Math 212 - Exam 2 SOLUTIONS

① (a) $y'' + 2y' + 5y = e^{-x} \sec 2x$

characteristic equation: $m^2 + 2m + 5 = 0$

$$\Longrightarrow m = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$$

particular solution: $y_p(x) = u_1 e^{-x} \cos(2x) + u_2 e^{-x} \sin(2x)$,

where u_1 and u_2 solve the system

$$\begin{cases} u_1' e^{-x} \cos 2x + u_2' e^{-x} \sin 2x = 0 \\ u_1' (-e^{-x} \cos 2x - 2e^{-x} \sin 2x) + u_2' (-e^{-x} \sin 2x + 2e^{-x} \cos 2x) \\ \qquad = e^{-x} \sec 2x \end{cases}$$

$$\Longrightarrow \begin{cases} u_1' \cos 2x + u_2' \sin 2x = 0 \\ u_1' (\underline{-\cos 2x} - 2\sin 2x) + u_2' (\underline{-\sin 2x} + 2\cos 2x) = \sec 2x \end{cases}$$

using the first equation,

$-u_1' \cos 2x - u_2' \sin 2x = 0$.

This yields the simpler system...

$$\begin{cases} u_1' \cos 2x + u_2' \sin 2x = 0 \\ -2u_1' \sin 2x + 2u_2' \cos 2x = \sec 2x \end{cases}$$

$$\Longrightarrow \quad \begin{matrix} 2u_1' \sin 2x \cos 2x + 2u_2' \sin^2 2x = 0 \\ -2u_1' \sin 2x \cos 2x + 2u_2' \cos^2 2x = 1 \end{matrix} \quad \Longrightarrow 2u_2' = 1 \Longrightarrow u_2' = \frac{1}{2}$$

$$\Longrightarrow u_2 = \frac{1}{2}x$$

Since $u_2' = \frac{1}{2}$, the first equation becomes

$$u_1' \cos 2x + \frac{1}{2}\sin 2x = 0 \Longrightarrow u_1' = -\frac{1}{2}\tan(2x)$$

$$\Longrightarrow u_1(x) = -\frac{1}{2}\int \tan(2x)\,dx = \frac{1}{4}\ln|\cos(2x)|$$

Thus, the particular solution is

$$y_p(x) = \frac{1}{4}\ln|\cos(2x)|\, e^{-x}\cos 2x + \frac{1}{2}x e^{-x}\sin(2x),$$

and the general solution is

$$y(x) = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x) + \frac{1}{4}\ln|\cos(2x)|\, e^{-x}\cos 2x$$
$$+ \frac{1}{2}x e^{-x}\sin(2x).$$

(b) $2y'' + 3y' + y = x^2 + 3\sin x$

characteristic equation: $2m^2 + 3m + 1 = 0$

$$\Longrightarrow \quad m = \frac{-3 \pm \sqrt{9-8}}{4} = \frac{-3 \pm 1}{4} = -1, -\frac{1}{2}$$

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-x} + c_2 e^{-\frac{1}{2}x}.$$

particular solution: $y_p(x) = ax^2 + bx + c + \alpha \sin x + \beta \cos x$

$$\Longrightarrow \quad y_p' = 2ax + b + \alpha \cos x - \beta \sin x,$$

$$y_p'' = 2a - \alpha \sin x - \beta \cos x.$$

Substitute these into the ODE:

$$2(2a - \alpha \sin x - \beta \cos x) + 3(2ax + b + \alpha \cos x - \beta \sin x)$$
$$+ ax^2 + bx + c + \alpha \sin x + \beta \cos x = x^2 + 3\sin x$$

$$\Longrightarrow \quad ax^2 + (6a + b)x + (4a + 3b + c)$$
$$+ (-2\alpha - 3\beta + \alpha)\sin x + (-2\beta + 3\alpha + \beta) = x^2 + 3\sin x$$

$$\Longrightarrow \quad a = 1, \quad b = -6, \quad c = 14$$

$$\left.\begin{aligned} -\alpha - 3\beta &= 3 \\ 3\alpha - \beta &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} -3\alpha - 9\beta &= 9 \\ 3\alpha - \beta &= 0 \end{aligned}$$

$$-10\beta = 9 \Rightarrow \beta = -\frac{9}{10}$$

$$\Rightarrow \alpha = -\frac{3}{10}$$

Thus, the general solution of the ODE is

$$y(x) = \underbrace{c_1 e^{-x} + c_2 e^{-\frac{1}{2}x}}_{y_c(x)} + \underbrace{x^2 - 6x + 14 - \tfrac{3}{10}\sin x - \tfrac{9}{10}\cos x}_{y_p(x)}$$

(c) $y'' + 2y' + y = e^{-x}\ln x$

characteristic equation: $m^2 + 2m + 1 = 0$

$$\Longrightarrow \quad (m+1)^2 = 0 \quad \Longrightarrow \quad m = -1 \quad \text{(double root)}$$

general solution of homogeneous equation:

$$y_c(x) = (c_1 + c_2 x)e^{-x}$$

particular solution via variation of parameters:

$$y_p(x) = u_1 e^{-x} + u_2 x e^{-x},$$ where u_1 and u_2

satisfy the system

$$\begin{cases} u_1' e^{-x} + u_2' x e^{-x} = 0 \\ -u_1' e^{-x} + u_2'(e^{-x} - xe^{-x}) = e^{-x}\ln x \end{cases}$$

$$\Longrightarrow \begin{cases} u_1' + u_2' x = 0 \\ -u_1' + u_2'(1-x) = \ln x \end{cases}$$

$$u_2' = \ln x \Longrightarrow u_2 = \int \ln x \, dx = x\ln x - x$$

Then $u_1' = -x\ln x \Longrightarrow u_1 = -\int x \ln x \, dx = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x$

The general solution is therefore

$$y(x) = (c_1 + c_2 x) e^{-x} + \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right) e^{-x} + (x \ln x - x)\, x e^{-x}$$

$$= (c_1 + c_2 x) e^{-x} + \frac{1}{2} x^2 \ln x \, e^{-x} - \frac{3}{4} x^2 e^{-x}$$

② $y_1(x) = \sin(x^2)$ solves $xy'' - y' + 4x^3 y = 0$;

second solution via reduction of order : $y_2(x) = u \cdot \sin(x^2)$

$$\Longrightarrow \quad y_2' = u' \sin(x^2) + 2xu \cos(x^2),$$

$$y_2'' = u'' \sin(x^2) + 2xu' \cos(x^2) + 2u \cos(x^2)$$
$$+ 2xu' \cos(x^2) - 4x^2 u \sin(x^2)$$

$$= u'' \sin(x^2) + 4xu' \cos(x^2) + 2u\left(\cos(x^2) - 2x^2 \sin(x^2)\right)$$

Substitute these into the ODE to get :

$$xu'' \sin(x^2) + 4x^2 u' \cos(x^2) + 2xu\left(\cos(x^2) - 2x^2 \sin(x^2)\right)$$
$$- u' \sin(x^2) - 2xu \cos(x^2) + 4x^3 u \sin(x^2) = 0$$

$$\Longrightarrow \quad u''\left(x \sin(x^2)\right) + u'\left(4x^2 \cos(x^2) - \sin(x^2)\right) = 0$$

$$\Longrightarrow \quad u'' + u'\left(4x \cot(x^2) - \frac{1}{x}\right) = 0$$

$$\Longrightarrow \quad u'' = u'\left(\frac{1}{x} - 4x\cot(x^2)\right) \Longrightarrow \frac{u''}{u'} = \frac{1}{x} - 4x\cot(x^2)$$

Integrate: $\ln|u'| = \ln x - \int 4x \cdot \frac{\cos(x^2)}{\sin(x^2)}\,dx$

$$= \ln x - 2\ln|\sin(x^2)|$$

$$= \ln\left|\frac{x}{(\sin(x^2))^2}\right|$$

$$\Longrightarrow \quad u' = \frac{x}{(\sin(x^2))^2}$$

Note that $\frac{d}{dx}\left(\frac{\cos(x^2)}{\sin(x^2)}\right) = \frac{-2x\sin(x^2)\cdot\sin(x^2) - \cos(x^2)\cdot 2x\cos(x^2)}{\sin^2(x^2)}$

$$= \frac{-2x}{\sin^2(x^2)}$$

(in other words, $\frac{d}{dx}(\cot(x^2)) = -2x\csc^2(x^2)$)

Thus, $u' = \frac{x}{\sin^2(x^2)} \Longrightarrow u = -\frac{1}{2}\cdot\frac{\cos(x^2)}{\sin(x^2)}$

$$\Longrightarrow \quad y_2(x) = -\frac{1}{2}\cdot\frac{\cos(x^2)}{\sin(x^2)}\cdot\sin(x^2) = -\frac{1}{2}\cos(x^2)$$

The general solution is therefore

$$y(x) = c_1\sin(x^2) + c_2\cos(x^2).$$

③ $y'' - xy = 0$

(a) $y = \sum_{n=0}^{\infty} a_n x^n$ (Maclaurin series)

$$\Longrightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Then get $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$;

Shift indices to combine these series:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Longrightarrow \quad 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

Thus, $\underline{\underline{a_2 = 0}}$ and the recurrence relation is

$(n+2)(n+1) a_{n+2} - a_{n-1} = 0$ for $n \geq 1$, i.e.,

$$\boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.}$$

$\underline{n=1}$: $a_3 = \frac{a_0}{6}$; $n=2$: $a_4 = \frac{a_1}{12}$

$\underline{n=3}$: $a_5 = \frac{a_2}{20} = 0$; $n=4$: $a_6 = \frac{a_3}{30} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$ ...

(since $a_2 = 0$)

From the recurrence relation, it follows that a_0 determines $a_3, a_6, a_9, \ldots$ etc. : a_0 determines a_{3n} for $n \geq 1$.

In fact :

$$a_3 = \frac{1}{3.2} a_0 ,$$

$$a_6 = \frac{1}{6.5.3.2} a_0 ,$$

$$a_9 = \frac{1}{9.8.6.5.3.2} a_0 , \ldots$$

In general :

$$a_{3n} = \frac{1}{(3n)(3n-1)(3n-3)(3n-4)\cdots 3.2} a_0 , \quad n \geq 1.$$

Similarly, a_1 determines $a_4, a_7, a_{10}, \ldots, a_{3n+1}$ for $n \geq 1$.

$$a_4 = \frac{1}{4.3} a_1 ,$$

$$a_7 = \frac{1}{7.6.4.3} a_1 ,$$

$$a_{10} = \frac{1}{10.9.7.6.4.3} a_1 , \ldots$$

In general :

$$a_{3n+1} = \frac{1}{(3n+1)(3n)(3n-2)(3n-3)\cdots 4.3} a_1 ,$$

for $n \geq 1$.

Since $a_2 = 0$, $a_5 = a_8 = \cdots = a_{3n+2} = 0$ for $n \geq 1$.

Thus, the general solution of the ODE is

$$y(x) = a_0 \underbrace{\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12{,}960}x^9 + \dots\right]}_{y_1(x)}$$

$$+ \; a_1 \underbrace{\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45{,}360}x^{10} + \dots\right]}_{y_2(x)}$$

(b) Here's an application of the Ratio Test to

$y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots$; $y_2(x)$ can be handled similarly.

Evaluate $\displaystyle \lim_{n\to\infty} \left| \frac{(3n)(3n-1)\dots(3)(2)\,x^{3n+3}}{(3n+3)(3n+2)(3n)(3n-1)\cdots(3)(2)\,x^{3n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^3}{(3n+3)(3n+2)} \right| = 0 \text{ for any } x.$$

Thus, the series defining y_1 converges for every x; same for y_2.

④ $(3-x^2)y'' - 3xy' - y = 0$

Maclaurin series : $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Longrightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these into the ODE :

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$- \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \quad \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$- \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \quad (6a_2 - a_0) + (18a_3 - 3a_1 - a_1)x$$

$$+ \sum_{n=2}^{\infty} \left[3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n a_n - a_n \right] x^n = 0$$

$$\Longrightarrow \quad (6a_2 - a_0) + (18a_3 - 4a_1)x$$

$$+ \sum_{n=2}^{\infty} \left[3(n+2)(n+1) a_{n+2} - (n^2 + 2n + 1) a_n \right] x^n = 0$$

Thus, $6a_2 = a_0 \implies a_2 = \frac{1}{6}a_0$,

$18a_3 = 4a_1 \implies a_3 = \frac{2}{9}a_1$, and

$$a_{n+2} = \frac{(n^2+2n+1)\,a_n}{3(n+2)(n+1)} = \frac{(n+1)\,a_n}{3(n+2)}$$

$$\implies \boxed{a_{n+2} = \frac{(n+1)\,a_n}{3(n+2)}}, \quad n \geq 2 \quad (\text{in fact, for } n \geq 0)$$

$\underline{n=2}$: $a_4 = \frac{3a_2}{3(4)} = \frac{1}{4}a_2 = \frac{1}{24}a_0$

$\underline{n=4}$: $a_6 = \frac{5a_4}{3(6)} = \frac{5}{18}\cdot\frac{1}{24}a_0$

$\underline{n=3}$: $a_5 = \frac{4a_3}{3\cdot 5} = \frac{4}{15}\cdot\frac{2}{9}a_1 = \frac{8}{135}a_1$

$\underline{n=5}$: $a_7 = \frac{6a_5}{3\cdot 7} = \frac{6}{21}\cdot\frac{8}{135}a_1$ etc. etc.

The general solution is therefore

$y_1(x)$

$$y(x) = a_0\left[1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \frac{5}{432}x^6 + \dots\right] +$$

$y_2(x)$

$$a_1\left[x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{945}x^7 + \dots\right].$$

(5) $2xy'' + (x+1)y' + 3y = 0$

Frobenius : $y(x) = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+m}$

$$\Longrightarrow \quad y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} ,$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

Substitute these into the ODE :

$$\sum_{n=0}^{\infty} 2(n+m)(n+m-1) a_n x^{n+m-1} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m}$$
$$+ \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + \sum_{n=0}^{\infty} 3 a_n x^{n+m} = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} 2(n+m)(n+m-1) a_n x^{n+m-1} + \sum_{n=1}^{\infty} (n+m-1) a_{n-1} x^{n+m-1}$$
$$+ \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + \sum_{n=1}^{\infty} 3 a_{n-1} x^{n+m-1} = 0$$

$$\Longrightarrow \left(2m(m-1) + m\right) a_0 x^{m-1} +$$
$$\sum_{n=1}^{\infty} \Big[2(n+m)(n+m-1) a_n + (n+m-1) a_{n-1} + (n+m) a_n + 3 a_{n-1} \Big] x^{n+m-1} = 0$$

indicial equation: $2m^2 - 2m + m = 0$

$$\Longrightarrow \quad 2m^2 - m = 0$$

$$\Longrightarrow \quad m(2m-1) = 0 \quad \Longrightarrow \quad \underline{\underline{m=0}}, \ \underline{\underline{m=\tfrac{1}{2}}}$$

recurrence relation:

$$(n+m)\Big(2n+2m-2+1\Big)a_n + \Big(n+m-1+3\Big)a_{n-1} = 0$$

$$\Longrightarrow \quad (n+m)(2n+2m-1)a_n + (n+m+2)a_{n-1} = 0$$

$$\Longrightarrow \quad \underline{\underline{a_n = \frac{-(n+m+2)a_{n-1}}{(n+m)(2n+2m-1)}, \quad n \geq 1}}$$

$\underline{\boxed{m=0}}$ Now have $a_n = \dfrac{-(n+2)a_{n-1}}{n(2n-1)}, \quad n \geq 1$

$$\Longrightarrow \quad a_1 = \frac{-3a_0}{1 \cdot 1} = -3a_0,$$

$$a_2 = \frac{-4a_1}{2(3)} = -\frac{2}{3} \cdot (-3a_0) = 2a_0,$$

$$a_3 = \frac{-5a_2}{3(5)} = -\frac{a_2}{3} = -\frac{2}{3}a_0 \quad \text{etc.}$$

Taking $a_0 = 1$, $m = 0$ yields the solution

$$y_1(x) = 1 - 3x + 2x^2 - \frac{2}{3}x^3 + \dots$$

$\boxed{m = \frac{1}{2}}$ Now have $$a_n = \frac{-(n+\frac{1}{2}+2)\,a_{n-1}}{(n+\frac{1}{2})(2n)}$$

$$\Longrightarrow \quad a_n = \frac{-(n+\frac{5}{2})\,a_{n-1}}{(2n+1)\cdot n}$$

$$\Longrightarrow \quad a_n = \frac{-(2n+5)\,a_{n-1}}{2n(2n+1)}, \quad n \geq 1$$

$$\Longrightarrow \quad a_1 = \frac{-7a_0}{6},$$

$$a_2 = \frac{-9a_1}{20} = \left(-\frac{9}{20}\right)\left(-\frac{7}{6}\right)a_0 = \frac{21}{40}a_0,$$

$$a_3 = \frac{-11a_2}{42} = \left(\frac{-11}{42}\right)\left(\frac{21}{40}\right)a_0 = \frac{-11}{80}a_0.$$

A second solution is therefore

$$y_2(x) = \left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \frac{11}{80}x^3 + \dots\right)x^{\frac{1}{2}}.$$