GSERM - St. Gallen 2019
Introduction to Survival Data

January 19, 2019 (afternoon session)

- Models for *time-to-event data*.
- Roots in biostats/epidemiology, plus engineering, sociology, economics.
- Examples...
  - Political careers, confirmation durations, position-taking, bill cosponsorship, campaign contributions, policy innovation/adoption, etc.
  - Cabinet/government durations, length of civil wars, coalition durability, etc.
  - War duration, peace duration, alliance longevity, length of trade agreements, etc.
  - Strike durations, work careers (including promotions, firings, etc.), criminal careers, marriage and child-bearing behavior, etc.

Characteristics of Time-To-Event Data

- *Discrete* events (i.e., not continuous),
- Take place over *time*,
- May not (or *never*) experience the event (i.e., possibility of *censoring*).

Survival Data Basics: Terminology

$$
\begin{aligned}
Y_i &= \text{the duration until the event occurs,} \\
Z_i &= \text{the duration until the observation is ``censored''} \\
T_i &= \min\{Y_i, Z_i\}, \\
C_i &= \text{0 if observation } i \text{ is censored, 1 if it is not.}
\end{aligned}
$$

Survival Data Basics: The Density

$$f(t) = \Pr(T_i = t)$$

Issues:

- $T_i = t$ iff $T_i > t - 1$, $t - 2$, etc.
- $C_i = 0$ (censoring)

Survival Data Basics: Survivor Function

$$\Pr(T_i \leq t) \equiv F(t) = \int_0^t f(t)\,dt$$

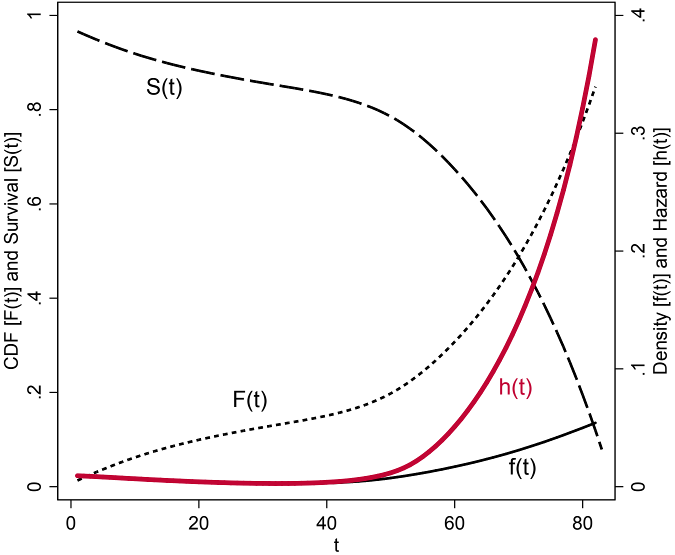
$$\begin{aligned}\Pr(T_i \geq t) \equiv S(t) &= 1 - F(t) \\ &= 1 - \int_0^t f(t)\,dt\end{aligned}$$

Survival Data Basics: The Hazard

$$\Pr(T_i = t | T_i \geq t) \equiv h(t) = \frac{f(t)}{S(t)}$$

$$= \frac{f(t)}{1 - \int_0^t f(t)\, dt}$$

Example: Human Mortality



Some Useful Equivalencies

$$f(t) = \frac{-\partial S(t)}{\partial t}$$

Implies

$$\begin{aligned} h(t) &= \frac{\frac{-\partial S(t)}{\partial t}}{S(t)} \\ &= \frac{-\partial lnS(t)}{\partial t} \end{aligned}$$

More Useful Things: Integrated Hazard

Define

$$H(t) = \int_0^t h(t)\, dt.$$

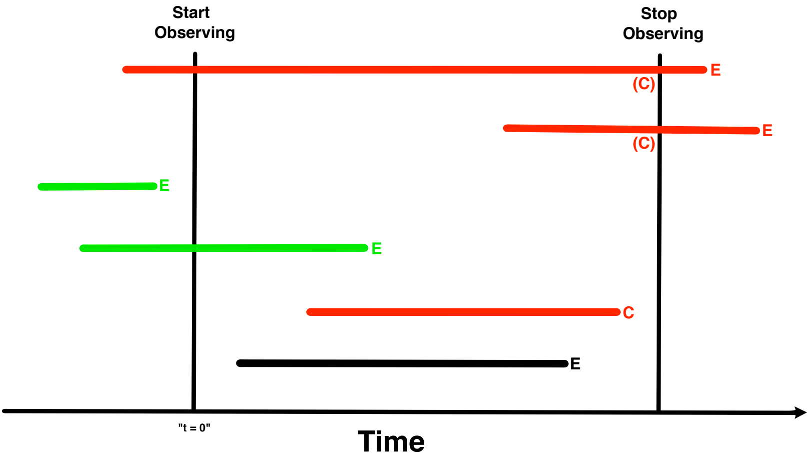
Implies

$$\begin{aligned} H(t) &= \int_0^t \frac{-\partial lnS(t)}{\partial t}\, dt \\ &= -ln[S(t)] \end{aligned}$$

and

$$S(t) = \exp[-H(t)]$$

Censoring and Truncation



Censoring

- Defined by the researcher
- Conditionally independent of both T_i and $\mathbf{X}_i$
- Doesn't mean that the observation provides no information

Estimating $S(t)$

Assume N observations, *absorbing* events, and no ties. Then define

$$
\begin{aligned}
n_t &= \text{number of observations ``at risk'' for the event at } t \text{, and} \\
d_t &= \text{number of observations which experience the event} \\
&\quad\ \text{at time } t.
\end{aligned}
$$

Then

$$\widehat{S(t_k)} = \prod_{t \leq t_k} \frac{n_t - d_t}{n_t}$$

Variance of $\widehat{S(t)}$

$$\text{Var}[\widehat{S(t_k)}] = \left[\widehat{S(t_k)}\right]^2 \sum_{t \le t_k} \frac{d_t}{n_t(n_t - d_t)}$$

Note:

- $\text{Var}[\widehat{S(t_k)}]$ is increasing in $S(t)$,
- is also increasing in d_t, but
- is decreasing in n_t.

Estimating $H(t)$

"Nelson-Aalen":

$$\widehat{H(t_k)} = \sum_{t \leq t_k} \frac{d_t}{n_t}$$

...which gives an alternative estimator for the survival function equal to:

$$\begin{aligned}
\widehat{S(t_k)} &= \exp[-\widehat{H(t_k)}] \\
&= \exp\left[-\sum_{t \leq t_k} \frac{d_t}{n_t}\right]
\end{aligned}$$

	Treatment	Placebo	Total
Event	d_{1t}	d_{0t}	d_t
No Event	$n_{1t} - d_{1t}$	$n_{0t} - d_{0t}$	$n_t - d_t$
Total	n_{1t}	n_{0t}	n_t

Log-Rank Test:

$$
\begin{aligned}
Q &= \frac{\left[\sum(d_{1t} - \frac{n_{1t}d_t}{n_t})\right]^2}{\left[\frac{n_{1t}n_{0t}d_t(n_t - d_t)}{n_t^2(n_t - 1)}\right]} \\
&\sim \chi_1^2
\end{aligned}
$$

A Diversion: Survival Models and Counting Processes

Assume

- Event is *absorbing*,
- Y_i is duration to the event
- Z_i is duration to censoring
- Observe $T_i = \min(Y_i, Z_i)$, and
- C_i:
    - $C_i = 0$ if $T_i = Z_i$,
    - $C_i = 1$ if $T_i = Y_i$.
- $T_i \neq T_j \; \forall \; i \neq j$ (no "ties")

Three Key Variables

1. *Counting Process* Indicator:

$$N_i(t) = I(T_i \leq t, C_i = 1)$$

2. *Risk* Indicator:

$$R_i(t) = I(T_i > t)$$

3. *Intensity Process*:

$$\lambda_i(t)\, dt = R_i(t) h(t)$$

With

$$\Lambda_i(t) = \int_0^t \lambda_i(t)\, dt$$

we can think of

$$N_i(t) = \Lambda_i(t) + M_i(t)$$

or

$$M_i(t) = N_i(t) - \Lambda_i(t).$$

Martingales!

$$E(X_{t+s}|X_0, X_1, ...X_i, ...X_t) = X_t \; \forall \; s > 0$$

Data Structure and Organization: Non-Time-Varying

id	durat	censor	timein	timeout	X
1	4	0	30	34	0.12
2	2	1	12	14	0.19
3	5	1	5	10	0.09
...	...	...	...	...	...
N	10	1	21	31	0.22

Time-Varying Data

id	durat	censor	timein	timeout	X	Z
1	1	0	30	31	0.12	331
1	2	0	31	32	0.12	412
1	3	0	32	33	0.12	405
1	4	0	33	34	0.12	416
2	1	0	12	13	0.19	226
2	2	1	13	14	0.19	296
3	1	0	5	6	0.09	253
3	2	0	6	7	0.09	311
3	3	0	7	8	0.09	327
3	4	0	8	9	0.09	344
3	5	1	9	10	0.09	301
...	...	...	...	...	...	...

survival object (non-time-varying):

```
library(survival)
NonTV<-read.csv(NonTVdata.csv)
NonTV.S<-Surv(NonTV$duration, NonTV$censor)
```

survival object (time-varying):

```
TV<-read.csv(TVdata.csv)
TV.S<-Surv(TV$starttime, TV$endtime, TV$censor)
```

An Example

OECD Cabinet survival [Strom (1985); King et al. (1990)],

$N = 314$ cabinets in 15 countries

Outcome: Duration of cabinet, in months

Covariates (all non-time varying):

- *Fractionalization*
- *Polarization*
- *Formation Attempts*
- **Investiture**
- *Numerical Status*
- *Post-Election*
- *Caretaker*

Also: Indicator for whether the cabinet ended within 12 months of the end of the "constitutional inter-election period" ($\rightarrow$ censored)

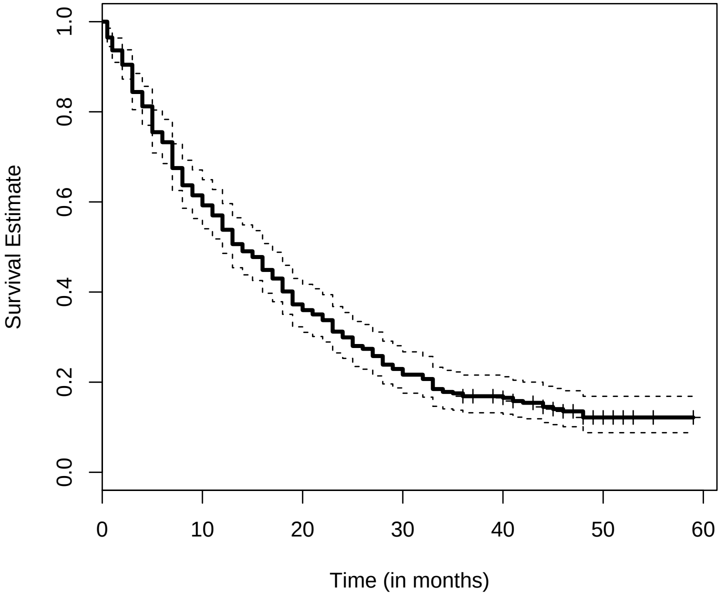
KABL Data

```
> head(KABL)
  id country durat ciep12 fract polar format invest numst2 eltime2 caretk2
1  1       1   0.5      1   656    11      3      1      0       1       0
2  2       1   3.0      1   656    11      2      1      1       0       0
3  3       1   7.0      1   656    11      5      1      1       0       0
4  4       1  20.0      1   656    11      2      1      1       0       0
5  5       1   6.0      1   656    11      3      1      1       0       0
6  6       1   7.0      1   634     6      4      1      1       1       0

> KABL.S<-Surv(KABL$durat,KABL$ciep12)

> KABL.S[1:50,]

 [1]  0.5   3.0   7.0  20.0   6.0   7.0   2.0  17.0  27.0  49.0+
[11]  4.0  29.0  49.0+  6.0  23.0  41.0+ 10.0  12.0   2.0  33.0
[21]  1.0  16.0   2.0   9.0   3.0   5.0   5.0   6.0  45.0+ 23.0
[31] 41.0   7.0  49.0+ 46.0   9.0  51.0+ 10.0  32.0  28.0   3.0
[41] 53.0+ 17.0  59.0+  9.0  52.0+  3.0  23.0  33.0   1.0  30.0
```

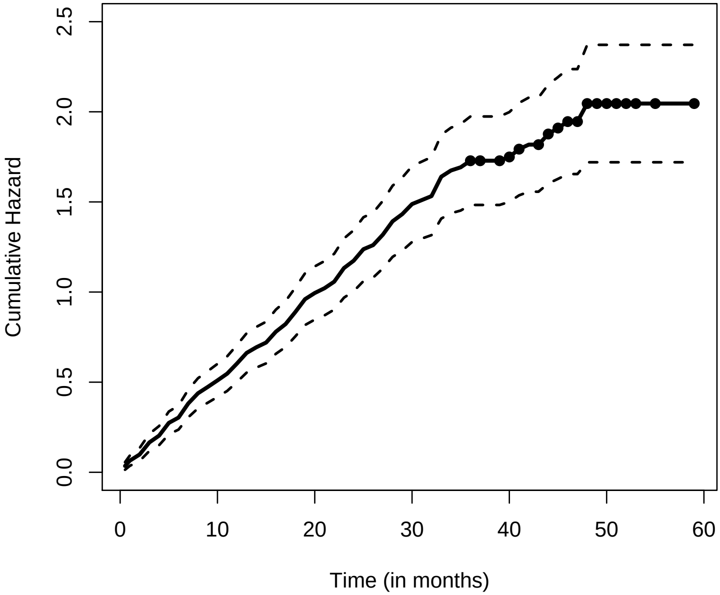
Example survfit Object

```
> KABL.fit<-survfit(KABL.S~1)

> str(KABL.fit)

List of 13
 $ n        : int 314
 $ time     : num [1:54] 0.5 1 2 3 4 5 6 7 8 9 ...
 $ n.risk   : num [1:54] 314 303 294 284 265 255 237 230 212 200 ...
 $ n.event  : num [1:54] 11 9 10 19 10 18 7 18 12 7 ...
 $ n.censor : num [1:54] 0 0 0 0 0 0 0 0 0 0 ...
 $ surv     : num [1:54] 0.965 0.936 0.904 0.844 0.812 ...
 $ type     : chr "right"
 $ std.err  : num [1:54] 0.0108 0.0147 0.0183 0.0243 0.0271 ...
 $ upper    : num [1:54] 0.986 0.964 0.938 0.885 0.856 ...
 $ lower    : num [1:54] 0.945 0.91 0.873 0.805 0.77 ...
 $ conf.type: chr "log"
 $ conf.int : num 0.95
 $ call     : language survfit(formula = KABL.S ~ 1)
 - attr(*, "class")= chr "survfit"
```

Plotting $\widehat{S(t)}$



Plotting $\widehat{H(t)}$



Log-rank test:

```
> survdiff(KABL.S~invest,data=KABL,rho=0)

Call:
survdiff(formula = KABL.S ~ invest, data = KABL, rho = 0)

           N Observed Expected (O-E)^2/E (O-E)^2/V
invest=0 172      137    178.7      9.72      30.5
invest=1 142      134     92.3     18.81      30.5

 Chisq= 30.5  on 1 degrees of freedom, p= 3.26e-08
```

Comparing $\widehat{S(t)}$s