$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' - x^2 y'' - xy' + \alpha^2 y = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{1}^{\infty} n a_n x^n + \sum_{0}^{\infty} \alpha^2 a_n x^n = 0$$

$$\underbrace{\left[2a_2 + \alpha^2 a_0\right]}_{n=0} + \left[\underbrace{3(2)a_3 - a_1 + \alpha^2 a_1}_{n=1}\right] x + \sum_{2}^{\infty} \Big[(n+2)(n+1) a_{n+2} - n(n-1) a_n - n a_n + \alpha^2 a_n\Big] x^n = 0$$

$$\left[2a_2 + \alpha^2 a_0\right] + \left[6a_3 + (\alpha^2 - 1)a_1\right]x + \sum_{2}^{\infty}\left[(n+2)(n+1)a_{n+2} - n^2 a_n + \alpha^2 a_n\right]x^n = 0$$

$$\Longrightarrow \quad a_2 = \frac{-\alpha^2}{2} a_0$$

$$a_3 = \frac{(1-\alpha^2)\, a_1}{6}$$

$$a_{n+2} = \frac{(n^2 - \alpha^2)}{(n+2)(n+1)} a_n, \quad n \geq 2$$

$$a_4 = \frac{4 - \alpha^2}{12} \cdot a_2 = \left(\frac{4-\alpha^2}{12}\right)\left(-\frac{\alpha^2}{2}\right) a_0$$

$$a_5 = \frac{(9-\alpha^2)}{5 \cdot 4} a_3 = \frac{(9-\alpha^2)(1-\alpha^2)}{120} a_1$$

$$y(x) = a_0 + a_1 x - \frac{\alpha^2}{2} a_0 x^2 + \frac{(1-\alpha^2)}{6} a_1 x^3 + \frac{\alpha^2(\alpha^2 - 4)}{24} a_0 x^4 + \frac{(9-\alpha^2)(1-\alpha^2)}{120} a_1 x^5 + \dots$$

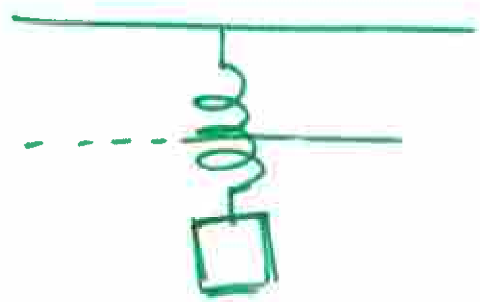
$$= a_0 \underbrace{\left[1 - \frac{\alpha^2}{2} x^2 + \frac{\alpha^2(\alpha^2-4)}{24} x^4 + \dots\right]}_{y_1} + a_1 \underbrace{\left[x + \frac{(1-\alpha^2)}{6} x^3 + \frac{(9-\alpha^2)(1-\alpha^2)}{120} x^5 + \dots\right]}_{y_2}$$

$\underline{\alpha = 2}$ $\quad y_1(x) = 1 - 2x^2$

$\underline{\alpha = 3}$ $\quad y_2(x) = x - \frac{4}{3}x^3 \quad \ldots$

$$y(x) = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n$$

Frobenius method



$y' = ky$

$y = e^{rt} \Longrightarrow y' = re^{rt} = ke^{rt}$

$\underline{\underline{r = k}}$

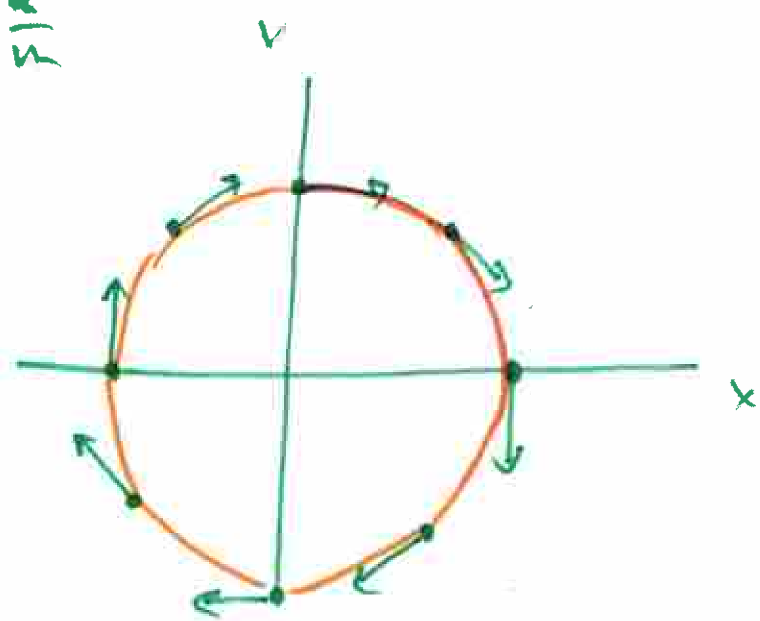
$F = ma$

$-kx = mx'' \Longrightarrow mx'' + kx = 0$

$\underline{\underline{x'' + \omega^2 x = 0}}, \quad \omega^2 = \frac{k}{m}$

$$\begin{cases} x' = v \\ v' = -\omega^2 x \end{cases}$$

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix} = [$$



$$\underbrace{\begin{bmatrix} x \\ v \end{bmatrix}'}_{\text{tangent vector}} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\text{state of system}}$$

$$\begin{bmatrix} x \\ v \end{bmatrix}' = A \begin{bmatrix} x \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

$$\boxed{\underline{y}' = A y}, \text{ where } y = \begin{bmatrix} x \\ v \end{bmatrix}$$

Look for $y(t) = e^{rt} \vec{v} = e^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

#1 $\quad x' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} x \qquad x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ 2x_1 - 2x_2 \end{bmatrix}$$

$$\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$$

Ansatz: $x(t) = e^{rt}\vec{v} = e^{rt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$x' = re^{rt}\vec{v} = e^{rt}\begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix}$

$$\Longrightarrow \cancel{e^{rt}}\begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \cancel{e^{rt}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{cases} rv_1 = 3v_1 - 2v_2 \\ rv_2 = 2v_1 - 2v_2 \end{cases}$$

$$(r-3)v_1 + 2v_2 = 0 \qquad\qquad (3-r)v_1 - 2v_2 = 0$$

$$\underline{\underline{\text{OR}}}$$

$$-2v_1 + (r+2)v_2 = 0 \qquad\qquad 2v_1 + (-2-r)v_2 = 0$$

$$\Longrightarrow (3-r)v_1 = 2v_2 \Longrightarrow v_2 = \frac{3-r}{2} v_1$$

$$\Longrightarrow 2v_1 + (-2-r)\left(\frac{3-r}{2}\right)v_1 = 0$$

$$\Longrightarrow 2 + (-2-r)\left(\frac{3-r}{2}\right) = 0$$

$$4 + (-2-r)(3-r) = 0$$

$$4 - 6 - r + r^2 = 0 \Longrightarrow r^2 - r - 2 = 0$$

$$\Longrightarrow r = 2,\ r = -1$$

$\underline{r=2}$: $v_2 = \left(\frac{3-2}{2}\right) v_1 \implies v_2 = \frac{1}{2} v_1 \implies \underline{2v_2 = v_1}$

$$\implies \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies x(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is a sol'n

$\underline{r=-1}$: $v_2 = \frac{4}{2} v_1 \implies v_2 = 2v_1 \implies \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\implies e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\implies x(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$