Two sums

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1 Sum of the first n integers

Here's a derivation of the formula for the sum of the first n positive integers, $1, 2, \ldots n$. Call the sum S, and write the expression in two equivalent ways:

$$1 + 2 + \ldots + n - 1 + n = S \ ,$$

$$n + (n-1) + \ldots + 2 + 1 = S$$

The sum of each column on the left is $(n+1)$; since there are n columns, it follows that

$$n(n+1) = 2S \ .$$

Solving for S yields the formula, namely

$$\sum_{i=1}^{n} i \ = \ \frac{n(n+1)}{2} \ .$$

This is an example of a direct proof.

2 Sum of the first n squares

What about the sum of the first n squares,

$$1 + 4 + 9 + \ldots + n^2 = \sum_{i=1}^{n} i^2 \ ?$$

A little experimentation does not seem to reveal the formula, but I claim that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

Proving this is a job for mathematical induction! Verifying the base case, $n = 1$, is easy, since

$$\frac{1(2)(3)}{6} = 1 = 1^2 \ .$$

Now suppose that the formula holds for some integer k, so that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \; .$$

This is the *induction hypothesis*. Using the induction hypothesis, we compute:

$$\begin{aligned}
\sum_{i=1}^{k} i^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right) \\
&= (k+1)\left(\frac{2k^2 + 7k + 6}{6}\right) \\
&= (k+1)\left(\frac{(k+2)(2k+3)}{6}\right) \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \; .
\end{aligned}$$

The proposed formula when $n = k+1$ therefore follows from the induction hypothesis, so the formula holds for all integers $n \geq 1$.