Binomial Theorem For $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + b^n.$$

Lemma $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

$$\boxed{\textbf{Recall: } \binom{n}{k} = \frac{n \cdot \ldots (n-k+1)}{k!}}$$

pf: compute

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot \overbrace{(n-(k-1)+1)}^{(n-k+2)}}{(k-1)!} + \frac{n \cdot \ldots \cdot (n-k+1)}{k!}$$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+2)}{(k-1)!} \left[1 + \frac{n-k+1}{k} \right]$$

$$= \frac{n \cdot \ldots \cdot (n-k+2)}{(k-1)!} \left[\frac{n+1}{k} \right] = \frac{(n+1) \cdot n \cdot \ldots \cdot (n+1-k+1)}{k!}.$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

pf. of binomial theorem

base case: $n=1 \Longrightarrow (a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = a^1 + b^1$ ✓

induction hypothesis: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for some integer n.

Then $(a+b)^{n+1} = (a+b)(a+b)^n$

$$= a(a+b)^n + b(a+b)^n$$

by IH

$$= a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^{k} + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^{k} + b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^{k} . \quad \blacksquare$$

$$\sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1}$$

$m = k+1$

$m-1 = k$

$$= \sum_{m=1}^{n} \binom{n}{m-1} a^{n-(m-1)} b^{m}$$

$$= \sum_{m=1}^{n} \binom{n}{m-1} a^{n+1-m} b^{m}$$

$$= \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^{k}$$