3(b)

$$(a+b)^3 = a^3 + \binom{3}{1} a^2 b + \binom{3}{2} a b^2 + b^3$$

$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & \boxed{1} & & \boxed{3} & & \boxed{3} & & \boxed{1} & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

etc.

$n \in \mathbb{N} \implies n = 3k$ or $n = 3k+1$ or $n = 3k+2$

$n = 3k$	$n = 3k+1$	$n = 3k+2$
$(3k)^3 = 27k^3 = 9m$, $m = 3k^3$.	$(3k+1)^3 = 27k^3 + 27k^2 + 9k + 1 = 9m+1$, $m = 3k^3 + 3k^2 + k$	$(3k+2)^3 = 27k^3 + 54k^2 + 36k + 8 = 9(3k^3 + 6k^2 + 4k) + 8$

$n \in \mathbb{N} \implies n = 3k$ or $n = 3k+1$ or $n = 3k-1$

$n = 3k$	$n = 3k+1$	$n = 3k-1$
$n^3 \equiv 0 \pmod 9$	$n^3 \equiv 1 \pmod 9$	$n^3 \equiv -1 \pmod 9$

Diophentine equations

$\boxed{2x + 3y = 17}$ YES!

$\boxed{4x + 6y = 25}$ **No!**

$\boxed{4x + 6y = 24}$ YES!

$\boxed{5x + 11y = 173}$ YES!

$(x = 28, y = 3)$

Are there integer solutions?

$(x, y \in \mathbb{Z})$

$$ax + by = c \quad , \quad a, b, c \in \mathbb{Z}$$

$\in \mathbb{N}$

$a, b \in \mathbb{N}$

Def'n a is a **divisor** of b iff $b = ac$ for some $c \in \mathbb{N}$. Notation: $a \mid b$.

Note: $a \nmid b$ means that a does not divide b.

Def'n $a, b \in \mathbb{N}$. The **greatest common divisor** of a and b, denoted $\gcd(a,b)$, is an integer $d \in \mathbb{N}$ satisfying

(1) $d \mid a$ and $d \mid b$, and

(2) $c \leq d$ for any divisor c of a and b.

n	Divisors of n
12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
15	1, 3, 5, 15
18	1, 2, 3, 6, 9, 18
48	1, 2, 3, 4, 6, 8, 12, 16, 24, 48

$3 = (-1)\cdot 12 + 1\cdot 15$

$\gcd(12, 15) = 3$

$\gcd(12, 18) = 6 \rightarrow 6 = (-1)\cdot 12 + (1)\cdot 18$

$\gcd(18, 48) = 6$

$6 = (3)\cdot 18 + (-1)\cdot 48$

<u>Theorem</u> If $d = \gcd(a,b)$, then $\exists\ x, y \in \mathbb{Z}$

s.t. $\quad d = ax + by$.

<u>pf</u>: Define $S = \{au + bv \mid u, v \in \mathbb{Z},\ au + bv > 0\}$.

S nonempty since $a \cdot \left(\frac{a}{|a|}\right) > 0$.

Thus, by the W.O.P., S has a minimal element, d.

Then $d = ax + by$ for some integers x, y.

To see that $d = \gcd(a,b)$, apply the Division Lemma:

$$a = qd + r, \text{ with } 0 \le r < d.$$

$$\Rightarrow r = a - qd = a - q(ax+by)$$

$$= a - qxa - qyb$$

$$= (1-qx)a + (-qy)b \in S \underbrace{\text{ if } r > 0}_{\text{can't happen!}}.$$

Thus, $r=0$ & $a=qd$, i.e., $d \mid a$.

Similarly, $d \mid b$.

If c is <u>any</u> common divisor of a and b, then

$$c \mid au+bv \text{ for any } u,v \in \mathbb{Z}.$$

Thus, if c is positive, $c \mid ax+by$, i.e., $c \mid d$, and $c \le d$. $\blacksquare$