Primes & Secrets

A gentle introduction to public-key cryptography

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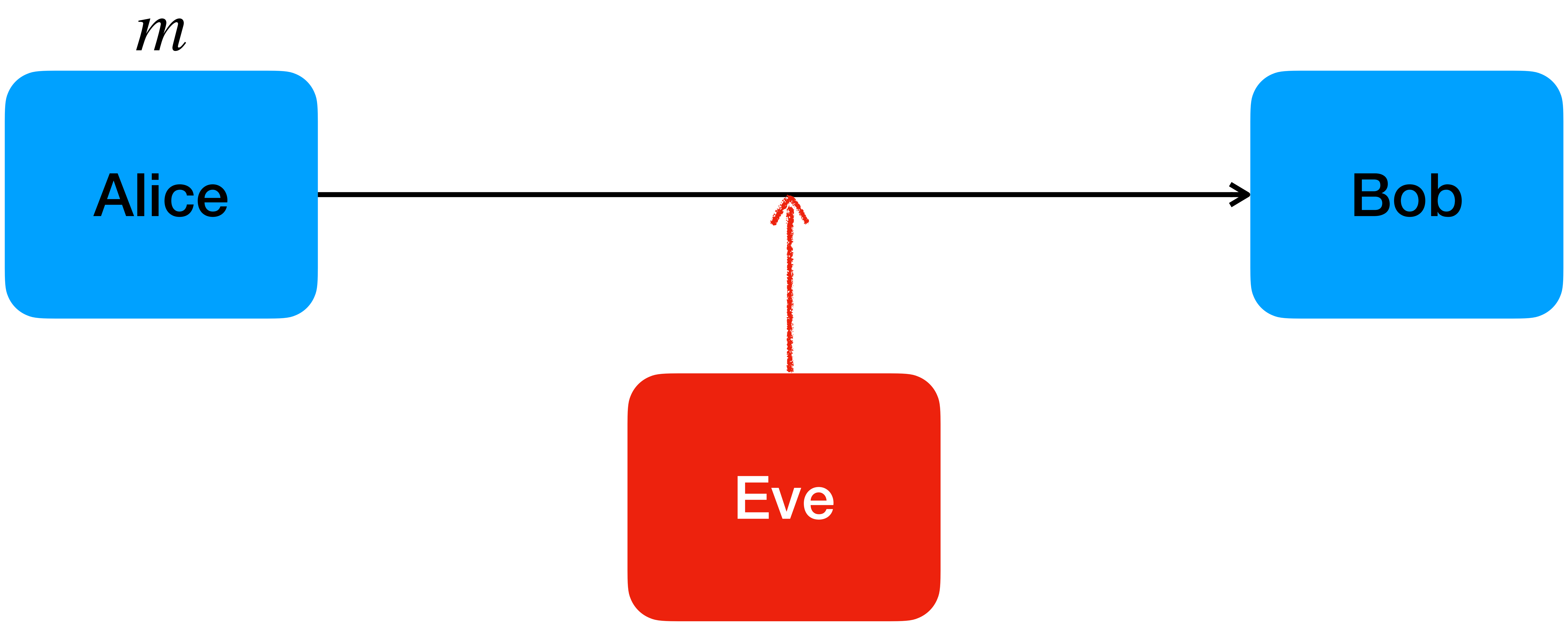
Introduction

The Plan

- What is public-key cryptography
- Some tools from Number Theory
- The Rivest-Shamir-Adleman encryption scheme
- Uses of the RSA scheme

Symmetric encryption

Alice, Bob, etc.



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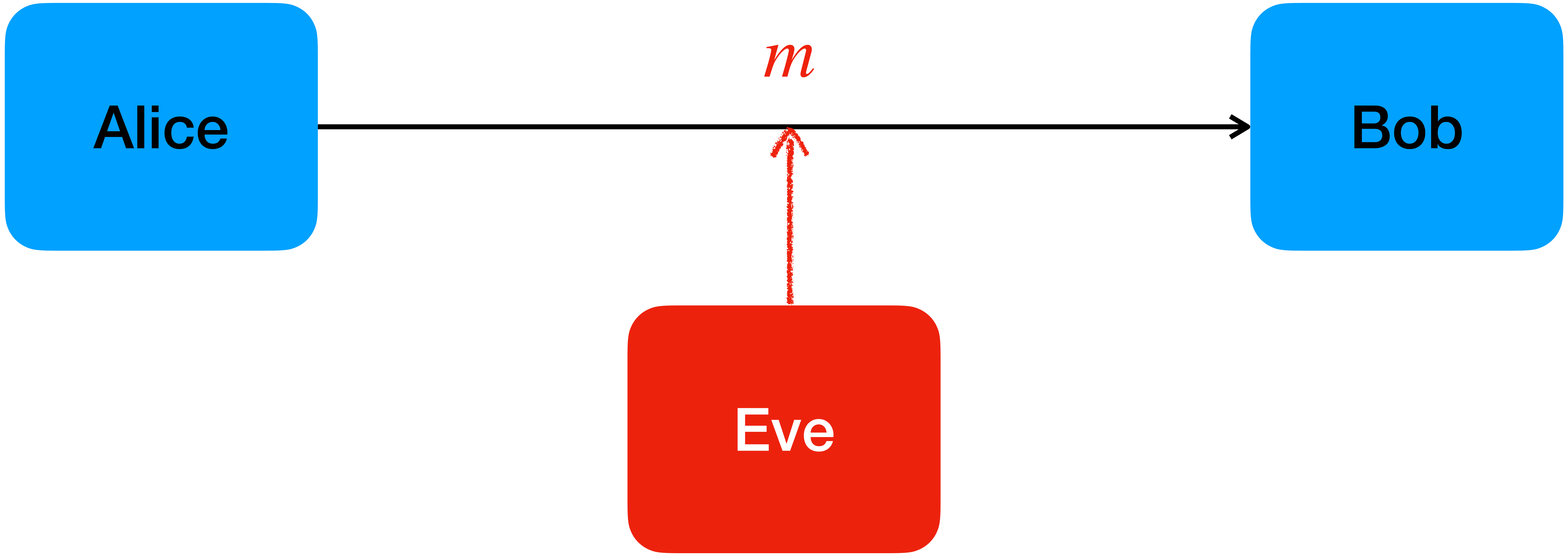

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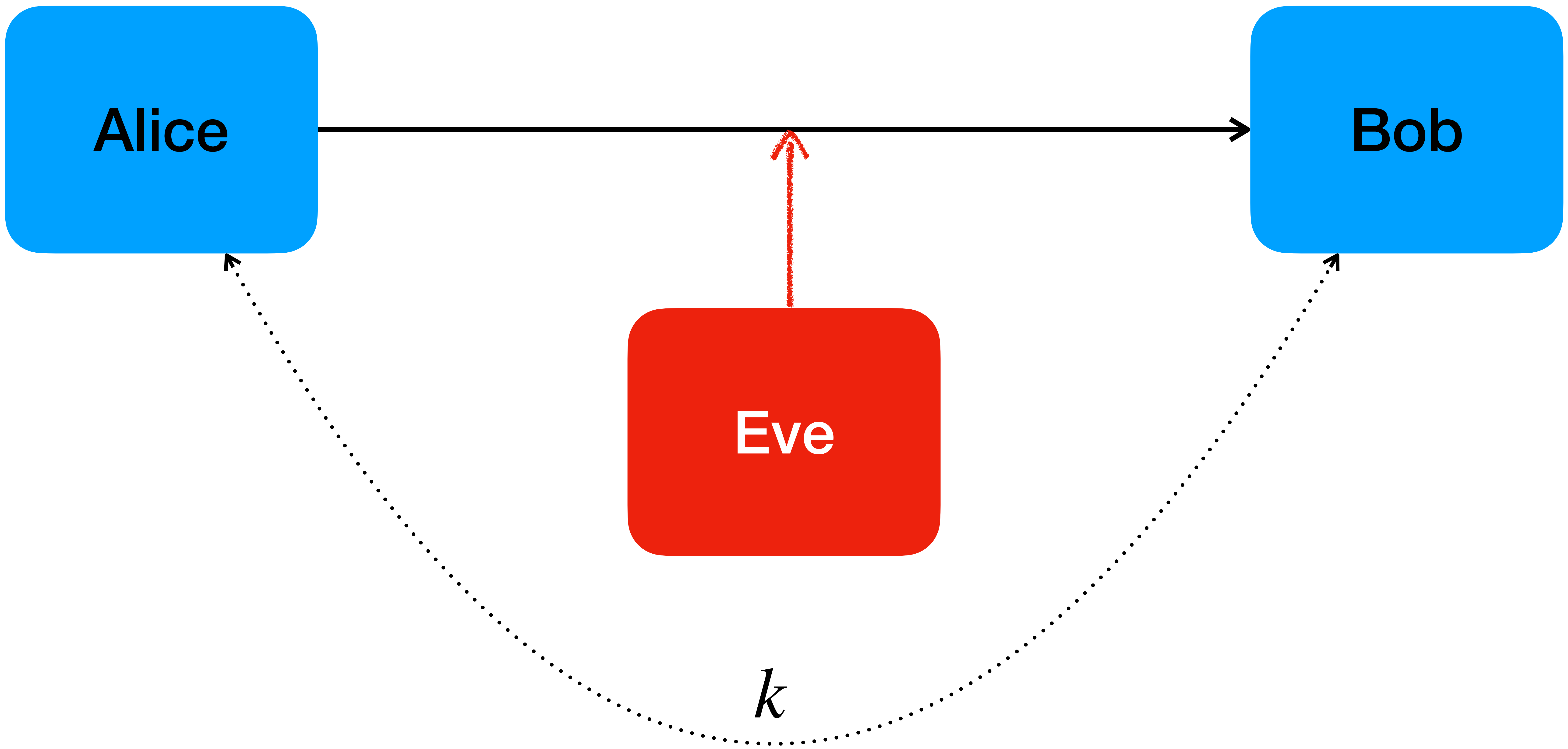

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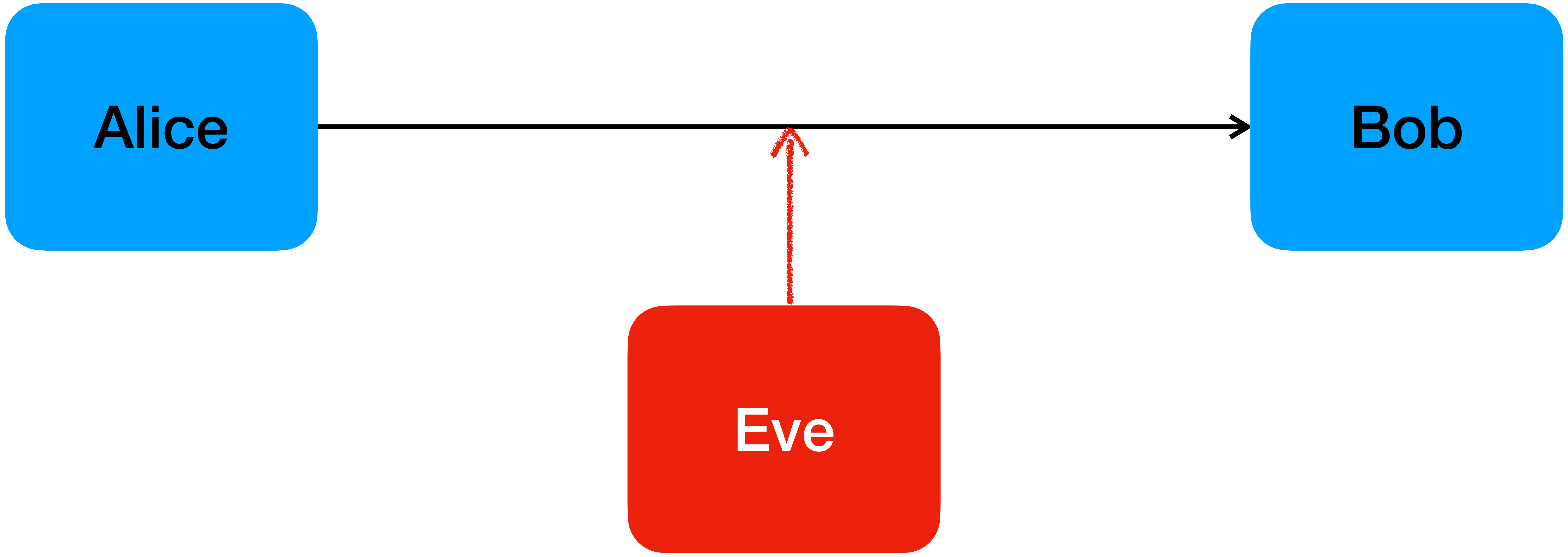
$$c \leftarrow Enc_k(m)$$



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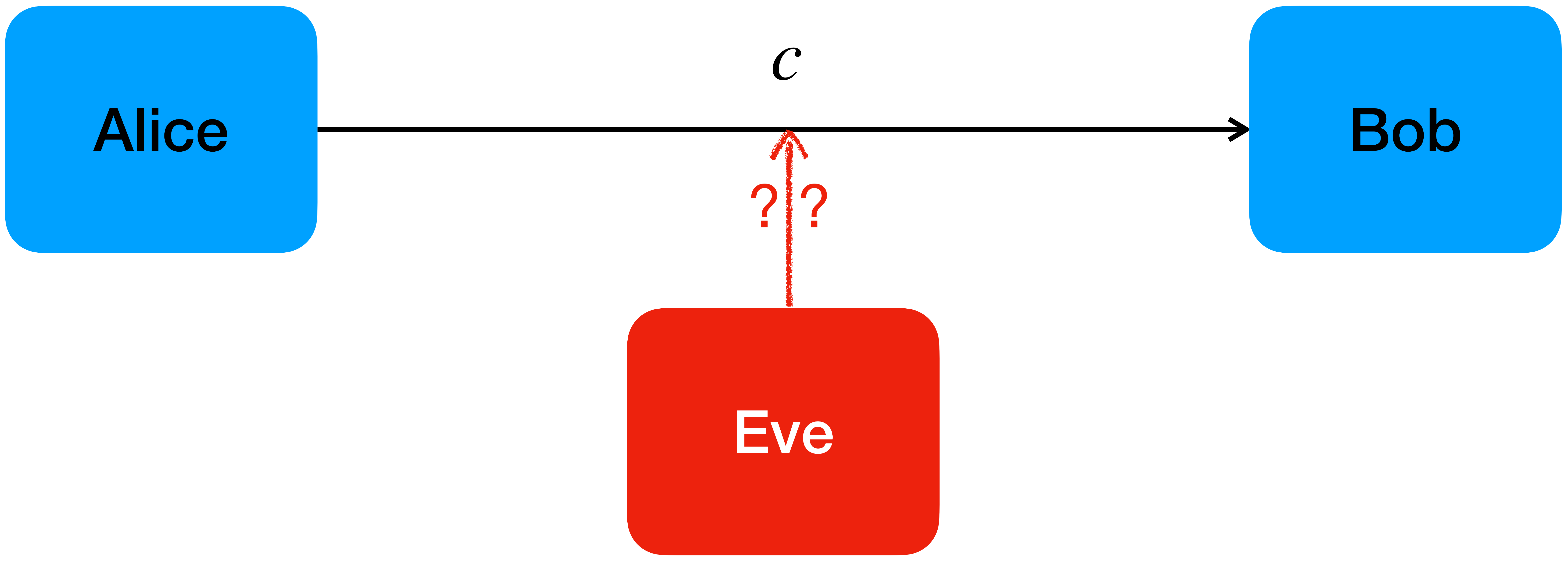

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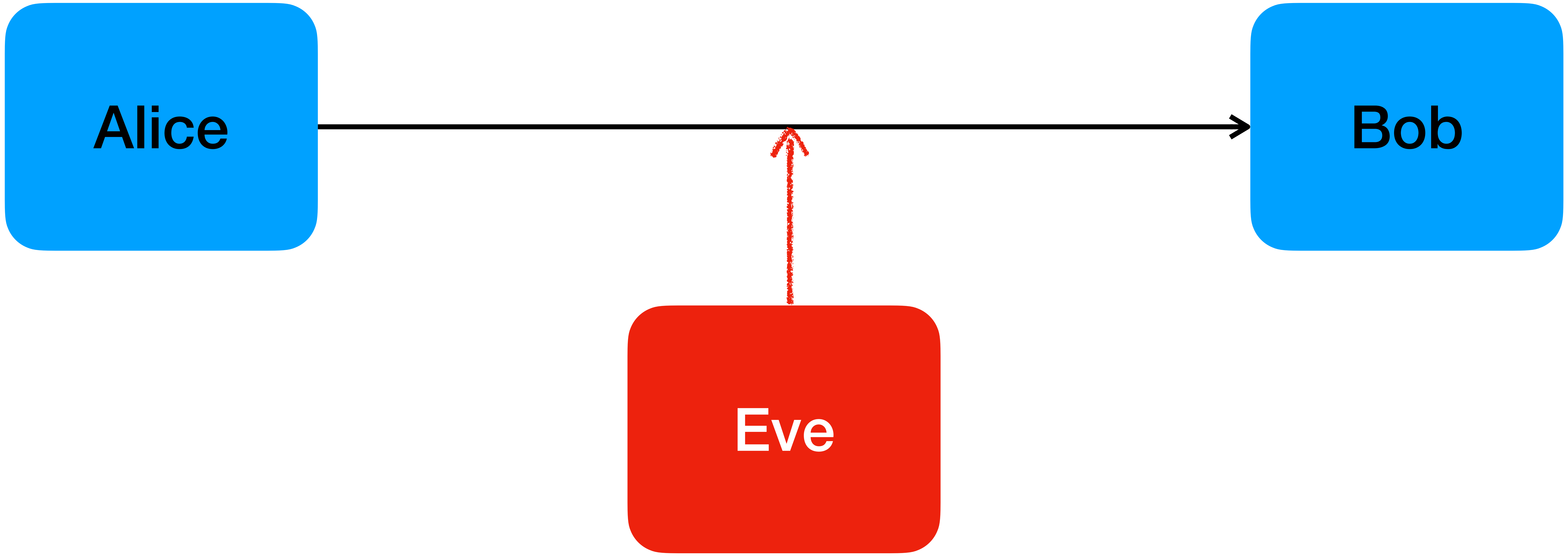
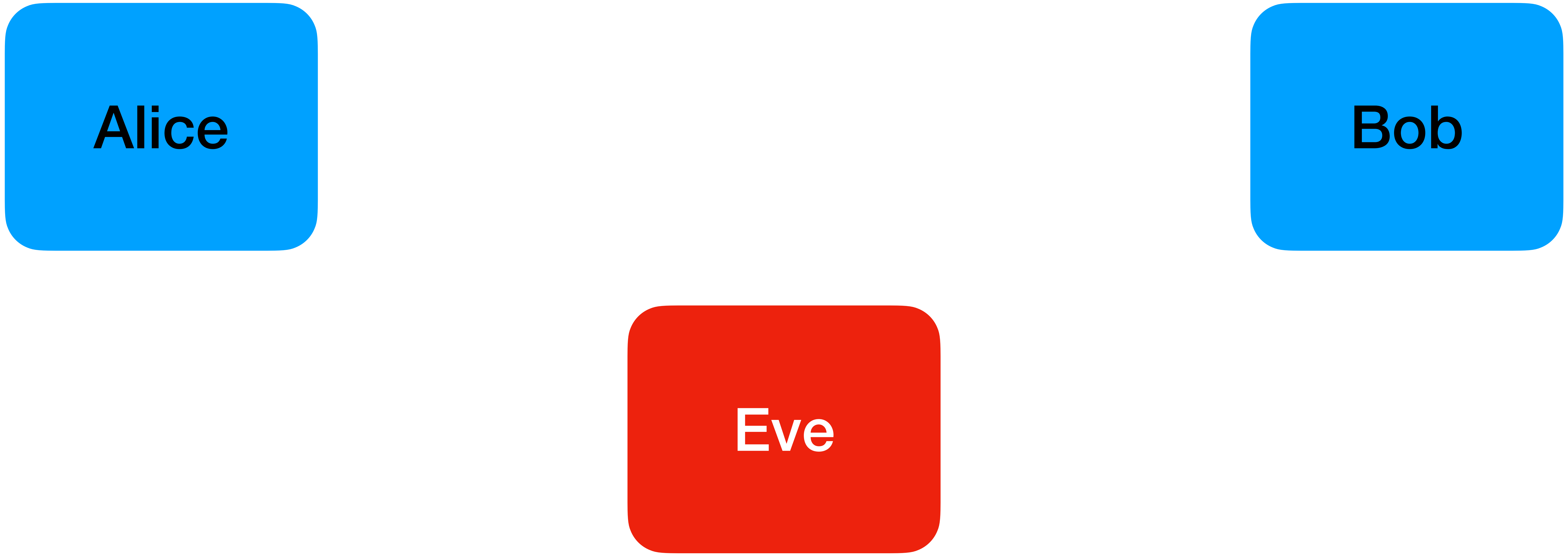
$m \leftarrow Dec_k(c)$



Public-key encryption

Or - *asymmetric* encryption

$(pk, sk) \leftarrow \mathrm{KeyGen}()$



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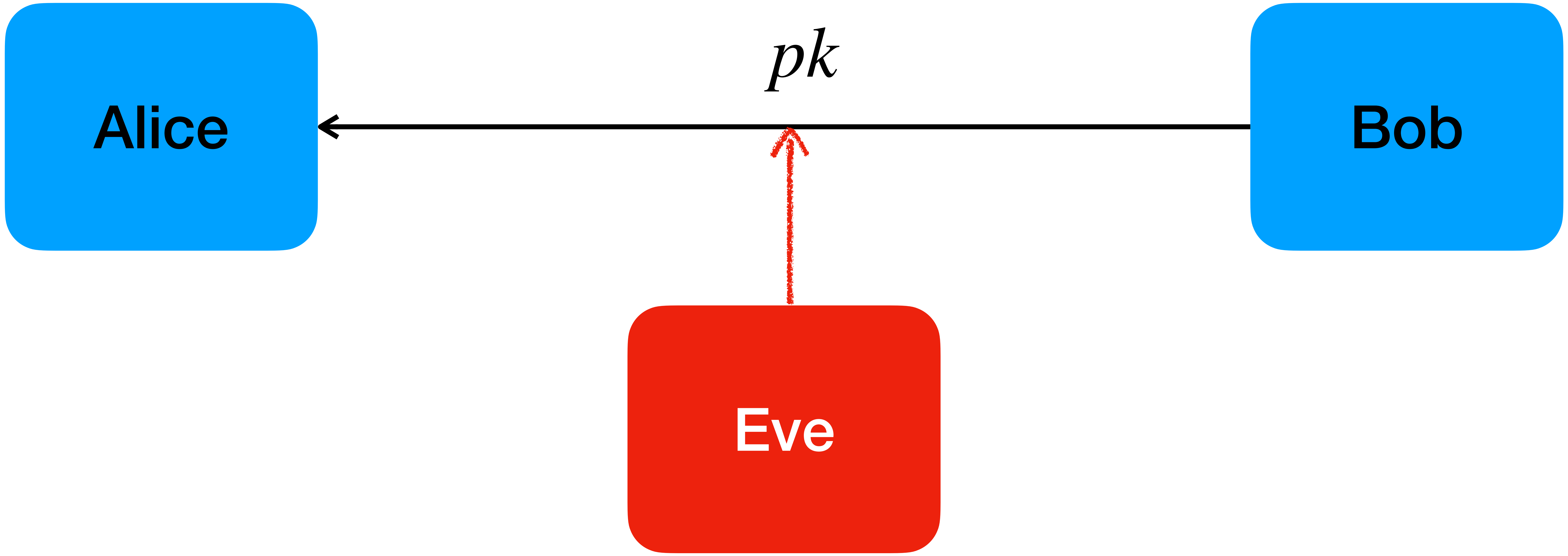

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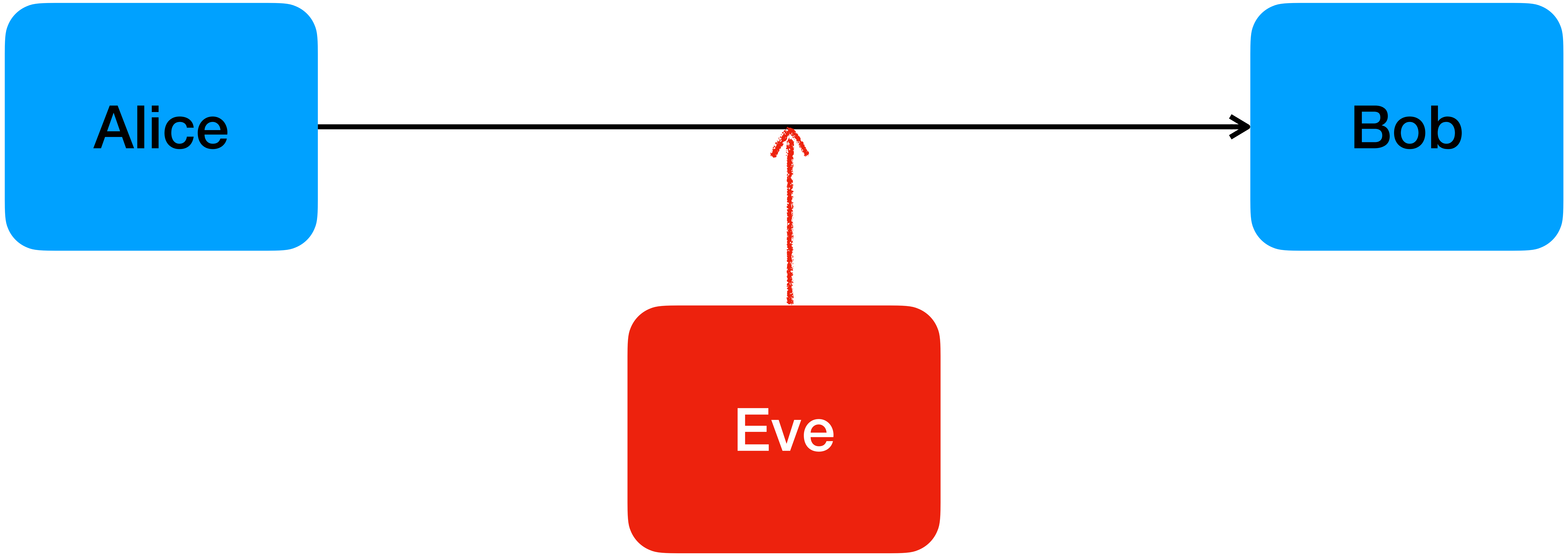
$c \leftarrow \mathsf{Enc}_{pk}(m)$



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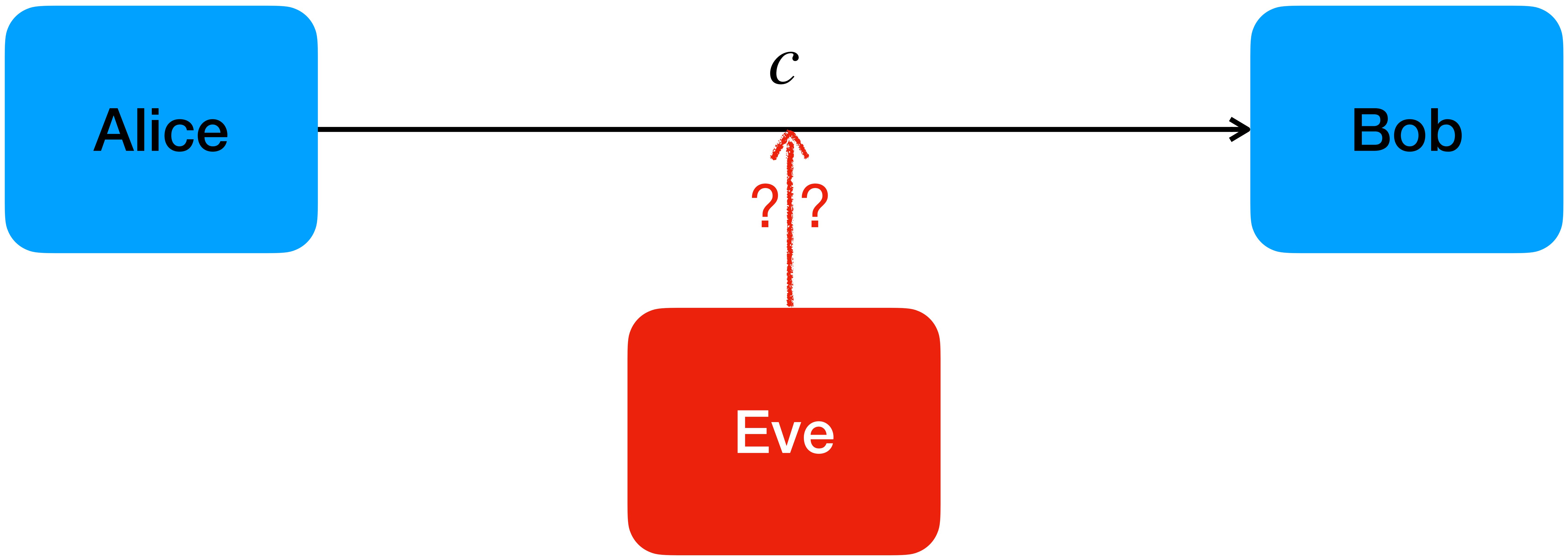

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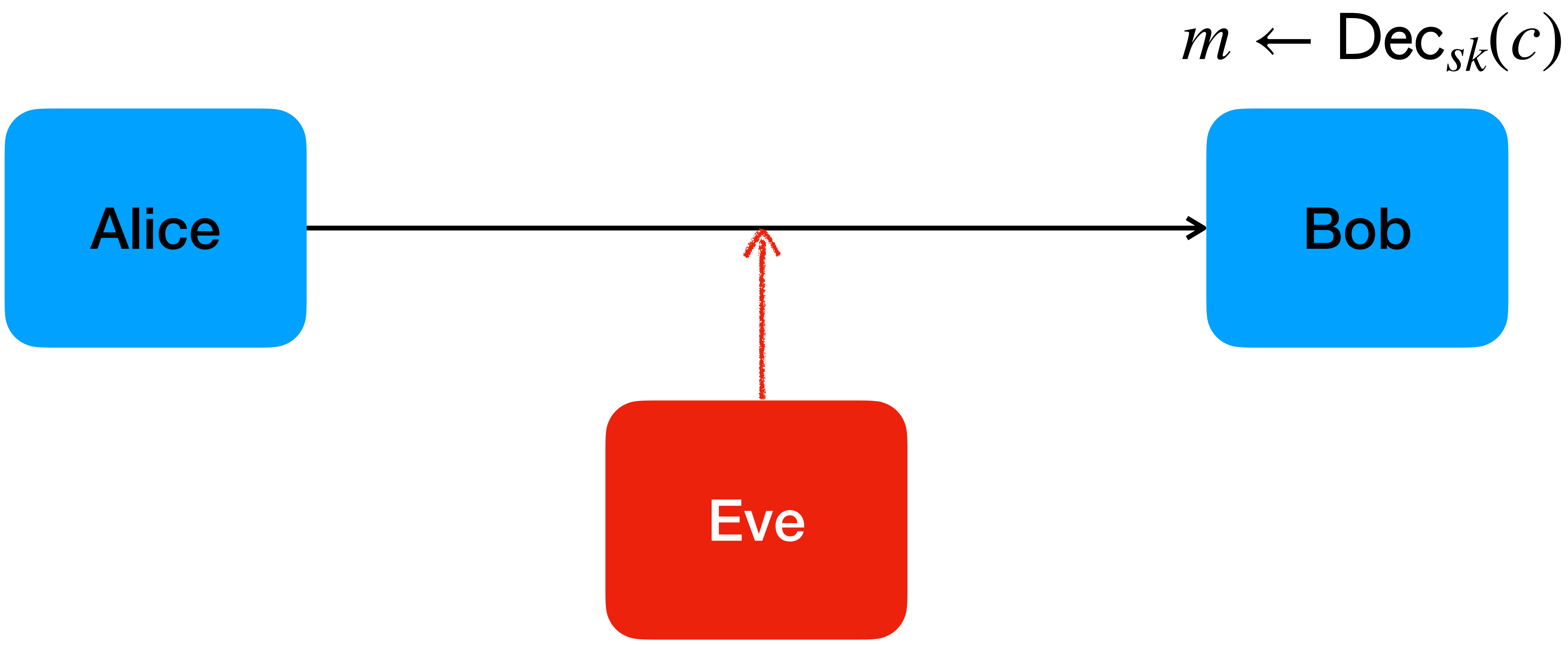

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- Take $n = 12$:
- $0 \mod 12 = 0, 1 \mod 12 = 1,\ldots,11 \mod 12 = 11,12 \mod 12 = 0,\ldots$
- Integers modulo n can be added and subtracted
- Formally: $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$ is a group under addition

Modular arithmetic

Prime moduli

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- In general, no.
- Take m in $\mathbb{Z}_n$. It has multiplicative inverse (another $m' \in \mathbb{Z}_n$ such that $m \cdot m' = 1 \mod n$) only if $\gcd(m, n) = 1$

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- How many elements in $\mathbb{Z}_n$ have multiplicative inverse?
- $\phi(n)$: counts the integers $\leq n$ coprimes with n
- p prime $\implies \phi(n) = p - 1$

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A fundamental theorem

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- Raising to the 27-power is the same as taking the cubic root (modulo 100)

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- In general:
- a coprime with n and $e \cdot d = 1 \mod \phi(n) \Rightarrow a^{e \cdot d} = a \mod n$
- That is, taking the d-th power is equal to taking the e-th root modulo n

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- We have all the ingredients for the RSA encryption scheme

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An application of Euler theorem

- Keys generation:
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  - $pk = (N, e)$ $sk = (d, p, q)$

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$$= m^{k(p-1)(q-1)+1} \mod N$$

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&= m^{ed} \mod N \\
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&= m \; m^{k(p-1)(q-1)} \mod N \\
&= m \mod N
\end{aligned}$$

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equal to 1 by Euler Thm

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Real-world RSA parameters

- $p =$
2852548559318792131996856928776907825119281774314763570567815857161180832220837519246298437849803897808133857804293068360720289689219293552787014914773341602205226485142445035331780826110954892820763749663847341009298597810475043222248992578332801443153983881114506335785938838763490070529273256179374411715693740352023288821141872517444832843142745124248167962975328995374423885701884560435346893852806427942025281363464456832501641858113503599539750773425462372913221908859606938360093958840350104476559208036630009503322492880333306863971638512561292424076321787354414847849812047565954709580803419355650482786511 7

- $q =$
2509785003812493366094206331512468181316605756771693905249883385880642709508878477068939652775190207339573525959319662037263317416923209740213635937018922141058811459917391399502361197229181706040899508893306387999326350207796710736265991703103379230835418488059153440114942935256332194856506022411021260589273249908630686846952779779617454980260997007245457420331241921253950673039150817147531381464668508844159047271084111532669050666200100584416669411441413444742531230931714539004213387169133262328065383797323397971665046699690044164577147301910153724302621814797515015487013570026365937192268613525458212916793

- $e = 65537$

Real-world use of RSA

A.k.a. certificates

- Alice and Bob may not know each other
- Bob puts its public key into a message that says:"hi, I'm Bob and this is my pk"
- Then this message is signed by an authority that is known and trusted by both Alice and Bob $\Rightarrow$ certificate of Bob

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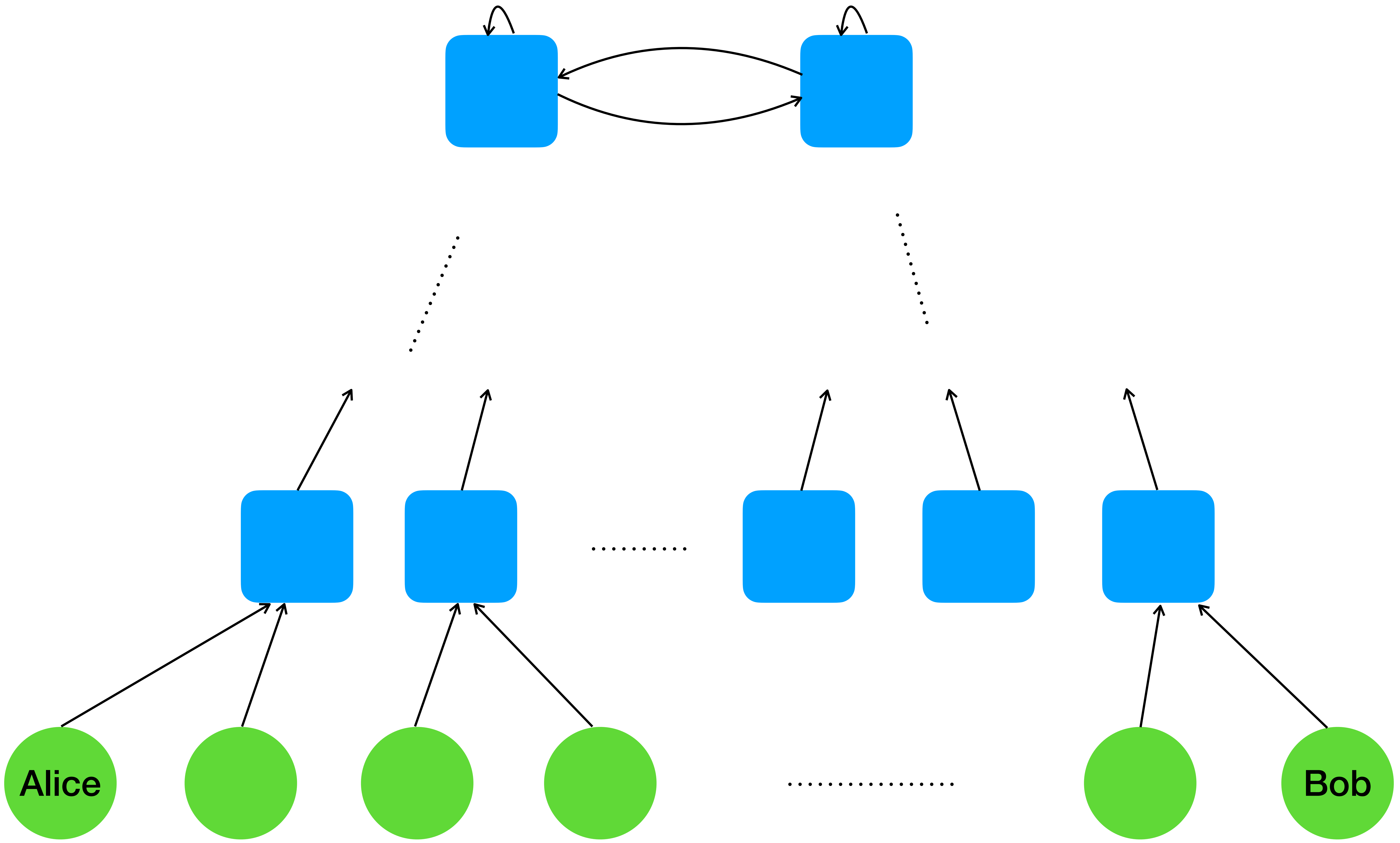
- Idea: local authorities have certificates signed by upper level certificates

Public Key Infrastructure



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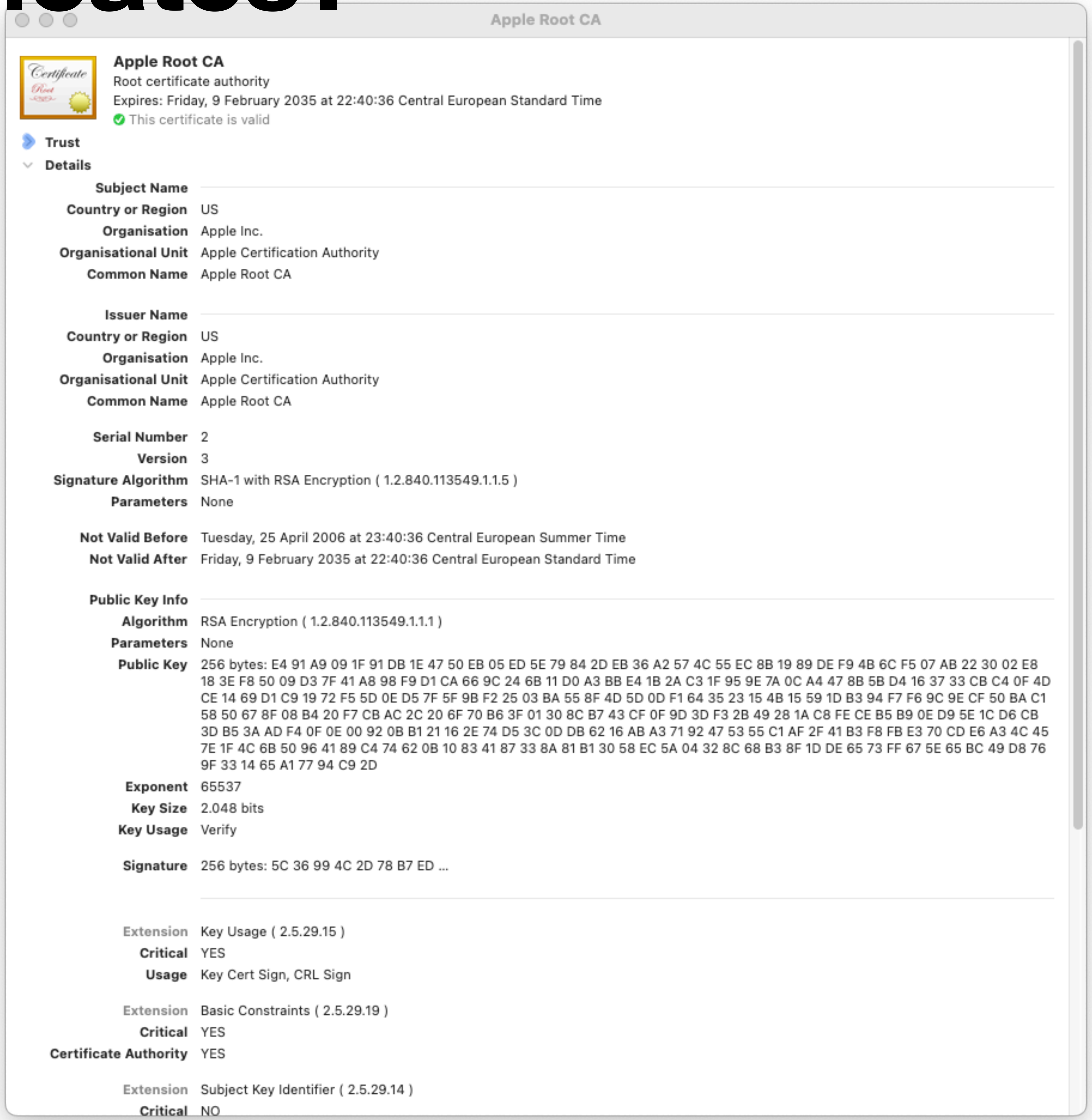
- $\mathsf{Sign}(sk, m) : \sigma \leftarrow m^{sk} \mod N$
- $\mathsf{Verify}(pk, \sigma) : m \leftarrow \sigma^{pk} \mod N$

Where are root CA certificates?

In your browser



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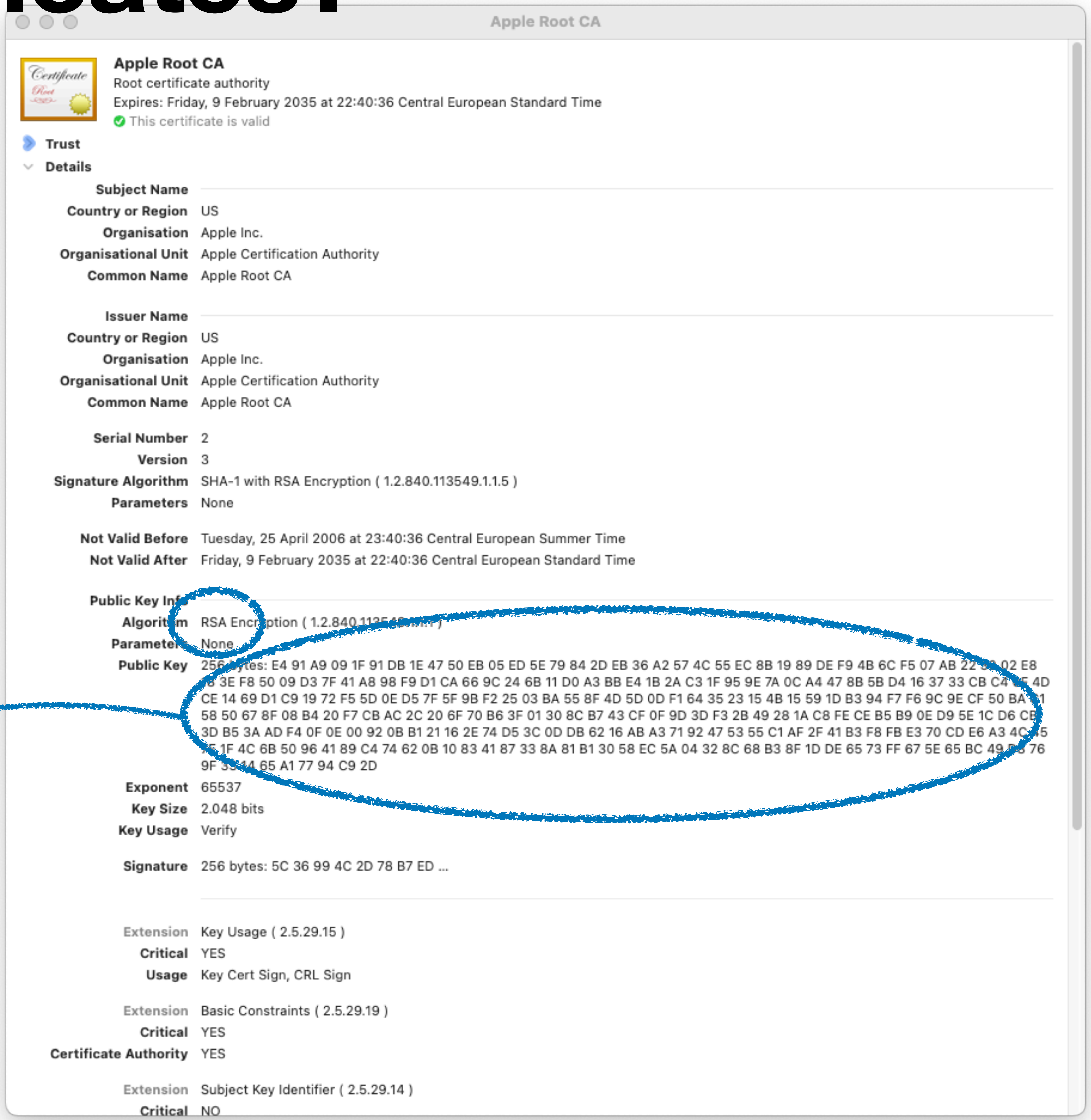
In your browser

RSA pk



Messages are numbers

Use Unicode, for example

- $'a' \to 97, \ldots, 'z' \to 122$
- ‘we meet at 11’ → 9459448527424017981640577921329

- In practice, RSA is used for encrypt short messages
- For example, 256 bits keys of symmetric encryption schemes

In the Lab

- Python basics
- Implementation of a simple symmetric cipher
- Brute force primality testing
- How to compute the greatest common divisor, efficiently
- Modular arithmetics
- Textbook RSA
- Diffie-Hellman (maybe)