PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1e^{-t/2} + c_2e^{-2t}$.
19. Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0, y(0) = \alpha, y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0, y(0) = 2, y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24 determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.

(c) Find the smallest value of β for which the solution has no minimum point.

note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution. Figure 3.3.3 shows the graph of two typical solutions of Eq. (28). In each case the solution is a pure oscillation whose amplitude is determined by the initial conditions. Since there is no exponential factor in the solution (29), the amplitude of each oscillation remains constant in time.

PROBLEMS

In each of Problems 1 through 6 use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-(\pi/2)i}$
5. 2^{1-i}
6. π^{-1+2i}

In each of Problems 7 through 16 find the general solution of the given differential equation.

7. $y'' - 2y' + 2y = 0$
8. $y'' - 2y' + 6y = 0$
9. $y'' + 2y' - 8y = 0$
10. $y'' + 2y' + 2y = 0$
11. $y'' + 6y' + 13y = 0$
12. $4y'' + 9y = 0$
13. $y'' + 2y' + 1.25y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
21. $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$
22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$
23. Consider the initial value problem
$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$
 (a) Find the solution $u(t)$ of this problem.
 (b) For $t > 0$ find the first time at which $|u(t)| = 10$.
24. Consider the initial value problem
$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$
 (a) Find the solution $u(t)$ of this problem.
 (b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.
25. Consider the initial value problem
$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$
 (a) Find the solution $y(t)$ of this problem.
 (b) Find α so that $y = 0$ when $t = 1$.
 (c) Find, as a function of α, the smallest positive value of t for which $y = 0$.
 (d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.

26. Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.
(c) Repeat part (b) for $a = 1/4, 1/2$, and 2.
(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

27. Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$.

28. In this problem we outline a different derivation of Euler's formula.
(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore

$$e^{it} = c_1 \cos t + c_2 \sin t \qquad \text{(i)}$$

for some constants c_1 and c_2. Why is this so?
(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.

29. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/2i.$$

30. If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.

31. If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt} e^{rt} = re^{rt}$$

for any complex number r.

32. Let the real-valued functions p and q be continuous on the open interval I, and let $y = \phi(t) = u(t) + iv(t)$ be a complex-valued solution of

$$y'' + p(t)y' + q(t)y = 0, \qquad \text{(i)}$$

where u and v are real-valued functions. Show that u and v are also solutions of Eq. (i).
Hint: Substitute $y = \phi(t)$ in Eq. (i) and separate into real and imaginary parts.

33. If the functions y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.
Hint: Suppose that t_1 and t_2 are two zeros of y_1 between which there are no zeros of y_2. Apply Rolle's theorem to y_1/y_2 to reach a contradiction.

Change of Variables. Sometimes a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \qquad \text{(i)}$$

can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 34 through 46. In particular, in Problem 34 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 35 through original second order equation for y. Although it is possible to write down a formula for $v(t)$, we will instead illustrate how this method works by an example.

EXAMPLE 3

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \tag{31}$$

find a fundamental set of solutions.

We set $y = v(t)t^{-1}$; then

$$y' = v't^{-1} - vt^{-2}, \qquad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting for y, y', and y'' in Eq. (31) and collecting terms, we obtain

$$\begin{aligned} 2t^2(v''t^{-1} &- 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ &= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ &= 2tv'' - v' = 0. \end{aligned} \tag{32}$$

Note that the coefficient of v is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq. (32) and solving for $v'(t)$, we find that

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \tfrac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \tfrac{2}{3}ct^{1/2} + kt^{-1}, \tag{33}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2(t) = t^{1/2}$. You can verify that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \tfrac{3}{2}t^{-3/2}, \qquad t > 0. \tag{34}$$

Consequently, y_1 and y_2 form a fundamental set of solutions of Eq. (31).

PROBLEMS

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $4y'' + 12y' + 9y = 0$
5. $y'' - 2y' + 10y = 0$
6. $y'' - 6y' + 9y = 0$
7. $4y'' + 17y' + 4y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $25y'' - 20y' + 4y = 0$
10. $2y'' + 2y' + y = 0$

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

11. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
13. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$
14. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.
(b) Determine where the solution has the value zero.
(c) Determine the coordinates (t_0, y_0) of the minimum point.
(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

17. Consider the initial value problem

$$4y'' + 4y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.
(b) Determine the coordinates (t_M, y_M) of the maximum point.
(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.
(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

19. If the roots of the characteristic equation are real, show that a solution of $ay'' + by' + cy = 0$ is either everywhere zero or else can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at}.
(b) Use Abel's formula [Eq. (22) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.
(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

21. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1t)$ and $\exp(r_2t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2t) - \exp(r_1t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \neq r_1$.

Then think of r_1 as fixed and use L'Hospital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

22. (a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}. \tag{i}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \tag{ii}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t \exp(r_1 t)$ is also a solution of $L[y] = 0$.

In each of Problems 23 through 30 use the method of reduction of order to find a second solution of the given differential equation.

23. $t^2 y'' - 4ty' + 6y = 0, \quad t > 0; \qquad y_1(t) = t^2$
24. $t^2 y'' + 2ty' - 2y = 0, \quad t > 0; \qquad y_1(t) = t$
25. $t^2 y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$
26. $t^2 y'' - t(t + 2)y' + (t + 2)y = 0, \quad t > 0; \qquad y_1(t) = t$
27. $xy'' - y' + 4x^3 y = 0, \quad x > 0; \qquad y_1(x) = \sin x^2$
28. $(x - 1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$
29. $x^2 y'' - (x - 0.1875)y = 0, \quad x > 0; \qquad y_1(x) = x^{1/4} e^{2\sqrt{x}}$
30. $x^2 y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2} \sin x$

31. The differential equation

$$xy'' - (x + N)y' + Ny = 0,$$

where N is a nonnegative integer, has been discussed by several authors.[6] One reason why it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1(x) = e^x$.

(b) Show that a second solution has the form $y_2(x) = ce^x \int x^N e^{-x}\, dx$. Calculate $y_2(x)$ for $N = 1$ and $N = 2$; convince yourself that, with $c = -1/N!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

Note that $y_2(x)$ is exactly the first $N + 1$ terms in the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

32. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

[6] T. A. Newton, "On Using a Differential Equation to Generate Polynomials," *American Mathematical Monthly 81* (1974), pp. 592–601. Also see the references given there.