# April 14

Thursday, April 14, 2016 9:00 AM

$$p_t = E_t[m_{t+1}(d_{t+1} + p_{t+1})]$$

$\{d_t\}$ - dividends, $\{p_t\}$

$m_{t+1}$ = stochastic discount factor

$$p_t = E_t m_{t+1} E_t(d_{t+1} + p_{t+1}) + \underline{\underline{cov_t}}(m_{t+1}, d_{t+1} + p_{t+1})$$

$\beta$

Markov setting

$S = [s_1, s_2, \ldots, s_n]$ $n$ states

$P = \text{Prob}\{\lambda_{t+1} = s_j \mid \lambda_t = s_i\}$

$\lambda_t \in S$.

examples: $m_t$ — depend only on $s_t$

$d_t$ ~ depend only on $s_t$

$p_t$ ~ dependent only on $s_t$

<u>no bubbles</u>

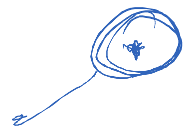
$$m_t \longrightarrow \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}, \quad m_i \equiv m_t \text{ when } s_t = s_i.$$

$$m_t \rightarrow \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}, \quad d_t \rightarrow \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}, \quad \lambda_t \rightarrow \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$\begin{pmatrix} \vdots \\ m_n \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ d_n \end{pmatrix}, \quad r_t \quad \begin{pmatrix} \vdots \\ r_n \end{pmatrix}$$

$$\pi_t = \begin{pmatrix} \pi_{1,t} \\ \pi_{2,t} \\ \vdots \\ \pi_{n,t} \end{pmatrix}, \quad \pi_0 = \begin{pmatrix} \pi_{1,0} \\ \vdots \\ \pi_{n,0} \end{pmatrix}$$

$$\pi_{j,t+1} = \sum_{i=1}^{n} P_{i,j}\, \pi_{i,t}$$



$$* \quad \pi'_{t+1} = P\pi'_t \quad \text{or} \quad \pi_{t+1} = P'\pi_t$$

verify

$$\text{Prob}\{m_{t+1} = m_j \mid s_t = s_i\} = P_{i,j}$$

What does

$E_t\, x_{t+j}$ mean & how compute it.

$$E_t\, x_{t+1} = \sum_{j=1}^{n} P_{ij}\, x_j$$

↑ a vector

whose $i$ th component is

$$E_t\, x_{t+1} = E[x_{t+1} \mid s_t = s_i], \quad i = 1, 2, \ldots, n$$

$$= \begin{pmatrix} \sum_{j=1}^{n} p_{1j}^{(h)} x_j \\ \sum_{j=1}^{n} p_{2j}^{(h)} x_j \\ \vdots \\ \sum_{j=1} p_{nj}^{(h)} x_j \end{pmatrix} \longleftarrow \text{row today's state.}$$

$P = \begin{matrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p & & & \end{matrix}$

$p_{ij}^{(h)} = ij$ th element of the $h$th power of $P$

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Examples: let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

let $\overset{n \times n}{P} = n \times n$ stochastic matrix

let $P^h$ be the $h$ ~~th~~ power of
$P^h = P P P \cdots P$

tell me what the following $n \times n$ vector means

$\underset{n \times n}{P} \underset{n \times 1}{X}$ — $i$th row is the mathematical expectation of $X_{t+1}$ conditional on $S_t = S_i$

$P^h X$ $\quad$ $i$th row of - - - - -

$x_{t+h}$

$$\underbrace{\overset{n\times n}{\left(I + \beta P + \beta^2 P^2 + \cdots\right)}} X$$

$\beta$ - scalar $\quad 0 < \beta < 1$

$$1 + c + c^2 + c^3 + \cdots = \frac{1}{1-c}$$

$$[I - \beta P]^{-1} = I + \beta P + \beta^2 P^2 + \cdots$$

Neuman series

$$\underset{n\times n}{(I - \beta P)^{-1}} \; \underset{n\times 1}{X} \quad \sim i\text{th row}$$

is the mathematical expectation of

$$(X_t + \beta X_{t+1} + \beta X_{t+2} + \cdots).$$

conditional on $s_t$ being $s_i$ at $t$.

$\Pi_t^{n\times 1}$

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Let $X_t$ be a random variable $\longrightarrow$ $X^{n\times 1}$

$y_t$ be a random variable $\qquad y^{n\times 1}$

Let $p^{n\times 1}$ be the distribution over states

$$p_i \geq 0 \qquad \sum p_i = 1$$

$E_p\, x = \sum_{i=1} p_i x_i$

random variable $\quad x_t\, y_t = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_n & y_n \end{pmatrix}$

$E\, xy = \sum_{i=1}^{n} p_i x_i y_i$

Python $\quad x = \begin{pmatrix} 1 \\ 2 \\ 2.5 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ $\quad$ math

$z = xy$

Python $\quad$ x = np.array((1, 2, 2.5))

y = np.array((2, 3, 0))

Want $\quad z = \begin{pmatrix} 1 \cdot 2 \\ 2 \cdot 3 \\ 2.5 \cdot 0 \end{pmatrix} \qquad z_i = x_i \cdot y_i$

$z = x * y$

want $E\, z$ $\quad$ where $E$ is w.r.t. $p$

$\sum_i p_i x_i y_i \qquad E\, xy$

Python: " sum( p * x * y ) — $E$

Conditional expectation

$i^{th}$ row of some $P$

or $P^h$

$$E_t\, m_{t+1} d_{t+1} = \text{sum}\left(P_{i,\cdot} * m * d\right)$$

$n \times 1$

$m$

$n \times 1$

$P_{i,\cdot}$

$i^{th}$ row of $P$

$$\text{sum}\left(P\, @\, (m * d)\right)$$

probability distribution

state space, $S = \{s_1, \ldots, s_n\}$

$\pi_1, \ldots, \pi_n$

$p_{11} \cdots p_{1n}$

$\vdots$

$p_{n1} \cdots p_{nn}$

random variable — vector

random variable

$E - \text{sum}(\pi * x)$

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We are acting as if we know

- the probabilities
- the random variables as functions of the state.

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Apply: - risk neutral

$$p_t = \beta E_t [d_{t+1} + p_{t+1}] \qquad (*)$$

$$d_{t+1} = \lambda_{t+1} d_t ,$$

$\lambda_{t+1} \sim$ Markov on state space $(s_1, s_2, \dots, s_n)$

$P$ - transition matrix

Problem: to solve $(*)$ for something that tells me $p_t$ as a function of stuff today.

Guess: $p_t = v(\lambda_t) d_t$ or

$$\frac{p_t}{d_t} = v(\lambda_t)$$

$d_{t+1} = \lambda_t d_t$

$p_t = \beta E_t [d_{t+1} + p_{t+1}]$

$$p_t = \beta E_t \left[ d_{t+1} + p_{t+1} \right]$$

$$v(\lambda_t)\cancel{d_t} = \beta E_t \left[ \lambda_{t+1} \cancel{d_t} + v(\lambda_{t+1}) \lambda_{t+1} \cancel{d_t} \right]$$

Cancel & Stare