MATH 210 Assignment 3

List Comprehensions and More Logic, Loops and Functions

INSTRUCTIONS

- Create a new Jupyter notebook and set the kernel to Python 3
- Present your solutions in a single Jupyter notebook and clearly label the solutions
- Your solutions should include clear explanations (including proper use of markdown language and LaTeX) and your functions should include comments
- There are 25 total points: each question is worth 4 points and 1 point will be awarded for the overall presentation of your notebook
- Each question is graded out of 4 points according to the rubric:
  - 4 - Solution is correct and written clearly (including comments if needed)
  - 3 - Solution is correct but is unclear
  - 2 - Solution is partly correct
  - 1 - Solution needs improvement
- Submit the `.ipynb` file to Connect by **11pm Friday January 29**
- You may work on these problems with others but you must write your solutions on your own

QUESTIONS

1. Use list comprehensions to create:
   (a) `[-1,3,-5,7,-9,11,-13,15,-17,19,-21,23,-25,27,-29,31,-33,35,-37,39,-41]`
   (b) `[[1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12], [13, 14, 15, 16]]`
2. Use list comprehensions and the `sum` function to:
   (a) Evaluate the 100th partial sum of the alternating series
   $$\sum_{n=1}^{100} \frac{(-1)^n}{n}$$
   (b) Evaluate the sum of divisors function $\sigma_2(3245)$ for $k = 2$ and $n = 3245$ where
   $$\sigma_k(n) = \sum_{d|n} d^k$$
   is the sum over all positive divisors of n.

3. Define a function called `sum_of_squares` which takes a positive integer n and returns the number of ways n can be expressed as the sum of two squares $n = a^2 + b^2$ with $1 \leq a \leq b$. For example:

   - ⋄ if $n = 2$, then `sum_of_squares` returns 1 since $2 = 1^2 + 1^2$
   - ⋄ if $n = 3$, then `sum_of_squares` returns 0 since 3 is not the sum of two squares
   - ⋄ if $n = 5$, then `sum_of_squares` returns 1 since $5 = 1^2 + 2^2$
   - ⋄ if $n = 85$, then `sum_of_squares` returns 2 since $85 = 2^2 + 9^2$ and $85 = 6^2 + 7^2$

   (Hint: Define `sum_of_squares` with a list comprehension and the `count` list method.)

4. Use the function `sum_of_squares` from the previous question to find:

   (a) the smallest integer that can be expressed as the sum of two squares in two ways

   (b) the smallest integer that can be expressed as the sum of two squares in three ways

   (c) the smallest integer that can be expressed as the sum of two squares in four ways

   In all cases (a), (b) and (c), find all of the representations $n = a^2 + b^2$ of the given n.

5. (a) Write the following theorem (including LaTeX to display the formula) in a markdown cell in your notebook:

   Let n be a positive integer and let $w(n)$ be the number of ways n can be expressed as the sum of two squares $n = a^2 + b^2$ with $1 \leq a \leq b$. Then

   $$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} w(n) = \frac{\pi}{8}$$

   In other words, the average number of representations of an integer n by a sum of two squares is $\pi/8$.

   (b) Use the function `sum_of_squares` from the previous questions to approximate the limit by evaluating the expression $\frac{1}{N} \sum_{n=1}^{N} w(n)$ at $N = 2000$. Compare your result to the true value $\pi/8$.

6. Define a function called `divisor_sort` which takes a list of positive integers and sorts them by the number of divisors (from least number of divisors to most), and integers with the same number of divisors are ordered in increasing order. For example, `divisor_sort([49,7,15,21,31,5])` should return the list `[5,7,31,49,15,21]`. (Warning: this is a challenging problem.)