Math 334 **Problem Set 2** **Due 9/25/19**

Problems 1 and 2 concern initial-value problems for the diffusion equation on $\mathbb{R}$,

$$u_t = u_{xx} \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0 .$$

1. Show that the solution satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \qquad \text{is} \qquad u(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-z^2}\, dz .$$

   Hint: Change variables in the heat kernel by letting $z = \dfrac{y-x}{\sqrt{4t}}$.

   What is $u(0,t)$ for $t > 0$? How does $u(x,t)$ behave as $t \to \infty$?

2. Let $u(x,t)$ be the unique solution satisfying the initial condition $u(x,0) = \varphi(x)$, where φ is a smooth function such that

   - $\varphi(x) > 0$ for $1 - \varepsilon < x < 1 + \varepsilon$, for some constant $\varepsilon > 0$, and
   - $\varphi(x) = 0$ otherwise.

   Determine, with justification, where $u(x,t) > 0$ when $t > 0$.

For the remaining problems, define

$$\varphi(x) = \begin{cases} 1 - |x| \ , & \text{for} \quad |x| < 1 \, , \\ 0 \ , & \text{otherwise} \, . \end{cases}$$

Solve the 1-dimensional wave equation, $u_{tt} = u_{xx}$, on the given domain and with the given auxiliary conditions, and plot the solution at the times $t = 0, \ \frac{1}{2}, \ 1, \ 2,$ and 3.

3. Domain: $-\infty < x < \infty$ Initial conditions: $u(x,0) = \varphi(x)\,,\ u_t(x,0) \equiv 0$
4. Domain: $-\infty < x < \infty$ Initial conditions: $u(x,0) \equiv 0\,,\ u_t(x,0) = \varphi(x)$
5. Domain: $0 < x < \infty$ Initial conditions: $u(x,0) = \varphi(x-2)\,,\ u_t(x,0) \equiv 0$

   Boundary condition: $u(0,t) = 0$ for all $t > 0$
6. Domain: $0 < x < 4$ Initial conditions: $u(x,0) = \varphi(x-2)\,,\ u_t(x,0) \equiv 0$

   Boundary conditions: $u(0,t) = 0 = u(4,t)$ for all $t > 0$

   For this last problem, you will obtain a Fourier series representation of the solution. To plot the solution at various times, it might help to use the trigonometric identity

$$\sin\alpha \cos\beta = \frac{\sin(\alpha - \beta) + \sin(\alpha + \beta)}{2}$$

   to express the solution as a sum of waves.