Quiz 10

Name: SOLUTIONS

1. Use Gauss-Jordan reduction to compute the inverses of the following matrices:

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

2. Compute the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

① inverse of A:

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\rho_1 \leftrightarrow \rho_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-\rho_1+\rho_2 \\ \rho_1+\rho_3}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 2 & 6 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-2\rho_2+\rho_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 10 & 1 & -2 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{10}\rho_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{10} & -\frac{1}{5} & \frac{3}{10} \end{array}\right] \xrightarrow[\substack{2\rho_3+\rho_2 \\ -3\rho_3+\rho_1}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{10} & \frac{3}{5} & \frac{1}{10} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{10} & -\frac{1}{5} & \frac{3}{10} \end{array}\right]$$

$\underline{A^{-1}}$

inverse of B:

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{3}\rho_3]{\frac{1}{2}\rho_1} \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow[\rho_3+\rho_2]{} \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow[(\frac{1}{2}\rho_2-\rho_3)+\rho_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

B^{-1}

(2) eigenvalues of A:

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)-2 = 0$$

$$\Longleftrightarrow \lambda^2 - 5\lambda + 4 = 0 \Longleftrightarrow (\lambda-1)(\lambda-4) = 0$$

eigenvalues: $\lambda = 1,\ \lambda = 4$

$\underline{\lambda = 1}$: solve $(A - \lambda I) v = \vec{0}$, i.e.,

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

with $v = \begin{bmatrix} x \\ y \end{bmatrix}$, this means that $2x - y = 0$, i.e.,

$y = 2x$; eigenvectors for $\lambda = 1$ are of the form $x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

any scalar $x \in \mathbb{R}$.

$$\underline{\underline{\lambda = 1, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}}$$

$\underline{\lambda = 4}$ now solve $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -2 & -2 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

with $v = \begin{bmatrix} x \\ y \end{bmatrix}$, need $x + y = 0$, i.e., $v = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

any $x \in \mathbb{R}$.

$$\underline{\underline{\lambda = 4, \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}}$$

eigenvalues of B:

$$\det(B-\lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & 2 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 0 & -1 \\ 0 & 3-\lambda \end{vmatrix}$$

$$- 2\begin{vmatrix} 0 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)(3-\lambda).$$

Thus, $\det(B-\lambda I) = 0$ when $\lambda=1,\ \lambda=2,\ \lambda=3$.

(3 distinct eigenvalues)

eigenvectors:

$\lambda = 1$
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, need $x-y=0$ & $z=0$ $\Longrightarrow$ $\vec{v} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 2}$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

with $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, need $y = 0$ & $z = 0$; no constraint on x,

i.e., any v of the form $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ solves this

homogeneous system.

$$\lambda = 2, \ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\underline{\lambda = 3}$

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, need $x - \frac{5}{2} z = 0$, $y + \frac{1}{2} z = 0$

so $v = \begin{bmatrix} 5/2\, z \\ -\frac{1}{2} z \\ z \end{bmatrix} = z \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix}$ $\qquad \lambda = 3, \ \vec{v} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix}.$