#17

$$x' = a_{11}x + a_{12}y$$
$$y' = a_{21}x + a_{22}y$$
$$\Longrightarrow \boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}}$$

$$\Longrightarrow \quad \vec{x}' = A\vec{x}, \quad \boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad \& \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{x} = e^{\lambda t}\vec{v} \Longrightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Longrightarrow \lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

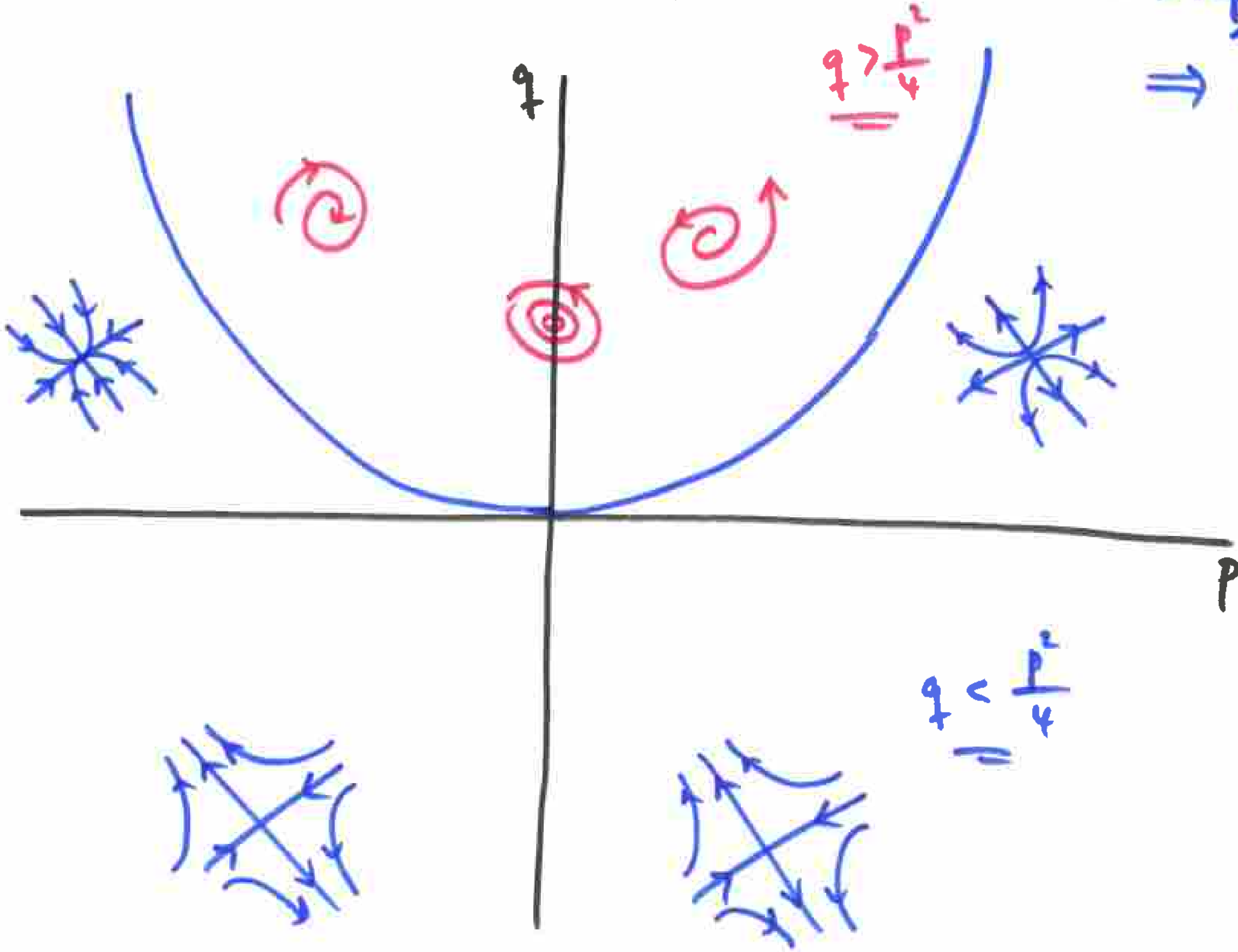
$$\boxed{\begin{aligned} &\lambda^2 - p\lambda + q = 0 \\ &p = tr(A),\ q = det(A) \end{aligned}} \Longleftarrow \lambda^2 - \underbrace{(a+d)}_{p = a+d = trace(A) = tr(A)}\lambda + \underbrace{(ad-bc)}_{\det A = q} = 0$$

$$\underline{\lambda^2 - p\lambda + q = 0} \implies \lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

$p^2 - 4q = 0$

$\implies$ repeated eigenvalues

$\implies \underline{\underline{q = \frac{p^2}{4}}}$



$$x' = x(3 - x - 2y)$$
$$y' = y(2 - x - y)$$

$\begin{pmatrix} x = \text{rabbits} \\ y = \text{sheep} \end{pmatrix}$ $\quad$ $\begin{pmatrix} \text{Lotka-} \\ \text{Volterra model} \end{pmatrix}$

Equilibria = <u>critical points</u>: $x' = y' = 0$

$$x(3 - x - 2y) = 0 \quad \& \quad y(2 - x - y) = 0$$

$x = 0, \quad y(2 - y) = 0 \Longrightarrow y = 0, \ y = 2 \qquad \underline{\underline{(0,0)}}, \ \underline{\underline{(0,2)}}$

$y = 0, \quad x(3 - x) = 0 \Longrightarrow x = 0, \ x = 3 \qquad \underline{\underline{(3,0)}}$

$x \neq 0, y \neq 0$: $\quad \begin{array}{l} 3 - x - 2y = 0 \\ 2 - x - y = 0 \end{array} \Longrightarrow \begin{array}{r} x + 2y = 3 \\ \underline{x + y = 2} \end{array} \qquad \underline{\underline{(1,1)}}$

$$y = 1, \ x = 1$$

$x' = 3x - x^2 - 2xy \approx 3x$ when (x,y) is close to $(0,0)$

$y' = 2y - xy - y^2 \approx 2y$ " " " " " "

Thus, near $(0,0)$:

$$\begin{aligned} x' &= 3x \\ y' &= 2y \end{aligned} \implies \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left\{ \begin{aligned} & e^{\lambda t}\vec{v} \implies \lambda = 3, 2 \\ & \lambda = 3 \implies \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\ & \lambda = 2 \implies \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \right.$$

$$x' = f(x,y)$$
$$y' = g(x,y)$$

critical point (x^*,y^*) : $f(x^*,y^*) = g(x^*,y^*) = 0$

near (x^*,y^*) : $f(x,y) = \underbrace{f(x^*,y^*)}_{0} + \frac{\partial f}{\partial x}(x^*,y^*)(x-x^*)$
$$+ \frac{\partial f}{\partial y}(x^*,y^*)(y-y^*) + \text{higher order terms}$$

$$\Longrightarrow f(x,y) \approx \frac{\partial f}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial f}{\partial y}(x^*,y^*)(y-y^*)$$

LINEARIZATIONS NEAR (x^*,y^*)

$$\& \quad g(x,y) \approx \frac{\partial g}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial g}{\partial y}(x^*,y^*)(y-y^*)$$

Thus, near (x^*, y^*):

$$x' \approx \frac{\partial f}{\partial x} \cdot (x - x^*) + \frac{\partial f}{\partial y} \cdot (y - y^*)$$

$$y' \approx \frac{\partial g}{\partial x} \cdot (x - x^*) + \frac{\partial g}{\partial y} \cdot (y - y^*)$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}$$

Jacobian matrix

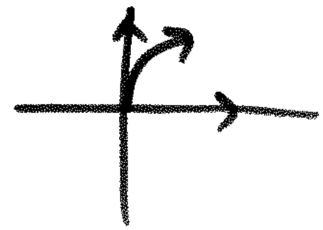
or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}$$

remember: each derivative is evaluated at (x^*, y^*)!

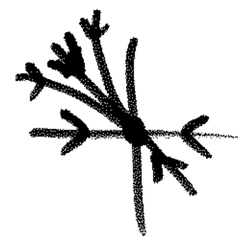
$\underline{\text{Here}}$: $$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

At $(0,0)$: $$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

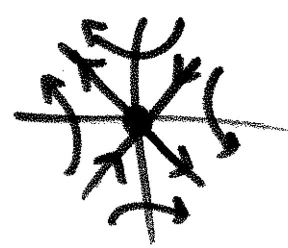


At $(0,2)$: $$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

At $(3,0)$: $$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$



At $(1,1)$: $$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$



At $(0,2)$:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$e^{\lambda t} \vec{v} \Longrightarrow \det \begin{pmatrix} -1-\lambda & 0 \\ -2 & -2-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow (-1-\lambda)(-2-\lambda) = 0 \Longrightarrow \lambda = -1, \ \lambda = -2$$

$\lambda = -1$:

$$\begin{pmatrix} 0 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -2v_1 - v_2 = 0$$

$$v_2 = -2v_1$$

$$v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\lambda = -2$:

$$\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow v_1 = 0$$

$$\longrightarrow v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

At $(3,0)$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\Longrightarrow \det \begin{pmatrix} -3-\lambda & -6 \\ 0 & -1-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow (-3-\lambda)(-1-\lambda) = 0 \Longrightarrow \lambda = -3,\ \lambda = -1$$

$\underline{\lambda = -3}$: $\begin{pmatrix} 0 & -6 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \underline{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\underline{\lambda = -1}$: $\begin{pmatrix} -2 & -6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow -2v_1 - 6v_2 = 0$$

$$v_1 = -3v_2 \qquad \underline{v = \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

At $(1,1)$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\Longrightarrow \det \begin{pmatrix} -1-\lambda & -2 \\ -1 & -1-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow \lambda^2 + 2\lambda + 1 - 2 = 0$$

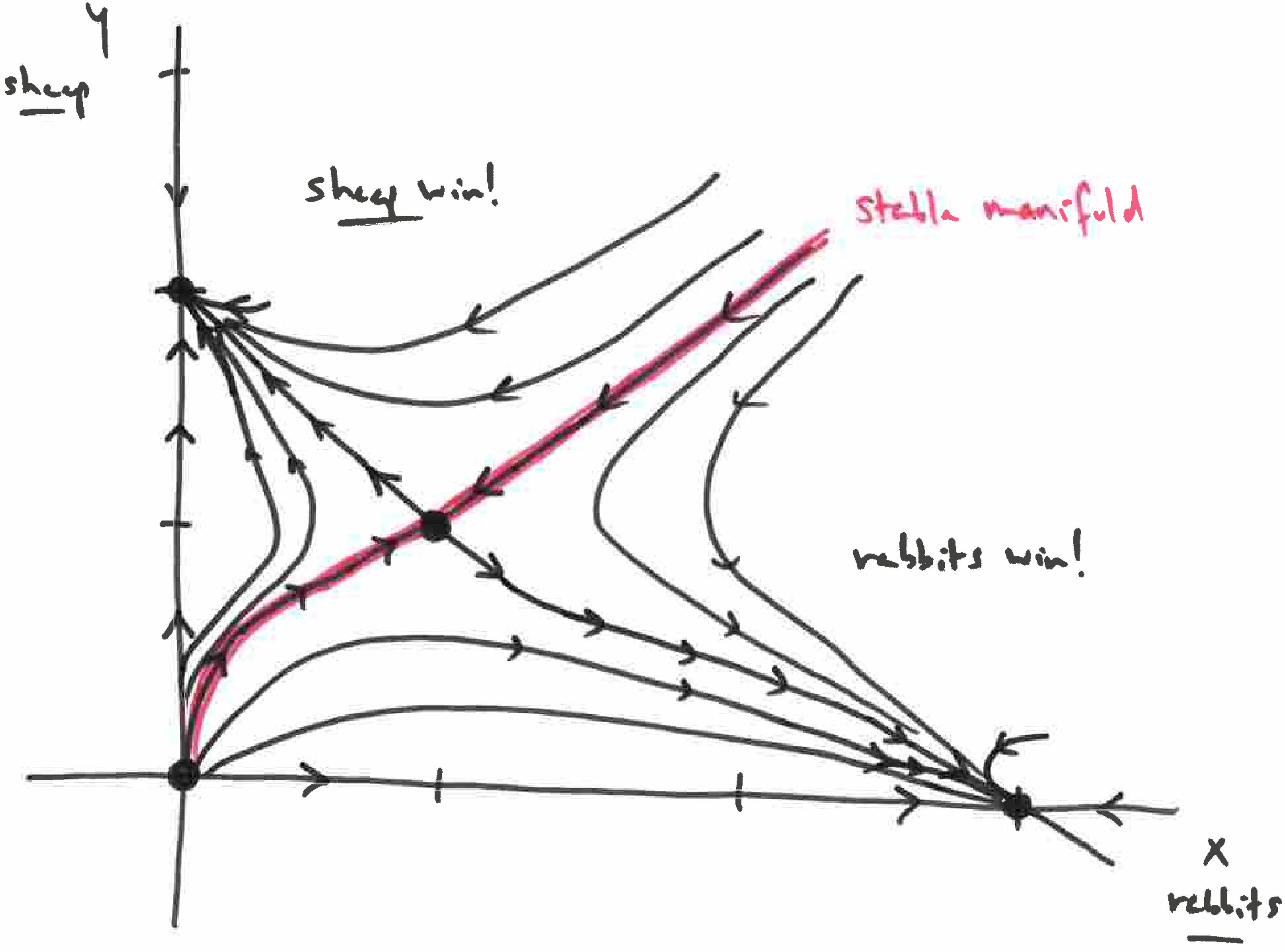
$$\Longrightarrow \lambda^2 + 2\lambda - 1 = 0 \Longrightarrow \lambda = \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \underline{\underline{-1 \pm \sqrt{2}}}$$

$\underline{\lambda = -1 + \sqrt{2}}$: $\begin{pmatrix} -\sqrt{2} & -2 \\ -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

$$\Longrightarrow -v_1 - \sqrt{2}\, v_2 = 0 \Longrightarrow v_1 = -\sqrt{2}\, v_2$$

$\lambda = -1 - \sqrt{2}$: $\begin{pmatrix} \sqrt{2} & -2 \\ -1 & +\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$ $\underline{\underline{v}} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$


y
sheep
sheep win!
stable manifold
rabbits win!
x
rabbits

#10 $x' = x - y^2 = f(x,y)$

$y' = y - x^2 = g(x,y)$

nullclines!

critical points: $x' = y' = 0$

$x = y^2$ & $y = x^2$

$x = x^4 \implies x^4 - x = 0$

$\implies x(x^3 - 1) = 0$

$\implies x = 0,\ x = 1$

$\implies (0,0)\ ,\ (1,1)$

J:

$$\begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}$$

At $(0,0)$:

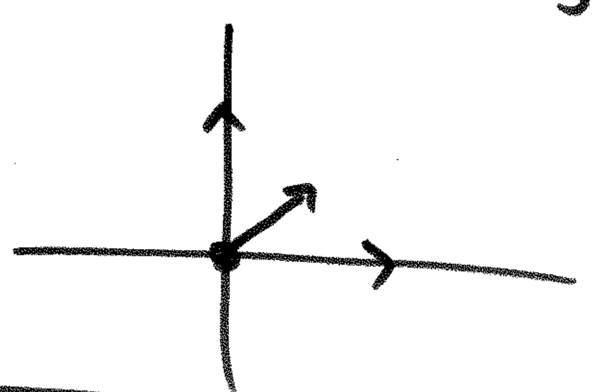
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

At $(1,1)$:

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

At $(0,0)$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\underline{\lambda = 1}$, $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors



At $(1,1)$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\det \begin{pmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 - 2\lambda + 1 - 4 = 0$

$\Longrightarrow \lambda^2 - 2\lambda - 3 = 0$

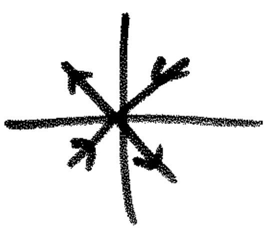
$\Longrightarrow (\lambda - 3)(\lambda + 1) = 0 \qquad \lambda = 3, \lambda = -1$

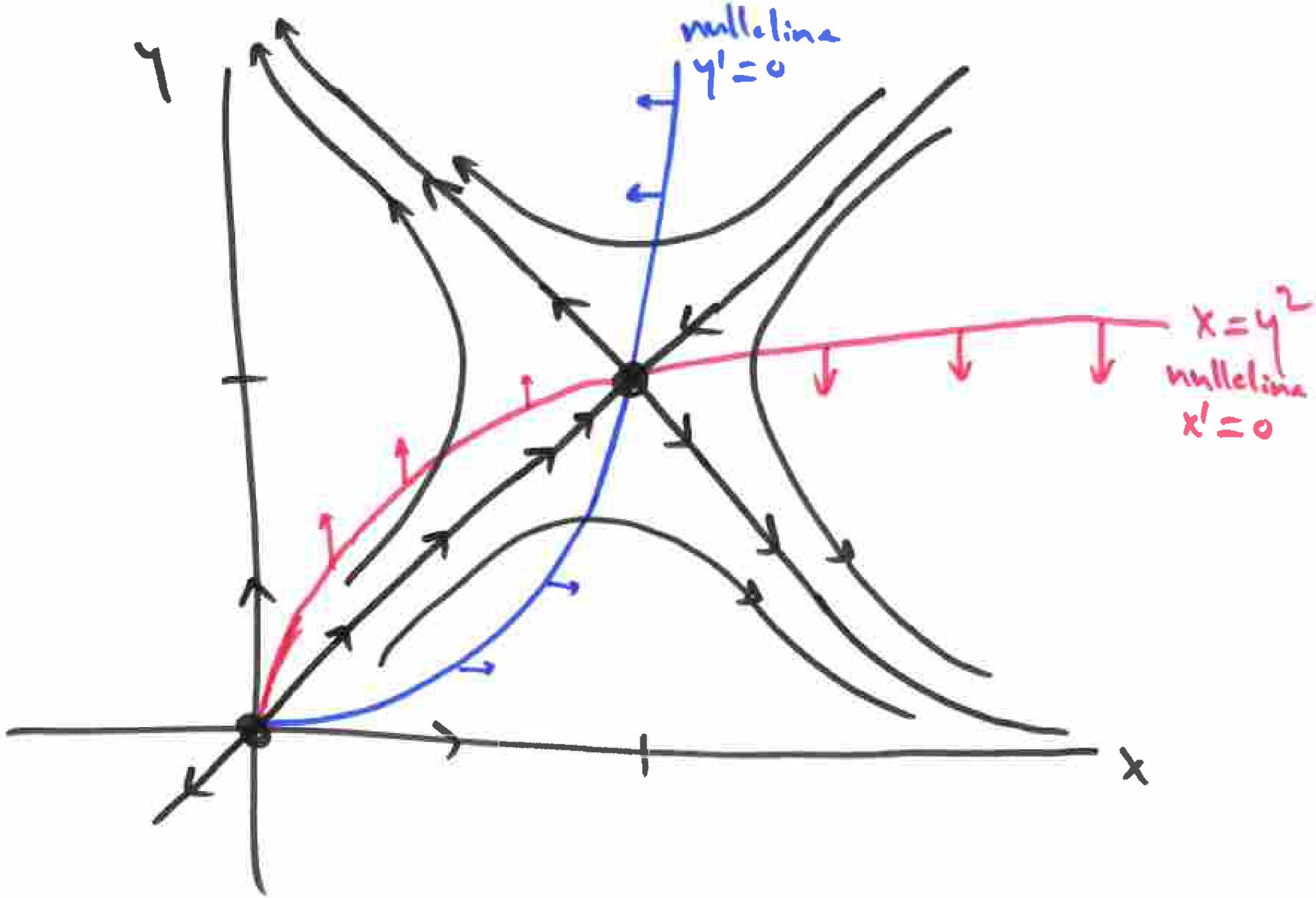
$\lambda = 3$: $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\vec{v} = \vec{0}$

$\underline{v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$\underline{\lambda = -1}$: $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\vec{v} = 0$

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$




nullcline
y' = 0
x = y^2
nullcline
x' = 0
y
x