attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

2.3 THE DIFFUSION EQUATION

In this section we begin a study of the one-dimensional diffusion equation

$$u_t = ku_{xx}. \tag{1}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations. Because (1) is harder to solve than the wave equation, we begin this section with a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we'll deduce the uniqueness of an initial-boundary problem. We postpone until the next section the derivation of the solution formula for (1) on the whole real line.

Maximum Principle. If $u(x, t)$ satisfies the diffusion equation in a rectangle (say, $0 \le x \le l$, $0 \le t \le T$) in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$) (see Figure 1).

In fact, there is a *stronger version* of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but *only on the bottom or the lateral sides* (unless u is a constant). The corners are allowed.

The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle, just apply the maximum principle to $[-u(x, t)]$.

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source, the hottest spot and the

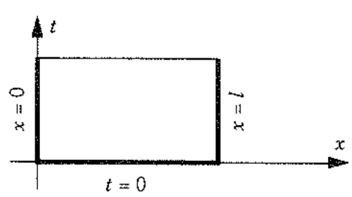

Figure 1

coldest spot can occur only initially or at one of the two ends of the rod. Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end). You can burn one of its ends but the maximum temperature will always be at the hot end, so that it will be cooler away from that end. Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube.

If we draw a "movie" of the solution, the maximum drops down while the minimum comes up. So the differential equation tends to smooth the solution out. (This is very different from the behavior of the wave equation!)

Proof of the Maximum Principle. We'll prove only the weaker version. (Surprisingly, its strong form is much more difficult to prove.) For the strong version, see [PW]. The idea of the proof is to use the fact, from calculus, that at an interior maximum the first derivatives vanish and the second derivatives satisfy inequalities such as $u_{xx} \le 0$. If we knew that $u_{xx} \neq 0$ at the maximum (which we do not), then we'd have $u_{xx} < 0$ as well as $u_t = 0$, so that $u_t \neq k u_{xx}$. This contradiction would show that the maximum could only be somewhere on the boundary of the rectangle. However, because u_{xx} could in fact be equal to zero, we need to play a mathematical game to make the argument work.

So let M denote the maximum value of $u(x,t)$ on the three sides $t = 0$, $x = 0$, and $x = l$. (Recall that any continuous function on any bounded closed set is bounded and assumes its maximum on that set.) We must show that $u(x,t) \le M$ throughout the rectangle R.

Let ϵ be a positive constant and let $v(x,t) = u(x,t) + \epsilon x^2$. Our goal is to show that $v(x,t) \le M + \epsilon l^2$ throughout R. Once this is accomplished, we'll have $u(x,t) \le M + \epsilon(l^2 - x^2)$. This conclusion is true for any $\epsilon > 0$. Therefore, $u(x,t) \le M$ throughout R, which is what we are trying to prove.

Now from the definition of v, it is clear that $v(x,t) \le M + \epsilon l^2$ on $t = 0$, on $x = 0$, and on $x = l$. This function v satisfies

$$v_t - k v_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - k u_{xx} - 2\epsilon k = -2\epsilon k < 0, \qquad (2)$$

which is the "diffusion inequality." Now suppose that $v(x,t)$ attains its maximum at an *interior* point (x_0, t_0). That is, $0 < x_0 < l$, $0 < t_0 < T$. By ordinary calculus, we know that $v_t = 0$ and $v_{xx} \le 0$ at (x_0, t_0). This contradicts the diffusion inequality (2). So there can't be an interior maximum. Suppose now that $v(x,t)$ has a maximum (in the closed rectangle) at a point on the *top* edge $\{t_0 = T$ and $0 < x < l\}$. Then $v_x(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \le 0$, as before. Furthermore, because $v(x_0, t_0)$ is bigger than $v(x_0, t_0 - \delta)$, we have

$$v_t(x_0, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \ge 0$$

as $\delta \to 0$ through positive values. (This is not an equality because the maximum is only "one-sided" in the variable t.) We again reach a contradiction to the diffusion inequality.

But $v(x,t)$ does have a maximum *somewhere* in the closed rectangle $0 \le x \le l$, $0 \le t \le T$. This maximum must be on the bottom or sides. Therefore $v(x, t) \le M + \epsilon l^2$ throughout R. This proves the maximum principle (in its weaker version).

UNIQUENESS

The maximum principle can be used to give a proof of *uniqueness for the Dirichlet problem for the diffusion equation*. That is, there is at most one solution of

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } 0 < x < l \text{ and } t > 0 \\ u(x,0) &= \phi(x) \\ u(0,t) &= g(t) \qquad u(l,t) = h(t) \end{aligned} \tag{3}$$

for four given functions f, ϕ, g, and h. Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of (3). Let $w = u_1 - u_2$ be their difference. Then $w_t - kw_{xx} = 0$, $w(x,0) = 0$, $w(0,t) = 0$, $w(l,t) = 0$. Let $T > 0$. By the maximum principle, $w(x, t)$ has its maximum for the rectangle on its bottom or sides—exactly where it vanishes. So $w(x,t) \le 0$. The same type of argument for the minimum shows that $w(x, t) \ge 0$. Therefore, $w(x,t) \equiv 0$, so that $u_1(x,t) \equiv u_2(x,t)$ for all $t \ge 0$.

Here is a second proof of uniqueness for problem (3), by a very different technique, the *energy method*. Multiplying the equation for $w = u_1 - u_2$ by w itself, we can write

$$0 = 0 \cdot w = (w_t - kw_{xx})(w) = \left(\tfrac{1}{2}w^2\right)_t + (-kw_x w)_x + kw_x^2.$$

(Verify this by carrying out the derivatives on the right side.) Upon integrating over the interval $0 < x < l$, we get

$$0 = \int_0^l \left(\tfrac{1}{2}w^2\right)_t dx - kw_x w\Big|_{x=0}^{x=l} + k\int_0^l w_x^2\,dx.$$

Because of the boundary conditions ($w = 0$ at $x = 0, l$),

$$\frac{d}{dt}\int_0^l \frac{1}{2}[w(x,t)]^2\,dx = -k\int_0^l [w_x(x,t)]^2\,dx \le 0,$$

where the time derivative has been pulled out of the x integral (see Section A.3). Therefore, $\int w^2\,dx$ is decreasing, so

$$\int_0^l [w(x,t)]^2\,dx \le \int_0^l [w(x,0)]^2\,dx \tag{4}$$

for $t \ge 0$. The right side of (4) vanishes because the initial conditions of u and v are the same, so that $\int [w(x,t)]^2\,dx = 0$ for all $t > 0$. So $w \equiv 0$ and $u_1 \equiv u_2$ for all $t \ge 0$.

(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1,\ 0 < t < \infty$.
(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x, t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \ge 0$ and $0 \le x \le 1$.
(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.
(b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty,\ 0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f,\ v_t - kv_{xx} = g,\ f \le g$, and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$.
(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi,\ 0 < t < \infty$, and if $v(0, t) \ge 0$, $v(\pi, t) \ge 0$ and $v(x, 0) \ge \sin x$, use part (a) to show that $v(x, t) \ge (1 - e^{-t}) \sin x$.

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$\boxed{\begin{aligned} &u_t = ku_{xx} \quad (-\infty < x < \infty,\ 0 < t < \infty) && (1)\\ &u(x, 0) = \phi(x). && (2)\end{aligned}}$$

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x - y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x - y, t)$ and so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x,t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\,x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x,t) = u(\sqrt{a}\,x, at)$. Then $v_t = [\partial(at)/\partial t]u_t = au_t$ and $v_x = [\partial(\sqrt{a}\,x)/\partial x]u_x = \sqrt{a}\,u_x$ and $v_{xx} = \sqrt{a}\cdot\sqrt{a}\,u_{xx} = a\,u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$\boxed{Q(x,0) = 1 \quad \text{for } x > 0 \qquad Q(x,0) = 0 \quad \text{for } x < 0.} \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x,t) = g(p) \quad \text{where } p = \frac{x}{\sqrt{4kt}} \tag{4}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that equation (1) doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, (3) doesn't change at all under the dilation. So $Q(x, t)$, which is defined by conditions (1) and (3), ought not see the dilation either. How could that happen? In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus let $p = x/\sqrt{4kt}$ and look for Q which satisfies (1) and (3) and has the form (4).

Step 2 Using (4), we convert (1) into an ODE for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p)$$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right].$$

Thus

$$g'' + 2pg' = 0.$$

This ODE is easily solved using the integrating factor $\exp \int 2p\, dp = \exp(p^2)$. We get $g'(p) = c_1 \exp(-p^2)$ and

$$Q(x, t) = g(p) = c_1 \int e^{-p^2} dp + c_2.$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(x, t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2} dp + c_2.$$

This formula is valid only for $t > 0$. Now use (3), expressed as a limit as follows.

$$\text{If } x > 0, \quad 1 = \lim_{t \searrow 0} Q = c_1 \int_0^{+\infty} e^{-p^2} dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

$$\text{If } x < 0, \quad 0 = \lim_{t \searrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2} dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

See Exercise 6. Here $\lim_{t \searrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp} \tag{5}$$

for $t > 0$. Notice that it does indeed satisfy (1), (3), and (4).

Step 4 Having found Q, we now *define* $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By property (b), S is also a solution of (1). Given any function ϕ, we also *define*

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad \text{for } t > 0.} \tag{6}$$

By property (d), u is another solution of (1). *We claim that u is the unique solution of* (1), (2). To verify the validity of (2), we write

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\,dy \\ &= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=+\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy \\ &= \int_{-\infty}^{x} \phi'(y)\,dy = \phi\Big|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition (2). We conclude that (6) is our solution formula, where

$$\boxed{S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad \text{for } t > 0.} \tag{7}$$

That is,

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy.} \tag{8}$$

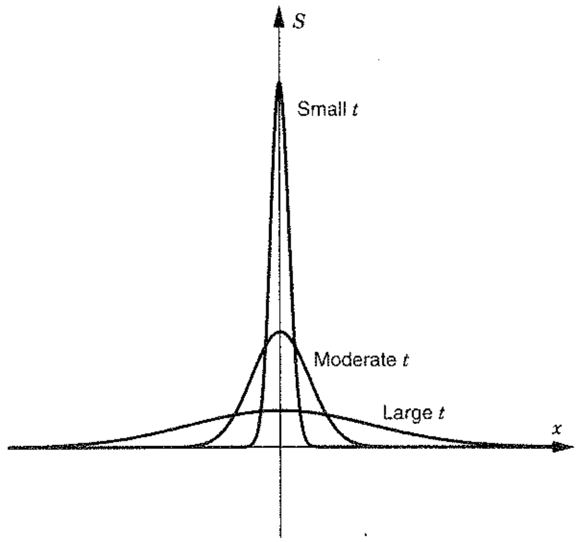

Figure 1

$S(x, t)$ is known as the *source function, Green's function, fundamental solution, gaussian*, or *propagator* of the diffusion equation, or simply the *diffusion kernel*. It gives the solution of (1),(2) with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$ it makes no sense. □

The *source function* $S(x, t)$ is defined for all real x and for all $t > 0$. $S(x, t)$ is positive and is even in x $[S(-x, t) = S(x, t)]$. It looks like Figure 1 for various values of t. For large t, it is very spread out. For small t, it is a very tall thin spike (a "*delta function*") of height $(4\pi kt)^{-1/2}$. The area under its graph is

$$\int_{-\infty}^{\infty} S(x, t)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} dq = 1$$

by substituting $q = x/\sqrt{4kt}$, $dq = (dx)/\sqrt{4kt}$ (see Exercise 7). Now look more carefully at the sketch of $S(x, t)$ for a very small t. If we cut out the tall spike, the rest of $S(x, t)$ is very small. Thus

$$\max_{|x|>\delta} S(x, t) \to 0 \qquad \text{as} \quad t \to 0 \tag{9}$$

Notice that the value of the solution $u(x, t)$ given by (6) is a kind of weighted *average* of the initial values around the point x. Indeed, we can write

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \simeq \sum_i S(x - y_i, t)\phi(y_i)\Delta y_i$$

approximately. This is the average of the solutions $S(x-y_i, t)$ with the weights $\phi(y_i)$. For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For *any* $t > 0$ the solution is a spread-out version of the initial values at $t = 0$.

Here's the physical interpretation. Consider diffusion. $S(x-y, t)$ represents the result of a unit mass (say, 1 gram) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances. For any initial distribution of concentration, the amount of substance initially in the interval Δy spreads out in time and contributes approximately the term $S(x-y_i, t)\phi(y_i)\Delta y_i$. All these contributions are added up to get the whole distribution of matter. Now consider heat flow. $S(x-y, t)$ represents the result of a "hot spot" at y at time 0. The hot spot is cooling off and spreading its heat along the rod.

Another physical interpretation is brownian motion, where particles move randomly in space. For simplicity, we assume that the motion is one-dimensional; that is, the particles move along a tube. Then the probability that a particle which begins at position x ends up in the interval (a, b) at time t is precisely $\int_a^b S(x-y, t)\,dy$ for some constant k, where S is defined in (7). In other words, if we let $u(x, t)$ be the probability density (probability per unit length) and if the initial probability density is $\phi(x)$, then the probability at all later times is given by formula (6). That is, $u(x, t)$ satisfies the diffusion equation.

It is usually impossible to evaluate integral (8) completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the *error function* of statistics,

$$\mathscr{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp. \tag{10}$$

Notice that $\mathscr{E}\mathrm{rf}(0) = 0$. By Exercise 6, $\lim_{x \to +\infty} \mathscr{E}\mathrm{rf}(x) = 1$.

Example 1.

From (5) we can write $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$ as

$$Q(x, t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$ □

Example 2.

Solve the diffusion equation with the initial condition $u(x, 0) = e^{-x}$. To do so, we simply plug this into the general formula (8):

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\,dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x,t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x,0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod. □

EXERCISES

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathscr{E}\mathrm{rf}(x)$.

2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.
3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation (1).
6. Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x,t)\,dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \qquad \text{as } t \to 0.$$