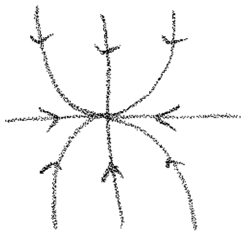
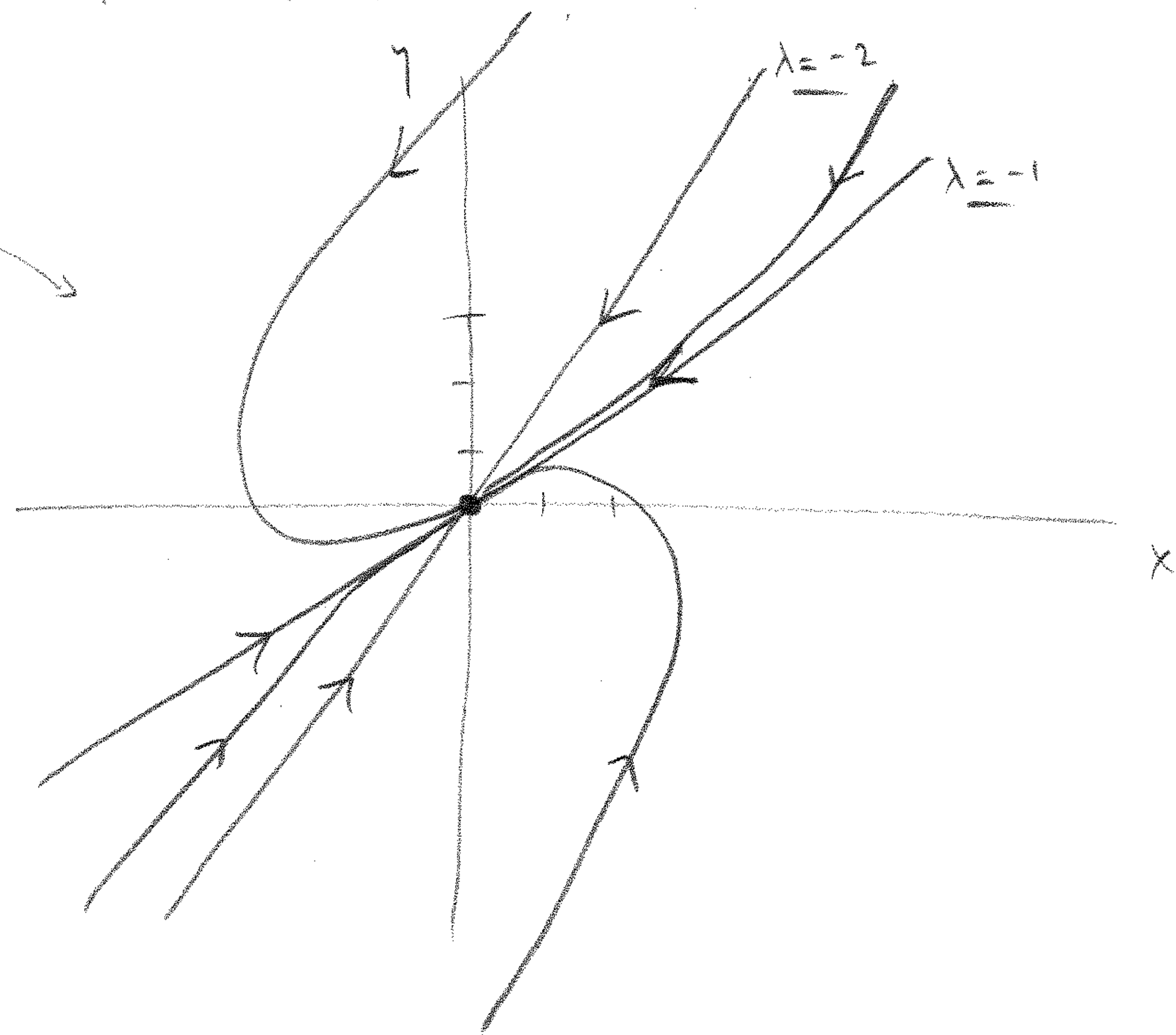
#2

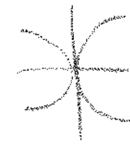
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

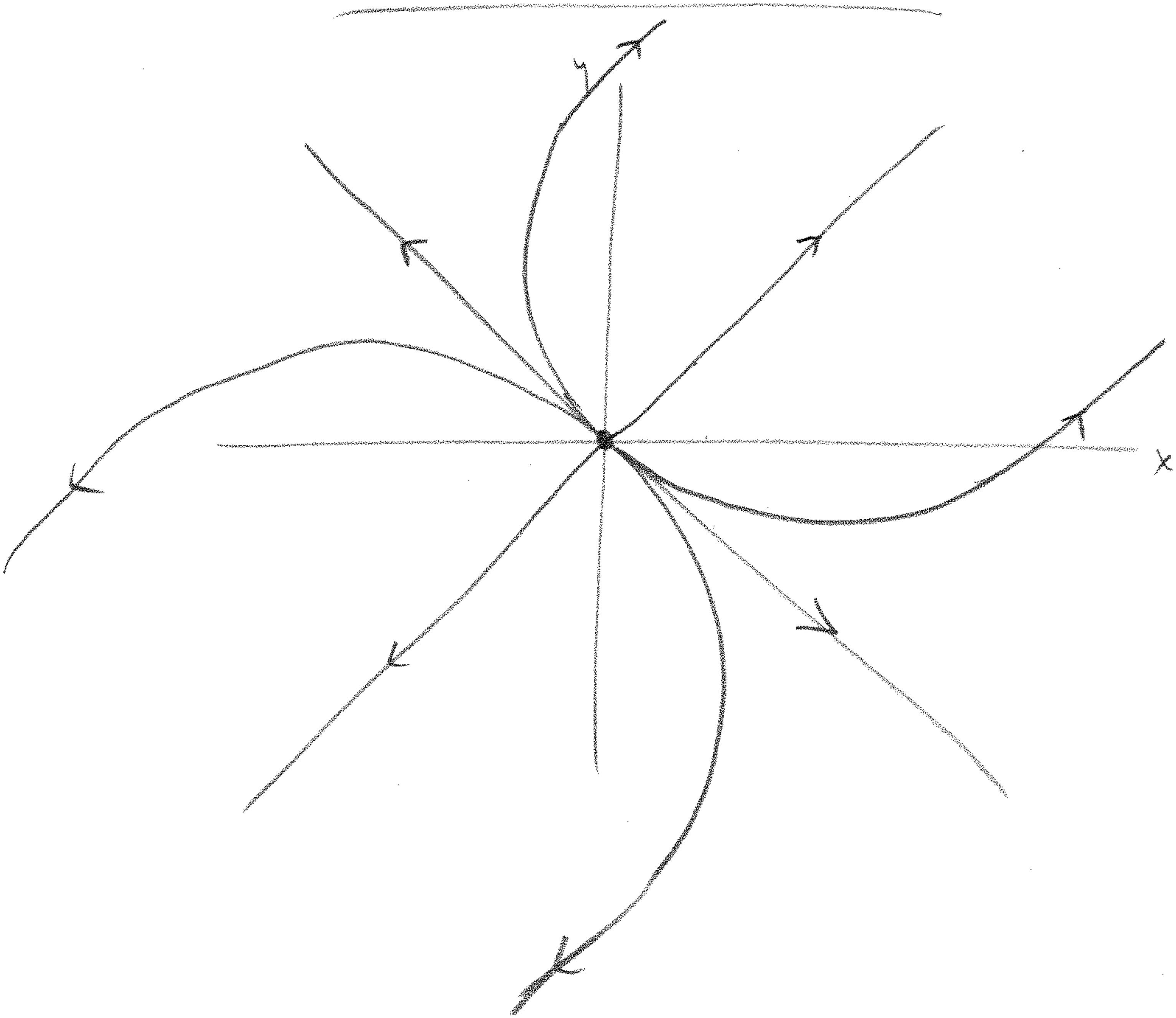




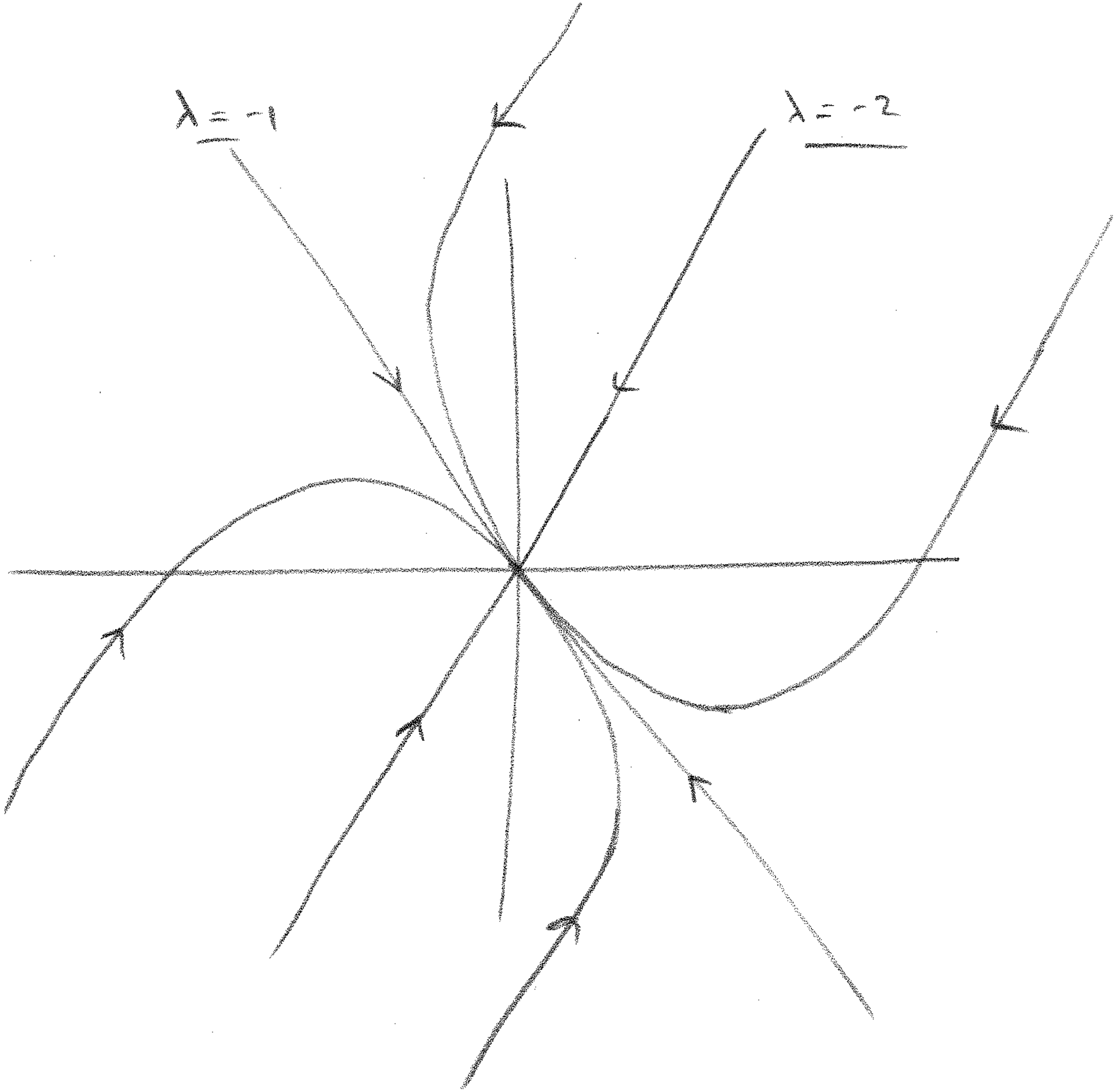
#4

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

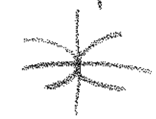
($x \sim y^4$ )



#17



$x \sim y^2$



$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{i.e.} \quad \underline{\underline{\begin{pmatrix} x \\ y \end{pmatrix}' = A}}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{\underline{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}}$$

$$\underline{\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t}\vec{v}} \implies \lambda e^{\lambda t}\vec{v} = A v \cdot e^{\lambda t}$$

$$\implies \boxed{\lambda \vec{v} = A v}$$

$$\implies Av - \lambda v = 0$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{v} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}}^{Av - \lambda v}$$

$$\implies \begin{pmatrix} a v_1 + b v_2 - \lambda v_1 \\ c v_1 + d v_2 - \lambda v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] v$$

$$\implies \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} v$$

$$\boxed{\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$\Longrightarrow$ To find λ, need $\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

i.e., $\underline{\underline{(a-\lambda)(d-\lambda) - bc = 0}}$

#1 $\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

λ : $\det(A - \lambda I) = 0 \implies \det\begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0$

$\implies (-1-\lambda)(-1-\lambda) + 4 = 0$

$\implies 1 + 2\lambda + \lambda^2 + 4 = 0$

$\implies \lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = \underline{\underline{-1 \pm 2i}}$

$\underline{\underline{\lambda = -1+2i, \quad \lambda = -1-2i}}$

$\underline{\lambda = -1+2i}$:

$$\begin{pmatrix} -1+1-2i & -4 \\ 1 & -1+1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad (-2i)v_1 - 4v_2 = 0$$

$$v_1 - 2i\,v_2 = 0 \Longrightarrow v_1 = 2i\,v_2$$

$$\Longrightarrow \quad v = \begin{pmatrix} 2iv_2 \\ v_2 \end{pmatrix} \Longrightarrow \underline{\underline{v = \begin{pmatrix} 2i \\ 1 \end{pmatrix}}} \text{ for } \lambda = \underline{\underline{-1+2i}}$$

$$e^{\lambda t} v = e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t} e^{2it}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$= e^{-t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

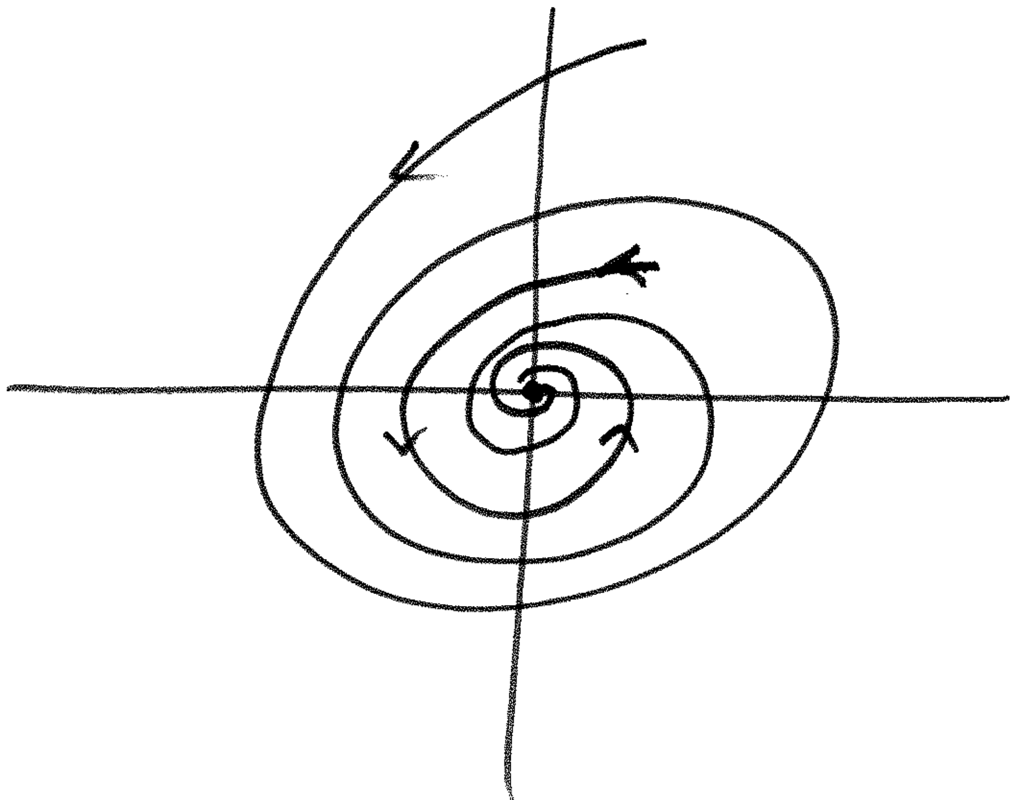
$$2i \cdot i = 2i^2 = \underline{\underline{-2}}$$

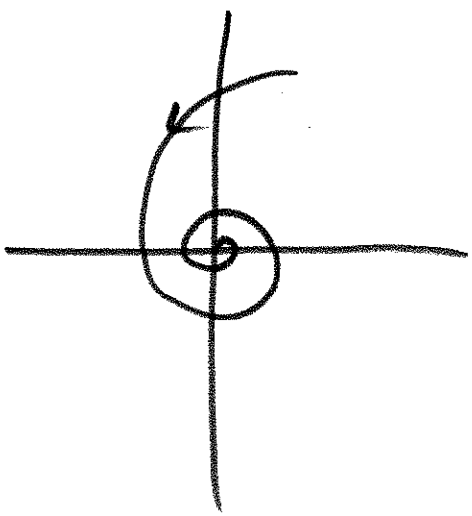
$$= \begin{pmatrix} e^{-t}\left[2i\cos(2t) - 2\sin(2t)\right] \\ e^{-t}\left[\cos(2t) + i\sin(2t)\right] \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -2e^{-t}\sin 2t \\ e^{-t}\cos(2t) \end{pmatrix}} + i \underbrace{\begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix}}$$

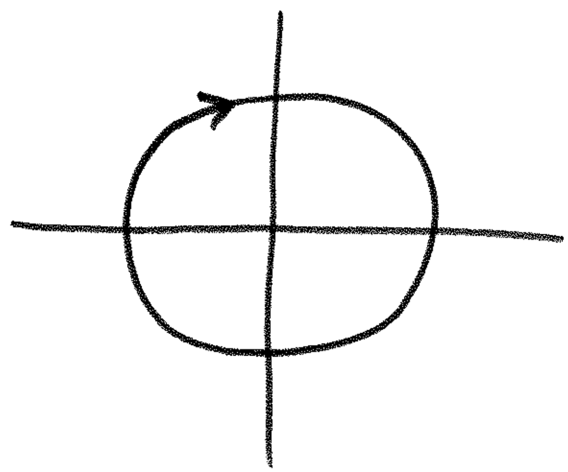
These are the 2 fundamental sol'ns!

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}$$





stable spiral



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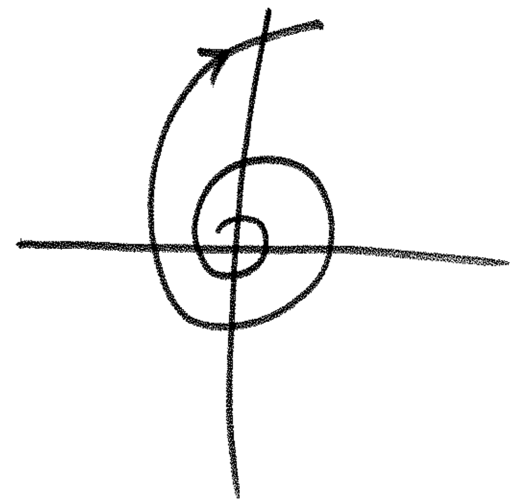


unstable spiral

#2

$$x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$$

$$\Longrightarrow \det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = 0 \Longrightarrow (2-\lambda)(-2-\lambda) + 5 = 0$$

$$\Longrightarrow \lambda^2 - 4 + 5 = 0$$

$$\Longrightarrow \lambda^2 + 1 = 0 \Longrightarrow \lambda^2 = -1 \Longrightarrow \lambda = \pm i$$

$\lambda = i$:

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

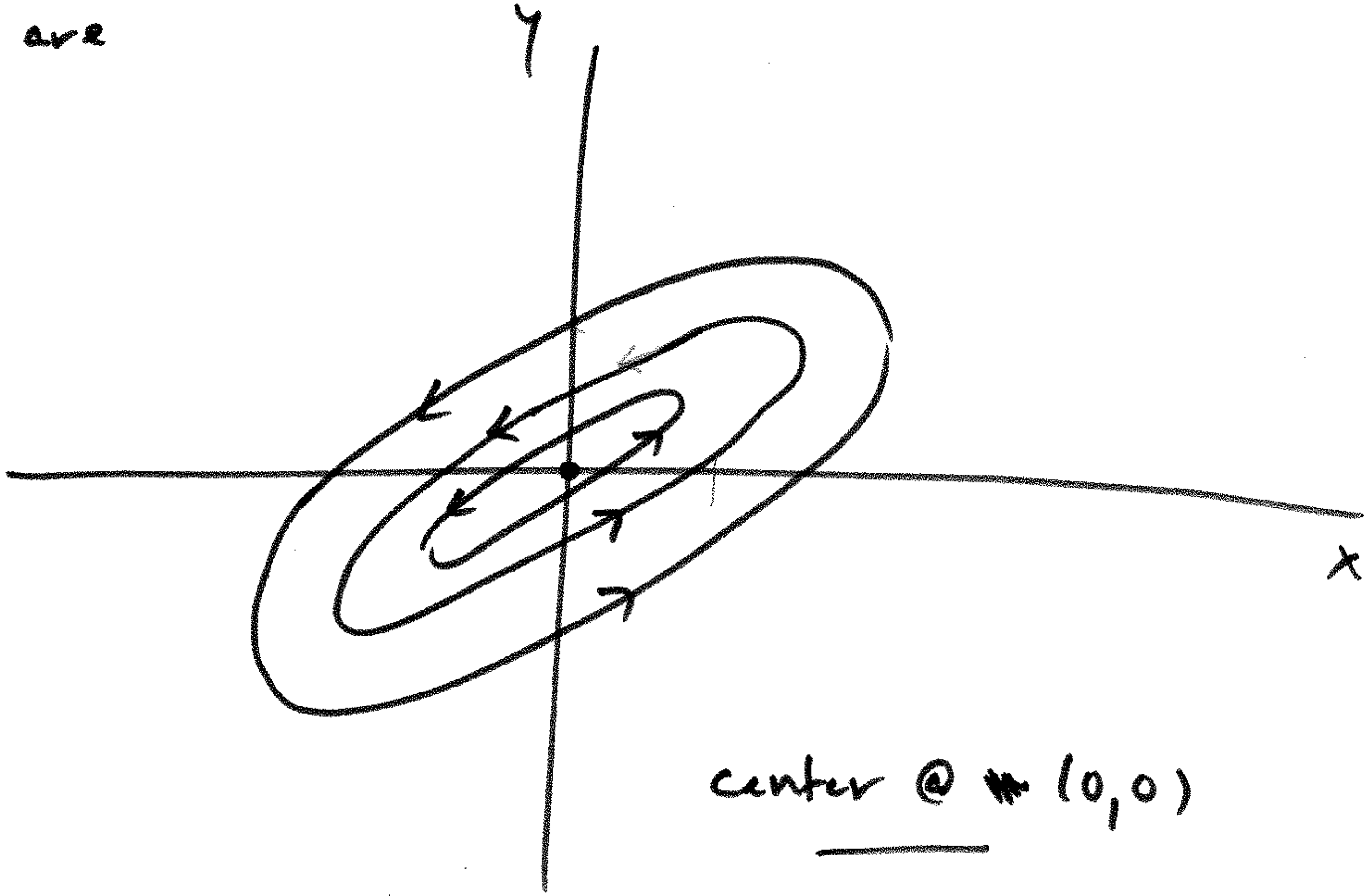
$$\Longrightarrow v_1 - (2+i) v_2 = 0 \Longrightarrow v_1 = (2+i) v_2 \ , \quad v = v_2 \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$e^{\lambda t} \vec{v} = e^{it} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = (\cos t + i \sin t) \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$ General Solution

phase portrait:

Trajectories are ellipses!



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