Lecture 1 - Simulation of Dynamical Systems

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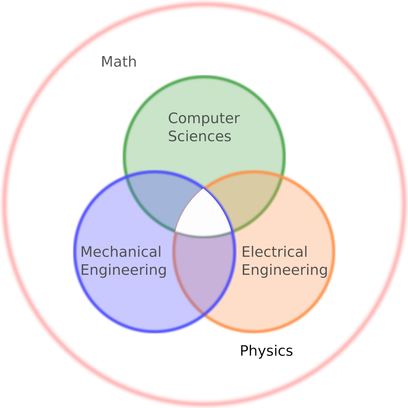
Robotics - An Interdisciplinary Science



Motivation

Simulation in the Context of Robotics

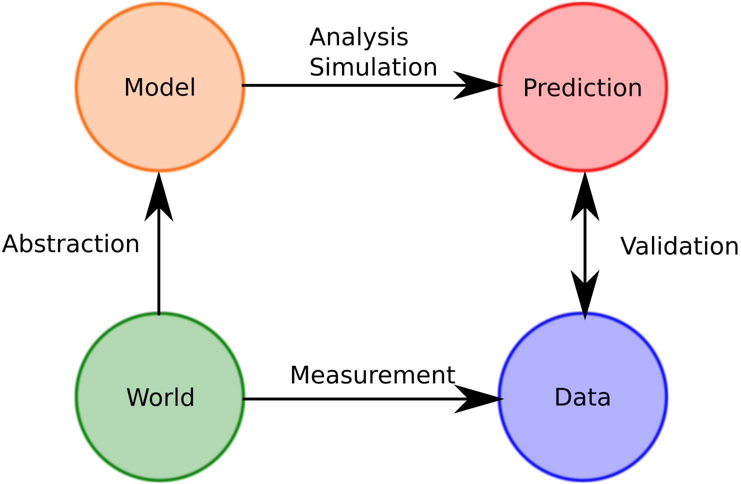
- Structural Optimization
- Mechanical Prototyping
- Testing of Controller
- Learning
- ...

Motivation

Lecture Targets

- Basics: Modeling techniques
- Basics: System dynamics
- Basics: (Numerical) Stability of models
- Example : DC Motor

Modeling of Systems



Basics of Systemdynamics

Initial Conditions
The system at begin of our simulation or observation.

Evolution
The system propagates through time and changes.

Next State
The system arrives at a new state.

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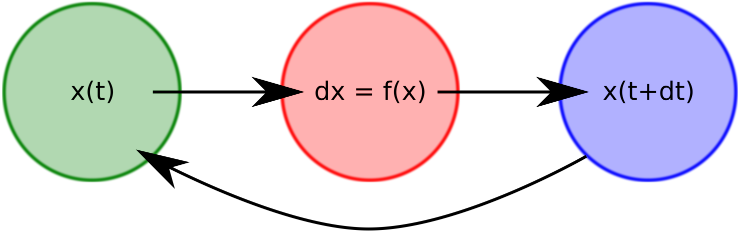


Basics of Systemdynamics

In general, the system is given as a set of nonlinear differential equations.

$$\dot{x} = f(x, u, t)$$
$$y = h(x, u, t)$$

Where $x \in \mathbb{R}^{n_x}$ is called the state , $u \in \mathbb{R}^{n_u}$ the inputs and $y \in \mathbb{R}^{n_y}$ the outputs of a system.
The function $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_x}$ evolves the system through time while $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_y}$ defines the output.

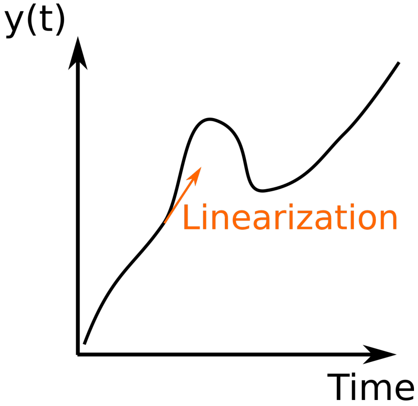
Basics of Systemdynamics

For many systems a set of of linear differential equations is sufficient

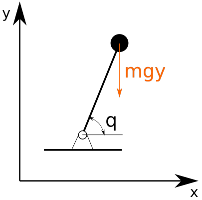
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

With the state matrix $A \in \mathbb{R}^{n_x \times n_x}$, the input matrix $B \in \mathbb{R}^{n_x \times n_u}$, the output matrix $C \in \mathbb{R}^{n_y \times n_x}$ and the feedthrough matrix $D \in \mathbb{R}^{n_y \times n_u}$

Basics of Systemdynamics



Pendulum



$$\dot{\phi} = \frac{p}{l^2 m}$$

$$\dot{p} = mgl\ cos(\phi)$$

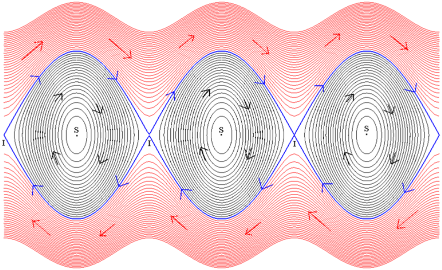


Figure: Phase space Wikipedia, 14/11/2018

Stability of Simulation

Status

Continoous model in form of a system of (nonlinear) equations.

Task

Simulate the system over time

Problem

Discrete calculation!

$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$
$$\downarrow$$
$$x_{i+1} = F(x_i, u_i)$$
$$y_{i+1} = G(x_i, u_i)$$

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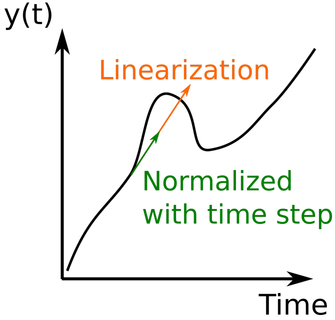
Stability of Simulation

Example : Forward Euler Scheme

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
&\approx \frac{x_{i+1} - x_i}{\delta t} \\
&= f(x_i, u_i) \\
x_{i+1} &= x_i + \delta t \; f(x_i, u_i)
\end{aligned}
$$



Stability of Simulation

A system is stable if trajectories starting near a final state end in this state or enter a closed orbit in its neighborhood.

$$x_S = \{x \in \mathbb{R}^{nx} | f(x_S) = 0\}$$
$$x_S = \{x \in \mathbb{R}^{nx} | f(x_S) \leq \epsilon\}$$

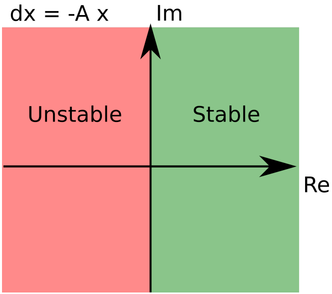
Image - Stability on Phase space

Stability - Direct Method of Lyapunov

A given system is stable of the form

$$\dot{x} = -Ax$$

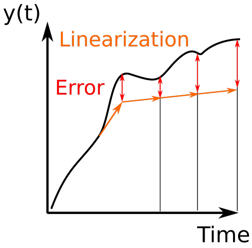
is stable if all eigenvalues λ_i are on the right half plane of the complex plane.



Stability of Simulation

Most physical system derived by first principles are stable due to conservation laws!

However, instability is possible due to wrong parametrization, the solver or the nature of the problem itself (stiffness of differential equations).



tic stability.

A-Stability

A solver is called asymptotic stable, if for a given problem of the form:

$$\dot{x} = -Ax,\ \ Re(\lambda_i(A)) > 0$$

The resulting calculation is monotonic decreasing for $\forall \delta t \in \mathbb{R}^+$

Stability of Simulation

Consider the Pendulum in Euler Forward Notation:

$$x_{i+1} = \begin{bmatrix} \phi_i \\ p_i \end{bmatrix} + \delta t \begin{bmatrix} \frac{p_i}{l^2 m} \\ m * g * l * cos(\phi_i) \end{bmatrix}$$
$$= \left[I + \delta t \begin{bmatrix} 0 & \frac{1}{l^2 m} \\ mglsin(\phi_i) & 0 \end{bmatrix} \right] \begin{bmatrix} \phi_i \\ p_i \end{bmatrix} + \mathcal{O}^2$$

The eigenvalues are then given as:

$$\lambda_i = 1 \pm \delta t \sqrt{\frac{g}{l} sin(\phi_i)}$$

Hence, the system is not alway stable, both in the sense of the step size or the formulation.

Stability of Simulation

Stiffness of the problem:

$$\mu = \frac{max(Re(\lambda_i))}{min(Re(\lambda_i))}$$
$$= \frac{1 + \delta t \ \sqrt{\frac{g}{l} sin(\phi_i)}}{1 - \delta t \ \sqrt{\frac{g}{l} sin(\phi_i)}}$$

Recapture

Modeling

A model can be derived via **first principles** (white-box), by **data-driven** techniques (black-box) or a combination (gray-box)

Systems

A system has characteristics like fixed points, saddle points, ... which **influence** the simulation

Simulation

Even **stable systems** can lead to **unstable simulations** due to wrong parametrization, poor choice of algorithms or high nonlinearities.

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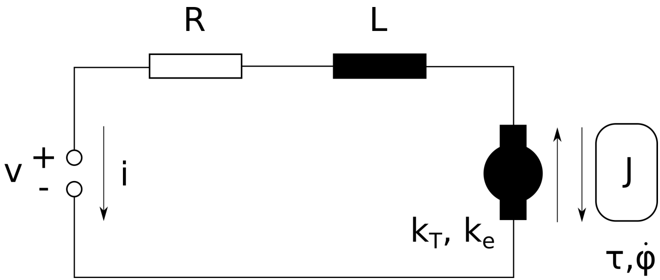
Example: Modeling a DC Motor

A simple DC Motor



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