2.1

11(d) $\underbrace{n \text{ triangular}}_{n = \frac{m(m+1)}{2} \text{ for some } m \in \mathbb{N}} \Longrightarrow$ so are $9n+1$, $25n+3$, $49n+6$

Then $9n+1 = \dfrac{9m(m+1)}{2} + 1 = \dfrac{9m^2+9m+2}{2} = \dfrac{(3m+1)(3m+2)}{2}$

$= \dfrac{N(N+1)}{2}$ for $N = 3m+1$, hence triangular.

Similarly, $25n+3 = \dfrac{25m(m+1)}{2} + 3 = \dfrac{25m^2+25m+6}{2} = \dfrac{(5m+2)(5m+3)}{2}$

$= \dfrac{N(N+1)}{2}$ for $N = 5m+2$

and $49n+6 = \dfrac{49m^2+49m+12}{2} = \dfrac{(7m+3)(7m+4)}{2} = \dfrac{N(N+1)}{2}$

for $N = 7m+3$. ∎

7 $$\left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{n+1}{2}\right)^2\left[(n+2)^2 - n^2\right]$$

$$= \frac{(n+1)^2}{4}\left(n^2+4n+4-n^2\right)$$

$$= \frac{(n+1)^2}{4}(4n+4) = (n+1)^3.$$

$\blacksquare$

8 $$t_n = \frac{n(n+1)}{2} \Longrightarrow \frac{1}{t_n} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Then $$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{t_n} = 1 + 2\left(\frac{1}{2} - \frac{1}{3}\right) + 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$+ \dots + 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 + 2\left(\frac{1}{2}\right) - \frac{2}{n+1}$$

$$< 2.$$

$\blacksquare$

10 pentagonal numbers: $p_1 = 1,\ p_n = p_{n-1} + (3n-2)$ for $n \geq 2$

Prove: $p_n = \dfrac{n(3n-1)}{2}$ for $n \geq 1$

proof: The base case $(n=1)$ obviously holds.

via induction

suppose that $p_n = \dfrac{n(3n-1)}{2}$ for some $n \in \mathbb{N}$.

Then $p_{n+1} = p_n + (3(n+1) - 2)$

$$= \frac{n(3n-1)}{2} + 3n + 1 \quad \text{via IH}$$

$$= \frac{3n^2 - n + 6n + 2}{2}$$

$$= \frac{3n^2 + 5n + 2}{2} = \frac{(3n+2)(n+1)}{2}$$

$$= \frac{(n+1)(3(n+1)-1)}{2}.$$

$\blacksquare$

2.2

31(c) $n \in \mathbb{N} \implies n^4 = 5k$ or $n^4 = 5k+1$:

By the Binomial Theorem, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

If $n \in \mathbb{N}$, then $n = 5q + r$ for $0 \le r < 5$, so

$$n^4 = (5q+r)^4 = 625q^4 + 4(125q^3)r + 6(25q^2)r^2 + 4(5q)r^3 + r^4$$

$$= \underbrace{5(125q^4 + 100q^3r + 30q^2r^2 + 4qr^3)}_{\text{multiple of } 5} + r^4 \, ;$$

it remains to analyze r^4 for $r = 0, 1, 2, 3, 4$:

$r = 0 \implies r^4 = 0$

$r = 1 \implies r^4 = 1$

$r = 2 \implies r^4 = 16 = 15 + 1$

$r = 3 \implies r^4 = 81 = 80 + 1$

$r = 4 \implies r^4 = 256 = 255 + 1$

In each case, $r^4 = 5k$ or $r^4 = 5k+1$. ∎

6 $n \in \mathbb{N} \Longrightarrow n^3 = 7k,\ 7k+1,$ or $7k-1$

pf: If $n = 7m$ for some $m \in \mathbb{N}$, then $n^3 = 7(49m^3) = 7k$.

otherwise, $n = 7m + r$ for some $m \in \mathbb{N}$, $0 < r < 7$, and

$$(7m+r)^3 = \underbrace{343m^3 + 3(49m^2)r + 3(7m)r^2}_{\text{multiple of } 7} + r^3 .$$

It suffices to analyze r^3 for $r = 1, 2, 3, 4, 5, 6$:

r	r^3
1	1
2	$8 = 7+1$
3	$27 = 28 - 1 = 7 \cdot 4 - 1$
4	$64 = 63 + 1 = 7 \cdot 9 + 1$
5	$125 = 126 - 1 = 7 \cdot 18 - 1$
6	$216 = 217 - 1 = 7 \cdot 31 - 1$

in each case, $r^3 = 7k+1$ or $r^3 = 7k-1$.

$\blacksquare$

<u>8</u> No integer in the sequence $11, 111, 1111, \ldots$ is a perfect square.

<u>pf.</u>: As in the hint, each of these numbers is of the form $4k+3$.

Given $n \in \mathbb{N}$, $n = 2k$ or $n = 2k+1$, so

$$n^2 = 4k^2 \quad \text{or} \quad n^2 = 4(k^2+k)+1 .$$

In other words, a perfect square cannot be of the form $4k+3$.

$\blacksquare$

10 For $n \geq 1$, the integer $n(7n^2+5)$ is a multiple of 6.

pf. Note that $7n^2+5 = 6n^2 + n^2 + 6 - 1 = 6(n^2+1) + n^2 - 1$, so

$$n(7n^2+5) = n\left(6(n^2+1) + n^2 - 1\right) = 6n(n^2+1) + n(n^2-1).$$

We therefore have to show that $n(n^2-1)$ is divisible by 6.

Consider cases:

n	$n(n^2-1)$
$6m$	$6m(n^2-1)$ ✓
$6m+1$	$n((6m+1)^2-1) = n(36m^2+12m)$ ✓
$6m+2$	$(6m+2)((6m+2)^2-1) = \underbrace{(6m+2)}_{\text{even}}\underbrace{(36m^2+24m+3)}_{\text{multiple of } 3}$ ✓
$6m+3$	$(6m+3)((6m+3)^2-1) = \underbrace{(6m+3)}_{\text{multiple of } 3}\underbrace{(36m^2+36m+8)}_{\text{even}}$ ✓
$6m+4$	$(6m+4)((6m+4)^2-1) = \underbrace{(6m+4)}_{\text{even}}\underbrace{(36m^2+48m+15)}_{\text{mult. of } 3}$ ✓

n	$n(n^2-1)$
$6m+5$	$(6m+5)((6m+5)^2-1)$
	$= (6m+5)(\underbrace{36m^2+60m+24}_{\text{multiple of } 6})$

∎