OLG with Production

Assumptions

- Timing:
  - $A_{t-1}, L_{t-1}, K_{t-1}$ are the technology stock, labour force and capital stock at the end of period $t-1$. As such they will be used in production in period t.
  - This means that in period t, the population of young agents is L_{t-1} and the population of old agents is L_{t-2}.
  - $Y_t, C_t^Y, C_t^O, I_t, S_t$ are the flows of output, consumption, investment, and savings in period t.
  - $W_t L_{t-1}, r_t K_{t-1}$ are the flows of payment to labour and capital in period t.

Aggegrate Equations

- Exogenous population and technology processes (where L_{t-1} is population of young agents in period t):

$$\begin{aligned} A_t &= (1+g)\,A_{t-1} \\ L_t &= (1+n)\,L_{t-1} \end{aligned}$$

- Expenditure in period t

$$Y_t = C_t^Y + C_t^O + I_t$$

- Income in period t

$$Y_t = W_t L_{t-1} + r_t K_{t-1}$$

- Equation of motion for capital

$$K_t = (1-\delta)\,K_{t-1} + I_t$$

- Production function

$$Y_t = A_{t-1} L_{t-1} \left(\frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^{\alpha}$$

- Interest rate

$$r_t = \frac{\partial Y_t}{\partial K_{t-1}} = \alpha \left(\frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^{\alpha - 1}$$

- Wage rate

$$W_t = \frac{\partial Y_t}{\partial L_{t-1}} = A_{t-1}\,(1-\alpha) \left(\frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^{\alpha}$$

- Old spend their savings

$$C_t^O = (1-\delta)\,K_{t-1} + r_t K_{t-1}$$

- Young spend and save their wages

$$C_t^Y + S_t = W_t L_{t-1}$$

Intensive form

- Let
  -
$$\begin{aligned}
C_t^Y &= A_{t-1} L_{t-1} c_t^Y \\
C_t^O &= A_{t-2} L_{t-2} c_t^O \\
I_t &= A_{t-1} L_{t-1} i_t \\
K_t &= A_t L_t k_t \\
Y_t &= A_{t-1} L_{t-1} y_t \\
S_t &= A_{t-1} L_{t-1} s_t \\
W_t &= A_{t-1} w_t
\end{aligned}$$

  - Then

$$\begin{aligned}
y_t &= c_t^Y + \frac{1}{(1+g)\,(1+n)} c_t^O + i_t \\
y_t &= w_t + r_t k_{t-1} \\
k_t &= \frac{1}{(1+g)\,(1+n)} \left[(1-\delta)\,k_{t-1} + i_t\right] \\
r_t &= \alpha k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha)\,k_{t-1}^{\alpha} \\
\frac{1}{(1+g)\,(1+n)} c_t^O &= (1 + r_t - \delta)\,k_{t-1} \\
c_t^Y + s_t &= w_t
\end{aligned}$$

Maximisation Problem

- Solve the young agents' consumption/savings decision problem:

$$U_t = \frac{\left(A_{t-1} c_t^Y\right)^{1-\theta} - 1}{1-\theta} + \beta \frac{\left(A_{t-1} c_{t+1}^O\right)^{1-\theta} - 1}{1-\theta}$$

- Use

$$c_t^Y = w_t - s_t$$

- And

$$\begin{aligned}
k_t &= \frac{1}{(1+g)(1+n)}\left[(1-\delta)k_{t-1}+i_t\right] \\
&= \frac{1}{(1+g)(1+n)}\left[(1-\delta)k_{t-1}+y_t-c_t^Y-\frac{1}{(1+g)(1+n)}c_t^O\right] \\
&= \frac{1}{(1+g)(1+n)}\left[(1-\delta)k_{t-1}+k_{t-1}^{\alpha}-c_t^Y-(1+r_t-\delta)k_{t-1}\right] \\
&= \frac{1}{(1+g)(1+n)}\left[k_{t-1}^{\alpha}-c_t^Y-r_t k_{t-1}\right] \\
&= \frac{1}{(1+g)(1+n)}\left[(1-\alpha)k_{t-1}^{\alpha}-c_t^Y\right] \\
&= \frac{1}{(1+g)(1+n)}\left[w_t-c_t^Y\right]=\frac{s_t}{(1+g)(1+n)}
\end{aligned}$$

i.e.

$$\frac{1}{(1+g)(1+n)}c_{t+1}^O=(1+r_{t+1}-\delta)k_t=(1+r_{t+1}-\delta)\frac{s_t}{(1+g)(1+n)}$$

So

$$c_{t+1}^O=(1+r_{t+1}-\delta)s_t$$

- Therefore write maximisation problem as function of s_t

$$U_t=\frac{(A_{t-1}w_t-A_{t-1}s_t)^{1-\theta}-1}{1-\theta}+\beta\frac{(A_{t-1}(1+r_{t+1}-\delta)s_t)^{1-\theta}-1}{1-\theta}$$

- Solving:

$$\frac{\partial U_t}{\partial s_t}=0\Longrightarrow$$

$$s_t=\frac{w_t}{1+\beta^{-\frac{1}{\theta}}(1+r_{t+1}-\delta)^{\frac{\theta-1}{\theta}}}$$

Problem fully specified

- Given initial state variable k_{t-1}, we have 8 equations (below) in 8 unknowns: $y_t, c_t^Y, c_t^O, i_t, s_t, k_t, r_t, w_t$ (NB r_{t+1} is a function of k_t which is already on this list):

$$
\begin{aligned}
y_t &= c_t^Y + \frac{1}{(1+g)(1+n)} c_t^O + i_t \\
y_t &= w_t + r_t k_{t-1} \\
k_t &= \frac{1}{(1+g)(1+n)} \left[(1-\delta) k_{t-1} + i_t \right] \\
r_t &= \alpha k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) k_{t-1}^{\alpha} \\
c_t^O &= (1+g)(1+n)(1+r_t-\delta) k_{t-1} \\
w_t &= c_t^Y + s_t \\
s_t &= \frac{w_t}{1+\beta^{-\frac{1}{\theta}} (1+r_{t+1}-\delta)^{\frac{\theta-1}{\theta}}}
\end{aligned}
$$

Steady state

- Reduce system to just the 3rd and 8th equations in only k_t and c_t^Y (k_{t-1} is known):

$$
\begin{aligned}
k_t &= \frac{1}{(1+g)(1+n)} \left[(1-\delta) k_{t-1} + i_t \right] \\
&= \frac{1}{(1+g)(1+n)} \left[(1-\delta) k_{t-1} + y_t - c_t^Y - \frac{1}{(1+g)(1+n)} c_t^O \right] \\
&= \frac{1}{(1+g)(1+n)} \left[(1+r_t-\delta) k_{t-1} + w_t - c_t^Y - \frac{1}{(1+g)(1+n)} c_t^O \right] \\
&= \frac{1}{(1+g)(1+n)} \left[(1+r_t-\delta) k_{t-1} + (1-\alpha) k_{t-1}^{\alpha} - c_t^Y - (1+r_t-\delta) k_{t-1} \right] \\
&= \frac{1}{(1+g)(1+n)} \left[(1-\alpha) k_{t-1}^{\alpha} - c_t^Y \right]
\end{aligned}
$$

$$
\begin{aligned}
s_t &= \frac{w_t}{1+\beta^{-\frac{1}{\theta}} (1+r_{t+1}-\delta)^{\frac{\theta-1}{\theta}}} \\
w_t - c_t^Y &= \frac{w_t}{1+\beta^{-\frac{1}{\theta}} \left(1+\alpha k_t^{\alpha-1}-\delta\right)^{\frac{\theta-1}{\theta}}} \\
(1-\alpha) k_{t-1}^{\alpha} - c_t^Y &= \frac{(1-\alpha) k_{t-1}^{\alpha}}{1+\beta^{-\frac{1}{\theta}} \left(1+\alpha k_t^{\alpha-1}-\delta\right)^{\frac{\theta-1}{\theta}}}
\end{aligned}
$$

- The reduced system is two equations in 2 unknowns: given initial condition k_{t-1}, there are two processes that relate the choice of c_t^Y and k_t: one from the equation of motion for capital and one from the agent's optimisation.

$$\begin{aligned} k_t &= \frac{1}{(1+g)(1+n)}\left[(1-\alpha)k_{t-1}^{\alpha} - c_t^Y\right] \\ (1-\alpha)k_{t-1}^{\alpha} - c_t^Y &= \frac{(1-\alpha)k_{t-1}^{\alpha}}{1+\beta^{-\frac{1}{\theta}}\left(1+\alpha k_t^{\alpha-1}-\delta\right)^{\frac{\theta-1}{\theta}}} \end{aligned}$$

There is a unique choice, c_t^Y, that satisfies both equations. This is the OLG model in its pure form: all the other equations basically just provide definitions for the other quantities in the model.

- Combine these equations to eliminate c_t^Y:

$$k_t = \frac{1}{(1+g)(1+n)}\left[\frac{(1-\alpha)k_{t-1}^{\alpha}}{1+\beta^{-\frac{1}{\theta}}\left(1+\alpha k_t^{\alpha-1}-\delta\right)^{\frac{\theta-1}{\theta}}}\right]$$

- Steady state intensive capital stock is the solution to:

$$k = \frac{1}{(1+g)(1+n)}\left[\frac{(1-\alpha)k^{\alpha}}{1+\beta^{-\frac{1}{\theta}}\left(1+\alpha k^{\alpha-1}-\delta\right)^{\frac{\theta-1}{\theta}}}\right]$$

No analytic solution for $\theta \neq 1$.Need to use numerical method.

- Other steady state variables:

$$\begin{aligned} c^Y &= (1-\alpha)k^{\alpha} - (1+g)(1+n)k \\ r &= \alpha k^{\alpha-1} \\ w &= (1-\alpha)k^{\alpha} \\ s &= w - c^Y \\ c^O &= (1+g)(1+n)(1+r-\delta)k \\ y &= w + rk \\ i &= y - c^Y - \frac{1}{(1+g)(1+n)}c^O \end{aligned}$$