#5

$$x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$$

$$e^{\lambda t} v \Longrightarrow \det \begin{pmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-3-\lambda) + 5 = 0$$

$$\underline{\lambda^2 + 2\lambda + 2 = 0} \Longrightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \underline{\underline{-1 \pm i}}$$

$$e^{\lambda t} v = e^{(-1 \pm i)t} v \Longrightarrow \underline{e^{-t} (\cos t \pm i \sin t) \vec{v}}$$

$\underline{\lambda = -1 + i}$:

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-i) v_1 - v_2 = 0 \Longrightarrow \underline{v_2 = (2-i) v_1}$$

$$\Longrightarrow v = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

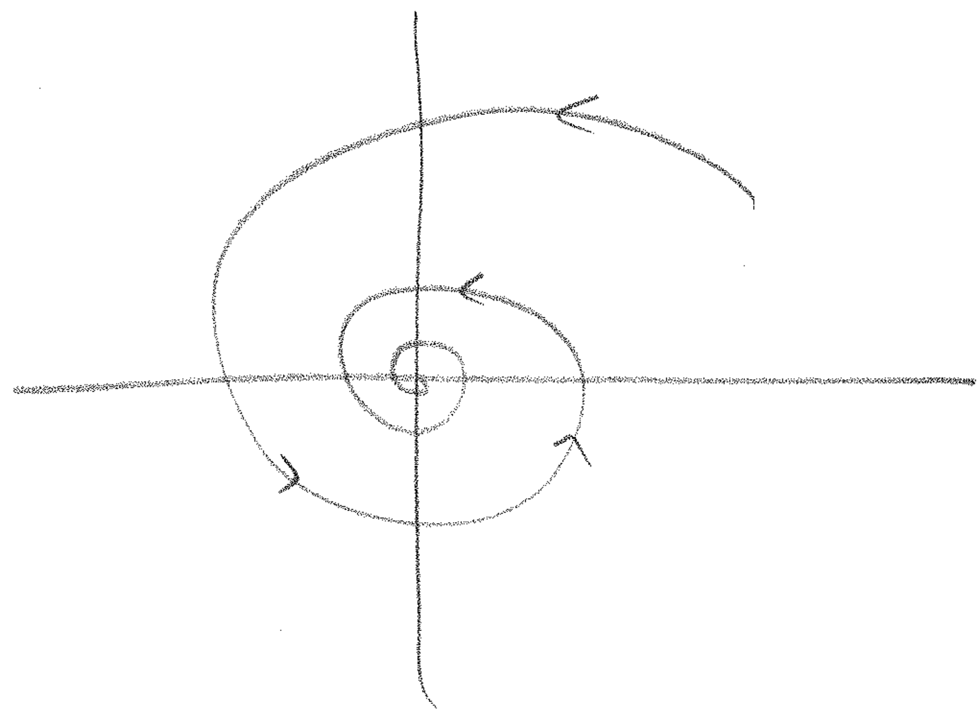
$$e^{(-1+i)t}\begin{pmatrix}1\\2-i\end{pmatrix} = \underline{e^{-t}(\cos t + i\sin t)}\begin{pmatrix}1\\2-i\end{pmatrix}$$

$$= \begin{bmatrix} \underline{e^{-t}(\cos t)} + i(e^{-t}\sin t) \\ (2e^{-t}\cos t + \underline{e^{-t}\sin t}) + i(2e^{-t}\sin t - e^{-t}\cos t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-t}\cos t \\ 2e^{-t}\cos t + e^{-t}\sin t \end{bmatrix}}_{x_1(t)} + i\underbrace{\begin{bmatrix} e^{-t}\sin t \\ 2e^{-t}\sin t - e^{-t}\cos t \end{bmatrix}}_{x_2(t)}$$

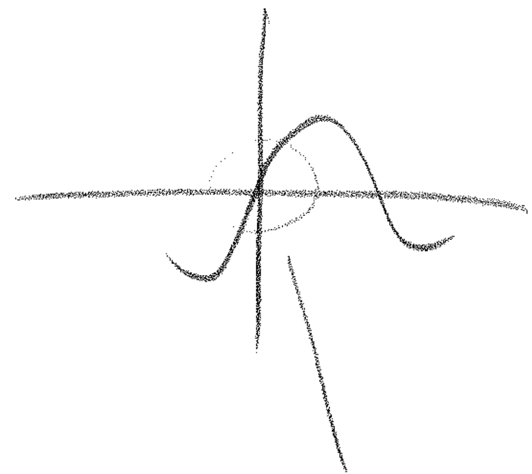
$$x(t) = c_1\begin{bmatrix} e^{-t}\cos t \\ 2e^{-t}\cos t + e^{-t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{-t}\sin t \\ 2e^{-t}\sin t - e^{-t}\cos t \end{bmatrix}$$

$$x(t) = c_1 e^{-t}\begin{bmatrix}\cos t\\ 2\cos t + \sin t\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}\sin t\\ 2\sin t - \cos t\end{bmatrix}$$



$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)} + \text{H.O.Ts}$$

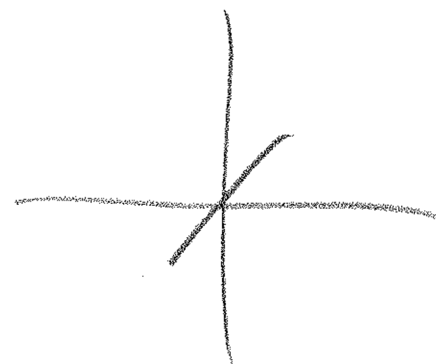
$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{LINEARIZATION}}$$



example : $f(x) = \sin x, \quad x_0 = 0$

$$\sin x \approx 0 + 1 \cdot (x - 0) = x$$

$$\sin x \approx x$$



$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

① Find <u>equilibria</u>, i.e., points (x^*, y^*) where

$$f(x^*, y^*) = g(x^*, y^*) = 0$$

② Linearize at equilibria

$$f(x,y) \approx \overbrace{f(x^*, y^*)}^{0} + \frac{\partial f}{\partial x}(x^*, y^*)(x - x^*) + \frac{\partial f}{\partial y}(x^*, y^*)(y - y^*)$$

$$= \frac{\partial f}{\partial x}(x^*, y^*)(x - x^*) + \frac{\partial f}{\partial y}(x^*, y^*)(y - y^*)$$

$$g(x,y) \approx \frac{\partial g}{\partial x}(x^*, y^*)(x - x^*) + \frac{\partial g}{\partial y}(x^*, y^*)(y - y^*)$$

3) Analyze the linear system

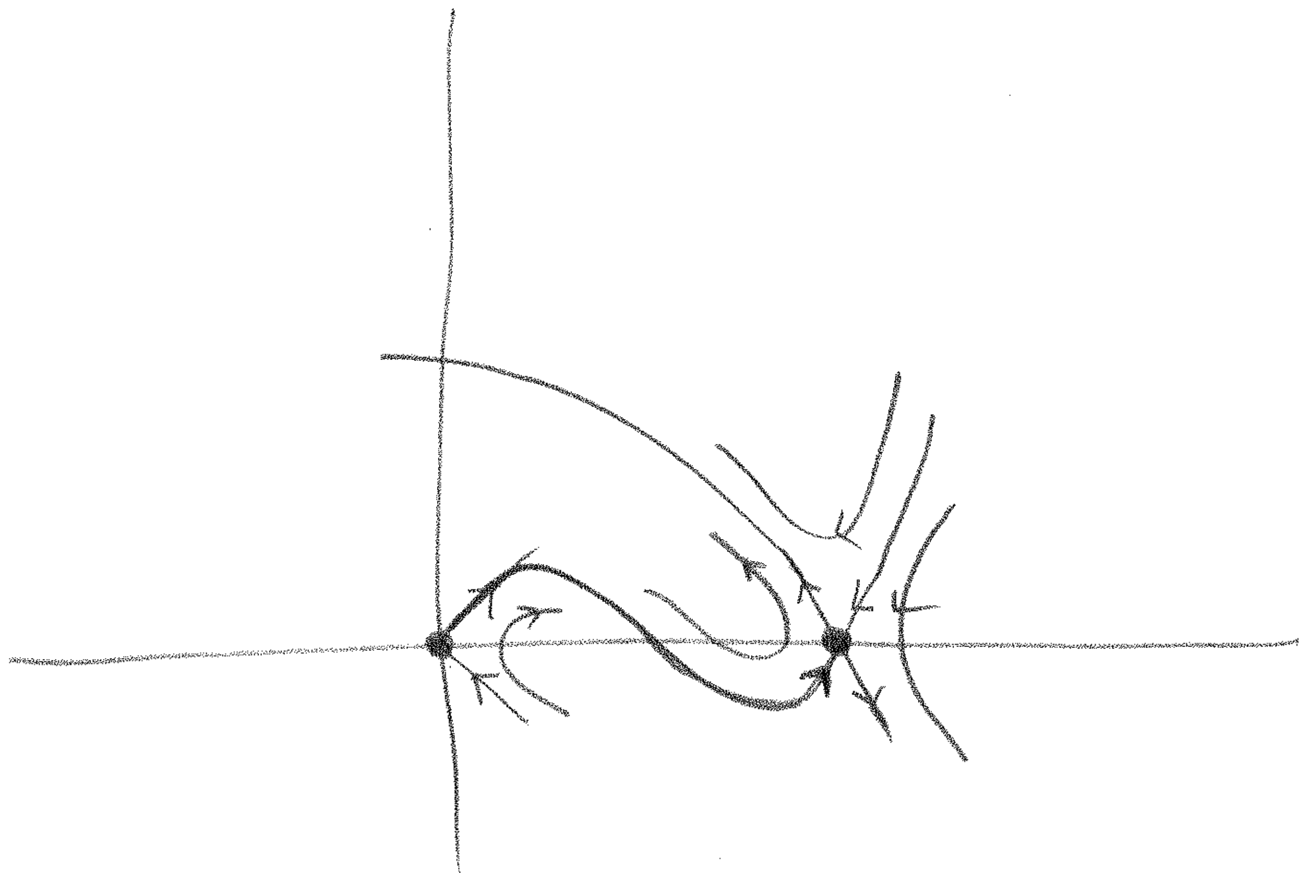
$$x' = \frac{\partial f}{\partial x} \cdot (x - x^*) + \frac{\partial f}{\partial y} \cdot (y - y^*)$$

$$y' = \frac{\partial g}{\partial x} \cdot (x - x^*) + \frac{\partial g}{\partial y} (y - y^*)$$

$$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}}_{A} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

$$(\vec{x})' = A \cdot \vec{x}$$

(4) Connect trajectories!



$$\begin{cases} x' = 4 - 2y = f(x,y) \\ y' = 12 - 3x^2 = g(x,y) \end{cases}$$

① $4 - 2y = 0 \Longrightarrow y = 2$

$12 - 3x^2 = 0 \Longrightarrow x = \pm 2$ $\longrightarrow$ $(2,2),\ (-2,2)$

② $f(x,y) \approx 4 - 2y = \underbrace{f(2,2)} + \frac{\partial f}{\partial x}(2,2)(x-2) + \frac{\partial f}{\partial y}(2,2)(y-2)$

$= 0 + 0\cdot(x-2) + (-2)(y-2)$

$g(x,y) \approx 12 - 3x^2 = \underbrace{g(2,2)} + \frac{\partial g}{\partial x}(2,2)(x-2) + \frac{\partial g}{\partial y}\cdot(y-2)$

$= 0 + (-12)(x-2) + 0\cdot(y-2)$

$= -12(x-2)$

At $(-2,2)$: $g(x,y) \approx 12(x-2)$

At $(-2,2)$:

$$x' = -2(y-2)$$
$$y' = 12(x-2) \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -2 \\ 12 & 0 \end{pmatrix} \begin{pmatrix} x-2 \\ y-2 \end{pmatrix}$$

eigenvalues: $\det \begin{pmatrix} -\lambda & -2 \\ 12 & -\lambda \end{pmatrix} = \lambda^2 + 24 = 0$

$$\underline{\lambda = \pm 2\sqrt{6}\, i}$$

At $(2,2)$:

$$x' = -2(y-2)$$
$$y' = -12(x-2) \implies \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x-2 \\ y-2 \end{bmatrix}$$

eigenvalues: $\det \begin{bmatrix} -\lambda & -2 \\ -12 & -\lambda \end{bmatrix} = 0 \implies \lambda^2 - 24 = 0$

$$\lambda = \pm 2\sqrt{6}$$

$\lambda = 2\sqrt{6}$ $\quad \begin{bmatrix} -2\sqrt{6} & -2 \\ -12 & -2\sqrt{6} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$-2\sqrt{6}\, v_1 - 2v_2 = 0 \implies v_2 = -\sqrt{6}\, v_1 \implies \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$$

$\lambda = -2\sqrt{6}$ $\quad \begin{bmatrix} 2\sqrt{6} & -2 \\ -12 & 2\sqrt{6} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$2\sqrt{6}\, v_1 - 2v_2 = 0 \implies v = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$$