for small-scale fading are illustrated.

4.1 Brief Review of Main Concepts

Small-scale fading comes from two sources:

1. Multipath effect which indicates that the received signal by the receiver in a wireless network is the superposition of multiple copies of the transmit signal which have been received with different delays. This comes from the fact, that the received signal is the superposition of all reflection of the transmit signals being received by the receiver through various paths available in the air.

2. Doppler effect which indicates that the carrier frequency of the transmit signal is being detected by some deviation at the receiver, once the receiver is moving.

These two effects cause the end-t-end channel to be changing through time and frequency. The fist effect defines the so-called *coherence bandwidth* for the channel and the second one, defines the *coherence time* for the wireless channel. We now illustrate these effects and definitions through two exercises.

4.2 Exercises

1. Consider a mobile station which is fixed at a certain distance from the base station. Assume that there exist *no path loss, shadowing and noise impairment*. The transmitter sends signal $x(t)$ with bandwidth B to the mobile station. This means that

$$X(f) = 0 \qquad \text{if } |f| > \frac{B}{2}$$

where $X(f) = \mathcal{F}\{x(t)\}$ is the Fourier transform of $x(t)$.

The mobile station receives $x(t)$ through two different paths. The delays of these paths are τ_0 and $\tau_0 + \Delta\tau$ and the channel gain of each path is $1/\sqrt{2}$.

(a) Write the received signal in the time domain $y(t)$, and determine the channel impulse response $h(t)$.

(b) Calculate the frequency response of this wireless channel $H(f)$.
Hint: Note that with $\mathcal{F}\{x(t)\} = X(f)$, we have

$$\mathcal{F}\{x(t-\tau)\} = \exp\{-j2\pi f\tau\} X(f).$$

We say that the signal is *faded* at the receiver, if the spectrum of the received signal, i.e., $Y(f) = \mathcal{F}\{y(t)\}$ is zero at some frequency $|f| < B/2$.

(c) Find the maximum bandwidth B_C for signal $x(t)$ by which we guarantee that signal at the receiver is not faded. Calculate B_C for $\Delta\tau = 0.5$ μsec and $\Delta\tau = 5$ μsec.

2. Consider a mobile station which moves towards a base station at the speed of v m/sec. The mobile station sees the base station at the angle $\theta = 60^\circ$. Assume that *no path loss, shadowing and noise impairment* exist. The base station sends a monotone pulse at frequency f_0 with duration T. This means that the transmit signal is $x(t) = \exp\{j2\pi f_0 t\} g_\mathrm{T}(t)$ with

$$g_\mathrm{T}(t) = \begin{cases} 1 & t \in [0, T) \\ 0 & t \in (-\infty, 0) \text{ and } t \in [T, \infty) \end{cases}.$$

For simplicity, ignore the propagation delay, and assume that the mobile station receives signal from $t = 0$.

(a) Calculate the received signal in the time domain $y(t)$ considering the Doppler effect.

The mobile station down-modulates its receive signal to the base band as

$$\hat{x}(t) = \Re\{y(t)\exp\{-j2\pi f_0 t\}\}.$$

It then samples the down-modulated signal at time $t = T_s < T$ and decides about the transmitted data based on the sign of this sample.

(b) Find the minimum value for the sampling time T_s at which the mobile station's sample is -1 instead of 1. Calculate this value for $v = 3$ m/sec and $v = 30$ m/sec assuming $f_0 = 900$ MHz.

4.3 Solutions to Exercises

1. Consider a mobile station which is fixed at a certain distance from the base station. Assume that there exist *no path loss, shadowing and noise impairment*. The transmitter sends signal $x(t)$ with bandwidth B to the mobile station. This means that

$$X(f) = 0 \qquad \text{if } |f| > \frac{B}{2}$$

where $X(f) = \mathcal{F}\{x(t)\}$ is the Fourier transform of $x(t)$.

The mobile station receives $x(t)$ through two different paths. The delays of these paths are τ_0 and $\tau_0 + \Delta\tau$ and the channel gain of each path is $1/\sqrt{2}$.

(a) Write the received signal in the time domain $y(t)$, and determine the channel impulse response $h(t)$.

(b) Calculate the frequency response of this wireless channel $H(f)$.
Hint: Note that with $\mathcal{F}\{x(t)\} = X(f)$, we have

$$\mathcal{F}\{x(t-\tau)\} = \exp\{-j2\pi f\tau\} X(f).$$

We say that the signal is *faded* at the receiver, if the spectrum of the received signal, i.e., $Y(f) = \mathcal{F}\{y(t)\}$ is zero at some frequency $|f| < B/2$.

(c) Find the maximum bandwidth B_C for signal $x(t)$ by which we guarantee that signal at the receiver is not faded. Calculate B_C for $\Delta\tau = 0.5$ μsec and $\Delta\tau = 5$ μsec.

♠ **Solution:**

(a) By ignoring the shadowing impact and noise, the receive signal can be written as

$$y(t) = \frac{1}{\sqrt{2}} x(t-\tau_0) + \frac{1}{\sqrt{2}} x(t-\tau_0-\Delta\tau)$$

where the first term is the signal received from the first path with delay τ_0 and the second term is the one received from the path with delay $\tau_0 + \Delta\tau$.

(b) The Fourier transform of $y(t)$ is given by

$$\begin{aligned} Y(f) &= \mathcal{F}\left\{\frac{1}{\sqrt{2}} x(t-\tau_0) + \frac{1}{\sqrt{2}} x(t-\tau_0-\Delta\tau)\right\} \\ &= \frac{1}{\sqrt{2}} \exp\{-j2\pi f\tau_0\} X(f) + \frac{1}{\sqrt{2}} \exp\{-j2\pi f(\tau_0+\Delta\tau)\} X(f) \\ &= \frac{1}{\sqrt{2}} \exp\{-j2\pi f\tau_0\} X(f)\left[1+\exp\{-j2\pi f\Delta\tau\}\right]. \end{aligned}$$

Consequently, the frequency response of the channel reads

$$\begin{aligned} H(f) &= \frac{Y(f)}{X(f)} \\ &= \frac{1}{\sqrt{2}} \exp\{-j2\pi f\tau_0\}\left[1+\exp\{-j2\pi f\Delta\tau\}\right]. \end{aligned}$$

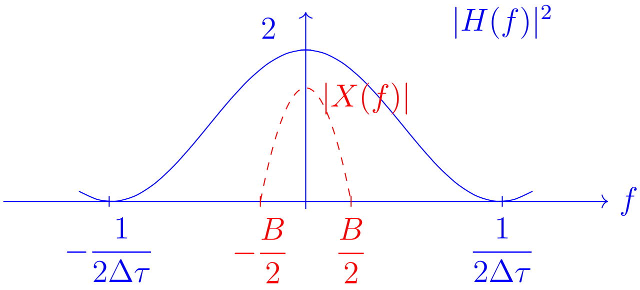


Figure 4.3.1: Frequency response of the wireless channel in black. Signal in the frequency domain in red.

As a result, the squared magnitude of the response is given by

$$\begin{aligned}|H(f)|^2 &= \frac{1}{2}\underbrace{|\exp\{-j2\pi f\tau_0\}|^2}_{1}|1+\exp\{-j2\pi f\Delta\tau\}|^2 \\ &= \frac{1}{2}(1+\cos 2\pi f\Delta\tau)^2 + \frac{1}{2}(\sin 2\pi f\Delta\tau)^2 \\ &= 1+\cos 2\pi f\Delta\tau\end{aligned}$$

(c) The frequency response is shown in Figure 3.1. As it is seen in the figure, $|H(f)|$ gets zero periodically each $1/\Delta\tau$ Hertz.

The signal is completely faded when $Y(f)=0$ at some frequency f for which we have $X(f)\neq 0$. To find the maximum value of B for which we can avoid complete fading, we have plotted a sample signal spectrum in the figure in red. As it is seen in the figure, regardless of the center frequency, the red spectrum meets one null of the frequency response, if

$$B > \frac{1}{\Delta\tau}.$$

Hence, to avoid complete fading, we need

$$B \leq \frac{1}{\Delta\tau}.$$

In other words,

$$B_{\mathrm{C}} = \frac{1}{\Delta\tau}.$$

As a result, $B_{\mathrm{C}} = 2$ MHz, when $\Delta\tau = 0.5$ μsec, and $B_{\mathrm{C}} = 200$ kHz, when $\Delta\tau = 5$ μsec.

B_{C} is known as the *coherence bandwidth* of the fading channel. For practical scenarios, the coherence bandwidth is determined from the power-delay profile, since the channel response is a random process.

2. Consider a mobile station which moves towards a base station at the speed of v m/sec. The mobile station sees the base station at the angle $\theta = 60°$. Assume that *no path loss, shadowing and noise impairment* exist. The base station sends a monotone pulse at frequency f_0 with duration T. This means that the transmit signal is $x(t) = \exp\{j2\pi f_0 t\}\, g_{\mathrm{T}}(t)$ with

$$g_{\mathrm{T}}(t) = \begin{cases} 1 & t \in [0, T) \\ 0 & t \in (-\infty, 0) \text{ and } t \in [T, \infty) \end{cases}.$$

For simplicity, ignore the propagation delay, and assume that the mobile station receives signal from $t = 0$.

(a) Calculate the received signal in the time domain $y(t)$ considering the Doppler effect.

The mobile station down-modulates its receive signal to the base band as

$$\hat{x}(t) = \Re\{y(t) \exp\{-j2\pi f_0 t\}\}.$$

It then samples the down-modulated signal at time $t = T_s < T$ and decides about the transmitted data based on the sign of this sample.

(b) Find the minimum value for the sampling time T_s at which the mobile station's sample is -1 instead of 1. Calculate this value for $v = 3$ m/sec and $v = 30$ m/sec assuming $f_0 = 900$ MHz.

♠ **Solution:**

(a) As the mobile station moves toward the base station with speed v with angle θ from the line connecting the base station and the mobile station, the *carrier frequency* of the signal observed by the mobile station is shifted[1]. This shifted frequency is approximately given by

$$\begin{aligned} f_{\mathrm{rel}} &= \frac{c + v\cos\theta}{c} f_0 \\ &= f_0 + f_{\mathrm{d}} \end{aligned}$$

where

$$f_{\mathrm{d}} = \frac{v}{c} f_0 \cos\theta$$

is the Doppler shift.

Consequently, the receive signal is shifted in the frequency domain by f_{d} meaning that the receiver observes

$$\begin{aligned} y(t) = \exp\{j2\pi f_{\mathrm{rel}} t\}\, g_{\mathrm{T}}(t) &= \exp\{j2\pi (f_0 + f_{\mathrm{d}}) t\}\, g_{\mathrm{T}}(t) \\ &= \exp\{j2\pi f_{\mathrm{d}} t\}\, x(t). \end{aligned}$$

By defining $h(t) = \exp\{j2\pi f_{\mathrm{d}} t\}$, one can write

$$y(t) = h(t)\, x(t).$$

[1] Some of students wonder about this fact. Simply google "Doppler effect" to get more insights.

(b) The shifted signal to the base-band is given by

$$\begin{aligned}\hat{x}(t) &= \Re\{y(t)\exp\{-j2\pi f_0 t\}\}\\ &= \Re\{h(t)\,x(t)\exp\{-j2\pi f_0 t\}\} = g_{\mathrm{T}}(t)\cos(2\pi f_{\mathrm{d}} t)\end{aligned}$$

The sample at T_s hence reads

$$\hat{x}(T_s) = g_{\mathrm{T}}(T_s)\cos(2\pi f_{\mathrm{d}} T_s) = \cos(2\pi f_{\mathrm{d}} T_s).$$

This sample becomes negative for the first time, when

$$\frac{1}{4f_{\mathrm{d}}} < T_s < \frac{3}{4f_{\mathrm{d}}}$$

Hence, the minimum sampling time at which the sign is wrongly detected is

$$T_{s,\min} = \frac{1}{4f_{\mathrm{d}}}.$$

For $v = 3$ m/sec and $v = 30$ m/sec, the Doppler frequency is $f_{\mathrm{d}} = 9$ Hz and 90 Hz, respectively, hence,

$$T_{s,\min} = \begin{cases} 56 \text{ msec} & v = 3 \text{ m/sec} \\ 5.6 \text{ msec} & v = 30 \text{ m/sec} \end{cases}.$$

$\rightsquigarrow$ REMARK:

The movement of the user in the network cause Doppler shift which results in time-varying channel response. The faster you move, the faster the channel changes.

4.4 Summary of the Small-Scale Fading Concept

In general, we have two effects which make small-scale fading in the channel:

- *Multi-path* which makes your channel response varying over the *frequency axis* (remember Exercise 1 in Section 3.3 of Tutorial 3). This effect is in general random, since the number of paths, their delays and their coefficients change. We quantify this effect by means of the so-called *coherence bandwidth* B_{C}. The coherence bandwidth is the bandwidth over which the frequency response of the channel is approximately constant. In this case,
  - If the bandwidth of the signal transmitted over this channel is less than B_{C}, we say that *the signal experiences "narrow-band" or "flat" fading*.
  - If the bandwidth of the signal transmitted over this channel is more than B_{C}, we say that *the signal experiences "wide-band" or "frequency selective" fading*.
- *Doppler* which makes your channel response varying over *time* (remember this exercise). This effect is in general random, since the speed of the mobile, its angle to the base station and its medium change. We quantify this effect by means of the so-called *coherence time* T_{C}. The coherence time is the time duration over which the channel response is approximately not changing. In this case,
  - If the time duration of the signal transmitted over this channel is less than T_{C}, we say that *the signal experiences "slow" fading*.
  - If the time duration of the signal transmitted over this channel is more than T_{C}, we say that *the signal experiences "fast" fading*.

It is very important to note that these two effects are *jointly coupled* which make the wireless channel response varying in both the time and frequency domain.

4.5 Homework

1. Consider a fading channel in which the base station and the mobile station are located in a fixed place. There exist two paths from the base station to the mobile station:
   - The first path has delay zero and channel coefficient h_1
   - The second path has delay τ and coefficient h_2

   The base station transmits a signal whose base-band representation is
   $$x(t) = \sqrt{PB}\,\mathrm{sinc}\,(\pi B t) = \sqrt{PB}\,\frac{\sin(\pi B t)}{\pi B t}.$$
   This signal is band-limited within interval $[-B/2, B/2]$, i.e.,
   $$X(f) = \mathcal{F}\{x(t)\} = \begin{cases} \sqrt{\dfrac{P}{B}} & |f| \leq \dfrac{B}{2} \\ 0 & |f| > \dfrac{B}{2} \end{cases}.$$

For this signal, we have

$$\int_{-\infty}^{+\infty} |x(t)|^2 \mathrm{d}t = P.$$

(a) Represent the signal $y(t)$, received by the mobile station, in the base band assuming additive white Gaussian noise at the receiving terminal.

(b) For given h_1 and h_2, determine the signal-to-noise ratio at the mobile station assuming that the power spectral density of noise is N_0 and that the receiver uses a low-pass filter in the base-band whose frequency response is as follows

$$G(f) = \begin{cases} 1 & |f| \leq \dfrac{B}{2} \\ 0 & |f| > \dfrac{B}{2} \end{cases}.$$

Hint: Use Parseval's theorem which indicates that for any signal $x(t)$ with limited energy P,

$$\int_{-\infty}^{+\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |X(f)|^2 \mathrm{d}f.$$

(c) Let the channel coefficients h_1 and h_2 be correlated complex Gaussian random variables with zero mean, variance 0.5 and real correlation ρ. Determine the average signal-to-noise ratio at the mobile station.

(d) Is the correlation between the channels necessarily beneficial?

♠ **Solution:** The received signal in the case-band reads

$$y(t) = h_1 x(t) + h_2 x(t - \tau) + n(t)$$

where $n(t)$ is noise. In the frequency domain, we have

$$Y(f) = H(f)X(f) + N(f)$$

where

$$H(f) = h_1 + h_2 \exp\{-j2\pi\tau f\}.$$

Denoting the signal after filtering at the mobile station with $r(t)$, we have in the frequency domain

$$R(f) = G(f)Y(f) = \underbrace{G(f)H(f)X(f)}_{S(f)} + \underbrace{G(f)N(f)}_{Z(f)}.$$

From Parseval's theorem, the signal-to-noise ratio is determined by

$$\begin{aligned}
\mathrm{SNR} &= \frac{\int_{-\infty}^{+\infty} |S(f)|^2 \mathrm{d}f}{\int_{-\infty}^{+\infty} |Z(f)|^2 \mathrm{d}f} \\
&= \frac{\frac{P}{B}\int_{-B/2}^{+B/2} |H(f)|^2 \mathrm{d}f}{N_0 B} \\
&= \frac{P}{N_0 B}\left[|h_1|^2 + |h_2|^2 + 2\mathrm{Re}\left\{h_1^* h_2\right\} \mathrm{sinc}(\pi B\tau)\right]
\end{aligned}$$

where $\mathrm{Re}\left\{\cdot\right\}$ denotes the real part and $\mathrm{sinc}(x) = \dfrac{\sin x}{x}$. By averaging over h_1 and h_2, we have

$$\overline{\mathrm{SNR}} = \mathcal{E}\left[\mathrm{SNR}\right] = \frac{P}{N_0 B}\left[1 + 2\rho\, \mathrm{sinc}\left(\pi B\tau\right)\right]$$

As $\mathrm{sinc}\left(\pi B\tau\right)$ takes both positive and negative values, the correlation could be constructive or destructive.

2. Consider a mobile user who is riding a bike with speed v toward the base station. The user is receiving signal from two paths

   - The line-of-sight with channel coefficient 1 and delay τ
   - A scatterer which is locate at angle θ with channel coefficient 0.5 and delay 2τ.

   For this scenario, do the following tasks:

   (a) Plot the Doppler spectral density of the channel from the base station.

   (b) Plot the amplitude and the real part of the channel transfer function in the frequency domain.

   (c) Determine these functions for some practical speed and delay in the GSM-900 band.

♠ **Solution:** There are only two scatterers. Therefore, the Doppler spectrum contains two impulses. One with amplitude one at the maximum Doppler shift $f_{\mathrm{d}} = v f_0/c$, and one with half amplitude at $f_{\mathrm{d}} \cos\theta$.

The transfer function further reads

$$H\left(f\right) = \exp\left\{-j2\pi\tau f\right\} + \frac{1}{2}\exp\left\{-j4\pi\tau f\right\}.$$

Chapter 5

Narrow-Band and Wide-Band Fading

This chapter covers the last part of Chapter 3 in the lecture-notes. As we mentioned in the previous chapter, given the characteristics of a communication channel, a transmitter can experience narrow-band or wide-band fading. In this chapter, we go through these concepts.

5.1 Brief Review of Main Concepts

As we mentioned in the previous chapter, depending on the coherence bandwidth of the channel, we have two scenarios:

- If the bandwidth of the signal transmitted over the channel is less than the coherence bandwidth, we say that *the signal experiences "narrow-band" or "flat" fading*.
- If the bandwidth of the signal transmitted over the channel is more than the coherence bandwidth, we say that *the signal experiences "wide-band" or "frequency selective" fading*.

The coherence time also specifies the cases of slow and fast fading. In this lecture, we do not go in details through the second categorization, as there is not so much to say about. We however go through the first categorization and some remaining concepts via some exercises.

5.2 Exercises

5.2.1 Doppler Spectrum

1. Consider a narrow-band fading wireless channel. The base station and the mobile station communicate at the carrier frequency $f_0 = 3$ GHz. Determine the Doppler spectral density for the following case:

   The multipath fading is Ricean with Rice factor $K = -6$ dB, and the receive power is 1 μW. The scatterers are uniformly distributed around the mobile station. The maximal velocity of the mobile station is $v = 36$ km/h and the line-of-sight toward the base-station is at $\alpha = 45°$.

5.2.2 Narrow-band Small-scale fading

1. Consider a *narrow-band* multipath fading channel between a transmitter and a receiver. The system operates at the carrier frequency $f_\mathrm{C} = 900\ \mathrm{MHz}$. The transmitter sees the receiver through a line-of-sight path with Rice factor $\log K = 4\ \mathrm{dB}$.
   The receiver moves with the maximum speed of $v_{\max} = 120$ km/h in a *rich* scattering environment, and the superposed power received through the scattered components is $\sigma_R^2 = 5$ mW.

   Assume that the transmitter sends the monotone signal

   $$s(t) = \sqrt{P} \exp\{-2\pi \mathrm{j} f_\mathrm{C} t\}$$

   where $P = 20$ W. The received signal is denoted by

   $$Y(t) = R(t) + N(t)$$

   where $N(t)$ is additive white Gaussian noise with zero mean and variance $\sigma_N^2 = 0.1$ mW, and $R(t)$ is the superposition of the signals received from multiple paths.

   (a) Calculate the probability density function of the time sample

   $$R_0 = R(t_0)$$

   for some arbitrary t_0. Specify the mean and the variance of this random variable.

   (b) We now sample $R(t)$ at some different time $t_1 \neq t_0$ and obtain the random variable

   $$R_1 = R(t_1).$$

   Do R_0 and R_1 have identical distributions?

   (c) Determine the average receive power at t_0 which is defined as

   $$P_\mathrm{R} = \mathcal{E}\left\{|Y(t_0)|^2\right\}.$$

   Here, $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation.

   (d) Assume that you are monitoring the random process $R(t)$ in the frequency domain by a spectrum meter which determines the average density of $|R(t)|^2$ at each frequency. You observe the frequency interval

   $$\Delta f = [f_\mathrm{C} - 200\ \mathrm{Hz}, f_\mathrm{C} + 200\ \mathrm{Hz}]$$

   by this spectrum meter. Plot the output of this spectrum meter in the interval Δf. Describe the distribution of the scatterers around the receiver that you assumed to plot the output.

5.2.3 Wide-band Transmission over Fading Channels

1. Consider a fading channel with *high* coherence time. The power-delay profile of this channel is given as follows:

$$\rho_h(\tau, 0) = \sum_{n=0}^{3} \frac{1}{2^n}\, \delta(\tau - n\tau_0)$$

where τ_0 is a constant.

(a) For this channel, calculate the following parameters:

i. the power transform factor

ii. the delay time

iii. the delay spread

iv. the excess delay

in terms of τ_0.

(b) Assume that GSM symbols are being transmitted over this wireless channel with pulses which occupy the bandwidth $B = 1/T$ with T being the duration of a symbol transmission. For the following values of τ_0, explain whether the signal experiences *narrow-band* or *wide-band* fading.

i. $\tau_0 = 0.1\ \mu$sec, and

ii. $\tau_0 = 5\ \mu$sec

5.3 Solutions to Exercises

5.3.1 Doppler Spectrum

1. Consider a narrow-band fading wireless channel. The base station and the mobile station communicate at the carrier frequency $f_0 = 3$ GHz. Determine the Doppler spectral density for the following case:

The multipath fading is Ricean with Rice factor $K = -6$ dB, and the receive power is $1\ \mu$W. The scatterers are uniformly distributed around the mobile station. The maximal velocity of the mobile station is $v = 36$ km/h and the line-of-sight toward the base-station is at $\alpha = 45°$.

♠ **Solution:**

In this case, there are two beams of wave being received. One through the line-of-sight, and the other one through the scatterers being uniformly distributed around the mobile station. Denoting the power received through the line-of-sight by P_{LOS} and the power received from the scatterers with P_{SC}, we have

$$K = \frac{P_{\text{LOS}}}{P_{\text{SC}}} = \frac{1}{4}$$

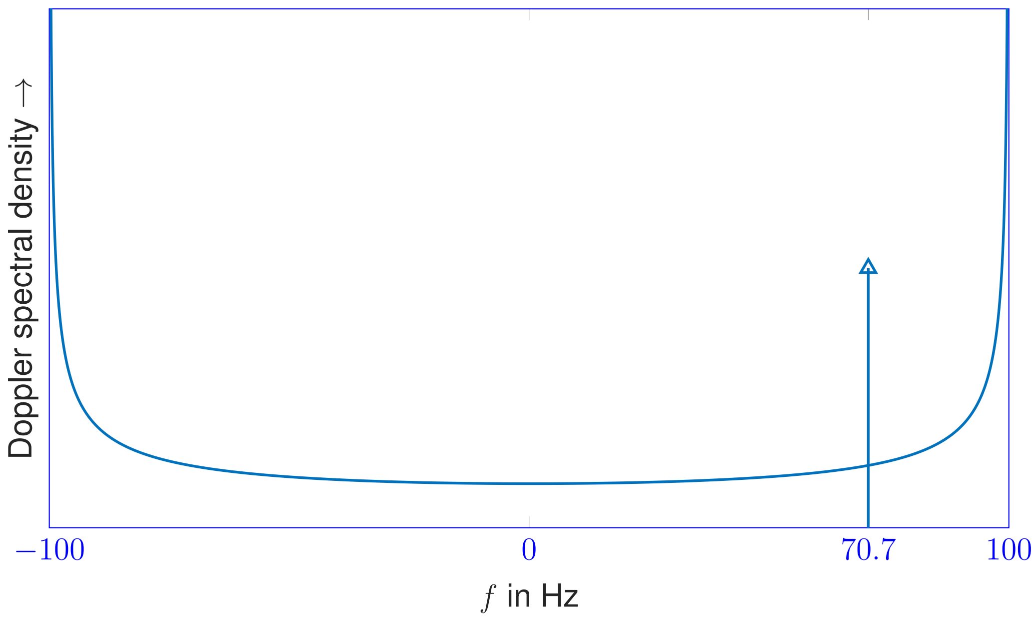


Figure 5.3.1: Doppler spectral density of Case (a).

Therefore, the receive power P_R reads

$$P_\mathrm{R} = P_\mathrm{LOS} + P_\mathrm{SC} = 5P_\mathrm{LOS}$$

As $P_\mathrm{R} = 1$ μW, we conclude that $P_\mathrm{LOS} = 0.2$ μW and $P_\mathrm{SC} = 0.8$ μW. For Doppler shifts of each beam we have

- As the line-of-sight is observed at $\alpha = 45°$, the Doppler shift caused by the line-of-sight transmission is

$$f_\mathrm{d,LOS} = \frac{v \cos\alpha}{c} f_0 = 70.7 \text{ Hz}.$$

- For the scatterers, the maximum Doppler shift in the signal received is

$$f_\mathrm{d,SC} = \frac{v}{c} f_0 = 100 \text{ Hz}.$$

The Doppler spectral density is the superposition of

- a Dirac impulse function at $f = 70.7$ Hz whose power is $P_\mathrm{LOS} = 0.2$ μW, and
- a standard Doppler shift in page 99 within $f \in [-100, 100]$ Hz whose power is $\sigma_\mathrm{R}^2 = P_\mathrm{SC} = 0.8$ μW.

The spectral density ha been shown in Figure 4.1.

5.3.2 Narrow-band Small-scale fading

1. Consider a *narrow-band* multipath fading channel between a transmitter and a receiver. The system operates at the carrier frequency $f_{\mathrm{C}} = 900 \text{ MHz}$. The transmitter sees the receiver through a line-of-sight path with Rice factor $\log K = 4 \text{ dB}$.
   The receiver moves with the maximum speed of $v_{\max} = 120$ km/h in a *rich* scattering environment, and the superposed power received through the scattered components is $\sigma_R^2 = 5$ mW.

   Assume that the transmitter sends the monotone signal

   $$s(t) = \sqrt{P} \exp\{-2\pi \mathrm{j} f_{\mathrm{C}} t\}$$

   where $P = 20$ W. The received signal is denoted by

   $$Y(t) = R(t) + N(t)$$

   where $N(t)$ is additive white Gaussian noise with zero mean and variance $\sigma_N^2 = 0.1$ mW, and $R(t)$ is the superposition of the signals received from multiple paths.

   (a) Calculate the probability density function of the time sample

   $$R_0 = R(t_0)$$

   for some arbitrary t_0. Specify the mean and the variance of this random variable.

   (b) We now sample $R(t)$ at some different time $t_1 \neq t_0$ and obtain the random variable

   $$R_1 = R(t_1).$$

   Do R_0 and R_1 have identical distributions?

   (c) Determine the average receive power at t_0 which is defined as

   $$P_{\mathrm{R}} = \mathcal{E}\left\{|Y(t_0)|^2\right\}.$$

   Here, $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation.

   (d) Assume that you are monitoring the random process $R(t)$ in the frequency domain by a spectrum meter which determines the average density of $|R(t)|^2$ at each frequency. You observe the frequency interval

   $$\Delta f = [f_{\mathrm{C}} - 200 \text{ Hz}, f_{\mathrm{C}} + 200 \text{ Hz}]$$

   by this spectrum meter. Plot the output of this spectrum meter in the interval Δf. Describe the distribution of the scatterers around the receiver that you assumed to plot the output.

♠ **Solution:**

(a) $R(t)$ is the received signal without considering the noise effect. Since the transmission is narrow-band, we can model $R(t)$ as

$$R(t) = H\ s(t)$$

where H is a random variable modelling the effect of multipath fading. Noting that the channel has a line-of-sight, we model the fading process by a Rice distribution. This means that H is a Gaussian random variable with mean μ_R and variance σ_R^2. As a result,

$$R_0 \sim \mathcal{CN}(M_R, V_R)$$

where

$$\begin{aligned} M_R &= \sqrt{P} \exp\{-2\pi \mathrm{j} f_{\mathrm{C}} t_0\} \mu_R \\ V_R &= P\sigma_R^2. \end{aligned}$$

From equation (3.39) in page 96 of the lecture notes, we know that

$$K = \frac{|\mu_R|^2}{\sigma_R^2}.$$

Since, $\log K = 4$ dB, we have $K = 2.5$ in linear scale; thus,

$$|\mu_R| = \sqrt{K\sigma_R^2} = \sqrt{2.5 \times 10^{-3}} = 0.112.$$

Assuming the initial phase to be $\varphi_0 = 0$, μ_R reads $\mu_R = |\mu_R| = 0.112$. As a result, mean of R_0 reads

$$M_R = \mathcal{E}[R_0] = 0.5 \exp\{-2\pi \mathrm{j} f_{\mathrm{C}} t_0\}.$$

The variance moreover is

$$V_R = \mathcal{E}\left[|R_0 - \mathcal{E}[R_0]|^2\right] = P\sigma_R^2 = 20 \times 5 \times 10^{-3} = 0.1.$$

This concludes that $\boxed{R_0 \sim \mathcal{CN}(0.5 \exp\{-2\pi \mathrm{j} f_{\mathrm{C}} t_0\}, 0.1)}$.

(b) Following the discussions in pages 93 – 96, the fading process in *narrow-band* transmission is modeled as a *white stationary process*. This means that each time sample of $R(t)$ is assumed to have a same distribution and is considered independent of other time samples. Hence, R_1 and R_0 have same distributions[1].

> ⇝ REMARK:
>
> In narrow band transmission, the fading process is simply a white Gaussian process.

[1] The only thing that they differ in is the phase of the mean which does not play any role in derivations.

(c) To calculate P_{R}, we write

$$\begin{aligned} P_{\mathrm{R}} = \mathcal{E}\left[|Y(t_0)|^2\right] &= \mathcal{E}\left[|R(t_0) + N(t_0)|^2\right] \\ &= \mathcal{E}\left[|R(t_0)|^2\right] + \mathcal{E}\left[|N(t_0)|^2\right] + 2\text{Real}\left\{\mathcal{E}\left[R(t_0)^* N(t_0)\right]\right\} \\ &= \mathcal{E}\left[|R(t_0)|^2\right] + \mathcal{E}\left[|N(t_0)|^2\right] + 2\text{Real}\left\{s(t_0)^* \mathcal{E}\left[H^* N(t_0)\right]\right\}. \end{aligned}$$

Noting that H and $N(t)$ are two independent processes, we have

$$\mathcal{E}\left[H^* N(t_0)\right] = \mathcal{E}\left[H^*\right] \underbrace{\mathcal{E}\left[N(t_0)\right]}_{0} = 0.$$

Therefore,

$$\begin{aligned} P_{\mathrm{R}} &= \mathcal{E}\left[|R(t_0)|^2\right] + \mathcal{E}\left[|N(t_0)|^2\right] \\ &= \left(V_R + |M_R|^2\right) + \mathcal{E}\left[\sigma_N^2\right] = (0.1 + 0.25) + 0.0001 = 0.3501 \text{ W} \end{aligned}$$

(d) The maximum Doppler shift in this system reads

$$\nu_{\max} = \frac{v_{\max}}{c} f_{\mathrm{C}}$$

where $v_{\max} = 10$ km/h = 33.33 m/sec and $c = 3 \times 10^8$ m/sec is the speed of light. Hence, we have

$$\nu_{\max} = \frac{33.33}{3 \times 10^8} \times 9 \times 10^8 = 100 \text{ Hz}.$$

Following the discussions in pages 98 – 101, the classical Doppler spectrum is given via the Jakes' spectrum whose *base-band* power density is given by equation (3.47) in page 99. Substituting the values of σ_R^2 and $\nu_{\max}$ into the equation, we have

$$\Psi_R(\nu) = \begin{cases} \dfrac{\sigma_R^2}{\pi\sqrt{\nu_{\max}^2 - \nu^2}} & |\nu| < \nu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

> ⇝ REMINDER:
>
> Remember that equation (3.47) in page 99 is the *base-band* spectrum. Hence, for the RF signal this spectrum should be shifted by the carrier frequency.

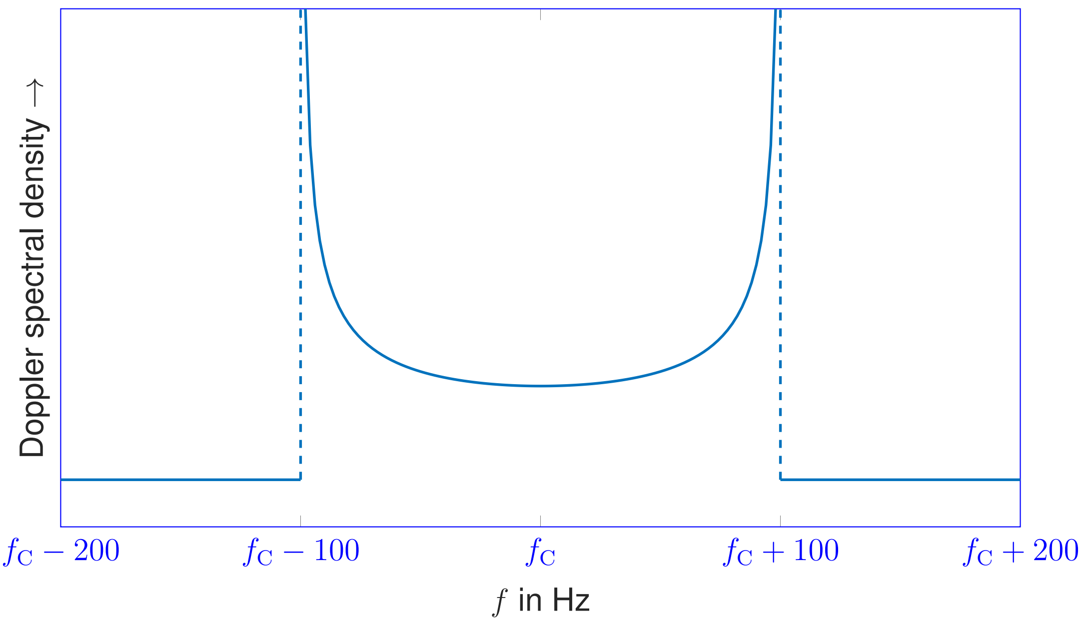
By shifting the spectrum with f_{C}, we have the RF Doppler spectrum as

$$\Psi_R(f) = \begin{cases} \dfrac{5 \times 10^{-3}}{\pi\sqrt{10^4 - |f - f_{\mathrm{C}}|^2}} & |f - f_{\mathrm{C}}| < 100 \text{ Hz} \\ 0 & \text{otherwise} \end{cases}$$

which in the interval

$$\Delta f = [f_{\mathrm{C}} - 200 \text{ Hz}, f_{\mathrm{C}} + 200 \text{ Hz}]$$

look like the following figure.



The above spectrum is given by considering the classical Doppler profile which means that *the scatterers are assumed to be uniformly distributed around the receiver.*

⇝ REMARK:

It is explicitly indicated that the classical Doppler spectrum is derived for uniform distribution of scatterers; see equation (3.46) in page 99 of the lecture notes. When the scatterers are distributed with some other distribution, one needs to replace $p(\alpha)$ with the new distribution in equation (3.45) and determine the corresponding spectrum. In the case that you have some confusion in this respect, please read pages 98 and 99 thoroughly.

5.3.3 Wide-band Transmission over Fading Channels

1. Consider a fading channel with *high* coherence time. The power-delay profile of this channel is given as follows:

$$\rho_h(\tau, 0) = \sum_{n=0}^{3} \frac{1}{2^n} \delta(\tau - n\tau_0)$$

where τ_0 is a constant.

(a) For this channel, calculate the following parameters:

i. the power transform factor

ii. the delay time

iii. the delay spread
iv. the excess delay

in terms of τ_0.

(b) Assume that GSM symbols are being transmitted over this wireless channel with pulses which occupy the bandwidth $B = 1/T$ with T being the duration of a symbol transmission. For the following values of τ_0, explain whether the signal experiences *narrow-band* or *wide-band* fading.

i. $\tau_0 = 0.1\ \mu$sec, and
ii. $\tau_0 = 5\ \mu$sec

♠ **Solution:**

(a) Using the formulas in pages 120 and 121, we have

i. The power transform factor is calculated from equation (3.86)

$$h_P = \frac{15}{8}.$$

ii. The mean delay follows equation (3.87)

$$\mu_\tau = \frac{11}{15}\tau_0.$$

iii. The delay response is given by equation (3.88) as

$$\sigma_\tau^2 = 0.93\tau_0,$$

iv. The excess delay is determined by equation (3.89)

$$\tau_{\text{ex}} = 3\tau_0.$$

(b) In the GSM system, the symbol rate is $R = 270$ ksymb/sec. Hence, the symbol duration reads

$$T = \frac{1}{270 \times 10^3} \approx 3.7\ \mu\text{sec}.$$

This means that the GSM signal occupies a bandwidth of

$$B = \frac{1}{T} = R = 270\ \text{kHz}$$

To decide whether the signal experiences narrow-band or wide-band fading, we require to calculate the *coherence bandwidth* of the channel. The coherence bandwidth of the channel is calculated from the *time-frequency* correlation function $\rho_H(\Delta f, \Delta t)$ at $\Delta t = 0$ via the definition given in page 117.
From Figure 3.39 in page 116, we know that

$$\rho_H(\Delta f, \Delta t) = \mathcal{F}_\tau\{\rho_h(\tau, \Delta t)\}$$

where $\mathcal{F}_\tau\left\{\cdot\right\}$ denotes Fourier transform with respect to τ. As a result, we have

$$\rho_H\left(\Delta f, 0\right) = \mathcal{F}_\tau\left\{\rho_h\left(\tau, 0\right)\right\}.$$

Noting that

$$\mathcal{F}_\tau\left\{\delta\left(\tau - n\tau_0\right)\right\} = \exp\left\{-\mathrm{j}2n\pi\Delta f\tau_0\right\},$$

we can write

$$\rho_H\left(\Delta f, 0\right) = \sum_{n=0}^{3} \frac{1}{2^n} \exp\left\{-\mathrm{j}2n\pi\Delta f\tau_0\right\}.$$

Using the identity

$$\cos\left(\alpha - \beta\right) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we finally have

$$\left|\rho_H\left(\Delta f, 0\right)\right|^2 = w_0 + 2\sum_{n=1}^{3} w_n \cos\left(2n\pi\Delta f\tau_0\right)$$

where[2]

$$w_n = \sum_{m=0}^{3-n} \frac{1}{2^{2m+n}}.$$

Substituting into the definition given in page 117, the coherence bandwidth B_c is a solution to the equation

$$\left|\rho_H\left(B_\mathrm{c}, 0\right)\right| = \frac{1}{2}\left|\rho_H\left(0, 0\right)\right| \Rightarrow \left|\rho_H\left(B_\mathrm{c}, 0\right)\right|^2 = \frac{1}{4}\left|\rho_H\left(0, 0\right)\right|^2$$

which for the derived function reads

$$w_0 + 2\sum_{n=1}^{3} w_n \cos\left(2n\pi B_\mathrm{c}\tau_0\right) = \frac{1}{4}\left(w_0 + 2\sum_{n=1}^{3} w_n\right).$$

The solution to this equation is not simple to be calculated analytically. We hence use the dominant term approximation and write

$$\rho_h\left(\tau, 0\right) \approx 1 + \delta\left(\tau - 3\tau_0\right).$$

Doing so, we can approximately write that

$$\left|\rho_H\left(\Delta f, 0\right)\right|^2 \approx 2 + 2\cos\left(6\pi\Delta f\tau_0\right)$$

Thus, B_c is approximately calculated when

$$\cos\left(6\pi B_\mathrm{c}\tau_0\right) \approx -\frac{1}{2} \Rightarrow B_\mathrm{c} \approx \frac{1}{9\tau_0} = \frac{1}{3\tau_\mathrm{ex}}.$$

[2] Don't worry if took time to calculate it. It is rather time-consuming!

⇝ REMARK:

A good approximation for the coherence bandwidth is

$$B_c \approx \frac{1}{2\tau_{ex}}.$$

Based on the formulation in page 122, the signal experiences *narrow-band* fading when $B < B_c$, and *wide-band* fading when $B > B_c$. Considering the particular given cases, we have

i. When $\tau_0 = 0.1$ μsec, $B_c \approx 1.11$ MHz which is more than the signal bandwidth $B = 270$ kHz. Hence, the signal experiences *narrow-band* fading in this case.

ii. When $\tau_0 = 5$ μsec, $B_c \approx 22.22$ kHz which is less than the signal bandwidth $B = 270$ kHz. Hence, the signal experiences *wide-band* fading in this case.

5.4 Homework

1. Consider a narrow-band fading wireless channel. The base station and the mobile station communicate at the carrier frequency $f_0 = 3$ GHz. Determine the Doppler spectral density for the following case:

   The multipath fading is Rayleigh. The scatterers are concentrated on the interval $[-\pi + \theta, -\theta] \cup [\theta, \pi - \theta]$ around the mobile station.

♠ **Solution:** In this case, as the scatterers have been concentrated within the interval $[-\pi + \theta, -\theta] \cup [\theta, \pi - \theta]$, the maximum Doppler shift is

$$f_{d,max} = \frac{v \cos\theta}{c} f_0 = f_d \cos\theta.$$

We can conclude that the Doppler spectral density in this case would be

- a normalized version of the standard spectral density in the interval $[-f_{d,max}, f_{d,max}]$, and
- zero in $[-f_d, -f_{d,max}]$ and $[f_{d,max}, f_d]$, since there is no signal received at the corresponding angles.

Figure 4.2 shows the spectral density.

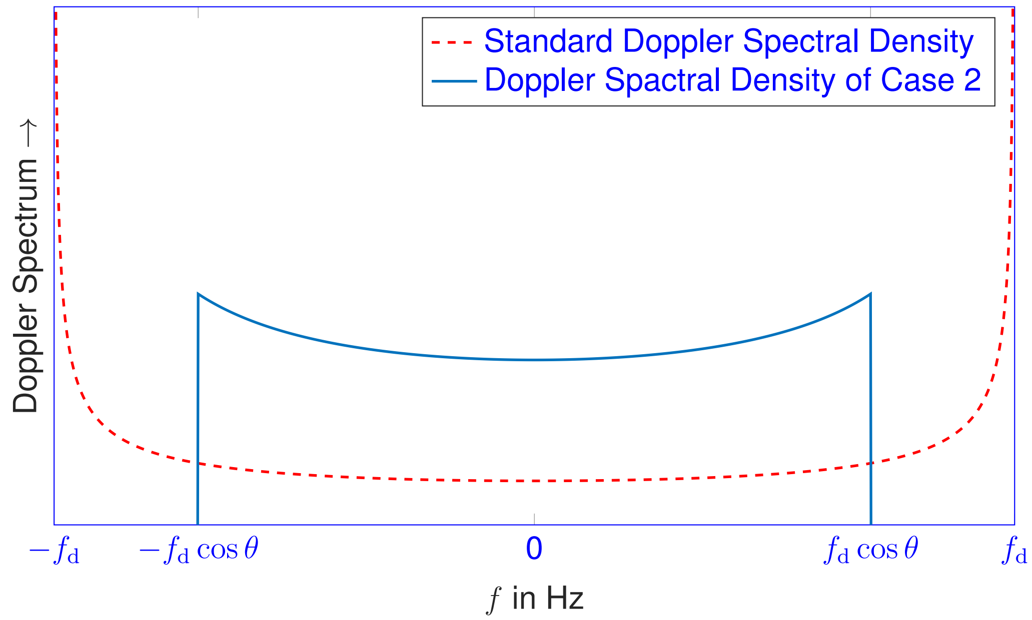


Figure 5.4.1: Doppler spectral density.

2. Consider the following two communications scenarios:

 SCENARIO A: A base station is located at the center of a cell in a countryside area. The area of the cell is $A_\mathrm{cell} = 50$ km^2. It is known that within distance of $D_0 = 5$ km from the base station, no large obstacle, such as tall buildings, exists. A mobile station in this cell, is $d = 2$ km far away from the base station, and receives signals from the base station.

 SCENARIO B: A base station is located on the roof of the Shanghai Grand Center with $h = 170$ m height in the middle of Shanghai. A mobile station is located on a bicycle which is being ridden $d = 1$ km far away from the base station in a narrow street. The base station transmits signals to the mobile station.

 Assume that the transmit signals are *narrow-band*. Name the distribution which models the *multipath* effect in each scenario.

♠ **Solution:** Since the transmission is *narrow-band*, we could model the fading process via a coefficient which is a random variable.

- SCENARIO A: In this case, there exist a direct path from the transmitter to the receiver, hence, we have line-of-sight in the channel. As the result, the multipath effect is modeled via the *Rice distribution*.
- SCENARIO A: In this case, there exist no direct path from the transmitter to the receiver, and whole the reception comes from the scattered paths. We hence have no line-of-sight in the channel. As the result, the multipath effect is modeled via the *Rayleigh distribution*.

Chapter 6

Diversity Principles

This chapter goes through the concept of diversity which is discussed in Chapter 4 of the lecture-notes. Similar to other chapters, we first quickly introduce the main concepts.

6.1 Brief Review of Main Concepts

Diversity is the key technique to combat fading in a wireless channel. It indicates that we send multiple copies of the same signal over *independent* realizations of the fading process. In this case, if one copy of signal is lost due to the deep fading of the channel, anther copy is most probably received properly.

The independent transmissions can be sent over various domains:

- time,
- frequency and
- space

are the most known examples. The received copies can also be combined with different techniques. The optimal technique is maximum ratio combining (MRC). Nevertheless, there are some other sub-optimal approaches which due to lack of side information at the receiver or lower complexity might be used.

In the sequel, we review these concepts and techniques through some exercises.

6.2 Exercises

6.2.1 Basics of Diversity

1. Consider a wireless channel which experiences *narrow-band* and slow fading. This means that the receive signal $y(t)$ is given in terms of the transmit signal $x(t)$ as

$$y(t) = H\,x(t) + n(t)$$

where $n(t)$ is additive white Gaussian noise with zero mean and variance σ^2, and H is the channel coefficient which models the fading process. We assume that the fading process follows the co-called *on-off* model:

> H is an α-Bernoulli random variable, i.e., $H \in \{0, 1\}$, and
> $$\Pr\{H = 0\} = 1 - \Pr\{H = 1\} = \alpha.$$

This is a toy-model which is commonly used in the literature.

Assume that $|x(t)|^2 = P$, and define the *base* signal-to-noise ratio (SNR) as follows:

$$\Gamma_0 = \frac{P}{\sigma^2}.$$

The receive SNR in this setup is calculated as

$$\mathrm{SNR} = \frac{|H|^2 P}{\sigma^2} = |H|^2 \Gamma_0.$$

(a) Calculate the average receive SNR defined as $\overline{\mathrm{SNR}} = \mathcal{E}\{\mathrm{SNR}\}$, where $\mathcal{E}\{\cdot\}$ denotes expectation.

(b) Determine the outage probability which is defined as

$$P_{\mathrm{Out}}(\Gamma) = \Pr\{\mathrm{SNR} \leq \Gamma\}$$

for some $\Gamma < \Gamma_0$.

Now, assume that we transmit M copies of $x(t)$ over M *independent* realizations of this fading channel. To keep the total transmit power after M transmissions P, we scale each copy with $1/\sqrt{M}$. Hence, in m-th transmission, we receive

$$y_m(t) = H_m \frac{x(t)}{\sqrt{M}} + n_m(t)$$

where $H_1, \dots, H_m$ are independent α-Bernoulli variables and $n_1(t), \dots, n_m(t)$ are independent white Gaussian noises with zero mean and variance σ^2. At the receiver, we superpose the receive signals by summing them.

(c) Determine the outage probability $P_{\mathrm{Out}}(\Gamma)$ for some $\Gamma < \Gamma_0$ considering this superposed receive signal.

(d) Compare the slope of $\log P_{\mathrm{Out}}(\Gamma)$ in terms of $\log \overline{\mathrm{SNR}}$ assuming that Γ_0 is very large. Compare the result to that of Part (b).

2. The distribution of the receive SNR after maximum ratio combining of M branches of diversity is given by

$$f(\Gamma) = \begin{cases} \dfrac{\Gamma^{M-1}}{\mu^M (M-1)!} \exp\left\{-\frac{\Gamma}{\mu}\right\} & \Gamma \geq 0 \\ 0 & \Gamma < 0 \end{cases}$$

where μ is the average receive SNR of each branch. When Differential-BPSK (DBPSK) modulation is used for signal transmission, the bit error rate is determined in terms of the receive SNR as

$$E_b(\Gamma) = \frac{1}{2}\exp\{-\Gamma\}.$$

For DBPSK transmission, calculate the *diversity gain* achieved by maximum ratio combining.

6.2.2 Diversity with Cost

1. In a wireless channel with transmit signal experiences *narrow-band* fading. To combat the fading, we employ M branches of diversity and use selection combining at the receiver. Let $P_{\mathrm{Out}}(\Gamma)$ denote the receive outage probability as defined in the first exercise of this tutorial.

   For a fixed Γ, compare the reduction in the $P_{\mathrm{Out}}(\Gamma)$ in terms of M in the following two scenarios:

   (a) When in each branch of diversity, the transmit signal power is fixed to P.

   (b) When in each branch of diversity, we transmit with same power, such that the total transmit power over all transmissions is P.

   What is your suggestion for choosing the value of M, in practice?

6.2.3 Further Discussions on Diversity

1. A *wide-band* transmission over a fading channel has the following power-delay profile:

$$\rho_h(\tau, 0) = \delta(\tau) + \frac{1+j}{\sqrt{2}}\delta(\tau - \tau_0)$$

   for some delay τ_0. The receiver knows this power-delay profile perfectly. Assume that transmission of a single symbol in this system takes T sec.

   (a) Consider a case in which $T \ll \tau_0$. Design a receiver which gives you a diversity gain of order 2 without any modification at the transmitter.

   (b) What is the diversity gain that you achieve with the receiver in Part (a), when $T > \tau_0$?

   (c) Discuss the expected diversity gain achieved by this receiver in the GSM system when the transmitter and the receiver are communicating in

      i. Rural area.

      ii. Typical urban channels.

6.3 Solutions to Exercises

6.3.1 Basics of Diversity

1. Consider a wireless channel which experiences *narrow-band* and slow fading. This means that the receive signal $y(t)$ is given in terms of the transmit signal $x(t)$ as

$$y(t) = H\, x(t) + n(t)$$

where $n(t)$ is additive white Gaussian noise with zero mean and variance σ^2, and H is the channel coefficient which models the fading process. We assume that the fading process follows the co-called *on-off* model:

H is an α-Bernoulli random variable, i.e., $H \in \{0,1\}$, and $$\Pr\{H=0\} = 1 - \Pr\{H=1\} = \alpha.$$

This is a toy-model which is commonly used in the literature.

Assume that $|x(t)|^2 = P$, and define the *bare* signal-to-noise ratio (SNR) as follows:

$$\Gamma_0 = \frac{P}{\sigma^2}.$$

The receive SNR in this setup is calculated as

$$\mathrm{SNR} = \frac{|H|^2 P}{\sigma^2} = |H|^2 \Gamma_0.$$

(a) Calculate the average receive SNR defined as $\overline{\mathrm{SNR}} = \mathcal{E}\{\mathrm{SNR}\}$, where $\mathcal{E}\{\cdot\}$ denotes expectation.

(b) Determine the outage probability which is defined as

$$P_{\mathrm{Out}}(\Gamma) = \Pr\{\mathrm{SNR} \leq \Gamma\}$$

for some $\Gamma < \Gamma_0$.

Now, assume that we transmit M copies of $x(t)$ over M *independent* realizations of this fading channel. To keep the total transmit power after M transmissions P, we scale each copy with $1/\sqrt{M}$. Hence, in m-th transmission, we receive

$$y_m(t) = H_m \frac{x(t)}{\sqrt{M}} + n_m(t)$$

where $H_1, \dots, H_m$ are independent α-Bernoulli variables and $n_1(t), \dots, n_m(t)$ are independent white Gaussian noises with zero mean and variance σ^2. At the receiver, we superpose the receive signals by summing them.

(c) Determine the outage probability $P_{\mathrm{Out}}(\Gamma)$ for some $\Gamma < \Gamma_0$ considering this superposed receive signal.

(d) Compare the slope of $\log P_{\text{Out}}(\Gamma)$ in terms of $\log \overline{\text{SNR}}$ assuming that Γ_0 is very large. Compare the result to that of Part (b).

♠ **Solution:**

(a) The average signal-to-noise ratio reads

$$\overline{\text{SNR}} = \mathcal{E}\{\text{SNR}\} = \Gamma_0 \mathcal{E}\left[|H|^2\right] = (1-\alpha)\,\Gamma_0.$$

Note that $\overline{\text{SNR}} \leq \Gamma_0$.

(b) From the definition, the outage probability reads

$$P_{\text{Out}}(\Gamma) = \Pr\{\text{SNR} \leq \Gamma\} = \Pr\left\{|H|^2 \leq \frac{\Gamma}{\Gamma_0}\right\}.$$

Since $\Gamma < \Gamma_0$, we have

$$\Pr\left\{|H|^2 \leq \frac{\Gamma}{\Gamma_0}\right\} = \Pr\left\{|H|^2 < 1\right\} = \Pr\{H = 0\} = \alpha$$

Thus, $P_{\text{Out}}(\Gamma) = \alpha$. Using the result in Part (a), we can write

$$\boxed{P_{\text{Out}}(\Gamma) = 1 - \frac{\overline{\text{SNR}}}{\Gamma_0}}$$

The extreme cases for the outage probability are as follows:

- When $\overline{\text{SNR}} \downarrow 0$, we have $P_{\text{Out}}(\Gamma) \to 1$ meaning that the receiver is always in outage.
- When $\overline{\text{SNR}} \uparrow \Gamma_0$, we have $P_{\text{Out}}(\Gamma) \to 0$ meaning that the receiver does not get into outage.

(c) The superposed signal reads

$$\begin{aligned} y_{\text{Sup}}(t) = \sum_{m=1}^{M} y_m(t) &= \sum_{m=1}^{M} H_m \frac{x(t)}{\sqrt{M}} + n_m(t) \\ &= H_{\text{Sup}} x(t) + n_{\text{Sup}}(t) \end{aligned}$$

where

- H_{Sup} is defined as

$$H_{\text{Sup}} := \sum_{m=1}^{M} \frac{H_m}{\sqrt{M}}$$

- $n_{\text{Sup}}(t)$ is superposed noise which is white Gaussian noise with zero mean and variance $\sigma^2_{\text{Sup}} = M\sigma^2$.

The signal-to-noise ratio of the superposed signal reads

$$\mathrm{SNR} = \frac{|H_{\mathrm{Sup}}|^2 P}{\sigma_{\mathrm{Sup}}^2} = \frac{|H_{\mathrm{Sup}}|^2 P}{M\sigma^2} \Rightarrow \boxed{\mathrm{SNR} = \frac{|H_{\mathrm{Sup}}|^2}{M}\Gamma_0}.$$

As a result, the outage probability reads

$$\begin{aligned} P_{\mathrm{Out}}(\Gamma) = \Pr\{\mathrm{SNR} \leq \Gamma\} &= \Pr\left\{\frac{|H_{\mathrm{Sup}}|^2}{M} \leq \frac{\Gamma}{\Gamma_0}\right\} \\ &\stackrel{\dagger}{=} \Pr\left\{\frac{1}{\sqrt{M}} H_{\mathrm{Sup}} \leq \sqrt{\frac{\Gamma}{\Gamma_0}}\right\} \end{aligned}$$

where $\dagger$ follows the fact that H_{Sup} is a non-negative real number. We further note that

$$\frac{1}{M} H_{\mathrm{Sup}} = \frac{1}{M}\sum_{m=1}^{M} H_m$$

which is a binomial random variable. Since $0 < \Gamma < \Gamma_0$, there exists always an integer $K \in \{0, \dots, M\}$, such that

$$\frac{K}{M} < \sqrt{\frac{\Gamma}{\Gamma_0}} \leq \frac{K+1}{M}.$$

In this case, the event

$$\frac{1}{M}\sum_{m=1}^{M} H_m < \sqrt{\frac{\Gamma}{\Gamma_0}}$$

holds if

$$\sum_{m=1}^{M} H_m \leq K,$$

or equivalently, at most k channel coefficients are non-zero. Therefore, the outage probability reads

$$\boxed{P_{\mathrm{Out}}(\Gamma) = \sum_{k=0}^{K} \binom{M}{k} \alpha^{M-k} (1-\alpha)^k}$$

(d) When $\Gamma_0 \uparrow \infty$, we have $\Gamma/\Gamma_0 \downarrow 0$. Therefore,

$$0 < \sqrt{\frac{\Gamma}{\Gamma_0}} \leq \frac{1}{M}$$

which means that $K = 0$. As a result,

$$P_{\mathrm{Out}}(\Gamma) = \alpha^M.$$

In terms of the average SNR at each branch, we can write

$$\boxed{P_{\text{Out}}\left(\Gamma\right) = \left(1 - \frac{\overline{\text{SNR}}}{\Gamma_0}\right)^M}$$

In the logarithmic scale, we have

$$\log P_{\text{Out}}\left(\Gamma\right) = M \log\left(1 - \frac{\overline{\text{SNR}}}{\Gamma_0}\right)$$

which decays with slope M. Comparing to Part (b), the slope increased by factor M which represents the diversity order.

2. The distribution of the receive SNR after maximum ratio combining of M branches of diversity is given by

$$f\left(-\Gamma\right) = \begin{cases} \dfrac{\Gamma^{M-1}}{\mu^M \left(M-1\right)!} \exp\left\{-\frac{\Gamma}{\mu}\right\} & \Gamma \geq 0 \\ 0 & \Gamma < 0 \end{cases}$$

where μ is the average receive SNR of each branch. When Differential-BPSK (DBPSK) modulation is used for signal transmission, the bit error rate is determined in terms of the receive SNR as

$$E_b\left(\Gamma\right) = \frac{1}{2} \exp\left\{-\Gamma\right\}.$$

For DBPSK transmission, calculate the *diversity gain* achieved by maximum ratio combining.

♠ **Solution:** To calculate the diversity gain, we need first to determine the *average error probability* which in this case is determined as

$$\begin{aligned} \bar{E}_b &= \int_{-\infty}^{+\infty} E_b\left(-\Gamma\right) f\left(-\Gamma\right) \mathrm{d}\Gamma \\ &= \int_0^{+\infty} \frac{1}{2} \exp\left\{-\Gamma\right\} \frac{\Gamma^{M-1}}{\mu^M \left(M-1\right)!} \exp\left\{-\frac{\Gamma}{\mu}\right\} \mathrm{d}\Gamma \\ &= \frac{1}{2\mu^M} \int_0^{+\infty} \frac{\Gamma^{M-1}}{\left(M-1\right)!} \exp\left\{-\Gamma\left(1+\frac{1}{\mu}\right)\right\} \mathrm{d}\Gamma \end{aligned}$$

We now multiply and divided the integral with the same constant term $\left(1+\frac{1}{\mu}\right)^M$. This results in

$$\begin{aligned}
\bar{E}_b &= \frac{1}{2\mu^M} \int_0^{+\infty} \frac{\left(1+\frac{1}{\mu}\right)^M}{\left(1+\frac{1}{\mu}\right)^M} \frac{\Gamma^{M-1}}{(M-1)!} \exp\left\{-\Gamma\left(1+\frac{1}{\mu}\right)\right\} \mathrm{d}\Gamma \\
&= \frac{1}{2\mu^M \left(1+\frac{1}{\mu}\right)^M} \int_0^{+\infty} \left(1+\frac{1}{\mu}\right)^M \frac{\Gamma^{M-1}}{(M-1)!} \exp\left\{-\Gamma\left(1+\frac{1}{\mu}\right)\right\} \mathrm{d}\Gamma \\
&= \frac{1}{2\left(1+\mu\right)^M} \int_0^{+\infty} \left(1+\frac{1}{\mu}\right)^M \frac{\Gamma^{M-1}}{(M-1)!} \exp\left\{-\Gamma\left(1+\frac{1}{\mu}\right)\right\} \mathrm{d}\Gamma
\end{aligned}$$

The *black term* under the integral is the chi-squared probability density function[1]. Hence, this integral equals to one, i.e.,

$$\int_0^{+\infty} \left(1+\frac{1}{\mu}\right)^M \frac{\Gamma^{M-1}}{(M-1)!} \exp\left\{-\Gamma\left(1+\frac{1}{\mu}\right)\right\} \mathrm{d}\Gamma = 1,$$

and we have

$$\bar{E}_b = \frac{1}{2\left(1+\mu\right)^M}.$$

The *diversity gain* is defined as the slope of the average error rate in logarithmic scale when the average SNR μ is large. To calculate the diversity gain, we write

$$\begin{aligned}
\log \bar{E}_b &= \log \frac{1}{2} - M \log\left(1+\mu\right) \\
&= -0.3 - M \log\left(1+\mu\right)
\end{aligned}$$

when μ is very large, one can write $\log\left(1+\mu\right) \approx \log \mu$, and thus

$$\log \bar{E}_b \approx -0.3 - M \log \mu$$

The slope of the $\log$-$\log$ curve in this case is M; hence, $\boxed{\text{diversity gain} = M}$. This result was obvious from the beginning, since we have M diversity branches.

6.3.2 Diversity with Cost

1. In a wireless channel with transmit signal experiences *narrow-band* fading. To combat the fading, we employ M branches of diversity and use selection combining at the receiver. Let $P_{\mathrm{Out}}\left(\Gamma\right)$ denote the receive outage probability as defined in the first exercise of this tutorial.

   For a fixed Γ, compare the reduction in the $P_{\mathrm{Out}}\left(\Gamma\right)$ in terms of M in the following two scenarios:

[1] In fact, it is the distribution of $\Gamma = \left[2\left(1+\frac{1}{\mu}\right)\right]^{-1} X$ where X is a chi-squared random variable with $2M$ degrees of freedom.

(a) When in each branch of diversity, the transmit signal power is fixed to P.

(b) When in each branch of diversity, we transmit with same power, such that the total transmit power over all transmissions is P.

What is your suggestion for choosing the value of M, in practice?

♠ **Solution:**

(a) From equation (4.20) in page 134 of the lecture notes, the outage probability for selection combining over M diversity branches with fixed transmit power in each transmission reads

$$P_{\text{Out}}(\Gamma) = \left(1 - \exp\left\{-\frac{\Gamma}{\mu}\right\}\right)^M$$

where μ is the average SNR received over each branch. In the logarithmic scale, the outage probability reads

$$\log P_{\text{Out}}(\Gamma) = M \log\left(1 - \exp\left\{-\frac{\Gamma}{\mu}\right\}\right).$$

Noting that $\Gamma/\mu \geq 0$, we have

$$0 < 1 - \exp\left\{-\frac{\Gamma}{\mu}\right\} \leq 1.$$

This indicates that

$$\log\left(1 - \exp\left\{-\frac{\Gamma}{\mu}\right\}\right) < 0.$$

This concludes that $P_{\text{Out}}(\Gamma)$ for a fixed Γ *linearly* decreases as M increases. In other words, the performance *in this case* always gets improved when the number of diversity branches is increased.

(b) In this case, the total transmit power is constrained to be constant. This means that the transmit power in each diversity branch is scaled by factor $1/M$, or equivalently, the power in each transmission is P/M. Consequently, the average SNR in this case is

$$\hat{\mu} = \frac{\mu}{M}$$

where μ is the average SNR in Part (a). As a result, the outage probability reads

$$\begin{aligned}\log P_{\text{Out}}(\Gamma) &= \left(1 - \exp\left\{-\frac{\Gamma}{\hat{\mu}}\right\}\right)^M \\ &= \left(1 - \exp\left\{-M\frac{\Gamma}{\mu}\right\}\right)^M\end{aligned}$$

In the logarithmic scale, we have

$$\log P_{\text{Out}}(\Gamma) = M \log \left(1 - \exp\left\{-M\frac{\Gamma}{\mu}\right\}\right)$$

In contrast to Part (a), $\log P_{\text{Out}}(\Gamma)$ does not decrease linearly in this case. In fact, it first decreases and then increases as M grows large. This means that for a fixed Γ, the outage probability does not necessarily decrease as M increases.

In Figure 6.3.1, $\log P_{\text{Out}}(\Gamma)$ is plotted against M for two cases $\Gamma = 0.1\mu$ and $\Gamma = 0.05\mu$.

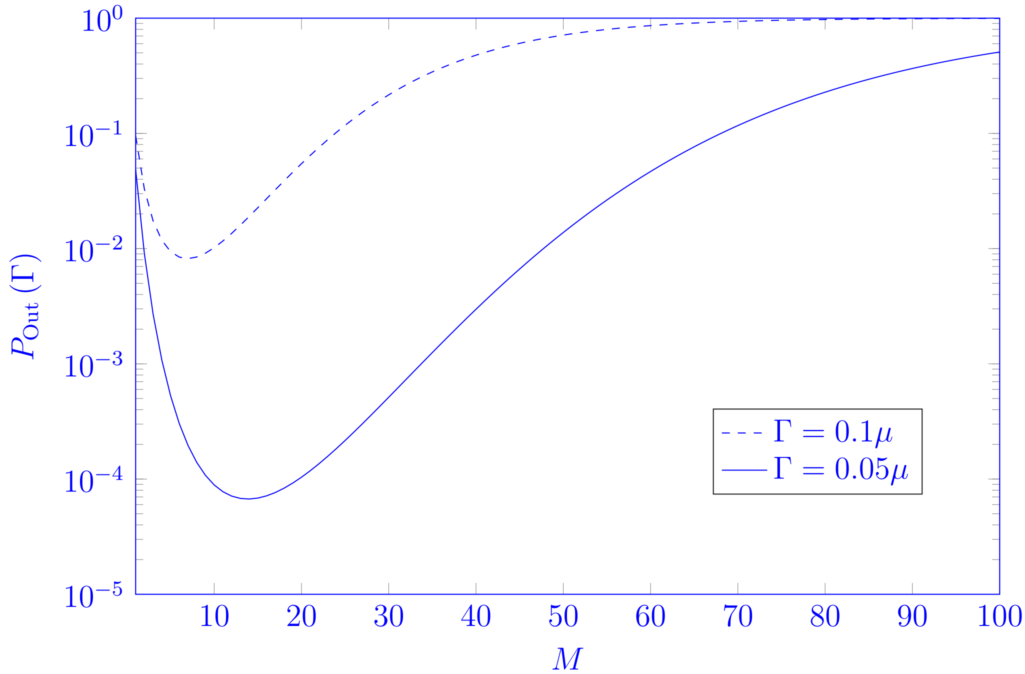


Figure 6.3.1: Outage probability for two different choices of γ/μ in Case 2.

As the figure shows by growing M, the outage probability decreases up to a minimum, and then starts to increase. In the extreme case of $M \to \infty$, the outage probability reads

$$\lim_{M\to\infty} P_{\text{Out}}(\Gamma) = 1.$$

From this observation, we can conclude that with fixed total power, the increase in the number of diversity branches does not always improve the performance. This is due to this fact that by increasing the number of branches, the transmit power over each branch decreases.

Noting that in practice, the total transmit power is usually constrained, practical systems are usually limited in the maximum order of diversity.

6.3.3 Further Discussions on Diversity

1. A *wide-band* transmission over a fading channel has the following power-delay profile:

$$\rho_h(\tau, 0) = \delta(\tau) + \frac{1+j}{\sqrt{2}}\delta(\tau - \tau_0)$$

for some delay τ_0. The receiver knows this power-delay profile perfectly. Assume that transmission of a single symbol in this system takes T sec.

(a) Consider a case in which $T \ll \tau_0$. Design a receiver which gives you a diversity gain of order 2 without any modification at the transmitter.

(b) What is the diversity gain that you achieve with the receiver in Part (a), when $T > \tau_0$?

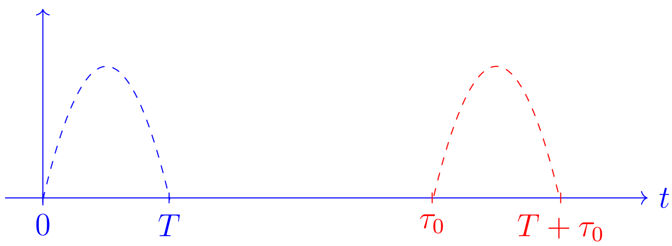
(c) Discuss the expected diversity gain achieved by this receiver in the GSM system when the transmitter and the receiver are communicating in

i. Rural area.

ii. Typical urban channels.

♠ **Solution:**

(a) From the power-delay profile, it is seen that there are two paths from the transmitter to the receiver. The first path has delay $\tau = 0$ with coefficient 1 and the second path has delay $\tau = \tau_0$ with the same coefficient. When $T \ll \tau_0$, the signals received from different paths are separable. Therefore, the receiver is able to distinguish between the two copies of the transmitted signal; see the following figure.

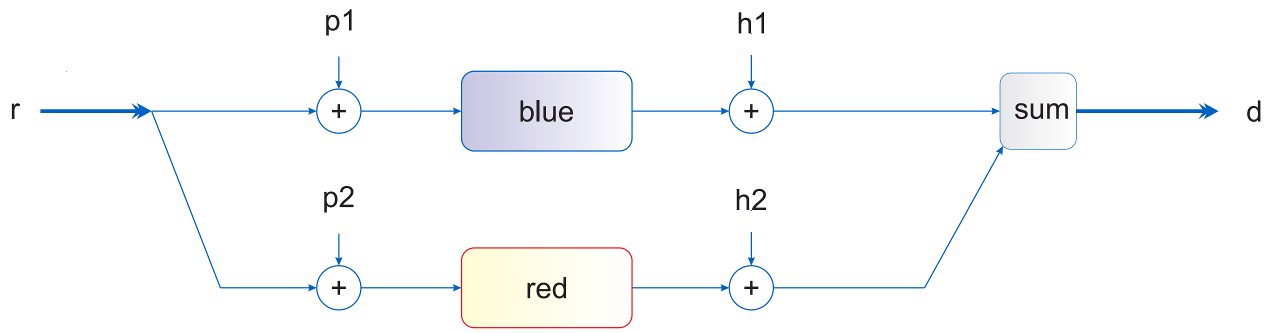


This can be also concluded by following the same approach as in the first exercise. Im fact, in this case, the signal bandwidth which is proportional to $1/T$ is significantly more than the coherence bandwidth which is proportional to $1/\tau_0$, and hence we have a *wide-band transmission*.

In this case, the receiver is receiving two independent copies of the transmit signal. It hence can combine them properly to get the diversity gain of order $M = 2$. This is shown in the following diagram.

In this receiver, we consider the following assumptions:

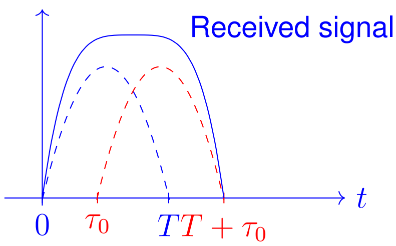
- The receive signal is $r(t)$



- The pulse shape is $g(t)$ which is zero out of the interval $[0, T]$

The upper branch separates the blue pulse in the initial diagram, and the red pulse distinguishes the red pulse. The pulses are then combined by maximum ratio combining approach (MRC).

(b) When $T > \tau_0$, the signal received through the second path overlaps with the one received from the first one. Therefore, the receiver cannot distinguish between the two copies; see the following figure.



As a result, in this case, the receiver is receiving only one distorted superposed copy of the signal. This concludes that the receiver in Part (a) has diversity gain of order $M = 1$ in this case.

(c) In GSM, the symbol rate is $R = 270.833$ kHz, this means that the pulse duration reads

$$T = \frac{1}{270.833 \times 10^3} \approx 3.7 \ \mu\text{sec}.$$

i. Considering the typical power-delay profiles for *rural area* in page 120, one can observe that the maximum delay in a rural area is $\tau_{\text{ex,RA}} \approx 0.5 \ \mu$sec. As $\tau_{\text{ex,RA}} < T = 3.7 \ \mu\text{sec}$, we conclude that the diversity gain of order $M = 1$ is only possible in a rural area.

ii. For a typical urban channel, the maximum delay is $\tau_{\text{ex,TU}} = 5 \ \mu$sec. In this case, we have $\tau_{\text{ex,TU}} > T = 3.7 \ \mu\text{sec}$ which indicates that $M > 1$ is possible in this case. In practice, it is usually possible to obtain $M = 2$ in urban channels with GSM signals.

6.4 Homework

1. Assume you are employing two branches of diversity in a fading channel. The corresponding coefficients to these diversity branches are h_1 and h_2. Furthermore, assume that your channel experiences *narrow-band Rayleigh fading*[2]. Unlike the standard case in diversity, the branches are *not* independent and have real correlation ρ, i.e.,

$$\mathsf{E}\left\{h_1 h_2^*\right\} = \mathsf{E}\left\{h_2 h_1^*\right\} = \rho.$$

Let the noise variance be σ^2 and the signal power be P, and define

$$\Gamma_0 = \frac{P}{\sigma^2}.$$

(a) item Assuming that Maximum-Ratio-Combining (MRC) is used at the receiver, determine the outage probability

$$\Pr\left\{\mathrm{SNR} \leq \Gamma\right\}$$

for some $\Gamma > 0$ where SNR indicates the signal-to-noise ratio (SNR) at the receiver.

(b) Discuss the order of diversity for the limiting cases $\rho \to 1$ and $\rho \to 0$.

Hint: By basic linear algebra, one can show that

$$|h_1|^2 + |h_2|^2$$

is distributed similar to

$$(1+\rho)\,|u_1|^2 + (1-\rho)\,|u_2|^2$$

where u_1 and u_2 are independent Gaussian with zero mean and unit variance.

♠ **Solution:**

(a) The SNR for a given realization of h_1 and h_2 reads

$$\mathrm{SNR} = \gamma_0 \left(|h_1|^2 + |h_2|^2\right).$$

As h_1 and h_2 are correlated with covariance ρ, one can use the hint and state that $|h_1|^2 + |h_2|^2$ is distributed similar to $(1+\rho)|u_1|^2 + (1-\rho)|u_2|^2$ where u_1 and u_2 are independent Gaussian with zero mean and unit variance. Thus, the received signal is similar to the case studied in Section 4.3.2.2 of the lecture notes and the outage probability can be derived via equation (4.30) of the lecture note.

(b) In the limiting case of $\rho \to 1$, the branches get completely correlated and thus the diversity order is $M \to 1$. When $\rho \to 0$, the branches become independent and hence $M \to 2$.

[2] Remmember that this means h_1 and h_2 are complex Gaussian random variables with zero mean and unit variance.

Chapter 7

Duplexing and Multiplexing

In this chapter, we go through the multiplexing and duplexing techniques in wireless networks. This session covers Chapter 5 of the lecture-notes.

7.1 Brief Review of Main Concepts

In wireless networks, duplexing is done ti establish a two-way communication. The main techniques for duplexing are

- Time division duplexing (TDD) in which the uplink and downlink transmissions are done in two separate time intervals.
- Frequency division duplexing (FDD) in which the uplink and downlink transmissions are done at two separate frequency bands.

Multiplexing or multiple-access techniques are on the other hand used to enable multiple users communicate simultaneously in the network. The main techniques for multiplexing are

- Time division multiplexing (TDM) where different users use the channel in different time intervals.
- Frequency division multiplexing (FDM) where different users use the channel at different frequency bands.
- Code division multiplexing (CDM) where different users sign their signals with a unique code sequence which is orthogonal to other code sequences.
- Space division multiplexing (SDM) where different users are served by different beams of array antennas.
- Orthogonal frequency division multiplexing (OFDM) where different users send orthogonal signals over overlapping carrier frequencies.
- Non-orthogonal multiple access (NOMA), where different users send their signals together and the receiver distinguishes among them by taking into account the difference of the received powers.

In this lecture, we go mainly through the first four techniques. Other techniques are briefly mentioned and will be given in more details in other lectures. Now, we start with the exercises.

7.2 Exercises

7.2.1 Frequency and Time Division Multiplexing

1. A communication channel of bandwidth $B = 5$ MHz is given. It is intended to multiplex this channel among K users for *two-way* communications with a base station. To this end, a combination of frequency division multiplexing (FDM) and time division multiplexing (TDM) is used. Moreover, the time division duplexing (TDD) technique is employed for establishing two-way communication.

   (a) Plot a *time-frequency* diagram for this system assuming the following specification:

$K = 32$ users are served in total and each FDM subchannel is share among 8 users via TDM.

   Assume that the parameters of this communication system are as follows

Parameter	Value
TDM slot duration	$T_{\mathrm{TDM}} = 150\ \mu$s
Guard time interval between two subsequent transmissions	$\Delta T_{\mathrm{G}} = 1\ \mu$s
Frequency guard interval between two adjacent subchannels	$\Delta B_{\mathrm{G}} = 10$ kHz

   In the table, the TDM slot duration indicates the time duration taken in each FDM subchannel to transmit the symbols of all users in both the transmit and receive modes.

   For frequency multiplexing, the given channel is divided into N_{C} FDM subchannels. Each subchannel is shared among 8 users via TDM. The users utilize a pulse shape with time duration T whose bandwidth is given by

   $$B = \frac{2}{T}. \tag{7.2.1}$$

   (b) Calculate the maximum value of N_{C}, such that no cross-talk happens among the users.

7.2.2 Code Division Multiplexing

1. Consider a cellular network with two users. The users utilize the code division multiplexing (CDM) technique with spreading factor N, chip duration T_{C} and symbol duration T_{S}. Denote the spreading sequence for user $k = 1, 2$ with $\{c_k[0], \dots, c_k[N-1]\}$. Assume that the spreading sequences are ideal and that the system uses rectangular pulses

for transmission. Moreover, let the impulse response of the uplink channel from user k to the base station be $h_k(t)$.

User k intends to transmit data symbol a_k.

(a) Determine the base-band receive signal at the base station.

Let the channel impulse response for these two users be

$$h_1(t) = \delta(t) + \frac{1+j}{\sqrt{2}}\delta(t-2T_{\mathrm{C}}) + \frac{1+2j}{\sqrt{5}}\delta(t-4T_{\mathrm{C}})$$
$$h_2(t) = \frac{1-j}{\sqrt{2}}\delta(t) + \delta(t-3T_{\mathrm{C}}) + \frac{3+j}{\sqrt{10}}\delta(t-4T_{\mathrm{C}})$$

(b) Assuming that the base station intends to employ a rake receiver, what is the optimal number of fingers for this receiver?

(c) Sketch the complete structure of the optimal rake receiver, and specify all the coefficients and blocks.

2. Consider a CDM system with K users whose spreading factor is N. In this system, the spreading sequences are generated randomly as follows:

For a given user k, the entries $c_k[0], \dots, c_k[N-1]$ are independent and identically distributed uniform bipolar random variables. This means that

$$\Pr\{c_k[n] = +1\} = \Pr\{c_k[n] = -1\} = 0.5.$$

(a) Determine the mean and variance of auto correlation $\rho_{mm}[d]$.

(b) Determine the mean and variance of cross correlation $\rho_{mk}[d]$.

(c) Discuss the performance of this system when N is large.

7.2.3 Space Division Multiplexing

1. Consider a multiple-input multiple-output (MIMO) system with K single antenna users. The base station is equipped with N antennas.

(a) Determine the maximum number of users supported in uplink transmission, when a zero-forcing receiver is employed at the base station.

(b) Determine this value, when a linear minimum mean squared error is employed.

Consider a case with $K = 2$ users and $N = 2$ antennas at the base station. Assume that the channel matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & -j \\ j & 2 \end{bmatrix},$$

and let the noise variance be $\sigma^2 = 0.1$.

(c) Calculate the weight factors of the receiver for both the techniques considered in the previous parts.

7.3 Solutions to Exercises

7.3.1 Frequency and Time Division Multiplexing

1. A communication channel of bandwidth $B = 5$ MHz is given. It is intended to multiplex this channel among K users for *two-way* communications with a base station. To this end, a combination of frequency division multiplexing (FDM) and time division multiplexing (TDM) is used. Moreover, the time division duplexing (TDD) technique is employed for establishing two-way communication.

   (a) Plot a *time-frequency* diagram for this system assuming the following specification:

$K = 32$ users are served in total and each FDM subchannel is share among 8 users via TDM.

   Assume that the parameters of this communication system are as follows

Parameter	Value
TDM slot duration	$T_{\mathrm{TDM}} = 150$ μs
Guard time interval between two subsequent transmissions	$\Delta T_{\mathrm{G}} = 1$ μs
Frequency guard interval between two adjacent subchannels	$\Delta B_{\mathrm{G}} = 10$ kHz

   In the table, the TDM slot duration indicates the time duration taken in each FDM subchannel to transmit the symbols of all users in both the transmit and receive modes.

   For frequency multiplexing, the given channel is divided into N_{C} FDM subchannels. Each subchannel is shared among 8 users via TDM. The users utilize a pulse shape with time duration T whose bandwidth is given by

   $$B = \frac{2}{T}.$$

   (b) Calculate the maximum value of N_{C}, such that no cross-talk happens among the users.

♠ **Solution:**

(a) We have $K = 32$ users. Since in each FDM sub-channel 8 users are multiplexed via TDM, we require

$$C = \frac{32}{8} = 4$$

FDM sub-channels. In each sub-channel, we have 8 TDM symbols.

Noting that the system uses TDD for duplexing, each of the TDM symbols is divided into two symbols: an *uplink* symbol and a *downlink* symbol. the symbols in time domain are spaced by some *guard time*. Similarly, the sub-channels in the frequency domain are separated by a *guard frequency band*. The time-frequency diagram for this system is hence given by Figure 6.1.

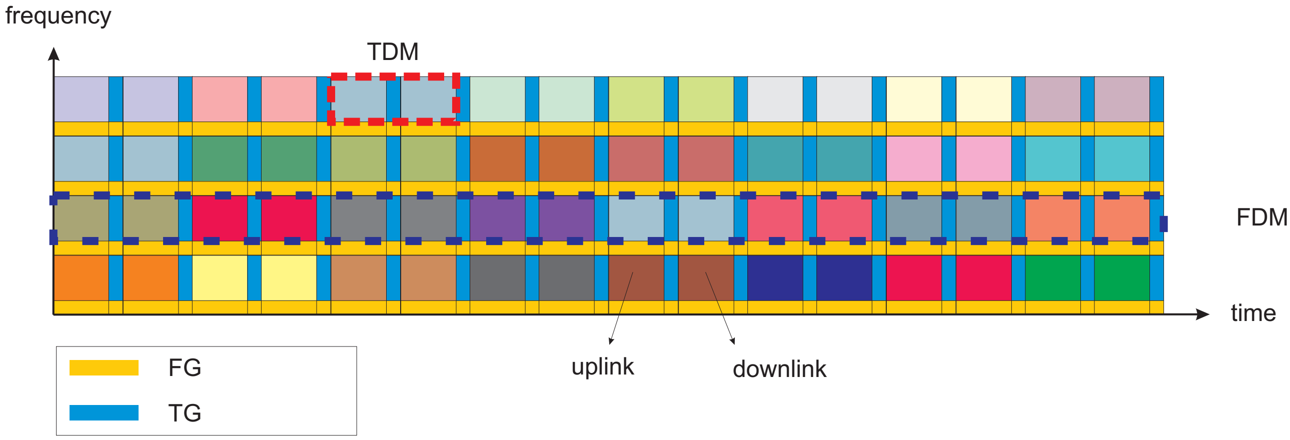


Figure 7.3.1: Time-frequency diagram for exercise 1.

(b) As Figure 6.1 shows, with 8 users in each FDM sub-channel, the TDM slot duration consists of 16 guard time intervals and 16 data transmission intervals. Assuming that for data transmission we use a pulse of time length T, we need

$$T_{\mathrm{TDM}} \geq 16\,(T + \Delta T_{\mathrm{G}})$$

which concludes

$$T \leq \frac{T_{\mathrm{TDM}}}{16} - \Delta T_{\mathrm{G}} = \frac{150}{16} - 1 = 8.375\ \mu\mathrm{sec}$$

This means that the bandwidth occupied by pulses in each FDM sub-channel reads

$$\begin{aligned} B + \mathrm{C} &= \frac{2}{T} \\ &\geq \frac{2}{8.375} = 228.6\ \mathrm{kHz} = 0.2286\ \mathrm{MHz} \end{aligned}$$

This means that the bandwidth of each FDM sub-channel should be at least $B_{\min} = 228.6$ kHz, to make sure that there is no cross-talk in the system.
From Figure 6.1, it is observed that the total bandwidth occupied by N_{C} FDM sub-channels is

$$B = N_{\mathrm{C}}\,(B_{\mathrm{C}} + \Delta B_{\mathrm{G}})\,.$$

By setting $B = 5$ MHz, we have

$$\begin{aligned} N_{\mathrm{C}} &= \frac{B}{B_{\mathrm{C}} + \Delta B_{\mathrm{G}}} = \frac{5}{B_{\mathrm{C}} + 0.01} \\ &\leq \frac{5}{0.2286 + 0.01} = 20.95. \end{aligned}$$

Thus, the maximum number of FDM sub-channels is $\boxed{N_{\mathrm{C,max}} = 20}$.

7.3.2 Code Division Multiplexing

1. Consider a cellular network with two users. The users utilize the code division multiplexing (CDM) technique with spreading factor N, chip duration T_{C} and symbol duration T_{S}. Denote the spreading sequence for user $k = 1, 2$ with $\{c_k[0], \dots, c_k[N-1]\}$. Assume that the spreading sequences are ideal and that the system uses rectangular pulses for transmission. Moreover, let the impulse response of the uplink channel from user k to the base station be $h_k(t)$.

   User k intends to transmit data symbol a_k.

   (a) Determine the base-band receive signal at the base station.

   Let the channel impulse response for these two users be

$$h_1(t) = \delta(t) + \frac{1+j}{\sqrt{2}}\delta(t - 2T_{\mathrm{C}}) + \frac{1+2j}{\sqrt{5}}\delta(t - 4T_{\mathrm{C}})$$
$$h_2(t) = \frac{1-j}{\sqrt{2}}\delta(t) + \delta(t - 3T_{\mathrm{C}}) + \frac{3+j}{\sqrt{10}}\delta(t - 4T_{\mathrm{C}})$$

   (b) Assuming that the base station intends to employ a rake receiver, what is the optimal number of fingers for this receiver?

   (c) Sketch the complete structure of the optimal rake receiver, and specify all the coefficients and blocks.

♠ **Solution:**

(a) Since we use rectangular pulse shape, the chip pulse is

$$g(t) = \begin{cases} 1 & 0 \le t \le T_{\mathrm{C}} \\ 0 & \text{Otherwise} \end{cases}.$$

Consequently, the pulse shape of user k is

$$p_k(t) = \sum_{n=0}^{N-1} c_1[n]\, g(t - nT_{\mathrm{C}}).$$

User k sends symbol a_k by transmitting a signal whose base-band representation is

$$x_k(t) = a_k p_k(t).$$

The base-band receive signal is then given by

$$
\begin{aligned}
y(t) &= x_1(t) * h_1(t) + x_2(t) * h_2(t) + z(t) \\
&= \int_{-\infty}^{+\infty} x_1(\tau) h_1(t-\tau)\,\mathrm{d}\tau + \int_{-\infty}^{+\infty} x_2(\tau) h_2(t-\tau)\,\mathrm{d}\tau + z(t) \\
&= \hat{x}_1(t) + \hat{x}_2(t) + z(t)
\end{aligned}
$$

where $z(t)$ is Gaussian noise and we define

$$
\hat{x}_k(t) = \int_{-\infty}^{+\infty} x_k(\tau) h_k(t-\tau)\,\mathrm{d}\tau.
$$

(b) The rake receiver takes the copied of the transmit signal received through various *distinguishable* paths and combine them via maximum ratio combining (MRC) to get some diversity gain. Each finger of this receiver takes the copy of a single paths.

Noting that there are three *distinguishable* paths from each user to the base station, the optimal number of fingers is $\boxed{L=3}$.

(c) The structure of the optimal rake receiver is shown in Figure 6.2. As the figure shows, the receiver consists of two blocks. The first block is the rake receiver for user 1 and the second block denotes the receiver which recovers signal of user 2.

In this receiver each finger corresponds to one of the paths. The coefficients w_ℓ and $\hat{w}_\ell$ are the combining weights. To specify the values of these weights, we need to determine the output symbol of each finger, namely s_ℓ and $\hat{s}_\ell$.

We start by calculating the first finger in the first block. For s_1, we have

$$
\begin{aligned}
s_1 &= \frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} y(t) p_1^*(t)\,\mathrm{d}t \\
&= \frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} \left[\hat{x}_1(t) + \hat{x}_2(t) + z(t)\right] p_1^*(t)\,\mathrm{d}t \\
&= \underbrace{\frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} \hat{x}_1(t) p_1^*(t)\,\mathrm{d}t}_{D_1} + \underbrace{\frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} \hat{x}_2(t) p_1^*(t)\,\mathrm{d}t}_{I_1} + \underbrace{\frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} z(t) p_1^*(t)\,\mathrm{d}t}_{Z_1}.
\end{aligned}
$$

The terms D_1, I_1 and Z_1 read

- Noting that $\hat{x}_1(t)$ is given by

$$
\hat{x}_1(t) = x_1(t) + \frac{1+j}{\sqrt{2}} x_1(t-2T_\mathrm{C}) + \frac{1+2j}{\sqrt{5}} x_1(t-4T_\mathrm{C}),
$$

  the term D_1 is calculated as

$$
\begin{aligned}
D_1 = &\frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} x_1(t) p_1^*(t)\,\mathrm{d}t + \frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} \frac{1+j}{\sqrt{2}} x_1(t-2T_\mathrm{C}) p_1^*(t)\,\mathrm{d}t \\
&+ \frac{1}{T_\mathrm{S}} \int_0^{T_\mathrm{S}} \frac{1+2j}{\sqrt{5}} x_1(t-4T_\mathrm{C}) p_1^*(t)\,\mathrm{d}t.
\end{aligned}
$$

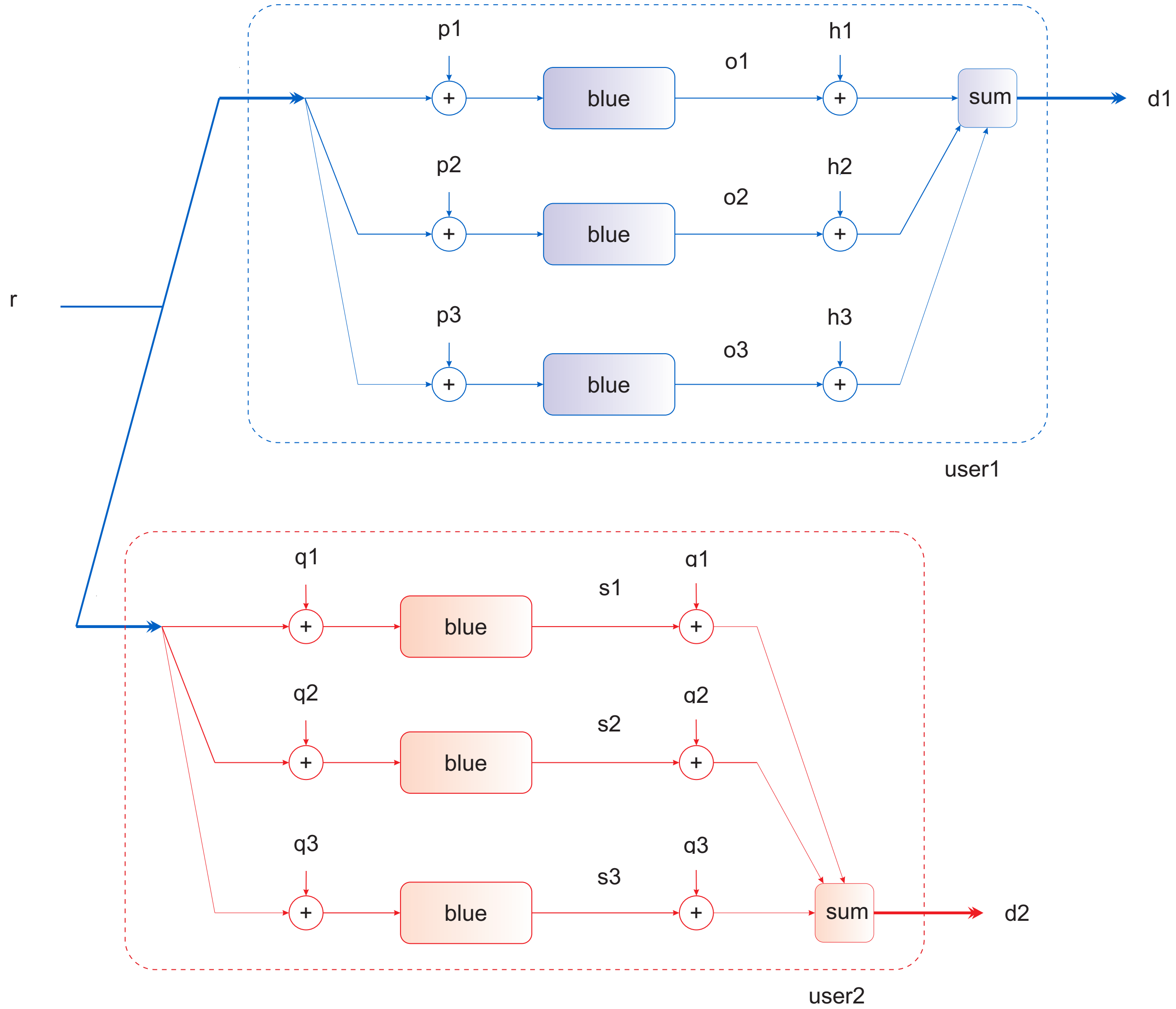


Figure 7.3.2: Structure of the rake receiver at the base station.

We now substitute $x_1(t) = a_1 p_1(t)$ which concludes

$$\begin{aligned} D_1 =& a_1 \left(\frac{1}{T_S} \int_0^{T_S} p_1(t)\, p_1^*(t)\, \mathrm{d}t \right) + \frac{1+j}{\sqrt{2}} a_1 \left(\frac{1}{T_S} \int_0^{T_S} p_1(t - 2T_C)\, p_1^*(t)\, \mathrm{d}t \right) \\ &+ \frac{1+2j}{\sqrt{5}} a_1 \left(\frac{1}{T_S} \int_0^{T_S} p_1(t - 4T_C)\, p_1^*(t)\, \mathrm{d}t \right) \end{aligned}$$

Since the spreading sequences are *ideal*, each sequence is orthogonal to its shifted version as well as other sequences; thus, we have

$$\begin{aligned} &p_1(t)\, p_1^*(t) = |p_1(t)|^2 = 1 \\ &p_1(t - 2T_C)\, p_1^*(t) = p_1(t - 4T_C)\, p_1^*(t) = 0. \end{aligned}$$

As a result, we can write

$$D_1 = a_1 \left(\frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} {\rm d}t \right) + 0 + 0.$$

This concludes that $\boxed{D_1 = a_1}$.

- Noting that $\hat{x}_2(t)$ is given by

$$\hat{x}_2(t) = \frac{1-j}{\sqrt{2}} x_2(t) + x_2(t - 3T_{\rm C}) + \frac{3+j}{\sqrt{10}} x_2(t - 4T_{\rm C}),$$

the term I_1 is calculated as

$$\begin{aligned} I_1 =& \frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} \frac{1-j}{\sqrt{2}} x_2(t) p_1^*(t)\,{\rm d}t + \frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} x_2(t - 3T_{\rm C}) p_1^*(t)\,{\rm d}t \\ &+ \frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} \frac{3+j}{\sqrt{10}} x_2(t - 4T_{\rm C}) p_1^*(t)\,{\rm d}t. \end{aligned}$$

We now substitute $x_2(t) = a_2 p_2(t)$ concluding that

$$\begin{aligned} I_1 =& \frac{1-j}{\sqrt{2}} a_2 \left(\frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} p_2(t) p_1^*(t)\,{\rm d}t \right) + a_2 \left(\frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} p_2(t - 3T_{\rm C}) p_1^*(t)\,{\rm d}t \right) \\ &+ \frac{3+j}{\sqrt{10}} a_2 \left(\frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} p_2(t - 4T_{\rm C}) p_1^*(t)\,{\rm d}t \right) \end{aligned}$$

Similar to previous calculations, we can invoke the *ideality* of the spreading sequences and write

$$p_2(t) p_1^*(t) = p_1(t - 3T_{\rm C}) p_1^*(t) = p_1(t - 4T_{\rm C}) p_1^*(t) = 0$$

which results in $\boxed{I_1 = 0}$.

- For Z_1, we further have

$$Z_1 = \frac{1}{T_{\rm S}} \int_0^{T_{\rm S}} z(t) p_1^*(t)\,{\rm d}t$$

which is a Gaussian random variable.

Considering the above derivations, we conclude that the output of the first finger in the first block is

$$s_1 = a_1 + Z_1.$$

By repeating exactly same steps, we will conclude that

$$\begin{aligned} s_2 &= \frac{1+j}{\sqrt{2}} a_1 + Z_2, \\ s_3 &= \frac{1+2j}{\sqrt{5}} a_1 + Z_3, \end{aligned}$$

where Z_2 and Z_3 are Gaussian noises, and

$$\hat{s}_1 = \frac{1-j}{\sqrt{2}} a_2 + \hat{Z}_1,$$
$$\hat{s}_2 = a_2 + \hat{Z}_2,$$
$$\hat{s}_3 = \frac{3+j}{\sqrt{10}} a_2 + \hat{Z}_3$$

with $\hat{Z}_1$, $\hat{Z}_2$ and $\hat{Z}_3$ being Gaussian random variables.

As it is observed, in each block we have three noisy copies of the transmitted symbol corresponding to user k received by different coefficients. This is similar to signals received through diversity branches. The optimal approach in this case is to combine these noisy copies of the signals via MRC. Hence, the weights w_1, w_2 and w_3 read

$$w_1 = 1$$
$$w_2 = \frac{1-j}{\sqrt{2}}$$
$$w_3 = \frac{1-2j}{\sqrt{5}}.$$

Similarly, the weights $\hat{w}_1$, $\hat{w}_2$ and $\hat{w}_3$ are given via MRC as

$$\hat{w}_1 = \frac{1+j}{\sqrt{2}}$$
$$\hat{w}_2 = 1$$
$$\hat{w}_3 = \frac{3-j}{\sqrt{10}}.$$

⇝ REMARK:

The rake receiver simply performs as follows:

(a) It finds the distinguishable paths from the transmitter to receiver.

(b) It takes out the noisy copy of each path by a dedicated finger. This finger simply uses a matched filter matched to the corresponding delayed pulse.

(c) It combines the outputs of the finger using MRC. By doing so, it exploits diversity from the multipath channel.

2. Consider a CDM system with K users whose spreading factor is N. In this system, the spreading sequences are generated randomly as follows:

For a given user k, the entries $c_k[0], \ldots, c_k[N-1]$ are independent and identically distributed uniform bipolar random variables. This means that

$$\Pr\{c_k[n] = +1\} = \Pr\{c_k[n] = -1\} = 0.5.$$

(a) Determine the mean and variance of auto correlation $\rho_{mm}[d]$.

(b) Determine the mean and variance of cross correlation $\rho_{mk}[d]$.

(c) Discuss the performance of this system when N is large.

♠ **Solution:**

(a) From the definitions in page 161 of the lecture notes, the auto correlation is given by

$$\rho_{mm}[d] = \frac{1}{N} \sum_{n=0}^{N-1} c_m^*[n] c_m[n+d]$$

where the sequence $c_m[n]$ is the periodic repetition of the spreading sequence for user m. Since the sequences $c_m[n]$ is real in our example, the conjugation is ineffective in this case.

As the entries of $c_m[n]$ are randomly generated with $\Pr\{c_m[n] = +1\} = \Pr\{c_m[n] = +1\} = 0.5$, the auto correlation function for each d is a random variable. To determine the mean value of $\rho_{mm}[d]$, we write

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \frac{1}{N}\mathsf{E}\left\{\sum_{n=0}^{N-1} c_m^*[n] c_m[n+d]\right\} = \frac{1}{N}\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^*[n] c_m[n+d]\right\}$$
$$\overset{(\mathrm{A})}{=} \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^*[n]\right\} \mathsf{E}\left\{c_m[n+d]\right\} & d \neq 0 \\ \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^2[n]\right\} & d = 0 \end{cases}$$

where (A) comes from the fact that $c_m[n]$ and $c_m[n+d]$ are independently generated for $d \neq 0$.

Here, one should note that, for all n

$$\mathsf{E}\left\{c_m[n]\right\} = 0.5 \times (+1) + 0.5 \times (-1) = 0,$$
$$\mathsf{E}\left\{c_m[n]^2\right\} = 0.5 \times (+1)^2 + 0.5 \times (-1)^2 = 1,$$
$$\mathsf{E}\left\{c_m[n]^4\right\} = 0.5 \times (+1)^4 + 0.5 \times (-1)^4 = 1.$$

Therefore, we can conclude that

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^*[n]\right\} \mathsf{E}\left\{c_m[n+d]\right\} & d \neq 0 \\ \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m[n]^2\right\} & d = 0 \end{cases}$$

which results in

$$\boxed{\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases} 0 & d \neq 0 \\ 1 & d = 0 \end{cases}.}$$

The variance of the auto correlation function is moreover given by

$$\mathrm{Var}\left\{\rho_{mm}[d]\right\} = \mathsf{E}\left\{\rho_{mm}[d]^2\right\} - \mathsf{E}\left\{\rho_{mm}[d]\right\}^2.$$

For $\mathsf{E}\left\{\rho_{mm}[d]^2\right\}$, we have

$$\begin{aligned} \mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \mathsf{E}\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]\right)^2\right\} \\ &= \frac{1}{N^2}\mathsf{E}\left\{\sum_{j=1}^{N-1}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} \\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\}. \end{aligned}$$

To determine the *black term*, we note that when $d \neq 0$ and $n \neq j$

$$\begin{aligned} \mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} &= \mathsf{E}\left\{c_m^*[n]\right\} \mathsf{E}\left\{c_m[n+d]\right\} \\ &\quad \times \mathsf{E}\left\{c_m^*[j]\right\} \mathsf{E}\left\{c_m[j+d]\right\} = 0 \end{aligned}$$

Moreover, for $d \neq 0$ and $n = j$

$$\begin{aligned} \mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} &= \mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\} \mathsf{E}\left\{c_m[n+d]c_m[j+d]\right\} \\ &= \mathsf{E}\left\{c_m^*[j]^2\right\} \mathsf{E}\left\{c_m[j+d]^2\right\} = 1 \times 1 = 1. \end{aligned}$$

Consequently, for $d \neq 0$ we can write

$$\begin{aligned} \mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\Bigg[\mathsf{E}\left\{c_m^*[j]^2 c_m[j+d]^2\right\} \\ &\qquad + \sum_{n=0, n\neq j}^{N-1} \mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\}\Bigg] \\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[1 + \sum_{n=0, n\neq j}^{N-1} 0\right] \\ &= \frac{1}{N^2} \times N = \frac{1}{N}. \end{aligned}$$

Similarly, when $d = 0$ and $n \neq j$

$$\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} = \mathsf{E}\left\{c_m^*[n]^2\right\}\mathsf{E}\left\{c_m^*[j]^2\right\} = 1 \times 1 = 0,$$

and when $d = 0$ and $n = j$

$$\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} = \mathsf{E}\left\{c_m[j]^4\right\} = 1.$$

As the result, for $d = 0$, we have

$$\begin{aligned}\mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1}c_m^*[n]c_m[n+d]\right)^2\right\} \\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}1 = \frac{1}{N^2}\times N^2 = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}\left\{\rho_{mm}[d]\right\} &= \mathsf{E}\left\{\rho_{mm}[d]^2\right\} - \mathsf{E}\left\{\rho_{mm}[d]\right\}^2 \\ &= \begin{cases}\frac{1}{N} & d \neq 0 \\ 1 & d = 0\end{cases} - \left[\begin{cases}0 & d \neq 0 \\ 1 & d = 0\end{cases}\right]^2.\end{aligned}$$

This concludes that

$$\boxed{\mathrm{Var}\left\{\rho_{mm}[d]\right\} = \begin{cases}\frac{1}{N} & d \neq 0 \\ 0 & d = 0\end{cases}.}$$

(b) For cross correlation, we follow a similar approach by starting from

$$\rho_{mk}[d] = \frac{1}{N}\sum_{n=0}^{N-1}c_m^*[n]c_k[n+d]$$

for some $k \neq m$. In this case, we have

$$\begin{aligned}\mathsf{E}\left\{\rho_{mk}[d]\right\} &= \frac{1}{N}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_k[n+d]\right\} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_k[n+d]\right\},\end{aligned}$$

which concludes that

$$\boxed{\mathsf{E}\left\{\rho_{mk}[d]\right\} = 0.}$$

For the variance, we note that

$$\begin{aligned}\mathsf{E}\left\{\rho_{mk}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_k[n+d]c_m^*[j]c_k[j+d]\right\}\\ &\overset{\text{(B)}}{=} \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\}\end{aligned}$$

where (B) follows the fact that $m \neq k$.
For the black term, we can follow the same approach as for the auto correlation: For $n \neq j$, we have

$$\begin{aligned}\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\} = \mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_m^*[j]\right\}\\ \times\,\mathsf{E}\left\{c_k[n+d]\right\}\mathsf{E}\left\{c_k[j+d]\right\} = 0,\end{aligned}$$

and for $n = j$, we have

$$\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\} = \mathsf{E}\left\{c_m[j]^2\right\}\mathsf{E}\left\{c_k[j+d]^2\right\} = 1 \times 1 = 1.$$

Consequently, for any d, we can write

$$\begin{aligned}\mathsf{E}\left\{\rho_{mk}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[\mathsf{E}\left\{c_m^*[j]^2\right\}\mathsf{E}\left\{c_k[j+d]^2\right\}\right.\\ &\qquad\left. + \sum_{n=0,n\neq j}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m[n+d]\right\}\mathsf{E}\left\{c_k^*[j]c_k[j+d]\right\}\right]\\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[1 + \sum_{n=0,n\neq j}^{N-1} 0\right]\\ &= \frac{1}{N^2}\times N = \frac{1}{N},\end{aligned}$$

and thus,

$$\boxed{\mathrm{Var}\left\{\rho_{mk}[d]\right\} = \mathsf{E}\left\{\rho_{mk}[d]^2\right\} - \mathsf{E}\left\{\rho_{mk}[d]\right\}^2 = \frac{1}{N}.}$$

(c) By taking the limit $N \to \infty$, we have

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases}0 & d \neq 0\\ 1 & d = 0\end{cases} \qquad \text{and} \qquad \mathrm{Var}\left\{\rho_{mm}[d]\right\} = 0$$

and

$$\mathsf{E}\left\{\rho_{mk}[d]\right\} = \mathrm{Var}\left\{\rho_{mk}[d]\right\} = 0.$$

This means that for large N, the random spreading sequences perform close to optimal codes. This is why *random scrambling* is employed in practical CDM systems[1].

[1]Those who have Mobile Communications Lab observe this result in Experiment 7.

7.3.3 Space Division Multiplexing

1. Consider a multiple-input multiple-output (MIMO) system with K single antenna users. The base station is equipped with N antennas.

   (a) Determine the maximum number of users supported in uplink transmission, when a zero-forcing receiver is employed at the base station.

   (b) Determine this value, when a linear minimum mean squared error is employed.

   Consider a case with $K = 2$ users and $N = 2$ antennas at the base station. Assume that the channel matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & -j \\ j & 2 \end{bmatrix},$$

   and let the noise variance be $\sigma^2 = 0.1$.

   (c) Calculate the weight factors of the receiver for both the techniques considered in the previous parts.

♠ **Solution:**

(a) From equation (5.53) in page 178, we know that the zero-forcing receiver matrix is

$$\mathbf{W} = \left(\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{H}}$$

where $\mathbf{H}^{\mathsf{H}}$ denotes the transposed and conjugated version of $\mathbf{H}$.
The zero-forcing receiver can recover the data transmitted by users, when the inverse term $\left(\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}$ exists. Note that $\mathbf{H}$ is an $N \times K$ matrix; hence, $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ is a $K \times K$ matrix whose rank[2] is at most $\min\{N, K\}$. To make $\mathbf{H}^{\mathsf{H}}\mathbf{H}$ invertible, we need to guarantee that this matrix is full rank[3]. Therefore, we need

$$K = \min\{N, K\}$$

which means

$$K \leq N.$$

As the result, this zero-forcing receiver can support up to N users.

(b) With linear minimum mean squared error, we have

$$\mathbf{W} = \left(\mathbf{H}^{\mathsf{H}}\mathbf{H} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{H}^{\mathsf{H}}$$

where σ^2 is the noise variance. Unlike the zero-forcing receiver, the inverse term in this case always exists, thus K could be any integer.

[2] Number of non-zero eigenvalues.

[3] A full rank matrix is a squared matrix whose rank equals to its dimension. Since determinant equals to the product of eigenvalues, a squared matrix is invertible if and only if it is full rank.

⇝ REMARK:

There is a trade-off between zero-forcing and linear minimum mean squared error receivers:

- In the former, we completely invert the channel at the cost of increasing noise variance. This limits the number of supported users, since for high number of users the noise variance is significantly amplified.
- In the latter, we allow some interference from other users. By paying this cost, we achieve noise resistance which enables us supporting a larger number of users.

(c) By substituting into the equations, we have

i. For the zero-forcing receiver the matrix of coefficients reads

$$\mathbf{W} = \begin{bmatrix} 2 & j \\ -j & 1 \end{bmatrix}.$$

ii. For the linear minimum mean squared error receiver, we have

$$\mathbf{W} = \begin{bmatrix} 1.228 & 0.526j \\ -0.526j & 0.7018 \end{bmatrix}.$$

7.4 Homework

1. Consider a multiuser system which uses Frequency-Division Multiplexing (FDM). In this system, a bandpass filter is employed whose 3 dB bandwidth is B and whose total bandwidth is

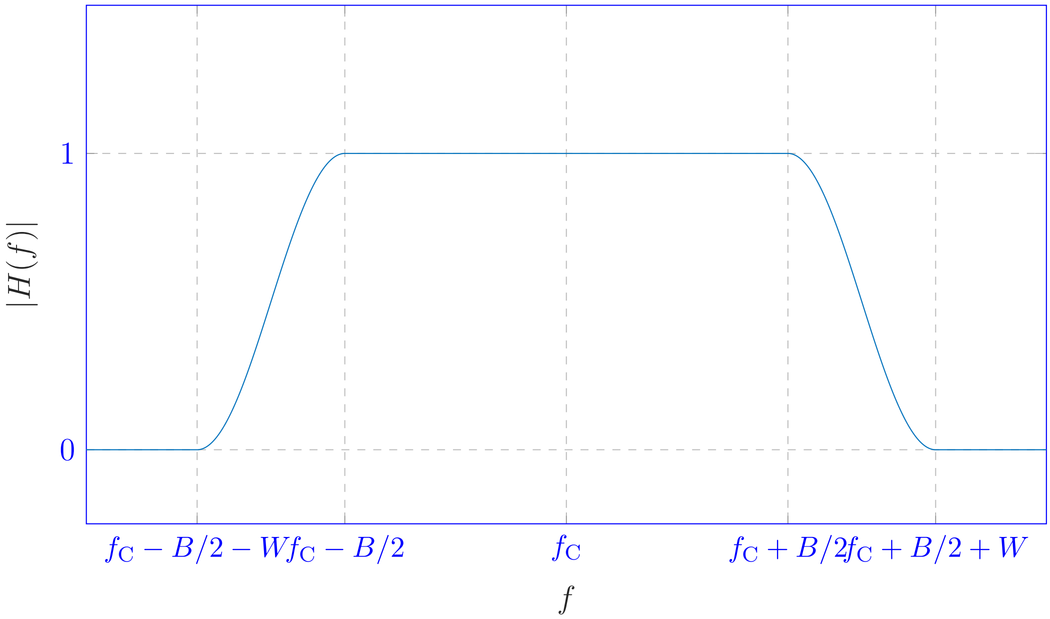
$$B_{\mathrm{T}} = B + 2W.$$

(a) Calculate the guard interval for this system assuming that 3 dB distortion in the signal is tolerable at the receive side.

(b) In practice, B and W increase as the carrier frequency increases. Assuming that K users are supported in this system, what is the choice for the guard interval in a practical scenario?

♠ **Solution:**

(a) The frequency response of a sample filter is shown in the following figure where f_{C} is the carrier frequency.



With such a filter, the transmit signal would have some spectral components within $[f_C + B/2, f_C + B/2 + W]$ and $[f_C - B/2 - W, f_C - B/2]$. Consequently, the guard interval B_G should satisfy

$$B_G > W$$

in order to make sure that the signal from the user in the adjacent channel does not interfere.

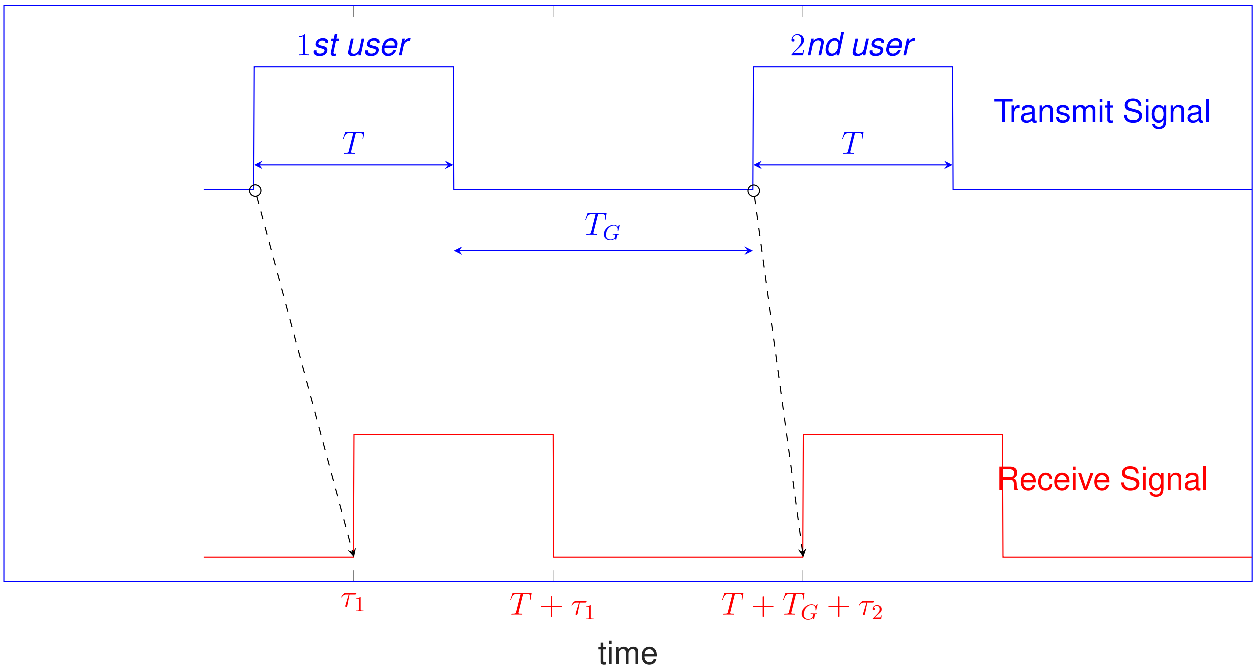
(b) When the filter parameters are functions of the carrier frequency, the tail region, i.e., the region that frequency response transits from one to zero, has the width of W_k for the channel of user k. Consequently, the guard interval has to satisfy

$$B_G > \max_{k \in \{1,\ldots,K\}} W_k$$

to make sure that none of the users interferes its adjacent.

2. Consider a Time-Division Multiplexing (TDM) system with two users. The delay from the first user to the base station is τ_1 and the delay to the second user is $\tau_2 < \tau_1$. Ignoring the circuit imperfection, calculate the guard interval when timing advance is employed.

♠ **Solution:** The figure below shows a sample transmit and receive signal of a TDM transmission with two users when timing advance is employed. Note that timing advance enables us to know the beginning of pulse transmission. The duration T_G shows the guard interval between the two users.



As the figure shows, to avoid interference from the second user's pulse to the signal of the first user, we need to set T_G, such that

$$T + \tau_1 < T + T_G + \tau_2$$

which concludes that

$$T_G > \tau_1 - \tau_2.$$

In practical scenarios with more than two users, the guard interval T_G should satisfy

$$T_G > \max_{k \neq j} (\tau_k - \tau_j)$$

to guarantee no multiuser interference in the system.

3. Consider a CDM system with K users whose spreading factor is N. In this system, the spreading sequences are generated randomly as follows:

> For a given user k, the entries $c_k[0], \dots, c_k[N-1]$ are independent and identically distributed uniform bipolar random variables. This means that
>
> $$\Pr\{c_k[n] = +1\} = \Pr\{c_k[n] = -1\} = 0.5.$$

(a) Determine the mean and variance of auto correlation $\rho_{mm}[d]$.

(b) Determine the mean and variance of cross correlation $\rho_{mk}[d]$.

(c) Discuss the performance of this system when N is large.

♠ **Solution:**

(a) From the definitions in page 161 of the lecture notes, the auto correlation is given by

$$\rho_{mm}[d] = \frac{1}{N}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]$$

where the sequence $c_m[n]$ is the periodic repetition of the spreading sequence for user m. Since the sequences $c_m[n]$ is real in our example, the conjugation is ineffective in this case.

As the entries of $c_m[n]$ are randomly generated with $\Pr\{c_m[n] = +1\} = \Pr\{c_m[n] = +1\} = 0.5$, the auto correlation function for each d is a random variable. To determine the mean value of $\rho_{mm}[d]$, we write

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \frac{1}{N}\mathsf{E}\left\{\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]\right\} = \frac{1}{N}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m[n+d]\right\}$$
$$\overset{(\mathrm{A})}{=} \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_m[n+d]\right\} & d \neq 0 \\ \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^2[n]\right\} & d = 0 \end{cases}$$

where (A) comes from the fact that $c_m[n]$ and $c_m[n+d]$ are independently generated for $d \neq 0$.

> Here, one should note that, for all n
>
> $$\begin{aligned} \mathsf{E}\left\{c_m[n]\right\} &= 0.5\times(+1) + 0.5\times(-1) = 0, \\ \mathsf{E}\left\{c_m[n]^2\right\} &= 0.5\times(+1)^2 + 0.5\times(-1)^2 = 1, \\ \mathsf{E}\left\{c_m[n]^4\right\} &= 0.5\times(+1)^4 + 0.5\times(-1)^4 = 1. \end{aligned}$$

Therefore, we can conclude that

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_m[n+d]\right\} & d \neq 0 \\ \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m[n]^2\right\} & d = 0 \end{cases}$$

which results in

$$\boxed{\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases} 0 & d \neq 0 \\ 1 & d = 0 \end{cases}.}$$

The variance of the auto correlation function is moreover given by

$$\mathrm{Var}\left\{\rho_{mm}[d]\right\} = \mathsf{E}\left\{\rho_{mm}[d]^2\right\} - \mathsf{E}\left\{\rho_{mm}[d]\right\}^2.$$

For $\mathsf{E}\left\{\rho_{mm}[d]^2\right\}$, we have

$$\begin{aligned}
\mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \mathsf{E}\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]\right)^2\right\} \\
&= \frac{1}{N^2}\mathsf{E}\left\{\sum_{j=1}^{N-1}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} \\
&= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1} \mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\}.
\end{aligned}$$

To determine the *black term*, we note that when $d \neq 0$ and $n \neq j$

$$\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} = \mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_m[n+d]\right\}\mathsf{E}\left\{c_m^*[j]\right\}\mathsf{E}\left\{c_m[j+d]\right\} = 0$$

Moreover, for $d \neq 0$ and $n = j$

$$\begin{aligned}
\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} &= \mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_m[n+d]c_m[j+d]\right\} \\
&= \mathsf{E}\left\{c_m^*[j]^2\right\}\mathsf{E}\left\{c_m[j+d]^2\right\} = 1 \times 1 = 1.
\end{aligned}$$

Consequently, for $d \neq 0$ we can write

$$\begin{aligned}
\mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[\mathsf{E}\left\{c_m^*[j]^2 c_m[j+d]^2\right\} + \sum_{n=0,n\neq j}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\}\right] \\
&= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[1 + \sum_{n=0,n\neq j}^{N-1} 0\right] \\
&= \frac{1}{N^2} \times N = \frac{1}{N}.
\end{aligned}$$

Similarly, when $d = 0$ and $n \neq j$

$$\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} = \mathsf{E}\left\{c_m^*[n]^2\right\}\mathsf{E}\left\{c_m^*[j]^2\right\} = 1 \times 1 = 0,$$

and when $d = 0$ and $n = j$

$$\mathsf{E}\left\{c_m^*[n]c_m[n+d]c_m^*[j]c_m[j+d]\right\} = \mathsf{E}\left\{c_m[j]^4\right\} = 1.$$

As the result, for $d = 0$, we have

$$\begin{aligned}
\mathsf{E}\left\{\rho_{mm}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1} c_m^*[n]c_m[n+d]\right)^2\right\} \\
&= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1} 1 = \frac{1}{N^2} \times N^2 = 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}\left\{\rho_{mm}[d]\right\} &= \mathsf{E}\left\{\rho_{mm}[d]^2\right\} - \mathsf{E}\left\{\rho_{mm}[d]\right\}^2\\ &= \begin{cases}\frac{1}{N} & d\neq 0\\ 1 & d=0\end{cases} - \left[\begin{cases}0 & d\neq 0\\ 1 & d=0\end{cases}\right]^2.\end{aligned}$$

This concludes that

$$\boxed{\mathrm{Var}\left\{\rho_{mm}[d]\right\} = \begin{cases}\frac{1}{N} & d\neq 0\\ 0 & d=0\end{cases}.}$$

(b) For cross correlation, we follow a similar approach by starting from

$$\rho_{mk}[d] = \frac{1}{N}\sum_{n=0}^{N-1} c_m^*[n]c_k[n+d]$$

for some $k\neq m$. In this case, we have

$$\begin{aligned}\mathsf{E}\left\{\rho_{mk}[d]\right\} &= \frac{1}{N}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_k[n+d]\right\}\\ &= \frac{1}{N}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_k[n+d]\right\},\end{aligned}$$

which concludes that

$$\boxed{\mathsf{E}\left\{\rho_{mk}[d]\right\} = 0.}$$

For the variance, we note that

$$\begin{aligned}\mathsf{E}\left\{\rho_{mk}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_k[n+d]c_m^*[j]c_k[j+d]\right\}\\ &\overset{\text{(B)}}{=} \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\}\end{aligned}$$

where (B) follows the fact that $m\neq k$.
For the black term, we can follow the same approach as for the auto correlation: For $n\neq j$, we have

$$\begin{aligned}\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\} &= \mathsf{E}\left\{c_m^*[n]\right\}\mathsf{E}\left\{c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]\right\}\mathsf{E}\left\{c_k[j+d]\right\}\\ &= 0,\end{aligned}$$

and for $n = j$, we have

$$\mathsf{E}\left\{c_m^*[n]c_m^*[j]\right\}\mathsf{E}\left\{c_k[n+d]c_k[j+d]\right\} = \mathsf{E}\left\{c_m[j]^2\right\}\mathsf{E}\left\{c_k[j+d]^2\right\} = 1 \times 1 = 1.$$

Consequently, for any d, we can write

$$\begin{aligned}
\mathsf{E}\left\{\rho_{mk}[d]^2\right\} &= \frac{1}{N^2}\sum_{j=0}^{N-1}\Big[\mathsf{E}\left\{c_m^*[j]^2\right\}\mathsf{E}\left\{c_k[j+d]^2\right\} \\
&\qquad + \sum_{n=0,n\neq j}^{N-1}\mathsf{E}\left\{c_m^*[n]c_m[n+d]\right\}\mathsf{E}\left\{c_k^*[j]c_k[j+d]\right\}\Big] \\
&= \frac{1}{N^2}\sum_{j=0}^{N-1}\left[1 + \sum_{n=0,n\neq j}^{N-1} 0\right] \\
&= \frac{1}{N^2}\times N = \frac{1}{N},
\end{aligned}$$

and thus,

$$\boxed{\mathrm{Var}\left\{\rho_{mk}[d]\right\} = \mathsf{E}\left\{\rho_{mk}[d]^2\right\} - \mathsf{E}\left\{\rho_{mk}[d]\right\}^2 = \frac{1}{N}.}$$

(c) By taking the limit $N \to \infty$, we have

$$\mathsf{E}\left\{\rho_{mm}[d]\right\} = \begin{cases} 0 & d \neq 0 \\ 1 & d = 0 \end{cases} \qquad \text{and} \qquad \mathrm{Var}\left\{\rho_{mm}[d]\right\} = 0$$

and

$$\mathsf{E}\left\{\rho_{mk}[d]\right\} = \mathrm{Var}\left\{\rho_{mk}[d]\right\} = 0.$$

This means that for large N, the random spreading sequences perform close to optimal codes. This is why *random scrambling* is employed in practical CDM systems[4].

[4]Those who have Mobile Communications Lab observe this result in one of the experiments.

Chapter 8

Modulation

In this chapter, we discuss various aspects of modulation in cellular networks. The contents of this tutorial covers Chapter 6 of the lecture notes.

8.1 Brief Review of Main Concepts

There are various techniques for modulation in a communication system. In wireless networks, the choice of the modulation technique depends on the required performance. The metrics which usually specify the modulation technique are

- Peak to average power ratio (PAPR) which specify how the output power of the time-domain signal varies over time.
- Spectral efficiency which is the number of information bits transmitted per time and frequency.
- Error probability which is given in bit or symbol error rate.

In this chapter, we review these metrics for some known and popular modulation schemes.

8.2 Exercises

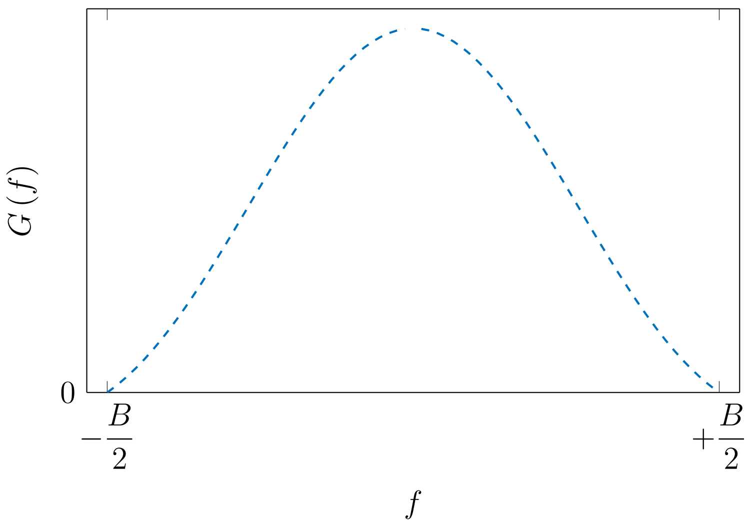
8.2.1 Linear Modulations

1. Consider a single-user system with no fading. We intend to transmit data stream

$$\mathrm{d} = \texttt{apple, banana}$$

in this system. The source encoder encodes `apple` to `0` and `banana` to `1`. The channel encoder then adds one bit with value `1` to the encoded data. This means that the codeword for `0` is `10`, and the codeword for `1` is `11`.

We transmit this stream using QPSK modulation with pulse shape $g(t)$ whose time duration is $T_\mathrm{s} = 5\mu\mathrm{sec}$. The spectrum of this pulse is given in the following figure:



The symbol duration is also $T_s = 5\mu\text{sec}$, and the carrier frequency is f_0.

(a) Determine the transmit symbols in this system.

(b) Write the base-band transmit signal and determine its spectrum. What is the bandwidth of this spectrum?

(c) Determine the radio-frequency signal in the time domain.

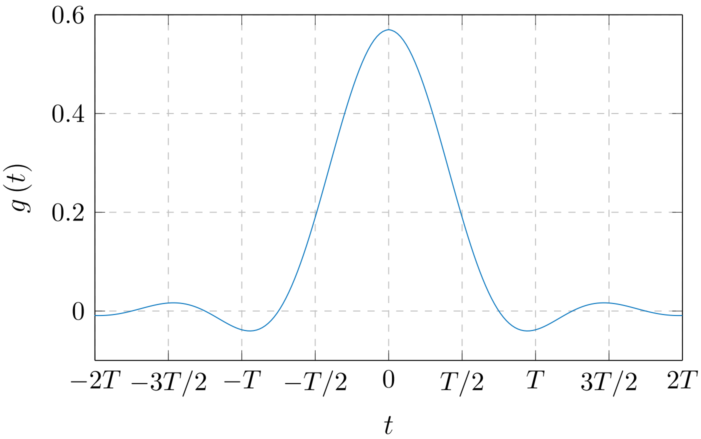
(d) Calculate the spectrum of the radio-frequency signal. Is the bandwidth different from the one calculated in Part (c)?

8.2.2 Pulse Shaping

1. Consider a system which uses a root-raised-cosine (RRC) pulse shape with roll-off factor $\alpha = 1$. To limit the time duration, the pulse has been truncated within $[-2T, 2T]$. This means that

$$g(t) = \begin{cases} g_1(t) & |t| \leq 2T \\ 0 & |t| > 2T \end{cases}$$

where $g_1(t)$ is RRC pulse with roll-off factor $\alpha = 1$; see the figure. Assume that the transmitter sends symbols $\{a[k]\} = \{\ldots, a[-1], a[0], a[1], \ldots\}$ over time which are taken from a known constellation, e.g., QPSK.



(a) Derive the value of the base-band signal $s(t)$ at $t = T/2$ in terms of the transmit symbols and pulse $g(t)$.

(b) Derive the maximum and minimum possible amplitude at $t = T/2$ when BPSK constellation is employed.

8.2.3 Comparison of Modulation Schemes

1. Sort the peak-to-average power ratio (PAPR) of the transmit signal when the following constellations and pulse shapes are employed:

   (a) QPSK and RRC pulse with $\alpha = 0.1$

   (b) QPSK and RRC pulse with $\alpha = 0.5$

   (c) $\pi/4$-QPSK and RRC pulse with $\alpha = 0.1$

   (d) $\pi/4$-QPSK and RRC pulse with $\alpha = 0.5$

   (e) Offset 4-QAM and RRC pulse with $\alpha = 0.1$

   (f) Offset 4-QAM and RRC pulse with $\alpha = 0.5$

   (g) 64-QAM and RRC pulse with $\alpha = 0.1$

   (h) QPSK and rectangular pulse

2. Sort the spectral efficiency of the following modulations when RRC pulse is employed:

   (a) $\pi/4$-QPSK and $\alpha = 0.1$

   (b) 16-QAM and $\alpha = 0.3$

   (c) 64-QAM and $\alpha = 1$

8.2.4 Matched Filter Receiver

1. Consider a single-user system in which the base station transmits the sequence $\{a[k]\}$ from the 4-QAM constellation to a mobile station. The wireless channel consists of a single line-of-sight path over which the Doppler shift of f_d occurs. The base station

uses rectangular pulses of duration T for transmission at carrier frequency f_C. The symbol duration is also T. The mobile station uses a matched filter to detect the transmitted symbol in each interval.

(a) Write the complex base-band signal received at the mobile station.

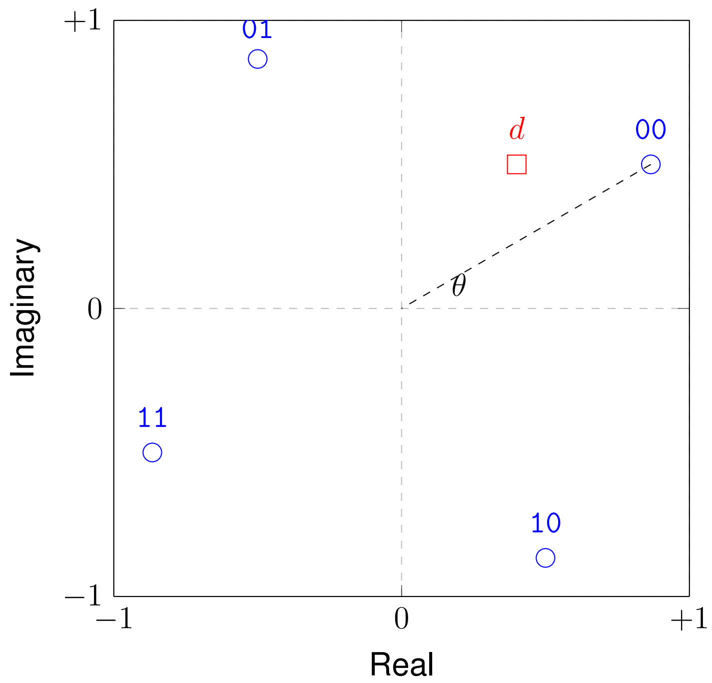
(b) Derive the signal-to-noise ratio (SNR) for the output of the matched filter in transmission interval k. Assuming that the noise power at the output of the matched filter is σ^2.

The commercial versions of GSM and LTE standards are employed at carrier frequency $f_C = 1900$ MHz and are required to support the maximum speed of 300 km/sec.

(c) Discuss the derivations in the previous part for the GSM and LTE standards.

8.2.5 Soft Output Matched Filter

1. A communication system uses a modulation scheme with the following constellation. The constellation is given by rotating the QPSK constellation with angle θ.



Assume that you receive a soft decision variable d shown in the figure.

(a) What is the hard decision of the information bits q_1 and q_2?

(b) Calculate the log likelihood ratio (LLR) for both the information bits q_1 and q_2.

8.3 Solutions to Exercises

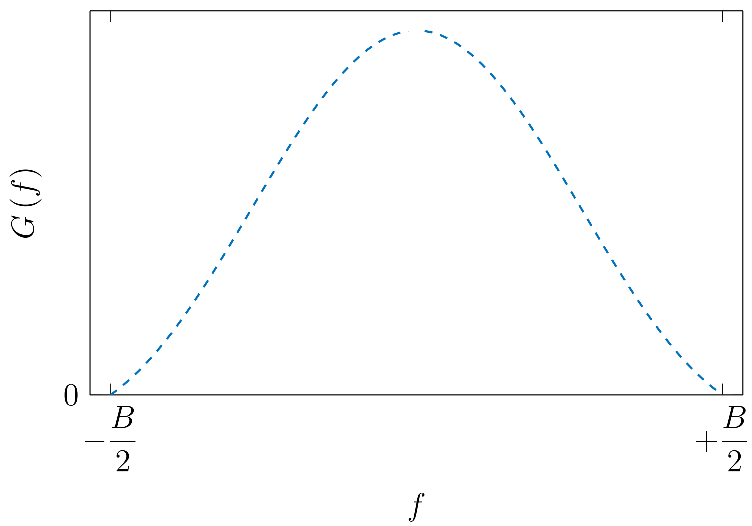
8.3.1 Linear Modulations

1. Consider a single-user system with no fading. We intend to transmit data stream

$$\mathrm{d} = \texttt{apple, banana}$$

in this system. The source encoder encodes `apple` to `0` and `banana` to `1`. The channel encoder then adds one bit with value `1` to the encoded data. This means that the codeword for `0` is `10`, and the codeword for `1` is `11`.

We transmit this stream using QPSK modulation with pulse shape $g(t)$ whose time duration is $T_{\mathrm{s}} = 5\mu\mathrm{sec}$. The spectrum of this pulse is given in the following figure:



The symbol duration is also $T_{\mathrm{s}} = 5\mu\mathrm{sec}$, and the carrier frequency is f_0.

(a) Determine the transmit symbols in this system.

(b) Write the base-band transmit signal and determine its spectrum. What is the bandwidth of this spectrum?

(c) Determine the radio-frequency signal in the time domain.

(d) Calculate the spectrum of the radio-frequency signal. Is the bandwidth different from the one calculated in Part (c)?

♠ **Solution:**

(a) The transmit data after *source encoding* is

$$d = 01$$

After the *channel encoding*, we have the bit stream

$$\{q[m]\} = 1011$$

Using the QPSK constellation, we can transmit two bits in each transmission. Hence, each two bits are mapped to a constellation point. Considering Figure 6.5 in page 192, we have

$$\begin{aligned} 10 &\mapsto j \\ 11 &\mapsto -1. \end{aligned}$$

Consequently, the transmit symbols are

$$\{a[k]\} = j, -1.$$

(b) The base-band transmit signal in the time domain is given by

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{+\infty} a[k]\, g(t - kT_{\mathrm{s}}) \\ &= jg(t) - g(t - T_{\mathrm{s}}). \end{aligned}$$

In the frequency domain, we use the property

$$\mathcal{F}\{g(t - T_{\mathrm{s}})\} = \exp\{-j2\pi f T_{\mathrm{s}}\}\, G(f),$$

we have

$$\begin{aligned} S(f) &= \mathcal{F}\{s(t)\} \\ &= jG(f) - \exp\{-j2\pi f T_{\mathrm{s}}\}\, G(f) \\ &= (j - \exp\{-j2\pi f T_{\mathrm{s}}\})\, G(f). \end{aligned}$$

Noting that for $|f| > B/2$, we have $G(f) = 0$, we conclude that the bandwidth of $S(f)$ is B.

> ⇝ REMARK:
>
> The bandwidth of the base-band transmit signal equals to the *pulse bandwidth*.

(c) The radio-frequency signal reads

$$s_{\mathrm{RF}}(t) = \text{Real}\{s(t)\exp\{j2\pi f_0 t\}\}.$$

Noting that $j = \exp\{j0.5\pi\}$, we conclude that

$$\begin{aligned} s_{\mathrm{RF}}(t) &= \text{Real}\{g(t)\exp\{j0.5\pi + j2\pi f_0 t\}\} - \text{Real}\{g(t - T_{\mathrm{s}})\exp\{j2\pi f_0 t\}\} \\ &= g(t)\,\text{Real}\{\exp\{j0.5\pi + j2\pi f_0 t\}\} - g(t - T_{\mathrm{s}})\,\text{Real}\{\exp\{j2\pi f_0 t\}\} \\ &= g(t)\sin(2\pi f_0 t) - g(t - T_{\mathrm{s}})\cos(2\pi f_0 t) \end{aligned}$$

(d) The spectrum of $s_{\mathrm{RF}}(t)$ is calculates as

$$
\begin{aligned}
S_{\mathrm{RF}}(f) &= \mathcal{F}\{s_{\mathrm{RF}}(t)\} \\
&= \mathcal{F}\{g(t)\sin(2\pi f_0 t)\} - \mathcal{F}\{g(t-T_{\mathrm{s}})\cos(2\pi f_0 t)\}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\mathcal{F}\{g(t)\sin(2\pi f_0 t)\} &= \frac{-j}{2}\left(G(f-f_0) - G(f+f_0)\right) \\
\mathcal{F}\{g(t)\cos(2\pi f_0 t)\} &= \frac{1}{2}\left(G(f-f_0) + G(f+f_0)\right)
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathcal{F}\{g(t-T_{\mathrm{s}})\cos(2\pi f_0 (t-T_{\mathrm{s}}))\} &= \exp\{-j2\pi f T_{\mathrm{s}}\}\,\mathcal{F}\{g(t)\cos(2\pi f_0 t)\} \\
&= \frac{\exp\{-j2\pi f T_{\mathrm{s}}\}}{2}\left(G(f-f_0) + G(f+f_0)\right).
\end{aligned}
$$

Assuming that $f_0 T_{\mathrm{s}} = i$ for some integer i, we conclude that

$$
S_{\mathrm{RF}}(f) = A(f)\,G(f-f_0) + B(f)\,G(f+f_0)
$$

where

$$
\begin{aligned}
A(f) &= \frac{-j + \exp\{-j2\pi f T_{\mathrm{s}}\}}{2}, \\
B(f) &= \frac{j + \exp\{-j2\pi f T_{\mathrm{s}}\}}{2}.
\end{aligned}
$$

Similar to Part (b), we note that $|f - f_0| > B/2$, we have $G(f-f_0) = 0$. This concludes that the bandwidth of $SRF(f)$ on the right hand side of the frequency axis is B.

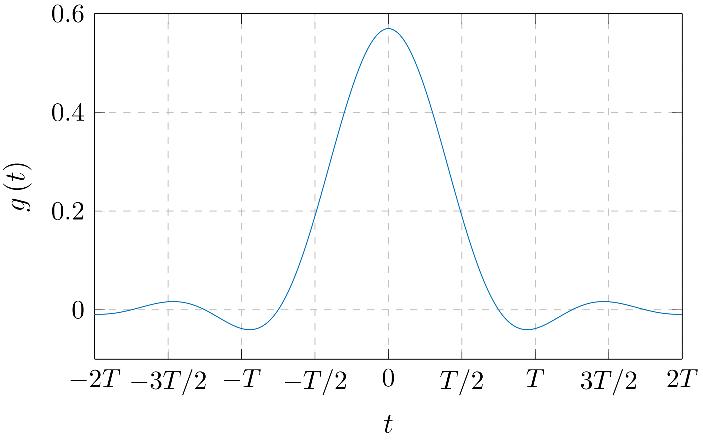
> ⇝ REMARK:
>
> The bandwidth of the radio-frequency transmit signal on the right hand side of the frequency axis equals to the *pulse bandwidth*.

8.3.2 Pulse Shaping

1. Consider a system which uses a root-raised-cosine (RRC) pulse shape with roll-off factor $\alpha = 1$. To limit the time duration, the pulse has been truncated within $[-2T, 2T]$. This means that

$$
g(t) = \begin{cases} g_1(t) & |t| \le 2T \\ 0 & |t| > 2T \end{cases}
$$

where $g_1(t)$ is RRC pulse with roll-off factor $\alpha = 1$; see the figure. Assume that the transmitter sends symbols $\{a[k]\} = \{\ldots, a[-1], a[0], a[1], \ldots\}$ over time which are taken from a known constellation, e.g., QPSK.



(a) Derive the value of the base-band signal $s(t)$ at $t = T/2$ in terms of the transmit symbols and pulse $g(t)$.

(b) Derive the maximum and minimum possible amplitude at $t = T/2$ when BPSK constellation is employed.

♠ **Solution:**

(a) The transmit signal is written as

$$s(t) = \sum_{n-\infty}^{+\infty} a[n]\; g(t - nT).$$

Therefore, the signal value at $t = T/2$ reads

$$s(\frac{T}{2}) = \sum_{n-\infty}^{+\infty} a[n]\; g(\frac{T}{2} - nT)$$
$$= \sum_{n=-\infty}^{+\infty} a[n]\; g(\frac{1-2n}{2}T).$$

By expanding the sum, we have

$$s(\frac{T}{2}) = \sum_{n=-\infty}^{-2} a[n]\; g(\frac{1-2n}{2}T)$$
$$+ a[-1]\; g(\frac{3T}{2}) + a[0]\; g(\frac{T}{2}) + a[1]\; g(-\frac{T}{2}) + a[2]\; g(-\frac{3T}{2})$$
$$+ \sum_{n=3}^{+\infty} a[n]\; g(\frac{1-2n}{2}T).$$

The summands in the red terms are all zero as the pulse is truncated within $[-2T, 2T]$, i.e.,

$$g(t) = 0 \qquad \text{for} \qquad |t| > 2T.$$

Moreover, due to the symmetry in the pulse shape,

$$g(\frac{3T}{2}) = g(-\frac{3T}{2}) \qquad \text{and} \qquad g(\frac{T}{2}) = g(-\frac{T}{2})$$

which concludes that

$$s(\frac{T}{2}) = (a[-1] + a[2])\, g(\frac{3T}{2}) + (a[0] + a[1])\, g(\frac{T}{2}).$$

(b) When we use BPSK constellation, $a[n] \in \{\pm 1\}$. In this case, $|s(\frac{T}{2})|$ is maximized when

$$a[-1] = a[2] = a[0] = a[1] = \pm 1.$$

and reads

$$|s(\frac{T}{2})|_{\max} = 2 \times |g(\frac{3T}{2}) + g(\frac{T}{2})| = 2 \times (0.2 + 0.016) = 0.432.$$

The minimum value of $|s(\frac{T}{2})|$ is moreover achieved when

$$a[-1] = -a[2] \qquad \text{and} \qquad a[0] = -a[1]$$

which reads $|s(\frac{T}{2})|_{\min} = 0$.

8.3.3 Comparison of Modulation Schemes

1. Sort the peak-to-average power ratio (PAPR) of the transmit signal when the following constellations and pulse shapes are employed:

   (a) QPSK and RRC pulse with $\alpha = 0.1$
   (b) QPSK and RRC pulse with $\alpha = 0.5$
   (c) $\pi/4$-QPSK and RRC pulse with $\alpha = 0.1$
   (d) $\pi/4$-QPSK and RRC pulse with $\alpha = 0.5$
   (e) Offset 4-QAM and RRC pulse with $\alpha = 0.1$
   (f) Offset 4-QAM and RRC pulse with $\alpha = 0.5$
   (g) 64-QAM and RRC pulse with $\alpha = 0.1$
   (h) QPSK and rectangular pulse

♠ **Solution:** Assume that PAPR_i denotes the PAPR for the constellation in item i = a, . . ., h. From Figure 6.12 in page 196, we observe that

$$\mathrm{PAPR_a} > \mathrm{PAPR_c} > \mathrm{PAPR_e} > \mathrm{PAPR_b} > \mathrm{PAPR_d} > \mathrm{PAPR_f}.$$

Noting that for constellations in (g) and (a) same pulse shape has been employed, one could write

$$\mathrm{PAPR_g} > \mathrm{PAPR_a},$$

since 64-QAM has more amplitude variations than QPSK.

Moreover, we know that QPSK with rectangular pulse has *no amplitude variation*. This means that $\mathrm{PAPR_h} = 1$ which is the minimum possible PAPR. Hence, we could conclude

$$\mathrm{PAPR_g} > \mathrm{PAPR_a} > \mathrm{PAPR_c} > \mathrm{PAPR_e} > \mathrm{PAPR_b} > \mathrm{PAPR_d} > \mathrm{PAPR_f}. > \mathrm{PAPR_h}.$$

2. Sort the spectral efficiency of the following modulations when RRC pulse is employed:
   (a) $\pi/4$-QPSK and $\alpha = 0.1$
   (b) 16-QAM and $\alpha = 0.3$
   (c) 64-QAM and $\alpha = 1$

♠ **Solution:** From equation (6.1) in page 185, we know that the spectral efficiency is defined by

$$\eta = \frac{R_\mathrm{b}}{B}$$

where R_b is the bit rate, and B is the band width equipped by the modulation. Assuming that the modulation has M constellation points and a signal duration of length T, the bit rate reads

$$R_\mathrm{b} = \frac{\text{\# of bits for one constellation point}}{\text{Pulse duration}} = \frac{\log_2 M}{T}.$$

To find the bandwidth B occupied with a single pulse, we note that a RRC pulse is used. Therefore, we use the spectrum in equation (6.15), page 190, which indicates that the bandwidth of a RRC pulse with roll-off factor α is

$$B = \frac{1+\alpha}{T}.$$

Therefore, the spectral efficiency for a given modulation with a RRC pulse reads

$$\eta = \frac{R_\mathrm{b}}{B} = \frac{\log_2 M}{1+\alpha}.$$

Using η_1, η_2 and η_3 to denote the spectral efficiency of the given schemes, we have

$$\eta_\mathrm{a} = \frac{\log_2 4}{1.1} = 1.82$$
$$\eta_\mathrm{b} = \frac{\log_2 16}{1.3} = 3.08$$
$$\eta_\mathrm{c} = \frac{\log_2 64}{2} = 3$$

which concludes that

$$\eta_{\mathrm{b}} > \eta_{\mathrm{c}} > \eta_{\mathrm{a}}.$$

8.3.4 Matched Filter Receiver

1. Consider a single-user system in which the base station transmits the sequence $\{a[k]\}$ from the 4-QAM constellation to a mobile station. The wireless channel consists of a single line-of-sight path over which the Doppler shift of f_{d} occurs. The base station uses rectangular pulses of duration T for transmission at carrier frequency f_{C}. The symbol duration is also T. The mobile station uses a matched filter to detect the transmitted symbol in each interval.

   (a) Write the complex base-band signal received at the mobile station.

   (b) Derive the signal-to-noise ratio (SNR) for the output of the matched filter in transmission interval k, assuming that the noise power at the output of the matched filter is σ^2.

   The commercial versions of GSM and LTE standards are employed at carrier frequency $f_{\mathrm{C}} = 1900$ MHz and are required to support the maximum speed of 300 km/sec.

   (c) Discuss the derivations in the previous part for the GSM and LTE standards.

♠ **Solution:**

(a) Considering the data sequence $\{a[k]\}$, the transmitted signal in the base-band reads

$$s(t) = \sum_{m=-\infty}^{+\infty} a[m]\, g(t - mT)$$

where $g(t)$ is a rectangular pulse with duration T, i.e.,

$$g(t) = \begin{cases} 1 & t \in [0, T] \\ 0 & \text{otherwise} \end{cases}.$$

Note that there exist only one path with no delay and Doppler shift f_{d}. From Exercise 1 in Section 4.1 of Tutorial 4, we know that this channel is described in the base-band with

$$h(t) = \exp\{-j2\pi f_{\mathrm{d}} t\}.$$

Hence, the signal received by the mobile station in the base-band is given by

$$y(t) = h(t)\, s(t) + z(t) = \sum_{m=-\infty}^{+\infty} a[m]\, h(t)\, g(t - mT) + z(t)$$

where $z(t)$ is the base-band noise.

(b) The matched filter receiver for this system in the base-band has the impulse response[1]

$$u(t) = \frac{1}{T} g^*(-t) \overset{(\mathrm{A})}{=} \frac{1}{T} g(-t)$$

where (A) follows the fact that $g(t)$ is real. For k-th transmission interval, the output of the matched filter is sampled at $t_k = kT$; thus, the output symbol at interval k, $d[k]$, reads

$$\begin{aligned}
d[k] &= y(t) * u(t)\,|_{t=kT} \\
&= \int_{-\infty}^{+\infty} y(\tau)\, u(t-\tau)\, \mathrm{d}\tau|_{t=kT} \\
&= \frac{1}{T}\int_{-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} a[m]\; h(\tau) g(\tau - mT) + z(\tau)\right] g(\tau - t)\, \mathrm{d}\tau|_{t=kT} \\
&= \sum_{m=-\infty}^{+\infty} a[m] \left(\frac{1}{T}\int_{-\infty}^{+\infty} h(\tau)\, g(\tau - mT)\, g(\tau - kT)\, \mathrm{d}\tau\right) \\
&\qquad + \underbrace{\int_{-\infty}^{+\infty} z(\tau)\, g(\tau - kT)\, \mathrm{d}\tau}_{z[k]} \\
&= \underbrace{\sum_{m=-\infty}^{+\infty} a[m] \left(\frac{1}{T}\int_{-\infty}^{+\infty} h(\tau)\, g(\tau - mT)\, g(\tau - kT)\, \mathrm{d}\tau\right)}_{r[k]} + z[k] \\
&= r[k] + z[k]
\end{aligned}$$

Here, we have

- $z[k]$ is output noise at transmission interval k. Noting that

$$g(t - kT) = \begin{cases} 1 & kT \le t \le (k+1)T \\ 0 & \text{otherwise} \end{cases},$$

  we can write

$$z[k] = \int_{-\infty}^{+\infty} z(\tau)\, g(\tau - kT)\, \mathrm{d}\tau = \int_{kT}^{(k+1)T} z(\tau)\, \mathrm{d}\tau.$$

  $z[k]$ is hence a zero-mean Gaussian random variable. As indicated in the question, we have

$$\mathsf{E}\left\{|z[k]|^2\right\} = \sigma^2.$$

[1]We add the factor $1/T$ to normalize the pulse power to 1, i.e., $\int u(t)\, g(t) = 1$. This guarantees that the signal processing tasks do not change the signal power.

- The term $r[k]$ is moreover the recovered symbol. In order to determine $r[k]$, we need to calculate the black term first. In this respect, we note that

$$g(t-mT)\,g(t-kT) = \delta_k[m] g(t-kT) = \begin{cases} g(t-kT) & m=k \\ 0 & \text{otherwise} \end{cases}$$

where we have defined the sequence $\delta_k[m]$ as

$$\delta_k[m] = \begin{cases} 1 & m=k \\ 0 & \text{otherwise} \end{cases}.$$

By replacing in the main equation, we have

$$\begin{aligned} \frac{1}{T}\int_{-\infty}^{+\infty} h(\tau) g(t-mT)\,g(t-nT)\,\mathrm{d}\tau &= \delta_k[m]\frac{1}{T}\int_{-\infty}^{+\infty} h(\tau) g(t-kT)\,\mathrm{d}\tau \\ &= \delta_k[m]\frac{1}{T}\int_{kT}^{(k+1)T} h(\tau)\,\mathrm{d}\tau \\ &= \delta_k[m]\frac{\exp\{-j2\pi f_\mathrm{d} kT\} - \exp\{-j2\pi f_\mathrm{d}(k+1)T\}}{j2\pi f_\mathrm{d} T} \\ &= \delta_k[m]\exp\{-j2\pi f_\mathrm{d}(k+0.5)T\} \\ &\quad \times \left[\frac{\exp\{j\pi f_\mathrm{d} T\} - \exp\{-j\pi f_\mathrm{d} T\}}{j2\pi f_\mathrm{d} T}\right]. \end{aligned}$$

Noting that $\exp\{j\theta\} - \exp\{-j\theta\} = 2j\sin\theta$, we finally conclude that

$$\begin{aligned} \frac{1}{T}\int_{-\infty}^{+\infty} h(\tau) g(\tau-mT)\,g(\tau-kT)\,\mathrm{d}\tau &= \delta_k[m]\frac{\sin \pi f_\mathrm{d} T}{\pi f_\mathrm{d}}\exp\{-j2\pi f_\mathrm{d}(k+0.5)T\} \\ &= \delta_k[m]\,\mathrm{sinc}(\pi f_\mathrm{d} T)\exp\{-j2\pi f_\mathrm{d}(k+0.5)T\}. \end{aligned}$$

By defining

$$c[k] := \mathrm{sinc}(\pi f_\mathrm{d} T)\exp\{-j2\pi f_\mathrm{d}(k+0.5)T\},$$

we have

$$\boxed{r[k] = \sum_{m=-\infty}^{+\infty} a[m]\delta_k[m] c[k] = a[k]\,c[k]}$$

As a result, the received sample $d[k]$ reads

$$d[k] = a[k]\,c[k] + z[k].$$

This concludes that the SNR at the output of the matched filter in transmission interval k is

$$\mathrm{SNR} = \frac{\mathsf{E}\{|r[k]|^2\}}{\mathsf{E}\{|z[k]|^2\}} = \frac{|c[k]|^2\mathsf{E}\{|a[k]|^2\}}{\sigma^2}$$

Noting that $a[k]$ is taken from the 4-QAM constellation, we have $|a[k]|^2 = 2$ which concludes that

$$\begin{aligned}\mathrm{SNR} &= \frac{2}{\sigma^2}|c[k]|^2 = \frac{2}{\sigma^2}\mathrm{sinc}^2(\pi f_\mathrm{d} T)\overbrace{|\exp\{-j2\pi f_\mathrm{d}(k+0.5)T\}|^2}^{1}\\ &= \mathrm{SNR}_0 \mathrm{sinc}^2(\pi f_\mathrm{d} T)\end{aligned}$$

where

$$\mathrm{SNR}_0 := \frac{2}{\sigma^2}$$

represents the SNR when the channel is not experiencing any fading.

(c) Note that

$$\mathrm{sinc}(\pi f_\mathrm{d} T) \leq 1$$

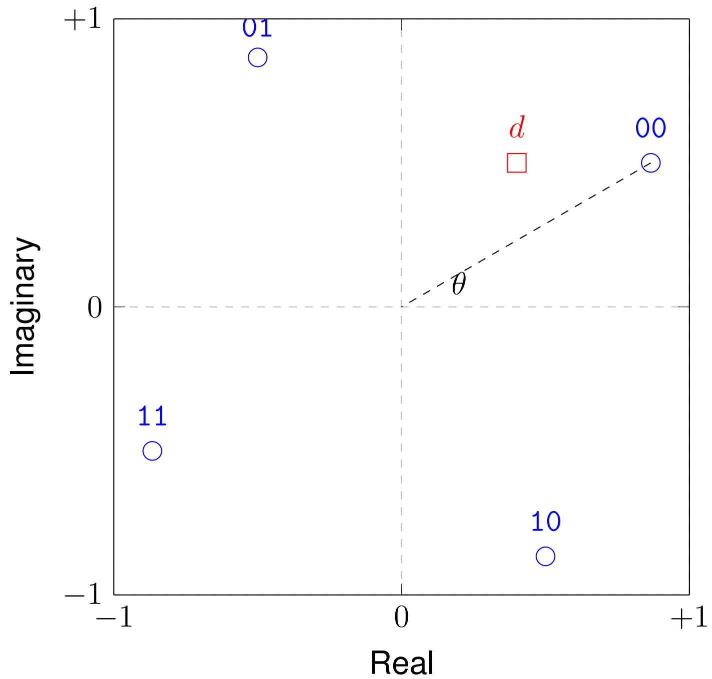
and it becomes 1 if and only if $f_\mathrm{d} = 0$. Therefore, the Doppler shift reduces the SNR at the receive side. As a result, for reasonable performance, we need in practice

$$|f_\mathrm{d} T| \ll 1.$$

This value for the standards GSM and LTE is $f_\mathrm{d} T = 0.002$ and $f_\mathrm{d} T = 0.035$ respectively which fulfills the constraint.

8.3.5 Soft Output Matched Filter

1. A communication system uses a modulation scheme with the following constellation. The constellation is given by rotating the QPSK constellation with angle θ.



Assume that you receive a soft decision variable d shown in the figure.

(a) What is the hard decision of the information bits q_1 and q_2?

(b) Calculate the log likelihood ratio (LLR) for both the information bits q_1 and q_2.

♠ **Solution:**

(a) Since the *shortest* distance of d is to the constellation point corresponding to 00, the hard decision is

$$q_1 = 0, \qquad \text{and} \qquad q_2 = 0.$$

(b) To calculate LLR, we note that by rotating the constellation with

$$\alpha = \frac{\pi}{4} - \theta$$

the given constellation lies on the 4-QAM constellation. This means that receiving d with the given constellation is equivalent to receiving

$$\hat{d} = d \exp\{j\alpha\}$$

with the 4-QAM constellation. As a result, we can use equations (6.88) and (6.89) in page 223 and write

$$\mathrm{LLR}(q_1) = 4\frac{\sqrt{E_\mathrm{b}/T}}{\sigma^2}\text{Real}\left\{\hat{d}\right\} = 4\frac{\sqrt{E_\mathrm{b}/T}}{\sigma^2}\text{Real}\left\{d \exp\{j\alpha\}\right\}$$
$$\mathrm{LLR}(q_2) = 4\frac{\sqrt{E_\mathrm{b}/T}}{\sigma^2}\text{Imag}\left\{\hat{d}\right\} = 4\frac{\sqrt{E_\mathrm{b}/T}}{\sigma^2}\text{Imag}\left\{d \exp\{j\alpha\}\right\}$$

where E_b/T is the power of the transmit signal per bit and σ^2 is the noise variance.

8.4 Homework

1. Consider a frequency-flat Rayleigh fading channel with 4 branches of diversity. Assume system uses root-raised-cosine pulses with roll-off factor α. Sort the following modulation schemes in terms of noise resistance at large signal-to-noise ratios.

   (a) $3\pi/8$-8PSK and $\alpha = 0.1$

   (b) 16-QAM and $\alpha = 0.3$

   (c) 256-QAM and $\alpha = 1$

   (d) DPSK with $\alpha = 0.8$

♠ **Solution:** The answer to this question is given by Figure 6.29 in page 221 of the lecture notes. The pulse shape and phase shift does not play any rule in noise resistance. Moreover, as 256-QAM has more constellation points than 64-QAM, it has larger bit-error-rate for a given signal-to-noise ratio. Thus,

$$(4) \succ (1) \succ (2) \succ (3).$$

Chapter 9

Channel Coding

This Chapter reviews on the fundamentals of channel coding which is the topic of Chapter 7 in the lecture-notes. The discussions in this chapter are very overall. More details shall be followed in the Channel Coding lecture.

9.1 Brief Review of Main Concepts

A general categorization of coding methods splits the techniques into two major groups:

- **Block Coding:** In these techniques, the codeword in each transmission block is determined from the information-word of that particular block. A well-known example is the linear block coding described by a generator matrix in which each codeword is simply determined by multiplying the information-word with the generator matrix.
- **Convolutional Coding:** These techniques generate a codeword, from the information word of the current transmission interval and those of previous intervals. As the result, these codes have memory which is specified by the *constraint length*. The well-known example of these codes is the turbo code.

The performance of coding schemes is often characterized by their *free distance* and measured by the coding gain which is defined to be the difference in signal-to-noise ratio (SNR) in dB by which we achieve a specific (usually very low) error rate with and without coding. For very low error rate this gain is always positive, indicating that coding is improving the performance. This is however only true for static channels, i.e., channels without fading. For channels with fading, we further need interleaving to make the channel coding effective. We study these points further in through exercises.

9.2 Exercises

9.2.1 Basics of Channel Coding

1. In the transmission interval t, a channel code encodes the binary information vector

$$\mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_K(t) \end{bmatrix}$$

to the codeword

$$\mathbf{c}(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_{2K}(t) \end{bmatrix}.$$

   (a) Determine the coding rate R_c.

   (b) Assume that the encoding procedure is as follows

$$\mathbf{c}(t) = \begin{bmatrix} \mathbf{b}(t) \\ \mathbf{b}(t) \oplus \mathbf{b}(t-1) \oplus \mathbf{b}(t-2) \end{bmatrix}.$$

   Classify this channel coding scheme, i.e., specify if it is a *block* or *convolutional* code.

   (c) Repeat the last part assuming that the following encoding scheme is employed

$$\mathbf{c}(t) = \mathbf{H}\, \mathbf{b}(t).$$

2. Consider a block code with K information bits and codewords of length N. For this code,

   (a) Determine the total number of codewords.

   (b) Assume that this code can correct T or less bits of error. Find a *necessary* condition on K, N and T.

   (c) Check this constraint for $K = 4$, $N = 7$ and $T = 1$.

   (d) Assume that N is odd. Determine the maximum possible coding rate R_c when $T = (N-1)/2$.

   Hint: Note that for odd N, we have

$$\sum_{n=0}^{(N-1)/2} \binom{N}{n} = 2^{N-1}.$$

3. Consider the repetition code whose coding rate is R_c and classify it as a convolutional code. Determine its constraint length L_c and the free distance d_{free}.

9.2.2 Performance of Coded Transmission

1. Consider BPSK transmission via a convolutional code of rate $R_c = 0.5$ and constraint length of $L_c = 7$ over a *static* additive white Gaussian noise channel.
   (a) Calculate the *maximum* possible coding gain.
   (b) Determine the coding gain at the bit error rate of $\mathrm{BER} = 10^{-5}$ for this scheme assuming that the decoder uses soft decisions.
   (c) Determine the bit error rate at which the coding gain is negative. What does it mean?
   (d) Compare the maximum coding gain in Part (a) to the gain achieved via the repetition code with the same coding rate.

2. Consider the coding scheme in the previous exercise and assume that BPSK transmission is performed over a *narrow-band fading* channel with additive white Gaussian noise.
   (a) Which component is crucial to be added at the transmitter?
   (b) Determine the maximum diversity gain of this coding scheme and compare it to the repetition code with $R_c = 0.5$.

   Hint: In the lecture slides, it has been indicated that under ideal interleaving

$$g_{\mathrm{d}} \approx 2R_c d_{\mathrm{free}}.$$

9.3 Solutions to Exercises

9.3.1 Basics of Channel Coding

1. In the transmission interval t, a channel code encodes the binary information vector

$$\mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_K(t) \end{bmatrix}$$

   to the codeword

$$\mathbf{c}(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_{2K}(t) \end{bmatrix}.$$

   (a) Determine the coding rate R_c.
   (b) Assume that the encoding procedure is as follows

$$\mathbf{c}(t) = \begin{bmatrix} \mathbf{b}(t) \\ \mathbf{b}(t) \oplus \mathbf{b}(t-1) \oplus \mathbf{b}(t-2) \end{bmatrix}.$$

   Classify this channel coding scheme, i.e., specify if it is a *block* or *convolutional* code.

(c) Repeat the last part assuming that the following encoding scheme is employed

$$\mathbf{c}(t) = \mathbf{H}\,\mathbf{b}(t).$$

♠ **Solution:**

(a) The given code encodes K information bits into codewords of length $N = 2K$. Hence, the coding rate reads

$$R_{\mathrm{c}} = \frac{K}{N} = \frac{K}{2K} = \frac{1}{2}.$$

(b) Considering the discussion in pages 236 and 237, the channel code in Part (b) is a *convolutional code*, since the codeword in time slot t depends on the information bits in the interval t, $t-1$ and $t-2$.

(c) The coding in Part (c) is a *linear block coding*, as it linearly encodes the messages in each time interval into codewords of the same interval with a fixed linear transform $\mathbf{H}$.

2. Consider a block code with K information bits and codewords of length N. For this code,

   (a) Determine the total number of codewords.

   (b) Assume that this code can correct T or less bits of error. Find a *necessary* condition on K, N and T.

   (c) Check this constraint for $K = 4$, $N = 7$ and $T = 1$.

   (d) Assume that N is odd. Determine the maximum possible coding rate R_c when $T = (N-1)/2$.

   Hint: Note that for odd N, we have

$$\sum_{n=0}^{(N-1)/2} \binom{N}{n} = 2^{N-1}.$$

♠ **Solution:**

(a) Since we have binary information vectors of length K, we have in general 2^K different information vectors. Each of these vectors is mapped by the encoder into a single codeword. Therefore, there exist in total 2^K codewords in the codebook.

(b) As the codewords are of length N, we have in general 2^N choices for codewords among which we only use 2^K of them. This means that we have $2^N - 2^K$ vectors in the N-dimensional space which are *unused* by the encoder. To make sure that we can correct T bits at the receiver, we should guarantee that all the possible vectors which differ from all the codewords in the codebook in T or less bits are not included in the codebook and are one of the *unused* N-dimensional vectors.

To formulate this argument, consider the i-th codeword $\mathbf{c}_i$ in the codebook, where $i = 1, \ldots, 2^K$. The number of N-dimensional vectors which differ from $\mathbf{c}_i$ in j bits is

$$E_j(\mathbf{c}_i) = \binom{N}{j}$$

Therefore, the total number of codewords which differ from $\mathbf{c}_i$ in T or less bits reads

$$E(\mathbf{c}_i) = \sum_{j=1}^{T} E_j(\mathbf{c}_i) = \sum_{j=1}^{T} \binom{N}{j}.$$

To guarantee that these vectors are *not* included in the codebook, we need to make sure that sum of all $E(\mathbf{c}_i)$ over i is less than the number of *unused* N-dimensional vectors, i.e.,

$$\sum_{i=1}^{2^K} E(\mathbf{c}_i) \leq 2^N - 2^K \Rightarrow 2^K E(\mathbf{c}_i) \leq 2^N - 2^K$$

which concludes that

$$2^K \left[1 + \sum_{j=1}^{T} \binom{N}{j} \right] \leq 2^N.$$

Noting that $1 = \binom{N}{0}$, the necessary condition reads

$$\boxed{\sum_{j=0}^{T} \binom{N}{j} \leq 2^{N-K}.}$$

(c) By setting $T = 1$, $K = 4$ and $N = 7$, we have

$$\binom{7}{0} + \binom{7}{1} \leq 2^3 \Rightarrow 7 + 1 \leq 8$$

which holds, and thus, there exists a channel code with $K = 4$ and $N = 7$ which can correct one bit of error. An example of such code is Hamming $(4, 7)$ code[1].

(d) When N is odd and $T = (N-1)/2$, we have

$$\sum_{j=0}^{(N-1)/2} \binom{N}{j} \leq 2^{N-K}$$

[1]Google it, if you do not know it.

which by using the hint in the question, reduces to

$$2^{N-1} \leq 2^{N-K}.$$

This concludes that $K \leq 1$. On the other hand, we know that K is integer and positive meaning that $K \geq 1$. Thus, we conclude that $K = 1$. This concludes that the coding rate in this case is

$$R_{\mathrm{c}} = \frac{1}{N}.$$

3. Consider the repetition code whose coding rate is R_c and classify it as a convolutional code. Determine its constraint length L_c and the free distance d_{free}.

♠ **Solution:** A repetition code of order M, repeats each information bit M times. This means that

$$R_c = \frac{1}{M}$$

This code can be considered as a block code or a convolutional code with constraint length

$$L_c = 1.$$

To find the free distance of this code, we consider the definition of d_{free} in equation (7.19) page 248. This definition indicates that the free distance is the minimum distance between two *different* codewords in the codebook. To find this value, consider to information vectors $\mathbf{b}$ and $\mathbf{b}'$ which differ in $D \leq K$ bit. As the corresponding codewords $\mathbf{c}$ and $\mathbf{c}'$ are constructed by repeating the bits in $\mathbf{b}$ and $\mathbf{b}'$ for M times, we can conclude that $\mathbf{c}$ and $\mathbf{c}'$ are different in $D \times M$ bits in this case, i.e.,

$$d_{\mathrm{H}}\left(\mathbf{c}, \mathbf{c}'\right) = DM.$$

Note that any two different information vectors are at least different in one bit. Hence, $D_{\min} = 1$ and

$$d_{\text{free}} = \min_{\mathbf{c} \neq \mathbf{c}'} d_{\mathrm{H}}\left(\mathbf{c}, \mathbf{c}'\right) = M.$$

9.3.2 Performance of Coded Transmission

1. Consider BPSK transmission via a convolutional code of rate $R_c = 0.5$ and constraint length of $L_c = 7$ over a *static* additive white Gaussian noise channel.

   (a) Calculate the *maximum* possible coding gain.

(b) Determine the coding gain at the bit error rate of $\mathrm{BER} = 10^{-5}$ for this scheme assuming that the decoder uses soft decisions.

(c) Determine the bit error rate at which the coding gain is negative. What does it mean?

(d) Compare the maximum coding gain in Part (a) to the gain achieved via the repetition code with the same coding rate.

♠ **Solution:**

(a) The maximum possible coding gain for a convolutional code is given in equation (7.23) page 251 and reads

$$G_{c,\max} = 10 \log R_c d_{\text{free}}.$$

From Table 7.1 in page 249, we observe that for $L_c = 7$ at rate $R_c = 1/2$, the free distance is

$$d_{\text{free}} = 10.$$

Therefore, the *maximum* coding gain is

$$G_{c,\max} = 10 \log R_c d_{\text{free}} = 10 \log 5 = 7 \text{ dB}.$$

(b) To find the exact coding gain achieved with this code via soft decision making at the receiver, we use Figure 7.12 in page 252. From the figure, we see that for $\mathrm{BER} = 10^{-}5$,

- the *uncoded* transmission needs $\mathrm{SNR} = 9.6$ dB SNR, and
- the given coding scheme requires $\mathrm{SNR} = 4.6$ dB.

This means that

$$G_c = 9.6 - 4.6 = 5 \text{ dB}$$

which is less than the upper bound $G_{c,\max} = 7$ dB.

(c) In Figure 7.12, we observe that as the SNR converges to 0 dB, the bit error rate under coded transmission becomes larger than the error rate achieved by uncoded BPSK. For soft decision making with $R_c = 1/2$ and $L_c = 7$ convolutional codes, this phenomenon occurs around $\mathrm{BER} = 10^{-}1$ which is a very large error rate and is not the case in practice.

> ⇝ REMARK:
>
> For large error rates, channel coding does not gain anything, and in fact can degrade the performance by the extra redundant bits.

(d) For a repetition code with rate $R_c = 1/2$, $M = 2$ and thus, we have $d_{\text{free}} = M = 2$. Consequently, the *maximum* coding gain reads

$$G_{c,\max} = 10 \log R_c d_{\text{free}} = 10 \log 1 = 0 \text{ dB}.$$

This means that the coding performs in terms of bit error rate similar to uncoded transmission. This is due to the fact that repetition code only gives *diversity* gain which in *static* channels is zero.

2. Consider the coding scheme in the previous exercise and assume that BPSK transmission is performed over a *narrow-band fading* channel with additive white Gaussian noise.

   (a) Which component is crucial to be added at the transmitter?

   (b) Determine the maximum diversity gain of this coding scheme and compare it to the repetition code with $R_c = 0.5$.

   Hint: In the lecture slides, it has been indicated that under ideal interleaving

$$g_{\text{d}} \approx 2R_c d_{\text{free}}.$$

♠ **Solution:**

(a) In fading channels, the *interleaver* is the crucial components. As it has been shown in the lecture, without interleaving channel coding cannot give any gain; see discussions in Section 7.2.4 of the lecture notes.

(b) Using the hint in the question, for the convolutional code with rate $R_c = 1/2$, constraint distance $L_c = 7$, and free distance $d_{\text{free}} = 10$, the maximum diversity gain under ideal interleaving reads

$$g_{\text{d}} \approx 2R_c d_{\text{free}} = 10.$$

For a repetition code of rate $R_c = 1/2$, we have $M = 2$ or equivalently, $d_{\text{free}} = 2$, and thus,

$$g_{\text{d}} \approx 2R_c d_{\text{free}} = 2.$$

which is only the diversity gain and is significantly smaller than 10.

Chapter 10

The GSM-System

In this chapter, we practice some exercises on the GSM system. This tutorial covers Chapter 8 of the lecture-notes.

10.1 Brief Review of Main Concepts

For a long time, GSM has been a basic standard for mobile communications. It has been further supported by mobile networks which operate with other standards, e.g., 3G and LTE. In this chapter, we study some basic designs of GSM. Given what we learned in the previous chapters, we are now capable of understanding the detailed design parameters. This chapter further discusses the fundamental challenges in design and describes the basic solutions proposed in GSM.

Clearly, next generations of mobile networks, i.e., 3G. LTE and nowadays 5G, are considering different designs and techniques. Nevertheless, the principles are the same, and hence being capable of dealing with GSM confirms our capability of going through other newer standards.

In this chapter, we go through few exercises to learn some fundamentals and practice what we learned earlier within GSM.

10.2 Exercises

10.2.1 Limitations on the GSM System

1. Consider a standard variation of the GSM system.
   (a) Name the parameter which quantifies the maximum number of users supported in this system.
   (b) Sort GSM-900, DCS-1800 and PCS-1900 in terms of the maximum number of users.
2. Consider a standard variation of the GSM system while keeping the following fact in mind:

In the first access to the system, in which users log on the system, *timing advance* is not employed. After the log on, the system uses timing advance to avoid collision.

(a) The maximum cell radius in the system is $R_{\max} = 35$ km. Name the factor which restricts the maximum possible radius of this system.

Assuming that the switches in the system are not ideal and have the on-off time $T_{\text{on}-\text{off}}$, the guard interval when we employ timing advance reads

$$T_{\text{G,TA}} \approx \Delta\tau_{\max} + T_{\text{on}-\text{off}}.$$

where $\Delta\tau_{\max}$ is the difference between the maximum and minimum propagation delays in the system.

In the absence of timing advance, the maximum propagation delay $\tau_{\max}$ should be also included into the guard interval to avoid collision. Thus, the guard interval in this case is

$$T_{\text{G},0} \approx 2\tau_{\max} + \Delta\tau_{\max} + T_{\text{on}-\text{off}}.$$

(b) With the above fact in mind and the answer to Part (a), show that the maximum cell radius in a standard GSM system is $R_{\max} = 35$ km.

3. The total modulation rate of a communication system is described by the number of bits transmitted, end-to-end, in the system per unit of time. In the GSM system, this rate in *full-rate traffic channels* is

$$R_{\text{GSM}} = 270.83 \text{ kbits/sec.}$$

In this exercise, we intend to use the structure of the GSM system and derive this rate.

(a) What is the data rate in the GSM system after encoding the source data, i.e., speech encoding?

(b) What is approximately the channel coding rate in the GSM system?

(c) Which channel is used for transmitting user data?

(d) Describe the data frame structure for a full-rate traffic channel.

(e) Calculate the data rate after construction of a *multiframe* in a full-rate traffic channel.

In the GSM system, 8 different TDM-frames are multiplexed in a given transmission interval. Keeping this in mind, answer further the following items:

(f) Determine the factor with which the data rate increase after multiplexing multiuser TDM-frames.

(g) Calculate the factor with which the data rate increases after formatting the date via normal burst.

(h) Compare the derived modulation rate with R_{GSM}.

10.2.2 Characteristics of GSM Systems

1. The theoretical limit given by the *channel coding theorem* of Shannon is derived by using the capacity expression for additive white Gaussian noise (AWGN) channels, i.e.,

$$C = \log_2(1 + \mathrm{SNR})\ \text{bits/sec/Hz}$$

where SNR is the signal-to-noise ratio (SNR) at the receive side. For a standard variation of the GSM system, we have the following information:

The average receive power in a *typical* GSM channel is $P = -90$ dBm and the noise level at the receive side is $N = -121$ dBm. In the case of perfect multiplexing, the multiuser interference is negligible.

   (a) Calculate the channel capacity for a typical GSM channel.

   (b) Derive the spectral efficiency of a standard GSM system and compare it to the channel capacity.

2. Consider the following approaches for transmission

   A_1 Coded transmission with hard decoding,

   A_2 Coded transmission with burst-wise soft decoding,

   A_3 Coded transmission with bit-wise soft decoding,

   in a standard traffic channel (TCH) of the GSM system. Assume that the GSM signal experiences *narrow-band* fading and *ideal frequency hopping* is employed.

   (a) Calculate the coding gain achieved by these approaches at the bit-error-rate 10^{-2}.

   (b) Calculate the diversity gain of these approaches.

   (c) Name the crucial module at the transmitter which makes these gains achievable.

10.3 Solutions to Exercises

10.3.1 Limitations on the GSM System

1. Consider a standard variation of the GSM system.

   (a) Name the parameter which quantifies the maximum number of users supported in this system.

(b) Sort GSM-900, DCS-1800 and PCS-1900 in terms of the maximum number of users.

♠ **Solution:**

(a) From Table 8.1 in page 262 of the lecture notes, as well as the discussion in Section 8.2, we know that the number of time slots in TDM frames of the GSM system *remains unchanged* in different standards.

Consequently, the maximum number of users being supported in these systems is quantified by the *number of FDM sub-channels*. When carrier separation is fixed (as is the case in Table 8.1), the number of FDM sub-channels is linearly proportional to the *bandwidth*.

The maximum number of users supported in this system is linearly proportional to the *bandwidth*.

(b) For GSM-900, DCS-1800 and PCS-1900, we use the values in Table 8.1 which indicate

- GSM-900: Downlink bandwidth = 25 MHz, and # of FDM sub-channels = 124
- DCS-1800: Downlink bandwidth = 75 MHz, and # of FDM sub-channels in = 374
- PCS-1900: Downlink bandwidth = 60 MHz, and # of FDM sub-channels in = 299

Therefore, in terms of the maximum number of supported users, we have

users in DCS-1800 $>$ # users in PCS-1900 $>$ # users in GSM-900.

2. Consider a standard variation of the GSM system while keeping the following fact in mind:

In the first access to the system, in which users log on the system, *timing advance* is not employed. After the log on, the system uses timing advance to avoid collision.

(a) The maximum cell radius in the system is $R_{\max} = 35$ km. Name the factor which restricts the maximum possible radius of this system.

Assuming that the switches in the system are not ideal and have the on-off time $T_{\text{on}-\text{off}}$, the guard interval when we employ timing advance reads

$$T_{\text{G,TA}} \approx \Delta\tau_{\max} + T_{\text{on}-\text{off}}.$$

where $\Delta\tau_{\max}$ is the difference between the maximum and minimum propagation delays in the system.

In the absence of timing advance, the maximum propagation delay $\tau_{\max}$ should be also included into the guard interval to avoid collision. Thus, the guard interval in this case is

$$T_{\text{G},0} \approx 2\tau_{\max} + \Delta\tau_{\max} + T_{\text{on}-\text{off}}.$$

(b) With the above fact in mind and the answer to Part (a), show that the maximum cell radius in a standard GSM system is $R_{\max} = 35$ km.

♠ **Solution:**

(a) From the discussions in page 285 of the lecture notes, the maximum cell radius in the GSM system is mainly limited by the *maximum propagation delay* in the system. In other words, the cell radius in the GSM system is limited such that the propagation delay be always less than an upper limit.

(b) The maximum propagation delay in the system can be derived from the difference in the guard intervals of the TDM frame used for the first access, i,e, random access burst (RAB) in page 285, and the normal TDM frame used during the call in GSM systems, i.e., normal burst in page 277.

More precisely, from $T_{\mathrm{G,TA}}$ and $T_{\mathrm{G},0}$, we have

$$\tau_{\max} = \frac{T_{\mathrm{G},0} - T_{\mathrm{G,TA}}}{2}$$

Noting that in the normal burst $T_{\mathrm{G,TA}} = 8.25$ bits and in the RAB $T_{\mathrm{G},0} = 68.25$ bits, the maximum propagation delay in a GSM system reads

$$\tau_{\max} = 30 \text{ bits}.$$

From Table 8.1 in page 265 of the lecture notes, we know that 156.25 bits take 577 μsec long. Therefore,

$$\tau_{\max} \approx 30 \times \frac{577}{156.25} = 110.78 \ \mu\text{sec}.$$

As a result, the cell radius need to be limited such that the wave traveling in the cell reaches the cell edge at most $\tau_{\max} = 110.78 \ \mu\text{sec}$ after transmission. Noting that the wave travels with speed $c = 3 \times 10^8$ m/sec, we have

$$R_{\max} \approx c\tau_{\max} = 3 \times 10^8 \times 110.78 \times 10^{-6} = 33.234 \text{ km}.$$

which by rounding up derives the upper bound $R_{\max} = 35$ given for the GSM system.

3. The total modulation rate of a communication system is described by the number of bits transmitted, end-to-end, in the system per unit of time. In the GSM system, this rate in *full-rate traffic channels* is

$$R_{\mathrm{GSM}} = 270.83 \text{ kbits/sec}.$$

In this exercise, we intend to use the structure of the GSM system and derive this rate.

(a) What is the data rate in the GSM system after encoding the source data, i.e., speech encoding?

(b) What is approximately the channel coding rate in the GSM system?

(c) Which channel is used for transmitting user data?

(d) Describe the data frame structure for a full-rate traffic channel.

(e) Calculate the data rate after construction of a *multiframe* in a full-rate traffic channel.

In the GSM system, 8 different TDM-frames are multiplexed in a given transmission interval. Keeping this in mind, answer further the following items:

(f) Determine the factor with which the data rate increase after multiplexing multiuser TDM-frames.

(g) Calculate the factor with which the data rate increases after formatting the date via normal burst.

(h) Compare the derived modulation rate with R_{GSM}.

♠ **Solution:**

(a) From Figure 8.5 and the discussions in pages 266 – 268, the data rate after speech encoding is 13 kbits/sec. At this stage, the data which is mainly a voice message has been compressed by a speech encoder such that the redundant parts have been removed.

(b) The compressed data is given into a channel encoder which encodes the message. The output data has now the rate of

$$R_0 = 22.8 \text{ kbits/sec}$$

Assuming a fixed time duration for this process, one can conclude that the coding rate is

$$R_c = \frac{13}{22.8} = 0.57.$$

(c) The data message is transmitted over a standard traffic channel (TCH) whose structure is shown in Figure 8.3 in page 266.

(d) In this channel, the whole data sequence is divided into 24 data frames. The TCH-multiframe is then constructed by gathering these 24 frames into two groups of 12 frames, and separating them with a slow associated control channel (SACCH) frame at index 12 and a dummy frame at index 25.

(e) The rate after TCH-multiframe construction is given by

$$R_1 = R_0 \times \frac{26}{24} = 22.8 \times \frac{26}{24} = 24.7 \text{ kbits/sec}.$$

The TCH-multiframe is then modified for encryption which does not add any extra bit to the frames and simply encrypts them in groups of 114 bits.

(f) The multiplexer constructs a TDM-frame for the channel by multiplexing TCH multiframes of 8 different users. This means that the total rate in a full-rate traffic channel reads

$$R_2 = R_1 \times 8 = 24.7 \times 8 = 197.6 \text{ kbits/sec}.$$

(g) The normal burst is constructed from the multiplexed bits as shown in Figure 8.15 in page 277. In this time slot frame, each packet of 114 bits is partitioned into two groups of 57 bits and structured by adding *two tail sequences each of length 3 bits*, *two stealing flags*, *a training sequence of length 26 bits* and *a guard interval of length 8.25 bits*. The total frame is thus of length 156.25. Consequently, the rate after construction of a normal burst reads

$$R_3 = R_2 \times \frac{156.25}{114} = 197.6 \times \frac{156.25}{114} = 270.83 \text{ kbits/sec.}$$

(h) As it is observed,

$$R_3 = R_{\text{GSM}}$$

10.3.2 Characteristics of GSM Systems

1. The theoretical limit given by the *channel coding theorem* of Shannon is derived by using the capacity expression for additive white Gaussian noise (AWGN) channels, i.e.,

$$C = \log_2 (1 + \text{SNR}) \text{ bits/sec/Hz}$$

where SNR is the signal-to-noise ratio (SNR) at the receive side. For a standard variation of the GSM system, we have the following information:

The average receive power in a *typical* GSM channel is $P = -90$ dBm and the noise level at the receive side is $N = -121$ dBm. In the case of perfect multiplexing, the multiuser interference is negligible.

(a) Calculate the channel capacity for a typical GSM channel.

(b) Derive the spectral efficiency of a standard GSM system and compare it to the channel capacity.

♠ **Solution:**

(a) Considering the provided information, the SNR at the receive side reads

$$\log \text{SNR} = P - N = 31 \text{ dB}$$

Thus, in linear scale $\text{SNR} = 10^{3.1} = 1258.9$. Consequently, the channel capacity reads

$$C = \log_2 (1 + 1258.9) = 10.3 \text{ bits/sec/Hz.}$$

(b) The total data rate in *a single* full-rate traffic channel of GSM systems is $R_{\mathrm{GSM}} = 270.83$ kbits/sec. The bandwidth of a single FDM sub-channel is moreover $B_{\mathrm{FDM}} = 200$ kHz; see table 8.1 in page 262. Therefore, the spectral efficiency η is given as

$$\begin{aligned}\eta &= \frac{R_{\mathrm{GSM}}}{B_{\mathrm{FDM}}}\\ &= \frac{270.83 \times 10^3}{200 \times 10^3} = 1.35 \text{ bits/sec/Hz.}\end{aligned}$$

which is significantly smaller than the theoretical capacity.

2. Consider the following approaches for transmission

   A_1 Coded transmission with hard decoding,

   A_2 Coded transmission with burst-wise soft decoding,

   A_3 Coded transmission with bit-wise soft decoding,

   in a standard traffic channel (TCH) of the GSM system. Assume that the GSM signal experiences *narrow-band* fading and *ideal frequency hopping* is employed.

   (a) Calculate the coding gain achieved by these approaches at the bit-error-rate 10^{-2}.

   (b) Calculate the diversity gain of these approaches.

   (c) Name the crucial module at the transmitter which makes these gains achievable.

♠ **Solution:**

(a) Considering Figure 8.12 in page 274, at the bit-error-rate $\mathrm{BER} = 10^{-2}$, the required signal-to-noise ratio for uncoded transmission reads

$$\log \frac{E_{\mathrm{b}}}{N_0} = 16 \text{ dB}$$

This value for each of the above approaches reads

A_1: $\log \frac{E_{\mathrm{b}}}{N_0} = 13$ dB

A_2: $\log \frac{E_{\mathrm{b}}}{N_0} = 9$ dB

A_3: $\log \frac{E_{\mathrm{b}}}{N_0} = 8$ dB

Consequently, the coding gain G_{c} in each approach is given by

- $G_{\mathrm{c,A_1}} = 3$ dB
- $G_{\mathrm{c,A_2}} = 7$ dB
- $G_{\mathrm{c,A_3}} = 8$ dB

(b) To determine the diversity order M, we remember from Chapter 4 that in the logarithmic scale, the bit-error-rate decays linearly with slope M at *large* SNRs. This means that the bit-error-rate BER changes in terms of the SNR in dB, SNR, as

$$10 \log \mathrm{BER} = K - M \log \mathrm{SNR}.$$

when SNR is large enough. For approach A_1, we see in Figure 8.12 that at $\mathrm{SNR} = 14$ dB, $\mathrm{BER} = 6 \times 10^{-3}$, and at $\mathrm{SNR} = 16$ dB we have $\mathrm{BER} = 2 \times 10^{-3}$. Thus,

$$M_{A_1} = \frac{10 \log(3)}{2} \approx 2$$

Similarly, it is straightforward to calculate

$$M_{A_2} = M_{A_3} \approx 4.$$

(c) The coding gain is achieved by channel coding plus *interleaving*, due to fading. Thus, the crucial component in Figure 8.5 which achieves these gains is the *interleaver*.

Chapter 11

Sample Exams with Solutions

In this chapter, some sample final exams of the Mobile Communications course in Friedrich-Alexander University of Erlangen are given. The solutions to the exams are further provided to help you preparing for the final exam.

11.1 Winter Semester 2014-2015

Exam Date: February 10, 2015
9 Problems with total of 90 Points.
Exam Duration: 90 Minutes

11.1.1 Problem 1 (9 Points)

Consider a cellular system with power controlled handover and a hysteresis threshold of 3 dB. Let the attenuation exponent and the variance of the uncorrelated shadowing be 3.5 and 49 dB^2, respectively.

How large need the shadowing margin be in order to ensure a service probability of 98$\%$ at the cell border?

♠ **Solution:**
Omitted.

11.1.2 Problem 2 (7 Points)

What is a punctured symbol and what is its purpose?

♠ **Solution:**
A punctured symbol is a symbol that is not transmitted (3p). The purpose is to increase the code rate (3p) without increasing the decoding complexity (1p).

11.1.3 Problem 3 (12 Points)

Consider a CDMA system with 8 users and spreading factor 8.

(a) Give a set of binary (antipodal) spreading sequences such that there is no interference on a memoryless channel.

(b) Let one of the spreading sequences be given by $+1, +1, -1, +1, -1, -1, -1, +1$. Construct a set of 8 orthogonal spreading sequences containing this sequence.

♠ **Solution:**

For (a), use an 8×8 Hadamard matrix (6p). For (b), take an 8×8 matrix and exchange column 3 with column 8 (6p).

11.1.4 Problem 4 (15 Points)

(a) The digital TV-channel no. 55 is transmitted with 50 kW at 746 MHz from the 292 m high Fernmeldeturm Nürnberg. Calculate the median received power level in the 19 km distant lecture room R4.15 predicted by the Okumura-Hata model.

♠ **Solution:** First the urban path loss is calculated with

$$f_0/\text{MHz} = 746, \quad h_{\text{BS}}/m = 292, \quad h_{\text{MS}}/m = 17.$$

Any MS-height between 11 m and 21 m is acceptable (1p). We find $a(h_{\text{MS}}) = 38.1$ dB (2p). The urban path loss is found to be $\text{PL}_{\text{urban}} = 109.3$ dB (2p). Next, we need to categorize the terrain. Urban does not fit at all (0p). Best fit is suburban (2p). Open terrain is questionable (0p, but gets points for the correct calculation of the correction). Including terrain correction, we get

$$\text{PL}_{\text{suburban}} = 99.8 \text{ dB} \quad \text{(2p)}$$

The transmitted power level is 77 dBm, yielding a received power level of -22.8 dBm (2p).

(b) For which reasons is it problematic to apply the Okumura-Hata model to this scenario?

♠ **Solution:** The BS height (2p) and the MS height (2p) violate the limitations of the model.

11.1.5 Problem 5 (12 Points)

Below is shown a measured time-variant power-delay profile.

(a) Calculate the relative speed between receiver and transmitter.

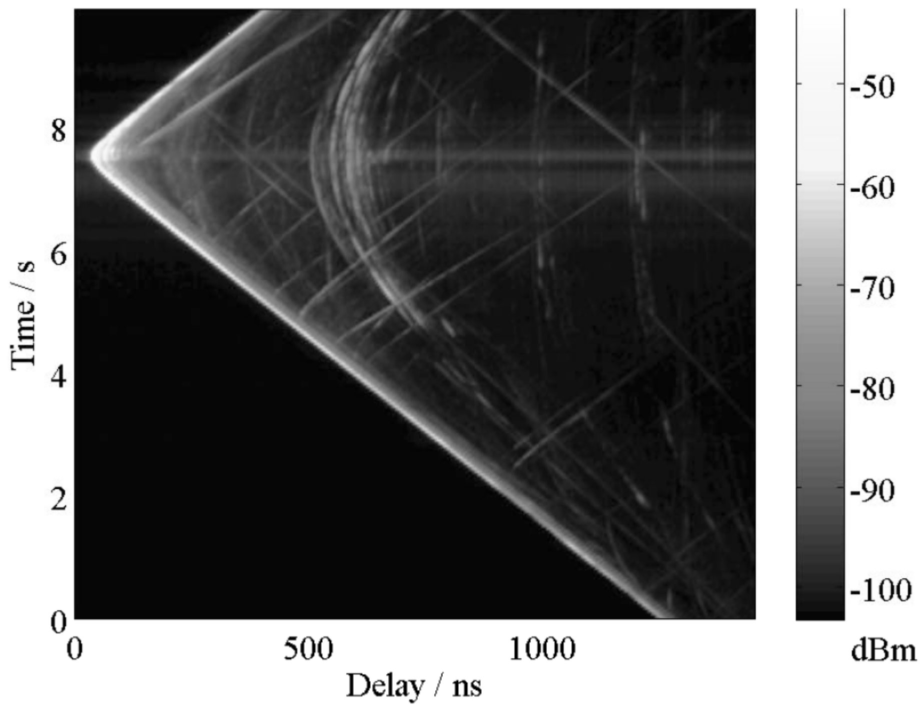
♠ **Solution:** A delay of 1000 ns corresponds to a distance of 300 m. Within 6 s, this distance shrink to almost zero. So, the speed is

$$300 \text{ m}/6 \text{ s} = 50\ m/s \quad (6p)$$

(b) Calculate the distance of the transmitter to the most dominant static scatterer (a factory building) at the time when transmitter and receiver are closest to each other.

♠ **Solution:**

The most dominant scatterer appears at 700 ns delay. This corresponds to a path length of 210 m (4p). The path length is much larger than the line-of-sight. Thus, the distance is half the path length, i.e, 105 m (2p).



11.1.6 Problem 6 (8 Points)

Calculate the time duration

(a) of the GSM training sequence on traffic channels,

♠ **Solution:** The training sequences contain 26 bits (page 281) (2p) at 270.833 kbit/s (Table 9.1, page 266) (2p). Thus, we find $26/270833$ s = $96\ \mu s$ (2p).

(b) of the delay of full rate data in GSM

♠ **Solution:** The solution to (b) is ambiguous:

On page 279 of the script, you find (2p): *"To evaluate the transmission delay of a* 20 *ms block, we must take into account that the burst carrying the first bits of the last sub-block can only be transmitted after completion of channel encoding. Furthermore, interleaving over* 19 *time slots can mean that up to two additional TDM-frames (SACCH and empty frame) can be contained taking the TCH frame structure into account. The first bits of the last sub-block can only be decoded when the last time slot has been received. This results in a delay for the last bit of about* $5 + (19 + 2) \cdot 4,615 \approx 102\ ms$*."*

On page 268, Table 8.2 of the script, you find (2p) either 62 ms or 131 ms depending on the data rate.

All solutions are recognized.

in milliseconds.

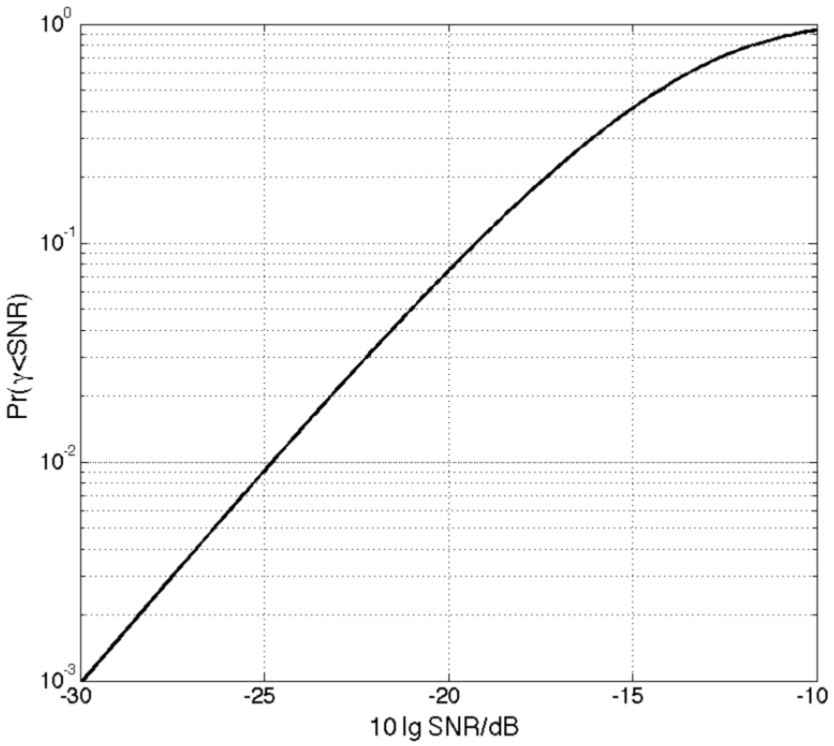
11.1.7 Problem 7 (6 Points)

The plot below shows the distribution of the received signal-to-noise ratio after combining M paths. Determine M.

♠ **Solution:** The diversity order is slope of the graph in double-lograithmic scale, i.e.,

$$10/5 = 2.$$

Thus, there are 2 paths.



11.1.8 Problem 8 (12 Points)

You want to serve a skyscraper with two towers like New York's former World Trade Center by cellular coverage. In order to maximize spectral efficiency you want to minimize the re-use factor. Make a drawing and indicate where in the towers you would place the base stations, which area they would serve, and what the frequency re-use factor would be. Which electromagnetic properties of the ceilings, floors and outer windows are required for your proposal to work? Give reasons for your answer.

♠ **Solution:** In such large office buildings, each tower should be subdivided into several cells. Ideally, one could use metal-plated windows and walls to fully isolate the building from the outside world. Metal-plated ceiling could further sub-divide each tower into several cells achieving re-use factor 1 (12p).

If such a subdivision does not work, e.g., for cost reasons, one could also go for re-use factor 2. In that case one would vertically subdivide each tower into cells that cover a certain number of floors. If the cells are indexed by $1, 2, 1, 2, \dots$ in the 1^{st} tower, they should be indexed by $2, 1, 2, 1, \dots$ in the 2^{nd} tower to minimize interference from one tower into the other (12p).

11.1.9 Problem 9 (9 Points)

Sort the following digital modulation formats

1. differential binary phase shift keying and roll-off factor 0.4,
2. $3\pi/8$-shifted 8-ary phase shift keying with Gray mapping and roll-off factor 0.4,
3. 16-ary quadrature amplitude modulation (16QAM) with roll-off factor 1.0,
4. 16QAM with roll-off factor 0.2,

with respect to

(a) noise resistance on a non-dispersive Rayleigh fading channel with diversity at high signal-to-noise ratio,

♠ **Solution:** Subproblem (a):

From pages 221 and 222, we get DPSK $>$ 8PSK $>$ 16QAM. The roll-off factor does not matter. Thus

$$(1) > (2) > (3) = (4)$$

(b) spectral efficiency,

♠ **Solution:** Subproblem (b):

- For (1), we get $1/1.4 = 0.71$.
- For (2), we get $3/1.4 = 2.14$.
- For (3), we get $4/2.0 = 2.00$.
- For (4), we get $4/1.2 = 3.33$.

Thus, we find

$$(4) > (2) > (3) > (1).$$

(c) amplitude variations (peak-to-average- power ratio).

♠ **Solution:** Subproblem (c): Amplitude modulation has more variations than phase modulation. $3\pi/8$-shifts reduce amplitude variations. Higher roll-off also slightly reduces amplitude variations. Thus,

$$(4) > (3) > (1) > (2).$$

Do **not** take into account any issues that come to matter due to all kinds of interference or channel estimation.

11.2 Winter Semester 2016-2017

Exam Date: February, 2017
9 Problems with total of 90 Points.
Exam Duration: 90 Minutes

11.2.1 Problem 1 (9 Points)

Calculate the equivalent noise bandwidth of a filter with the transfer function

$$H(f) = \frac{5}{\pi} \exp(-|f|/5\text{kHz}).$$

♠ **Solution:** From page 34, Eq. (1.8):

$$B_{\text{noise}} = \frac{\int_{-\infty}^{+\infty} |H(f)|^2 \, \mathrm{d}f}{|H(0)|^2}$$

which is a simple exponential integral.

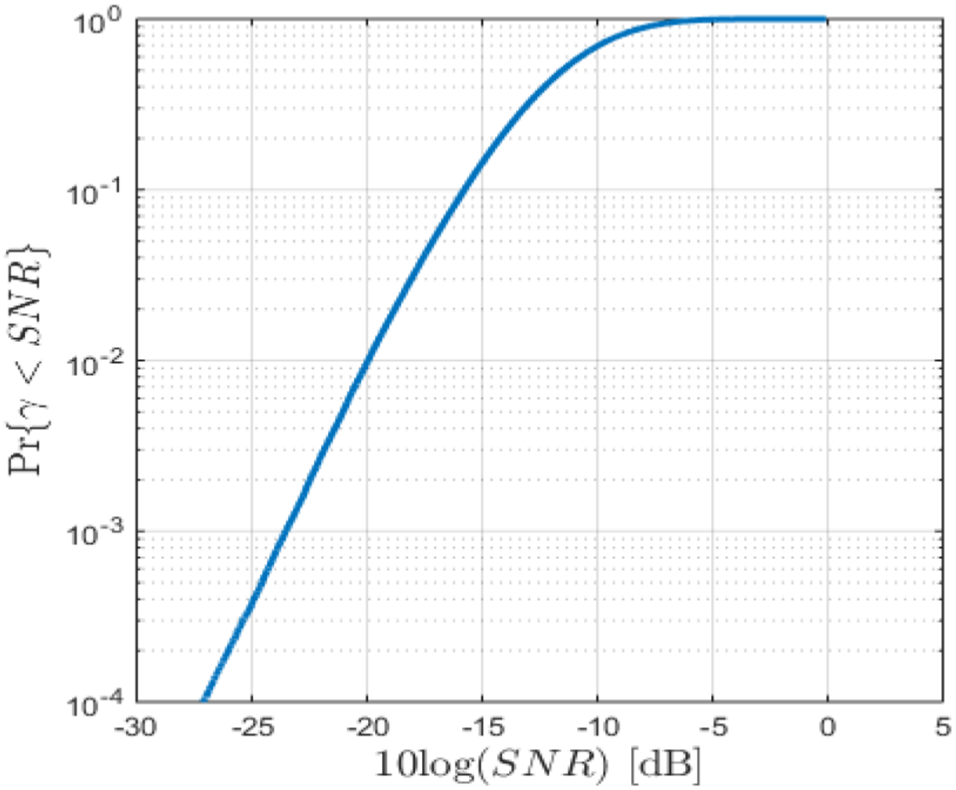
11.2.2 Problem 2 (8 Points)

Why there is a big difference between the guard intervals of a normal burst and that of a random access burst in GSM?

♠ **Solution:** From Chapter 8, we know that this difference is because of "timing advance", which <u>is</u> used in normal burst but <u>NOT</u> is RAB.

11.2.3 Problem 3 (9 Points)

The plot below shows the distribution of the received signal-to-noise ratio after combining 4 paths. Write down a valid set of correlations between the 4 paths. (Hint: compute the diversity order.)



♠ **Solution:** From the figure, we observe that,

$$\Pr\{\gamma < \text{SNR}\} = 10^{-4} \text{ at SNR } \approx -27 \text{ dB}$$
$$\Pr\{\gamma < \text{SNR}\} = 10^{-2} \text{ at SNR } \approx -20 \text{ dB}$$

Therefore, the slope of $\log - \log$ Figure is:

$$\frac{10 \times \log\left(10^{-2}\right) - 10 \times \log\left(10^{-4}\right)}{-20 - (-27)} = \frac{-20 + 40}{-20 + 27} \approx \boxed{3}$$

This means that diversity order is

$$\boxed{M = 3}$$

Since we have 4 branches, we could conclude that one of branches is completely dependent on the other three branches. Thus, one possible case is

$$\begin{cases} h_1 \text{ and } h_2 \text{ and } h_3 & \Longrightarrow \text{ independent} \\ h_4 = h_3 \end{cases}$$

where h_j for $j = 1, \ldots, 4$ are the channel coefficient of diversity branches.

11.2.4 Problem 4 (12 Points)

A mobile is moving through an environment with several trees and buildings on all sides. Give answers for the following:

1. Draw a typical Doppler power spectrum for an omnidirectional receive antenna

♠ **Solution:** It is Jake's spectrum: Fig 3.25, Page 102.

2. Describe how does the Doppler power spectrum change for a directional receive antenna

♠ **Solution:** Starting from Eq. (3.45) in page 102, we see that the spectrum for a general angular pattern $p(\alpha)$ is given. The particular pattern in Eq. (3.46) is for an omnidirectional antenna.

When we replace, $p(\alpha)$ in Eq. (3.46) with the pattern of a directive antenna, the spectrum for a directive antenna is derived.

3. How does the delay spread, coherence bandwidth and coherence time change when the base station transmits using a high directivity antenna? Give reasons.

♠ **Solution:** Directive antenna reduces the impact of path in null angels. This means that the path with long distances are cancelled out. Thus,

$$\text{Delay spread } \downarrow, \text{ Coherence Bandwidth } \propto \frac{1}{\Delta\tau} \uparrow$$

However, coherence time does not change.

11.2.5 Problem 5 (12 Points)

A worker on standing on the roof of Erlangen Rathaus (city hall) is sending a GSM 900 SMS to a base station located in Erlangen 3 km away.

1. Calculate the pathloss between the mobile and the base station using Okumura-Hata model. Choose reasonable values for relevant parameters.

♠ **Solution:** Use equations in pages 92 and 93. Don't forget that Erlangen is a sub-urban region in Okumura-Hata model.

2. Is there a problem in using the above model? Give reasons.

♠ **Solution:** Yes, Rathaus is 14 floors.

$$\text{Each floor } > 2\ m \implies h_{MS} > 28\ m$$

This does not agree with the limit

$$1\ m \leq h_{MS} \leq 10\ m$$

in page 93.

11.2.6 Problem 6 (9 Points)

Consider a CDMA transmission system over a multipath channel that consists of "M" resolvable paths. A user's rake receiver consists of "L" fingers.

1. For a single user case, what will determine the performance of the detector and what is desirable?

♠ **Solution:** Auto-correlation function

$$\rho_{kk}[d] = \frac{1}{N} \sum_{<N>} c_k[n] c_k^*[n-d]$$

Ideally:

$$\rho_{kk}[d] = \begin{cases} 1 & d = 0 \\ 0 & d \neq 0 \end{cases}$$

2. For a multiple user case, what will determine the performance of the detector and what is desirable?

♠ **Solution:**

Auto-correlation $+$ Cross-correlation (CC)

$$\text{CC: } \rho_{kj}[d] = \frac{1}{N} \sum_{<N>} c_k[n] c_j^*[n-d]$$

Ideally,

$$\rho_{kj}[d] = 0 \quad \forall d$$

3. Compare the performance of the rake receiver and the performance of another receiver that uses a filter matched to the convolution of the channel and the spreading signal for $L < M$, $L = M$ and $L > M$.

♠ **Solution:**

$$(L < M) \underbrace{<}_{\text{worse}} (L = M) \underbrace{\equiv}_{\text{equivalent}} (L > M)$$

11.2.7 Problem 7 (10 Points)

Compare the importance of power control for

1. CDMA downlink transmissions

♠ **Solution:** Downlink does not need power control: There is only one transmitter, and thus, near-far effect does not exist.

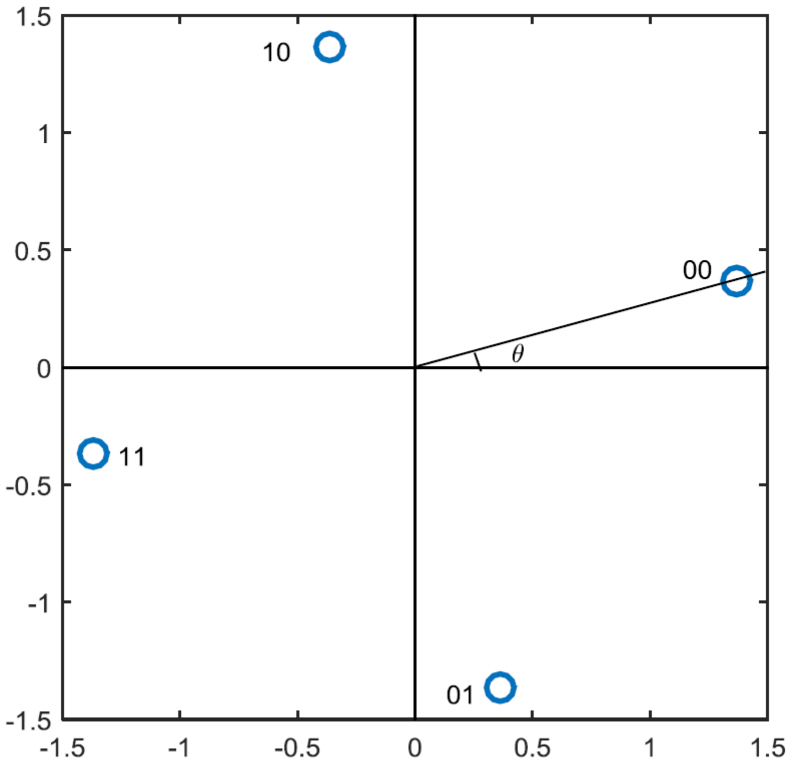
2. CDMA uplink transmissions.

♠ **Solution:** Uplink needs power control: Multiple users can cause near-far effect.

Give reasons for your answer.

11.2.8 Problem 8 (12 Points)

A transmission scheme uses a digital modulation scheme with SRRC pulse shaping. The roll-off factor is $\alpha = 0.3$. The bit-mapping and signal constellation employed are shown below. The noise in the system can be approximated by an additive white Gaussian noise (AWGN).



1. Design the optimum receive filter for this system.

♠ **Solution:** Assume $g_\alpha(t)$ denotes RRC pulse shape (see page 193, Eq. (6.16)).

Then, the impulse response of the matched filter would be,

$$\boxed{h(t) = g_\alpha^*(-t)}$$

2. What should be done to the filter output to make the soft-bit computation (LLR) as simple as possible?

♠ **Solution:** Soft-bit computation (LLR) is simple for QAM. Looking at the constellation, we observe that here the constellation is a QAM constellation which has been shifted with angle

$$-\left(\frac{\pi}{4} - \theta\right) = \boxed{\theta - \frac{\pi}{4}}$$

Therefore, one could multiply the output of filter with $e^{j\left(\frac{\pi}{4}-\theta\right)}$ to compensate the phase-shift.

3. Give the values of the LLRs in its simplest form (accurate up to a proportionality constant is enough).

♠ **Solution:** After the phase-shift in Part (2), the LLR is given by Eq. (6.88) and Eq. (6.89) in page 226. Thus,

$$\begin{aligned}\text{LLR}(q_1) &= 4\frac{\sqrt{E_b/T}}{\sigma^2} \times \text{Re}\{d\, e^{j\left(\frac{\pi}{4}-\theta\right)}\} \\ &= 4\frac{\sqrt{E_b/T}}{\sigma^2} \times \left[\text{Re}(d)\cos\left(\theta - \frac{\pi}{4}\right) + \text{Im}(d)\sin\left(\frac{\pi}{4} - \theta\right)\right]\end{aligned}$$

Same for $\text{LLR}(q_2)$:

$$\text{LLR}(q_2) = 4\frac{\sqrt{E_b/T}}{\sigma^2} \times \left[\text{Im}(d)\cos\left(\theta - \frac{\pi}{4}\right) - \text{Re}(d)\sin\left(\theta - \frac{\pi}{4}\right)\right]$$

11.2.9 Problem 9 (9 Points)

Sort the following modulation schemes

1. $\pi/4$-shifted QPSK using a SRRC pulse shape with roll-off factor $\alpha = 0.3$
2. DBPSK using rectangular pulse shape
3. 16-QAM using a SRRC pulse shape with roll-off factor $\alpha = 0.3$.

with respect to

a. Noise resistance in a non-dispersive Rayleigh fading channel with diversity at high signal-to- noise ratio

♠ **Solution:** Since number of diversity branches is indicated, we can use either Fig. 6.28 in page 220 or Fig 6.29 in page 221.

From the figure:

$$(2) > (3)$$

However, scheme (1) is not available!

b. Spectral efficiency

♠ **Solution:**

$$\eta = \frac{\log M}{1+\alpha} \implies \begin{cases} (1) : \frac{2}{1\cdot 3} \\ (2) : \frac{1}{\infty} = 0 \\ (3) : \frac{4}{1\cdot 3} \end{cases} \implies (3) > (1) > (2)$$

Note that bandwidth of a rectangular pulse is ∞, i.e., Fourier transform of rectangular function is sinc function, and sinc function has infinite support.

c. Amplitude variations (peak-to-average ratio)

♠ **Solution:** PAPR: Fig. 6.12, Page 199:

$$\text{Always}: \boxed{\text{16-QAM} > \text{PSK}}$$

Hence,

$$(3) > (1) > \underbrace{(2)}_{\text{With rectangular pulse, we have PAPR=0 for DPSK}}$$

As a final result,

$$(3) > (1) > (2)$$

Do **not** take into account any issues that come to matter due to all kinds of interference or channel estimation.

11.3 Summer Semester 2017

Exam Date: August 11, 2017
6 Problems with total of 70 Points.
Exam Duration: 90 Minutes

11.3.1 Question 1 (13 Points)

Answer the following items.

(a) Briefly illustrate the concept of hand-over, and explain why hand-over algorithms are needed in cellular networks.

♠ **Solution:** Handover is illustrated in Chapter 1. We need hand-over as the users are movable in wireless system and change their cells.

(b) What is the proper distribution to model the narrow-band fast fading coefficient of a wireless radio channel when the line-of-sight is blocked?

♠ **Solution:** Rayleigh Fading.

(c) Consider the Maximum Ratio Combining (MRC) and Equal Gain Combining (EGC) methods.

(c-1) Briefly explain these combining methods.

♠ **Solution:** In MRC, the combining coefficients are conjugate of channel gains:

$$\boxed{g_i = h_i^*}$$

In EGC, the combining coefficients only in-phase the signals:

$$g_i = \frac{h_i^*}{|h_i|}$$

(c-2) Compare the performance of these methods when the Channel State Information (CSI) is perfectly available

♠ **Solution:**

$$\text{MRC} > \text{EGC}$$

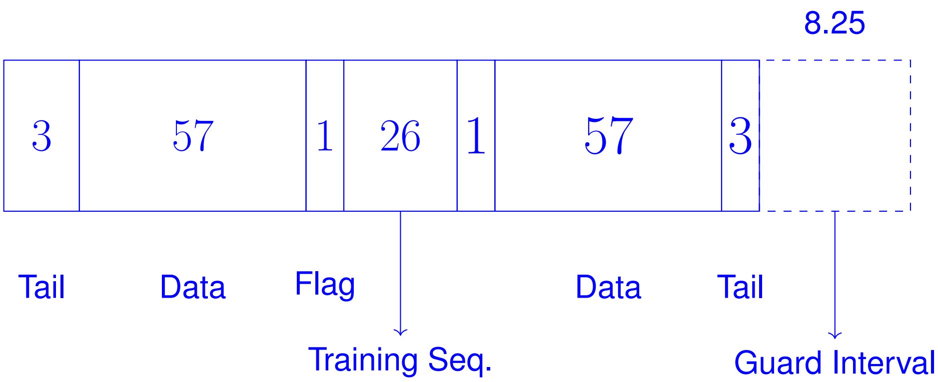
(c-3) Now assume that the CSI is estimated at the receiver, how does the performance of the MRC method change as the estimation error increases. Justify your answer by giving a reason.

♠ **Solution:** The performance degrades, as MRC depends on the CSI.

(d) Consider the GSM system.

(d-1) Explain the structure of the Normal Burst by making a drawing.

♠ **Solution:** Page 280, Fig. 8.15



(d-2) Calculate the duration of the guard interval in a Normal Burst in **seconds**.

♠ **Solution:**

$$\text{Guard interval } = 8.25 \times \frac{577}{156.25} = 30.46\ \mu s$$

(d-3) Calculate the duration of the training sequence in a Normal Burst in **seconds**.

♠ **Solution:**

$$\text{Training sequence } = 26 \times \frac{577}{156.25} = 96.01\ \mu s$$

11.3.2 Question 2: Convolutional Codes (7 Points)

Consider a binary convolutional code with $K = 1$ bit of input and $N = 2$ bits of output. Let the constraint length be $L_c = 3$.

(a) Define the free distance d_{free} of a convolutional code.

♠ **Solution:** Page 251, Eq. (7.19)

$$d_{\text{free}} = \min_{c \neq c'} d_H(c; c')$$

(b) Assume that $d_{free} = 5$. Calculate the maximum possible coding gain G_c for this code considering an Additive White Gaussian Noise (AWGN) channel.

♠ **Solution:** Page 254, Eq. (7.23)

$$G_c = 10 \times \log R_c \times d_{\text{free}} \overset{R_c=1/2}{\underset{d_{\text{free}=5}}{\Longrightarrow}} G_c = 4 \text{ dB}$$

(c) Now assume that the convolutional code is intended to be used for channel coding over a fading channel. Name an additional block which we need to add in this case, in order to obtain gains from channel coding.

♠ **Solution:** Interleaver.

11.3.3 Question 3: Comparison of Modulation Schemes (9 Points)

Consider the following modulation schemes, and assume that Root Raised Cosine (RRC) pulses with the roll-off factor α are used for transmission.

(1) $3\pi/8$-shifted 8-Phase Shift Keying (PSK); $\alpha = 0.5$

(2) Differential Binary PSK (DBPSK); $\alpha = 0.1$

(3) 16-Quadrature Amplitude Modulation (QAM) with Gray mapping; $\alpha = 0.2$

(4) 64-QAM with Gray mapping; $\alpha = 0.1$

Sort these schemes with respect to

(a) Noise resistance at **high** Signal-to-Noise Ratios (SNRs) considering transmission over a non-dispersive (frequency flat) Rayleigh fading channel with two-branch antenna diversity

♠ **Solution:** To compare noise resistance for "Fading Channel" with "2-branch of diversity" we use,

$$\boxed{\text{Page 220, Fig. 6.28}} \implies (2) > (1) > (3) > (4)$$

(b) Spectral (bandwidth) efficiency

♠ **Solution:** Spectral Efficiency $\implies$ From Tutorial 7:

$$\eta = \frac{\log M}{1+\alpha}$$

$$\implies (4) > (3) > (1) > (2)$$

(c) Peak-to-Average Power Ratio (PAPR)

♠ **Solution:** PAPR $\implies$ We use Fig. 6.12, page 199.

We also know that:

- With root-raised-cosine pulses PAPR would be fixed and less than $\sqrt{2}$ after $\alpha > 0.4$.

- QAM constellations have always more PAPR, due to amplitude change.
- For $\alpha \leq 0.1$, PAPR significantly increase

Using all these lines of justifications:

$$(4) > (3) > (2) > (1)$$

11.3.4 Question 4: Mobile Radio Channels (15 Points)

Assume that a base station is placed in Tennenlohe at the height of 160 m and serves a mobile station whose height is 2 m. The carrier frequency is set to be 900 MHz, and the gains of the antennas at both the base and mobile stations are $G_{BS} = G_{MS} = 0$ dB$_i$. At the distance of d from the base station, the mobile station receives a signal whose power is $P_r(d)$.

(a) Using the free-space propagation model, calculate the path-loss in dB when the mobile station stands $d = 8$ km far from the base station at Konrad-Zuse-Straße in Erlangen.

♠ **Solution:** Page 82, Eq. (3.9)

$$a = \left(\frac{1}{g_{\text{MS}}\, g_{\text{BS}}}\right)\left(4\pi \frac{f\, d}{c}\right)^2$$

$$g_{\text{MS}} = g_{\text{BS}} = 1 \quad (\text{or } 0\ \text{dB}_i)$$

$$\Longrightarrow\ a = \left(4\pi \frac{f\, d}{c}\right)^2 \Longrightarrow\ \text{PL}_{\text{free}} = 10 \log a = 32.44 + 20 \log \underbrace{f}_{\text{(MHz)}} + 20 \log \underbrace{d}_{\text{km}}$$

$$\begin{cases} d = 8 \\ f = 900 \end{cases} \Longrightarrow \boxed{\text{PL}_{\text{free}} = 109.5\ \text{dB}}$$

(b) Now consider the Okumura-Hata model. Calculate the path-loss in dB again for the setup illustrated in Part (b) using the Okumura-Hata model, and compare it with the path-loss determined in Part (a). Which model results in a larger path-loss, and what is your conclusion?

♠ **Solution:** Erlangen is sub-urban in Okumura-Hata model.

Page 92, Section 3.2.3.2:

$$\text{PL}_{\text{urban}} = 142.8\ \text{dB and } \boxed{\text{PL}_{\text{sub}} = 142.8 - 9.94 = 132.64\ \text{dB}}$$

$\text{PL}_{\text{sub}} > \text{PL}_{\text{free}}$: Okumura-Hata is a practical model considering the reflection and other realistic impairments.

(c) Consider Part (a) and assume that the mobile station has been moved to the roof of a building with 15 floors. Do you choose the Okumura-Hata model to calculate the path-loss?Justify your answer by giving a reason.

♠ **Solution:** No. 15 floors, each floor has more then 2 m height

$$\implies h_{\text{MS}} \geq 15 \times 2\ m \implies h_{\text{MS}} \geq 30\ m$$

Okumura-Hata model is valid only for

$$1\ m \leq h_{\text{MS}} \leq 10\ m \quad (\text{see Page 93})$$

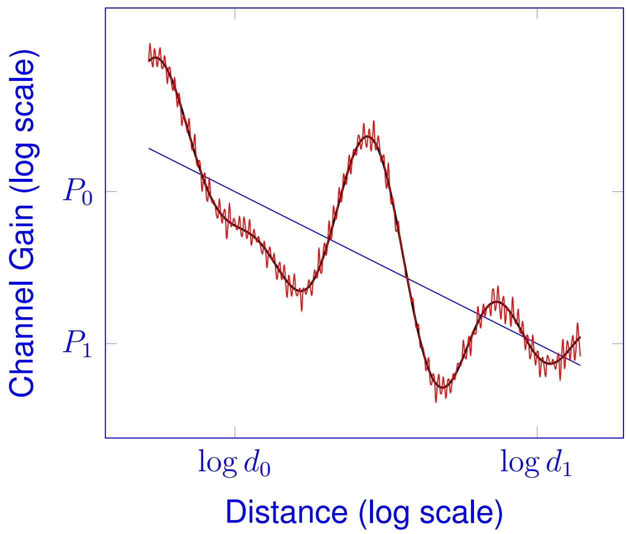
(d) Consider the free-space propagation model again, and let $P_r(d_0) = P_0$ and $P_r(d_1) = P_1$ for some distances d_0 and d_1 which are given in kilometer. Without doing any calculations, plot a figure which describes the variation of the received power $P_r(d)$ in dB in terms of $\log(d/1\text{km})$ by taking the impact of path-loss, shadowing and small-scale fading into account.

♠ **Solution:**

$$\text{Pr}(d) \propto \frac{1}{d^2} \implies \text{Pr}(d) = \frac{c}{d^2}$$

where "c" is some constant.

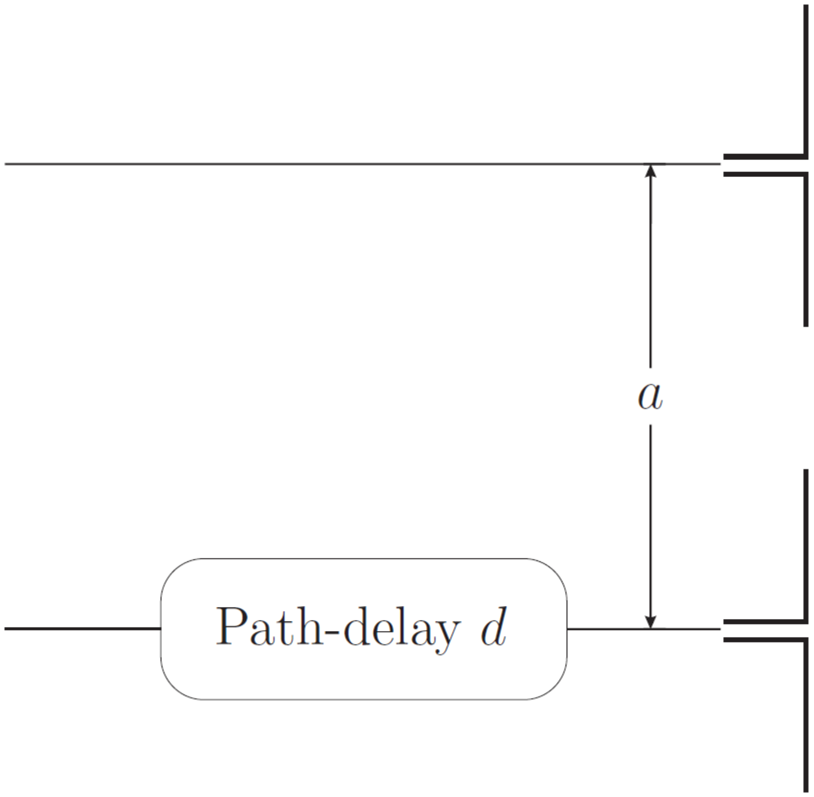
$$\text{Pr}(d)\ [\text{dB}] = 10 \log C + 20 \log(d)$$



In the figure, the "Path-Loss" is denoted with a straight blue line. The relatively low frequency change in the loss is due to the "Shadowing" and shown with a thick black line in the figure. The high frequency volatility is caused by "Small-scale Fading (Multipath)" and shown with red color.

11.3.5 Question 5: Antenna Arrays (10 Points)

Consider the following antenna array consisting of two $\lambda/2$-dipole elements. The antennas are located on a vertical array and spaced with the distance a from each other. Moreover, they are fed by the same source, and the feed into the second antenna is delayed with a path-delay d. This means that the wave at the second antenna has travelled an extra path of length d compared to the wave at the first antenna element.



(a) Determine the vertical radiation pattern of the antenna array $g(\alpha)$.

♠ **Solution:**

{ All "verticals" should be "horizontal". }

$$g(\alpha) = g_A(\alpha) \cdot g_0(\alpha)$$

where

$$g_0(\alpha): \text{ horizontal pattern for } \frac{\lambda}{2} - \text{dipole}$$

and

$$g_0(\alpha) = 1.$$

Page 59, Eq.(2.14):

$$g_A(\alpha) = \frac{|\mathsf{AF}(\alpha)|^2}{\max_\alpha |\mathsf{AF}(\alpha)|^2}$$

$$\mathsf{AF}(\alpha) = \Big\{ w \quad w_m e^{-j2\pi \frac{a}{\lambda} m \sin\alpha}$$

Here, we have

$$\begin{cases} M = 2 \\ w_0 = 1, w_1 = e^{-j2\pi \frac{d}{\lambda}} \end{cases}$$

$$\Longrightarrow \boxed{g_A(\alpha) = \frac{1}{2} + \frac{1}{2} \cdot \cos\left(\frac{2\pi}{\lambda}\left[d + a\sin\alpha\right]\right)}$$

$$g(\alpha) = g_A(\alpha) \cdot 1 = g_A(\alpha)$$

(b) Now assume that the antenna elements are replaced with isotropic antennas, and let $a = \lambda/4$. Calculate the value of d in terms of the vertical angle θ, such that the main beam (main direction) of the antenna array points to the vertical angle θ.

♠ **Solution:**

To have the main beam at $\alpha = \theta$, we need $g(\alpha)$ be maximized at $\alpha = \theta \implies$ The cosine term should be one at $\alpha = \theta$ which means that

$$\frac{2\pi}{\lambda}\left[d + a\sin\alpha\right] = 2\pi k, \qquad \text{for some integer } k$$

$$\Longrightarrow \ d = k\lambda - a\sin\alpha$$

$$\left(a = \frac{\lambda}{4}\right) \Longrightarrow \boxed{d = \lambda - \frac{\lambda}{4}\sin\alpha}$$

11.3.6 Question 6: CDMA Systems (16 Points)

Consider a single cell with K users in which the base station and mobile stations are equipped with a single antenna. The base station employs the Code-Division-Multiple-Access (CDMA) technique with spreading factor N and rectangular pulses of magnitude 1. The symbol and chip duration are T_s and T_c, respectively. For a given user $k \in \{1, \ldots, K\}$, denote the spreading sequence with $c_k[n]$, and let $\rho_{kk}[m]$ and $\rho_{k\ell}[m]$ represent the auto-correlation of the sequence of user k and the cross-correlation between the sequences of users k and ℓ, respectively.
Consider the specific user j. The channel between the base station and user j consists of

two paths. One direct path with channel factor 1, and a delayed path with delay $\tau = dT_c$ where d is an integer and the complex channel factor α. The impulse response of the channel is thus given by

$$h_j(t) = \delta(t) + \alpha\delta(t - dT_c).$$

Assume that the base station transmits the independent symbols $\{s_1, \ldots, s_K\}$ for the users where $s_k \in \{\pm 1\}$ for all k. The received signal by user j is then written as

$$r_j(t) = h_j(t) * \sum_{k=1}^{K} s_k x_k(t) + z(t)$$

where $x_k(t)$ is the pulse shape of user k constructed from the spreading sequence $c_k[n]$, and $z(t)$ is additive white Gaussian noise with zero mean and variance σ^2.

(a) Assume that user j employs a rake receiver with a single finger. The receiver considers only the direct path. Derive the Signal-to-Interference-and-Noise Ratio (SINR) at user j in terms of α, σ^2, $\rho_{kk}[m]$ and $\rho_{k\ell}[m]$

♠ **Solution:** The simplest way is to solve the problem in digital domain. It means that we consider discrete-time signals.

The receive signal in digital domain at user j:

$$r_j[n] = s[n] + \alpha s[n - d] + z[n]$$

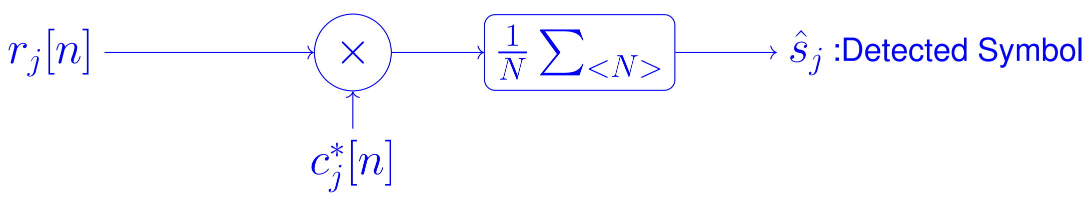
Here:

- $s[n]$ is the transmit signal:

$$s[n] = \sum_{k=1}^{K} s_k c_k[n] \qquad : c_k[n] = \text{ Spreading sequence of user } k$$

- d is the delay in digital domain (see $h_j(t)$)
- $z[n]$ is noise in digital domain.

Rake receiver with one finger matched to first path:



$$\Longrightarrow \hat{s}_j = \frac{1}{N} \sum_{<N>} r_j[n] c_j^*[n] \triangleq \langle r_j[n], c_j[n] \rangle_N$$

We define this notation of $< \cdot, \cdot >_N$ for simplicity.

$$\hat{s}_j = \langle s[n], c_j[n] \rangle_N + \alpha \langle s[n-d], c_j[n] \rangle_N + \underbrace{\langle z[n], c_j[n] \rangle_N}_{z_j}$$

$$\hat{s}_j = \sum_{k=1} s_k \langle c_k[n], c_j[n] \rangle_N + \alpha \sum_{k=1} s_k \langle c_k[n-d], c_j[n] \rangle_N + z_j$$

Here,

→ z_j is the output noise term: $z_j \sim CN(0, \sigma^2)$

→ $\langle c_k[n], c_j[n-d] \rangle_N = \frac{1}{N} \sum_{<N>} c_k[n] c_j^*[n-d] = \rho_{kj}[d]$

Thus,

$$\hat{s}_j = s_j \overbrace{\langle c_j[n], c_j[n] \rangle}^{\rho_{jj}[0]} + \sum_{\substack{k=1 \\ k \neq j}}^{K} s_k \overbrace{\langle c_k[n], c_j[n] \rangle}^{\rho_{kj}[0]}$$

$$+ \alpha \sum_{k=1}^{K} s_k \underbrace{\langle c_k[n-d], c_j[n] \rangle_N}_{\rho_{kj}[d]} + z_j$$

$$= s_j \rho_{jj}[0] + \sum_{\substack{k \neq j \\ k=1}}^{K} s_k \rho_{kj}[0] + \alpha s_j \rho_{jj}[d]$$

$$+ \alpha \sum_{\substack{k \neq j \\ k=1}}^{K} s_k \rho_{kj}[d] + z_j$$

$$= s_j \rho_{jj}[0] + \underbrace{\alpha s_j \rho_{jj}[d]}_{\substack{\text{Self Interference} \\ (I_j)}} + \underbrace{\sum_{\substack{k=1 \\ k \neq j}}^{K} s_k \left(\rho_{kj}[0] + \alpha \rho_{kj}[d] \right)}_{\substack{\text{Multi-user interference} \\ (I_0)}} + z_j$$

$$\text{SINR}_j = \frac{\mathcal{E}\left\{ |s_j \rho_{jj}[0]|^2 \right\}}{\mathcal{E}\left\{ |I_j + I_0 + z_j|^2 \right\}}$$

$$\Longrightarrow \mathcal{E}\left\{ |s_j \rho_{jj}[0]|^2 \right\} = \boxed{\rho_{jj}^2[0]}$$

$$\mathcal{E}\left\{ |I_j + I_0 + z_j|^2 \right\} = \mathcal{E}\left\{ |I_j|^2 \right\} + \mathcal{E}\left\{ |I_0|^2 \right\} + \mathcal{E}\underbrace{\left\{ |z_j|^2 \right\}}_{\sigma^2}$$

Here,

$$\mathcal{E}\left\{|I_j|^2\right\} = \rho_{jj}^2[d] \cdot |\alpha|^2$$

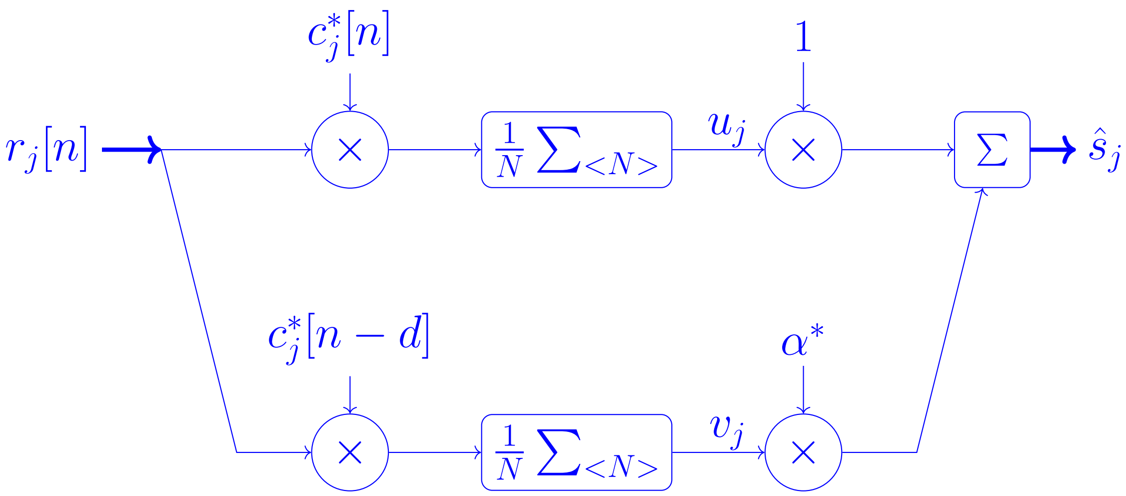
$$\mathcal{E}\left\{|I_0|^2\right\} = \sum_{\substack{k\neq j\\k=1}}^{K} |\rho_{kj}[0] + \alpha\rho_{kj}[d]|^2$$

$$\implies \text{SINR}_j = \frac{\rho_{jj}^2[0]}{\sigma^2 + |\alpha|^2\rho_{jj}^2[d] + \sum_{\substack{k=1\\k\neq j}}^{K} |\rho_{kj}[0] + \alpha\rho_{kj}[d]|^2}$$

(b) Now, assume that user j employs a rake receiver with two fingers which considers both the direct and the delayed paths. Derive the SINR at user j in this case.

♠ **Solution:**

For two fingers:



From Part (a), we know that,

$$u_j = s_j\rho_{jj}[0] + \alpha s_j\rho_{jj}[d] + \sum_{\substack{k=1\\k\neq j}}^{K} s_k\left(\rho_{kj}[0] + \alpha\rho_{kj}[d]\right) + z_j$$

$$v_j = \langle s[n], c_j[n-d]\rangle_N + \alpha\left\langle s[n-d], c_j[n-d]\right\rangle + \underbrace{\langle z[n], c_j[n-d]\rangle}_{\hat{z}_j}$$

Here,

$$\hat{z}_j \sim CN(0,\sigma^2), \qquad \hat{z}_j \text{ is independent of } z_j$$

$$\Longrightarrow\ v_j = s_j\rho_{jj}[d] + \sum_{\substack{k\neq j\\k=1}}^{K} s_k\rho_{kj}[d] + \alpha s_j\rho_{jj}[0] + \alpha\sum_{\substack{k\neq j\\k=1}}^{K} s_k\rho_{kj}[d] + \hat{z}_j$$

As the result,

$$\begin{aligned}\hat{s}_j &= u_j + \alpha^* v_j\\ &= s_j\rho_{jj}[0]\left(1+|\alpha|^2\right)\\ &+ \underbrace{s_j\rho_{jj}[d](\alpha+\alpha^*)}_{\substack{\text{Self Interference}\\ \hat{I}_j}}\\ &+ \underbrace{\sum_{\substack{k\neq j\\k=1}}^{K} s_k\left[\rho_{kj}[d] + \left(\alpha+|\alpha|^2\right) + \rho_{kj}[0]\left(1+\alpha^*\right)\right]}_{\substack{\text{Multi-user Interference}\\ \hat{I}_0}}\\ &+ \underbrace{z_j + \alpha^*\hat{z}_j}_{\substack{\text{NOISE}\\ \tilde{z}_j}}\end{aligned}$$

$$\Longrightarrow\ \text{SINR}_j = \frac{\left(\rho_{jj}[0]\left(1+|\alpha|^2\right)\right)^2}{\rho_{jj}^2[d]\left|\alpha+\alpha^*\right|^2 + \sum_{\substack{k\neq j\\k=1}}^{K}\left|\left[\alpha\rho_{kj}[d]+\rho_{kj}[0]\right]\left(1+\alpha^*\right)\right|^2 + \left(1+|\alpha|^2\right)\sigma^2}$$

(c) Calculate the SINR in Parts (a) and (b) and for $K = 8$, $\sigma^2 = 0.1$, $\alpha = 0.5$ and the correlation functions

$$\rho_{kk}[m] = \begin{cases}1 & m=0\\ 0.2 & m\neq 0\end{cases} \qquad \rho_{k\ell}[m] = \begin{cases}0 & m=0\\ 0.05 & m\neq 0\end{cases} \quad (k\neq\ell).$$

Compare the SINRs and justify your result.

♠ **Solution:** Setting into the results in Part (a) and (b):

$$\begin{aligned}\text{SINR}_j \text{ in case (a)} &= \frac{1^2}{0.1+(0.5)^2\times 0.2+(8-1)\times(0.05)^2}\\ &= \frac{1}{0.1675} = 5.97\end{aligned}$$

$$\text{SINR}_j \text{ in case (b)} = \frac{[1 \times (1 \times 0.5^2)]^2}{(0.2)^2 \times (1)^2 + 7 \times (0.5 \times 0.05)^2 \times (1.5)^2 + (1 + 0.5^2) \times 0.1} = \frac{1.5625}{0.1748} = 8.94$$

$$\text{SINR}_j(b) > \text{SINR}_j(a)$$

Since we have two path and two fingers in case (b): $L = M$.

But, we have two path and one finger in case (a): $L < M$.

11.4 Summer Semester 2018

Exam Date: July 27, 2018
6 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.4.1 Question 1: Short Questions (20 Points)

Answer briefly the following questions.

(a) Why is frequency reused in cellular networks?

♠ **Solution:** From *page 21* of the lecture notes, we know that frequency is reused because in practice mobile networks are limited in terms of available spectrum. By reusing the frequency, the mobile networks can handle this issue which makes service providing in mobile networks economic.

(b) Name **at least one** technique to overcome the different propagation delays in mobile networks with near-far effects.

♠ **Solution:** From *page 41*, we know that the effect of different propagation delays is addressed either by sufficiently large guard intervals or timing advance.

(c) What is the difference between an omnidirectional and an isotropic antenna?

♠ **Solution:** An isotropic antenna is an imaginary antenna whose pattern is uniform in all directions; see *page 51*. However, an omnidirectional antenna has a uniform pattern only in the horizontal plane, e.g., $\lambda/2$-dipole antennas; see *page 53*.

(d) Assume that a receiver with bandwidth $B_{RX} = 200$ kHz is employed for communication over a radio channel. Give a number for the coherence bandwidth B_C of the channel, such that it is classified as a frequency-flat (narrow-band) fading channel.

♠ **Solution:** A fading channel is classified as a frequency-flat (narrow-band) fading channel, if

$$B_{\text{Coherence}} > B_{\text{RX}}$$

See *page 128*. Hence, a good choice for the coherence bandwidth would be a value which is enough larger than B_{RX}, e.g., $B_{\text{Coherence}} = 1$ Mhz or more. However, any number larger than 200 kHz is considered correct.

(e) When using time diversity over a fading channel, what is the cost being paid in exchange for the lower bit error rate?

♠ **Solution:** From the discusstions in *Section 4.1.6, page 132,* we can conclude that:

- With time diversity, a given data signal is transmitted multiple times. This means that the time duration for transmitting the data symbols is multiplied by the order of diversity; hence, the cost is paid by *reducing the data rate.*

- In the case that the total time duration is kept fixed, time diversity requires shorter pulse duration for each signal transmission. In this case, the data rate is kept fixed; however, *the bandwidth is increased.*

Therefore, we pay either by lower data rate, or larger bandwidth. (Either of these items would be considered correct.)

(f) Name the quantitative measure which determines the reliability of a binary decision at the demodulator. Define this measure.

♠ **Solution:** From the discussions in *Section 6.4, pages 222 and 223,* we know that the quantitative measure for determining the reliability of a binary decision is the "Log-Likelihood-Ratio (LLR)".

Assume that symbol d is detected at the receiver. Let d be used to decide for binary sequence $q_1, \ldots, q_I$. Then, the LLR for each binary bit q_i, $i = 1, \ldots, I$ is defined as

$$\text{LLR}(q_i) = \ln\left(\frac{\Pr\{q_i = +1|d\}}{\Pr\{q_i = -1|d\}}\right).$$

See the definition in *equation (6.79), page 223.*

(g) You intend to achieve Peak-to-Average-Ratio (PAR) PAR $= 2$ via a Root-Raised-Cosine (RRC) pulse shape. How much do you save in terms of the roll-off factor, if you use the $\pi/4$-shifted Quadrature-Phase-Shift-Keying (QPSK) constellation instead of the standard QPSK constellation?

♠ **Solution:** *Figure 6.12 in page 199* indicates that PAR $= 2$ is achieved via an RRC pulse and QPSK constellation, when the roll-off factor is set to $\alpha = 0.185$. The required roll-off factor when we use $\pi/4$-shifted QPSK is $\alpha = 0.135$. Hence, one saves by this change around $\Delta\alpha = 0.05$ in terms of the roll-off factor.

(h) How can we realize a convolutional channel code with $K = 1$ (according to the notation in the lecture notes) and a code rate of 0.6?

♠ **Solution:** As indicated in *Section 7.2.2, page 253,* a convolutional code with $K = 1$ and coding rate greater than 0.5 is constructed by means of *puncturing.*

(i) Which type of channel codes are used in GSM for error detection? Briefly illustrate why it differs from the type of channel codes used for error correction.

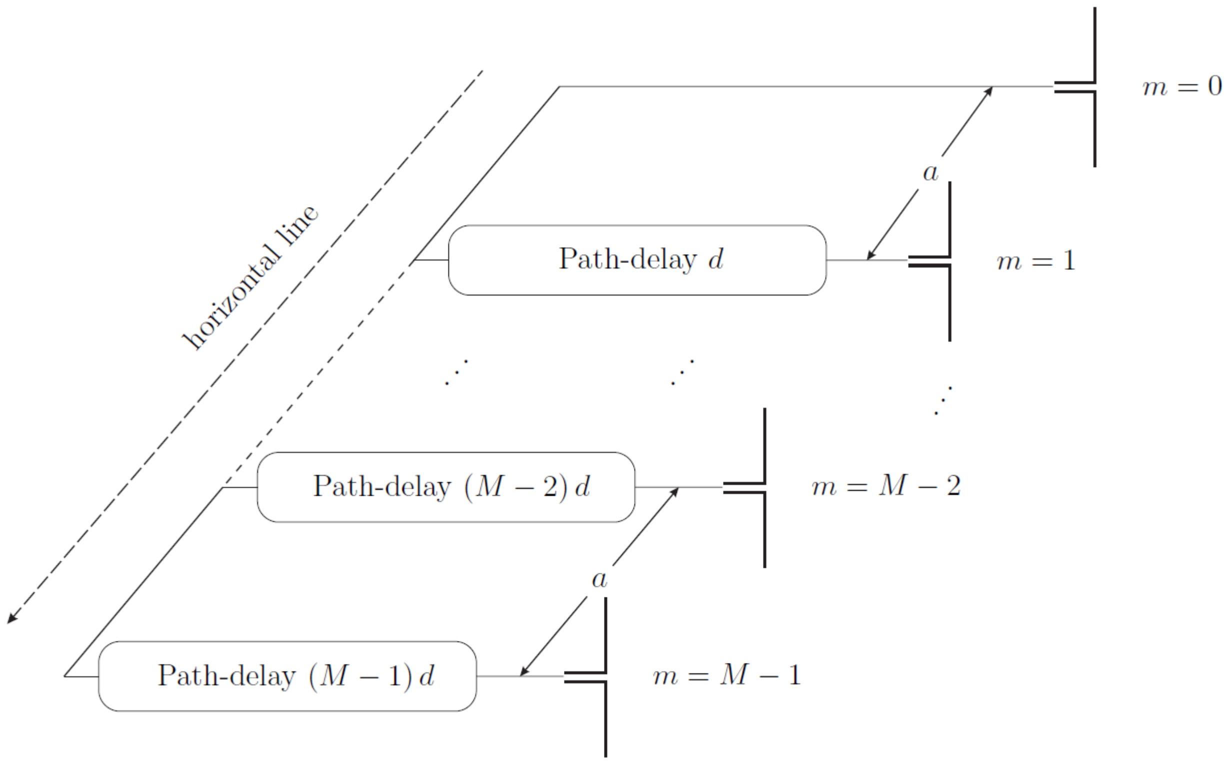
♠ **Solution:** From *Section 8.3.1, page 271,* we know that GSM uses *block codes* for error detection while convolutional codes are used for error correction. The difference comes from the experimental observation showing that block codes are superior with respect to error detection, and convolutional codes correct errors better.

(j) Why is the training sequence in the GSM synchronization burst longer than the training sequence in the GSM normal burst?

♠ **Solution:** The synchronization burst is provided by a longer training sequence to achieve high noise suppression for channel estimation, even when the time position of the synchronization burst is roughly known; see *page 284.*

11.4.2 Question 2: Antenna Arrays (16 Points)

Consider a horizontal antenna array consisting of M identical $\lambda/2$-dipole antennas. The distance between each two neighbouring antennas is a; see the figure below. The antenna are indexed by $m = 0, \ldots, M-1$. The signal def into m-th antenna is delayed with path-delay md.



(a) Determine the radiation pattern $g(v, \alpha)$ of the antenna array.
Hint: The following identity holds

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

♠ **Solution:**

The antenna array lies on the horizontal plane. Hence, the array gain would be only a function of the horizontal angle α. On the other hand, the array pattern of a $\lambda/2$-dipole is only a function of the vertical angle ϑ. Consequently, we can write

$$g(\vartheta, \alpha) = g_d(\vartheta)\, g_A(\alpha)$$

where $g_A(\alpha)$ is the array patten, and $g_d(\vartheta)$ is the radiation pattern for $\lambda/2$-dipole given in *page 52* of the lecture notes, i.e.,

$$g_d(\vartheta) = \frac{\cos^2\left(\frac{\pi}{2}\cos\vartheta\right)}{\sin^2\theta}.$$

To calculate the array gain, we invoke the formulation in *page 59* where the array factor AF(α) is given by

$$\text{AF}(\alpha) = \sum_{m=0}^{M-1} w_m \exp\left\{-j2\pi m\frac{a}{\lambda}\sin\alpha\right\}.$$

Considering the path delays at the feeding stage of each antenna in the array, the weighting factor w_m reads

$$w_m = \exp\left\{-j2\pi m\frac{d}{\lambda}\right\}.$$

Thus, we have

$$\begin{aligned}
\text{AF}(\alpha) &= \sum_{m=0}^{M-1} \exp\left\{-j\frac{2\pi}{\lambda}m(d+a\sin\alpha)\right\} \\
&= \sum_{m=0}^{M-1} \left(\exp\left\{-j\frac{2\pi}{\lambda}(d+a\sin\alpha)\right\}\right)^m \\
&= \frac{1-\exp\left\{-j\frac{2\pi}{\lambda}M(d+a\sin\alpha)\right\}}{1-\exp\left\{-j\frac{2\pi}{\lambda}(d+a\sin\alpha)\right\}}. \qquad \text{(from the hint)} \\
&= \exp\left\{j(M-1)\Delta\psi/2\right\}\frac{\sin(M\Delta\psi/2)}{\sin(\Delta\psi/2)}.
\end{aligned}$$

where $\Delta\psi$ is

$$\Delta\psi = \frac{2\pi}{\lambda}(d+a\sin\alpha).$$

AF(α) is a shifted version of the array factor derived for linear arrays with equal weights; see *Section 2.3.1.1 in page 59*. As the result, the array pattern reads

$$\begin{aligned}
g_A(\alpha) &= \frac{|\text{AF}(\alpha)|^2}{\max_\alpha |\text{AF}(\alpha)|^2} \\
&= \frac{\sin^2\left(M\frac{\pi}{\lambda}(d+a\sin\alpha)\right)}{M^2\sin^2\left(\frac{\pi}{\lambda}(d+a\sin\alpha)\right)}.
\end{aligned}$$

We can thus conclude that the pattern is

$$g(\vartheta,\alpha) = \frac{\cos^2\left(\frac{\pi}{2}\cos\vartheta\right)\sin^2\left(M\frac{\pi}{\lambda}(d+a\sin\alpha)\right)}{M^2\sin^2\vartheta\sin^2\left(\frac{\pi}{\lambda}(d+a\sin\alpha)\right)}.$$

where ϑ and α are the vertical and horizontal angles.

(b) For antenna spacing $a = \lambda/4$, find d such that the main beam of the antenna array in the horizontal plane points to $\alpha = 45°$.

♠ **Solution:** The maximum of the pattern on the horizontal plane occurs when

$$\Delta\psi/2 = k\pi$$

for some integer k. This means that when the main beam points towards the angle $\alpha = \alpha^\star$, we have

$$d + a \sin \alpha^\star = k\lambda$$

Hence, when $\alpha = \lambda/4$, to have the main beam at $\alpha = 45°$, we should set

$$d = k\lambda - \frac{\lambda}{4\sqrt{2}}.$$

One choice would thus be $d = 0.8232\,\lambda$.

11.4.3 Question 3: Mobile Radio Channels(16 Points)

Consider a base-station with 10 m height placed at the top of a skyscraper with height 100 m in New York City. This base station is employed for downlink transmission to a mobile station located on the roof of a van (automobile) in the city. The distance of the mobile station from the roof of the van is 1 m. The base-station operates at a carrier frequency which is exactly at the middle of the downlink band in PCS1900 standard. The gain of the transmit antenna at the base-station g_{BS} and the receive antenna at the mobile station g_{MS} are both one, i.e., $G_{BS} = G_{MS} = 0$ dB$_i$.

(a) Why is such a scenario impossible in practice?

♠ **Solution:** Such a scenario is not possible in practice, since the antenna at the base station and the mobile station are assumed to be *isotropic*.

The mobile station is driving within the city such that its distance from the base-station is approximately fixed to $d = 9$ km during the drive.

(b) Using the Okumura-Hata model, calculate the median path-loss in dB.

♠ **Solution:** The Okumura-Hata model is illustrated in *page 92* of the lecture notes. New York City is an urban area in this model. The carrier frequency is more than 1.5 GHz. The median math-loss in logarithmic scale is therefore given by *equation (3.31)*. For this scenario, the parameters are

- → $f_0 = (1930 + 1990)/2 = 1960$ MHz.
- → $h_{\text{BS}} = 100 + 10 = 110$ m.
- → $h_{\text{MS}} = 1 + 1.5 = 2.5$ m, where we have assumed the height of the van to be 1.5 m. Any assumption between 1 m and 3 m is however reasonable.

$\rightarrow\ d = 9$ km.

The median path-loss is therefore given by

$$\text{PL} = 161.38 - 2.9\, h_{\text{MS}} = 154.13 \text{ dB}.$$

For other assumptions on the height of the van, $2 \leq h_{\text{MS}} \leq 4$; hence,

$$149.78 \leq \text{PL} \leq 155.58$$

is reasonable for the median path-loss.

The base-station transmits signals with power $P_T = 200$ mW. The equivalent noise bandwidth at the receiver is $B_{noise} = 200$ kHz, and the noise figure is $F = 1$. Furthermore, the noise power spectral density is $N_0 = 4 \times 10^{-18}$ mW/Hz.

(c) Assuming that the standard deviation of the shadowing process is $\sigma_\Lambda = 1.6$, determine the probability that the signal-to-noise ratio at the received side is less than SNR $= 1$ in terms of the Q-function $Q(x)$.
Hint: For Gaussian random variable X with mean μ and standard deviation σ,

$$\text{Pr}\{X > u\} = Q\left(\frac{u-\mu}{\sigma}\right).$$

♠ **Solution:** From *Section 3.3 in page 94,* we know that due to shadowing in the channel, the received power is distributed normally in the logarithmic scale. The transmit power is $P_T = 200$ mW or equivalently $P_T = 23$ dBm. From Part (b), we know that the median path-loss is PL $= 154.13$ dB. Hence, the average received power is

$$\begin{aligned}\mu_\Lambda &= P_T - \text{PL} \\ &= 23 - 154.13 = -131.13 \text{ dBm}.\end{aligned}$$

For other assumptions on h_{MS}, the average received power would read

$$-132.58 \text{ dBm} \leq \mu_\Lambda \leq -126.78 \text{ dBm}.$$

The standard deviation of the received power in the linear scale is moreover $\sigma_\Lambda = 1.6$ which in logarithmic scale is $\sigma = 2$ dB.

The minimum required signal-to-noise ratio is SNR$= 1$, i.e., SNR$= 0$ dB. Furthermore, the noise power at the receive side is

$$P_N = N_0\, B_{\text{noise}}\, F = 8 \times 10^{-13} \text{ mW},$$

or equivalently, $P_N = -120.97$ dBm; *see equation (1.10) in page 34.* Thus, to have the minimum signal-noise-ratio of SNR$= 0$ dB, the received power should read

$$\Lambda \geq -120.97 \text{ dBm}.$$

Noting that Λ is normally distributed, the outage probability is given by

$$\begin{aligned}\text{Pr}_{\text{Out}} = \text{Pr}\{\Lambda < -120.97\} &= 1 - \text{Pr}\{\Lambda > -120.97\} \\ &= 1 - Q\left(\frac{-120.97 + 131.13}{2}\right) \\ &= 1 - Q(5.08)\,.\end{aligned}$$

For other choices of h_{MS}, one would conclude

$$1 - Q(2.905) \leq \text{Pr}_{\text{Out}} \leq 1 - Q(5.805)$$

Which depends on the assumed height for the van.

11.4.4 Question 4: Fading Channels (12 Points)

Consider a **time invariant** wireless channel whose impulse response is given by

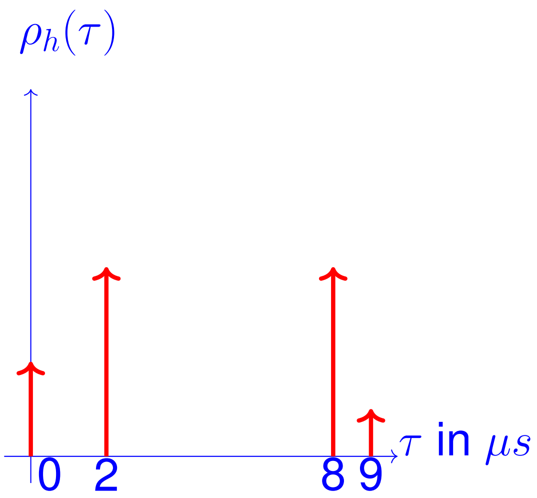
$$h(\tau) = \frac{1+\sqrt{2}j}{2\sqrt{3}}\delta(\tau) + \frac{2-j}{\sqrt{10}}\delta(\tau - 2\mu\text{s}) + \frac{1+j}{\sqrt{4}}\delta(\tau - 8\mu\text{s}) + \frac{\sqrt{2}-j}{2\sqrt{6}}\delta(\tau - 9\mu\text{s})$$

(a) Determine the power-delay profile of this channel and plot it.

♠ **Solution:** The power-delay profile of the channel is defined as delay-time auto-correlation function of the channel at $\Delta t = 0$; see *page 120*. From *equation (3.79) in page 118,* the auto-correlation function reads

$$\begin{aligned}\rho_h(\tau, \Delta t) &= \int_{-\infty}^{+\infty} \mathbb{E}\{h^*(\tau_1, t)\, h(\tau, \Delta t)\}\, \mathrm{d}\tau_1 \\ &= \frac{1}{4}\delta(\tau) + \frac{1}{2}\delta(\tau - 2\;\mu s) + \frac{1}{2}\delta(\tau - 8\;\mu s) + \frac{1}{8}\delta(\tau - 9\;\mu s) = \rho_h(\tau)\end{aligned}$$

which is not a function of Δt, since the channel is time invariant.



(b) Calculate the mean delay time of this channel.

♠ **Solution:** Using the power-delay profile in the previous part, the transfer factor from *equation (3.86)* in *page 123* reads

$$h_P = \int_0^\infty \rho_h(\tau)\mathrm{d}\tau = \frac{11}{8} = 1.375.$$

Substituting in *equation (3.87), page 123,* the mean delay of the channel would be

$$\mu_\tau = \frac{1}{h_P}\int_0^\infty \tau\rho_h(\tau)\ \mathrm{d}\tau = \frac{8}{11}\times\left(\frac{1}{4}\times 0+\frac{1}{2}\times 2+\frac{1}{2}\times 8+\frac{1}{8}\times 9\right) = \frac{49}{11}.$$

Hence, the mean delay is $\mu_\tau = 4.45\ \mu s$.

(c) Determine the delay-Doppler spectrum of this channel $S(\tau,\nu)$.
Hint: The Fourier transform of a constant function $h(t) = C$ is

$$H(f) = \mathcal{F}\{C\} = 2\pi C\ \delta(2\pi f).$$

♠ **Solution:** From *Figure 3.35* in *page 112,* we know that

$$S(\tau,\nu) = \mathcal{F}_t\{h(\tau,t)\}$$

Since the channel is time invariant, $h(\tau,t) = h(\tau)$ meaning that it is constant in terms of t. Hence, from the hint we conclude that

$$S(\tau,\nu) = 2\pi h(\tau)\delta(2\pi\nu)$$

where $h(\tau)$ is the one given in Part (a).

11.4.5 Question 5: Code-Division-Multiplexing (20 Points)

Consider a cellular network with K users. The base-station employs the Code-Division-Multiplexing (CDM) technique with spreading factor N to transmit data $a_1,\ldots,a_K$ simultaneously to the users in downlink. The spreading sequence for user k is

$$\{c_k[0],\ldots,c_k[N-1]\}.$$

Moreover, the base-station uses the rectangular chip pulse

$$g(t) = \begin{cases} 1 & t\in[0,T_C] \\ 0 & \text{Otherwise} \end{cases}$$

where T_C denotes the chip duration and is related to the symbol duration T_S via $T_S = NT_C$. In this system, the channel from the base-station to user k is time invariant, and its impulse response is given by

$$h_k(\tau) = \frac{1}{\sqrt{3}}\delta(\tau) + \frac{1+2j}{\sqrt{15}}\delta(\tau-3T_C) + \frac{1-j}{\sqrt{6}}\delta(\tau-5T_C)$$

for all $k = 1,\ldots,K$.

(a) Write the complex base-band representation for the k-th transmit signal $s_k(t)$ in terms of the data symbol a_k, the k-th spreading sequence and the chip pulse $g(t)$.

♠ **Solution:** From *Section 5.3* in *page 156,* we can write the pulse shape for the k-th user as

$$p_k(t) = \sum_{n=0}^{N-1} c_k[n]\, g(t - nT_C).$$

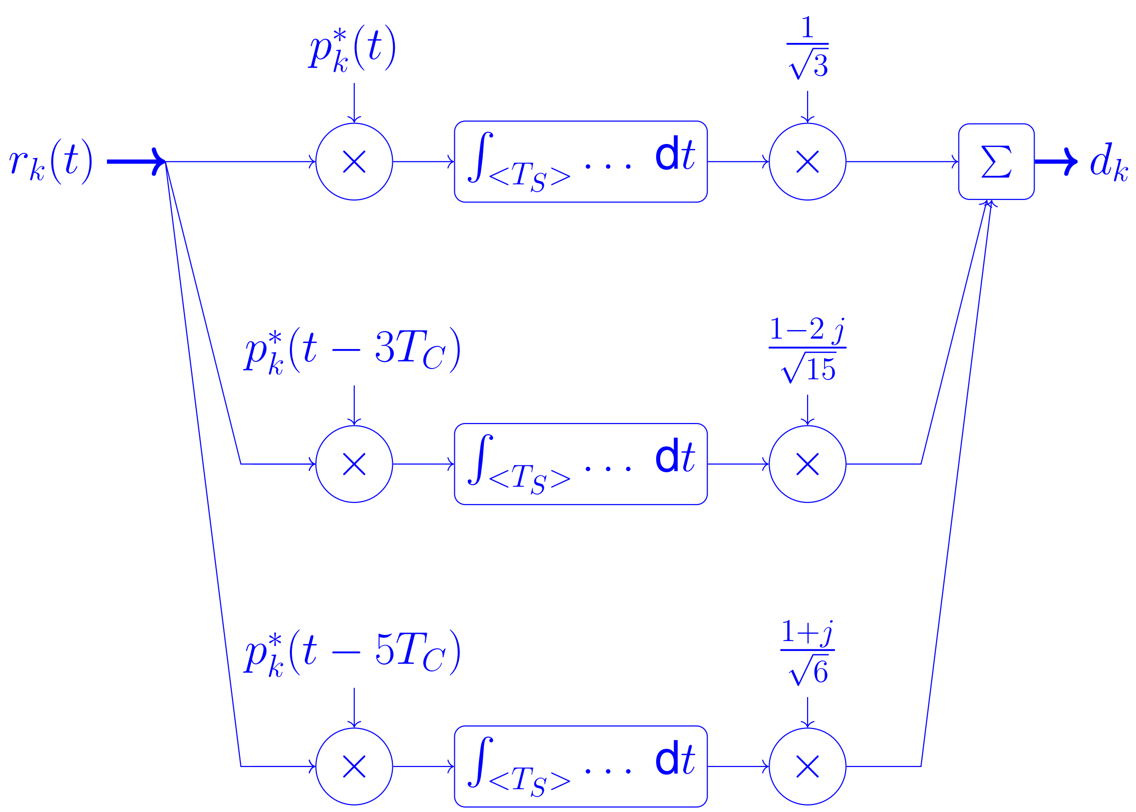
The transmit signal for user k is then

$$s_k(t) = a_k\, p_k(t) = \sum_{n=0}^{N-1} a_k\, c_k[n]\, g(t - nT_C)$$

where a_k is the data symbol for user k.

(b) Design the rake receiver for user k assuming that the user knows the channel impulse response $h_k(t)$ exactly. Specify all the parameters and coefficients in the receiver.

♠ **Solution:** Let us denote the receive signal at user k with $r_k(t)$ and the estimated symbol with d_k. Following the discussions in *Section 5.3.1, page 160,* the rake receiver for user k has three fingers. Since the user is provided with the exact channel impulse response, the block diagram of the receiver reads

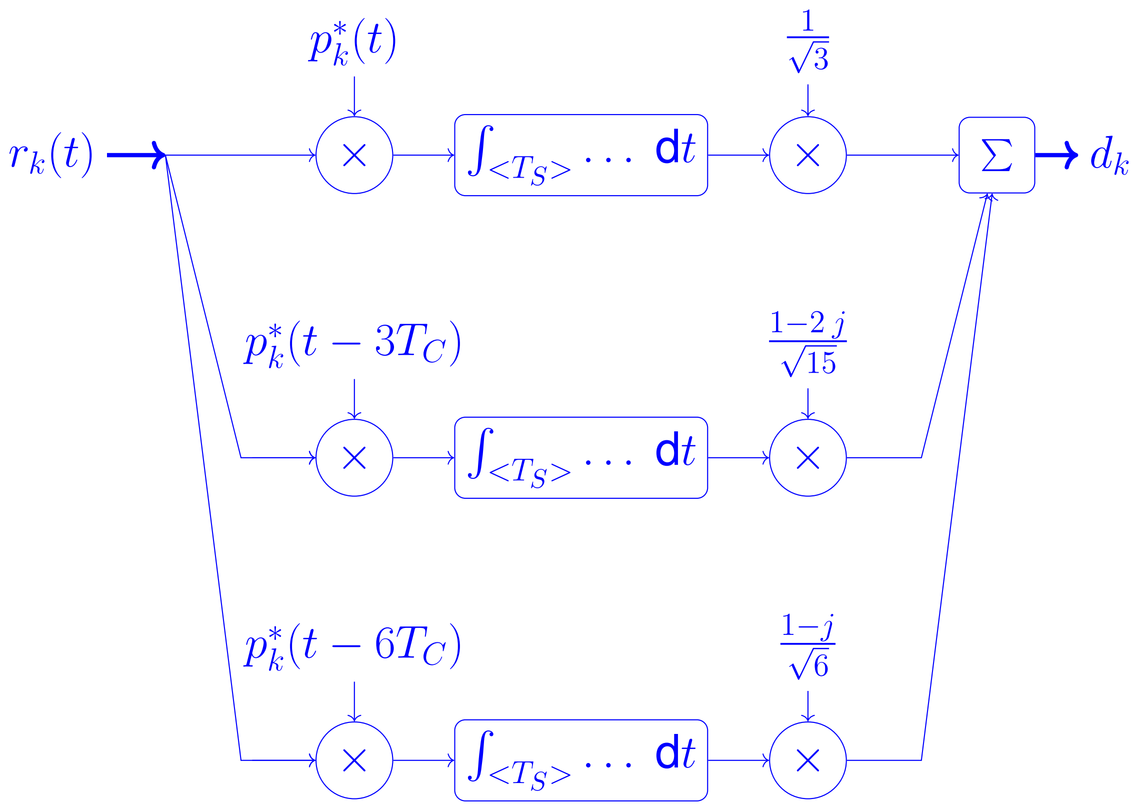


(c) Now assume that user k estimates the channel impulse response as

$$\hat{h}_k(\tau) = \frac{1}{\sqrt{3}}\delta(\tau) + \frac{1+2j}{\sqrt{15}}\delta(\tau - 3T_C) + \frac{1+j}{\sqrt{6}}\delta(\tau - 6T_C).$$

Considering this estimated impulse response, design the rake receiver for user k, and specify all the parameters and coefficients in the receiver.

♠ **Solution:** Here, the last channel tap is estimated wrongly. Hence, the rake receiver in this case differs from the one in the previous part, in the last finger. The block diagram is



(d) Without doing any calculations, compare the performances of the rake receivers in Parts (b) and (c). Justify your answer by giving reasons.

♠ **Solution:** As in the Part (c) the delay of the third finger is not as the exact delay in the third path, the output of the matched filter would be only noise [1]. Therefore, the receiver in Part (b) outperforms the one in Part (c).

[1] This holds, when we assume perfect spreading sequences.

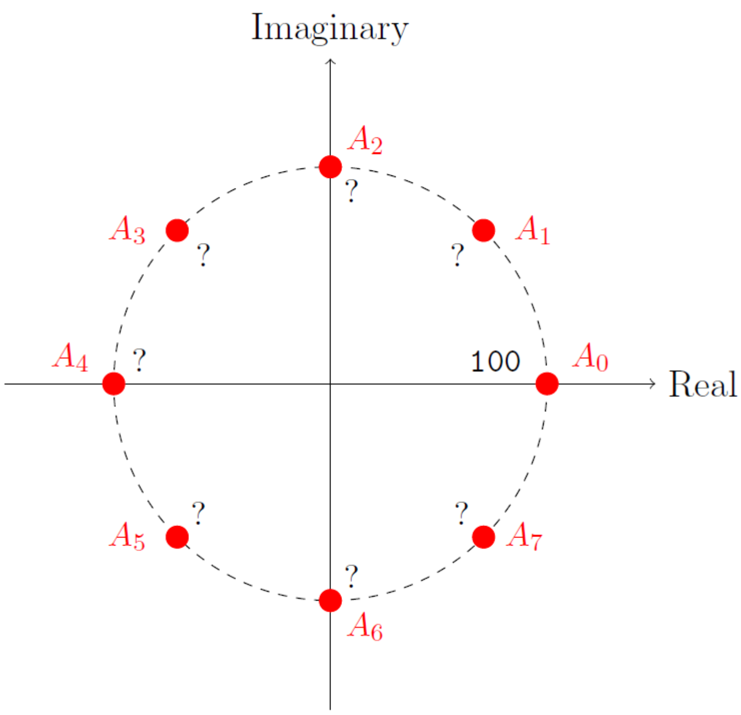
11.4.6 Question 6: Modulation Schemes (16 Points)

Consider the 8-Phase-Shift-Keying (8-PSK) constellation. The eight constellation points are located at

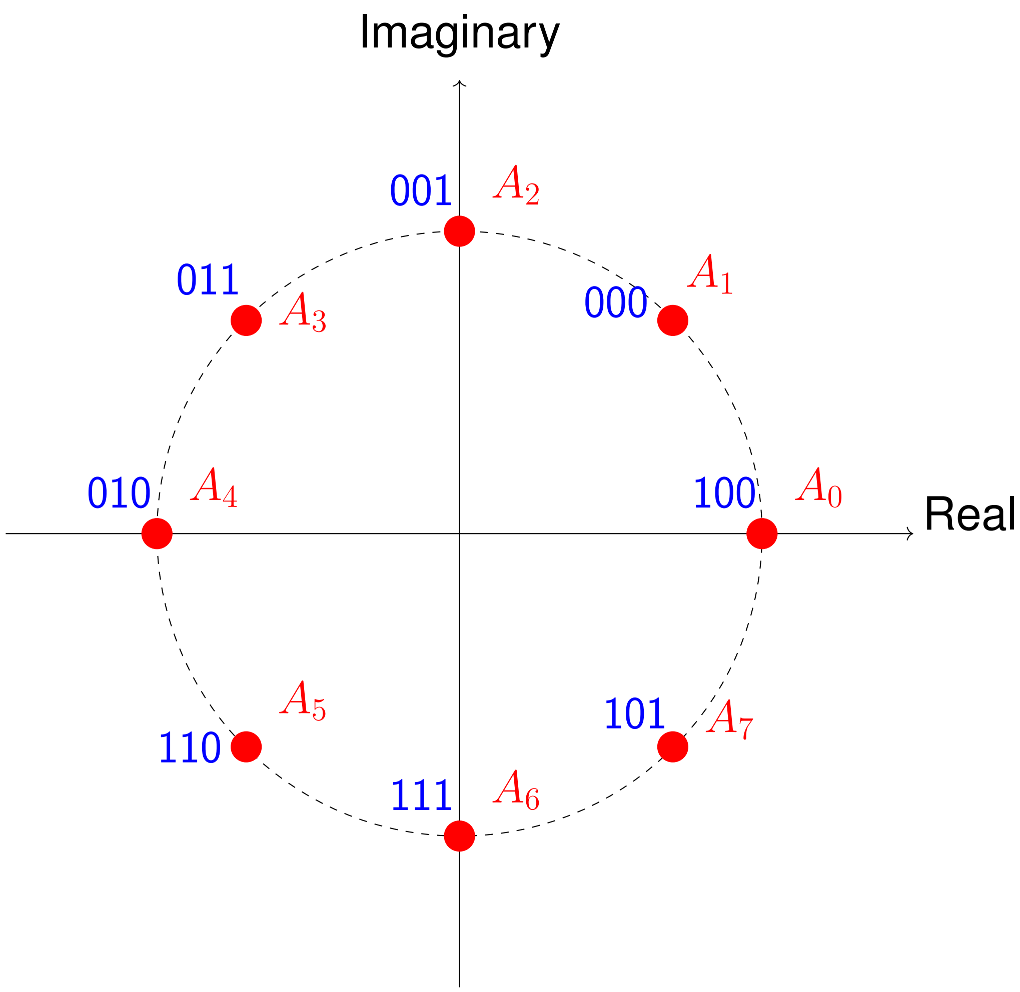
$$A_m = e^{jm\frac{\pi}{4}} \qquad \text{for } m = 0, \ldots, 7.$$

Gray mapping is used to map the data bits onto the constellation points.

(a) Assuming the data bits 100 are mapped onto A_0, find a choice of data bits corresponding to the other constellation points; see the figure below.
Hint: Remember that you are using Gray mapping.



♠ **Solution:** Since Gray mapping is used, one should map the information bits, such that for any two neighboring constellation points, the mapped sequences be different only in one bit. One choice is to shift the mapping in *Figure 6.20, page 212* by $\pi/4$ which concludes the following mapping. Other solutions are also possible.



(b) Given the data stream

$$110101001011$$

find the sequence of the corresponding constellation points.

♠ **Solution:** Since we use an 8-PSK constellation, each transmit symbol represents three bits. Therefore, the transmit symbols would be

$$110\ 101\ 001\ 011 \implies A_5\ A_7\ A_2\ A_3$$

Clearly, other choices for the mapping conclude different sequences.

(c) Now assume that the transmission is done over a frequency-flat (narrow-band) fading channel. The average bit error rate at the signal-to-noise ratio SNR $= 10$ is desired to be less than 10^{-2}.
You are allowed to choose between **two or four** branches of diversity. Which one would you choose? Justify your answer by giving a reason.

♠ **Solution:** From *Figures 6.28 and 6.29* in *page 220 and 221* of the lecture notes, we see that at SNR$=$ 10 (or equivalently SNR$=$ 10 dB), the bit error rates with two branches of diversity BER_2 and four branches of diversity BER_4 read

$$\begin{aligned}\text{BER}_2 &\approx 5 \times 10^{-3} < 10^{-2}\\ \text{BER}_4 &\approx 10^{-4} < 10^{-2}.\end{aligned}$$

This indicates that both setups achieve the desired bit error rate at $\text{SNR} = 10$. Consequently, one should choose two branches, to avoid further costs, such as lower data rate or higher bandwidth, caused by higher order of diversity.

11.5 Summer Semester 2019

Exam Date: August 9, 2019
5 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.5.1 Question 1: Short Questions (26 Points)

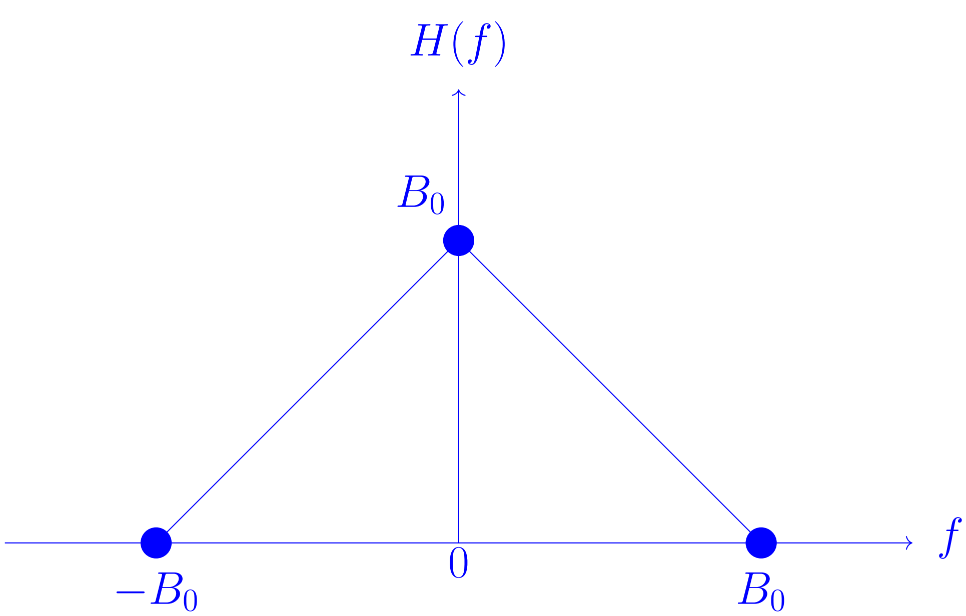
Answer the following questions briefly.

(a) A receiver consists of a bandpass filter and a low noise amplifier (LNA). The transfer function of the filter is

$$H(f) = \begin{cases} B_0 + f & -B_0 \leq f \leq 0 \\ B_0 - f & 0 < f \leq B_0 \\ 0 & \text{otherwise} \end{cases}$$

where $B_0 = 3$ MHz. Furthermore, the noise figure of the LNA is $\log F = 3$ dB. Calculate the *effective* noise power. Give your final answer *in dBm*.

♠ **Solution:**



From Eq. (1.8) in page 32:

$$\begin{aligned}
B_{\text{noise}} &= \frac{1}{|H(0)|^2} \cdot \int |H(f)|^2 \, \mathrm{d}f \\
&= \frac{1}{B_0^2} \cdot \left[\int_{-B_0}^{0} (B_0 + f)^2 \, \mathrm{d}f + \int_{0}^{B_0} (B_0 - f)^2 \, \mathrm{d}f \right] \\
&= \frac{1}{B_0^2} \cdot \left[\frac{1}{3}(B_0 + f)^3 \Big|_{-B_0}^{0} + \frac{1}{3}(B_0 - f)^3 \Big|_{0}^{B_0} \right] \\
&= \frac{1}{B_0^2} \left(\frac{1}{3} B_0^3 + \frac{1}{3} B_0^3 \right) = \boxed{\frac{2}{3} B_0 = 2 \times 10^6 \text{ Hz}}
\end{aligned}$$

From Eq. (1.10) in page 32:

$$P_N = N_0 \, B_{\text{noise}} \, F \qquad \text{(linear scale)}$$

$$\begin{aligned}
\Longrightarrow \quad \log P_N \text{ [dBm]} &= \overbrace{\log N_0}^{\text{dBm/Hz}} + \overbrace{\log B_{\text{noise}}}^{\text{Hz}} + \overbrace{\log F}^{\text{dB}} \\
&= -174 + 63 + 3 = \boxed{-108 \text{ dBm}}
\end{aligned}$$

(b) Consider the following two statements:

(A) An isotropic antenna is an omnidirectional antenna.

(B) An omnidirectional antenna is an isotropic antenna.

Which of these statements is *correct*? Explain your answer by giving a reason.

♠ **Solution:** (A) is correct.

Reason: An isotropic antenna is an imaginary antenna which radiates the same in all directions. As a result, it also radiates the same "in all directions in the horizontal plane" which makes it also an "omnidirectional antenna".

$$\Longrightarrow \boxed{\text{(A) is correct.}}$$

A dipole antenna is also an omnidirectional antenna which is <u>not</u> isotropic.

$$\Longrightarrow \boxed{\text{(B) is wrong.}}$$

(c) The coherence bandwidth of a mobile channel is defined as frequency B_c at which

$$|\rho_H(B_c, 0)| = \frac{1}{2} |\rho_H(0, 0)|$$

where $\rho_H(\Delta f, \Delta t)$ denotes the *frequency-time* correlation function of the channel. Calculate the coherence bandwidth for a mobile channel whose power delay profile is

$$\rho_h(\tau, 0) = \sqrt{2}\delta(\tau) + \sqrt{2}\delta(\tau - T_{Burst})$$

where T_{Burst} denotes the time duration of the *GSM normal burst*.
Hint: Note that

$$\mathcal{F}\{x(t-\tau)\} = \exp\{-j2\pi f\tau\}X(f),$$

where $X(f) = \mathcal{F}\{x(t)\}$ is the Fourier transform of $x(t)$.

♠ **Solution:** From Figure 3.39 in Page 116:

$$\rho_H(\Delta f, \Delta t) = \mathcal{F}_\tau\left\{\rho_h(\tau, \Delta t)\right\} \implies \rho_H(\Delta, 0) = \mathcal{F}_\tau\left\{\rho_h(\tau, 0)\right\}$$

As a result:

$$\begin{aligned}\implies \rho_H(\Delta f, 0) &= \sqrt{2} + \sqrt{2}\, e^{-j2\pi\Delta f T_{\text{Burst}}} \\ &= \sqrt{2}\left[(1+\cos\left(2\pi T_{\text{Burst}}\Delta f\right)) - j\sin\left(2\pi T_{\text{Burst}}\Delta f\right)\right]\end{aligned}$$

To find B_c, we write:

$$|\rho_H(B_c, 0)| = \frac{1}{2}\,|\rho_H(0,0)| \implies |\rho_H(B_c, 0)|^2 = \frac{1}{4}\,|\rho_H(0,0)|^2$$

Using the derived frequency-time correlation, we have:

$$\begin{aligned}|\rho_H(\Delta f, 0)|^2 &= \left(\sqrt{2}\right)^2\left[(1+\cos\left(2\pi T_{\text{Burst}}\Delta f\right))^2 + \sin^2\left(2\pi T_{\text{Burst}}\Delta f\right)\right] \\ &= 2\left[1 + 2\cos\left(2\pi T_{\text{Burst}}\Delta f\right) + 1\right] \\ &= 4 + 4\cos\left(2\pi T_{\text{Burst}}\Delta f\right)\end{aligned}$$

Hence, B_c reads:

$$4 + 4\cos\left(2\pi T_{\text{Burst}} B_c\right) = 2$$

$$\implies \cos\left(2\pi T_{\text{Burst}} B_c\right) = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$$

From page 277:

$$\boxed{T_{\text{Burst}} = 156.25 \text{ bits} = 577\ \mu s}$$

$$\implies T_{\text{Burst}}\, B_c = \frac{1}{3} \implies B_c = \frac{1}{3T_{\text{Burst}}} = \frac{10^6}{3\times 577} = 577.7 \text{ Hz}$$

Thus,

$$\boxed{B_c \approx 0.58 \text{ kHz}}$$

(d) Consider a BPSK transmission over a *static* additive white Gaussian noise (AWGN) channel which is encoded via a convolutional code. The code rate is $R_c = 0.5$ and the constraint length is $L_c = 7$. The decoder uses the *Viterbi* decoding algorithm with *hard decision*. Determine the *maximum* signal-to-noise ratio (SNR) at which channel coding is *detrimental (disadvantageous)*.
Explain why channel coding degrades the performance for SNRs less than determined value.

♠ **Solution:** From Figure 7.12 in page 252:

The figures meet at some point between 3 and 4 dB.

$$\Longrightarrow \text{SNR}_{\text{max}} \approx 3.3 \text{ dB} \quad \text{(any \# between 3 and 4 is correct.)}$$

Reason: In page 253 at the 3rd bullet is said:

"At low SNRs, the decoder introduces additional errors instead of correcting them."

This makes the coding disadvantageous in low SNRs.

(e) Assume 15 users intend to establish *two-way* communication in the GSM system *simultaneously*. What is the *minimum* number of frequency division multiplexing (FDM) channels required to support this device?

♠ **Solution:**

- GSM uses FDD for duplexing. (Page 262).
- In each FDM channel, 8 users are multiplexed via TDM. (Check Table 8.1 in page 262.)

$\Longrightarrow$ Number of FDM channels needed for "One-way" communications.

$$\left\lceil \frac{15}{8} = 2 \right\rceil$$

$\Longrightarrow$ For "Two-way" communications:

We need double number of channels, one for uplink and one for down link. Therefore:

$$\boxed{\text{Required number of channels} = 2 \times 2 = 4}$$

11.5.2 Question 2: Mobile Radio Channels (16 Points)

A radio transmitter is located on the top of the Berliner Funkturm in Berlin whose height is 146 m. A mobile receiver is located in Schlossgarten Charlottenburg about 4 km away from the radio tower in the first floor of a palace. The transmitter and the receiver are equipped with *two* $\lambda/2$-dipole antennas each.

The radio tower transmits a signal with power

$$P_{TX} = 100 \text{ mW}$$

in two different transmission time intervals:

- In the first interval, it uses a carrier frequency which is exactly in the middle of the downlink band of GSM900.

- In the second interval, it switches to the center frequency of the PCS1900 downlink band.

Let P_1 and P_2 denote the *median received power* in the first and second time interval, respectively.

(a) Calculate the power ratio

$$\xi = \frac{P_1}{P_2}.$$

♠ **Solution:** The center frequency of GSM900-Downlink $= \frac{935+960}{2} = 947.5$ MHz.

$$\Longrightarrow \boxed{f_1 = 947.5 \text{ MHz}}$$

The center frequency of PCS1900-Downlink $= \frac{1930+1990}{2}$.

$$\Longrightarrow \boxed{f_2 = 1960 \text{ MHz}}$$

Given the scenario, we use Okumura-Hata model:

$$\begin{aligned} h_{\text{BS}} &= 146 \text{ m} \\ h_{\text{MS}} &= 3 \text{ m (any number between 1 m and 5 m is OK)} \\ d &= 4 \text{ km} \end{aligned}$$

The receiver power P_1 is given by:

$$P_1 = \frac{P_{\text{TX}}}{a_1} \cdot g_{\text{BS}} \cdot g_{\text{MS}} \Longrightarrow \begin{cases} a_1: & \text{attenuation in interval 1} \\ g_{\text{BS}}: & \text{gain of BS antenna} \\ g_{\text{MS}}: & \text{gain of MS antenna} \end{cases}$$

$$P_2 = \frac{P_{\text{TX}}}{a_2} \cdot g_{\text{BS}} \cdot g_{\text{MS}}$$

$$\Longrightarrow \xi = \frac{a_2}{a_1} \Longrightarrow \log \xi \text{ [dB]} \begin{aligned}[t] &= \log a_2 - \log a_1 \\ &= \text{PL}_2 - \text{PL}_1 \end{aligned}$$

PL_i denotes the path-loss in interval i. The path-loss in the first interval is given by Eq. (3.27) in page 89, and PL_2 is calculated via Eq. (3.31) in page 90, due to their carrier frequency. Hence,

$$\begin{aligned} \log \xi &= \text{PL}_2 - \text{PL}_1 \\ &= \big(46.3 + 33.9 \log f_2 - \underline{13.82 \log h_{\text{BS}}} - a_2(h_{\text{MS}}) + \underline{10n \log d}\big) \\ &\quad - \big(69.55 + 26.16 \log f_1 - \underline{13.82 \log h_{\text{BS}}} - a_1(h_{\text{MS}}) + \underline{10n \log d}\big) \end{aligned}$$

The underlined terms cancel out. Thus,

$$\log \xi \text{ [dB]} = -23.25 + 33.9 \log f_2 - 26.16 \log f_1 - a_2(h_{\mathsf{MS}}) + a_1(h_{\mathsf{MS}})$$

$$\Longrightarrow \boxed{\log \xi = 9.9409 \text{ dB}}$$

(b) Now assume that the *same scenario* happens in *Erlangen*. This means that the mobile station and the base station have exactly the same heights and are in the same distance from each other. How does the power ratio ξ change in this case?

♠ **Solution:** This ratio does not change, because the change of area impacts the receiver power P_1 and P_2 the same.

11.5.3 Question 3: Fading Channels (14 Points)

A signal is transmitted over a multipath fading channel in a *rich scattering* environment. The transmitter transmits the signal

$$s(t) = \sqrt{P} \exp\{j2\pi f_0 t\}$$

where f_0 is the carrier frequency and $P = 10$ mW.

Due to the small bandwidth of the signal, the receiver is *unable* to distinguish among the signals received from different paths. Therefore, it models the received signal $Y(t)$ as

$$Y(t) = H(t)\, s(t) + N(t)$$

where $H(t)$ is a stochastic process modelling the fading, and $N(t)$ is additive white Gaussian noise (AWGN) with zero mean and variance $\sigma^2 = 0.05$ mW.

The measurements show that the average power of the received signal is 8.55 mW. This means that

$$\mathcal{E}\{|Y(t)|^2\} = 8.55 \text{ mW}$$

where $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation. For stochastic process $H(t)$, the following information is given:

▷ The variance and average of $H(t)$ are related as follows:

$$\text{Variance of } H(t) = \mathcal{E}\{|H(t) - \mathcal{E}\{H(t)\}|^2\} = \frac{1}{2}|\mathcal{E}\{H(t)\}|^2.$$

(a) Does the transmitter experience a *narrow-band* fading process? Give a reason for your answer.

♠ **Solution:** Yes, because it cannot distinguish between different paths. This means that the max delay is significantly smaller than transmission time ($\tau_{\text{ex}} << T$).

$\Longrightarrow$ The coherence bandwidth which is proportional to $\frac{1}{\tau_{\text{ex}}}$ is significantly larger than the signal bandwidth B which is proportional to $\frac{1}{T}$.

(b) What is the distribution of $H(t)$? Explicitly specify all the parameters of the distribution.

♠ **Solution:** Since

$$\mathcal{E}\left\{|(H(t)|^2\right\} \neq \text{ Variance of } H(t),$$

We conclude that

$$\mathcal{E}\{H(t)\} \neq 0$$

This means that the fading process is Rice.

As a result, we have:

$$H(t) \sim CN(\mu_H, \sigma_H^2)$$

To calculate μ_H,σ_H^2, we note that:

$$\begin{aligned}
\mathcal{E}\{|Y(t)|^2\} &= \mathcal{E}\left\{|H(t)s(t) + N(t)|^2\right\} \\
&= \mathcal{E}\left\{|H(t)|^2\right\} \underbrace{|s(t)|^2}_{P} + \underbrace{\mathcal{E}\left\{|N(t)|^2\right\}}_{\sigma_N^2} \\
&\quad + 2\text{Re}\left\{ s^*(t) \underbrace{\mathcal{E}\left\{H^*(t)N(t)\right\}}_{\mathcal{E}\{N(t)\}\mathcal{E}\{H(t)\}=0} \right\} \\
&= \underbrace{P}_{10 \text{ mW}} \mathcal{E}\left\{|H(t)|^2\right\} + \underbrace{\sigma_N^2}_{0.05 \text{ mW}} = 8.55 \text{ mW} \\
&\Longrightarrow 10 \cdot \mathcal{E}\left\{|H(t)|^2\right\} = 8.5 \Longrightarrow \mathcal{E}\left\{|H(t)|^2\right\} = 0.85
\end{aligned}$$

Thus,

$$\sigma_H^2 = \mathcal{E}\left\{|H(t) - \mu_H|^2\right\} = \frac{1}{3}\mathcal{E}\left\{|H(t)|^2\right\} \boxed{= 0.283}$$

$$|\mu_H|^2 = \frac{2}{3} \times 0.85 = 0.567 \Longrightarrow \mu_H = 0.753$$

Furthermore,

$$\Longrightarrow \boxed{H(t) \sim CN(0.753\, e^{j\psi}, 0.283)}$$

For some $\psi \in [0, 2\pi]$.

The rice factor is further:

$$K = \frac{|\mu_H|^2}{\sigma_H^2} = 2 \Longrightarrow \boxed{\log K = 3 \text{ dB}}$$

11.5.4 Question 4: Diversity Techniques (14 Points)

Consider a downlink transmission in a cellular network with *only a single active user.* The base station employs *code division multiplexing (CDM)* technique with spreading factor $N = 128$ and *chip duration* T_C for transmission.

Let the spreading sequence of the active user be $c_m[n]$ for $n = 0, \ldots, N-1$ whose auto-correlation function is given as follows:

$$\rho_{mm}[d] = \begin{cases} 1 & d = 0 \\ 0 & d \neq 0 \end{cases}.$$

The base station transmits a signal to the user. Within the transmission time interval, the realization of the downlink channel impulse response from the base station to the receiver is as follows:

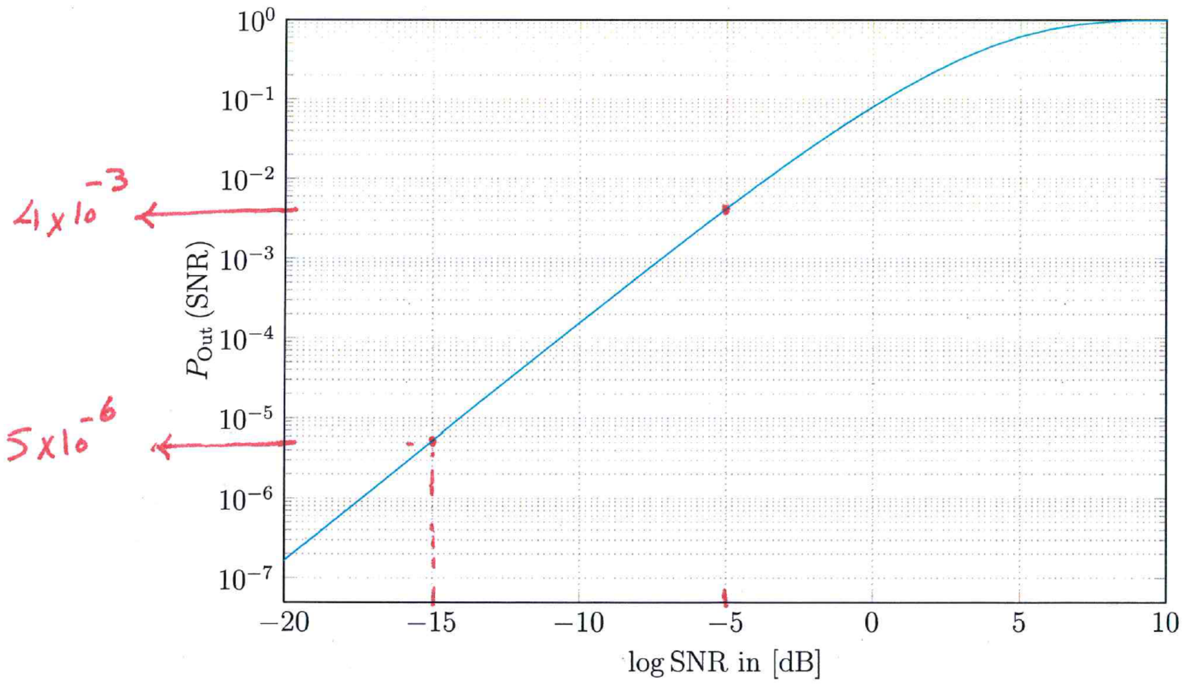
$$h(t) = \sum_{m=0}^{7} \exp\left\{-j\frac{m\pi}{14}\right\} \delta(t - 10mT_C),$$

where $j = \sqrt{-1}$ denotes the imaginary unit. The user knows $h(t)$ perfectly. To receive the transmitted signal, the user utilizes a *rake receiver with L fingers.*

Let Γ denote the signal-to-noise ratio (SNR) at the output of the rake receiver. The outage probability for a given SNR is defined as

$$P_{\text{Out}}(\text{SNR}) = \Pr\{\Gamma < \text{SNR}\},$$

and if plotted against $\log$ SNR in the following figure:



(a) Given the outage probability curve, calculate the number of fingers L in the rake receiver.

♠ **Solution:** First, we calculate the diversity gain: By definition, in Chapter 4, the diversity gain is the slope of the "$\log - \log$" Figure; thus,

$$G_D = \frac{\log P_{\text{out}}(-5 \text{ dB}) - \log P_{\text{out}}(-15 \text{ dB})}{-0.5 + 1.5} = \log(4 \times 10^{-3}) - \log(5 \times 10^{-6})$$
$$\boxed{\approx 3}$$

Since $G_D = 3 \implies$ There are three paths being combined by the rake receiver.

$$\boxed{L = 3}$$

(b) Sketch *one possible* rake receiver whose outage probability looks like the one given in the figure.
Specify all the parameters in the receiver.

♠ **Solution:** One can choose any three paths and combine them via MRC. Here is an example: I choose paths 0,1 and 2.

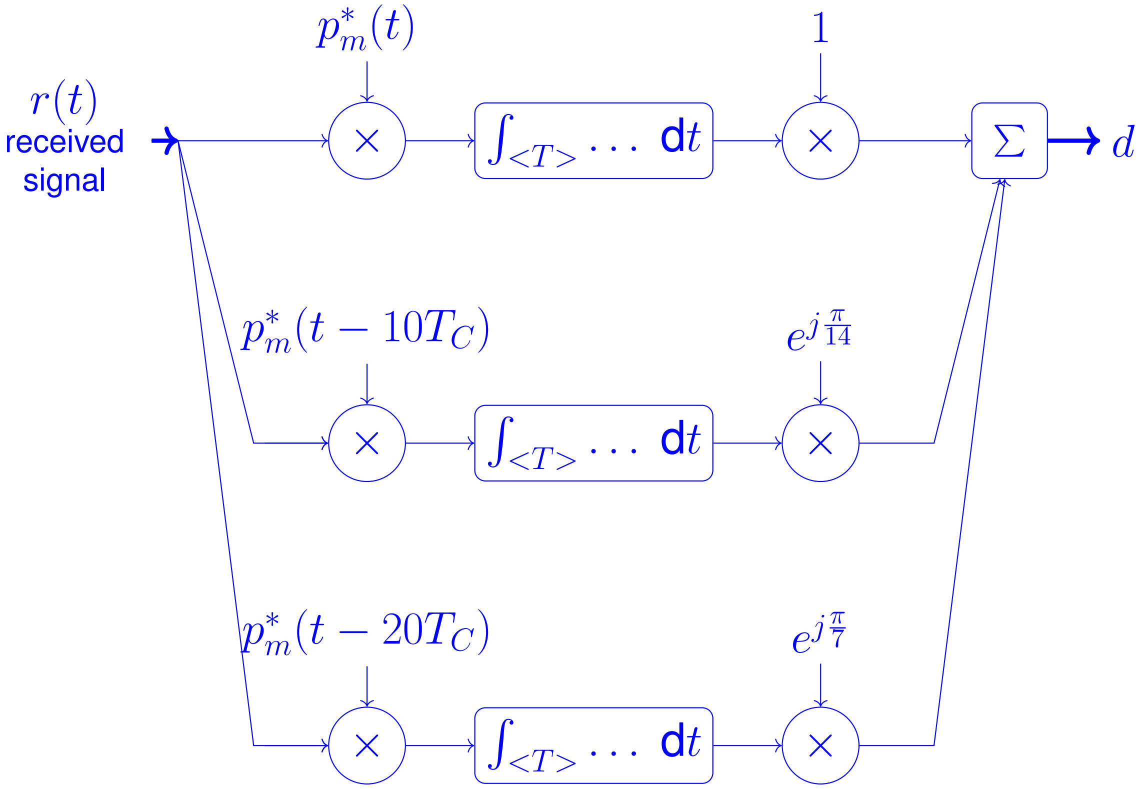
Let

$$p_m(t) = \sum_{n=0}^{N-1} c_m[n] g_{T_C}(t - nT_C)$$

where T_C is the chip duration, and

$$g_T(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}.$$

Then the receiver looks like:

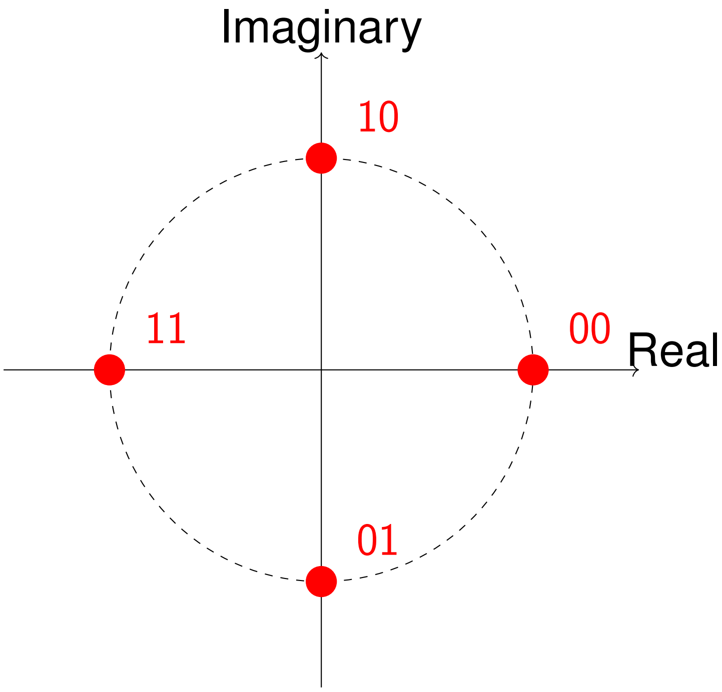


11.5.5 Question 5: Modulation Schemes (30 Points)

A communication system transmits messages which consist of two letters a and b. For a given message, the transmitter encodes letter a with 0 and letter b with 1. After encoding the message, the transmitter adds one parity check bit, such that the encoded message has an even number of bits 1.

> **Example:** *The message* aaabb *is first encoded to* 00011*. Since the number of bits* 1 *is even, the parity check bit is set to* 0*; hence, the sequence* 000110 *is transmitted.*

To transmit the encoded message, the transmitter uses a phase-shift keying (PSK) modulation with gray mapping whose symbol constellations are shown in the following figure:



For transmission, a *rectangular pulse* $g(t)$ with time domain duration is $T = 1$ μsec and power $P = 10$ mW is used. This means that

$$g(t) = \begin{cases} \sqrt{P} & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

The symbol duration is further set to T.

We intend to transmit the message

baa

with this system. First, answer the following items:

(a) Determine the *base-band* transmit signal $x(t)$.

♠ **Solution:** Since we want to transmit baa, we have source codeword 100 whose parity is 1. Thus, the channel codeword is

1001

This means that we transmit symbols:

$$a[0] : \text{"10"} \leftrightarrow j \quad \text{and} \quad \text{"01"} \leftrightarrow -j \;\; : a[1]$$

The baseband signal is thus:

$$x(t) = \sum_{k=0}^{1} a[k]g(t - kT) = \boxed{j\, g(t) - j\, g(t - T)}$$

$$\Longrightarrow \boxed{x(t) = j\,(g(t) - g(t - T))}$$

(b) What is peak-to-average power ratio (PAPR) of the transmit signal?

♠ **Solution:** PAPR $= 1$ or $\log$PAPR $= 0$ dB.

Because it's PSK transmission via rectangular pulse.

Now, assume that the base-band transmit signal $x(t)$, which is determined in Part (a), is transmitted over a fading additive white Gaussian noise (AWGN) channel. Hence, the base-band received signal is written as

$$y(t) = H\ x(t) + n(t)$$

where $n(t)$ is noise. The coefficient H is a random variable whose mean and variance are

$$\text{Mean of } H = \mathcal{E}\{H\} = 0.5 \exp\{-j\frac{3\pi}{7}\}$$

$$\text{Variance of } H = \mathcal{E}\{|H - \mathcal{E}\{H\}|^2\} = 0.25,$$

where $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation.

To detect the transmitted message, the receiver employs a *matched filter* whose output is sampled at $t = 0$ and $t = T$. Let

$$r_k = d_k + n_k$$

be the outputs of the matched filter sampled at $t = kT$ for $k = 0, 1$, where n_k denotes the output sample *when only noise* $n(t)$ *is given as the input*. Assume that n_0 and n_1 are independent complex random variables each distributed Gaussian with *zero mean* and *variance* $\sigma_N^2 = 0.25$ *mW*.

Answer the following items:

(c) Explicitly calculate d_k for $k = 0, 1$ in terms of H.

♠ **Solution:** By standard derivations, as in the tutorial, we have

$$\begin{aligned} d_k = \int_{-\infty}^{+\infty} H\ x(\tau) g^*(-kT+\tau)\ \mathrm{d}\tau = j\ H \Bigg[&\int_{-\infty}^{+\infty} g(\tau) g^*(-kT+\tau)\ \mathrm{d}\tau \\ &- \int_{-\infty}^{+\infty} g(\tau - T) g^*(-kT+\tau)\ \mathrm{d}\tau \Bigg] \end{aligned}$$

$$d_k = j\ H \left[\begin{cases} P, & k=0 \\ 0, & k=1 \end{cases} - \begin{cases} 0, & k=0 \\ P, & k=1 \end{cases} \right]$$

$$\Longrightarrow\ d_k = \begin{cases} jHP, & k=0 \\ -jHP, & k=1 \end{cases}$$

or equivalently

$$\boxed{d_k = HP\ a[k]}$$

(d) Calculate the *average* signal-to-noise ratio (SNR) which is defined as

$$\text{SNR}_k = \frac{\mathcal{E}\{|d_k|^2\}}{\mathcal{E}\{|n_k|^2\}}$$

for $k = 0, 1$.

♠ **Solution:**

$$d_k = HP\, a[k]$$

Thus,

$$\begin{aligned} \Longrightarrow\ \text{SNR}_k &= \frac{\mathcal{E}|H|^2 \cdot P^2 \cdot |a[k]|^2}{\sigma_N^2} = \frac{0.25 \times P^2 \times 1}{0.25} \\ &= P^2 = \boxed{100} \end{aligned}$$

$$\Longrightarrow\ \boxed{\log \text{SNR}_k = 20 \text{ dB}}$$

11.6 Winter Semester 2019-2020

Exam Date: February 19, 2020
5 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.6.1 Question 1: Short Questions (25 Points)

Answer the following questions briefly.

(a) You are asked to design a cellular network in an area with *high* subscriber density.

(a-1) What is your choice for the shape of cells?

♠ **Solution:** Hexagone.
As they are optimal shape to have coverage.

(a-2) What is your choice for the cite of base stations (BSs)?

♠ **Solution:** Cloverleaf.
For areas with high subscriber density, cloverleaf is better than classical.

For *each* of the above choices, give *at least one reason.*

(b) Consider an antenna whose normalized radiation pattern reads

$$g(\vartheta, \alpha) = |cos(\vartheta)|.$$

Calculate the *vertical* directivity D_v and the *horizontal* directivity D_h of this antenna.
Hint: You can use the following identity:

$$\sin 2\vartheta = 2 \sin\vartheta \cos\vartheta$$

♠ **Solution:**

$$g(\vartheta, \alpha) = |\cos\vartheta| \implies \begin{cases} g_v(\vartheta) = |\cos\vartheta| \\ g_h(\alpha) = 1 \end{cases}$$

$$D_h = 1$$

$$\begin{aligned} D_v &= \frac{2}{\int_0^\pi |\cos\vartheta| \sin\vartheta \, \mathrm{d}\vartheta} \\ &= \frac{2}{\int_0^{\frac{\pi}{2}} \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta - \int_{\frac{\pi}{2}}^{\pi} \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta} \\ &= \frac{4}{\int_0^{\frac{\pi}{2}} \sin 2\vartheta \, \mathrm{d}\vartheta - \int_{\frac{\pi}{2}}^{\pi} \sin 2\vartheta \, \mathrm{d}\vartheta} = \frac{2}{\underbrace{\int_0^{\frac{\pi}{2}} \sin 2\vartheta \, \mathrm{d}\vartheta}_{=1}} = \boxed{2} \end{aligned}$$

(c) To combat fading in a wireless channel, we utilize the *multipath diversity* technique with $M = 5$ *independent paths*. At the receiver side, we use *selection combining (SC)* to combine the signals received through different paths. Calculate the *additional* diversity gain you would have, if you use *maximum ratio combining (MRC)* instead of SC at the receiver.

♠ **Solution:** From (4.26):

$$\Delta G = G_{\text{MRC}} - G_{\text{SC}} \approx \frac{10}{M} \log M$$

$$M = 5 \implies \Delta G = \frac{10}{5} \log 120 = \boxed{4.1584 \text{ dB}}$$

(d) You want to transmit a *coded* binary sequence over a *static* additive white Gaussian noise (AWGN) channel, such that your bit error rate p_b is *less than* 10^{-5}. Assume that you intend to achieve a coding gain G_c *more than* 3.5 *dB*. Give the coding rate, constraint length and free distance of a *convolutional* channel code which achieves the required coding rate.

♠ **Solution:** To have $p_b < 10^{-5}$ with uncoded transmission, we need:

$$\text{SNR} \approx 9.6$$

To have coding gain > 3.5 dB, we should have $p_b < 10^{-5}$

At SNR≤ 6.1. From Figure 7.12, we see this happens for:

A $\rightarrow$ $R = \frac{1}{2}$, $L_c = 7$, $d_{\text{free}} = 10$, soft decoding.

B $\rightarrow$ $R = \frac{1}{3}$, $L_c = 7$, $d_{\text{free}} = 15$, hard decoding.

C $\rightarrow$ $R = \frac{1}{3}$, $L_c = 7$, $d_{\text{free}} = 15$, soft decoding.

(e) Consider your answer to Part (d). Determine the *maximum* possible coding gain achieved by this code.

♠ **Solution:**

$$\text{For A} \implies G_{c,\max} = 10 \log R_c d_{\text{free}} = 10 \log 5 \approx 7 \text{ dB}$$

$$\text{For B, C} \implies G_{c,\max} = 10 \log 5 \approx 7 \text{ dB}$$

Same!

11.6.2 Question 2: Mobile Radio Channels (22 Points)

A radio transmitter is located in Tennelohe. The height of the transmitter is 132 m. A mobile station (MS) is in the Technical Faculty of Friedrich-Alexander University about 3.6 km away from the base station (BS) site. The MS is located in the balcony of an office which is 7 m away from the ground. The transmitter is equipped with a $\lambda/4$-*linear antenna on metal plate*, and the MS has a $\lambda/2$-*dipole antenna*. The system operates at a carrier frequency which is exactly in the middle of the uplink band of GSM900.

(a) Which path-loss model gives the most *accurate* value for the path-loss in this scenario?

♠ **Solution:** Of course Okumura-Hata model.

(b) Calculate the path-loss using you answer to Part (a).

♠ **Solution:**

- $h_{\text{BS}} = 132$ m.
- $d = 3.6$ km.
- $h_{\text{MS}} = 7$ m.
- $G_{\text{BS}} = 5.15$ dB$_i$.
- $G_{\text{MS}} = 2.15$ dB$_i$.
- $f_0 = \frac{890+915}{2} = 902.5$ MHz.

$$\boxed{\text{Area} = \text{Sub-urban}}$$

$$\begin{aligned} \text{Eq}(3.28) &\implies a(h_{\text{MS}}) = 14.0464 \\ \text{Eq}(3.27) &\implies 10n = 44.9 - 6.55\log_{10}(132) = 31.01 \end{aligned}$$

$$\text{PL}_0 = 103.5118$$

$$\text{PL}_{\text{urban}} = 103.5118 + 31.01\log_{10}(3.6) = 120.7627 \text{ dB}$$

Correction term for sub-urban:

$$2\left(\log_{10}\frac{f_0}{28}\right)^2 + 5.4 = 9.95.$$

$$\text{PL}_{\text{sub}} = 120.7627 - 9.95 = \boxed{110.8128}$$

This is for omnidirectional antennas. When antenna gain are not zero, we have:

$$
\begin{aligned}
\text{PL} &= \text{PL}_{\text{sub}} - G_{\text{BS}} - G_{\text{MS}} \\
&= 110.8128 - 5.15 - 2.15 \\
&= 110.8128 - 7.3 \\
&= \boxed{103.5128}
\end{aligned}
$$

(c) Calculate the path-loss using the *free space propagation* model.

♠ **Solution:** By free-space model, we have:

$$
\begin{aligned}
\text{Eq. (3.10)} \implies \text{PL}_{\text{free}} &= 32.44 + 20\log_{10} d + 20\log_{10} f_0 \\
&\quad - 5.15 - 2.15 \\
&= 95.375
\end{aligned}
$$

(d) Compare the results of Parts (b) and (c) *explain* your observation.

♠ **Solution:**

$$\text{PL}_{\text{Okumura}} > \text{PL}_{\text{free}}$$

This is because free-space model ignores impact of reflection, absorption, and defraction in the medium.

11.6.3 Question 3: Fading Channels (16 Points)

Consider a wireless channel in which the mobile station (MS) moves with constant velocity v. The MS receives signal from different directions. The power angular density at the antenna foot point of the MS is as follows:

$$p(\alpha) = \frac{\sigma_R^2}{4}|\cos\alpha|.$$

(a) *Determine* the Doppler spectrum $\Psi_R(\nu)$ in this case.
Hint: Note that

$$\cos(-\alpha) = \cos(\alpha)$$

♠ **Solution:** From (3.49), we have

$$\Psi_R(\nu) = \frac{p(\alpha) + p(-\alpha)}{\sqrt{\nu_{\text{max}}^2 - \nu^2}} = \frac{\sigma_R^2\,(2|\cos\alpha|)}{4\sqrt{\nu_{\text{max}}^2 - \nu^2}}$$

$$\implies \Psi_R(\nu) = \frac{\sigma_R^2|\cos\alpha|}{2\sqrt{\nu_{\text{max}}^2 - \nu^2}}$$

Furthermore, we have:

$$\cos\alpha = \frac{\nu}{\nu_{\text{max}}}$$

Thus,

$$\boxed{\Psi_R(\nu) = \frac{\sigma_R^2|\nu|}{2\nu_{\text{max}}\sqrt{\nu_{\text{max}} - \nu^2}}} \qquad , |\nu| < \nu_{\text{max}}$$

(b) The MS moves with velocity

$$v = 120 \text{ km/h}.$$

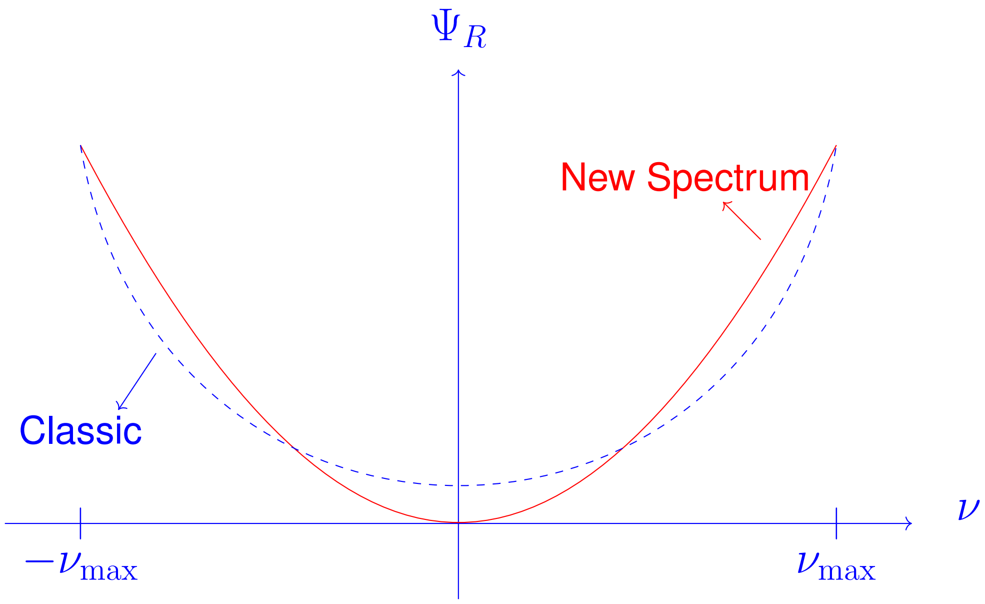
Moreover, the received power at the MS is $\sigma_R^2 = 10\ \mu$W. *Plot* the Doppler spectrum determined in Part (a), and *compare* it to the classical Doppler spectrum.

♠ **Solution:**

$$v = 120\text{km/h} \implies \nu_{\text{max}} = f_0\frac{v}{c}$$

$$v = 120\text{km/h} = \frac{100}{3}\frac{\text{m}}{\text{s}} \implies \nu_{\text{max}} = f_0\frac{\frac{100}{3}}{3\times 10^8} = \frac{10^{-6}}{9}.$$

$$\implies \boxed{\nu_{\text{max}} = \frac{10^{-6}}{9}f_0}$$



$$\Psi_R(\nu) = \frac{5|\nu|}{\frac{10^{-6}}{9}f_0\sqrt{\left(\frac{10^{-6}}{9}\right)^2 f_0^2 - \nu^2}}$$

11.6.4 Question 4: Multiplexing Techniques (15 Points)

Consider a cellular network with one base station (BS) and K mobile stations (MSs). This system utilizes code-division-multiplexing (CDM) technique with $N = 10$ chips to support multiple users. The *symbol* interval in this system is $T_S = 20\ \mu$ sec.

In this system, the spreading sequence of the first MS, $c_1[n]$ for $n = 0, \ldots, 9,$ is as follows:

$$c_1[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1, \ldots, 9 \end{cases}$$

The impulse response of the channel between the BS and this MS is further given by

$$h_1(\tau) = H_0\delta(\tau - 10\ \mu\,\text{sec}) + H_1\delta(\tau - 10.5\ \mu\,\text{sec}) + H_2\delta(\tau - 120\ \mu\,\text{sec})$$

where H_0, H_1 and H_2 are *independent* random variables and $\delta(\tau)$ denotes the Dirac impulse function.

(a) Determine the duration of the chip interval T_C.

♠ **Solution:**

$$T_C = \frac{T_S}{N} = \frac{20}{10} = 2\ \mu\text{sec}$$

(b) We want to make sure that the spreading sequences of different MSs are *orthogonal*. What is the maximum value of K for which this constraint can be fulfilled?

♠ **Solution:**

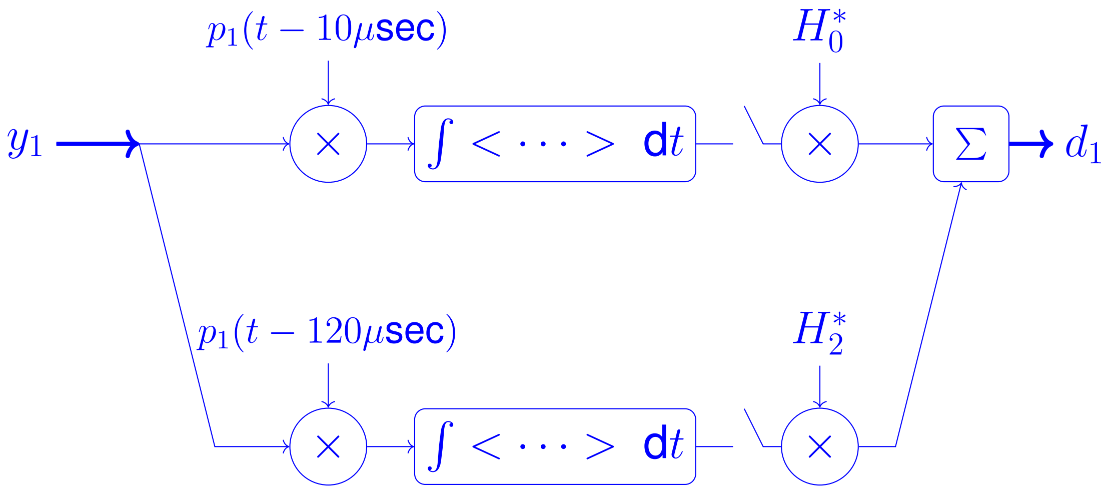
$$K = N = 10$$

(c) The first MS intends to use *rake* receiver with L fingers. What is the *optimal* choice for L? *Explain* your answer by giving a reason.

♠ **Solution:** $T_C = 2\mu$sec and we have three paths. One with delay $\tau = 10\mu$sec, another with $\tau = 10.5\mu$sec, last one with $\tau = 120\mu$sec. Since the first and second paths are not distinguishable $(\Delta\tau < T_C = 2\mu\text{sec}) \implies$ We have only 2 distinguishable paths. Therefore $\boxed{L = 2}$ is optimal.

(d) Let $L = L_0$ be your answer to Part (c). Sketch a rake receiver for the first MS with L_0 fingers whose diversity gain is exactly L_0. Specify all parameters in the receiver.

♠ **Solution:** $L_0 = 2$. Thus: $y_1 =$ received by MS_1,



Here,

$$p_1(t) = \sum_{n=0}^{9} c_1[n] g(t - nT_C)$$
$$= g(t)$$

where $g(t)$ is the pulse-shape.

11.6.5 Question 5: Modulation Schemes (22 Points)

Consider a pulse amplitude modulation (PAM) scheme in which the modulation symbols, i.e., $a[k]$ for $k \in \mathbb{Z}$, are taken from the following *modulation alphabet*:

$$\mathcal{A} = \{-3\sqrt{E}, -\sqrt{E}, +\sqrt{E}, +3\sqrt{E}\}.$$

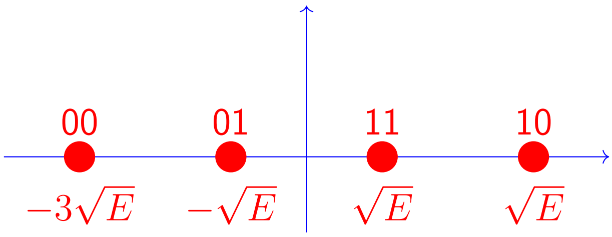
The modulation scheme uses a *rectangular pulse* $g(t)$ with time duration $T/2$, i.e.,

$$g(t) = \begin{cases} \sqrt{\frac{2}{T}} & 0 \leq t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The modulation symbol interval is further set to T.

(a) Draw a symbol *constellation* for this modulation scheme with *Gray-mapping*.

♠ **Solution:** One solution:



(b) Write the transmit signal $s(t)$ in terms of the modulation symbols and the rectangular pulse.

♠ **Solution:**

$$s(t) = \sum_{k=-\infty}^{+\infty} a[k]g(t - kT)$$

(c) Is this modulation linear? *Explain* your answer.

♠ **Solution:** Yes. Because the signal elements of consecutive symbols superimpose additively.

(d) Let $S(t)$ denote the modulated signal given for random modulation symbols $A[k]$ for $k \in \mathbb{Z}$ which have the following stochastic properties:

- $A[k]$ for $k \in \mathbb{Z}$ are jointly statistically independent and identically distributed (i.i.d).
- Each modulation symbol is uniformly distributed over alphabet $\mathcal{A}$.

Write the *average* power spectral density of $S(t)$. *Determine* all the parameters explicitly.

♠ **Solution:** From (6.10), we have:

$$\phi_S(f) = \frac{1}{E_g}|G(f)|^2$$

$$E_g = \frac{TP_S}{\mathcal{E}|A|^2}$$

where:

$$\begin{aligned} P_S &= \frac{1}{T}\int_0^T \mathcal{E}|S(t)|^2 \,\mathrm{d}t \\ &= \frac{1}{T}\mathcal{E}|A|^2 \int_0^T |g(t)|^2 \,\mathrm{d}t \\ &= \frac{1}{T}\mathcal{E}|A|^2 \underbrace{\frac{2}{T}\int_0^{\frac{T}{2}} \mathrm{d}t}_{=1} \\ &= \frac{1}{T}\mathcal{E}|A|^2 \end{aligned}$$

$$\Longrightarrow \; E_g = \frac{T \cdot \frac{1}{T}\mathcal{E}|A|^2}{\mathcal{E}|A|^2} = \boxed{1}$$

$$|G(f)|^2 = 2\frac{\frac{T}{2}\sin^2\left(\pi f \frac{T}{2}\right)}{\left(\pi f \frac{T}{2}\right)^2} = \boxed{T\frac{\sin^2\left(\frac{\pi}{2}Tf\right)}{\left(\frac{\pi}{2}Tf\right)^2}}$$

(e) Let ζ_0 denote the peak-to-average ratio (PAR) of this modulation scheme. Moreover, let ζ_1 represent the PAR of the quadrature phase shift keying (QPSK) scheme when a rectangular pulse of duration T is used. Compare ζ_0 to ζ_1. *Explain* your answer by giving a reason.

♠ **Solution:**

$$\zeta_0 \geq \zeta_1$$

QPSK with rectangular pulse-shape has

$$\text{PAR} = 1$$

which is minimal PAR possible. So any PAR reads

$$\text{PAR} \geq \zeta_1.$$

(f) Calculate the *bandwidth efficiency* of this modulation scheme.

♠ **Solution:**

$$\eta = \frac{R_b}{B} = \frac{\frac{\log 4}{T}}{B} = \frac{2/T}{B}$$

Since $g(t)$ is rectangular:

$$B \approx \frac{1}{\frac{T}{2}} = \frac{2}{T}$$

$$\Longrightarrow \; \eta = \frac{\frac{2}{T}}{\frac{2}{T}} = \boxed{1}$$

(g) How could you *increase* the bandwidth efficiency?

♠ **Solution:** By using RRC-pulses.

11.7 Summer Semester 2020

Exam Date: August 21, 2020
6 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.7.1 Question 1: Short Questions (20 Points)

Answer the following questions briefly.

(a) Explain why cell-partitioning is required in cellular networks by giving at least two reasons.

♠ **Solution:** Possible answers:

- Limited bandwidth.
- Limited transmit power.
- Limited transmit delay.
- Limited user density per transmitter.

(b) Name two most important *near-far* effects in cellular networks, and explain each effect briefly.

♠ **Solution:**

(1) Significant difference in propagation delays:
The user which is far from the BS has a significantly higher delay compared to the one which is close. This could be problematic for TDM systems.

(2) Significant difference in path loss:
The far user experiences higher loss. In this case, the user close to the BS could cause large interference on the signal of the far user.

(c) Explain the difference between the methods of selection combining and maximum ratio combining (MRC). For each scheme name an advantage and disadvantage.

♠ **Solution:** In selection combining, one chooses the strongest signal.
In MRC, we combine all received signals.

MRC $\Longrightarrow$ Advantage: Best error probability. Disadvantage: CSI needed.
SC $\Longrightarrow$ Advantage: Less CSI needed. Disadvantage: Higher error rate.

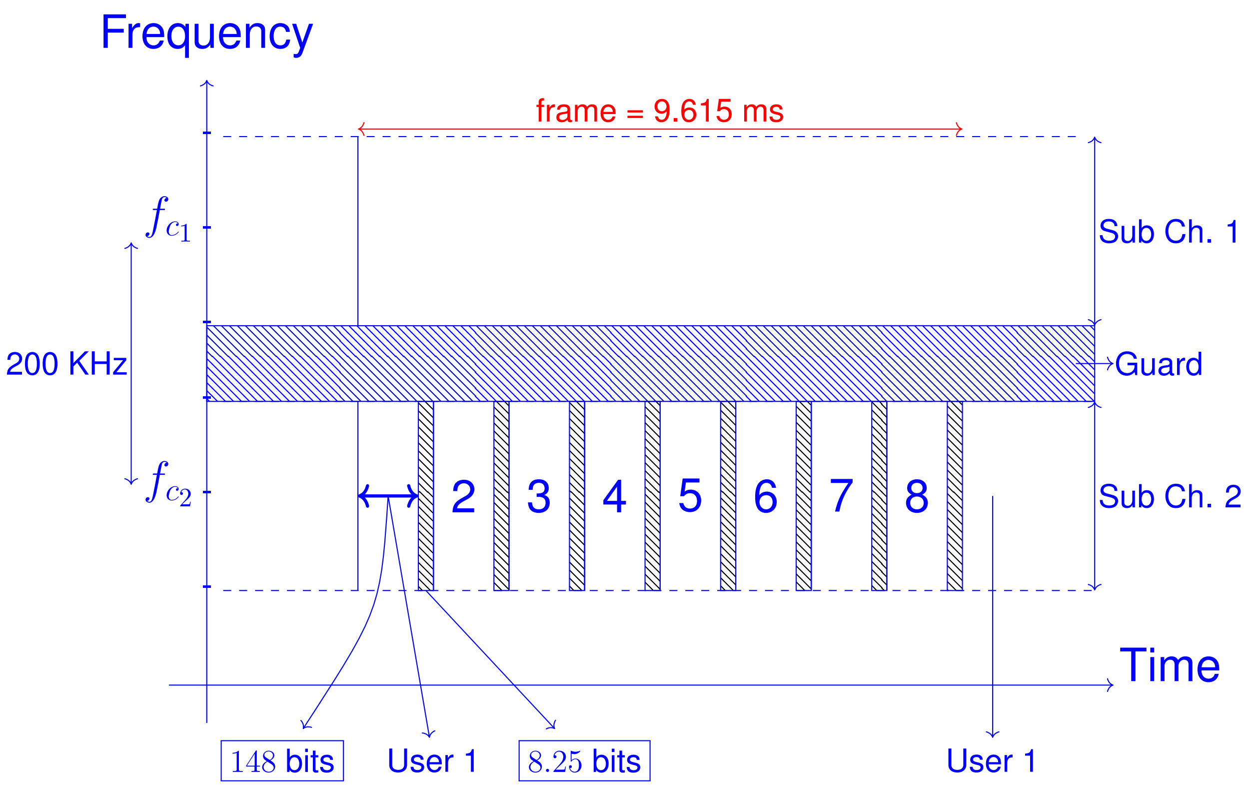
(d) A convolutional code has the free distance $d_{free} = 13$. What is the smallest possible constraint length of this code, if you know that the code rate is $R_c = 1/3$? What about a code with rate $R_c = 1/4$?

♠ **Solution:** From Table 7.1,page 249,

- At $R = \frac{1}{3}$, $d_{\text{free}} = 13 \implies \min L_c = 6$.
- At $R = \frac{1}{4}$, $d_{\text{free}} = 13 \implies \min L_c = 4$.

(e) Plot the time-frequency allocation diagram for *only two* neighboring frequency division multiplexing (FDM) sub-channels in a normal traffic channel of a GSM system. Specify the carrier separation between the FDM sub-channels, and the duration of each time division frame.

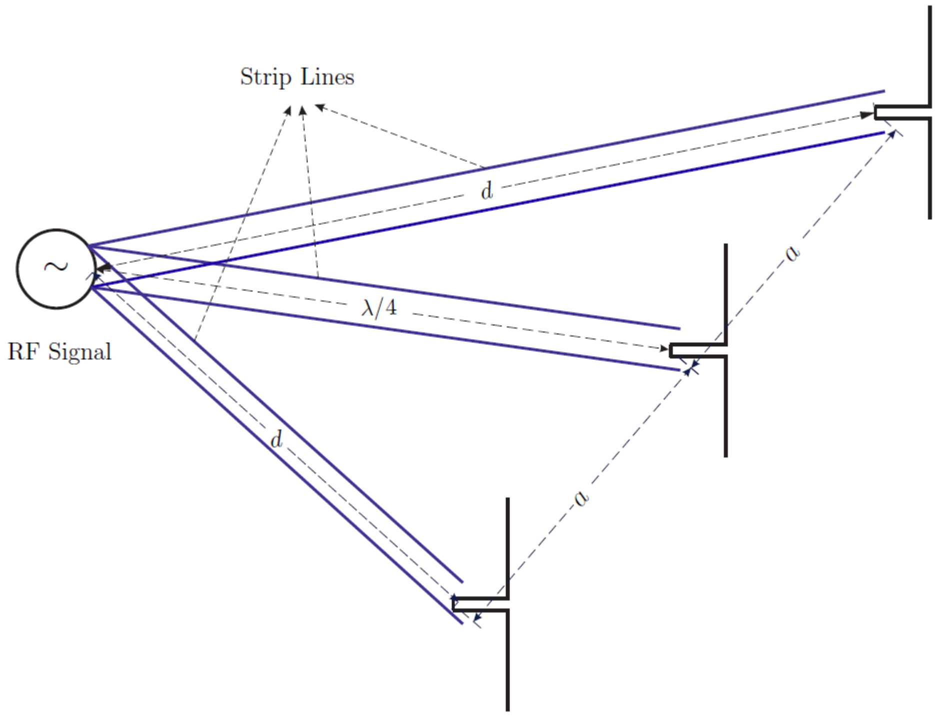
♠ **Solution:**



From Table 8.1, Page 262 and Figure 8.15, Page 277.

11.7.2 Question 2: Array Antennas (15 Points)

Three $\lambda/2$-dipole antennas are arranged with equal distances *horizontally* as observed in the figure below. The distance between two neighboring antennas is a. All the distances in the figure are given in the *horizontal plane*. As it is observed in the figure, these three antennas are fed by the same radio frequency (RF) signal, with carrier wave-length λ, through three different strip lines. The strip line connecting the RF signal generator to the middle antenna is of length $\lambda/4$, and the two other strip lines which connect the right and left antennas to the RF signal generator are of length d.



(a) Determine the *horizontal* radiation pattern of this array antenna in terms of a.

Hint: Note that a strip line only adds a *path-delay* to the signal.

♠ **Solution:** Enumeration of the antennas is as follows:

- Top-right is 0th antenna.
- Middle one is 1st antenna.
- Bottom one is 2nd antenna.

A dipole antenna has a uniform horizontal pattern: Thus, the array pattern is equivalent to the array gain.

This is a ULA with gains:

$$w_0 = \frac{1}{\sqrt{3}} e^{-j\frac{2\pi}{\lambda}d} = w_2$$

$$w_1 = \frac{1}{\sqrt{3}} e^{-j\frac{2\pi}{\lambda}\frac{\lambda}{4}} = \frac{1}{\sqrt{3}} e^{-j\frac{\pi}{2}} = \frac{-j}{\sqrt{3}}$$

Using Eq. (2.14) and Eq. (2.15) in page 56, we have:

$$\begin{aligned}
\mathsf{AF}(\alpha) &= \frac{1}{\sqrt{3}}e^{-j\frac{2\pi}{\lambda}d} + \frac{1}{\sqrt{3}}e^{-j\frac{\pi}{2}} \cdot e^{-j\frac{2\pi}{\lambda}a\sin\alpha} \\
&\quad + \frac{1}{\sqrt{3}}e^{-j\frac{2\pi}{\lambda}d}e^{-j\frac{2\pi}{\lambda}a2\sin\alpha} \\
&= \frac{1}{\sqrt{3}}\left\{e^{-j\left[\frac{\pi}{2}+\frac{2\pi}{\lambda}a\sin\alpha\right]} + e^{-j\frac{2\pi}{\lambda}d}\left[1+e^{-j\frac{4\pi}{\lambda}a\sin\alpha}\right]\right\} \\
&= \frac{e^{-j\left(\frac{2\pi}{\lambda}d+\frac{2\pi}{\lambda}a\sin\alpha\right)}}{\sqrt{3}}\left\{e^{-j\left(\frac{\pi}{2}-\frac{2\pi}{\lambda}d\right)} + \underbrace{\left(e^{+j\frac{2\pi}{\lambda}a\sin\alpha}+e^{-j\frac{2\pi}{\lambda}a\sin\alpha}\right)}_{=2\cos\left(\frac{2\pi}{\lambda}a\sin\alpha\right)}\right\} \\
&= \frac{e^{-j\left(\frac{2\pi}{\lambda}d+\frac{2\pi}{\lambda}a\sin\alpha\right)}}{\sqrt{3}}\left\{e^{-j\frac{2\pi}{\lambda}\vartheta} + 2\cos\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\right\}
\end{aligned}$$

where

$$\vartheta = \frac{\lambda}{4} - d = \frac{\lambda}{4} - \sqrt{\frac{\lambda^2}{16} - a^2}.$$

This implies,

$$\begin{aligned}
|\mathsf{AF}(\alpha)|^2 &= \frac{1}{3}\left[\cos\left(\frac{2\pi}{\lambda}\vartheta\right) + 2\cos\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\right]^2 + \sin^2\left(\frac{2\pi}{\lambda}\vartheta\right) \\
&= \frac{1}{3}\left[1 + 4\cos^2\left(\frac{2\pi}{\lambda}a\sin\alpha\right) + 4\cos\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\cos\left(\frac{2\pi}{\lambda}\vartheta\right)\right]
\end{aligned}$$

$$\begin{aligned}
\implies g_A(\alpha) &= \frac{|\mathsf{AF}(\alpha)|^2}{\max_\alpha |\mathsf{AF}(\alpha)|^2} \\
&= \frac{1 + 4\cos^2\left(\frac{2\pi}{\lambda}a\sin\alpha\right) + 4\cos\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\cos\left(\frac{2\pi}{\lambda}\vartheta\right)}{5 + 4\cos\left(\frac{2\pi}{\lambda}\vartheta\right)}
\end{aligned}$$

(b) Calculate the *three-dimensional* radiation pattern of the array antenna for $a = \sqrt{3}\lambda/4$.

♠ **Solution:**

$$a = \frac{\sqrt{3}\lambda}{4} \implies d = \sqrt{\frac{\lambda^2}{16} + \frac{3\lambda^2}{16}} = \boxed{\frac{\lambda}{2}} \implies \vartheta = \frac{-\lambda}{4}$$

$$\begin{aligned}
&\implies \cos\left(\frac{2\pi}{\lambda}\vartheta\right) = \cos\left(\frac{-\pi}{2}\right) = 0 \\
&\implies g_h(\alpha) = g_A(\alpha) = \frac{1}{5}\left(1 + 4\cos^2\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\right)
\end{aligned}$$

The vertical pattern is given in page 49, Eq. (2.5).

$$g_v(\vartheta) = \frac{\cos^2\left(\frac{\pi}{2}\cos\vartheta\right)}{\sin^2\vartheta}$$

$$\implies g(\alpha,\vartheta) = g_h(\alpha)g_v(\vartheta) = \frac{\left(1+4\cos^2\left(\frac{2\pi}{\lambda}a\sin\alpha\right)\right)\cos^2\left(\frac{\pi}{2}\cos\vartheta\right)}{5\sin^2\vartheta}$$

(c) For $a = \sqrt{3}\lambda/4$, find the main beam of the array in the *horizontal* plane. What is the main beam of the array in three-dimensional space?

♠ **Solution:** α changes from $-180°$ to $180°$.

At $a = \frac{\sqrt{3}\lambda}{4} \implies g_h(\alpha)$ is calculated in Part (b).

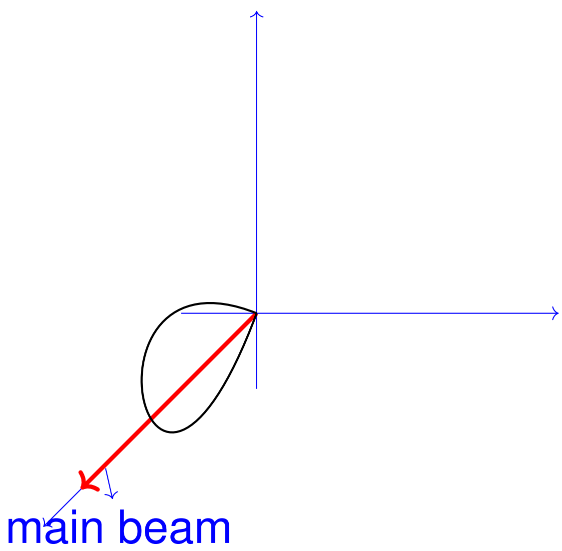
The horizontal beam $\implies$ where $g_h(\alpha)$ is maximal.

$$\alpha^* = \underset{-180°\le\alpha\le180°}{\operatorname{argmax}} \; g_h(\alpha)$$

This happens when $\frac{2\pi}{\lambda}a\sin\alpha = 0 \implies \boxed{\alpha = 0°}$ which is intuitive as the array is symmetric.

Given the properties of $\frac{\lambda}{4}$-dipole (Page 49)

$$\implies \text{3-dimensional beam is at} \begin{bmatrix}\alpha = 0° \\ \vartheta = 90°\end{bmatrix}$$



11.7.3 Question 3: Mobile Radio Channels (18 Points)

Centre Point is a tower with 117 meters height located in Tottenham Court Road in London. Assume a *long term evolution (LTE)* transmitter is located on the top of this tower. The transmitter operates at carrier frequency $f_0 = 1.8$ GHz and is equipped with an array antenna whose *array gain* is $G_{Array} = 8$ dB. Each antenna element of this array has further the *antenna gain* $G_0 = 5$ dB$_i$. You are having your dinner in the first floor loggia of a restaurant located at Flood Street, Chelsea approximately 5 kilometers away from Centre Point.

Assume that your cell phone is being served by the transmitter on the Centre Point. Your cell phone is using only a single $\lambda/2$-dipole antenna for the signal reception.

(a) Name the best model which describes the path-loss of the communication channel.

♠ **Solution:** The Okumura-Hata model.

(b) Using the model named in Part (a), determine the path-loss of the communication channel.

♠ **Solution:**

- $f_0 =$ Carrier $= 1.8$ GHz $= 1800$ MHz.
- $h_{\text{BS}} = 117$ m
- $h_{\text{MS}} = 3$ m : Any number between 2 m and 5 m is correct.
- $d = 5$ km.
- $G_{\text{BS}} = 5$ dB$_i$, $G_{\text{MS}} = 2.15$ dB$_i$.

London is <u>urban</u>; given the frequency $\Longrightarrow$ Eq. (3.31) in page 90 is used.

$$\begin{aligned} a(h_{\text{MS}}) &= (1.1 \log 1800 - 0.7) \times 3 - 1.56 \log 1800 + 0.8 \\ &= 4.36 \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \text{PL}_0 &= 46.3 + 33.9 \log 1800 - 13.82 \log 117 - a(h_{\text{MS}}) \\ &\quad + (44.9 - 6.55 \log 117) \log d \\ &= 46.3 + 110.35 - 28.58 - 4.36 + 21.92 \\ &= 145.63 \text{ dB} \end{aligned}$$

This is the path-loss for "isotropic transmit & receive antennas". As we have gains, we can calculate path-loss as:

$$\begin{aligned} \text{PL} = \text{PL}_0 - G_{\text{BS}} - G_{\text{MS}} &= 145.63 - 13 - 2.15 \\ &= 130.48 \text{ dB} \end{aligned}$$

$$\boxed{\text{PL} = 130.48 \text{ dB}}$$

For G_{BS}, we have:

$$G_{\text{BS}} = G_0 + G_{\text{Array}} = 8 + 5 = 13 \text{ dB}_i$$

(c) Assume that the transmitter transmits with power $P_{\text{Tx}} = 45$ dB$_m$. Calculate the *median* power received by your cell phone.

♠ **Solution:** For attenuation a we have $\implies$ $P_R = \frac{P_{\text{Tx}}}{a}$

$\implies$ In logarithmic scale:

$$\begin{aligned} P_R \; [\text{dB}_m] &= P_{\text{Tx}} \; [\text{dB}_m] - \text{PL} \; [\text{dB}] \\ &= 45 \text{ dB}_m - 130.48 \text{ dB} \\ &= -85.48 \text{dB}_m \end{aligned}$$

$$\boxed{P_R = -85.48 \text{ dB}_m}$$

(d) You now use a power analyzer to measure the *instantaneous received power*. Do you expect to see the same value as you calculated in Part (c)? Justify you answer by giving a reason.

♠ **Solution:** No.

The channel experiences "fading" in practice. Therefore, the instantaneous power changes frequently.

(e) Given the scenario, do you think it is realistic to assume that your cell phone is served by the transmitter on Centre Point? Justify your answer by giving a reason.

Hint: Pay attention to the fact that you are in London!

♠ **Solution:** No.

The cellphone is 5 km away from the BS in "London". In crowded areas like London's city center, it is expected that the cell size be significantly small.

11.7.4 Question 4: Fading Channels (18 Points)

Consider the communication channel between a base station (BS) and a mobile (MS) distanced from each other with $d = 3$ kilometers.

First, assume that the communication between the BS and MS is described by the imaginary *free space propagation model*. This means that the signal is transmitted by the BS is received *without any distortion* at the MS after the *propagation delay.*

(a) Describe the communication channel given by the free space propagation model as a *linear time-invariant* system. Explicitly determine the impulse response of this system, i.e., $h(\tau)$.

♠ **Solution:**

$$d = 3 \text{ km} \implies \text{delay} = \tau_0 = \frac{d}{c} = \frac{3 \times 10^3}{3 \times 10^8} = 10^{-5} s = 10 \mu s$$

$$x(t) \longrightarrow \boxed{h(\tau)} \longrightarrow y(t)$$

$$h(\tau) = \delta(\tau - \tau_0) = \boxed{\delta(\tau - 10)}$$

(b) The delay-time correlation function for a linear time-invariant channel is defined as

$$\rho_h(\tau) = \int_{-\infty}^{+\infty} \mathcal{E}\{h^*(\alpha)h(\tau)\} d\alpha$$

where $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation. Determine the delay-time correlation function for the channel described in Part (a).

♠ **Solution:**

$$\rho_h(\tau) = \int_{-\infty}^{+\infty} \delta(\tau - \tau_0)\delta(\alpha - \tau_0) \, \mathrm{d}\alpha = \boxed{\delta(\tau - \tau_0)}$$

Now, consider the *real case*: In this case, the channel *is not* described via the free space propagation model anymore. Assume that the channel in this case is *time-invariant* and its delay-time correlation function is given by

$$\rho_h(\tau, \Delta t) = \frac{1}{2}\delta(\tau - \Delta t) + \frac{1}{2}\delta(\tau - 0.1 \, \mu \, \text{sec})$$

(c) Explain why the delay-time correlation function of the channel is different from what you determined via the free propagation model in Part (b)?

♠ **Solution:** In real case, it experiences "multipath" effect.

Therefore, multiple copies are received through different paths.

Also, we have the "Doppler effect", which makes the impulse response time-variant.

(d) For this channel, write the *power-delay* profile.

♠ **Solution:** Power-delay profile is given by $\rho_h(\tau, 0)$.

$$\implies \rho_h(\tau, 0) = \frac{1}{2}\delta(\tau) + \frac{1}{2}\delta(\tau - 0.1)$$

(e) Using the power-delay profile determined in Part (d), calculate

i. power transfer factor
ii. mean delay
iii. delay spread
iv. excess delay

of this channel.

♠ **Solution:** Using Eqs. (3.86)-(3.89) in page 120, we have,

i.

$$h_P = \int_0^{\infty} \rho_h(\tau, 0)\ \mathrm{d}\tau = \frac{1}{2} + \frac{1}{2} = 1$$

ii.

$$\mu_\tau = \frac{1}{h_P} \int_0^{\infty} \tau \rho_h(\tau, 0)\ \mathrm{d}\tau = \frac{1}{2} \times 0 + \frac{1}{2} \times 0.1 = \boxed{0.05\ \mu s}$$

iii.

$$\sigma_\tau = \sqrt{\frac{1}{h_P} \int_0^{\infty} (\tau - \mu_\tau)^2 \rho_h(\tau, 0)\ \mathrm{d}\tau} \qquad = \sqrt{\frac{1}{2}\mu_\tau^2 + \frac{1}{2}(0.1 - \mu_\tau)^2}$$
$$= \boxed{0.05\ \mu s}$$

iv.

$$\tau_{\text{ex}} = \tau_{\max} - \tau_{\min} = 0.1 - 0 = \boxed{0.1\ \mu s}$$

11.7.5 Question 5: Multiple Access Techniques (18 Points)

Consider a base station (BS) which is serving *multiple* mobile stations (MSs) in the network. The number of *active* MSs in the network is denoted by K.

The BS intends to choose a multiplexing technique, in order to serve all the MSs in the network, simultaneously. To this end, it considers the following constraint:

Constraint: The *average* signal-to-interference-and-noise ratio (SINR) of each MS must be more than or equal by 0 dB.

The first choice of the BS is the code division multiplexing (CDM) technique with random spreading: In this approach, the BS generates independent and indentically distributed (i.i.d) uniform sequences of $\{\pm 1\}$ with lengths N at random and uses them as the spreading sequences. These sequences are used to transmit phase shift keying (PSK) symbols with unit amplitude to the active MSs, i.e., $|a_k[i]|^2 = 1$ for each symbol interval i. The duration of each symbol interval is $T_s = 5$ msec. The BS further superposes the signals of active MSs using *equal* amplitude factors $w_1 = \cdots = w_k = 1$.

(a) Assuming that the BS uses its first choice to serve $K = 6$ active MSs, calculate the minimum spreading factor N for the CDM system, such that the given constraint is satisfied. For calculations, assume that the spectral density of noise at the MSs is $N_0 = 4 \times 10^{-18}$ mW/Hz.

♠ **Solution:** The SINR of the first choice is given in Eq. (5.29) in page 167:

$$\text{SINR} = \frac{N}{K - 1 + N\frac{N_0}{E_S}}$$

Here, $K = 6$, $N_0 = 4 \times 10^{-18}$, $E_S = w_K^2 T_S = 5$ ms.

$$\implies \text{SINR} = \frac{N}{5 + N \times 8 \times 10^{-19}} \geq \underbrace{1}_{0\text{ dB}}$$

$$\implies N \geq 5 + N \times 8 \times 10^{-19} \implies N \geq \frac{5}{1 - 8 \times 10^{-19}} \approx 5$$

$$\boxed{N_{\min} = 5}$$

(b) The bandwidth of the signature waveform of an MS is shown by B and determined in terms of its chip interval T_c as

$$B = \frac{2}{T_c}$$

Given your answer to Part (a), calculate the minimum bandwidth occupied by the system.

♠ **Solution:**

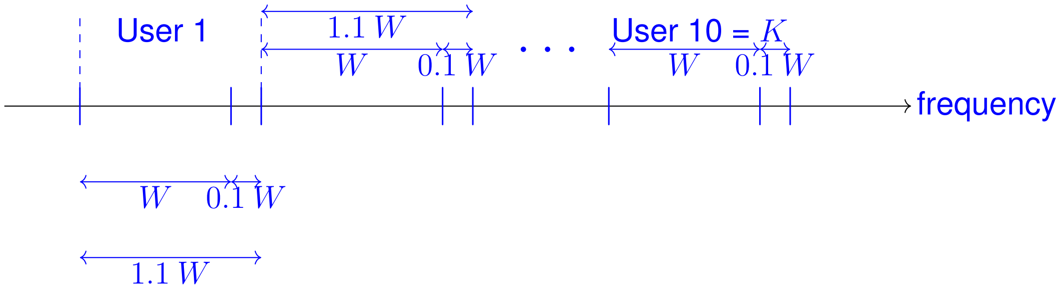
$$N_{\min} = 5 \implies T_{c,\max} = \frac{T_S}{N_{\min}} = \frac{5}{5} = 1 \text{ msec}$$

$$\implies B_{\min} = \frac{2}{T_{c,\max}} = \boxed{2 \text{ kHz}}$$

Now, assume that the BS chooses to use the frequency division multiplexing (FDM) technique. To this end, it divides the available bandwidth among the active MSs *equally*. To avoid multiple access interference (MAI), it further sets a guard interval between two neighboring FDM sub-channels whose width is $B_{\text{guard}} = 0.1\ W$ with W being the *sub-channel bandwidth*. The signal of each MS is transmitted over a dedicated FDM sub-channel using a band-limited pulse. The effective noise bandwidth of the MS filter matched to the band-limited pulse is W. The transmit signal in each FDM sub-channel is amplified such that the signal power received at the MS be $P_s = 1$ mW.

(c) Assume the available bandwidth of the system is given by the value calculated in Part (b). Determine the bandwidth of each FDM sub-channel, when $K = 10$ MSs are active.

♠ **Solution:** FDM frequency diagram:



$$\implies \text{Total Bandwidth } = K \times 1.1W = 1.1KW = B$$

For $K = 10 \implies B = 11W$.

When $B = B_{\min}$ [in Part (b)] $= 2$ kHz, we have

$$W = \frac{2}{11}\text{ kHz} = \boxed{0.18\text{ kHz}}$$

(d) The maximum spectral efficiency of a multiuser system is approximated with

$$\eta_{max} = W_m \log_2(1 + \text{SINR}_m)$$

where W_m denotes the bandwidth occupied by each MS, and SINR_m is the average SINR of each MS.

Using your answers to Parts (a) and (c), calculate $\eta_{\max}$ for the CDM and FDM systems when $K = 10$ MSs are active. For the CDM system, assume that the BS uses exactly the spreading factor calculated in Part (a). As in Part (a), assume that the spectral density of noise at the MSs is $N_0 = 4 \times 10^{-18}$ mW/Hz, wherever needed.

♠ **Solution:** For CDM system:

$$\text{SINR} = \frac{N}{K - 1 + N\frac{N_0}{E_S}} = \frac{5}{9 + 5 \times \frac{4\times10^{-18}}{5}}$$
$$= \frac{5}{9 + 4 \times 10^{-18}} \approx 0.55$$

In CDM, each MS uses the whole BW; thus,

$$W_m = B_{\min} = 2\text{ kHz}.$$

$$\eta_{\max} = 2 \times 10^3 \cdot \log_2(1.55) = 1264.54\text{ bit/sec}$$

For FDM system: We have no interference; thus,

$$\text{SINR} = \frac{P_S}{N_0 \underbrace{B_{\text{noise}}}_{\text{effective bandwidth} = W}} = \frac{1}{4 \times 10^{-18}\ \text{mW}} = \frac{1}{4 \times 10^{-18} \times 0.18 \times 10^3}$$

$$= \frac{10^{15}}{0.72} = 1.39 \times 10^{15}$$

$$\implies \eta_{\max} \approx 0.18 \times 10^3 \log_2 \left(1.39 \times 10^{15} + 1\right) = 9054.72 \text{ bits/sec}$$

11.7.6 Question 6: Modulation Schemes (11 Points)

Consider the following list of modulation schemes:

(S_1) Quadrature phase shift keying (QPSK) via root raised cosine (RRC) pulses with roll-off factor $\alpha = 0.3$

(S_2) $\pi/4$-shifted QPSK via RRC pulses with roll-off factor $\alpha = 0.1$

(S_3) $\pi/4$-shifted binary phase shift keying (BPSK) via RRC pulses with roll-off factor $\alpha = 0.4$

(S_4) 64-Quadrature amplitude modulation (QAM) via RRC pulses with roll-off factor $\alpha = 0.1$

(S_5) QPSK via rectangular pulses

(S_6) BPSK via rectangular pulses

(a) Sort the given modulation schemes in terms of spectral efficiency.

♠ **Solution:** For spectral efficiency, we have

$$\eta = \frac{\log_2 M/T}{B} = \begin{cases} \frac{\log_2 M/T}{\frac{(1+\alpha)}{T}} = \frac{\log_2 M}{1+\alpha} & \text{for RRC} \\ \frac{\log_2 M}{\infty} = 0 & \text{for Rectangular Pulse.} \end{cases}$$

with M being the constellation size.

$$\implies \eta_6 = \eta_7 = 0$$

$$\eta_1 = \frac{2}{1 \cdot 3} = 1.54$$
$$\eta_2 = \frac{2}{1 \cdot 1} = 1.82$$
$$\eta_4 = \frac{1}{1 \cdot 4} = 0.71$$
$$\eta_5 = \frac{6}{1 \cdot 1} = 5.45$$

$$\eta_5 > \eta_2 > \eta_1 > \eta_4 > \eta_6 = \eta_7$$

(b) Sort the given modulation schemes in terms of peak-to-average ratio (PAR).

♠ **Solution:** From Figure 6.12; page 196:

$$\text{PAR}_2 > \text{PAR}_1 > \text{PAR}_4$$

(S_5) has very high PAR. and given that $\alpha = 0.1$; the PAR is higher than (S_2). Thus,

$$\text{PAR}_5 > \text{PAR}_2 > \text{PAR}_1 > \text{PAR}_4$$

(S_6) and (S_7) have unit PAR, since they use rectangular pulses and are PSK. Hence,

$$\text{PAR}_5 > \text{PAR}_2 > \text{PAR}_1 > \text{PAR}_4 > \text{PAR}_6 = \text{PAR}_7$$

Hint: If someone drops (S_4) out, the answer is still right as it has not been explicitly indicated in the lecture notes. It is however concluded straight forwardly.

(c) Name the modulation scheme with maximum error probability when used over a static additive white Gaussian channel (AWGN).

♠ **Solution:** (S_5) as it has highest number of constellation points.

11.8 Winter Semester 2020-2021

Exam Date: February 23, 2021
6 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.8.1 Question 1: Short Questions (22 Points)

Answer the following items:

ITEM 1: A *hexagonal* cellular network has a *regular* frequency reuse pattern. The outer radius of the cells is $R = 5$ km. The following constraint is required to be satisfied in this network:

> There should be *at least* a distance of $D_{\min} = 25$ km between a pair of co-channel cells

(a) Determine the *minimum* cluster size, such that this constraint is fulfilled.

♠ **Solution:** From (1.3):

$$K = \frac{1}{3}\left(\frac{D}{R}\right)^2 \implies K_{\min} \geq \frac{1}{3}\left(\frac{D_{\min}}{R}\right)^2$$
$$= \frac{1}{3}5^2 = \frac{25}{3}$$

From Table 1.1

$$\implies \boxed{K_{\min} = 9}$$

(b) For the determined cluster size in Part (a), calculate the *reuse distance* D.

♠ **Solution:** Reuse distance:

$$D = \sqrt{3KR^2}$$
$$= \sqrt{3 \cdot 9 \cdot 25} = 15\sqrt{3} \approx 26 \text{ km}$$

ITEM 2: Let G_d and G_l denote the gains of a $\lambda/2$-*dipole antenna* and a $\lambda/4$-*linear antenna on a metal plate* in logarithmic scale dB$_i$, respectively. Define ΔG as $\Delta G = G_l - G_d$.

(c) Determine ΔG and *explain* why you conclude this value for ΔG.

> **Hint:** You could use the *radiation patterns* of the antennas to explain your observation.

♠ **Solution:** From Eq. (2.4) and (2.5), we have:

$$G_d = 2.15 \text{ dB}_i \qquad G_l = 5.15 \text{ dB}_i$$

Thus,

$$\boxed{\Delta G = 3 \text{ dB}}$$

The received power in the latter is doubled due to the reflection of the metal surface.

$\boxed{\text{ITEM 3:}}$ A mobile user receives signals through a *fading* channel. For a long period of time the user is *not moving*. Now, the user starts to move.

(d) Does the coherence time of the channel change after the user starts to move? Justify your answer by giving a reason.

♠ **Solution:** Yes.
By moving, the user observes Doppler effect. Thus, the coherence time interval of the channel could get impacted.

$\boxed{\text{ITEM 4:}}$ Consider a traffic channel (TCH) in the GSM system which is used for full-rate speech (FS) transmission. We know that for this service, the output of each *multiframe multiplexer* at the base-station passes a data stream of rate 24.7 kbits/s to its corresponding *encryption module*.

(e) Starting from the outputs of *multiframe multiplexers*, explain why the *time slot formatter*, in the case of TCH-FS, transmits a data stream of rate 270.833 kbits/s.

♠ **Solution:** The output of the encryption module is of rate 24.7 kbits/s.
The encrypted streams are multiplexed for 8 users. Thus, after MUX, we have

$$R_{\text{TDM}} = 24.7 \times 8 \text{ kbits/s}$$

We then give the stream to the time slot formatter which maps each 114 bits to 156.25 bits. Thus, the rate after time formatting would be

$$R_{\text{GSM}} = 8 \times 24.7 \times \frac{156.25}{114} = 270.833 \text{ kbits/s}$$

11.8.2 Question 2: Mobile Radio Channels (18 Points)

A transmitter is installed on the top of a tower with height 85 m in a farm field with a plain landscape. The farm filed is of size 900 km^2 and the tower is located exactly at the middle of the field. The carrier frequency at which the transmitter operates is set to be exactly at the middle of the *downlink* frequency band of GSM900. The transmitter uses an array of *isotropic* antenna with the *array gain* $G_{\text{Array}} = 8$ dB.

A wireless link is established between the transmitter and a receiver which is 15 km away from the transmit tower. The receiver is equipped with a single $\lambda/2$-dipole antenna and is held in the hand of a person who is standing on the ground.

(a) Explain why considering such a transmitter is not *realistic*.

♠ **Solution:** Isotropic antenna is not realized.

(b) Determine the path-loss of this communication link using the Okumura-Hata model.

♠ **Solution:**

$$f_0 = \frac{935 + 960}{2} = 947.5 \text{ MHz}$$

$$h_{\text{BS}} = 85 \text{ m}, \qquad h_{\text{MS}} = 1.5 \text{ m (any \# between 1 m and 3 m is correct)}$$

$$d = 15 \text{ km}$$

We use equation (3.27).

The medium is <u>"open"</u> $\Longrightarrow$ Correction by Eq. (3.30).

From Eq. (3.28):

$$\begin{aligned} a(h_{\text{MS}}) &= (1.1 \log 947.5 - 0.7) \cdot 1.5 \\ &\quad - 1.56 \log 947.5 + 0.8 \\ &= 0.018 \end{aligned}$$

From Eq. (3.27):

$$\begin{aligned} \text{PL}_{\text{urban}} &= 69.55 + 26.16 \log 947.5 - 13.82 \log 85 \\ &\quad - 0.018 + (44.9 - 6.55 \log 85) \log 15 \\ &= 69.55 + 77.96 - 26.66 - 0.018 + 37.94 \\ &= 158.772 \text{ dB} \end{aligned}$$

This path loss is corrected by Eq. (3.30):

$$\begin{aligned} \text{PL}_{\text{open}} &= \text{PL}_{\text{urban}} - 4.78 (\log 947.5)^2 + 18.33 \log 947.5 - 40.98 \\ &= 158.772 - 42.45 + 54.62 - 40.98 \\ &= 129.96 \text{ dB} \end{aligned}$$

This PL is given for $G_{\text{out}} = 0$ dB. For the given scenario, we further should correct with antenna gain, which leads to:

$$\text{PL} = 129.96 - \underbrace{G_{\text{Array}}}_{=8 \text{ dB}} = \boxed{121.96 \text{ dB}}$$

The person who holds the receiver is now lifted 8 m up by a lifter. Let ΔL denote the difference between the path-loss determined in Part (b) and the path-loss after the person is lifted.

(c) Determine ΔL.

♠ **Solution:** Note that h_{MS} only appears in $a(h_{\text{MS}})$.Thus;

$$\begin{aligned}
\text{PL}_2 - \text{PL}_1 &= -a(h_{\text{MS}}^{(2)}) - \left(-a(h_{\text{MS}}^{(1)})\right) \\
&= a(h_{\text{MS}}^{(1)}) - a(h_{\text{MS}}^{(2)}) \\
&= (1.1 \log 947.5 - 0.7) \left(h_{\text{MS}}^{(1)} - h_{\text{MS}}^{(2)}\right) \\
&= 2.58 \times (-8 \text{ m}) \\
&= -20.64 \text{ dB}
\end{aligned}$$

This means that by moving 8 m up, PL drops by 20.64 dB.

11.8.3 Question 3: Fading Channels (23 Points)

A wireless link is established between a transmitter, which is located on the top of a tower, and a receiver, which is located on the roof of a house and does *not* move. The transmitter sends rectangular pulses with duration $T = 2$ ms. The bandwidth of the band-pass filter used at the receiver is

$$B_{\text{RX}} = \frac{2}{T}.$$

There exist *only* two paths between the transmitter and the receiver. The delays of the first and second path are $\tau_1 = 1.2$ ms and $\tau_2 = \alpha$ ms, respectively, and do *not* change in time. This means that for a given transmitted signal $x(t)$, the receiver receives

$$y(t) = h_1 x(t - \tau_1) + h_2 x(t - \tau_2) + z(t)$$

where h_1 and h_2 are the gains of the first and second path, respectively, and $z(t)$ is additive white Gaussian noise (AWGN).

Due to fading, the path gains h_1 and h_2 are modelled as *independent* Gaussian random variables with zero mean and variance θ^2, i.e., $h_1, h_2 \sim \mathcal{N}(0, \theta^2)$.

(a) Determine the *delay-time* and *frequency-time* correlation functions of this channel.

> **Hint:** You could use the following property:
> *Let the Fourier transform of $x(t)$ be $X(f) = \mathcal{F}\{x(t)\}$. Then we have*
>
> $$\mathcal{F}\{x(t - \tau)\} = \exp\{-j2\pi f \tau\} X(f)$$
>
> *where $j = \sqrt{-1}$ is the imaginary unit.*

♠ **Solution:** By definition from Table 3.2, we have:

$$\rho_h(\tau, \Delta t) = \int_R \mathcal{E}\left\{h^*(\beta, t) h(\tau, t + \Delta t)\right\} \, \mathrm{d}\beta$$

Here, we have LTI system; thus,

$$h(\beta, t) = h(\beta) = h_1\delta(\beta - \tau_1) + h_2\delta(\beta - \tau_2) = h(\beta)$$

Thus,

$$\rho_h(\tau, \Delta t) = \int_R \mathcal{E}\left\{h^*(\beta)h(\tau)\right\} \,\mathrm{d}\beta$$

Noting that h_1,h_2 are independent and zero mean, we have

$$\mathcal{E}\left\{h^*(\beta)h(\tau)\right\} = \mathcal{E}\left\{|h_1|^2\right\}\delta(\beta - \tau_1)\delta(\tau - \tau_1) + \mathcal{E}\left\{|h_2|^2\right\}\delta(\beta - \tau_2)\delta(\tau - \tau_2)$$

and since $\mathcal{E}\left\{|h_1|^2\right\} = \mathcal{E}\left\{|h_2|^2\right\} = \theta^2$, we have:

$$\rho_h(\tau, \Delta t) = \theta^2\delta(\tau - \tau_1) + \theta^2\delta(\tau - \tau_2)$$

The frequency-time correlation function is further given by:

$$\begin{aligned}\rho_H(\Delta f, \Delta t) &= \mathcal{F}_\tau\left\{\rho_h(\tau, \Delta t)\right\} \\ &= \theta^2 e^{-j2\pi\Delta f\tau_1} + \theta^2 e^{-j2\pi\Delta f\tau_2} \\ &= \theta^2 e^{-j\pi\Delta f(\tau_1+\tau_2)}\cos\left(\pi\Delta f(\tau_2 - \tau_1)\right)\end{aligned}$$

(b) Determine the *Doppler-frequency* correlation and the *scattering* function of this channel.

> **Hint:** You could use the following property:
> *For a constant function $x(t) = C$, we have*
>
> $$\mathcal{F}\{C\} = C\delta(f)$$
>
> *where $\delta(f)$ is the Dirac impulse function.*

♠ **Solution:** The Doppler-frequency correlation:

$$\begin{aligned}\rho_T(\Delta f, \nu) &= \mathcal{F}_{\Delta t}\left\{\rho_h(\Delta f, \Delta t)\right\} \\ &= \rho_h(\Delta f, \Delta t)\cdot\delta(\nu) \\ &= \theta^2 e^{-j\pi\Delta f(\tau_1+\tau_2)}\cos\left(\pi\Delta f(\tau_2 - \tau_1)\right)\delta(\nu)\end{aligned}$$

The scattering function is:

$$\begin{aligned}\rho_S(\tau, \nu) &= \mathcal{F}_{\Delta t}\left\{\rho_h(\tau, \Delta t)\right\} \\ &= \rho_h(\tau, \Delta t)\delta(\nu) \\ &= \theta^2\left(\delta(\tau - \tau_1) + \delta(\tau - \tau_2)\right)\delta(\nu)\end{aligned}$$

(c) Determine the *coherence time* and the *coherence bandwidth* of the channel.

♠ **Solution:** The coherence time is given by: (Eqs. in page 117)

$$|\rho_H(0, T_c)| = \frac{1}{2}|\rho_H(0,0)|$$

Since $\rho_H(\tau, \Delta t)$ is not a function of Δt, we could conclude that this never happens. This means that

$$T_c \to \infty.$$

The coherence bandwidth is:

$$|\rho_H(B_c, 0)| = \frac{1}{2}|\rho_H(0,0)|$$

$$\theta^2 \left|\cos\left(\pi \underbrace{\Delta f}_{B_c}(\tau_2 - \tau_1)\right)\right| = \frac{\theta^2}{2}\underbrace{|\cos(0)|}_{=1}$$

Thus, we have:

$$|\cos(\pi B_c(\tau_2 - \tau_1))| = \frac{1}{2}$$

or equivalently:

$$\pi B_c(\tau_2 - \tau_1) = \pm\frac{\pi}{3} \implies \boxed{B_c = \left|\frac{1}{3(\tau_2 - \tau_1)}\right|}$$

Given the values of τ_1 and τ_2, we have

$$B_c = \left|\frac{1}{3(\alpha - 1.2)}\right|$$

(d) Find α such that the channel is classified as a *non frequency selective* (narrow-band fading) channel.

♠ **Solution:** For non-frequency selective fading, we have (page 122).

$$B_c > B_{\text{RX}}$$

Hence,

$$\frac{1}{3|\alpha - 1.2|} > \frac{1}{T}$$

$$T = 2\text{ ms} \implies \frac{1}{3|\alpha - 1.2|} > 1 \implies |\alpha - 1.2| < \frac{1}{3}$$

We could hence have the following solutions:

$$1.2 < \alpha < 1.53$$
$$0.87 < \alpha < 1.2$$

Thus,

$$\boxed{0.87\text{ ms} < \alpha < 1.53\text{ ms}}$$

11.8.4 Question 4: Diversity Principles (17 Points)

A single-antenna transmitter communicates with a single-antenna receiver over a *narrow-band* fading channel with additive white Gaussian noise (AWGN). Let h denote the channel coefficient which is described by a *Rayleigh* fading model. This means that $|h|$ is distributed with a Rayleigh distribution.

The transmitter uses *time diversity* technique with $M = 5$ branches. For a articular signal transmission, the realizations of the channel coefficient for the diversity branches are as follows:

$$h_1 = 1 + 0.8j$$
$$h_2 = 0.3 + 0.03j$$
$$h_3 = 1.6 - 2.8j$$
$$h_4 = 0.01 + 0.2j$$
$$h_5 = -1.3 - 0.6j$$

where $j = \sqrt{-1}$ is the imaginary unit. Here, h_m for $m \in \{1, \ldots, 5\}$ denotes the realization of the channel coefficient in the diversity branch m.

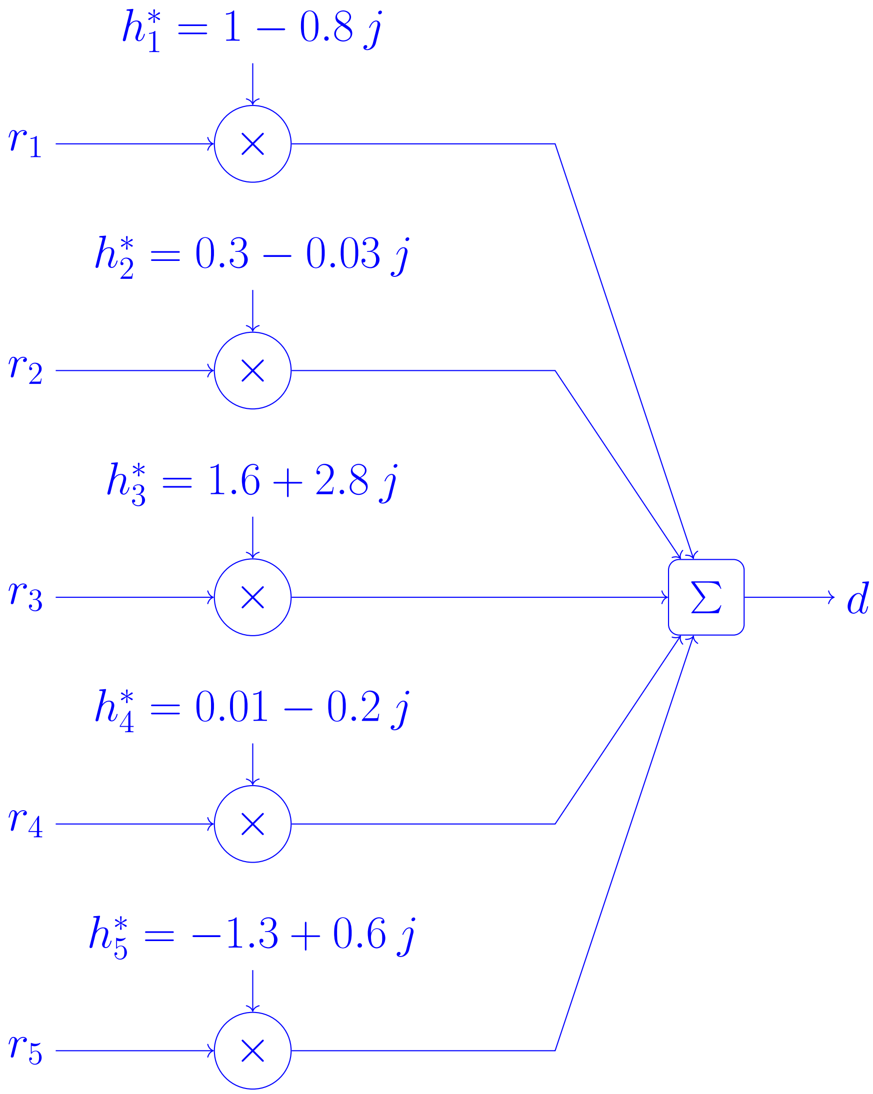
The receiver applies matched filtering to the signal received from each branch, and combines the output symbols with a combining technique. The noise power in all branches after matched filtering is $\log P_n = -20$ dBm. Moreover, the power of the transmitted signal is $\log P_S = 10$ dBm.

(a) Assume that the *maximum ratio combining (MRC)* technique is used. How does the receiver combine the outputs of the matched filters? Clarify your answer by drawing a diagram. Specify the receiver gain of each branch in the diagram.

♠ **Solution:** The signal received at branch "m" after matched filtering:

$$r_m = h_m x + z_m$$

MRC:



(b) Calculate the *instantaneous* signal-to-noise ratio (SNR) of the combined symbol in Part (a).

♠ **Solution:**

$$d = \sum_{m=1}^{5} h_m^* r_m = \left(\sum_{m=1}^{5} |h_m|^2\right) x + \left(\sum_{m=1}^{5} h_m^* z_m\right)$$

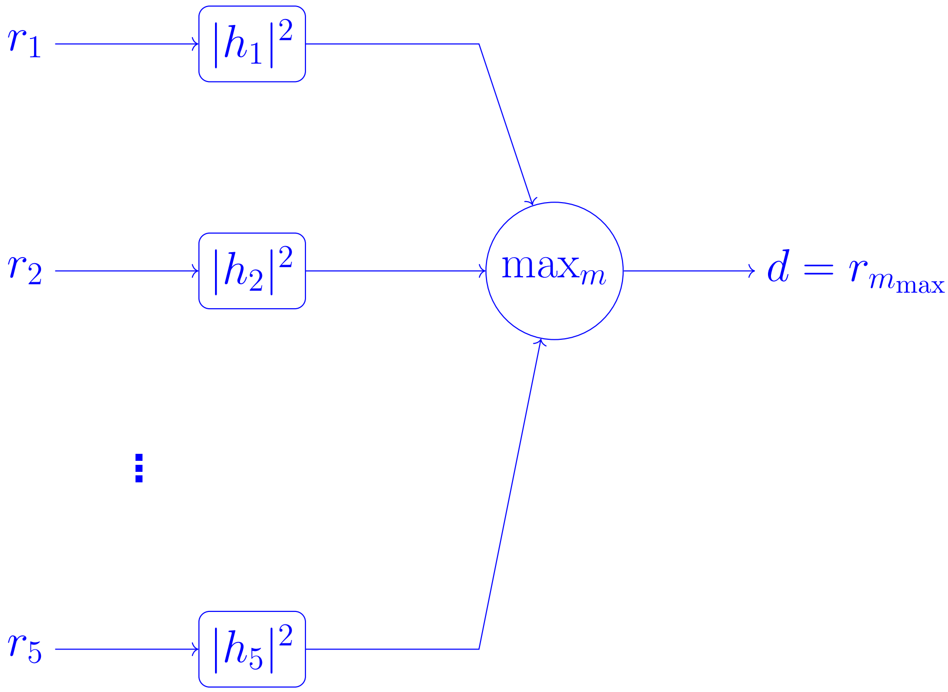
$$P_X = \left(\sum_{m=1}^{5} |h_m|^2\right)^2 \cdot P_S = 10\log\left(\sum_{m=1}^{5} |h_m|^2\right)^2 + 10$$
$$= 23.06 + 10 = 33.06 \text{ dbm}$$

$$P_N = \left(\sum_{m=1}^{5} |h_m|^2\right) \cdot P_n \implies \log P_N = 10\log\left(\sum_{m=1}^{5} |h_m|^2\right) - 20 = -8.76 \text{ dBm}$$

$$\boxed{\text{SNR} = 41.53 \text{ dB}}$$

(c) Repeat Parts (a) and (b) for the case in which the receiver uses *selection combining* technique.

♠ **Solution:** Diagram:



⋆ Note:: Since the noise power is the same, once could replace $|h_m|^2$ with $|r_n|^2$.

Since h_3 has maximum amplitude $\Longrightarrow$ $\boxed{d = r_3}$.

SNR is calculated as:

$$\text{SNR} = \frac{|h_3|^2 P_S}{P_n} \Longrightarrow 10 \log \text{SNR} = 10 \log |h_3|^2 + 30 \text{ dB}$$
$$= 10.17 + 30$$
$$= 40.17 \text{ dB}$$

Clearly:

$$\text{SNR}_{\text{Sel}} < \text{SNR}_{\text{MRC}}$$

Now assume that the receiver uses the following naive selection technique:

After matched filtering, it selects one of the branches at random and ignores the symbols of the other diversity branches. The probability of each branch being selected is *equal* .

For this technique, we define $\overline{\text{SNR}}$ as

$$\overline{\text{SNR}} = \sum_{m=1}^{M} \text{Pr}\{\text{Branch } m \text{ being selected}\}\text{SNR}_m$$

where SNR_m is the *instantaneous* SNR, when branch m is selected.

(d) Determine $\overline{\text{SNR}}$.

♠ **Solution:**

$$\text{SNR}_m = \frac{|h_m|^2 P_S}{P_n} = |h_m|^2\ \text{SNR}_0 \qquad [\text{define: SNR}_0 = \frac{P_S}{P_n}]$$

Thus,

$$\overline{\text{SNR}} = \sum_{m=1}^{5} \frac{1}{5}|h_m|^2\ \text{SNR}_0 = \frac{\text{SNR}_0}{5}\left(\sum_{m=1}^{5}|h_m|^2\right)$$

$$\begin{aligned} \Longrightarrow\ 10\log\overline{\text{SNR}} &= 10\log\text{SNR}_0 - 10\log 5 + 10\log\sum_{m=1}^{5}|h_m|^2 \\ &= 30 - 7 + 11.53 \\ &= 34.53\ \text{dB} \end{aligned}$$

⋆ NOTE: Once could directly conclude:

$$\overline{\text{SNR}} = \frac{\text{SNR}_{\text{MRC}}}{5} \Longrightarrow 10\log\overline{\text{SNR}} = 10\log\text{SNR}_{\text{MRC}} - 7$$

(e) Compare $\overline{\text{SNR}}$ in Part (d) to the instantaneous SNRs determined in Parts (b) and (c). Explain your observation.

♠ **Solution:** Clearly, it is worse than both cases:

$$\overline{\text{SNR}} < \text{SNR}_{\text{Sel}} < \text{SNR}_{\text{MRC}}$$

This is due to the fact that bad and good branches contribute the same in this case.

11.8.5 Question 5: Modulation Schemes (10 Points)

Consider the following list of modulation schemes:

(S_1) Quadrature phase shift keying (QPSK) via root raised cosine (RRC) pulses with roll-off factor $\alpha = 0.7$

(S_2) $\pi/4$-shifted QPSK via RRC pulses with roll-off factor $\alpha = 0.4$

(S_3) 64-quadrature amplitude modulation (QAM) via RRC pulses with roll-off factor $\alpha = 0.05$

(S_4) Offset-QAM via RRC pulses with roll-off factor $\alpha = 0.1$

(S_5) 8-phase shift keying (8-PSK) via rectangular pulses

(a) Sort the given modulation schemes in terms of spectral efficiency.

♠ **Solution:** For RRC, we have:

$$R_b = (\log_2 M)/T$$
$$B = \frac{1}{T}(1+\alpha)$$

$$\eta = \frac{\log M}{1+\alpha} \implies \begin{cases} (S_1) \to \eta_1 = \frac{2}{1.7} \\ (S_2) \to \eta_2 = \frac{2}{1.4} \\ (S_3) \to \eta_3 = \frac{6}{1.05} \\ (S_4) \to \eta_4 = \frac{2}{1.1} \end{cases}$$

$$S_3 > S_4 > S_2 > S_1$$

For rectangular pulses, $B \to \infty \implies \eta = 0 \implies S_5$ has minimum η:

$$\boxed{S_3 > S_4 > S_2 > S_1 > S_5}$$

(b) Sort the given modulation schemes in terms of peak-to-average ratio (PAR).

♠ **Solution:** Figure 6.12, page 196: From the figure, we have

$$S_4 > S_1 > S_2$$

Since S_5 uses rectangular pulses with PSK constellation

$$\text{PAR}_5 = 1 = \text{ minimum possible value}$$

$$S_4 > S_1 > S_2 > S_5$$

S_3 uses 64-QAM constellation whose PAR is larger than QPSK. Given that the roll-off factor is also smaller than all others, we could conclude that is has maximum PAR:

$$S_3 > S_4 > S_1 > S_2 > S_5$$

(c) You use these schemes to transmit data over a *Rayleigh* fading channel with 4 branches of diversity. The mean E_b/N_0 is set to

$$\log E_b/N_0 = 5 \text{ dB}.$$

Name two schemes whose mean bit error rates are less than 10^{-2}.

♠ **Solution:** From Figure 6.28, page 217:

S_5 and 16-QAM have $P_e < 10^{-2}$.

Since S_4 uses offset-QAM with smaller constellation, we could say

$$P_e(S_4) < P_e(\text{16-QAM}) < 10^{-2} \implies \boxed{S_5 \text{ and } S_4}$$

11.8.6 Question 6: Channel Coding (10 Points)

For binary phase shift keying (BPSK) transmission over a *static* channel with additive white Gaussian noise (AWGN), a convolutional code with the code rate

$$R_c = \frac{1}{3}$$

is used. By *static*, we mean that the channel does *not* experience fading.

It is desired to have a coding gain G_c which satisfies

$$G_c \geq 6.5 \text{ dB}.$$

(a) Given the desired constraint on G_c, find a lower bound on the free distance of the code.

♠ **Solution:** From Eq. (7.23), we know

$$\max G_c = 10 \log R_c d_{\text{free}} > 6.5$$
$$\implies \boxed{d_{\text{free}} > 13.4}$$

(b) Specify a constraint length for which a convolutional code exists whose free distance satisfies the lower bound in Part (a).

♠ **Solution:** From Table 7.1, we have for $R_c = \frac{1}{3}$:

$$\boxed{L_c \geq 7} \implies d_{\text{free}} \geq 15 : \text{ Any solution more than or equal to 7 is correct.}$$

Assume that the constraint length is set to the one specified in Part (b). We now use a convolutional code whose free distance for the given constraint length and code rate is *maximized*. This code is used to achieve the bit error probability

$$p_b = 10^{-5}.$$

(c) Specify the free distance of the code.

♠ **Solution:** Table 7.1: If we set $L_c = 7 \implies d_{\text{free}} = 15$.

(d) Find the coding gain when the decoder uses the Viterbi algorithm with soft decisions.

♠ **Solution:** From Fig. 7.12 at page 252, we have

$$\text{At } P_e = 10^{-5} \implies \begin{cases} \text{coded: SNR} \approx 4.2 \text{ dB} \\ \text{uncoded: SNR} \approx 9.6 \text{ dB} \end{cases} \implies \boxed{G_c = 5.4 \text{ dB}}$$

(e) Repeat Part (d) for the case in which the decoder replaces the soft decisions with hard decisions.

♠ **Solution:** Again Fig. 7.12:

$$\text{At } P_e = 10^{-5} \implies \begin{cases} \text{coded: SNR} \approx 4.6 \text{ dB} \\ \text{uncoded: SNR} \approx 9.6 \text{ dB} \end{cases} \implies \boxed{G_c = 5 \text{ dB}}$$

(f) Compare the results of Parts (d) and (e). Explain your observation.

♠ **Solution:**

$$G_{c,\text{soft}} > G_{c,\text{hard}}$$

This is because soft estimation keeps soft information whereas hard estimation kills them.

(g) What is the coding gain when the code is used in a fading channel with *no* interleaving? Explain your answer.

♠ **Solution:**

$$G_c = 0 \text{ dB}$$

In presence of fading, channel coding without interleaving results in no coding gain.

11.9 Summer Semester 2021

Exam Date: July 30, 2021
6 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.9.1 Question 1: Short Questions (18 Points)

Answer the following items briefly.

(a) What is the difference between *external* noise and *thermal* noise in a mobile communication system? Name the parameter which models the impacts of *external* noise.

♠ **Solution:** Page 35 of the Lecture Notes: External noise is mainly generated by external resources like electrical machines, athmosphere or space. It is manmade. However, thermal noise comesfrom movement of particles in electronic devices. Parameter to model external noise: Noise Figure

(b) Which arrangement of array antennas is mostly used at the base stations of cellular networks?

♠ **Solution:** Page 53: Uniform planer array antennas

(c) A mobile system uses code division multiplexing (CDM) technique. Let L_1 and L_2 be numbers of fingers used in two rake receivers which detect signals in two different areas: L_1 is the number of fingers for the receiver which is used in a *typical urban* area with excess delay $\tau_{\text{ex}} = 5\ \mu$sec, and L_2 denotes the number of fingers which is used in the receiver designed for *hilly terrain* area with excess delay $\tau_{\text{ex}} = 32\ \mu$sec. You know that $L_1 \neq L_2$. Which one do you expect to be larger? Justify your answer.

♠ **Solution:** Since T_{ex} for second area is much higher, we have much more copies to distinguish @ the Rake receiver. Therefore, $L_2 \geq L_1$.

(d) Name an advantage and a disadvantage of code division multiplexing (CDM) technique.

♠ **Solution:** You could choose any from those given in page 169 of the Lecture Notes.

(e) What is the benefit of using the offset-quadrature amplitude modulation (QAM) instead of standard QAM.

♠ **Solution:** We could reduce PAPR.

(f) Explain why *puncturing* is used in convolutional channel codes?

♠ **Solution:** Page 250 of Lecture Notes: To realize codes with notes $R_c > \frac{1}{2}$ via convolutional codes with K = 1.

11.9.2 Question 2: Mobile Radio Channels (18 Points)

A base station (BS) is operating at the middle of downlink band of the GSM standard PCS1900. The BS is installed on top of a tower of height 180 meters in the Museum Park, Chicago, USA and transmits signals of power $P_s = 10$ W. The mobile station (MS) is at the distance of 9 kilometers in the backpack of a person who is riding a bike.
The BS uses an array antenna which consists of eight $\lambda/2$-diploe antenna elements. The gain of the array antenna is $\log G_{\text{BS}} = 12$ dB$_i$. The MS uses a $\lambda/2$-*dipole antenna*.

(a) Using the Okumura-Hata model, calculate the *median* power received by the MS.

♠ **Solution:** We need to use formula in page 90:

$$f_0 = \frac{1990 + 1930}{2} = 1960$$
$$d = 9\ km$$
$$h_{BS} = 180\ m$$
$$h_{MS} = 1.5\ m \text{(any number between 1-3 is correct)}$$

Area: Chicago = Urban

$$a(h_{MS}) = 0.0463$$
$$PL_{urban} = 155.443\ dB \text{ for isotropic antennas}$$

Considering the antenna gains:

$$\begin{aligned} PL &= 155.443 - \log G_{BS}[dBi] - \log G_{MS}[dBi] \\ &= 155.443 - 12 - 2.15 = 141.293\ dB \end{aligned}$$

The median receive power is:

$$\begin{aligned} \log P_r &= \log P - PL = 10[dB] - 141.293[dB] \\ &= -131.293\ dB \end{aligned}$$

in [dBm] it would be:

$$\log P_r = -101.293\ dBm$$

The noise power at the MS is -147 dBm and the *variance* of the shadowing process is $\sigma_\Lambda^2 = 4\text{dB}^2$. It is known that for successful communication, the signal-to-noise ratio (SNR) at the MS should be more than $\log \text{SNR}_{min} = 48$ dB.

(b) Determine the probability of having a successful communication.

Hint: You can give the final result in terms of the Q-function.

♠ **Solution:**

$$SNR = \frac{P_r}{P_N} : \log SNR = \log P_r - \log P_N$$

We want $\log SNR \ > \log SNR_{min}$:

$$\log P_r - (-147) > 48\ dB \rightarrow \log P_r - 99dBm$$

Define $\Lambda = \log P_r \rightarrow \Lambda \sim \mathcal{N}(-101.293, \sigma_\Lambda^2 = 4)$. Thus, we have

$$\begin{aligned} Pr\{\Lambda > -99\} &= Q(\frac{-99 + 101.293}{2}) \\ &= Q(\frac{2.293}{2}) = Q(1.1465) \end{aligned}$$

(c) We now replace the array antenna of the BS with a single $\lambda/2$-dipole antenna. Does this change decrease the probability of successful communication? Explain your answer by giving a reason.

♠ **Solution:** Yes. It decreases, since the PL increases as the antenna gain decreases.

11.9.3 Question 3: Fading Channels (21 Points)

Consider a wireless channel between a base station (BS) and a mobile station (MS). The MS is located in a middle of an area in which many reflectors are available. There exists *no line-of-sight* link between the BS and the MS. This means that the signal received by the MS is a superposition of reflections from many reflectors.
The reflectors are located such that the MS receives reflections *only* in incident angles between $-60°$ and $+60°$. As the result, the distribution of the received power over the incident angle is

$$p(\alpha) = \begin{cases} \frac{3}{2\pi} & -\frac{\pi}{3} \leq \alpha \leq +\frac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}.$$

It is further known that the transmission experiences a *slow narrow-band* fading process. Let $R(t)$ be the stochastic processes which describes the base-band received signal.

(a) What is the distribution of $R(t)$? Specify the mean and the variance.

Hint: You can use $p(\alpha)$ to calculate the average received power.

♠ **Solution:** $R(t)$ is a complex normal (Gaussian) random variable. Since we have no line-of-sight, the fading process is Rayleigh. Hence, the mean value is 0. The variance of $R(t)$ is the average receive power. Given the power distribution $p(\alpha)$, we have

$$P_R = \sigma_R^2 = \int_{-\pi}^{\pi} p(\alpha) d\alpha = \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \frac{3}{2\pi} d\alpha = \frac{3}{2\pi}\frac{2\pi}{3} = 1$$

Thus,

$$R(t) \sim CN(0,1)$$

(b) Let $s(t_0)$ be the *base-band* representation of the transmit signal at $t = t_0$. Write down a simple model which describes $R(t_0)$ in terms of $s(t_0)$. Specify all components of this model.

♠ **Solution:**

$$R(t) = \int h(\tau,t)s(t-\tau)d\tau + z(t)$$

Since it is a slow and narrow-band fading:

$$R(t_0) = Hs(t_0) + z(t_0)$$

H : channel coefficient : Complex normal
$Z(t)$: AWGN process

(c) Determine the power spectral density of $R(t)$.

Hint: Pay attention to the received power distribution over the incident angle.

♠ **Solution:** Given the formula (3.45) in page 99 of the Lecture notes, we have

$$\Psi_R(\nu) = \frac{p(\alpha)+p(-\alpha)}{\sqrt{\nu_{max}^2 - \nu^2}}$$

Thus, by substituting p(α), we have

$$\Psi_R(\nu) = \begin{cases} \frac{3}{\pi\sqrt{\nu_{max}^2-\nu^2}} & |\alpha| \leq \frac{\pi}{3} \\ 0 & otherwise \end{cases}$$

To represent $|\alpha| \leq \frac{\pi}{3}$, in terms of ν, we note $cos(\alpha) = \frac{\nu}{\nu_{max}}$. Thus:

$$|\alpha| \leq \frac{\pi}{3}$$
$$\frac{1}{2} \leq |cos\alpha| \leq 1$$
$$\nu_{max}/2 \leq |\nu| \leq \nu_{max}$$

Therefore,

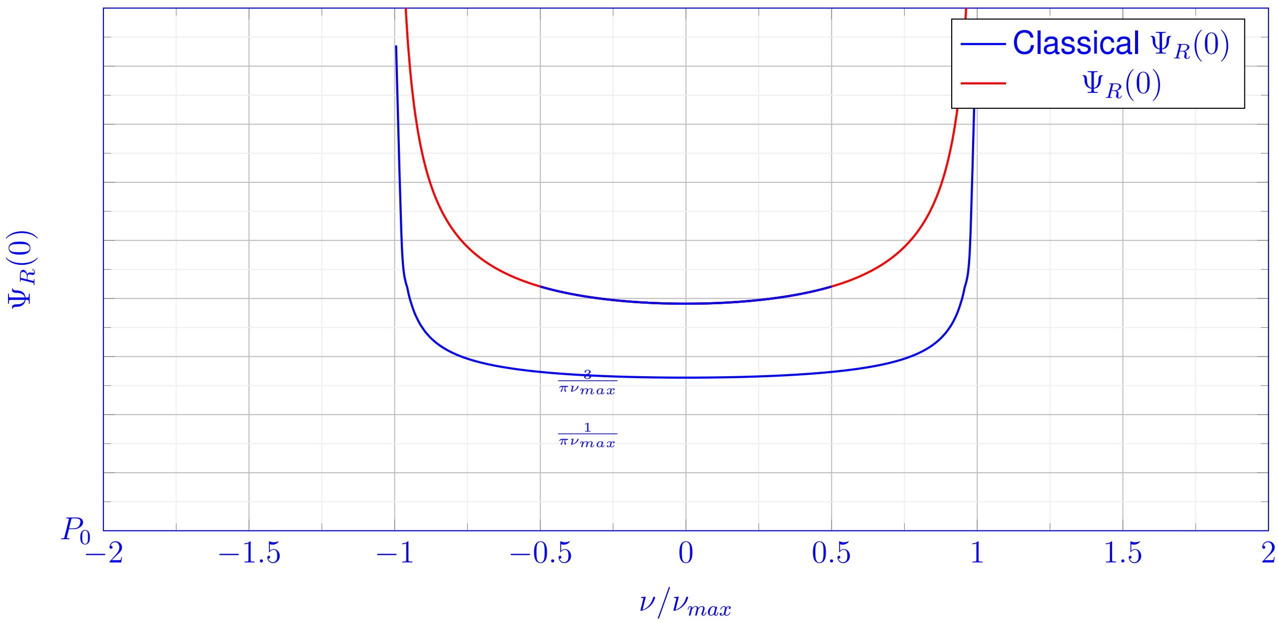
$$\Psi_R(0) = \begin{cases} \frac{3}{\pi\sqrt{\nu_{max}^2-\nu^2}} & \nu_{max}/2 \leq |\nu| \leq \nu_{max} \\ 0 & otherwise \end{cases}$$

(d) Plot the power spectral density of $R(t)$ and compare it to the classical Doppler spectrum.

♠ **Solution:** The classical Doppler spectrum with σ_R^2 = 1 is classical

$$\Psi_R(0) = \begin{cases} \frac{1}{\pi\sqrt{\nu_{max}^2 - \nu^2}} & |\nu| \leq \nu_{max} \\ 0 & otherwise \end{cases}$$

Thus:



The area under the both curves are similar = 1.

11.9.4 Question 4: Diversity Techniques (13 Points)

A base station (BS) is communicating with a mobile station (MS) over a wireless channel. The channel between the BS and the MS experiences a *slow narrow-band* fading with Rayleigh distribution. To combat fading, the BS transmits the same signal *two times.* The delay between two transmissions is long enough, such that the corresponding channels are *stochastically independent.*

(a) What is the diversity order achieved in this system? Give a reason for your answer.

♠ **Solution:** Since we have two independent transmissions, M = 2.

Let the mean signal power received in each transmission be $\sigma_R^2 = 1$, and the variance of additive white Gaussian noise (AWGN) be $\sigma^2 = 0.1$. The MS combines the two received signals via the *equal gain combining (EGC)* technique. For the combined signal, the *average* received signal-to-noise ratio (SNR) is defined as the expectation of the received SNR *with respect to the channel coefficients.*

(b) Determine the average received SNR of the combined signal.

♠ **Solution:** The received signals are:

$$y_1 = h_1 s(t_0) + z_1$$
$$y_2 = h_2 s(t_0) + z_2$$

$s(t_0)$: sampled transmit signal
z_1, z_2 : AWGN samples $z_1, z_2 \sim N(0, 0.1)$
h_1, h_2 : Channel coefficients $h_1, h_2 \sim CN(0, \sigma_h^2)$
The received power is $\sigma_R^2 = 1$: Since $R(t_0) = h_i s(t_0)$

$$\sigma_R^2 = E\{|R(t_0)|^2\} = |s(t_0)|^2 E\{|h_i|^2\} = P_T \sigma_h^2$$

where P_T is the transmit power. Thus:

$$1 = \sigma_R^2 = P_T \sigma_h^2 \rightarrow \sigma_h^2 = \frac{1}{P_T}$$

After EGC: d = Combined sample = $(|h_1| + |h_2|)s(t_0) + \hat{z}_1 + \hat{z}_2$

$$SNR = \frac{E\{(|h_1| + |h_2|)^2\}|s(t_0)|^2}{E\{|\hat{z}_1|^2\} + E\{|\hat{z}_2|\}} = \frac{P_T E\{(|h_1|^2 + |h_2|^2)\}}{0.2}$$

$$E\{(|h_1| + |h_2|)^2\} = E\{|h_1|^2\} + E\{|h_2|^2\} + 2E\{|h_1||h_2|\}$$
$$= \sigma_h^2 + \sigma_h^2 + 2E\{|h_1|\}E\{|h_2|\}$$

Using the properties of Rayleigh distribution in page 96, (3.41) - (3.43); we have

$$E\{|h_1|\} = E\{|h_2|\} = \frac{\sqrt{\pi}}{2}\sigma_h$$

$$E\{(|h_1| + |h_2|)^2\} = 2\sigma_h^2 = 2\frac{\pi}{2}\sigma_h^2 = (2+\pi)\sigma_h^2 = \frac{2+\pi}{P_T}$$

$$SNR = \frac{P_T}{0.2}\frac{2+\pi}{P_T} = \frac{2+\pi}{0.2} = 25.7$$

(c) Determine the average received SNR for the case *without* diversity, i.e., when the BS transmits *only once*. Compare your answer to the result of Part (b).

♠ **Solution:** Without diversity:

$$y = hs(t_0) + z \rightarrow SNR = \frac{E\{|h|^2\}|s(t_0)|^2}{\sigma_h^2} = \frac{\sigma_h^2 P_T}{0.1} = \frac{1}{0.1} = 10$$
$$10 < 25.7$$

This gain is due to diversity.

(d) What is the diversity order, if we reduce the delay between the two submissions to the half of a coherence time interval? Explain your answer by giving a reason.

♠ **Solution:** The diversity order is less than 2: M $<$ 2 (One could also say M = 1). This is due to the fact that in this case the diversity branches are not independent.

11.9.5 Question 5: Multiplexing Techniques and Modulation Schemes (17 Points)

Consider a cellular network with *hexagonal* cells. The area of each cell is $A = 742300\ m^2$. In this network, the following geometrical constraint is needed to be satisfied:

> The distance between each two co-channel cells should be more than 2 kilometers.

The network is provided with the total bandwidth of $B = 30$ MHz. To serve multiple users, it uses a combination of *time division multiplexing (TDM) and frequency division multiplexing (FDM):*

> The total bandwidth is divided into *equal-width* FDM sub-channels. In each FDM sub-channel, 12 users are multiplexed via the TDM technique.

To prevent cross-talk, the following separation between neighboring sub-channels is considered:

> Each two neighboring FDM sub-channels are separated with a guard band whose bandwidth is 25% of the bandwidth of an FDM sub-channel.

The network provides two-way communication via the *frequency division duplexing (FDD)* technique: Each FDM sub-channel is divided into two *equal* sub-bands for uplink and downlink transmissions. There is *no guard band* between the uplink and downlink sub-bands in an FDM sub-channel.
The network uses the quadrature phase shift keying (QPSK) modulation via root raised cosine (RRC) pulses of duration $T = 190\ \mu$sec with the roll-off factor $\alpha = 0.9$.

(a) What are the valid cluster sizes in this network?

♠ **Solution:** For a hexagonal network, we have

$$A = \frac{3\sqrt{3}}{2}R^2 \rightarrow R = \sqrt{\frac{2}{3\sqrt{3}}A} = 534.52m = 0.534km$$

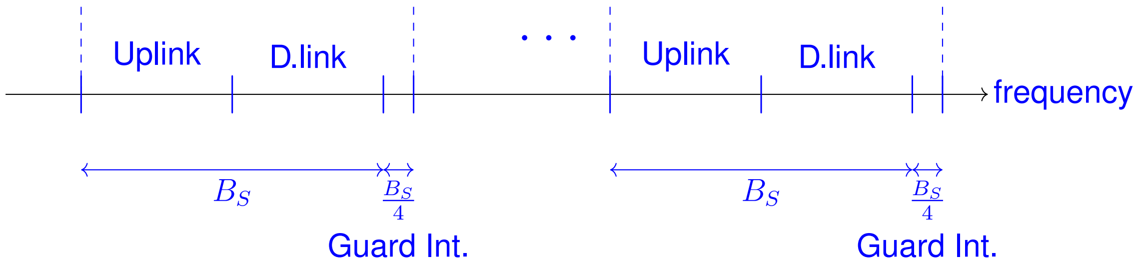
From Eq 1.3 in page 35, we can write

$$K = \frac{1}{3}(\frac{D}{R})^2 = \frac{1}{3}(\frac{D}{0.534})^2$$

Since min value for D is D_{min} = 2 km, we have : $K \geq \frac{1}{3}(\frac{2}{0.534})^2 = 4.68$. Given Table 1.1, in page 26, we have K = 7, 9, 12, 13, ...

(b) Determine the maximum number of users which can be served in this network.

♠ **Solution:** The frequency band is divided as below



The bandwidth for one way transmission is the pulse BW:

$$\frac{B_s}{2} = \frac{1+a}{T} = \frac{1.9}{190 \times 10^{-6}} = 1 \times 10^4 = 10\, kHz$$

$$B_S + \frac{B_S}{4} = 20 + 5 = 25\, kHz \text{(For each FDM sub-channel + guard)}$$

Thus, the maximum number of FDM sub channels :

$$N_{FDM}^{max} = \frac{B}{B_S + \frac{B_S}{4}} = \frac{30\, MHz}{25\, kHz} = \frac{30}{25} \times 10^3 = 1200$$

Each FDM sub channel is shared among 12 users by TDM. Thus, we have

$$\text{max number of users } = N_{FDM}^{max} = 1200 \times 12 = 14400$$

Remark: One may ignore the last guard BW. In this case: $N_{FDM}^{max} = 1202$. Max number of users: 14424. This is also correct.

(c) Let C be the cell capacity in [user/cell/Hz]. Determine the maximum value of C.

♠ **Solution:**

$$C_{max} = \frac{N_{user}^{max}}{K_{min} B} = \frac{14400}{7 \times 30} = 68.57$$

with alternative solution:

$$C_{max} = \frac{14424}{7 \times 30} = 68.69$$

(d) Let η be the spectral efficiency in [bits/sec/Hz]. Determine the maximum value of η.

♠ **Solution:** We transmit in each sub-band of FDM sub-channel QPSK symbols which take T=190 μsec time. Thus,

$$\eta = \frac{2\, bits}{190 \mu sec\, B_{user}}$$

$$B_{user} = \frac{B}{N_{FDM}} = \frac{30\, MHz}{1200} = 25 \times 10^3 Hz \eta = \frac{2}{190 \times 10^{-6} \times 25 \times 10^3} = 0.4211$$

Alternative solution η = 0.4217

(e) What is the peak to average power ratio (PAPR) of the transmitted signals in this network?

♠ **Solution:** Given the figure 6.12 in page 196, for α = 0.9, QPSK constellation leads to PAPR = 1.5.

(f) Explain how you can increase the cell capacity in this network, *without changing the PAPR?*

♠ **Solution:** By reducing α from 0.9 to 0.4, PAPR does not change (see Fig 6.12 in page 196). So, we could reduce α. This decreases the bandwidth of pulse, and thus increases the number of served users. This leads to higher cell capacity.

11.9.6 Question 6: GSM System (13 Points)

Consider a 20 msec data block which is sent over the *full-rate speech* channel of the GSM system.

(a) What is the coding rate of the *block code* used to encode the 50 most important bits of this data block (class 1a)?

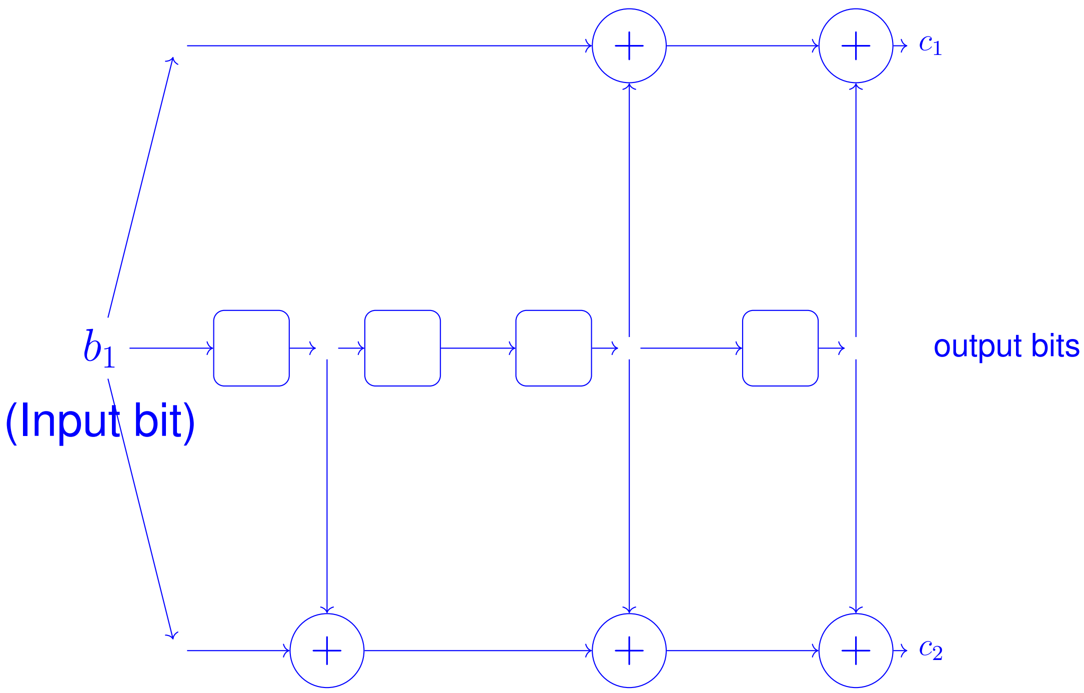
♠ **Solution:** Check Fig 8.7, in page 270: For this code K=50, N=58, $R_c = \frac{K}{N} = \frac{50}{58} = 0.94$

(b) What is the coding rate of the *convolutional code* used to encode the encoded class 1a block and the *class 1b*?

♠ **Solution:** From Fig 8.7, we have: $R_c = \frac{1}{2}$

(c) Plot the diagram of the encoder for the convolutional code of Part (b) and explain why its constraint length is $L_c = 5$.

♠ **Solution:** Check Fig 8.8 in page 270:



Since each output depends on input bit on the given interval and 4 delayed versions, then t_0 = 5.

(d) Calculate the *uncoded* data bit rate in [bits/sec] before channel coding and the *coded* bit rate in [bits/sec] after the channel coding in the full-rate speech channel.

♠ **Solution:** A 20 msec block has 260 bits: uncoded bitrate = $\frac{260}{20\times10^{-3}} = 13$ kbit/sec. After coding we have $378 + 78bits = 456$ bits. Coded bitrate = $\frac{456}{20\times10^{-3}} = 22.8$ kbit/sec

Assume that we use *bit-wise soft decoding* to decode the received signal in the full-rate speech channel. We want to achieve the bit error rate (BER) of

$$\text{BER} = 4 \times 10^{-3}.$$

(e) Determine the coding gain.

♠ **Solution:** Given Figure 8.12, page 274: Coded transmission with bitwise soft decoding leads to (Class 1) BER = 4 $\times 10^{-3}$ @ SNR = 9dB. For uncoded transmission (class 2) we need SNR = 20 dB. Thus G_c = 20 - 9 = 11dB

11.10 Winter Semester 2021-2022

Exam Date: February 22, 2021
6 Problems with total of 100 Points.
Exam Duration: 90 Minutes

11.10.1 Question 1: Short Questions (18 Points)

Answer the following items briefly.

(a) A *regular hexagonal* cellular network consists of cells with area 5.85 km^2. The frequency reuse distance in this setting needs to be at least 5.35 km. Find the minimum required cluster size for this network.

♠ **Solution:** A = 5.85km^2: from page 21,

$$5.85 = \frac{3\sqrt{3}}{2}R^2$$
$$R = \sqrt{\frac{11.7}{3\sqrt{3}}} = 1.5\,\text{km}$$

From page 25:

$$K = \frac{1}{3}(\frac{D}{R})^2 \geq \frac{1}{3}(\frac{5.35}{1.5})^2$$
$$K \geq 4.24$$

From Table 1.1 $\rightarrow K_{min} = 7$

(b) An antenna with gain $\log G$ = 3dB radiates 90% of its input electrical power and burns the remaining 10% to heat. Assume that this antenna is operating at the frequency f = 3 GHz. Determine the effective area of this antenna.

♠ **Solution:** Efficiency of antenna $\rightarrow \eta = 0.9$

$$\log G = 3dB \rightarrow G = 2$$
$$G = D\eta \rightarrow D = \frac{G}{\eta} = \frac{2}{0.9} = 2.22$$
$$Eq(2.3)pg.48 \rightarrow A_w = \frac{\lambda^2}{4\pi}D$$
$$= \frac{c^2}{4\pi f^2}D$$
$$= (\frac{3\times 10^8}{3\times 10^9})^2\frac{2.22}{4\pi} = 1.7\times 10^{-3}m$$

(c) A GSM system transmits signals in two different scenarios:

(A) A wireless fading channel with coherence bandwidth B_{Coh} = 20kHz.

(B) A wireless fading channel with coherence bandwidth B_{Coh} = 2 MHz.

Specify each scenario as a *narrow-band* or *wide-band* fading. Give a reason for your answer.

♠ **Solution:**

$$B_{GSM} = 200 \text{ kHz(From Table 8.1)}$$

(A) $B_{Coh} = 20kHz << B_{GSM}$: Wide-band fading
Reason: A GSM subchannel observes multiple coherence intervals.
(B) $B_Coh = 2\ MHz >> B_{GSM}$: narrow-band
Reason: A GSM subchannel observes only one coherence interval.

(d) Assume that we use the code division multiplexing (CDM) scheme for transmission and employ a rake receiver for signal reception. We know that the channel between the transmitter and the receiver consists of five distinguishable paths.
How many fingers do you suggest for the rake receiver? Give a reason for your answer. Also explain what happens, if a larger number of fingers compared to what you suggested is used?

♠ **Solution:** We need 5 fingers.
Reason: each finger is to detect ane distinguishable path.
With larger number, we,
1) either only detect noise which degrades the performance.
2) or we can ignore them (block them) and we perform as with 5 fingers.
So, we are either "worse" or "same".

(e) Name the parameter which specifies the number of least erroneous bits that can be corrected by

(A) a block code,

(B) a convolutional code.

Define both parameters.

♠ **Solution:**

(A) minimum (Hamming) distance
Eq(7.2) pg238 $\rightarrow$

$$d_{min} = \min_{0 \neq C'} d_H(C, C')$$

where $d_H(.,.)$ is the Hamming distance.

(B) free distance
Eq(7.19) pg248 which is similar to d_{min}

11.10.2 Question 2: Mobile Radio Channels (18 Points)

A base station (BS) is located on the top of Erlangen' s city hall (Rathaus Erlangen) which is of 61 meters height. It serves a mobile station (MS) which is located in an office on the second floor of the EEI tower at Cauerstrasse 7, Erlangen. The distance between the BS and the MS is approximately 3.9 km. The BS operates at a carrier frequency which lies exactly at the middle of the uplink band of the GSM standard DCS1800.

The BS uses an array antenna whose gain is $\log G_{BS}$ = 15dB$_\text{i}$. The MS uses an array of $\frac{\lambda}{2}$-dipole antennas whose *array gain* is $\log G_{array}$ = 5dB.

(a) Determine the path-loss using the *free space propagation* model.

♠ **Solution:** From (3.10) in pg. 79, we have

$$\begin{aligned} PL &= 2\log\frac{4\pi}{0.3} + 2\log d + 2\log f_0 - G_{BS} - G_{MS} \\ &= 32.44 + 2\log 3.9 + 2\log f_0 - 15 - G_{MS} \\ f_0 = \text{center of DCS1800}_{\text{ uplink}} &= \frac{1710+1785}{2} = 1747.5MHz \\ \log G_{MS} = \log G_{\lambda/2} + \log G_{array} &= 2.15 + 5 = 7.15 \\ PL = 3244 + 11.82 + 64.85 - 15 - 7.15 &= 86.96\ dB \end{aligned}$$

(b) Determine the path-loss, using the Okumura-Hata model.

♠ **Solution:** The correct model is the one given is Eq(3.31) pg. 90, due to the carrier frequency. Let each floor be 2.5m $\rightarrow h_{MS} = 5$m (any number between 4 and 8 is fine)

$$\begin{aligned} h_{BS} &= 61m \\ PL_0 &= 46.3 + 33.9\log 1747.5 - 13.82\log 61 - a(h_{\mu s}) \\ a(h_{\mu s}) &= (1.1\log 1747.5 - 0.7)5 - 1.56\log 1747.5 + 0.8 \\ &= 3.94\log 1747.5 - 2.7 = 10.08 \\ PL_0 &= 121.4647dB \\ 10\log d &= 19.63 \\ PL_{urban,isotropic} &= 141.09dB \end{aligned}$$

Correction from (3.29):

$$\begin{aligned} 2(\log\frac{1747.5}{28})^2 + 5.4 &= 11.85 \\ \text{Antenna gains} &= 15 + 7.15 = 22.15 \\ &\rightarrow 11.85 + 22.15 = 34dB \end{aligned}$$

$$PL = PL_{urban,isotropic} - \text{correction}$$
$$= 141.09 - 34 = 107.09\, dB$$

(c) Compare the results in Parts (a) and (b). give a reason for your observation.

♠ **Solution:** Part(b) $>$ Part(a), Free space is a simplified model which ignores various properties of the channel like ground-reflection, ... It is therefore optimistic.

11.10.3 Question 3: Fading Channels (14 Points)

Consider the following two wireless communication scenarios:

(A) A wireless transmitter is sending signals over the air to a receiver which is in its sight.

(B) A wireless transmitter is located on top of the city hall building (Rathaus) in Erlangen and is sending signals to a receiver which is located in an apartment in the neighborhood Bruck.

Assume that the large-scale impacts of the channel; that means path-loss and the shadowing, can be ignored. You are asked to only consider the multipath fading.

(a) Assume that the transmitted signal is narrow-band and the number of multipath components between the transmitter and the receiver is *large*. Moreover, the receiver is located in a *rich* scattering environment. For each of the above scenarios, suggest a stochastic model for the channel coefficient. Explain your answer by giving a reason.

♠ **Solution:** Scenario (A): $y = \underline{H}x + n$
$\underline{H}$ is the channel coefficient which is a complex Gaussian random variable with mean $\mu \neq 0$, and variance σ_H^2. μ represents the line-of-sight (LoS).
Scenario (B): Here, $\underline{H}$ is a zero mean complex Gaussian random variable since LoS is blocked in this case.

Now, consider another multipath fading channel between a transmitter and a receiver. The delay time correlation function of the channel is given by

$$\rho_h(\tau, \Delta t) = e^{-\frac{\Delta t}{T_0}} \sum_{n=1}^{4} \delta(\tau - 2n\tau_0),$$

where $\delta(\tau)$ is Dirac's impulse function, and T_0 = τ_0 = 1 msec.
For this channel, answer the following items:

(b) Determine the coherence bandwidth.

Hint: You may find the following items useful:

(a) The Fourier transform of $x(t) = \delta(t)$ is $X(f) = \mathcal{F}\{\delta(t)\} = 1$.

(b) $\mathcal{F}\{x(t-t_0)\} = e^{-j2\pi f t_0}\mathcal{F}\{x(t)\}$, with $j = \sqrt{-1}$ being the imaginary unit.

♠ **Solution:** From pg 117, we need to calculate freq-time func.

$$\begin{aligned}\rho_H(\Delta f, \Delta t) &= \mathcal{F}_\tau\{\rho_h(\tau, \Delta t)\} \\ &= \mathcal{F}_\tau\{e^{-\frac{\Delta t}{T_0}}\sum_{n=1}^{4}\delta(\tau - 2n\tau_0)\} \\ &= e^{-\frac{\Delta t}{T_0}}\sum_{n=1}^{4}e^{-j4\pi n f\tau_0}\end{aligned}$$

The coherence BW is thus given by

$$|\rho_H(B_C, 0)| = \frac{1}{2}|\rho_H(0,0)|$$

$$\rightarrow 2 = |\sum_{n=1}^{4}e^{-j4\pi n B_c\tau_0}| \rightarrow B_C$$

(c) Determine the power transfer factor of the channel.

♠ **Solution:** From pg 120, power delay profile:

$$\rho_h(\tau, 0) = \sum_{n=1}^{4}\delta(\tau - 2n\tau_0)$$

$$h_\rho = \rho_H(0,0) = 4$$

(d) Determine the delay spread of the channel.

♠ **Solution:** Pg 121, Eq(3.88)

$$\sigma_\tau^2 = \frac{1}{h_p}\int_0^\infty(\tau - \mu_\tau)^2\rho_h(\tau, 0)d\tau$$

$$\begin{aligned}\mu_\tau &= \frac{1}{h_p}\int_0^\infty \tau\rho_h(\tau, 0)d\tau \\ &= \frac{(2\tau_0 + 4\tau_0 + 6\tau_0 + 8\tau_0)}{4} \\ &= 5\tau_0 = 5ms\end{aligned}$$

$$\begin{aligned}\sigma_\tau^2 &= \frac{1}{4}[(2\tau_0 - 5\tau_0)^2 + (4\tau_0 - 5\tau_0)^2 + (6\tau_0 - 5\tau_0)^2 + (8\tau_0 - 5\tau_0)^2] \\ &= \frac{1}{4}[18\tau_0^2 + 2\tau_0^2] = 5\tau_0^2 \rightarrow \sigma_\tau = \sqrt{5}\tau_0 = \sqrt{5}ms\end{aligned}$$

(e) Determine the excess delay of the channel.

♠ **Solution:** From page 121, Eq (3.89),

$$T_{ex} = 8\tau_0 - 2\tau_0 = 6\tau_0 = 6ms$$

11.10.4 Question 4: Diversity Techniques (20 Points)

A base station (BS) with a single antenna is transmitting signal $s(t)$ to a mobile station (MS) with two antennas. The signal received by antenna i at the MS is described as

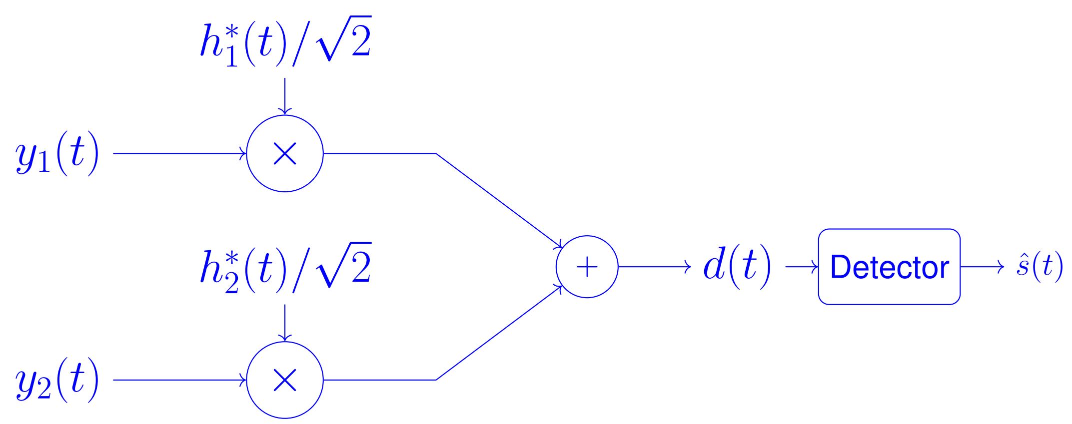
$$y_i(t) = h_i(t)s(t) + n_i(t),$$

for i ∈ 1, 2, where $h_i(t)$ is the channel coefficient and $n_i(t)$ denotes additive white Gaussian noise (AWGN) with mean zero and variance σ^2 = 0.1.

The receiver uses the maximum ratio combining (MRC) technique to combine $y_1(t)$ and $y_2(t)$, and then uses the combined signal to detect $s(t)$.

(a) Plot the diagram for MRC technique. Specify all coefficients.

♠ **Solution:**



(b) Let γ denote the signal-to-noise ratio (SNR) after MRC and γ_1 denotes the SNR at the first receive antenna before MRC. We know that for Γ, $\Gamma_1 >$ 0, we have

$$Pr\{\gamma \leq \Gamma\} = Pr\{\gamma_1 \leq \Gamma_1\} = 0.1.$$

Compare Γ and Γ_1; that means, fill □ in $\Gamma \square \Gamma_1$ with $>$ or $\geq$ or $<$ or $\leq$ or $=$.

♠ **Solution:**

$$\Gamma_1 \leq \Gamma$$

Because γ_1 is the non-combined version which leads to poorer performance. "=" holds in static channels.

First, assume that $h_1(t)$ and $h_2(t)$ are two *correlated* complex circularly symmetric Gaussian processes whose means are zeros and whose *covariance* matrix is given by

$$\boldsymbol{R} = \mathcal{E}\{\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \begin{bmatrix} h_1^*(t) h_2^*(t) \end{bmatrix}\} = \begin{bmatrix} 1 & 0.25 + j0.17 \\ 0.25 - j0.17 & 1 \end{bmatrix}$$

where $h_i^*(t)$ denotes the conjugate of $h_i(t)$, and $j = \sqrt{-1}$ is the imaginary unit.

(c) Determine the diversity gain.

♠ **Solution:** From Eq(4.61) in page 143:

$$Pr\{\gamma \leq \Gamma\} = \frac{1}{2(1 - |p|^2)}(\frac{\Gamma}{\mu})^2$$

$$\begin{aligned} \text{Diversity Gain} &= \frac{1}{2}\log 2(1 - |p|^2) \\ &= \frac{1}{2}\log 2(1 - \sqrt{(0.25)^2 + (0.17)^2}) \\ &= \frac{1}{2}\log 2(1 - 0.3) \\ &= \frac{1}{2}\log 1.4 \\ &= 0.73dB \text{ (correlated)} < 1.5dB \text{ (uncorrelated)} \end{aligned}$$

Consider now the case in which $h_1(t)$ and $h_2(t)$ are two *independent complex circularly symmetric Gaussian processes with means zero and variance 1.*

(d) Determine the diversity gain for this case.

♠ **Solution:** From page 137, Eq(4.27),

$$\text{Diversity} = \frac{1}{2}\log 2 = 1.5dB$$

As the last case, consider the scenario in which $h_1(t) = h_2(t) = 1$; that means, the channel coefficients are *deterministic.*

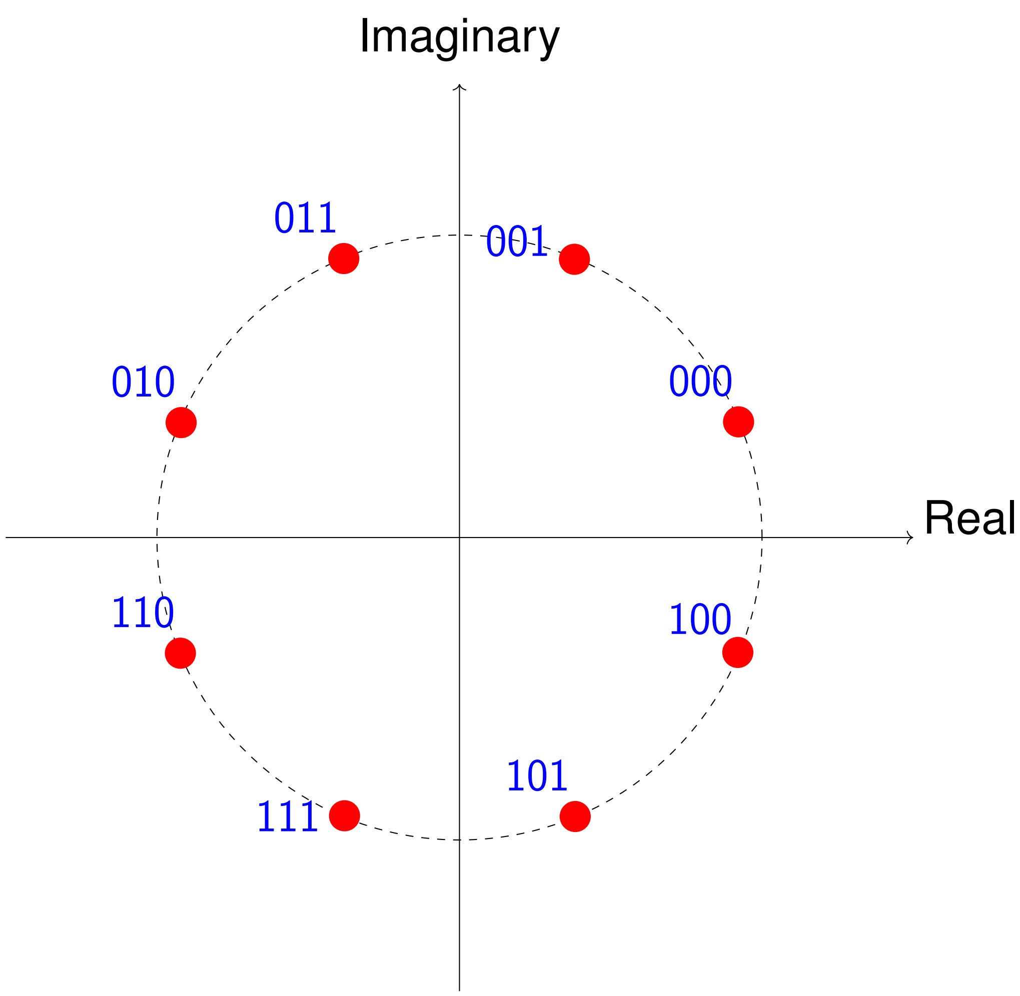
(d) Determine the diversity gain for this case.

Hint: You can give the answer directly without any derivations.

♠ **Solution:** Diversity = 0dB, because in static channels diversity does not have any input.

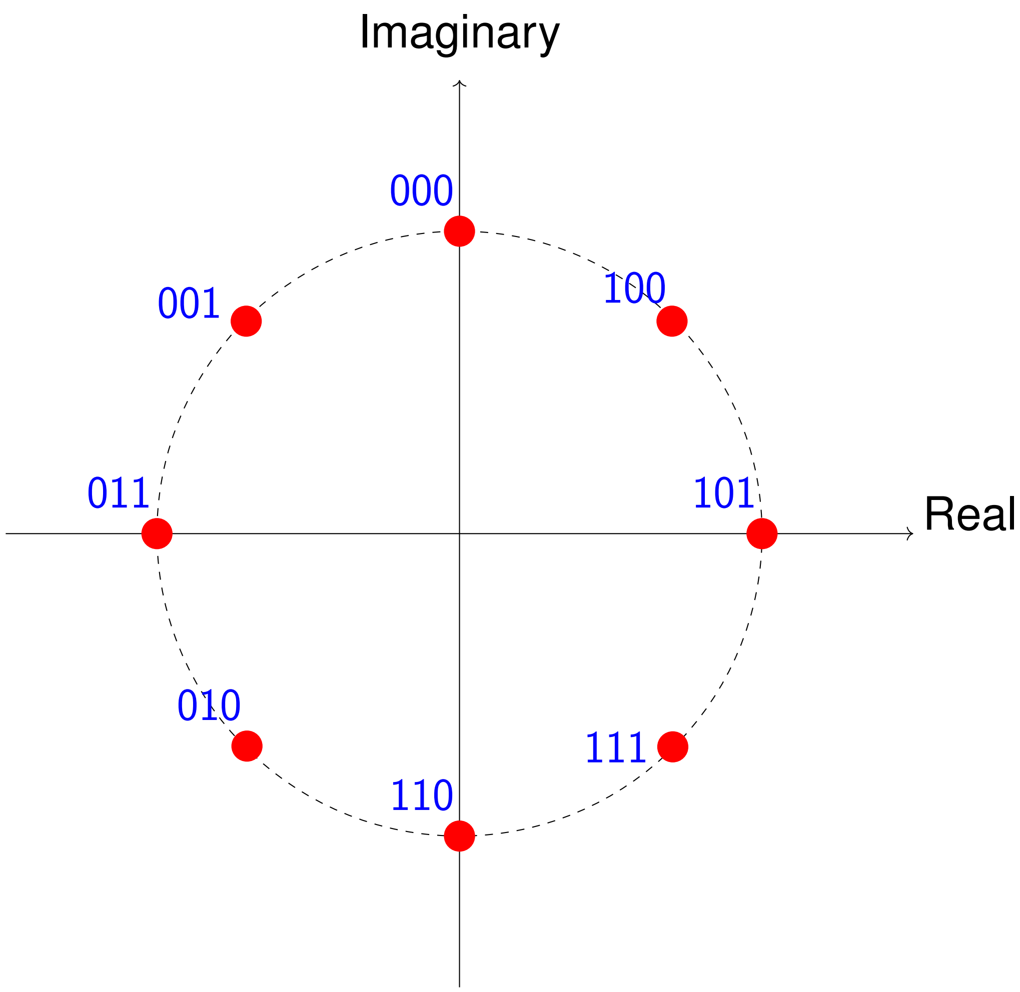
11.10.5 Question 5: Modulation Schemes (17 Points)

A communication system, refered to as System (0), uses $\frac{\pi}{8}$-shifted 8-phase shift keying (PSK) with *rectangular* pulses for signaling. The constellation at the symbol time interval k = 3 is as follows:



(a) Draw the constellation for the symbol time interval k = 6.

♠ **Solution:** @ k = 6, the constellation is rotated for $\Delta\phi = (6-3)\frac{\pi}{8} = \frac{3\pi}{8}$



(b) Does System (0) gain anything by using the $\pi/8$-shifted 8-PSK scheme, instead of using the standard 8-PSK scheme? Give a reason for your answer.

♠ **Solution:** Theoretically no, because with rectangular pulses we have PAPR = 0dB (1) and thus by using $\frac{\pi}{M}$-shift we do not gain.

You are now asked to compare System (0) to four other systems which use the following techniques for signaling:

1. Quaternary PSK (QPSK) and root-raised-cosine (RRC) pulses with roll-off factor α = 0.5.
2. $\pi/2$-shifted binary PSK (BPSK) and RRC pulses with roll-off factor α = 0.2.
3. 16-quadrature amplitude modulation (QAM) and RRC pulses with roll-off factor α = 0.5.
4. 64-QAM and RRC with roll-off factor α = 0.1.

(c) Sort Systems (0) to (4) based on their peak-to-average ratio (PAPR).

♠ **Solution:** Based on Fig. 6.12:

$$Sys(2) < Sys(1)$$

We also know that Sys(0) has minimum PAPR. Also we know that,

$$\begin{aligned} Sys(1) < Sys(3) < Sys(4) \\ \rightarrow Sys(0) < Sys(2) < Sys(1) < Sys(3) < Sys(4) \end{aligned}$$

(d) Find the system whose spectral efficiency is maximum.

♠ **Solution:**

$$\begin{aligned} \text{For } Sys(0) \rightarrow \eta_0 = \frac{3}{\infty} \approx 0 \\ Sys(1) \rightarrow \eta_1 = \frac{2}{1.5} \\ Sys(2) \rightarrow \eta_2 = \frac{1}{1.2} \\ Sys(3) \rightarrow \eta_3 = \frac{4}{1.5} \\ Sys(4) \rightarrow \eta_4 = \frac{6}{1.1} \\ \rightarrow (4) > (3) > (1) > (2) > (0) \end{aligned}$$

(e) Find the system with highest mean bit error probability when they are used for uncoded communication over Rayleigh fading channel with two branches of diversity. If needed, you can assume that the mean E_b/N_0 is

$$\log \tilde{E}_b/N_0 = 10dB.$$

♠ **Solution:** Sys(4) has the most dense constellation; therefore it has the higest error prob.

11.10.6 Question 6: GSM System (13 Points)

Consider the full-rate traffic channel used for full rate speech transmission (TCH/FS) in the standard GSM system.

(a) What is the modulation scheme used at the base station (BS)?

♠ **Solution:** Based on Fig 8.5, GSM uses GMSK modulation.

(b) Start from the fact that the input bit stream to the channel interleaver at the BS is of the rate 22.8 kbit/sec, and explain why the input of the modulator is of rate 270.833 kbit/sec. Indicate explicitly all required calculations.

♠ **Solution:** Consider Fig 8.5:

(A) The interleaver does not change rate: R_{out} = 22.8

(B) Multiframe multiplexer adds two dummy bits to every frame of 24 bits. Thus

$$R_{out} = 22.8 \times \frac{26}{24} = 24.7kbps$$

(C) Encryption does not change the rate: $R_{out} = 24.7$

(D) TDMMax combines 8 TDM-frames $R_{out} = 8 \times 24.7$ kbps

(E) time slot formatter adds signaling and control bit to each frame of 114 bits, we add control bits and make it a frame of 156.25: Thus,

$$R_{out} = 8 \times 24.7 \times \frac{156.25}{114} = 270.83kbps$$

Now consider the random access burst (RAB) used for the first access is the GSM system and the normal burst.

(c) Determine the difference between the guard intervals of these two time slot formats in seconds.

♠ **Solution:** Guard interval of RAB is 68.25 bits = 252μs. Normal burst has time guard of: 8.25 bits

$$\Delta t = 60bits = 60\frac{252}{68.25} = 221.54\mu s$$

(d) Explain why the guard intervals of these two time slot formats are different.

♠ **Solution:** This is because in the first access, GSM cannot use "timing advance".

Chapter 12

Sample Exams Without Solutions

To have further practice, some older sample exams are given in this chapter. These exams have no solutions and hence can be used to test your self.

12.1 Winter Semester 2017 Exam

Exam Date: February 20, 2018
6 Problems with total of 90 Points.
Exam Duration: 90 Minutes

12.1.1 Question 1: Short Questions (13 Points)

Answer the following items.

(a) What is the difference between the classical and cloverleaf sectorization?

(b) What is the proper distribution to model the narrow-band fast fading coefficient of a wireless radio channel when the direct signal component is **not** zero?

(c) Define the main beam of an antenna array.
At which vertical angle is the main beam of a linear array with equal weight factors?

(d) Name a necessary block which we need to add in order to get coding gain when we use channel coding over a fading channel.

(e) Calculate the duration of the training sequence in a Synchronization Burst of the GSM system in **seconds**.

12.1.2 Question 2: Modulation Schemes (12 Points)

Consider the following modulation schemes, and assume that Root Raised Cosine (RRC) pulses with the roll-off factor α are used for transmission.

(1) Quadrature Phase Shift Keying (QPSK); $\alpha = 0.1$

(2) $3\pi/8$-shifted 8-ary PSK; $\alpha = 0.5$

(3) 16-Quadrature Amplitude Modulation (QAM) with Gray mapping; $\alpha = 0.1$

(4) 64-QAM with Gray mapping; $\alpha = 0.1$

Sort these schemes with respect to

(a) Noise resistance at **high** Signal-to-Noise Ratios (SNRs) assuming transmission over a non-dispersive (frequency flat) Rayleigh fading channel with four-branch antenna diversity.

(b) Spectral (bandwidth) efficiency.

(c) Peak-to-Average Power Ratio (PAPR).

12.1.3 Question 3: Fading Channels (15 Points)

The power-delay profile of a wireless channel is given by

$$\rho_h(\tau, 0) = \sum_{n=0}^{3} \frac{1}{2^n} \delta(\tau - n\tau_0)$$

where $\tau_0 = 5\ \mu s$.

(a) For this channel, calculate

(a-1) the power transform factor

(a-2) the delay time

(a-3) the delay spread

(a-4) the excess delay

(b) Assume that GSM symbols are being transmitted over this wireless channel. Is the channel in this case a narrow-band fading channel? Justify your answer by giving reasons.

12.1.4 Question 4: Mobile Radio Channels (18 Points)

Consider a communication tower at height of 110 m located in Fürth. This tower is employed to serve a mobile station on the roof of a van (automobile) in Erlangen. The tower operates at a carrier frequency which is exactly at the middle of the downlink frequency band of GSM900. The gain of the transmit antenna at the communication tower g_{BS} and the receive antenna at the mobile station g_{MS} are both one, i.e., $G_{BS} = G_{MS} = 0$ dB$_i$. Moreover, the distance of the mobile station from the roof of the van is 1 m.

(a) Consider the Okumura-Hata model. Calculate the path-loss in dB when the van is parked 16 km far from the tower close to Rathaus in Erlangen.

(b) Assume that one models the path-loss of this communication channel as the following

$$a = \frac{P_s}{P_r} = \frac{4\pi}{g_{BS} g_{MS}} \left(\frac{d}{\lambda_0} \right)^n$$

for some given n, where P_s and P_r denote the transmit and receive power over the channel, respectively, d is the distance between the mobile station and the communications tower and λ_0 is the wavelength. Determine n, such that the assumed model fits the Okumura-Hata model for the scenario in Part (a).

(c) Compare the value of n in Part (b) with the free-space attenuation model. What is your conclusion?

12.1.5 Question 5: Cellular Design (20 Points)

You are employed by a Mobile Service Provider (MSP) as an engineer to design a cellular network in a city. The area of the city is 80 km^2, and there are around $100,000$ subscribers. The MSP has also hired a company to provide you with experimental measurements and statistics about the subscribers and the city architecture. The measuring company has reported that in this city with probability β, less than $100\beta^9\%$ of the subscribers are simultaneously making a call. Moreover, based on your calculations you know that you can support 200 calls simultaneously in each cell. When all cells are occupied, a new call is blocked. This event is called call-block, and its probability is denoted by p_{block}.

(a) Calculate approximately the maximum cell area A_c in which you can assure the MSP that $p_{block} < 0.9$.
Hint: Assume that the cells are equally sized and perfectly cover the city area.

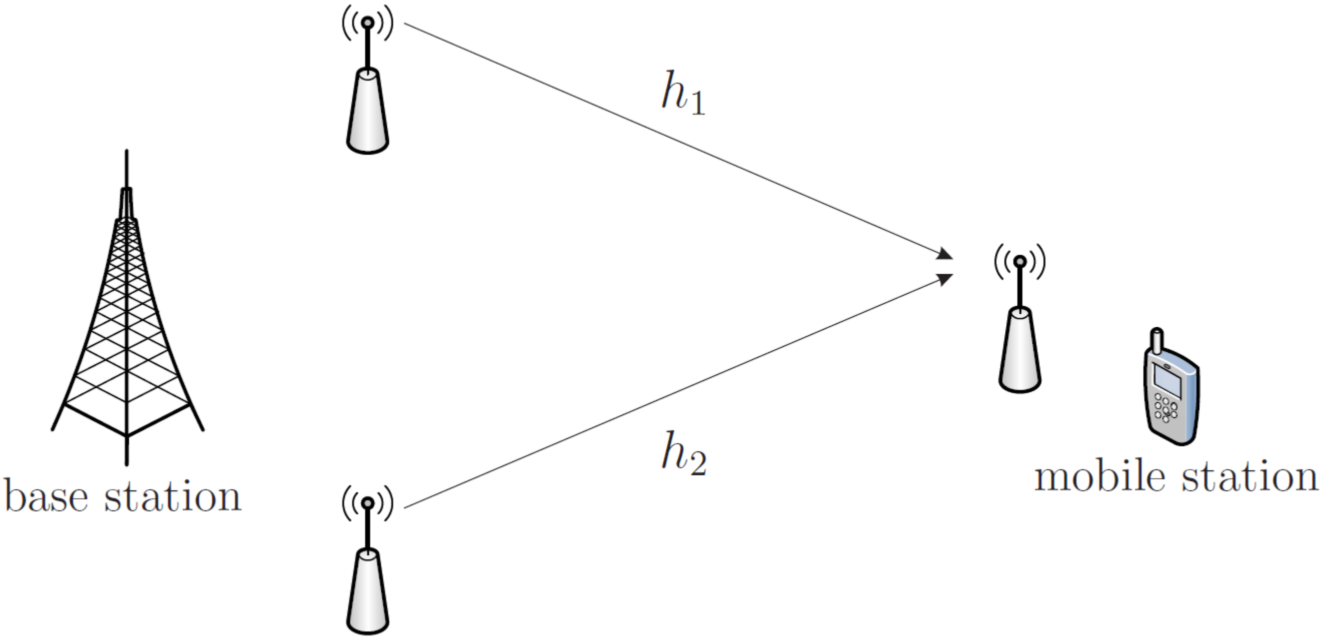
(b) Assuming that you consider hexagonal cells, calculate the radius of the cell.

(c) Name two parameters other than the subscribers statistics which you need from the measuring company to be provided for you.

(d) Name two parameters other than the cell area which you need to specify in your design.

12.1.6 Question 6: MIMO Transmission (20 Points)

Consider a base station with two transmit antennas and a mobile station with a single receive antenna. The channel coefficients between the transmit and receive antennas are denoted by h_1 and h_2 which satisfy $|h_1|^2 + |h_2|^2 = \alpha$ and are assumed to be constant within the transmission period. See the figure below.



In two subsequent time intervals the base station does the following: In the first interval, it transmits the symbol s_1 over the first antenna and s_2 over the second antenna. In the second interval, $-s_2^*$ is transmitted over the first antenna and s_1^* is transmitted over the second antenna. The received symbols by the mobile station are then written as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where r_1 and r_2 are the received symbols in the first and the second intervals and n_1 and n_2 are independent complex Gaussian noise with variance σ^2.

1. Find the 2×2 matrix **H** such that

$$\begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} = \mathbf{H} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

2. Use the matrix **H** in Part (a) and derive the Minimum Square Error (MMSE) receiver **W**.

3. Calculate the estimated symbols $\tilde{s}_1$ and $\tilde{s}_2$, such that

$$\begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} = \mathbf{W} \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix}$$

12.2 Winter Semester 2018 Exam

Exam Date: February 20, 2019
5 Problems with total of 100 Points.
Exam Duration: 90 Minutes

12.2.1 Question 1: Short Questions (25 Points)

Answer the following questions briefly.

(a) You are asked to design a cellular network. **Name** at least **four** parameters you should specify by your calculations and design.

(b) You want to design a radiation pattern via a linear array antenna with M **omnidirectional** antenna elements and equal weight factors. The array is arranged **horizontally**, such that the distance between each two neighboring antennas is $a = 10$ cm. It is desired that at the carrier frequency $f_0 = 600$ MHz, the 3 dB beamwidth of the radiation patten be less than $10.5°$, i.e., $\Delta\alpha < 10.5°$. Calculate **approximately** the minimum number of antenna elements required to achieve such 3 dB beamwidth with this array.

(c) Consider the following two communication scenarios:

- ▷ SCENARIO A: *A base station is located at the center of a cell in a countryside area. The area of the cell is $A_{cell} = 50\ km^2$. It is known that within distance of $D_0 = 5; km$ from the base station, no large obstacle, such as tall buildings, exists. A mobile station in this cell, is $d = 2$ km far away from the base station, and receives signals from the base station.*
- ▷ SCENARIO B: *A base station is located on the roof of the Shanghai Grand Center with $h = 170$ m height in the middle of Shanghai. A mobile station is located on a bicycle which is being ridden $d = 1$ km far away from the base station in a narrow street. The base station transmits signals to the mobile station.*

Assume that the transmit signals are **narrow-band**. Name the distribution which models the **multipath** effect in each scenario. Give a **reason** for your choice in each scenario.

(d) Consider transmission over a fading channel in which a diversity technique with **M diversity branches** is used. The receiver has a simple hardware which can select **only two branches** of diversity from the M available branches and combine them. Assume that the receiver **can** calculate the instantaneous signal-to-noise ratio (SNR) of each branch before selection.
Given this hardware restriction, design an effective receiver module, such that the SNR at the receiver's output is maximized. Namely, explain how the receiver selects two branches out of M available branches, and how it combines the signals of the selected branches?

(e) Which control channel (CCH) is initially used by a mobile station in a GSM system, when it attempts to setup a call with the base station?

12.2.2 Question 2: Mobile Radio Channels (16 Points)

Consider a base station with 100 m height placed in the middle of a plain landscape in Baden-Wüttemberg, Germany. The land contains only farm fields and trees. The base station transmits data to receiver installed on the roof of a farmhouse. The distance between the base station and the farmhouse is $d = 4$ km. The base-station operates at a carrier frequency which is exactly at the middle of the downlink band in PCS1900 standard. The gains of the transmit antenna at the base station g_{BS} and the receive antenna at the mobile station g_{MS} are both one, i.e., $G_{BS} = G_{MS} = 0$ dB$_i$.

(a) What type of antennas is assumed to be used at the base station and the receiver? Is it a practical assumption?

(b) Using the Okumura-Hata model, calculate the median path-loss in dB.

Now assume that we replace the antennas at both the base station and receiver with dipole antennas. The base station transmits signals with power $P_T = 100$ mW. The equivalent noise bandwidth at the receiver is $B_{noise} = 200$ kHz, and the noise figure is $F = 1$. Furthermore, the noise power spectral density is $N_0 = 4 \times 10^{-18}$ mW/Hz.

(c) Calculate the receive power at the receiver.

12.2.3 Question 3: Stochastic Model for Fading Channels (20 Points)

Consider a **narrow-band** multipath fading channel between a transmitter and a receiver. The system operates at the carrier frequency $f_C = 900$ MHz. The transmitter sees the receiver through a line-of-sight path with Rice factor $\log K = 4$ dB.
The receiver moves with the maximum speed of $v_{max} = 120$ km/h in a **rich** scattering environment, and the superposed power received through the scattered components is $\sigma_R^2 = 5$ mW.

Assume that the transmitter sends the monotone signal

$$s(t) = \sqrt{P} \exp\{-2\pi f_C t\}$$

where $P = 20$ W and $j = \sqrt{-1}$ is the imaginary unit. The received signal is denoted by

$$Y(t) = R(t) + N(t)$$

where $N(t)$ is additive white Gaussian noise with zero mean and variance $\sigma_N^2 = 0.1$ mW, and $R(t)$ is the superposition of the signals received from multiple paths.

(a) Calculate the probability density function of the time sample

$$R_0 = R(t_0)$$

for some arbitrary t_0. Specify the mean and the variance of this random variable.

(b) We now sample $R(t)$ at some different time $t_1 \neq t_0$ and obtain the random variable

$$R_1 = R(t_1).$$

Do R_0 and R_1 have identical distributions? Give a reason for your answer.

(c) Determine the average receiver power at t_0 which is defined as

$$P_R = \mathcal{E}\{|Y(t_0)|^2\}.$$

Here, $\mathcal{E}\{\cdot\}$ denotes the mathematical expectation.

(d) Assume that you are monitoring the random process $R(t)$ in the frequency domain by a spectrum meter which determines the average density of $|R(t)|^2$ at each frequency. You observe the frequency interval

$$\Delta f = [f_C - 200 \text{ Hz}, f_C + 200 \text{ Hz}]$$

by this spectrum meter. Plot the output of this spectrum meter in the interval Δf. Describe the distribution of the scatterers around the receiver that you assumed to plot the output.

12.2.4 Question 4: Multiplexing Techniques (16 Points)

A communication channel of bandwidth $B = 5$ MHz is given. It is intended to multiplex this channel among the multiple users for **two-way** communications with a base station. To this end a combination of frequency division multiplexing (FDM) and time division multiplexing (TDM) is used for multiplexing and time division duplexing (TDD) is employed for establishing a two-way communication.

The parameters of this communication system are as follows

Parameter	Value
TDM slot duration	$T_{TDM} = 150\ \mu$s
Guard time interval between two subsequent transmissions	$\Delta T_G = 1\ \mu$s
Frequency guard interval between two adjacent subchannels	$\Delta B_G = 10$ kHz

In the above table, the TDM slot duration indicates the time duration taken in each FDM subchannel to transmit the symbols of all users in both the transmit and receive modes.

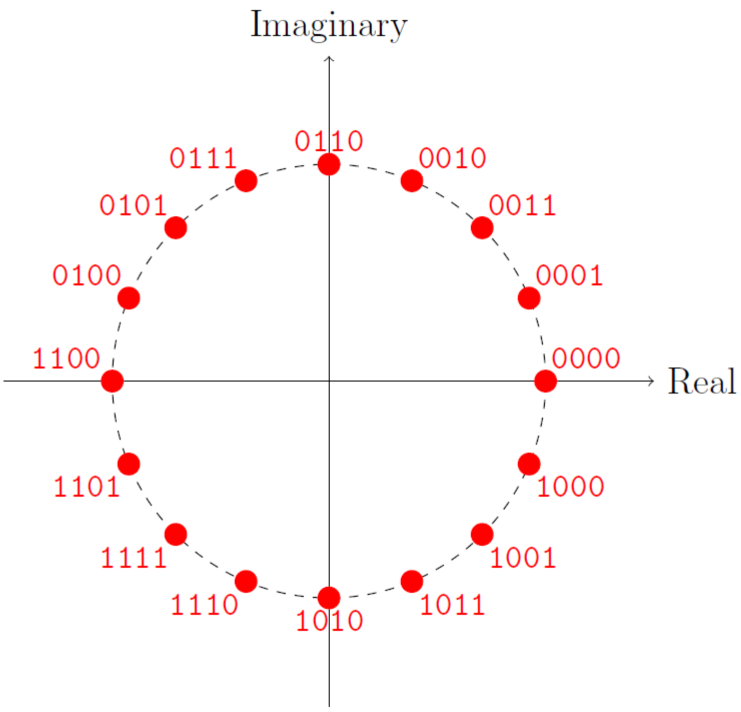
For multiplexing, the given channel is divided into N_C FDM subchannels. Each subchannel is shared among 8 users via TDM. The users utilize root raised cosine (RRC) pulses with roll-off factor $\alpha = 0.5$ for signal transmission.

(a) Calculate the maximum value of N_C, such that no cross-talk happens among the users.

(b) What is the major difference of this setting to the GSM system?

12.2.5 Question 5: Modulation Schemes (23 Points)

A communication system uses $\pi/16$-shifted 16-ary phase-shift-keying (PSK) modulation with root raised cosine (RRC) for data transmission over a frequency selective Rayleigh fading channel. The transmitter uses gray mapping according to the following diagram.



The message which has been transmitted in the time index $k = 6$ is $M = 0001$. To transmit this message, the transmitter sends the constellation point

$$a[6] = \exp\left\{j\frac{3\pi}{8}\right\}$$

where $j = \sqrt{-1}$ is the imaginary unit.

First, assume that the transmitter sends the messages **uncoded**. Answer the following items:

(a) Plot the transmit constellation for the time index $k = 8$.

(b) Why $\pi/16$-shifted 16-ary PSK is used instead of the standard 16-ary PSK constellation?

Now, assume that the transmitter uses a linear block code for channel coding to combat the destructive effects of multipath fading. This block code encodes each block of four information bits into a codeword of length $N = 6$ using the following generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The transmitter uses this block code to encode the two subsequent messages

$$M_1 = 0011 \qquad \text{and} \qquad M_2 = 0101.$$

The two subsequent codewords are then transmitted via the $\pi/16$-shifted 16-ary PSK modulation in three subsequent time intervals **without interleaving.**

(c) Assume that the transmission is started at the time index $k = 8$ with the same constellation determined in Part (a). Calculate the constellation points that are transmitted in the time indices $k = 8$, $k = 9$ and $k = 10$, i.e., $a[8]$, $a[9]$ and $a[10]$.

(d) Does the transmitter reduce the bit-error rate (BER) by this channel coding approach? Explain your answer by giving a reason.