16. (b) $u_n \to -\infty$ as $n \to \infty$
19. (a) 4.7263 (b) 1.223% (c) 3.5643 (e) 3.5699

Miscellaneous Problems, page 132

1. $y = (c/x^2) + (x^3/5)$
2. $2y + \cos y - x - \sin x = c$
3. $x^2 + xy - 3y - y^3 = 0$
4. $y = -3 + ce^{x-x^2}$
5. $x^2y + xy^2 + x = c$
6. $y = x^{-1}(1 - e^{1-x})$
7. $x^4 + x - y^2 - y^3 = c$
8. $y = (4 + \cos 2 - \cos x)/x^2$
9. $x^2y + x + y^2 = c$
10. $x + \ln|x| + x^{-1} + y - 2\ln|y| = c$; also $y = 0$
11. $(x^3/3) + xy + e^y = c$
12. $y = ce^{-x} + e^{-x}\ln(1 + e^x)$
13. $y = \tan(x + x^2 + c)$
14. $x^2 + 2xy + 2y^2 = 34$
15. $y = c/\cosh^2(x/2)$
16. $e^{-x}\cos y + e^{2y}\sin x = c$
17. $y = ce^{3x} - e^{2x}$
18. $y = e^{-2x}\displaystyle\int_0^x e^{-s^2}\,ds + 3e^{-2x}$
19. $2xy + xy^3 - x^3 = c$
20. $e^x + e^{-y} = c$
21. $2xy^2 + 3x^2y - 4x + y^3 = c$
22. $y^3 + 3y - x^3 + 3x = 2$
23. $y = \dfrac{e^{2t}}{3t} + c\dfrac{e^{-t}}{t}$
24. $\sin y \sin^2 x = c$
25. $(x^2/y) + \arctan(y/x) = c$
26. $e^{-y/x} + \ln|x| = c$
27. $(x^2 + y^2 + 1)e^{-y^2} = c$
28. $x^3 + x^2y = c$
29. $\arctan(y/x) - \ln\sqrt{x^2 + y^2} = c$
30. $(y^2/x^3) + (y/x^2) = c$
31. $x^3y^2 + xy^3 = -4$
32. $\dfrac{1}{y} = -x\displaystyle\int_1^x \frac{e^{2s}}{s^2}\,ds + \frac{x}{2}$
34. (a) $y = t + (c - t)^{-1}$ (b) $y = t^{-1} + 2t(c - t^2)^{-1}$
    (c) $y = \sin t + (c\cos t - \frac{1}{2}\sin t)^{-1}$
35. (a) $v' - [x(t) + b]v = b$
    (b) $v = \left[b\int \mu(t)\,dt + c\right]/\mu(t), \quad \mu(t) = \exp[-(at^2/2) - bt]$
36. $y = c_1t^{-1} + c_2 + \ln t$
37. $y = c_1\ln t + c_2 + t$
38. $y = (1/k)\ln|(k - t)/(k + t)| + c_2$ if $c_1 = k^2 > 0$; $y = (2/k)\arctan(t/k) + c_2$ if $c_1 = -k^2 < 0$; $y = -2t^{-1} + c_2$ if $c_1 = 0$; also $y = c$
39. $y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2$; also $y = c$ *Hint:* $\mu(v) = v^{-3}$ is an integrating factor.
40. $y = c_1e^{-t} + c_2 - te^{-t}$
41. $c_1^2y = c_1t - \ln|1 + c_1t| + c_2$ if $c_1 \neq 0$; $y = \frac{1}{2}t^2 + c_2$ if $c_1 = 0$; also $y = c$
42. $y^2 = c_1t + c_2$
43. $y = c_1\sin(t + c_2) = k_1\sin t + k_2\cos t$
44. $\frac{1}{3}y^3 - 2c_1y + c_2 = 2t$; also $y = c$
45. $t + c_2 = \pm\frac{2}{5}(y - 2c_1)(y + c_1)^{1/2}$
46. $y\ln|y| - y + c_1y + t = c_2$; also $y = c$
47. $e^y = (t + c_2)^2 + c_1$
48. $y = \frac{4}{3}(t + 1)^{3/2} - \frac{1}{3}$
49. $y = 2(1 - t)^{-2}$
50. $y = 3\ln t - \frac{3}{2}\ln(t^2 + 1) - 5\arctan t + 2 + \frac{3}{2}\ln 2 + \frac{5}{4}\pi$
51. $y = \frac{1}{2}t^2 + \frac{3}{2}$

CHAPTER 3

Section 3.1, page 144

1. $y = c_1e^t + c_2e^{-3t}$
2. $y = c_1e^{-t} + c_2e^{-2t}$
3. $y = c_1e^{t/2} + c_2e^{-t/3}$
4. $y = c_1e^{t/2} + c_2e^t$
5. $y = c_1 + c_2e^{-5t}$
6. $y = c_1e^{3t/2} + c_2e^{-3t/2}$
7. $y = c_1\exp[(9 + 3\sqrt{5})t/2] + c_2\exp[(9 - 3\sqrt{5})t/2]$
8. $y = c_1\exp[(1 + \sqrt{3})t] + c_2\exp[(1 - \sqrt{3})t]$
9. $y = e^t$; $y \to \infty$ as $t \to \infty$
10. $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$; $y \to 0$ as $t \to \infty$
11. $y = 12e^{t/3} - 8e^{t/2}$; $y \to -\infty$ as $t \to \infty$
12. $y = -1 - e^{-3t}$; $y \to -1$ as $t \to \infty$
13. $y = \frac{1}{26}(13 + 5\sqrt{13})\exp[(-5 + \sqrt{13})t/2] + \frac{1}{26}(13 - 5\sqrt{13})\exp[(-5 - \sqrt{13})t/2]$; $y \to 0$ as $t \to \infty$

14. $y = (2/\sqrt{33})\exp[(-1+\sqrt{33})t/4] - (2/\sqrt{33})\exp[(-1-\sqrt{33})t/4]$; $y \to \infty$ as $t \to \infty$
15. $y = \frac{1}{10}e^{-9(t-1)} + \frac{9}{10}e^{t-1}$; $y \to \infty$ as $t \to \infty$
16. $y = -\frac{1}{2}e^{(t+2)/2} + \frac{3}{2}e^{-(t+2)/2}$; $y \to -\infty$ as $t \to \infty$
17. $y'' + y' - 6y = 0$
18. $2y'' + 5y' + 2y = 0$
19. $y = \frac{1}{4}e^t + e^{-t}$; minimum is $y = 1$ at $t = \ln 2$
20. $y = -e^t + 3e^{t/2}$; maximum is $y = \frac{9}{4}$ at $t = \ln(9/4)$, $y = 0$ at $t = \ln 9$
21. $\alpha = -2$
22. $\beta = -1$
23. $y \to 0$ for $\alpha < 0$; y becomes unbounded for $\alpha > 1$
24. $y \to 0$ for $\alpha < 1$; there is no α for which all nonzero solutions become unbounded
25. (a) $y = \frac{1}{5}(1+2\beta)e^{-2t} + \frac{1}{5}(4-2\beta)e^{t/2}$
    (b) $y \cong 0.71548$ when $t = \frac{2}{5}\ln 6 \cong 0.71670$ (c) $\beta = 2$
26. (a) $y = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$
    (b) $t_m = \ln[(12+3\beta)/(12+2\beta)]$, $y_m = \frac{4}{27}(6+\beta)^3/(4+\beta)^2$
    (c) $\beta = 6(1+\sqrt{3}) \cong 16.3923$ (d) $t_m \to \ln(3/2)$, $y_m \to \infty$
27. (a) $y = d/c$ (b) $aY'' + bY' + cY = 0$
28. (a) $b > 0$ and $0 < c < b^2/4a$ (b) $c < 0$ (c) $b < 0$ and $0 < c < b^2/4a$

Section 3.2, page 155

1. $-\frac{7}{2}e^{t/2}$
2. 1
3. e^{-4t}
4. x^2e^x
5. $-e^{2t}$
6. 0
7. $0 < t < \infty$
8. $-\infty < t < 1$
9. $0 < t < 4$
10. $0 < t < \infty$
11. $0 < x < 3$
12. $2 < x < 3\pi/2$
14. The equation is nonlinear.
15. The equation is nonhomogeneous.
16. No
17. $3te^{2t} + ce^{2t}$
18. $te^t + ct$
19. $5W(f,g)$
20. $-4(t\cos t - \sin t)$
21. y_3 and y_4 are a fundamental set of solutions if and only if $a_1b_2 - a_2b_1 \neq 0$.
22. $y_1(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$, $y_2(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t$
23. $y_1(t) = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)}$, $y_2(t) = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}$
24. Yes
25. Yes
26. Yes
27. Yes
28. (b) Yes (c) $[y_1(t), y_3(t)]$ and $[y_1(t), y_4(t)]$ are fundamental sets of solutions; $[y_2(t), y_3(t)]$ and $[y_4(t), y_5(t)]$ are not
29. ct^2e^t
30. $c\cos t$
31. c/x
32. $c/(1-x^2)$
34. $2/25$
35. $3\sqrt{e} \cong 4.946$
36. $p(t) = 0$ for all t
40. If t_0 is an inflection point, and $y = \phi(t)$ is a solution, then from the differential equation $p(t_0)\phi'(t_0) + q(t_0)\phi(t_0) = 0$.
42. Yes, $y = c_1e^{-x^2/2}\displaystyle\int_{x_0}^{x} e^{t^2/2}\,dt + c_2e^{-x^2/2}$
43. No
44. Yes, $y = \dfrac{1}{\mu(x)}\left[c_1\displaystyle\int_{x_0}^{x}\frac{\mu(t)}{t}\,dt + c_2\right]$, $\mu(x) = \exp\left[-\displaystyle\int\left(\frac{1}{x} + \frac{\cos x}{x}\right)dx\right]$
45. Yes, $y = c_1x^{-1} + c_2x$
47. $x^2\mu'' + 3x\mu' + (1 + x^2 - \nu^2)\mu = 0$
48. $(1-x^2)\mu'' - 2x\mu' + \alpha(\alpha+1)\mu = 0$
49. $\mu'' - x\mu = 0$
51. The Legendre and Airy equations are self-adjoint.

Section 3.3, page 163

1. $e\cos 2 + ie\sin 2 \cong -1.1312 + 2.4717i$
2. $e^2\cos 3 - ie^2\sin 3 \cong -7.3151 - 1.0427i$
3. -1
4. $e^2\cos(\pi/2) - ie^2\sin(\pi/2) = -e^2 i \cong -7.3891i$
5. $2\cos(\ln 2) - 2i\sin(\ln 2) \cong 1.5385 - 1.2779i$
6. $\pi^{-1}\cos(2\ln\pi) + i\pi^{-1}\sin(2\ln\pi) \cong -0.20957 + 0.23959i$
7. $y = c_1e^t\cos t + c_2e^t\sin t$
8. $y = c_1e^t\cos\sqrt{5}\,t + c_2e^t\sin\sqrt{5}\,t$
9. $y = c_1e^{2t} + c_2e^{-4t}$
10. $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$
11. $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$
12. $y = c_1\cos(3t/2) + c_2\sin(3t/2)$
13. $y = c_1e^{-t}\cos(t/2) + c_2e^{-t}\sin(t/2)$
14. $y = c_1e^{t/3} + c_2e^{-4t/3}$
15. $y = c_1e^{-t/2}\cos t + c_2e^{-t/2}\sin t$
16. $y = c_1e^{-2t}\cos(3t/2) + c_2e^{-2t}\sin(3t/2)$
17. $y = \frac{1}{2}\sin 2t$; steady oscillation
18. $y = e^{-2t}\cos t + 2e^{-2t}\sin t$; decaying oscillation
19. $y = -e^{t-\pi/2}\sin 2t$; growing oscillation
20. $y = (1+2\sqrt{3})\cos t - (2-\sqrt{3})\sin t$; steady oscillation
21. $y = 3e^{-t/2}\cos t + \frac{5}{2}e^{-t/2}\sin t$; decaying oscillation
22. $y = \sqrt{2}\,e^{-(t-\pi/4)}\cos t + \sqrt{2}\,e^{-(t-\pi/4)}\sin t$; decaying oscillation
23. (a) $u = 2e^{t/6}\cos(\sqrt{23}\,t/6) - (2/\sqrt{23})e^{t/6}\sin(\sqrt{23}\,t/6)$
    (b) $t = 10.7598$
24. (a) $u = 2e^{-t/5}\cos(\sqrt{34}\,t/5) + (7/\sqrt{34})\,e^{-t/5}\sin(\sqrt{34}\,t/5)$
    (b) $T = 14.5115$
25. (a) $y = 2e^{-t}\cos\sqrt{5}\,t + [(\alpha+2)/\sqrt{5}]\,e^{-t}\sin\sqrt{5}\,t$ (b) $\alpha = 1.50878$
    (c) $t = \{\pi - \arctan[2\sqrt{5}/(2+\alpha)]\}/\sqrt{5}$ (d) $\pi/\sqrt{5}$
26. (a) $y = e^{-at}\cos t + ae^{-at}\sin t$ (b) $T = 1.8763$
    (c) $\alpha = \frac{1}{4}$, $T = 7.4284$; $\alpha = \frac{1}{2}$, $T = 4.3003$; $\alpha = 2$, $T = 1.5116$
35. $y = c_1\cos(\ln t) + c_2\sin(\ln t)$
36. $y = c_1t^{-1} + c_2t^{-2}$
37. $y = c_1t^{-1}\cos(\frac{1}{2}\ln t) + c_2t^{-1}\sin(\frac{1}{2}\ln t)$
38. $y = c_1t^6 + c_2t^{-1}$
39. $y = c_1t^2 + c_2t^3$
40. $y = c_1t\cos(2\ln t) + c_2t\sin(2\ln t)$
41. $y = c_1t + c_2t^{-3}$
42. $y = c_1t^{-3}\cos(\ln t) + c_2t^{-3}\sin(\ln t)$
44. Yes, $y = c_1\cos x + c_2\sin x$, $x = \int e^{-t^2/2}\,dt$
45. No
46. Yes, $y = c_1e^{-t^2/4}\cos(\sqrt{3}\,t^2/4) + c_2e^{-t^2/4}\sin(\sqrt{3}\,t^2/4)$

Section 3.4, page 171

1. $y = c_1e^t + c_2te^t$
2. $y = c_1e^{-t/3} + c_2te^{-t/3}$
3. $y = c_1e^{-t/2} + c_2e^{3t/2}$
4. $y = c_1e^{-3t/2} + c_2te^{-3t/2}$
5. $y = c_1e^t\cos 3t + c_2e^t\sin 3t$
6. $y = c_1e^{3t} + c_2te^{3t}$
7. $y = c_1e^{-t/4} + c_2e^{-4t}$
8. $y = c_1e^{-3t/4} + c_2te^{-3t/4}$
9. $y = c_1e^{2t/5} + c_2te^{2t/5}$
10. $y = e^{-t/2}\cos(t/2) + c_2e^{-t/2}\sin(t/2)$
11. $y = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$, $y \to -\infty$ as $t \to \infty$
12. $y = 2te^{3t}$, $y \to \infty$ as $t \to \infty$
13. $y = -e^{-t/3}\cos 3t + \frac{5}{9}e^{-t/3}\sin 3t$, $y \to 0$ as $t \to \infty$
14. $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}$, $y \to 0$ as $t \to \infty$
15. (a) $y = e^{-3t/2} - \frac{5}{2}te^{-3t/2}$ (b) $t = \frac{2}{5}$
    (c) $t_0 = 16/15$, $y_0 = -\frac{5}{3}e^{-8/5} \cong -0.33649$
    (d) $y = e^{-3t/2} + (b+\frac{3}{2})te^{-3t/2}$; $b = -\frac{3}{2}$
16. $y = 2e^{t/2} + (b-1)te^{t/2}$; $b = 1$
17. (a) $y = e^{-t/2} + \frac{5}{2}te^{-t/2}$ (b) $t_M = \frac{8}{5}$, $y_M = 5e^{-4/5} \cong 2.24664$
    (c) $y = e^{-t/2} + (b+\frac{1}{2})te^{-t/2}$
    (d) $t_M = 4b/(1+2b) \to 2$ as $b \to \infty$; $y_M = (1+2b)\exp[-2b/(1+2b)] \to \infty$ as $b \to \infty$

18. (a) $y = ae^{-2t/3} + (\frac{2}{3}a - 1)te^{-2t/3}$ (b) $a = \frac{3}{2}$
23. $y_2(t) = t^3$
24. $y_2(t) = t^{-2}$
25. $y_2(t) = t^{-1}\ln t$
26. $y_2(t) = te^t$
27. $y_2(x) = \cos x^2$
28. $y_2(x) = x$
29. $y_2(x) = x^{1/4}e^{-2\sqrt{x}}$
30. $y_2(x) = x^{-1/2}\cos x$
32. $y = c_1e^{-\delta x^2/2}\displaystyle\int_0^x e^{\delta s^2/2}\,ds + c_2e^{-\delta x^2/2}$
33. $y_2(t) = y_1(t)\displaystyle\int_{t_0}^t y_1^{-2}(s)\exp\left[-\int_{s_0}^s p(r)\,dr\right]ds$
34. $y_2(t) = t^{-1}\ln t$
35. $y_2(t) = \cos t^2$
36. $y_2(x) = x$
37. $y_2(x) = x^{-1/2}\cos x$
39. (b) $y_0 + (a/b)y_0'$
41. $y = c_1t^2 + c_2t^2\ln t$
42. $y = c_1t^{-1/2} + c_2t^{-1/2}\ln t$
43. $y = c_1t + c_2t^{5/2}$
44. $y = c_1t^{-1} + c_2t^{-1}\ln t$
45. $y = c_1t^{3/2} + c_2t^{3/2}\ln t$
46. $y = c_1t^{-2}\cos(3\ln t) + c_2t^{-2}\sin(3\ln t)$

Section 3.5, page 183

1. $y = c_1e^{3t} + c_2e^{-t} - e^{2t}$
2. $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$
3. $y = c_1e^{3t} + c_2e^{-t} + \frac{3}{16}te^{-t} + \frac{3}{8}t^2e^{-t}$
4. $y = c_1 + c_2e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$
5. $y = c_1\cos 3t + c_2\sin 3t + \frac{1}{162}(9t^2 - 6t + 1)e^{3t} + \frac{2}{3}$
6. $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$
7. $y = c_1e^{-t} + c_2e^{-t/2} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$
8. $y = c_1\cos t + c_2\sin t - \frac{1}{3}t\cos 2t - \frac{5}{9}\sin 2t$
9. $u = c_1\cos\omega_0t + c_2\sin\omega_0t + (\omega_0^2 - \omega^2)^{-1}\cos\omega t$
10. $u = c_1\cos\omega_0t + c_2\sin\omega_0t + (1/2\omega_0)t\sin\omega_0t$
11. $y = c_1e^{-t/2}\cos(\sqrt{15}\,t/2) + c_2e^{-t/2}\sin(\sqrt{15}\,t/2) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$
12. $y = c_1e^{-t} + c_2e^{2t} + \frac{1}{6}te^{2t} + \frac{1}{8}e^{-2t}$
13. $y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}$
14. $y = \frac{7}{10}\sin 2t - \frac{19}{40}\cos 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$
15. $y = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4$
16. $y = e^{3t} + \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$
17. $y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t$
18. $y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$
19. (a) $Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + D\sin 3t + E\cos 3t$
    (b) $A_0 = 2/15$, $A_1 = -2/9$, $A_2 = 8/27$, $A_3 = -8/27$, $A_4 = 16/81$, $B_0 = -1/9$, $B_1 = -1/9$, $B_2 = -2/27$, $D = -1/18$, $E = -1/18$
20. (a) $Y(t) = A_0t + A_1 + t(B_0t + B_1)\sin t + t(D_0t + D_1)\cos t$
    (b) $A_0 = 1$, $A_1 = 0$, $B_0 = 0$, $B_1 = 1/4$, $D_0 = -1/4$, $D_1 = 0$
21. (a) $Y(t) = e^t(A\cos 2t + B\sin 2t) + (D_0t + D_1)e^{2t}\sin t + (E_0t + E_1)e^{2t}\cos t$
    (b) $A = -1/20$, $B = -3/20$, $D_0 = -3/2$, $D_1 = -5$, $E_0 = 3/2$, $E_1 = 1/2$
22. (a) $Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}\cos t + t(D_0t^2 + D_1t + D_2)e^{-t}\sin t$
    (b) $A = 3$, $B_0 = -2/3$, $B_1 = 0$, $B_2 = 1$, $D_0 = 0$, $D_1 = 1$, $D_2 = 1$
23. (a) $Y(t) = A_0t^2 + A_1t + A_2 + t^2(B_0t + B_1)e^{2t} + (D_0t + D_1)\sin 2t + (E_0t + E_1)\cos 2t$
    (b) $A_0 = 1/2$, $A_1 = 1$, $A_2 = 3/4$, $B_0 = 2/3$, $B_1 = 0$, $D_0 = 0$, $D_1 = -1/16$, $E_0 = 1/8$, $E_1 = 1/16$
24. (a) $Y(t) = t(A_0t^2 + A_1t + A_2)\sin 2t + t(B_0t^2 + B_1t + B_2)\cos 2t$
    (b) $A_0 = 0$, $A_1 = 13/16$, $A_2 = 7/4$, $B_0 = -1/12$, $B_1 = 0$, $B_2 = 13/32$
25. (a) $Y(t) = (A_0t^2 + A_1t + A_2)e^t\sin 2t + (B_0t^2 + B_1t + B_2)e^t\cos 2t + e^{-t}(D\cos t + E\sin t) + Fe^t$
    (b) $A_0 = 1/52$, $A_1 = 10/169$, $A_2 = -1233/35{,}152$, $B_0 = -5/52$, $B_1 = 73/676$, $B_2 = -4105/35{,}152$, $D = -3/2$, $E = 3/2$, $F = 2/3$
26. (a) $Y(t) = t(A_0t + A_1)e^{-t}\cos 2t + t(B_0t + B_1)e^{-t}\sin 2t + (D_0t + D_1)e^{-2t}\cos t + (E_0t + E_1)e^{-2t}\sin t$
    (b) $A_0 = 0$, $A_1 = 3/16$, $B_0 = 3/8$, $B_1 = 0$, $D_0 = -2/5$, $D_1 = -7/25$, $E_0 = 1/5$, $E_1 = 1/25$