1(b), p. 15

T is triangular iff $8T+1$ is a perfect square.

pf: T triangular $\Longrightarrow T = \frac{n(n+1)}{2}$ for some $n \in \mathbb{N}$.

$$\begin{aligned} \Longrightarrow 8T+1 &= 4n(n+1)+1 \\ &= 4n^2+4n+1 \\ &= (2n+1)^2. \end{aligned}$$

$8T+1$ a perfect square $\Longrightarrow 8T+1 = (2n+1)^2$ for some $n \in \mathbb{N}$.

Then reverse the steps to see that T is triangular.

$\blacksquare$

1(c)

$$\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{n+1}{2}\left[n + n+2\right]$$

$$= \left(\frac{n+1}{2}\right)(2n+2)$$

$$= (n+1)^2.$$

$\blacksquare$

Euclid's Division Lemma

If $a > b$ $(a, b \in \mathbb{N})$, then $\exists!$ $q, r \in \mathbb{N}$ such that

$$a = qb + r, \quad \text{with} \quad 0 \leq r < b.$$

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example: $a = 17$, $b = 5 \Longrightarrow q = 3$, $r = 2$,

since $17 = 3 \cdot 5 + 2$

q = quotient, b = divisor

r = remainder, a = dividend

If $r = 0$, then b is a **divisor** of a.

Notation: $b \mid a$

If $r \neq 0$, then $b \nmid a$.

<u>pf</u> : Consider the set $S = \{a - nb \mid n \in \mathbb{N},\ a - nb \geq 0\}$.

Since $a > b$, $a - b \in S$; thus, S is not empty.

The <u>Well-Ordering Principle</u> guarantees that <u>any</u> nonempty set of positive integers has a smallest element.

<u>Define</u> r to be the smallest element of S. Thus,

$$r = a - qb \text{ for some } q \in \mathbb{N}, \text{ and}$$

$$a = qb + r.$$

To see that $r < b$, suppose not : then

$$r = a - qb \geq b$$

$$\Longrightarrow \quad a - qb - b \geq 0$$

$$\Longrightarrow \quad \underbrace{a - (q+1)b}_{\in S} \geq 0,$$ contradicting the minimality of r.

∎

#3(a) n^2 is either $3k$ or $3k+1$:

$n \in \mathbb{N} \implies n = 3m$ or $n = 3m+1$ or $n = 3m+2 = 3m+3-1$

$n^2 = 9m^2$	$n^2 = 9m^2 + 6m + 1$	$n^2 = 9m^2 + 12m + 4$
$= 3(3m^2)$	$= 3\underbrace{(3m^2+2m)}_{k} + 1$	$= 3\underbrace{(3m^2+4m+1)}_{k} + 1$
$k = 3m^2$		

$n = 3m$ means that n is congruent to 0 modulo 3,

$n \equiv 0 \pmod 3$

$n = 3m+1$ means that n is congruent to 1 mod 3,

$n \equiv 1 \pmod 3$

$n = 3m+2$ means that n is congruent to 2 mod 3,

$n \equiv 2 \pmod 3$