To solve the one-dimensional heat and wave equations with separation of variables, we look for a solution of the form

$$u(x,t) \;=\; X(x)T(t)\,,$$

subject to certain auxiliary conditions. In all of the problems we will consider, the spatial component, $X(x)$, must satisfy a two-point boundary-value problem of the form

$$X'' + \lambda^2 X = 0\,, \quad \text{for} \quad 0 < x < L\,, \tag{1}$$

$$\alpha_0 X(0) + \beta_0 X'(0) = 0\,, \quad \alpha_L X(L) + \beta_L X'(L) = 0\,, \tag{2}$$

for given constants $\alpha_0, \beta_0, \alpha_L, \beta_L$ such that

$$|\alpha_0| + |\beta_0| > 0 \quad \text{and} \quad |\alpha_L| + |\beta_L| > 0\;.$$

Note that

- The length L of the underlying interval is often either 1 or π, as these values simplify some calculations.
- The conditions in (2) are ***Sturm-Liouville boundary conditions***. They include homogeneous Dirichlet boundary conditions ($X(0) = X(L) = 0$) and no-flux boundary conditions ($X'(0) = X'(L) = 0$) as special cases.

To solve this boundary-value problem, we have to find both a function $X(x)$ and a constant λ ; having done so, the function is called an ***eigenfunction***, the constant is called an ***eigenvalue***, and the two together form an ***eigenpair***. In the problems we consider, we will obtain a countable sequence $\{(X_n, \lambda_n)\}$ of eigenpairs, either for $n = 0, 1, 2, \ldots$ or for $n = 1, 2, 3, \ldots$. This note provides a proof that eigenfunctions corresponding to different eigenvalues are ***orthogonal***, which means that

$$\int_0^L X_n X_m\,dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m\;.$$

The advantage of this proof is its generality; the following argument eliminates the use of various trigonometric identities or other random trivia.

Suppose that X_n and X_m are eigenfunctions corresponding to the distinct eigenvalues λ_n and λ_m, respectively. Then, by the definition above,

$$X_n'' + \lambda_n^2 X_n = 0\,, \tag{3}$$

$$X_m'' + \lambda_m^2 X_m = 0\,. \tag{4}$$

Multiply (3) by X_m and multiply (4) by X_n to obtain

$$X_m X_n'' + \lambda_n^2 X_m X_n = 0\,, \tag{5}$$

$$X_m'' X_n + \lambda_m^2 X_m X_n = 0\,. \tag{6}$$

Subtract (6) from (5) to get

$$X_m X_n'' - X_m'' X_n + \left(\lambda_n^2 - \lambda_m^2\right) X_m X_n = 0 \tag{7}$$

and integrate this equation from $x = 0$ to $x = L$:

$$\int_0^L \left(X_m X_n'' - X_m'' X_n\right)\, dx + \left(\lambda_n^2 - \lambda_m^2\right) \int_0^L X_m X_n \, dx = 0 \ .$$

Rearrange a bit and use integration by parts to get the following:

$$\begin{aligned}
\left(\lambda_n^2 - \lambda_m^2\right) \int_0^L X_m X_n \, dx &= \int_0^L \left(X_m'' X_n - X_m X_n''\right)\, dx \\
&= \int_0^L X_m'' X_n \, dx - \int_0^L X_m X_n'' \, dx \\
&= \left(X_n X_m' \Big|_0^L - \int_0^L X_m' X_n' \, dx \right) - \left(X_n' X_m \Big|_0^L - \int_0^L X_m' X_n' \, dx \right) \\
&= \left(X_n X_m' - X_n' X_m\right) \Big|_0^L = 0 \ .
\end{aligned}$$

The fact that this final boundary term vanishes is a consequence of the Sturm-Liouville boundary conditions above. (Be sure to check this for yourself!) Finally, since $\lambda_n \neq \lambda_m$, we see that

$$\int_0^L X_m X_n \, dx = 0$$

as claimed.