Lake Model: Workers' Dynamics and Markov Process

Quantitative Economics with Python

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Overview

- A worker's employment dynamics are governed by a *Markov process*
- The worker can be in one of two states:
  - $s = 0$ means that the worker is unemployed
  - $s = 1$ means that the worker is employed
- The transition matrix between the two states

$$P_{ij} = \text{Prob}(s_{t+1} = j | s_t = i)$$

and

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- P_{ij}: probability for worker moves from state i at t to state j at $t+1$

Questions

- What is the average fraction of time a worker being employed over time?
- What is the fraction of workers who are employed at a particular time?
- What is the link between the answers to the above questions?

To answer these questions, we need some knowledge about Markov chain.

Finite Markov Chains

- Stochastic Matrix and Markov Chain
  - Definition
  - Irreducibility
  - Aperiodicity
- Marginal Distribution
  - Definition
  - Evolution of marginal distribution
  - Stationary marginal distribution
  - Unique stationary distribution
- Ergodicity

Stochastic Matrix (Markov Matrix)

- Let $S = \{s_1, s_2, \ldots, s_n\}$
- A stochastic matrix is an $n \times n$ square matrix $P = P[s, s']$ such that
  - each element $P[s, s']$ is nonnegative, and
  - each row $P[s, \cdot]$ sums to one
- Each row $P[s, \cdot]$ can be regarded as a distribution on S
- Remark: if P is a stochastic matrix, so is k-th power P^k for all $k \in \mathbb{N}$

Stochastic Matrix: Example

- Let's look at the worker's employment dynamics example

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- All elements are positive
- Each row sums to one

Stochastic Matrix: Example

- Stochastic matrix about U.S. economy state (monthly frequency)

$$P = \begin{pmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{pmatrix}$$

  - the first state represents "normal growth"
  - the second state represents "mild recession"
  - the third state represents "severe recession"

- For example, when the state is normal growth, the state will again be normal growth next month with probability 0.97

- Large values on the main diagonal indicate persistence in the process

Stochastic Matrix: Multiple Step Transition Probability

- The probability of transitioning from s to s' in m step: $P^m[s, s']$.
- A two state example

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

- Probability of moving from state i to state j in one period is p_{ij}
- Probability of moving from state 1 to state 1 in two periods is

$$\begin{pmatrix} p_{11}p_{11} + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{21}p_{12} + p_{22}p_{22} \end{pmatrix} = P \times P = P^2$$

Markov Chain

- A Markov Chain $\{X_t\}$ is a stochastic process that has the Markov property

$$\mathbb{P}\{X_{t+1} = s' \mid X_t\} = \mathbb{P}\{X_{t+1} = s' \mid X_t, X_{t-1}, \ldots\}$$

- Knowing current state is enough to understand probabilities for future states

- The dynamics of a Markov chain are fully determined by the set of values

$$P[s, s'] = \mathbb{P}\{X_{t+1} = s' \mid X_t = s\} \qquad (s, s' \in S)$$

  - $P[s, s']$ is the probability of going from s to s' in one unit of time (one step)
  - $P[s, \cdot]$ is the conditional distribution of X_{t+1} given $X_t = s$

- It's clear that P is a stochastic matrix

Markov Chain

- With a stochastic matrix P, we can generate a Markov chain $\{X_t\}$
  - draw X_0 from some specified distribution
  - draw X_{t+1} from $P[X_t, \cdot]$
- Let us move to Python to generate some...

Irreducibility

- Let P be a fixed stochastic matrix
- Two states s and s are said to communicate with each other if there exist positive integers j and k such that

$$P^j[s, s'] > 0 \quad \text{and} \quad P^k[s', s] > 0$$

- This means that
  - state s can be reached eventually from state s'
  - state s' can be reached eventually from state s
- The stochastic matrix P is called irreducible if all states communicate

Irreducibility

- Worker's employment dynamics example is irreducible

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- Suppose worker has probability d to die

$$P = \begin{pmatrix} (1-d)(1-\alpha) & (1-d)\alpha & d \\ (1-d)\lambda & (1-d)(1-\lambda) & d \\ 0 & 0 & 1 \end{pmatrix}$$

This is not irreducible. Death is an absorbing state.

Aperidocity

- Loosely speaking, a Markov chain is called periodic if it cycles in a predictible way and aperiodic otherwise

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The period of a state s is the greatest common divisor of the set of integers

$$D(s) := \{j \geq 1 : P^j[s, s] > 0\}$$

- A stochastic matrix is called aperiodic if the period of every state is 1, and periodic otherwise

Aperidocity

- Consider the following example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

All the states have period 2, which is a periodic Markov chain.

- Consider the following example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

All the states have period 1, which is a aperiodic Markov chain.

Marginal Distribution

- Suppose that
  - $\{X_t\}$ is a Markov chain with stochastic matrix P
  - the distribution of X_t is known to be ψ_t
- What is the distribution of X_{t+1}, or, more generally, of X_{t+m}?

Marginal Distribution

- Let ψ_t be the distribution of X_t . We are looking for ψ_{t+1} given ψ_t and P

- To begin, pick any $s' \in S$. The probability that $X_{t+1} = s'$ is:

$$\mathbb{P}\{X_{t+1} = s'\} = \sum_{s \in S} \mathbb{P}\{X_{t+1} = s' \mid X_t = s\} \cdot \mathbb{P}\{X_t = s\}$$

- We account for all ways this can happen and sum their probabilities.

- More compactly

$$\psi_{t+1}[s'] = \sum_{s \in S} P[s, s'] \psi_t[s]$$

- There are n such equations, and the matrix expression is

$$\psi_{t+1} = \psi_t P$$

Marginal Distribution

- To move the distribution forward one unit of time, we postmultiply by P
- By repeating this m times we move forward m steps into the future
- Hence, $\psi_{t+m} = \psi_t P^m$ is also valid
- If ψ_0 is the initial distribution from which X_0 is drawn, then $\psi_0 P^m$ is the distribution of X_m

$$X_0 \sim \psi_0 \quad \Longrightarrow \quad X_m \sim \psi_0 P^m$$

$$X_t \sim \psi_t \quad \Longrightarrow \quad X_{t+m} \sim \psi_t P^m$$

Stationary Distribution

- A distribution ψ^* is called stationary for P if $\psi^* = \psi^* P$

Theorem

Every stochastic matrix P has at least one stationary distribution

Theorem

If P is irreducible and aperiodic, then

1. *P has a unique stationary distribution*
2. *For any initial distribution ψ_0, $\|\psi_0 P^t - \psi^*\| \to 0$ as $t \to \infty$*

Ergodicity

- Under irreducibility, for all $s \in S$,

$$\frac{1}{n} \sum_{t=1}^{n} \mathbf{1}\{X_t = s\} \to \psi^*[s] \quad \text{as } n \to \infty$$

  - $\mathbf{1}\{X_t = s\} = 14$ if $X_t = s$ and zero otherwise
  - convergence is with probability one
  - the result does not depend on the distribution (or value) of X_0

- The fraction of time the chain spends at state s converges to $\psi^*[s]$ as time goes to infinity