$$\underline{\underline{y' = y}} \qquad y = y(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \underbrace{\sum_{n=0}^{\infty} a_n t^n}$$

Taylor series = Maclaurin series

$$\underline{\underline{a_n = ?}}$$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \implies y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y' - y = 0 \implies \sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\begin{cases} m = n-1 \\ m+1 = n \end{cases}$$

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} t^m - \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\implies \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n - \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\implies \boxed{\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_n\right] t^n = 0}$$

$$\Longrightarrow \quad (n+1)a_{n+1} - a_n = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow \quad a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 0 \quad \underline{\text{recurrence relation}}$$

$$a_1 = \frac{a_0}{1} \Longrightarrow a_1 = a_0$$

$$a_2 = \frac{a_1}{2} \Longrightarrow a_2 = \frac{a_0}{2!}$$

$$a_3 = \frac{a_2}{3} \Longrightarrow a_3 = \frac{a_0}{3!}$$

$$\vdots$$

$$\underline{\underline{a_n = \frac{a_0}{n!}}}$$

$$y(t) = \sum_{n=0}^{\infty} \frac{a_0}{n!} t^n$$

$$\Longrightarrow y(t) = a_0 \underbrace{\left(\sum_{n=0}^{\infty} t^n / n! \right)}_{\underline{\underline{e^t}}}$$

$$\Longrightarrow \underline{\underline{y(t) = a_0 e^t}}$$

$$y'' + y = 0$$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \quad , \quad y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$m = n-2$$
$$m+2 = n$$

$$\Longrightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] t^n = 0$$

$$\Longrightarrow \quad (n+2)(n+1)a_{n+2} + a_n = 0 \quad , \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow \quad a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \quad , \quad \text{recurrence relation}$$

<u>even n</u>

$$a_2 = \frac{-a_0}{2 \cdot 1} = \frac{-a_0}{2!}$$

$$a_4 = \frac{-a_2}{4 \cdot 3} = \frac{-1}{12} \cdot \left(\frac{-a_0}{2}\right) = \frac{a_0}{4!}$$

$$a_6 = \frac{-a_4}{6 \cdot 5} = -\frac{a_0}{6!} \quad \ldots \quad \boxed{a_{2k} = \frac{(-1)^k a_0}{(2k)!}}$$

<u>odd n</u>

$$a_3 = \frac{-a_1}{3 \cdot 2} = \frac{-a_1}{3!} \quad , \quad a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5!} \quad \ldots \quad \boxed{a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}}$$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \ldots$$

$$= a_0 + a_1 t - \frac{a_0}{2} t^2 - \frac{a_1}{3!} t^3 + \frac{a_0}{4!} t^4 + \frac{a_1}{5!} t^5 + \ldots$$

$$= a_0 \left[1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \ldots \right] + a_1 \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots \right]$$

$$= \underline{\underline{a_0 \cos t + a_1 \sin t}} .$$

$\underline{3}$ $y'' - xy' - y = 0$, $\underline{\underline{x_0 = 0}}$

$$y = \sum_{n=0}^{\infty} a_n x^n \Longrightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Longrightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$m = n-2, \quad n = m+2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{n=1}^{\infty} [n a_n + a_n] x^n - a_0 = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} [(n+1) a_n] x^n - a_0 = 0$$

$$\sum_{n=1}^{\infty} [\underbrace{(n+2)(n+1) a_{n+2} - (n+1) a_n}_{0}] x^n + \underbrace{2a_2 - a_0}_{0} = 0$$

$$\Longrightarrow 2a_2 - a_0 = 0 \Longrightarrow a_2 = \frac{a_0}{2}$$

recurrence relation

$$\& \quad (n+2)\cancel{(n+1)}\, a_{n+2} - \cancel{(n+1)}\, a_n = 0 \Longrightarrow \boxed{a_{n+2} = \frac{a_n}{n+2}}$$

$\underline{\underline{n \geq 0}}$

<u>even n</u>

$$a_2 = \frac{a_0}{2}$$

$$a_4 = \frac{a_2}{4} = \frac{a_0}{8}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{48}$$

$\vdots$

<u>odd n</u>

$$a_3 = \frac{a_1}{3}$$

$$a_5 = \frac{a_3}{5} = \frac{a_1}{15}$$

$$a_7 = \frac{a_5}{7} = \frac{a_1}{105} \quad \ldots$$

replace t with x !

$$\Longrightarrow y(t) = a_0\Big[\overbrace{1 + \tfrac{1}{2}t^2 + \tfrac{1}{8}t^4 + \tfrac{1}{48}t^6 + \ldots}^{y_1}\Big]$$

$$+\, a_1\Big[\underbrace{t + \tfrac{1}{3}t^3 + \tfrac{1}{15}t^5 + \tfrac{1}{105}t^7 + \ldots}_{y_2}\Big]$$

14 $(1-x)y'' + xy' - y = 0, \quad y(0) = -3, \quad y'(0) = 2$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=1}^{\infty} (n-1) a_n x^n - a_0 = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} \underbrace{\Big[(n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} + (n-1) a_n \Big]} x^n + \underbrace{2a_2 - a_0} = 0$$

$$\Longrightarrow a_2 = \frac{a_0}{2}$$

$$\Longrightarrow (n+2)(n+1) a_{n+2} = n(n+1) a_{n+1} - (n-1) a_n$$

$\underline{n=1}$: $\quad 6a_3 = 2a_2 \Longrightarrow a_3 = \frac{1}{3} a_2 = \frac{1}{6} a_0$

$\underline{n=2}$: $\quad 12 a_4 = 6a_3 - a_2 \Longrightarrow a_4 = \frac{1}{2} a_3 - \frac{1}{12} a_2$

$$= \frac{1}{12} a_0 - \frac{1}{24} a_0$$

$$\Longrightarrow \quad y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{1}{6} a_0 x^3 + \left(\frac{1}{12} a_0 - \frac{1}{24} a_0\right) x^4 + \ldots$$

<u>Note</u>: $y(0) = a_0$, $y'(0) = a_1$ $\Longrightarrow$ $a_0 = -3$, $a_1 = 2$, and

$$y(x) = -3 + 2x - \frac{3}{2} x^2 - \frac{1}{2} x^3 - \frac{1}{8} x^4 + \ldots$$