2.3 14(a) $\gcd(2a+1, 9a+4) = 1$

pf: Notice that $9(2a+1) - 2(9a+4) = 1$.

$\blacksquare$

2.4 1 $\gcd(272, 1479)$:

$$1479 = \underbrace{5 \cdot 272}_{1360} + 119$$

$$272 = \underbrace{2 \cdot 119}_{238} + 34$$

$$119 = \underbrace{3 \cdot 34}_{102} + \underline{\underline{17}}$$

$$34 = 2 \cdot 17 + 0$$

$$\begin{aligned} 17 &= 119 - 3 \cdot 34 \\ &= 119 - 3(272 - 2 \cdot 119) \\ &= -3 \cdot 272 + 7 \cdot 119 \\ &= -3 \cdot 272 + 7(1479 - 5 \cdot 272) \\ &= 7 \cdot 1479 - 38 \cdot 272 \end{aligned}$$

$17 = \gcd(272, 1479)$

Euclidean algorithm Given $a, b \in \mathbb{N}$ with $a > b$, this algorithm produces $\gcd(a,b)$.

$$a = q_1 b + r_1 \quad \text{with} \quad 0 \le r_1 < b$$

$$r_1 > 0 \implies b = q_2 r_1 + r_2 \quad \text{with} \quad 0 \le r_2 < r_1$$

$$r_2 > 0 \implies r_1 = q_3 r_2 + r_3 \quad \text{with} \quad 0 \le r_3 < r_2$$

$$\vdots$$

$$r_{n-1} > 0 \implies r_{n-2} = q_n r_{n-1} + r_n \quad \text{with} \quad 0 \le r_n < r_{n-1}$$

$$\& \quad r_{n-1} = q_{n+1} r_n .$$

THEN $r_n = \gcd(a,b)$.

why does this work?

<u>FACT</u>: If $a = qb + r$, then $\gcd(a,b) = \gcd(b,r)$.

<u>pf.</u>: Any d that divides a and b must divide r, since $r = a - qb$.

Let $d = \gcd(a,b)$. Then d divides b, r.

If $c|b$ and $c|r$, then $c|qb+r$

so $c|a$ & $c|b$ $\Longrightarrow$ $c|d$ and $c \leq d$.

Thus, $d = \gcd(b,r)$. $\blacksquare$

FACT: $a, b \in \mathbb{N}$ $\quad \gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$.

pf.: $\underbrace{d = \gcd(a,b)}$, define $m = \frac{ab}{d}$.

$d \mid a,\ d \mid b \implies a = dr,\ b = ds \quad r, s \in \mathbb{N}$

and $d = ax + by$ for some $x, y \in \mathbb{Z}$.

Let c be any multiple of a, b, and

compute

$$\frac{c}{m} = \frac{cd}{ab} = \frac{c(ax+by)}{ab}$$

$$= \underbrace{\left(\frac{c}{b}\right)}_{\in \mathbb{N}} x + \underbrace{\left(\frac{c}{a}\right)}_{\in \mathbb{N}} y \in \mathbb{N}$$

$\implies m \mid c$. ☒

pf. Know that $ax_0 + by_0 = c$.

Suppose that $ax + by = c$ also.

$$\Longrightarrow \quad ax + by = ax_0 + by_0$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$b(y - y_0) = -a(x - x_0)$$

$$\left(\frac{b}{d}\right)(y - y_0) = \left(-\frac{a}{d}\right)(x - x_0)$$

$$\frac{y - y_0}{-\frac{a}{d}} = \frac{x - x_0}{\frac{b}{d}} = t \in \mathbb{Z}$$

$$\Longrightarrow \quad x = x_0 + \left(\frac{b}{d}\right)t, \quad y = y_0 - \left(\frac{a}{d}\right)t.$$

$\blacksquare$

recall: $d = \gcd(a,b)$ $\Longrightarrow$

$d = au + bv$ for $u, v \in \mathbb{Z}$

$\Longrightarrow 1 = \left(\frac{a}{d}\right)u + \left(\frac{b}{d}\right)v$

$\Longrightarrow \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

Linear Diophantine equations

$$ax + by = c \qquad a, b, c \in \mathbb{N}$$

Goal: find $x, y \in \mathbb{Z}$ that solve

Theorem The Diophantine equation $ax + by = c$ is solvable iff $d \mid c$, where $d = \gcd(a, b)$.

If (x_0, y_0) is a solution, i.e., $ax_0 + by_0 = c$, then every solution is of the form

$$x = x_0 + t\left(\frac{b}{d}\right), \quad y = y_0 - t\left(\frac{a}{d}\right)$$

for $t \in \mathbb{Z}$.

3(b) $54x + 21y = 906$

s.l'n:

divide by 3 to get

$18x + 7y = 302$

$x_0 = 16, \ y_0 = 2$

gcd(54, 21):

$54 = 2 \cdot 21 + 12$

$21 = 12 + 9$

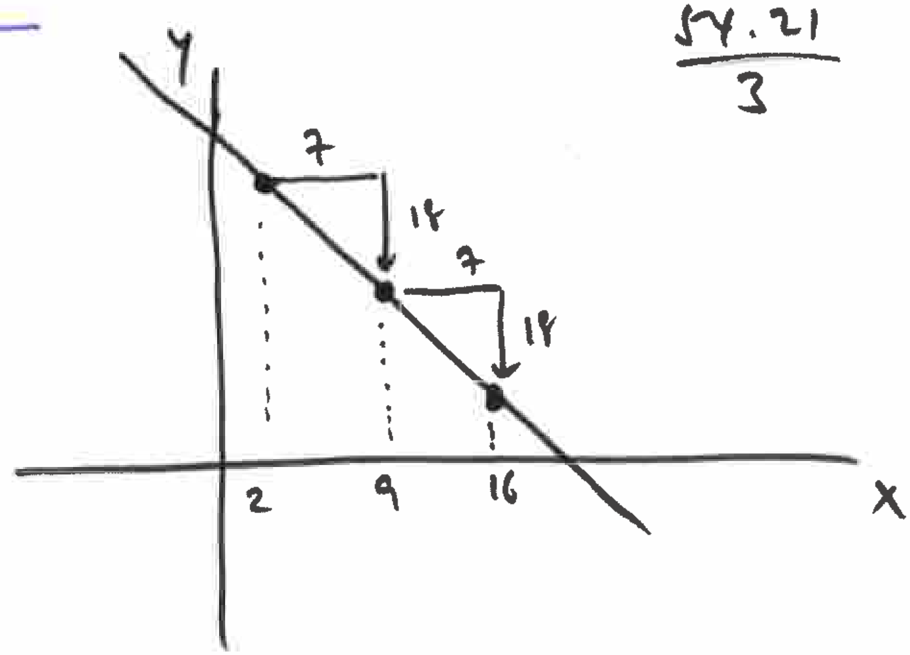
$12 = 9 + 3$

$9 = 3 \cdot 3$. $3 = \gcd$.

$$54(16 \pm t \cdot 7) + 21(2 \mp t \cdot 18) = 906$$

$\operatorname{lcm}(54, 21) = \dfrac{54 \cdot 21}{3}$

x	y
16	2
9	20
2	38



Since $3 = -14 \cdot 360 + 41 \cdot 123$,

$$99 = (-14 \cdot 33) \cdot 360 + (41 \cdot 33) \cdot 123$$

$$= \underbrace{(-462)}_{y_0} \cdot 360 + \underbrace{(1353)}_{x_0} \cdot 123$$

Then $\underline{\underline{\text{all}}}$ solutions are given by $\boxed{123x + 360y = 99}$

$$\begin{cases} x = (-462) + t(120) \\ y = 1353 - t(41) \end{cases}$$

<u>Positive</u> solutions: need $-462 + 120t > 0$

$1353 - 41t > 0$

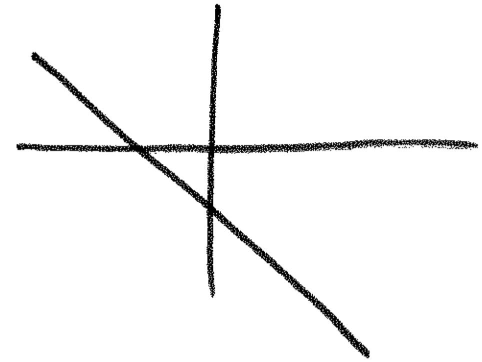
$$\Longrightarrow t > \frac{462}{120} \quad \& \quad t < \frac{1353}{41}$$

$$t > 3.85 \quad \& \quad t < 33$$

$$\Longrightarrow 4 \leq t \leq 32$$

$$\boxed{\begin{array}{c} \underline{\text{ex}}: \\ \underline{t = 4} \\ x = 18 \\ y = 1189 \end{array}}$$

$$x = 1353 + t(120)$$
$$y = -462 - t(41)$$



Positive solutions: $1353 + 120t > 0$

$-462 - 41t > 0$

$$\Longrightarrow \quad t > \frac{-1353}{120} \quad \& \quad t < \frac{-462}{41}$$

$$t > -11.275 \quad \& \quad t < -11.27\ldots$$

No positive sol'ns!

3(c) $123x + 360y = 99$

$\gcd(123, 360)$:

$$360 = \underbrace{2 \cdot 123}_{246} + 114$$
$$123 = 114 + 9$$
$$114 = 12 \cdot 9 + 6$$
$$9 = 6 + 3 \qquad \underline{\underline{\gcd = 3}}$$
$$6 = 2 \cdot 3.$$

Then $3 = 9 - 6$

$$= 9 - (114 - 12 \cdot 9) = -114 + 13 \cdot 9$$
$$= -114 + 13(123 - 114)$$
$$= 13 \cdot 123 - 14 \cdot 114$$
$$= 13 \cdot 123 - 14(360 - 2 \cdot 123)$$
$$= \underline{\underline{-14 \cdot 360 + 41 \cdot 123}}.$$

5(a)

$10x + 25y = 455$ $\qquad$ $\underline{\gcd(10, 25) = 5}$

$x_0 = 43, \quad y_0 = 1$

$x = 43 - t(5), \quad y = 1 + t(2)$

$t = 0, 1, 2, 3, 4, 5, 6, 7, 8$

t	x	y
0	43	1
1	38	3
2	33	5
3	28	7
4	23	9
5	18	11
6	13	13
7	8	15
8	3	17