Homogenous Functions

- For any scalar k, a real valued function $f(x_1, ..x_n)$ is homogenous of degree k if

$$f(tx_1, ...tx_n) = t^k f(x_1, ...x_n) \quad \text{for all } t > 0$$

  - $k = 1$, Constant returns to scale (Linear Homogenous)
  - $k < 1$, Decreasing returns to scale
  - $k > 1$, Increasing returns to scale

Properties of Homogenous Functions

- Theorem: If f is homogenous of degree k,its first order partial derivatives are homogenous of degree $k-1$

  Proof: $f(tx_1, ...tx_n) = t^k f(x_1, ...x_n)$.

  Take partials wrt to $x_1...x_n$.By the chain rule

  $t\frac{\partial f}{\partial x_1}(tx_1, ...tx_n) = t^k \frac{\partial f}{\partial x_1}(x_1, ...x_n)$

  divide both sides by t and get

  $\boxed{\frac{\partial f}{\partial x_1}(tx_1, ...tx_n) = t^{k-1} \frac{\partial f}{\partial x_1}(x_1, ...x_n)}$

Level sets of Homogenous Functions

- If $q = f(x)$ is a homogenous function then the level sets of f (isoquants in the case of a production function) have constant slope along each ray from the origin.

  Proof:

  – Bundles (L_0, K_0) and $(L_1, K_1) = (tL_0, tK_0)$ lie along same ray. $MRTS$ at (L_1, K_1) is $\frac{f_L(L_1,K_1)}{f_K(L_1,K_1)}$

  but $\frac{f_L(L_1,K_1)}{f_K(L_1,K_1)} = \frac{f_L(tL_0,tK_0)}{f_K(tL_0,tK_0)} = \frac{t^{k-1}f_L(L_0,K_0)}{t^{k-1}f_K(L_0,K_0)} = \frac{f_L(L_0,K_0)}{f_K(L_0,K_0)}$

  the $MRTS$ at (L_0, K_0)

Homothetic Functions

- A function v is homothetic if it is a monotone transformation of a homogenous function.

- Every homothetic function v can be written $v(x)=g(u(x))$ where $u(x)$ is homogenous and g is a monotonic transformation.

  Since 1 is a monotone transformation, every homogenous function is homothetic (converse not always true).

- Slopes of level curves of Homothetic functions are constant along rays from the origin.

  This was a property of homogenous functions but this extends the property to a broader class of functions.

Homothetic Functions level sets (proof)

- Slopes of level curves of Homothetic functions are constant along rays from the origin.

- G is homothetic so can be written $G(K,L) = g(F(K,L))$ where g is a monotonic function and $F(K,L)$ is homogenous of degree k

- $G_K(tK,tL) = tg'(F)F_K(tK,tL)) = g'(F)t^k F_K(K,L))$

- And hence $\frac{G_K(tK,tL)}{G_L(tK,tL)} = \frac{F_K(K,L)}{F_L(K,L)} = \frac{G_K(K,L)}{G_L(K,L)}$

Euler's Theorem

- If $f(\mathsf{x}) = f(x_1 ... x_n)$ is a homogenous function of degree k then

$$x_1 f_1(\mathsf{x}) + x_2 f_2(\mathsf{x}) + ... + x_n f_n(\mathsf{x}) = k f(x_1 ... x_n)$$

  Proof: differentiate both sides of the definition of a homogenous function, $f(tx_1, ... tx_n) = t^k f(x_1, ... x_n)$ with respect to t to get:

$$x_1 f_1(t\mathsf{x}) + ... + x_n f_n(t\mathsf{x}) = k t^{k-1} f(x_1 ... x_n)$$

  Then set $t = 1$ to get the result.

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Euler's Theorem Application (CRS)

- If $f(K, L)$ is a homogenous of degree 1 (CRS) then "Factor Payments exhaust the value of output" and the firm earns zero profits.

- To see this note that Euler's Theorem states that for all K and L

$$F(K, L) = F_K K + F_L L$$

  But at a cost-minimization optimum $P \cdot F_K(K^*, L^*) = r$ and $P \cdot F_L(K^*, L^*) = w$. Multiply both sides above by P and use the FOCs to obtain:

$$\begin{aligned} PF(K^*, L^*) &= PF_K(K^*, L^*)K^* + PF_L(K^*, L^*)L^* \\ PF(K^*, L^*) &= rK^* + wL^* \end{aligned}$$

Euler's Theorem Application (DRS)

- If $f(K, L)$ is a homogenous of degree $k < 1$ (DRS) then "Factor Payments do NOT exhaust the value of output" and the firm earns positive profits.

- Euler's Theorem states that for all K and L

$$kF(K, L) = F_K K + F_L L$$

$$\begin{aligned} kPF(K^*, L^*) &= rK^* + wL^* \\ \Pi^* &= (1-k)PF(K^*, L^*) \end{aligned}$$

Euler's Theorem Application (DRS CD example)

- Cobb-Douglas $F(K, L) = AK^{\alpha}L^{\beta}$ and $\alpha + \beta < 1$

$$\Pi^* = (1 - k)PF(K^*, L^*)$$

  - But note that if we think of $A = S^{1-\alpha-\beta}$ then we can write $G(S, K, L) = S^{1-\alpha-\beta}K^{\alpha}L^{\beta}$ which is hom. of deg 1 (CRS in S, K, L)
  - And profit of the firm can be thought of as payment to factor S
  - $PG(S, K, L) = G_S S + G_L L + G_K K$.
  - $PAF(K, L) = G_S S + F_L L^* + F_K K^*$
  - So $G_S S = \Pi^*$

Properties of Homogeneous Production Functions:

- – (1) If ratio K/L is optimal for $q = 1$ then also optimal for any q

  (2) Optimal input choice L or K will be a linear function of cost.

  (3) cost function will be homogenous of degree $1/k$

(2) Optimal input choice L or K will be a linear function of cost.

– Proofs: Homotheticity implies we can write $K(C) = a_K C$ and $L(C) = a_L C$

(e.g. suppose you spend $C = 10$ of which $a_K 10 = \$6$ spent on capital and $a_L 10 = \$4$ spent on labor. Now double cost expenditure to $C = 20$. Since K and L are used in same proportions you will now spend $a_K 20 = \$12$ on capital and $a_L 20 = \$8$ on labor)

This means can write:

$$q(C) = f(L(C), K(C)) = f(a_K C, a_L C) = C^k f(a_K, a_L) = C^k a^*.$$

– So $C(q) = bq^{1/k}$ where $b = (a^*)^{-1/k}$ (proving (3))

– $k = 1$ (CRS) then $C(q) = bq$ (straight line, constant MC and AC)

– $k = 2$ (IRS) then $C(q) = bq^{1/2}$ (falling MC and AC)

Cobb-Douglas Cost Function

$q = F(K, L) = K^{\alpha} L^{\beta}$

- From FOC: $\frac{w}{r} = \frac{\beta}{\alpha}\frac{K}{L}$ and so $K = \frac{\alpha}{\beta}\frac{w}{r}L.$
- $q = (\frac{\alpha}{\beta}\frac{w}{r}L)^{\alpha} L^{\beta} \rightarrow L = q^{1/(\alpha+\beta)} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} w^{-\alpha/(\alpha+\beta)} r^{\alpha/(\alpha+\beta)}$
- $K = q^{1/(\alpha+\beta)} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} w^{\beta/(\alpha+\beta)} r^{-\beta/(\alpha+\beta)}$
- $C(w, r, q) = wK + rL$
- $\boxed{C(w, r, q) = q^{1/(\alpha+\beta)} B r^{\alpha/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}}$

  where $B = (\alpha + \beta)\alpha^{-\alpha/(\alpha+\beta)} \beta^{-\beta/(\alpha+\beta)}$
- As predicted has form $C(q) = bq^{1/k}$