2.3

8 (a) The sum of the squares of 2 odd integers cannot be a perfect square.

pf. Let the 2 odd integers be $(2n+1)$ and $(2m+1)$, for some $m,n \in \mathbb{Z}$. Then compute:

$$(2n+1)^2 + (2m+1)^2 = 4n^2+4n+1+4m^2+4m+1$$
$$= 4(n^2+n+m^2+m)+2, \quad \text{i.e., } 4k+2.$$

Recall from earlier calculations (e.g., #8 in 2.2) that a perfect square cannot be of the form $4k+2$. ∎

8(b) The product of 4 consecutive integers is 1 less than a perfect square.

Preliminary calculations:

integers	product
2, 3, 4, 5	$120 = 11^2 - 1$
3, 4, 5, 6	$360 = 19^2 - 1$
4, 5, 6, 7	$840 = 29^2 - 1$
5, 6, 7, 8	$1680 = 41^2 - 1$
6, 7, 8, 9	$3024 = 55^2 - 1$
⋮	⋮
$n, (n+1), (n+2), (n+3)$	? $\left(n(n+3)+1\right)^2 - 1$?

These calculations suggest the result below!

Now for the proof:

$$
\begin{aligned}
n(n+1)(n+2)(n+3) &= (n^2+n)(n^2+5n+6) \\
&= n^4+5n^3+6n^2+n^3+5n^2+6n \\
&= n^4+6n^3+11n^2+6n
\end{aligned}
$$

and

$$
\begin{aligned}
(n(n+3)+1)^2-1 &= (n^2+3n+1)(n^2+3n+1) - 1 \\
&= n^4+3n^3+n^2+3n^3+9n^2+3n+n^2+3n+1-1 \\
&= n^4+6n^3+11n^2+6n.
\end{aligned}
$$

They're the same! $\blacksquare$

<u>14</u> All 3 parts rely on Thm. 2.4: $\gcd(a,b)=1$ iff $ax+by=1$ for some $x,y \in \mathbb{Z}$.

(a) $\gcd(2a+1, 9a+4)=1$: Note that $9(2a+1) - 2(9a+4) = 1$.

(b) $\gcd(5a+2, 7a+3)=1$: Note that $-7(5a+2) + 5(7a+3) = 1$.

(c) If a is odd, then $\gcd(3a, 3a+2)=1$:

a odd $\Longrightarrow a = 2n+1$ for some n.

Then note that $(3n+2)(3a) - (3n+1)(3a+2) = 1$.

(You can get this by applying the Euclidean algorithm!)

2.3 **20(f)** $\gcd(a,b)=1 \Longrightarrow \gcd(a^2,b^2)=1$:

Since $\gcd(a,b)=1$, $ax+by=1$ for some integers x,y.

Then $\underbrace{a^2x+aby=a}$ & $\underbrace{abx+b^2y=b}$.

if $d \mid a^2$ & $d \mid b$, $d \mid a \Longrightarrow d=1$	if $d \mid a$ and $d \mid b^2$, $d \mid b \Longrightarrow d=1$.

Thus, $\gcd(a^2,b)=\gcd(a,b^2)=1$, and

$a^2u+bv=1$, $ar+b^2s=1$ for some $u,v,r,s \in \mathbb{Z}$.

$\Longrightarrow$ $a^2bu+b^2v=b$ & $a^2r+ab^2s=a$. Then

$d \mid a^2$ & $d \mid b^2 \Longrightarrow d \mid b$ & $d \mid a \Longrightarrow d=1$.

∎

2.4

2(a) $\gcd(56,72)$: $72 = 56 + 16$

$56 = 3\cdot 16 + 8$

$16 = 2\cdot 8 \implies 8 = \gcd(56,72)$

and then $8 = 56 - 3\cdot 16 = 56 - 3(72-56) = -3\cdot 72 + 4\cdot 56$

$\implies x = 4,\ y = -3$

2(b) $\gcd(24,138)$: $138 = 5\cdot 24 + 18$

$24 = 18 + 6$

$18 = 3\cdot 6 \implies 6 = \gcd(24,138)$

and then $6 = 24 - 18 = 24 - (138 - 5\cdot 24) = 6\cdot 24 - 138$

$\implies x = 6,\ y = -1$

2(c) $\gcd(119,272)$: $272 = 2\cdot 119 + 34$

$119 = 3\cdot 34 + 17$

$34 = 2\cdot 17 \implies 17 = \gcd(119,272)$

and $17 = 119 - 3\cdot 34 = 119 - 3\cdot(272 - 2\cdot 119) = 7\cdot 119 - 3\cdot 272$

$\implies x = 7,\ y = -3$

2(d) $\gcd(1769, 2378)$:

$$2378 = 1769 + 609$$

$$1769 = \underbrace{2 \cdot 609}_{1218} + 551$$

$$609 = 551 + 58$$

$$551 = \underbrace{9 \cdot 58}_{522} + 29$$

$$58 = 2 \cdot 29 \implies 29 = \gcd(1769, 2378)$$

Then $29 = 551 - 9 \cdot 58 = 551 - 9 \cdot (609 - 551)$

$= 10 \cdot 551 - 9 \cdot 609 = -9 \cdot 609 + 10 \cdot (1769 - 2 \cdot 609)$

$= 10 \cdot 1769 - 29 \cdot 609 = 10 \cdot 1769 - 29(2378 - 1769)$

$= 39 \cdot 1769 - 29 \cdot 2378.$

$\implies x = 39, \ y = -29.$

4 Assume throughout that $\gcd(a,b)=1$.

(a) $\gcd(a+b, a-b) = 1$ or 2 :

Let $d = \gcd(a+b, a-b)$. Then $d \mid a+b$ & $d \mid a-b$, so

$$d \mid (a+b)+(a-b) = 2a \quad \& \quad d \mid (a+b)-(a-b) = 2b.$$

Since $d \mid 2a$ & $d \mid 2b$, $d \leq \gcd(2a, 2b) = 2\gcd(a,b)$, so

$d = 1$ or $d = 2$. $\blacksquare$

(b) $\gcd(2a+b, a+2b) = 1$ or 3 :

Let $d = \gcd(2a+b, a+2b)$. Then $d \mid 2(2a+b) - (a+2b) = 3a$

and $d \mid -(2a+b) + 2(a+2b) = 3b$. As in (a), this implies

that $d \leq \gcd(3a, 3b) = 3\gcd(a,b) = 3$. Finally, observe that

d cannot be 2; a and b ~~must have opposite parity~~ cannot both be even, so ~~I~~ the same is true of $2a+b$ and $a+2b$. $\blacksquare$

(c) $\gcd(a+b, a^2+b^2) = 1$ or 2 :

Let $d = \gcd(a+b, a^2+b^2)$. Then $d \mid a+b$ & $d \mid a^2+b^2 = (a+b)(a-b)+2b^2$

$\Longrightarrow d \mid 2b^2$. Similarly, $d \mid 2a^2$. Thus,

$$d \le \gcd(2a^2, 2b^2) = 2\underbrace{\gcd(a^2,b^2)}_{= 1 \text{ by } 20(f) \text{ on p. 25}} = 2 \Longrightarrow d = 1 \text{ or } 2.$$

$\blacksquare$

(d) $\gcd(a+b, a^2-ab+b^2) = 1$ or 3 :

Let $d = \gcd(a+b, a^2-ab+b^2)$, so $d \mid a+b$ & $d \mid a^2-ab+b^2$.

Then $d \mid (a+b)^2 - 3ab \Longrightarrow d \mid 3ab. \Longrightarrow d \mid 3ab$ & $d \mid 3(a+b)$

$\Longrightarrow d \le \gcd(3ab, 3(a+b)) = 3\gcd(ab, a+b) = 3$,

since $\gcd(ab, a+b) = 1$ by problem 6.

Thus, $d = 1, 2,$ or 3 ; d cannot be 2 since $\gcd(a,b)=1$ guarantees that $a+b$, a^2-ab+b^2 cannot both be even.

$\Longrightarrow d = 1$ or $d = 3$.

$\blacksquare$

6 $\gcd(a,b)=1 \Longrightarrow \gcd(a+b, ab)=1$:

Let $d = \gcd(a+b, ab)$, so that $d \mid (a+b)^2 - 2ab = a^2+b^2$.

Then, by 4(c), $d \leq \gcd(a+b, a^2+b^2) = 1$ or 2 ; d cannot be 2, so $d=1$. $\boxed{\text{ff}}$

$\boxed{2.5}$ 2(a) $56x + 72y = 40$

By 2(a) in 2.2, $\gcd(56,72)=8$ & $4\cdot 56 - 3\cdot 72 = 8$.

Thus, $20\cdot 56 - 15\cdot 72 = 40$ and all solutions are given by $x = 20+9t,\ y = -15-7t,\ t\in\mathbb{Z}$.

2(b) $24x + 138y = 18$

By 2(b) in 2.2, $\gcd(24, 138) = 6$ & $6 = 24 \cdot 6 - 138$.

Thus, $24 \cdot 18 + 138 \cdot (-3) = 18$ and all solutions are given by

$$x = 18 + 23t, \quad y = -3 - 4t.$$

$\blacksquare$

2(c) $221x + 35y = 11$

$\gcd(221, 35)$:
$$221 = 6 \cdot 35 + 11$$
$$35 = 3 \cdot 11 + 2$$
$$11 = 5 \cdot 2 + 1 \implies 1 = \gcd(221, 35),$$

and
$$1 = 11 - 5 \cdot 2 = 11 - 5 \cdot (35 - 3 \cdot 11) = 16 \cdot 11 - 5 \cdot 35$$
$$= -5 \cdot 35 + 16 \cdot (221 - 6 \cdot 35)$$
$$= -101 \cdot 35 + 16 \cdot 221. \qquad (!)$$

Thus, $221 \cdot 176 + 35 \cdot (-1111) = 11$, and all solutions are given by

$$x = 176 + 35t, \quad y = -1111 - 221t.$$

$\blacksquare$

3(a) $18x + 5y = 48$

Since $18 \cdot 2 - 5 \cdot 7 = 1$, $18 \cdot 96 + 5(-336) = 48$,

and all solutions are given by

$$x = 96 + 5t, \quad y = -336 - 18t, \quad t \in \mathbb{Z}.$$

Positive solutions correspond to t satisfying

$$96 + 5t > 0 \quad \& \quad -336 - 18t > 0$$

$$\iff \quad t > \frac{-96}{5} = -19.2 \quad \& \quad t < -\frac{336}{18} = -18.66\ldots$$

$\iff$ $\underline{t = -19}$ yields the only positive solution,

namely $\underline{\underline{x = 1, y = 6}}$.

■

31(b) $54x + 21y = 906$

$\gcd(54, 21)$:

$$54 = 2 \cdot 21 + 12$$
$$21 = 12 + 9$$
$$12 = 9 + 3 \implies \gcd(54, 21) = 3, \text{ and}$$
$$9 = 3 \cdot 3$$

$$3 = 12 - 9 = 12 - (21 - 12)$$
$$= 2 \cdot 12 - 21 = -21 + 2 \cdot (54 - 2 \cdot 21)$$
$$= 2 \cdot 54 - 5 \cdot 21 .$$

Thus, $604 \cdot 54 - 1510 \cdot 21 = 906$ and all solutions are given by $x = 604 + 7t$, $y = -1510 - 18t$, $t \in \mathbb{Z}$.

Positive solutions: $604 + 7t > 0$ & $-1510 - 18t > 0$

$$\implies t > \frac{-604}{7} = -86.28\ldots \quad \& \quad t < -\frac{1510}{18} = -83.8\ldots$$

$$\implies t = -86, -85, -84 \implies \exists\ 3 \text{ positive solutions:}$$

$x = 2, y = 38$; $x = 9, y = 20$; $x = 16, y = 2$.

$\blacksquare$

31(c) $123x + 360y = 99$

$\gcd(123, 360)$: $360 = \underbrace{2 \cdot 123}_{246} + 114$

$123 = 114 + 9$

$114 = 12 \cdot 9 + 6$

$9 = 6 + 3$

$6 = 2 \cdot 3$

$\Longrightarrow$ $\gcd(123, 360) = 3$, and

$$\begin{aligned} 3 &= 9 - 6 = 9 - (114 - 12 \cdot 9) \\ &= -114 + 13 \cdot 9 \\ &= -114 + 13 \cdot (123 - 114) \\ &= 13 \cdot 123 - 14 \cdot (360 - 2 \cdot 123) \\ &= -14 \cdot 360 + 41 \cdot 123. \end{aligned}$$

Since $-14 \cdot 360 + 41 \cdot 123 = 3$,

$$(-462) \cdot 360 + (1353) \cdot 123 = 99$$

and one solution is $x_0 = 1353$, $y_0 = -462$. All solutions are given by

$$x = 1353 + 120t, \quad y = -462 - 41t, \quad t \in \mathbb{Z};$$

positive solutions require $1353 + 120t > 0$ & $-462 - 41t > 0$

$$\Longleftrightarrow \quad t > \frac{-1353}{120} = -11.275 \quad \& \quad t < -\frac{462}{41} = -11.26\ldots$$

$\Longrightarrow$ There are <u>No</u> positive solutions! ∎

3(d) $158x - 57y = 7$

$\gcd(158, 57)$:

$$\left.\begin{aligned} 158 &= \underbrace{2\cdot 57}_{114} + 44 \\ 57 &= 44 + 13 \\ 44 &= 3\cdot 13 + 5 \\ 13 &= 2\cdot 5 + 3 \\ 5 &= 3 + 2 \\ 3 &= 2 + 1 \end{aligned}\right\}$$

$\gcd(158, 57) = 1$, and

$$\begin{aligned} 1 = 3 - 2 &= 3 - (5 - 3) \\ &= -5 + 2\cdot(13 - 2\cdot 5) \\ &= 2\cdot 13 - 5\cdot(44 - 3\cdot 13) \\ &= -5\cdot 44 + 17\cdot(57 - 44) \\ &= 17\cdot 57 - 22\cdot(158 - 2\cdot 57) \\ &= -22\cdot 158 + 61\cdot 57 \quad (!) \end{aligned}$$

Since $-22\cdot 158 + 61\cdot 57 = 1$,

$22\cdot 158 - 57\cdot 61 = -1$,

and $(-154)\cdot 158 - 57\cdot(-427) = 7$. One solution is $x_0 = -154$, $y_0 = -427$

and all solutions are given by $x = -154 - 57t$, $y = -427 - 158t$.

Positive solutions: $-154 - 57t > 0$ & $-427 - 158t > 0$

$\Longrightarrow \quad t < -\frac{154}{57} = -2.7\ldots$ & $t < -\frac{427}{158} = -2.7\ldots$

$\Longrightarrow$ Any $t \leq -3$ $(t \in \mathbb{Z})$ yields a positive solution.

5(a) $10x + 25y = 455$

$\gcd(10, 25)$: $\left.\begin{array}{l} 25 = 2\cdot 10 + 5 \\ 10 = 2 \cdot 5 \end{array}\right\}$ $\gcd(10, 25) = 5$, and $5 = 25 - 2\cdot 10$

Since $-2\cdot 10 + 25 = 5$, $-182 \cdot 10 + 91 \cdot 25 = 455$ and all solutions are given by $x = -182 + 5t$, $y = 91 - 2t$.

Positive solutions: $-182 + 5t > 0$ & $91 - 2t > 0$

$$\Longrightarrow \quad t > \frac{182}{5} \quad \& \quad t < \frac{91}{2}$$

$$\Longrightarrow \quad t > 36.4 \quad \& \quad t < 45.5$$

$$\Longrightarrow \quad t = 37, \dots, 45 \implies 9 \text{ positive solutions:}$$

t	x	y	
37	3	17	$\longrightarrow$ min # coins
38	8	15	
39	13	13	$\longrightarrow$ same # of each
40	18	11	
41	23	9	

t	x	y	
42	28	7	
43	33	5	
44	38	3	
45	43	1	$\longrightarrow$ max # coins

5(b) $180x + 75y = 9000, \quad x > y$

Equivalent equation: $36x + 15y = 1800$

$\gcd(36, 15) = 3$, and $-2 \cdot 36 + 5 \cdot 15 = 3$

Thus, $(-1200) \cdot 36 + (3000)(15) = 1800$ and all solutions are

given by $x = -1200 + 5t, \; y = 3000 - 12t$.

Positive solutions: $-1200 + 5t > 0$ & $3000 - 12t > 0$

$\Longrightarrow \; t > 240$ & $t < 250$

$\Longrightarrow \; t = 241, \ldots, 249$

Also, require $-1200 + 5t > 3000 - 12t$

$\Longrightarrow \; 17t > 4200 \Longrightarrow t > 247.05\ldots$

Thus, $t = 248$ gives the desired solution: $\underline{\underline{x = 40, \; y = 24}}$.

& $t = 249$: $x = 45, \; y = 12$.

5(c)

$$\begin{cases} 6x + 9y = 126 \\ 6y + 9x = 114 \end{cases} \Longrightarrow \begin{array}{l} 6x + 9y = 126 \\ 9x + 6y = 114 \end{array}$$

$$\Longrightarrow \begin{array}{r} 18x + 27y = 378 \\ \underline{18x + 12y = 228} \\ 15y = 150 \end{array}$$

$$\Longrightarrow y = 10, \ x = 6 .$$

∎

(Easier than using the same techniques as in (a), (b)!)