$$\int \tan(t)\,dt = \int \frac{\sin(t)}{\cos(t)}\,dt = \int -\frac{1}{u}\,du = -\ln|u| + c$$

$$= -\ln|\cos(t)| + c$$

$$= \ln|\sec(t)| + c$$

$u = \cos(t)$

$du = -\sin(t)\,dt$

$-du = \sin(t)\,dt$

$$\int \sec(t)\,dt = \ln|\sec t + \tan t| + c$$

#10 $t^2 y'' - 2y = 3t^2 - 1$, $y_1 = t^2$, $y_2 = t^{-1}$

$y_1 = t^2 \Longrightarrow \underline{y_1' = 2t, \; y_1'' = 2} \Longrightarrow t^2 y_1'' - 2y_1 = t^2 \cdot 2 - 2t^2 = 0$

$y_2 = t^{-1} \Longrightarrow \underline{y_2' = -t^{-2}, \; y_2'' = 2t^{-3}} \Longrightarrow t^2 y_2'' - 2y_2 = 2t^{-1} - 2t^{-1} = 0$

$y_p = u_1 y_1 + u_2 y_2 \Longrightarrow \begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = 3t^2 - 1 \end{array}$

$\Longrightarrow \begin{array}{l} u_1' t^2 + u_2' t^{-1} = 0 \\ u_1' \cdot 2t + u_2' (-t^{-2}) = 3t^2 - 1 \end{array}$

$$\Longrightarrow \quad 2u_1' t^2 - 2u_2' t^{-1} = 0$$

$$u_1'(2t^2) - u_2'(-t^{-1}) = 3t^3 - t$$

$$\Longrightarrow \quad \frac{3}{t} u_2' = 3t^3 - t \Longrightarrow 3u_2' = 3t^4 - t^2$$

$$\Longrightarrow \quad u_2' = t^4 - \frac{1}{3} t^2$$

$$\Longrightarrow \quad u_2 = \frac{1}{5} t^5 - \frac{1}{9} t^3$$

$$\Longrightarrow \quad u_1' t^2 + \left(t^4 - \frac{1}{3} t^2\right) \cdot t^{-1} = 0$$

$$\Longrightarrow \quad u_1' t^2 + t^3 - \frac{1}{3} t = 0 \Longrightarrow u_1' = \frac{1}{3} t^{-1} - t$$

$$\Longrightarrow \quad u_1 = \frac{1}{3} \ln|t| - \frac{1}{2} t^2$$

$$\Longrightarrow \quad y(t) = c_1 t^2 + c_2 t^{-1} + \left(\frac{1}{3} \ln t - \frac{1}{2} t^2\right) t^2 + \left(\frac{1}{5} t^5 - \frac{1}{9} t^3\right) t^{-1}$$

$$t^2 y'' + \alpha t y' + \beta y = 0 \quad \text{\underline{Euler} equation}$$

$$y(t) = t^r \Longrightarrow y' = r t^{r-1}, \quad y'' = r(r-1) t^{r-2}$$

$$t^2 \cdot r(r-1) t^{r-2} + \alpha t \cdot r t^{r-1} + \beta t^r = 0$$

$$\Longrightarrow \left[r(r-1) + \alpha r + \beta \right] t^r = 0$$

$$\underline{r^2 - r + \alpha r + \beta} = 0$$

<u>Example</u>: $t^2 y'' - 2y = 0$ $\qquad y_1 = t^2, \; y_2 = t^{-1}$

$$y(t) = t^r \Longrightarrow \underbrace{\left[r(r-1) - 2 \right]}\, t^r = 0$$

$$r^2 - r - 2 = 0 \Longrightarrow (r-2)(r+1) = 0$$

$$\Longrightarrow r = 2, \; r = -1$$

#28 $t^2y'' - 4ty' - 6y = 0$

$y(t) = t^r \implies \left[r(r-1) - 4r - 6 \right] t^r = 0$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$$r^2 - r - 4r - 6 = 0$$

$$r^2 - 5r - 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r = 2,\ r = 3 \implies y_1 = t^2,\ y_2 = t^3$$

$$\boxed{y(t) = c_1 t^2 + c_2 t^3}$$

#33 $t^2y'' + 3ty' + y = 0$

$y = t^r \Longrightarrow [r(r-1) + 3r + 1]t^r = 0$

$\Longrightarrow r^2 - r + 3r + 1 = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \Longrightarrow \underline{\underline{r = -1}} \Longrightarrow \underline{\underline{t^{-1}}} = y_1$

$y_2 = uy_1 \Longrightarrow y_2' = u'y_1 + uy_1'$, $y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$y_2 = ut^{-1} \Longrightarrow y_2' = u't^{-1} - t^{-2}u$

$y_2'' = u''t^{-1} - t^{-2}u' - t^{-2}u' + 2t^{-3}u$

$= \underline{u''t^{-1} - 2t^{-2}u' + 2t^{-3}u}$

$\Longrightarrow u''t - 2u' + \cancel{2t^{-1}u} + 3u' - \cancel{3t^{-1}u} + \cancel{ut^{-1}} = 0$

$\Longrightarrow u''t + u' = 0 \Longrightarrow$ Let $v = u'$: $tv' + v = 0$

$$tv' + v = 0 \Rightarrow tv' = -v$$

$$\Rightarrow \frac{v'}{v} = -\frac{1}{t}$$

$\Rightarrow$ Integrate: $\ln|v| = -\ln|t| = \ln\left(\frac{1}{t}\right)$

$$v = \frac{1}{t} \Rightarrow u' = \frac{1}{t} \Rightarrow \underline{\underline{u = \ln t}}$$

$$\Rightarrow y_2 = \frac{\ln|t|}{t}$$

$\Rightarrow$ General sol'n: $\underline{\underline{y(t) = c_1 t^{-1} + c_2 \frac{\ln|t|}{t}}}$