6.1 For any integer $n \geq 1$, $\tau(n) \leq 2\sqrt{n}$.

Let the $\tau(n)$ divisors of n be $d_1 = 1 < d_2 < \ldots < d_{\tau(n)} = n$.

If $\tau(n) = 2k$, then $1 = d_1 < d_2 < \ldots < d_k < d_{k+1} < \ldots < d_{\tau(n)} = n$ and

$d_k < \sqrt{n} < d_{k+1}$, since $d_k d_{k+1} = n$. Pair up the divisors:

$$1 \times d_{\tau(n)} = n$$

$$d_2 \times d_{\tau(n)-1} = n$$

$$\vdots$$

$$\underbrace{d_k}_{<\sqrt{n}\text{ divisors here}} \times \underbrace{d_{k+1}}_{<\sqrt{n}\text{ divisors here}} = n$$

$< \sqrt{n}$ divisors here $\Longrightarrow$ $< \sqrt{n}$ divisors here $\Longrightarrow$ fewer than $2\sqrt{n}$ divisors in all!

If $\tau(n) = 2k+1$, then $1 = d_1 < \ldots < d_k < d_{k+1} < d_{k+2} < \ldots < d_{\tau(n)} = n$, and pairing up divisors shows that $d_{k+1}^2 = n$. In this case, $\tau(n) \leq 2\sqrt{n}$.

$\blacksquare$

7(a) $\tau(n)$ is odd iff n is a perfect square.

pf.: Pair up divisors as in #6!

7(b) $\sigma(n)$ is odd iff $n = m^2$ or $n = 2m^2$, some $m \in \mathbb{N}$.

pf.: If $n = p^k$ for some prime p and $k \geq 1$,

$$\sigma(n) = \sigma(p^k) = 1 + p + p^2 + \dots + p^k .$$

If p is odd, this sum is odd iff k is even, $k = 2r \Longrightarrow n = (p^r)^2$.

If $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, $\sigma(n) = \sigma(p_1^{k_1} \cdots p_s^{k_s}) = \sigma(p_1^{k_1})\sigma(p_2^{k_2}) \cdots \sigma(p_s^{k_s})$

since σ is multiplicative, and this product is odd iff $\sigma(p_i^{k_i})$ is odd for each i; $\sigma(p_i^{k_i})$ is odd iff k_i is even when p_i is odd, and $\sigma(2^{k_i})$ is odd no matter what. If all of the p_i are odd, it follows that $\sigma(n)$ is odd iff $k_i = 2r_i \; \forall i \Longrightarrow n = (p_1^{r_1} \cdots p_s^{r_s})^2$.

If 2 is one of the divisors of n, then $n = 2(p_2^{r_2} \cdots p_s^{r_s})^2$. ∎

<u>8</u> $\displaystyle\sum_{d \mid n} \frac{1}{d} = \frac{\sigma(n)}{n}$ for every $n \in \mathbb{N}$.

Note that $f(n) = n^{-1} = \frac{1}{n}$ is multiplicative. It follows that

$F(n) := \displaystyle\sum_{d \mid n} \frac{1}{d}$ is also multiplicative, so we just need to analyze

$F(p^k)$ for a prime p and $k \geq 1$. Compute:

$$F(p^k) = \sum_{d \mid p^k} \frac{1}{d} = 1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^k} = \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}}$$

$$= \frac{p^{k+1} - 1}{p^{k+1} - p^k} = \frac{p^{k+1} - 1}{p^k(p-1)} = \frac{1}{p^k} \cdot \frac{p^{k+1} - 1}{p - 1}$$

$$= \frac{\sigma(p^k)}{p^k} = \frac{\sigma(n)}{n}.$$

In general: $n = p_1^{k_1} \cdots p_r^{k_r} \implies F(p_1^{k_1} \cdots p_r^{k_r}) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r})$

$$= \frac{\sigma(p_1^{k_1})}{p_1^{k_1}} \cdot \frac{\sigma(p_2^{k_2})}{p_2^{k_2}} \cdot \ldots \cdot \frac{\sigma(p_r^{k_r})}{p_r^{k_r}} = \frac{\sigma(n)}{n}.$$

$\boxtimes$

<u>21</u> For any $n \in \mathbb{N}$, $\sum_{d|n} (\tau(d))^3 = \left(\sum_{d|n} \tau(d) \right)^2$.

Since τ is multiplicative, so are $(\tau(d))^3$, $\sum_{d|n} (\tau(d))^3$, $\sum_{d|n} \tau(d)$, and $\left(\sum_{d|n} \tau(d) \right)^2$. It therefore suffices to prove this identity for $n = p^k$, p prime & $k \geq 1$. Compute:

$$\sum_{d|p^k} (\tau(d))^3 = 1 + (\tau(p))^3 + (\tau(p^2))^3 + \dots + (\tau(p^k))^3$$
$$= 1 + 2^3 + 3^3 + \dots + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2} \right)^2 \quad \text{(see 1.1(e), page 7)}$$

and

$$\left(\sum_{d|p^k} \tau(d) \right)^2 = \left(\tau(1) + \dots + \tau(p^k) \right)^2 = \left(1 + 2 + \dots + (k+1) \right)^2$$
$$= \left(\frac{(k+1)(k+2)}{2} \right)^2 .$$

$\boxtimes$

6.2 **2** $\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^k \text{ for a prime } p \text{ and } k \geq 1 \\ 0, & \text{otherwise} \end{cases}$

Then $\Lambda(n) = \sum_{d|n} \mu(\frac{n}{d}) \log d = - \sum_{d|n} \mu(d) \log d$.

pf.: Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. For any divisor d of n, $\Lambda(d) \neq 0$ iff $d = p_i^{j_i}$ for some i, $1 \leq j_i \leq k_i$. Then

$$\sum_{d|n} \Lambda(d) = \sum_{d|p_1^{k_1}} \Lambda(d) + \sum_{d|p_2^{k_2}} \Lambda(d) + \cdots + \sum_{d|p_r^{k_r}} \Lambda(d)$$

$$= 0 + \underbrace{\log p_1 + \cdots + \log p_1}_{k_1 \text{ times}} + \cdots + 0 + \underbrace{\log p_r + \cdots + \log p_r}_{k_r \text{ times}}$$

$$= \log(p_1^{k_1}) + \cdots + \log(p_r^{k_r}) = \log(p_1^{k_1} \cdots p_r^{k_r}) = \log n.$$

Since $\log n = \sum_{d|n} \Lambda(d)$, Möbius inversion $\Longrightarrow$ $\Lambda(n) = \sum_{d|n} \mu(\frac{n}{d}) \log d$

and $\Lambda(n) = \sum_{d|n} \mu(d) \log(\frac{n}{d}) = \sum_{d|n} \mu(d)(\log n - \log d) = (\log n) \underline{\sum \mu(d)} - \sum \mu(d) \log d$

3 $n = p_1^{k_1} \cdots p_r^{k_r}$, f multiplicative $\Longrightarrow \sum_{d|n} \mu(d) f(d) = (1 - f(p_1)) \cdots (1 - f(p_r))$.

Since μ and f are multiplicative, so are μf and $\sum_{d|n} \mu(d) f(d)$.

It therefore suffices to verify this formula for $n = p^k$, p prime.

When $n = p^k$, we have:

$$\sum_{d|p^k} \mu(d) f(d) = \mu(1) f(1) + \mu(p) f(p) + \underbrace{\mu(p^2) f(p^2) + \ldots + \mu(p^k) f(p^k)}_{0 \text{ since } \mu(p^2) = \ldots = \mu(p^k) = 0}$$

$$= f(1) + (-1) f(p)$$

$$= 1 - f(p).$$

* For any multiplicative f, $f(1) = 1$. *

$f(n) = f(1 \cdot n) = f(1) \cdot f(n)$

$\Longrightarrow f(1) = 1$.

The general result then follows. $\boxtimes$

<u>4</u> $n = p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_r^{k_r}$. Use #3 for each part:

(a)
$$\sum_{d \mid n} \mu(d)\tau(d) = (1-\tau(p_1))(1-\tau(p_2)) \cdot \ldots \cdot (1-\tau(p_r))$$
$$= \underbrace{(1-2)(1-2) \cdot \ldots \cdot (1-2)}_{r \text{ times}} = (-1)^r. \checkmark$$

(b)
$$\sum_{d \mid n} \mu(d)\sigma(d) = (1-\sigma(p_1))(1-\sigma(p_2)) \cdot \ldots \cdot (1-\sigma(p_r))$$
$$= (1-p_1-1)(1-p_2-1) \cdot \ldots \cdot (1-p_r-1)$$
$$= (-1)^r p_1 p_2 \cdot \ldots \cdot p_r. \checkmark$$

(c)
$$\sum_{d \mid n} \frac{\mu(d)}{d} = \left(1-\frac{1}{p_1}\right) \cdot \ldots \cdot \left(1-\frac{1}{p_r}\right) \checkmark$$

(d)
$$\sum_{d \mid n} \mu(d) d = (1-p_1)(1-p_2) \cdot \ldots \cdot (1-p_r). \checkmark$$

7 $\quad \lambda(n) := \begin{cases} 1, & n=1 \\ (-1)^{k_1+\dots+k_r}, & n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \end{cases}$

(a) λ is multiplicative:

$m, n \in \mathbb{N}$ with $\gcd(m,n)=1 \Longrightarrow m = p_1^{k_1} \cdots p_r^{k_r}$, $n = q_1^{j_1} \cdots q_s^{j_s}$,

distinct primes p_i, q_i. Then

$$\lambda(mn) = (-1)^{k_1+\dots+k_r+j_1+\dots+j_s} = (-1)^{k_1+\dots+k_r}(-1)^{j_1+\dots+j_s} = \lambda(m)\lambda(n).$$

$\boxed{\checkmark}$

(b) Given $n \in \mathbb{N}$, $\sum_{d|n} \lambda(d) = \begin{cases} 1, & n = m^2 \text{ for some } m \\ 0, & \text{otherwise} \end{cases}$

Just check when $n = p^k$, some prime p and $k \geq 1$:

$$\sum_{d|p^k} \lambda(d) = \lambda(1) + \lambda(p) + \dots + \lambda(p^k) = 1 + (-1) + (-1)^2 + \dots + (-1)^k$$
$$= \begin{cases} 1, & k \text{ even} \Longrightarrow n = (p^r)^2 \\ 0, & k \text{ odd} \Longrightarrow n \text{ not a perfect square.} \end{cases}$$

The general case follows since λ is multiplicative. (and thus so is $\sum_{d|n} \lambda(d)$!) $\boxed{\checkmark}$