Quiz 7

Name: SOLUTIONS

1. Let $T\colon V \longrightarrow W$ be a homomorphism.

(a) Define the *range space* and *rank* of T.

range space : $\{w \in W \mid T(v) = w \text{ for some } v \in V\} = T(V) =$ image

rank = dim (range space)

(b) Define the *null space* and *nullity* of T.

null space : $\{v \in V \mid T(v) = 0\}$ = kernel of T = $N(T)$

nullity = dim(null space)

(c) State the Rank–Nullity Theorem.

$\dim V = \text{rank}(T) + \text{nullity}(T)$

2. Consider the specific homomorphism $T\colon \mathcal{M}_{2\times 2} \longrightarrow \mathbb{R}^2$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} a \\ d \end{bmatrix}.$$

(a) Determine the range space and rank of T.

The arbitrary vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is the image of, say, $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ in $M_{2\times 2}$; in other words, every element of $\mathbb{R}^2$ is in the range space. $\underline{R(T) = \mathbb{R}^2}$, so $\underline{\text{rank}(T) = 2}$.

(b) Determine the null space and nullity of T.

if $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $a = 0$ & $d = 0$; thus,

$N(T) = \left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

and $\underline{\text{nullity}(T) =}$ 2.