1(c) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for $n \geq 1$.

<u>base case</u> $n=1$: $1\cdot 2 = \frac{1\cdot 2\cdot 3}{3}$ ✓

Suppose that $1\cdot 2 + 2\cdot 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

for some particular integer n. Then

$$\underbrace{1\cdot 2 + 2\cdot 3 + \ldots + n(n+1)}_{\text{IH}} + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= (n+1)(n+2)\left[\frac{n}{3} + 1\right]$$

$$= (n+1)(n+2)\left(\frac{n+3}{3}\right)$$

$$= \frac{(n+1)\big((n+1)+1\big)\big((n+1)+2\big)}{3}.$$

∎

1 (d) $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$.

base case: $n=1 \implies 1^2 = \dfrac{1\cdot(1)(3)}{3}$ ✓

Suppose that $1^2 + 3^2 + \dots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

for some integer n. Then

$$\underbrace{1^2 + 3^2 + \dots + (2n-1)^2}_{IH} + (2(n+1)-1)^2 = \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2$$

$$= (2n+1)\left[\frac{n(2n-1)}{3} + 2n+1\right]$$

$$= (2n+1)\left[\frac{2n^2-n+6n+3}{3}\right]$$

$$= (2n+1)\left[\frac{2n^2+5n+3}{3}\right]$$

$$= \frac{(2n+1)(n+1)(2n+3)}{3}.$$

$\blacksquare$

$$\boxed{\begin{aligned}&\underline{\text{NOTE}}:\\ &(2n+1)(n+1)(2n+3)\\ &\quad = (n+1)(2(n+1)-1)(2(n+1)+1)\end{aligned}}$$

1(e) $1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n \geq 1$.

base case: $n = 1 \implies 1 = \left(\frac{1 \cdot 2}{2}\right)^2$ ✓

Suppose that $1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for some n.

Then
$$\underbrace{1^3 + 2^3 + \dots + n^3}_{\text{IH}} + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$
$$= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right]$$
$$= (n+1)^2\left(\frac{n^2+4n+4}{4}\right)$$
$$= \left(\frac{(n+1)(n+2)}{2}\right)^2.$$
$\blacksquare$

7 $1\cdot(1!) + 2\cdot(2!) + 3\cdot(3!) + \dots + n\cdot(n!) = (n+1)! - 1$

<u>base case</u>: $n=1 \implies 1\cdot(1!) = 2! - 1$ ✓

Suppose that $1\cdot(1!) + 2\cdot(2!) + \dots + n\cdot(n!) = (n+1)! - 1$ for some n. Then

$$\underbrace{1\cdot(1!) + 2\cdot(2!) + \dots + n\cdot(n!)}_{\text{IH}} + (n+1)\cdot(n+1)!$$

$$= \big((n+1)! - 1\big) + (n+1)\cdot(n+1)!$$

$$= (n+1)!\,(n+1+1) - 1$$

$$= (n+2)! - 1.$$

$\blacksquare$

9 The Bernoulli inequality : if $(1+a) > 0$, then

$$(1+a)^n \geq 1+na \ , \text{ for any } n \geq 1.$$

base case : $n=1 \Longrightarrow 1+a = 1+a$ ✓

suppose that $(1+a)^n \geq 1+na$ for some n. Then

$$\begin{aligned}(1+a)^{n+1} &= \underbrace{(1+a)^n}_{\text{IH}} \cdot (1+a) \geq (1+na)\cdot(1+a) \\ &= 1+a+na+na^2 \\ &\geq 1+(n+1)a, \text{ since } na^2 \geq 0.\end{aligned}$$

$\boxtimes$

10(a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for $n \geq 1$.

First, we establish an auxiliary result: if $n \geq 1$, then

$$-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1}.$$

<u>proof</u>: $n \geq 1 \implies n+1 \geq 0 \implies n^2 + n \leq n^2 + 2n + 1$ (add n^2+n to both sides)

$$\implies \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\implies -\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1}. \quad \blacksquare$$

Now for the main result: the base case, $n=1$, obviously holds.

Suppose that $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for some n.

Then $\underbrace{\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}}_{IH} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$

$$\leq 2 - \frac{1}{n+1} \text{ by above.}$$

$\blacksquare$

10(b) $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$.

base case: $n=1 \Longrightarrow \frac{1}{2} = 2 - \frac{3}{2} = \frac{1}{2}$ ✓

suppose that $\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for some n.

Then
$$\underbrace{\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{n}{2^n}}_{IH} + \frac{n+1}{2^{n+1}} = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}}$$
$$= 2 - \frac{1}{2^{n+1}}\left[2(n+2) - (n+1)\right]$$
$$= 2 - \frac{1}{2^{n+1}}(n+3)$$
$$= 2 - \frac{(n+1)+2}{2^{n+1}}. \quad \blacksquare$$

11 $\dfrac{(2n)!}{2^n \cdot n!} \in \mathbb{N}$ for all $n \geq 0$.

By #8 (done in class), $\dfrac{(2n)!}{n!} = 2 \cdot 6 \cdot \ldots \cdot (4n-2)$.

$$= (2\cdot 1) \cdot (2 \cdot 3) \cdot \ldots \cdot (2)(2n-1)$$

$$= 2^n \cdot (1 \cdot 3 \cdot \ldots \cdot (2n-1)).$$

Thus, $\dfrac{(2n)!}{2^n \cdot n!} = 1 \cdot 3 \cdot \ldots \cdot (2n-1) \in \mathbb{N}$.

$\blacksquare$