Math 212 Problem Set #1 - Solutions

① (a) $y(x) = e^{mx} \implies y' = me^{mx}, \; y'' = m^2 e^{mx}, \; y''' = m^3 e^{mx}$

Substitute into the given ODE:

$$m^3 e^{mx} - 3m^2 e^{mx} - 4me^{mx} + 12e^{mx} = 0$$

Cancel e^{mx}:

$$m^3 - 3m^2 - 4m + 12 = 0$$

Fortunately, it's not too hard to find the solution $\underline{m=2}$ by inspection: $8 - 3(4) - 4(2) + 12 = 0$.

Factoring out $(m-2)$ yields

$$(m-2)(m^2 - m - 6) = 0,$$ from which we get

$$(m-2)(m+2)(m-3) = 0.$$

So the values of m that make $y = e^{mx}$ a solution of the given ODE are $\underline{\underline{m = 2, -2, 3}}$.

① (b) $x(t) = t^m \Longrightarrow x' = mt^{m-1}$, $x'' = m(m-1)t^{m-2}$,

$$x''' = m(m-1)(m-2)t^{m-3}$$

Substitute these into the ODE :

$$t^3\left(m(m-1)(m-2)t^{m-3}\right) + 2t^2\left(m(m-1)t^{m-2}\right)$$
$$-10t\left(mt^{m-1}\right) - 8t^m = 0$$

$$\Longrightarrow \left(m(m-1)(m-2) + 2m(m-1) - 10m - 8\right)t^m = 0$$

$$\Longrightarrow m(m-1)(m-2) + 2m(m-1) - 10m - 8 = 0$$

$$\Longrightarrow m(m^2 - 3m + 2) + 2m^2 - 2m - 10m - 8 = 0$$

$$\Longrightarrow m^3 - 3m^2 + 2m + 2m^2 - 2m - 10m - 8 = 0$$

$$\Longrightarrow m^3 - m^2 - 10m - 8 = 0.$$

Now notice that $\underline{m = -1}$ is a solution and factor out $(m+1)$ to get

$$(m+1)(m^2 - 2m - 8) = 0$$

$$\Longrightarrow (m+1)(m-4)(m+2) = 0$$

The values of m that make t^m a solution are thus $\underline{\underline{m = -1, 4, -2}}$.

② $y' + y = 2xe^{-x}$

(a) $y(x) = (x^2+c)e^{-x}$ is a solution:

$$y' = 2xe^{-x} + (x^2+c)(-e^{-x}) = (2x - x^2 - c)e^{-x},$$

so $y' + y = (2x - x^2 - c)e^{-x} + (x^2+c)e^{-x} = 2xe^{-x}$,

as necessary.

(b) solution with $y(0) = 2$; find c so that

$$y(0) = (0+c)e^{-0} = c = 2 \implies \underline{\underline{c=2}}$$

The solution is $\underline{\underline{y(x) = (x^2+2)e^{-x}}}$.

solution with $y(-1) = e+3$;

$$y(-1) = (1+c)e = \underbrace{e + ce = e+3}_{ce = 3 \implies c = \frac{3}{e}}$$

The solution is $\underline{\underline{y(x) = (x^2 + \frac{3}{e})e^{-x}}}$.

③ $x'' + x = 0$

(a) $x(t) = c_1 \sin t + c_2 \cos t$ is a solution:

$$x' = c_1 \cos t - c_2 \sin t, \quad x'' = -c_1 \sin t - c_2 \cos t,$$

so $x'' + x = -c_1 \sin t - c_2 \cos t + c_1 \sin t + c_2 \cos t = 0$,

as necessary.

(b) solution with $x(0) = 0$ and $x(\frac{\pi}{2}) = 1$:

$$x(0) = c_1 \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} = \underline{\underline{c_2 = 0}}.$$

$$x(\tfrac{\pi}{2}) = c_1 \underbrace{\sin(\tfrac{\pi}{2})}_{1} + c_2 \underbrace{\cos(\tfrac{\pi}{2})}_{0} = \underline{\underline{c_1 = 1}}$$

The solution is $\underline{\underline{x(t) = \sin t}}$.

solution with $x(0) = 1$ and $x'(\frac{\pi}{2}) = -1$:

$x(0) = c_2$ (by above), so $c_2 = 1$.

$$x'(\tfrac{\pi}{2}) = c_1 \underbrace{\cos(\tfrac{\pi}{2})}_{0} - c_2 \underbrace{\sin(\tfrac{\pi}{2})}_{1} = -c_2 = -1 \Rightarrow c_2 = 1$$

The solution is $\underline{\underline{x(t) = c_1 \sin t + \cos t}}$, $\underline{\text{any}}$ c_1.

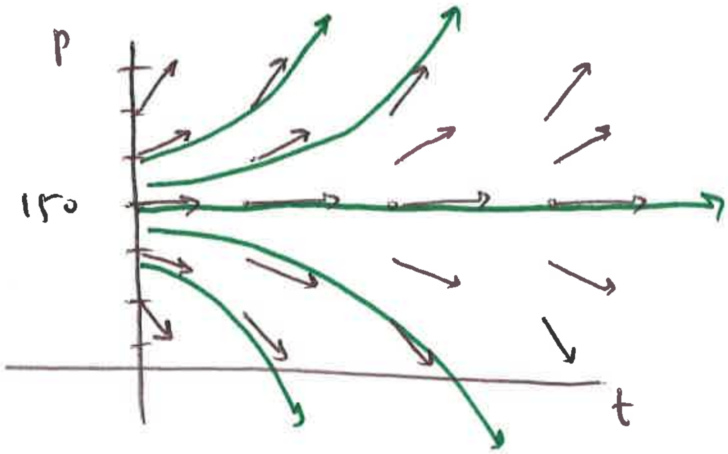
solution with $x(0) = 0$ and $x(\pi) = 1$:

$x(0) = c_2 = 0$ and $x(\pi) = -c_2 = 1$ } No $\underline{\text{solution}}$.

inconsistent!

④ $p' = 2p - 300, \quad p(0) = p_0$

(a)



(b) When $p_0 = 150$, $p(t) = 150$ for all t.

If $p_0 < 150$, $p(t)$ decreases forever; if $p_0 > 150$, $p(t)$ increases forever.

(c) Solution: $\dfrac{p'}{2p-300} = 1 \implies \displaystyle\int \frac{p'}{2p-300}\,dt = t + c$

$$\implies \tfrac{1}{2} \ln|2p - 300| = t + c$$

$$\implies \ln|2p-300| = 2t + c$$

$$\implies 2p - 300 = ce^{2t}$$

$$\implies 2p = 300 + ce^{2t}$$

$$\implies p(t) = 150 + ce^{2t}$$

when $t = 0$: $p_0 = 150 + c \implies c = p_0 - 150$.

The solution is therefore

$$p(t) = 150 + (p_0 - 150)\, e^{2t}.$$

(Note: This behaves as described in (b).)

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(5) $y' = -k(y-T)$, $y(0) = y_0$

(a) The coefficient of $(y-T)$ is negative because y decreases if $y_0 > T$ (something hot cools off) and y increases if $y_0 < T$ (something cold heats up).

This coefficient gives y' the appropriate sign.

(b) Solution:
$$\frac{y'}{y-T} = -k \implies \int \frac{y'}{y-T} = -kt + c$$

$$\implies \ln|y-T| = -kt + c$$

$$\implies y - T = ce^{-kt}$$

$$\implies y = T + ce^{-kt}$$

when $t=0$: $y_0 = T + c \implies c = y_0 - T$.

The solution is thus

$$y(t) = T + (y_0 - T)e^{-kt}.$$

(c) t^* is the time at which

$$y(t^*) - T = \frac{1}{2}(y_0 - T)$$

$$\Longrightarrow \quad y(t^*) = T + \frac{1}{2}(y_0 - T).$$

By the formula above, this means that

$$T + (y_0 - T)e^{-kt^*} = T + \frac{1}{2}(y_0 - T).$$

$$\Longrightarrow \quad (y_0 - T)e^{-kt^*} = \frac{1}{2}(y_0 - T)$$

$$\Longrightarrow \quad e^{-kt^*} = \frac{1}{2}$$

$$\Longrightarrow \quad -kt^* = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\Longrightarrow \quad kt^* = \ln 2$$