Week 1

Reading assignment

The oxford solid state Basics, Steven H.Simon(Part1, 3,Ch15-16. Ch22, Ch19-20)

Introduction to Quantum Mechanics格里菲斯（Q4.51/Q4.52-3）

Advanced Quantum Mechanics (Sakurai Ch1 section 5)

Classical Mechanics Herbert Goldstein (Vector potential in classical EM theory)

Quantum Mechanics (Cohen-Tannoudji, Diu Complement Dvil)

Meissner effect

The expulsion of a magnetic field from a superconductor during its transition to the superconducting state when it is cooled below the critical temperature.

Breakdown of Meissner effect

A superconductor with little or no magnetic field within it is said to be in the Meissner state. The Meissner state breaks down when the applied magnetic field is too strong.

超导体由非超导态进入超导态，将磁场排除体外。当磁场较强时，迈斯纳效应将被破坏。

超导体的分类：

Tpye -I(Pippard)

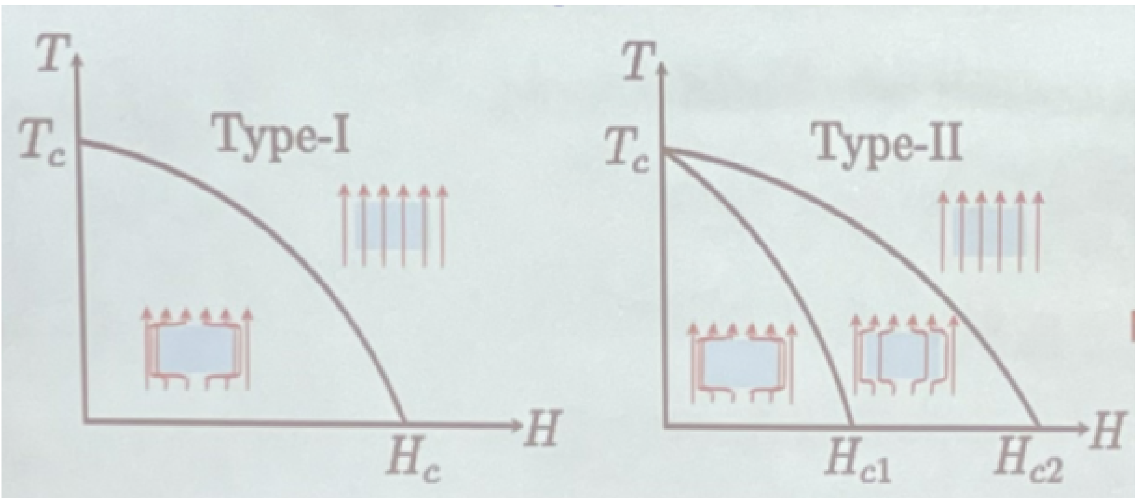

图 **1.** Nb is type-II superconductor(cf., V&T_c) Abrikosov vortices under magnetic field

$$\boldsymbol{k} < \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$$

Tpye - II(London)

$$\boldsymbol{k} > \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$$

$$\Delta H = \lambda^{-2} H$$

λ为伦敦贯穿深度

Meissner order

1)For $T>T_c$，place a magnet on the YBCO pellet. No levitation shows

2)Cooling YBCO below T_c fluxed

Meissner effect VS　Flux pinning

Aharonoc-Bohm effect (AB效应)

Double-slit experiment

1. Magnetic field confined in whisker/solenoid. No fields on the paths of two beams.

2. Shift of finges is related to the flux inside whisker/solenoid

3. Shift of fringes is attributed to the **vector potential** outside the whisker/solenoid.

4. Vector potential seems a more fundamental quantity(e.g. QED action)

Maxwell equation

无磁单极子： 1982年BlasCabera利用超导电流环检测到电流，认为是磁单极子。无人重现。

Gauge invariance

1　Incorporation of EM fields to Hamiltonian

First quantization

the equation of motion of a charged particle in ME fields read

$$m\frac{d\boldsymbol{v}}{\mathrm{dt}}=q\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)=\boldsymbol{F}_{\mathrm{Lorentz}}$$

Define a momentum $\boldsymbol{P}=m\,\boldsymbol{v}+q\,\boldsymbol{A}$

$$\frac{d\boldsymbol{P}}{\mathrm{dt}}=q\left(\frac{d\boldsymbol{A}}{\mathrm{dt}}-\frac{\partial\boldsymbol{A}}{\partial t}\right)+qv\times\left(\nabla\times\boldsymbol{A}\right)-q\nabla\phi$$

$$=q\nabla\left(\boldsymbol{v}\cdot\boldsymbol{A}\right)-q\nabla\phi$$

Make use of the Hamiton equation $\frac{d\,\boldsymbol{P}}{\mathrm{dt}}=-\frac{\partial H}{\partial r}$， and $\frac{dr}{\mathrm{dt}}=\frac{\partial H}{\partial\boldsymbol{P}}$,the Hamilton adopts the form

$$H=K\left(\boldsymbol{P}-q\,\boldsymbol{A}\right)+q\phi$$

K represents the kinetic energy. In QM course, we usually consider $K=\frac{P^2}{2m}$.

After first quantization, we obtain the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}(-i\hbar\nabla - q\boldsymbol{A})^2 + q\phi$$

Second quantization(B=0)

The tight-binding theory tells us that the band structure forms from the overlap of electron wave of functions in solids. For example, a crystal with simple cubic lattice is characterized by the following tight-binding Hamiltonian

$$\hat{H} = -\sum_{r,\delta} t\, \hat{c}_r^\dagger \hat{c}_{r+\delta} + H.c.$$

$\hat{c}_r^\dagger$ and $\hat{c}_r$ and creation and annihilation operatiors, repectively.

Perform Fourier transform $\hat{c}_r = \frac{1}{\sqrt{N}}\Sigma_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\hat{c}_k$, the Hamiltonian reads

$$\hat{H} = \sum_k \epsilon_k \hat{c}_r^\dagger \hat{c}_r$$

The band structure then reads

$$\epsilon_k = -2t[\cos(k_x a) + \cos(k_y a) + \cos(k_z a)]$$

In the vicinity of the Γ point (i.e., $k=0$), the dispersion becomes

$$\epsilon_k = -6t + t a^2 k^2$$

which exhibits a quadratic form (cf., $K = \frac{p^2}{2m}$).

Second quantization($\boldsymbol{B} \neq 0$)

In the presence of a magnetic field, an electron will acquire an Aharonov-Bohm phase when hops from $\boldsymbol{r}+\boldsymbol{\delta}$ to $\boldsymbol{r}$. Therefore, the hopping parameter becomes

$$\hat{H} = -\sum_{\boldsymbol{r},\boldsymbol{\delta}} t \cdot \exp\left(i\frac{e}{\hbar}\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\delta}} \boldsymbol{A}\cdot d\boldsymbol{l}\right) \hat{\boldsymbol{c}}_{\boldsymbol{r}}^\dagger \hat{c}_{\boldsymbol{r}+\boldsymbol{\delta}} + H.c.$$

The translational symmetry is at least broken in one direction because $\nabla \times A \neq 0$.

For $\boldsymbol{A} = (-B_y, 0, 0)$, perform Fourier transform in x and z directions.

$$\hat{H} = \sum_k \hat{c}_{\boldsymbol{k_x},\boldsymbol{y},\boldsymbol{k_z}}{}^\dagger H_{k_x,y,k_z} c_{\boldsymbol{k_x},\boldsymbol{y},\boldsymbol{k_z}}$$

In other words, k_x and k_z and still good quantum number.

The kernel (i.e. Bloch Hamiltonian) operator reads

$$\hat{H}_{k_x,y,k_z} = -2t\cos\left[\left(k_x - \frac{e}{\hbar}By\right)a\right] - 2t\cos(k_z a) - t(\hat{s}_+ + \hat{s}_-)$$

Here $\hat{s}_\pm$ are shift operators along y direction satisfying $\hat{s}_\pm f(y) = f(y \pm a)$. In the quadratic order, $f(y\pm a) = f(y) \pm a\partial_y f(y) + \frac{1}{2}a^2\partial_y^2 f(y)$. Thus we estimate that

$$\hat{s}_\pm = 1 \pm a\partial_y + \frac{1}{2}a^2\partial_y^2$$

Expanding the momenta to the quadratic order, the Bloch Hamiltonian reads

$$\hat{H}_{k_x,y,k_z} = -6t + t\,a^2\left(k_x - \frac{e}{\hbar}By\right)^2 + t\,a^2k_z^2 - t\,a^2\partial_y^2$$

which can also be obtained from its $B=0$ counterpart by

i. Peierls substitution in momentum space $k_x \to k_x - \frac{e}{\hbar}B_y$

ii. First quantization $k_z \longrightarrow -i\partial_y$

2 Why second quantization?

Partical number representation

In the context of condensed matter physics, a many-body system is characteriszed by

$$|n_1, n_2, \ldots\rangle$$

which means there are n_j particals on the jth quantum state.

The creation (annihilation) operator increases (decrease) the quantum number n_j when acting on the many-body wave vector. For fermion,

$$c_\alpha^\dagger|\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle = (-1)^{\sum_{\beta<\alpha} n_\beta}\sqrt{1-n_\alpha}\,|\ldots, n_\beta, n_{\alpha+1}, n_\gamma, \ldots\rangle$$

where the creation and annihilation operators obey the anti-commutation relations

$$\{c_\alpha, c_\beta^\dagger\} = c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta} \quad \{c_\alpha^\dagger, c_\beta^\dagger\} = \{c_\alpha, c_\beta\} = 0$$

Quiz. (i) Show the necessity of the Jordan-Wigner string $(-1)^{\sum_{\beta<\alpha} n_\beta}$

(ii) Explain why $c_\alpha^\dagger c_\alpha$ is called the fermion "number operator"

Particle number representation continued

For bosons, action creation/annihilation operations to the many-body states yields

$$b_\alpha^\dagger|\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle = \sqrt{n_\alpha+1}\,|\ldots, n_\beta, n_\alpha+1, n_\gamma, \ldots\rangle$$

$$b_\alpha|\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle = \sqrt{n_\alpha}\,|\ldots, n_\beta, n_\alpha-1, n_\gamma, \ldots\rangle$$

The creation and annihilation operations obey commutation relation

$$\{b_\alpha, b_\beta^\dagger\} = b_\alpha b_\beta^\dagger - b_\beta^\dagger b_\alpha = \delta_{\alpha\beta} \quad \{b_\alpha^\dagger, b_\beta^\dagger\} = \{b_\alpha, b_\beta\} = 0$$

Particle number non-conservation

The particle number representation and second quantization are particularly suitable to systems with variable particle numbers. Remarkably, superconductors are such systems with *non-conserved* electron numbers. Per Nother's theorem, the electron (charge) conservation is associated with the $U(1)$ symmetry.

$$c \to e^{i\theta}c \quad c^\dagger \to e^{-i\theta}c^\dagger$$

Under $U(1)$, the superconductor order parameter $\Delta \sim \langle cc\rangle$ transforms to

$$\Delta \to e^{2i\theta}\Delta$$

3 Remarks on non-conservation

3.1 Cooper pair (BCS)

In BCS theory, Cooper pairs are formed and conduct supercurrent. In a Cooper pair, two electrons are paired, because the lattice distortion(phonon) provides a net attraction on the order of milli-electron volt. The size of the Cooper pair is on the order of hundreds of nanometers. A Copper pair is effectively boson and condense into the ground state.

3.2 Pictorial understanding

Saying that the number of electrons is not conserved is more like an abuse of language. The electrons are simply paired to form Cooper pairs, but the total number of electrons has to be unchanged. Another example is the Andreev reflection, the electrons are conserved globally but not locally.

4 Landau levels

4.1 Band structure

In the context of condensed matter physics, band structure calculation is often a prerequisite to a variety of tasks (e.g., evaluating transport coefficients). We now consider the band structure arising from the following Hamiltonian

$$\hat{H} = \frac{1}{2m}[(P_x - eB_y)^2 - \hbar^2\partial_y + P_z^2]$$

which characterized an electron subjected to a magnetic field $\boldsymbol{B} = B\hat{z}$.Rewrite

$$\hat{H} = \frac{P_z^2}{2m} + \frac{\hbar e B}{2m}(\xi^2 - \partial_\xi^2)$$

where $\xi = \frac{y}{\ell_B}$ with magnetic length $\ell_B = \sqrt{\hbar / eB}$. In terms of the ladder operators $\hat{a} = \xi + \partial_\xi$ and $\hat{a}^\dagger = \xi - \partial_\xi$

$$\hat{H} = \frac{P_z^2}{2m} + \hbar\omega_c\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right)$$

The eigenvalue of $\hat{a}^\dagger\hat{a}$ is integer n, known as the Landau level index. The eigenenergy

$$E_n = \frac{P_z^2}{2m} + \hbar\omega_c\left(n + \frac{1}{2}\right)$$

5 Semiclassical transport theory in metals

5.1 Electrical and heat current

In the framework of the relaxation time approximation, the electron distribution function in the presence of E and ∇T reads (see Ashcroft & Mermin)

$$g(\boldsymbol{k}) = f[\epsilon(\boldsymbol{k})] + \tau[\epsilon(\boldsymbol{k})]\left(-\frac{\partial f}{\partial \epsilon}\right)\boldsymbol{v}\cdot\left[-e\cdot\boldsymbol{E} + \frac{\epsilon(\boldsymbol{k}) - \mu}{T}(-\nabla T)\right]$$

where f is the equilibrium Fermi-Dirac distribution. The energy and particle currents read:

$$\begin{bmatrix} \boldsymbol{j}^\epsilon \\ \boldsymbol{j}^m \end{bmatrix} = \sum_n \int \frac{d\boldsymbol{k}}{4\pi^3}\begin{bmatrix} \epsilon_n(\boldsymbol{k}) \\ 1 \end{bmatrix} v_n(\boldsymbol{k})\, g_n(\boldsymbol{k})$$

where we have considered all energy bands. Make use of $TdS = dU - \mu dN$, the heat and electrical currents respectively read

$$\boldsymbol{j}^{\epsilon} = -e\boldsymbol{j}^m$$

$$\boldsymbol{j}^q = \boldsymbol{j}^{\epsilon} - \mu \boldsymbol{j}^m$$

which also enclose contribution from all energy bands.

5.2 Electrical and heat current continued

With $g_n(\boldsymbol{k})$ plugged in, the two currents are rewritten as

$$\boldsymbol{j}^e = \boldsymbol{L_{11}E} + \boldsymbol{L_{12}}(-\boldsymbol{\nabla T})$$

$$\boldsymbol{j}^q = \boldsymbol{L_{21}E} + \boldsymbol{L_{22}}(-\boldsymbol{\nabla T})$$

The tensor coefficients are $\boldsymbol{L_{11}} = \boldsymbol{\sigma}$, $\boldsymbol{L}_{21} = T\boldsymbol{L_{12}} = -\frac{\pi^2}{3e}(k_B T)^2\boldsymbol{\sigma'}$,and $\boldsymbol{L_{22}} = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\boldsymbol{\sigma}$

5.3 Wiededmann-Franz law

Thermal conductivity is defined as the proportionality coefficient between $\boldsymbol{j}^q$ and $-\nabla T$

$$\kappa = \frac{\pi^2}{3}\frac{k_B^2}{e^2}T\boldsymbol{\sigma}$$

5.4 Seebeck coefficient

The seebeck effect is the electromotive force that develops across two points of an electrically conducting material when there is a temperature difference between them. The Seebeck coefficient reads

$$\boldsymbol{S} = \boldsymbol{L}_{11}^{-1}\boldsymbol{L}_{12} \Leftarrow E = \boldsymbol{S}\nabla T$$

5.5 Peltier effect

If an electric current is driven in a bimetallic circuit that is maintained at a uniform temperature, then heat will be evolved at one junction and absorbed at the other. This is because an isothermal electric current in a metal is accompanied by a thermal current,

$$\boldsymbol{j}^e = \boldsymbol{L_{11}E} \quad \boldsymbol{j}^q = \boldsymbol{L_{21}E} \quad \Longrightarrow j^q = \boldsymbol{L_{21}L_{11}^{-1}}\boldsymbol{j}^e$$

The Peltier coefficient reads

$$\boldsymbol{\Pi} = \boldsymbol{L_{21}L_{11}^{-1}}$$

5.6 Peltier effect in superconductors

The Boltzmann formalism applies to metals. In superconductors, the (super) current is driven by Cooper pairs, which do not conduct heat.

$$TdS = dU - \mu \mathrm{dN}$$

For $T = 0$, all electrons are paired. For $T < T_c$,

$$\boldsymbol{j} = \boldsymbol{j_n} + \boldsymbol{j_s}$$