Math 212
Quiz 2 1/28/20 Name: SOLUTIONS

1. Use an integrating factor to determine the general solution $x(t)$ of the linear ODE

$$x' + \left(\frac{2}{t}\right) x = \frac{\cos t}{t^2},$$

then find the unique solution satisfying $x(\pi) = 0$.

Multiply by μ, the integrating factor to be determined:

$$\underbrace{\mu x' + \mu \cdot \left(\frac{2}{t}\right) x} = \mu \cdot \left(\frac{\cos t}{t^2}\right)$$

Must equal $(x\mu)' = \mu x' + \mu' x$, so

need $\mu' = \mu \cdot \left(\frac{2}{t}\right) \Longrightarrow \frac{\mu'}{\mu} = \frac{2}{t}$

Integrate: $\ln|\mu| = 2\ln|t| = \ln(t^2)$,

so $\underline{\mu = t^2}$.

$\longrightarrow$ with $\mu = t^2$, this becomes

$$(t^2 x)' = \cos t$$

General solution:

Integrate: $t^2 x = \sin t + c \Longrightarrow x(t) = \underline{\underline{\frac{\sin t + c}{t^2}}}$

$x(\pi) = 0$: $0 = \frac{\sin(\pi) + c}{\pi^2} = \frac{c}{\pi^2} \Longrightarrow c = 0$,

$\underline{x(t) = \frac{\sin t}{t^2}}$, solution of IVP

2. Use an integrating factor to determine the general solution $x(t)$ of the linear ODE

$$tx' + x = t^2 ,$$

then find the unique solution satisfying the initial condition $x(1) = 1$.

Divide by t to put the ODE in standard form:

$$x' + \left(\frac{1}{t}\right)x = t$$

Multiply by μ: $\underbrace{\mu x' + \mu \cdot \left(\frac{1}{t}\right)x} = \mu t$

$$(\mu x)' = \mu x' + \mu' x, \quad \text{so} \quad \mu' = \mu \cdot \left(\frac{1}{t}\right)$$

integrate:

$$\Longrightarrow \frac{\mu'}{\mu} = \frac{1}{t} \Longrightarrow \ln|\mu| = \ln|t|$$

$$\Longrightarrow \mu(t) = t.$$

Then the ODE returns to its original form (sneaky!):

$$(tx)' = t^2$$

General:

Integrate $\Longrightarrow$ $tx = \frac{1}{3}t^3 + c \Longrightarrow \underline{\underline{x(t) = \frac{1}{3}t^2 + \frac{c}{t}}}$

Specific solution: $x(1) = 1 \Longrightarrow 1 = \frac{1}{3} + c \Longrightarrow c = \frac{2}{3},$

$$\underline{\underline{x(t) = \frac{1}{3}t^2 + \frac{2}{3t}}}$$