Math 212 - Problem Set #4 SOLUTIONS

① Given family: $y^2 = cx \implies 2yy' = c \implies y' = \frac{c}{2y}$

orthogonal family: $y' = \frac{-2y}{c}$, $c = \frac{y^2}{x}$

$$\implies y' = \frac{-2y}{\frac{y^2}{x}} = \frac{-2xy}{y^2} = \frac{-2x}{y}$$

$$\implies \frac{dy}{dx} = \frac{-2x}{y} \implies \underline{\underline{2x\,dx + y\,dy = 0}}$$

This is a separable ODE that can be integrated directly to yield the solution:

$$x^2 + \tfrac{1}{2}y^2 = k \quad \text{or} \quad \underline{\underline{2x^2 + y^2 = k}}.$$

② $cx^2 + y^2 = 1 \implies 2cx + 2yy' = 0$ since $c = \frac{1-y^2}{x^2}$

$$\implies cx + yy' = 0$$

$$\implies y' = -\frac{cx}{y} = \frac{-\left(\frac{1-y^2}{x^2}\right)x}{y} = \frac{y^2-1}{xy}$$

orthogonal family:

$$y' = \frac{xy}{1-y^2} \implies \frac{dy}{dx} = \frac{xy}{1-y^2}$$

$$\implies xy\,dx + (y^2-1)\,dy = 0$$

$$xy\,dx + (y^2-1)\,dy = 0$$

check for exactness:

$$\left.\begin{aligned} \frac{\partial}{\partial y}(xy) &= x \\ \frac{\partial}{\partial x}(y^2-1) &= 0 \end{aligned}\right\} \quad \underline{\text{Not}} \text{ exact!}$$

Since

$$\frac{\frac{\partial}{\partial x}(y^2-1) - \frac{\partial}{\partial y}(xy)}{xy} = \frac{-x}{xy} = \frac{-1}{y}$$

is a function of y only, $h = h(y)$ will be an integrating factor as long as

$$\frac{h'}{h} = \frac{-1}{y} \Rightarrow \ln|h| = -\ln|y| = \ln\left|\frac{1}{y}\right|$$

$$\Rightarrow h(y) = \frac{1}{y}.$$

using $h(y) = \frac{1}{y}$, we get

$$\frac{1}{y}(xy)\,dx + \frac{1}{y}(y^2-1)\,dy = 0$$

$$\Rightarrow \quad x\,dx + \left(y - \frac{1}{y}\right)dy = 0$$

This is exact, since $\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}\left(y - \frac{1}{y}\right)$.

Now find $f(x,y)$:

$$\frac{\partial f}{\partial x} = x \implies f(x,y) = \frac{1}{2}x^2 + R(y)$$

$$\implies \frac{\partial f}{\partial y} = R' = y - \frac{1}{y} \implies R(y) = \frac{1}{2}y^2 - \ln y$$

Thus, the general solution is

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - \ln|y| = k, \quad \text{or}$$

$$x^2 + y^2 - 2\ln|y| = k, \quad \text{or}$$

$$x^2 + y^2 - \ln(y^2) = k. \quad \text{(Any of these is fine!)}$$

③ Given family: $x = \frac{y^2}{4} + \frac{c}{y^2} = \frac{y^2}{4} + cy^{-2}$

$$\implies 1 = \frac{1}{2}yy' - 2cy^{-3}y' = y'\left(\frac{1}{2}y - 2cy^{-3}\right)$$

$$\implies y' = \frac{1}{\frac{1}{2}y - 2cy^{-3}} = \frac{2y^3}{y^4 - 4c}$$

$$\implies y' = \frac{2y^3}{y^4 - 4\left(xy^2 - \frac{y^4}{4}\right)} \quad \text{since } c = xy^2 - \frac{y^4}{4}$$

$$\implies y' = \frac{2y^3}{y^4 - 4xy^2 + y^4} = \frac{2y^3}{2y^4 - 4xy^2}$$

Finally, we have the ODE for the given family in simplified form: $y' = \dfrac{y}{y^2 - 2x}$.

For the orthogonal family:

$$y' = \frac{2x - y^2}{y}$$

$$\Longrightarrow \quad y\,dy = (2x - y^2)\,dx$$

$$\Longrightarrow \quad (y^2 - 2x)\,dx + y\,dy = 0$$

Not exact, since $\frac{\partial}{\partial y}(y^2 - 2x) = 2y \neq \frac{\partial}{\partial x}(y) = 0$.

However,

$$\frac{\frac{\partial}{\partial y}(y^2 - 2x) - \frac{\partial}{\partial x}(y)}{y} = \frac{2y}{y} = 2,$$

which can be interpreted as a function of x only, we can find an integrating factor $h = h(x)$:

$$\frac{h'}{h} = 2 \Longrightarrow \ln|h| = 2x \Longrightarrow h(x) = e^{2x}$$

Using this, we get the new ODE

$$e^{2x}(y^2-2x)\,dx + e^{2x}y\,dy = 0$$

$$\Rightarrow (y^2e^{2x} - 2xe^{2x})\,dx + (ye^{2x})\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}(y^2e^{2x} - 2xe^{2x}) &= 2ye^{2x} \\ \frac{\partial}{\partial x}(ye^{2x}) &= 2ye^{2x} \end{aligned}\right\} \underline{\text{exact!}}$$

Find $f(x,y)$:

$$\frac{\partial f}{\partial x} = y^2e^{2x} - 2xe^{2x}$$

$$\Rightarrow f(x,y) = \frac{1}{2}y^2e^{2x} - \underbrace{\int 2xe^{2x}\,dx}_{\substack{u=x \quad dv = 2e^{2x} \\ du = dx \quad v = e^{2x}}} = \frac{1}{2}y^2e^{2x} - \left[xe^{2x} - \int e^{2x}dx\right]$$

$$= \frac{1}{2}y^2e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + R(y)$$

Since $f(x,y) = \frac{1}{2}y^2e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + R(y)$,

$$\frac{\partial f}{\partial y} = ye^{2x} + R' \Rightarrow R' = 0, \quad R = 0.$$

Thus, the general solution is

$$\frac{1}{2}y^2e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} = k, \quad \text{or}$$

$$\underline{\underline{y^2e^{2x} - 2xe^{2x} + e^{2x} = k.}}$$

(4) 4 pound stone : $4 = mg$, $g = 32 \text{ ft}/s^2$,

so $4 = 32m \Longrightarrow \underline{\underline{m = \frac{1}{8}}}$ (in slugs).

Newton's 2nd Law then becomes

$$\underbrace{mv'}_{ma} = \underbrace{mg}_{\text{gravity}} - \underbrace{\tfrac{1}{2}v}_{\text{air resistance}} \Longrightarrow \tfrac{1}{8}v' = 4 - \tfrac{1}{2}v$$

Thus, the ODE is $v' = 32 - 4v \Longrightarrow v' + 4v = 32$.

Integrating factor : $\underbrace{\mu v' + 4\mu v}_{} = 32\mu$

$$\frac{d}{dt}(\mu v) = \mu v' + \mu' v \text{ , so } \mu' = 4\mu$$
$$\Longrightarrow \mu = e^{4t}$$

with $\mu = e^{4t}$, we get

$$\frac{d}{dt}(e^{4t}v) = 32e^{4t}$$

$$\Longrightarrow e^{4t}v = 8e^{4t} + c \Longrightarrow v(t) = 8 + ce^{-4t}$$

$v(0) = 0$, so $0 = 8 + c \Longrightarrow c = -8$.

Thus, $\underline{\underline{v(t) = 8 - 8e^{-4t}}}$ for $t \geq 0$.

The terminal velocity is $\lim_{t \to \infty} (8 - 8e^{-4t}) = \underline{\underline{8 \text{ ft}/s}}$.

The distance fallen is

$$y(t) = 8t + 2e^{-4t} + c \; ; \quad y(0) = 0, \text{ so}$$

$$0 = 0 + 2 + c \implies c = -2, \text{ and}$$

the distance fallen at time t is

$$y(t) = 8t + 2e^{-4t} - 2.$$

(5) $m = 100$ g ; $y_0 = 1000$ m ; $g = 9.8$ m/s^2 (down = positive);

initial height, not initial distance fallen!

air resistance $= kv$, for k to be determined

Newton's 2nd Law in this case:

$$\underbrace{mv'}_{ma} = \underbrace{(9.8)(100)}_{\text{gravity}} - \underbrace{kv}_{\text{air resistance}}$$

Thus,

$$100v' = 980 - kv$$

$$\implies 100v' + kv = 980$$

$$\implies v' + \frac{k}{100}v = 9.8$$

Integrating factor:

$$\underbrace{\mu v' + \mu \cdot \frac{k}{100} v}\; = 9.8\mu$$

$$\frac{d}{dt}(\mu v) = \mu v' + \mu' v, \quad \text{so} \quad \mu' = \mu \cdot \frac{k}{100} \implies$$

$$\frac{\mu'}{\mu} = \frac{k}{100} \implies \ln|\mu| = \frac{kt}{100} \implies \mu = \exp\left(\frac{kt}{100}\right)$$

using this, we get

$$\frac{d}{dt}\left(e^{\frac{kt}{100}} v\right) = 9.8\, e^{\frac{kt}{100}}$$

$$\implies e^{\frac{kt}{100}} v = \frac{980}{k} e^{\frac{kt}{100}} + c$$

$$\implies v(t) = \frac{980}{k} + c e^{-\frac{kt}{100}}$$

Since $v(0) = 0$, $c = -\frac{980}{k}$ and

$$v(t) = \frac{980}{k} - \frac{980}{k} e^{-\frac{kt}{100}}$$

The terminal velocity is $\frac{980}{k} = 245$,

so $k = \frac{980}{245} = 4$; thus,

$$v(t) = 245 - 245\, e^{-\frac{t}{25}}$$

For the rest of (a), integrate to get the distance fallen at time t:

$$y(t) = 245t + 6125\, e^{-t/25} + c.$$

Note that $y(0) = 0$, so $y(0) = 0 = 6125 + c \implies c = -6125.$

Thus, the distance fallen at time t is

$$y(t) = 245t + 6125e^{-t/25} - 6125.$$

Note: for part (b), we would need to determine t such that $y(t) = 1000$ — that's too hard!

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6 Newton's Law here:

$$mv' = -kv + \lambda v^3, \quad \text{or}$$

$$mv' + kv = \lambda v^3, \quad \text{with } v(0) = v_0 > 0.$$

(a) If $\lambda = 0$, the ODE is

$$mv' + kv = 0 \implies v' + \frac{k}{m}v = 0$$

For this ODE, $e^{\frac{k}{m}t}$ is an integrating factor, so we get $\frac{d}{dt}(e^{\frac{k}{m}t}v) = 0 \implies e^{\frac{k}{m}t}v = c$

$$\implies v(t) = ce^{-\frac{k}{m}t},$$

with $c = v(0)$. Thus, $v(t) = v_0 e^{-\frac{k}{m}t}$.

This is weird because $v(t) > 0$ for all time!

(b) with $\lambda > 0$, the ODE is

$$mv' + kv = \lambda v^3 \ , \text{ i.e.,}$$

$$v' + \frac{k}{m} v = \frac{\lambda}{m} v^3 .$$ This is a Bernoulli ODE.

Divide by v^3:

$$v^{-3} v' + \frac{k}{m} v^{-2} = \frac{\lambda}{m}$$

Multiply by -2 to get

$$\underbrace{-2 v^{-3} v'}_{} - \frac{2k}{m} v^{-2} = \frac{-2\lambda}{m}$$

Now this is $\frac{d}{dt}(v^{-2})$, so the ODE is

$$\frac{d}{dt}(v^{-2}) - \frac{2k}{m} v^{-2} = \frac{-2\lambda}{m}$$

Integrating factor:

$$\underbrace{\mu \frac{d}{dt}(v^{-2}) - \frac{2k}{m} \mu v^{-2}}_{} = \frac{-2\lambda}{m} \mu$$

$$\frac{d}{dt}(\mu v^{-2}) = \mu \frac{d}{dt}(v^{-2}) + \mu' v^{-2},$$

so need $\mu' = \frac{-2k}{m} \mu \Longrightarrow \frac{\mu'}{\mu} = \frac{-2k}{m}$

$$\Longrightarrow \mu = \exp\left(\frac{-2k}{m} t\right).$$

Using $\mu = \exp(-\frac{2k}{m} t)$, we get

$$\frac{d}{dt}\left(e^{-\frac{2k}{m}t} v^{-2}\right) = \frac{-2\lambda}{m} e^{-\frac{2k}{m}t}$$

$$\Longrightarrow \quad e^{-\frac{2k}{m}t} v^{-2} = \frac{\lambda}{k} e^{-\frac{2k}{m}t} + c$$

$$\Longrightarrow \quad v^{-2} = \frac{\lambda}{k} + c e^{+\frac{2k}{m}t}$$

$$\Longrightarrow \quad v_0^{-2} = \frac{\lambda}{k} + c \Longrightarrow c = v_0^{-2} - \frac{\lambda}{k}.$$

Thus, $v^{-2} = \frac{\lambda}{k} + \left(v_0^{-2} - \frac{\lambda}{k}\right) e^{\frac{2k}{m}t}$

$$\Longrightarrow \quad v^2 = \frac{1}{\frac{\lambda}{k} + \left(v_0^{-2} - \frac{\lambda}{k}\right)\left(e^{\frac{2k}{m}t}\right)}$$

$$\Longrightarrow \quad v(t) = \frac{1}{\sqrt{\frac{\lambda}{k} + \left(v_0^{-2} - \frac{\lambda}{k}\right) e^{\frac{2k}{m}t}}}.$$

(7) Newton's Law of Cooling says that

$$y' = -k(y - T),$$ where T is the constant ambient temperature. Integrate to find the solution:

$$\frac{y'}{y - T} = -k$$

$$\Longrightarrow \ln|y - T| = -kt + c$$

$$\Longrightarrow y - T = ce^{-kt}$$

$$\Longrightarrow y = T + ce^{-kt}$$

$$\Longrightarrow y_0 = T + c \Longrightarrow c = y_0 - T.$$

Thus, $\underline{\underline{y(t) = T + (y_0 - T)e^{-kt}}}$.

Now for the given problem:

when $y_0 = 60$, $T = 30$, $y(15) = 50$; thus,

$$50 = 30 + (60 - 30)e^{-15k}$$

$$\Longrightarrow 20 = 30e^{-15k} \Longrightarrow \ln\left(\frac{2}{3}\right) = -15k$$

$$\Longrightarrow k = .027.$$

Thus, $y(t) = T + (y_0 - T)e^{-.027t}$

when $y_0 = 100$, $T = 50$, we have

$$y(t) = 50 + (100 - 50)e^{-.027t}$$

$$\Longrightarrow y(t) = 50 + 50e^{-.027t}$$

we want t so that $y(t) = 80$:

$$80 = 50 + 50e^{-.027t}$$

$$\Longrightarrow \frac{3}{5} = e^{-.027t} \Longrightarrow \ln\left(\frac{3}{5}\right) = -.027t$$

$$\Longrightarrow \underline{t = 18.9 \text{ minutes}}.$$

(8) initially: 100 gallons of brine, 10 pounds salt

At $t=0$: pure water (no salt!) flows in at 5 gal/min; mixture leaves at 2 gal/min.

The volume of the tank at time t is thus $100 + 3t$ gallons, and the amount of salt in the tank at time t, denoted $y(t)$ (in pounds), satisfies

$$\frac{dy}{dt} = \underbrace{0}_{\substack{\text{rate}\\ \text{entering}}} - \underbrace{\left(\frac{y}{100+3t}\right)}_{\text{pounds/gal}} \underbrace{(2)}_{\text{gal/min}}$$

$$\text{pounds/gal} \times \text{gal/min} = \text{pounds/min}$$

$$\Longrightarrow \quad y' = \frac{-2y}{100+3t}, \quad y(0) = 10$$

$$\Longrightarrow \quad \frac{y'}{y} = \frac{-2}{100+3t}$$

$$\Longrightarrow \quad \ln|y| = -\frac{2}{3}\ln(100+3t) + c$$

$$= \ln|100+3t|^{-\frac{2}{3}} + c$$

$$\Longrightarrow \quad y(t) = c(100+3t)^{-2/3}$$

$$y(0) = c(100)^{-2/3} = 10 \Longrightarrow \underline{c = 215.44}$$

Thus, $\underline{y(t) = 215.44\,(100+3t)^{-2/3}}$.

(a) when $t = 15$, $y(15) = (215.44)(145)^{-2/3} = 7.8$ pounds,

and the concentration is $\dfrac{7.8}{145} = .054$ pounds/gal.

(b) The tank overflows when $t = 50$, at which time the concentration is

$$\frac{(215.44)(250)^{-2/3}}{250} = .022 \text{ pounds/gal}.$$