Set Point Weighting for Decoupling

Julius Martensen
Written as a part of the Master Thesis

May 2, 2017

The Transfer Function of the Closed Loop of a Plant with a Set Point Weighted Proportional Intergral Controller (PI-Controller), described in Astrm, Haggalund, Advanced PID p.145 ff, is given by:

$$y = [I + G\ (C_p + C_i)]^{-1}\ G(b\ C_p + C_i)\ y_{SP}$$

Where $G \in \mathbf{C}^{n \times n}$ is the transfer function of the plant, $C_i \in \mathbf{C}^{n \times n}$ the transfer function of the integral controller, $C_p \in \mathbf{C}^{n \times n}$ the transfer function of the proportional controller, $y \in \mathbf{R}^n$ the output of the plant, $y_{SP} \in \mathbf{R}^n$ the target value and $b \in \mathbf{R}$ the set point weight which describes the influence of the set point on the closed loop.

Since we design decentralized controller, restricting the plant to a Single Input Single Output (SISO) transfer function without loss of generality.

From above, it is clear that the denominator is not influenced by the weight. Hence we can assume a constant characteristic equation of the closed loop.

Assuming a first order transfer function for the plant given by:

$$G = \frac{K}{T\ s + 1}$$

where $T \in \mathbf{R}^+$ is the Time Constant of the Process Model with a gain $K \in \mathbf{R}^+$.

Computing the characteristic equation holds - see e.g. Astrm, Haggalund, Advanced PID p.174 Eq. (6.4) -(6.5) :

$$\begin{aligned} s^2 + 2\zeta\omega_0 s + \omega_0^2 &= s^2 + \frac{1 + Kk_p}{T} s + \frac{Kk_p}{TT_I} \\ &= s^2 + \frac{1 + Kk_p}{T} s + \frac{Kk_i}{T} \end{aligned}$$

Using Astrm, Haggalund, Advanced PID p.175 Eq.(6.6), extracting the roots of the characteristic equation holds:

$$\omega_0 = \sqrt{\frac{k_i\ K}{T}}$$

The proportional gain k_p can be calculated via:

$$k_p = \frac{2\ \zeta\ T\ \omega_0 - 1}{K}$$

Where $\omega_0 \in \mathbf{R}^+$ are also called the critical frequency. $k_i \in \mathbf{R}^+$ is the integral gain and $k_p \in \mathbf{R}^+$ the proportional gain of the controller of the form:

$$\begin{aligned} C_i &= \frac{1}{T_i\ s} \quad = \quad \frac{k_i}{k_p\ s} \\ C_p &= k_p \end{aligned}$$

$\zeta \in [0, 1]$ is the damping of the function.

Regarding the Paper by Astrm et al. the interaction from input M to output N for a controller of the given form can be written as:

$$|\kappa_N| \leq |k_{NM}||k_{p,M}\ b_M\ \omega + k_{i,M}||M_{s,N}\ M_{s,M}|$$

Where the interaction $\kappa_N \in \mathbf{R}$ and the Maximum Sensitivity $M_S \in \mathbf{R}$ are given while the interaction indexes $k_{NM} \in \mathbf{R}$ from N to M is given by the Taylor series approximation of the Matrix Q.

Assume $I_N, M_{s,N}, M_{s,M}$ to be given we can rewrite the equation to be:

$$|k_p\ b\ \omega + k_i| \leq \underbrace{\frac{|\kappa_N|}{|k_{NM}\ M_{s,N}\ M_{s_M}|}}_{\gamma}$$

We know the Gain to be maximal until the critical frequency, hence we can assume ω_0 to be an upper bound for the interaction. Using the already established equation for k_p

$$\begin{aligned} |k_p\ b\ \omega + k_i| &\leq \gamma \\ |k_p\ b\ \omega + k_i| &= |\frac{2\ \zeta\ T\ \omega_0 - 1}{K}\ b\ \omega_0 + k_i| \\ &= |\frac{2\ \zeta\ T}{K}\ b\ \omega_0^2 - \frac{1}{K}\ b\ \omega_0 + k_i| \end{aligned}$$

and for ω_0:

$$
\begin{aligned}
|\frac{2\ \zeta\ T}{K}\ b\ \omega_0^2 - \frac{1}{K}\ b\ \omega_0 + k_i| \leq \gamma \\
&= |\frac{2\ \zeta\ T}{K}\ b\ \frac{k_i\ K}{T} - \frac{1}{K}\ b\ \sqrt{\frac{k_i\ K}{T}} + k_i| \\
&= |2\ \zeta\ b\ k_i - \ b\ \sqrt{\frac{k_i}{K\ T}} + k_i|
\end{aligned}
$$

We can substitute $\sqrt{k_i} = x$ and rewrite:

$$
|(1 + 2\ \zeta\ b)\ x^2 - \sqrt{\frac{b^2}{T\ K}}\ x| - |\gamma| \geq 0
$$

And solve the quadratic equation:

$$
x_{1,2} = \frac{\sqrt{\frac{b^2}{T\ K}} \pm \sqrt{\frac{b^2}{T\ K} + 4\ (1 + 2\ \zeta\ b)\ \gamma}}{2\ (1 + 2\ \zeta\ b)}
$$