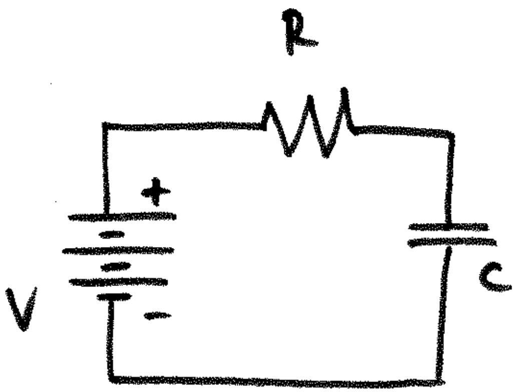
RC circuits — charging & discharging a capacitor

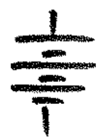
The basic circuit:



components:

 resistor, resistance R (ohms)

 capacitor, capacitance C (farads)

 battery, voltage V (volts)

Essential facts:

(1) $V = V_R + V_c$ (Kirchhoff's voltage law)

V_R = voltage across resistor

V_c = voltage across capacitor

(2) $V_R = IR$, I = current (in amps) (Ohm's Law)

(3) $I = CV_c'$ How capacitors charge/discharge

Putting these essential facts together, we get a 1^{st}-order ODE:

$$V_R + V_c = V \quad \text{(Kirchhoff)}$$

$$\Longrightarrow \quad IR + V_c = V \quad \text{(Ohm's Law)}$$

$$\Longrightarrow \quad (CV_c')R + V_c = V \quad \text{(capacitor behavior)}$$

$$\Longrightarrow \quad RC\,V_c' + V_c = V$$

__Note:__ R, C, V are known constants

$V_c = V_c(t)$ = voltage across the capacitor

is the only unknown.

$$\Longrightarrow \quad V_c' + \frac{1}{RC} V_c = \frac{1}{RC} V$$

$$\Longrightarrow \quad \boxed{\left.\begin{array}{l} V_c' = \frac{1}{RC}(V - V_c) \\ \text{initially, } V_c = V_c(0) = 0 \end{array}\right\} \text{ initial-value problem to be solved}}$$

$$\begin{cases} V_c' = \frac{1}{RC}(V - V_c) \\ V_c(0) = 0 \end{cases}$$

Solution:

$$\frac{V_c'}{V - V_c} = \frac{1}{RC}$$

$$\Longrightarrow \frac{d}{dt}\left(-\ln(|V - V_c|)\right) = \frac{1}{RC}$$

integrate $\Longrightarrow$ $$-\ln(|V - V_c|) = \frac{t}{RC} + k$$

$$\ln(|V - V_c|) = \frac{-t}{RC} + k$$

exponentiate $\Longrightarrow$ $$V - V_c = k e^{-t/RC}$$

Then solve for V_c: $V_c(t) = V - k e^{-t/RC}$

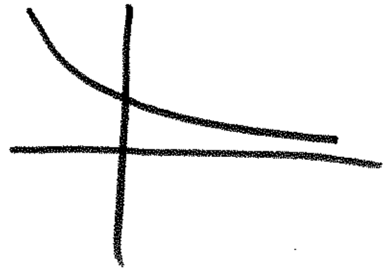
Since $V_c(0) = 0$, $k = V$.

Thus, $$V_c(t) = V - V e^{-t/RC} = \underline{\underline{V\left(1 - e^{-t/RC}\right)}}$$

<u>for $t \geq 0$</u> !

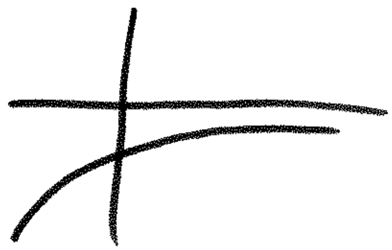
Plot of $V_c(t) = V(1 - e^{-t/RC})$:

step 1 : $e^{-t/RC}$



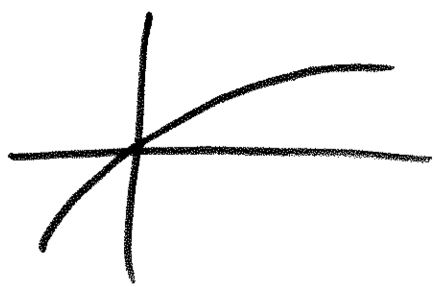
$\Downarrow$

step 2 : $-e^{-t/RC}$



$\Downarrow$

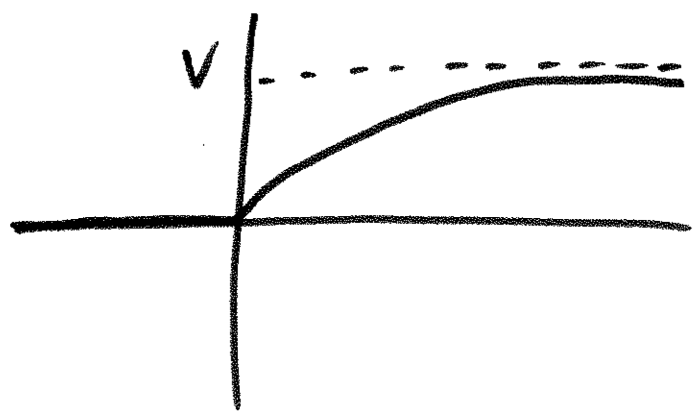
step 3 : $1 - e^{-t/RC}$



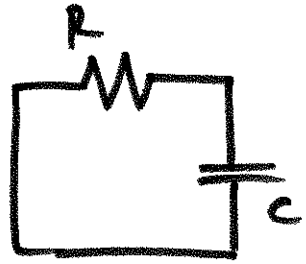
$\Downarrow$

step 4 : remember that $V_c(t) \equiv 0$ for $t \leq 0$ and multiply $(1 - e^{-t/RC})$ by V

This is the voltage $V_c(t)$ on the capacitor as it charges.



Discharging the capacitor: remove the power source



Now $V_R + V_c = 0$, $V_c(0) = V$ (fully charged)

$$\Longrightarrow IR + V_c = 0$$

$$\Longrightarrow (CV_c')R + V_c = 0$$

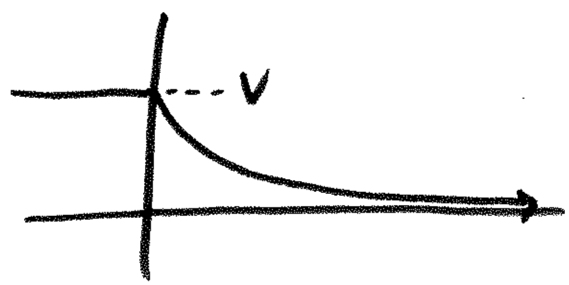
$$\Longrightarrow RC\, V_c' + V_c = 0$$

$$\Longrightarrow V_c' = -\frac{1}{RC} V_c$$

$$\Longrightarrow V_c(t) = k e^{-t/RC}$$

$$\Longrightarrow V_c(t) = V e^{-t/RC} \quad \text{since } V_c(0) = V.$$

Discharging:



Together: