#2 $y'' - xy' - y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Longrightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Longrightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$m = n-2, \; n = m+2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{n=1}^{\infty} (n a_n + a_n) x^n - a_0 = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1) a_n x^n - a_0 = 0$$

$$\Longrightarrow \underbrace{\left[-a_0 + 2a_2 \right]}_{0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1) a_{n+2} - (n+1) a_n \right]}_{0} x^n = 0$$

$$\Longrightarrow \boxed{a_2 = \frac{1}{2} a_0}$$

$$(n+2)\cancel{(n+1)} a_{n+2} = \cancel{(n+1)} a_n$$

$$\Longrightarrow \boxed{a_{n+2} = \frac{a_n}{n+2}, \quad n \geq 1}$$

$n=0 \Longrightarrow a_2 = \frac{1}{2} a_0$	$n=1 \Longrightarrow a_3 = \frac{a_1}{3}$
$n=2 \Longrightarrow a_4 = \frac{a_2}{4} = \frac{1}{8} a_0$	$n=3 \Longrightarrow a_5 = \frac{a_3}{5} = \frac{1}{15} a_1$
$n=4 \Longrightarrow a_6 = \frac{1}{6} \cdot \frac{1}{8} a_0 = \frac{a_0}{48}$	$n=5 \Longrightarrow a_7 = \frac{1}{7} \cdot \frac{1}{15} a_1 = \frac{a_1}{105}$
etc.	etc.

$$y(x) = \left(a_0 + \frac{1}{2} a_0 x^2 + \frac{1}{8} a_0 x^4 + \frac{1}{48} a_0 x^6 + \dots\right) + \left(a_1 x + \frac{1}{3} a_1 x^3 + \frac{1}{15} a_1 x^5 + \frac{1}{105} a_1 x^7 + \dots\right)$$

$$= a_0 \underbrace{\left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \dots\right)}_{y_1(x)} + a_1 \underbrace{\left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \frac{1}{105} x^7 + \dots\right)}_{y_2(x)}$$

#5 $(1-x)y'' + y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Longrightarrow (1-x)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \underbrace{\left[2a_2 + a_0\right]}_{0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n\right]}_{0} x^n = 0$$

$$\boxed{a_2 = -\frac{1}{2} a_0}$$

$$(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n = 0, \quad n \geq 1$$

$$\Longrightarrow \boxed{a_{n+2} = \frac{n(n+1)a_{n+1} - a_n}{(n+2)(n+1)}, \quad n \geq 1}$$

$\underline{n=0}$: a_0 , $\underline{n=1}$: a_1

$\underline{n=2}$: $a_2 = -\frac{1}{2}a_0$, $\underline{n=3}$: $a_3 = \frac{(1)(2)a_2 - a_1}{(3)(2)}$

$$a_3 = \frac{2a_2 - a_1}{6} = \frac{-a_0 - a_1}{6}$$

$$a_4 = \frac{(2)(3)a_3 - a_2}{(4)(3)} = \frac{6a_3 - a_2}{12} = \frac{1}{2}a_3 - \frac{1}{12}a_2$$

$$= \frac{1}{2}\left(\frac{-a_0 - a_1}{6}\right) - \frac{1}{12}\left(-\frac{1}{2}a_0\right) = -\frac{1}{12}a_0 - \frac{1}{12}a_1 + \frac{1}{24}a_0$$

$$\underline{a_4 = -\frac{1}{24}a_0 - \frac{1}{12}a_1}$$

$$y(x) = a_0 + a_1 x - \frac{1}{2}a_0 x^2 + \left(\frac{-a_0 - a_1}{6}\right)x^3 + \left(-\frac{1}{24}a_0 - \frac{1}{12}a_1\right)x^4 + \dots$$

$$= a_0\underbrace{\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right)}_{y_1} + a_1\underbrace{\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \dots\right)}_{y_2}$$