3.3 #18 $y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \geq 0$

(a) $e^{rt} \Longrightarrow r^2 + 2r + 6 = 0 \Longrightarrow r = \dfrac{-2 \pm \sqrt{4 - 24}}{2} = -1 \pm \sqrt{5}\, i$

$\Longrightarrow y_1(t) = e^{-t}\cos(\sqrt{5}\, t), \quad y_2(t) = e^{-t}\sin(\sqrt{5}\, t)$

general solution: $y(t) = c_1 e^{-t}\cos(\sqrt{5}\, t) + c_2 e^{-t}\sin(\sqrt{5}\, t)$

$y(0) = 2 \Longrightarrow 2 = c_1$ (since $\cos(0) = 1$, $\sin(0) = 0$)

$\Longrightarrow y(t) = 2e^{-t}\cos(\sqrt{5}\, t) + c_2 e^{-t}\sin(\sqrt{5}\, t)$

$y'(t) = -2e^{-t}\cos(\sqrt{5}\, t) - 2\sqrt{5}\, e^{-t}\sin(\sqrt{5}\, t)$

$\qquad - c_2 e^{-t}\sin(\sqrt{5}\, t) + c_2\sqrt{5}\, e^{-t}\cos(\sqrt{5}\, t)$

$y'(0) = \alpha \Longrightarrow \alpha = -2 + c_2\sqrt{5} \Longrightarrow c_2 = \dfrac{\alpha + 2}{\sqrt{5}}$

Solution: $y(t) = e^{-t}\left(2\cos(\sqrt{5}\, t) + \dfrac{\alpha + 2}{\sqrt{5}}\sin(\sqrt{5}\, t)\right)$

(b) $y = 0$ when $t = 1$:

$$0 = e^{-1}\left(2\cos(\sqrt{5}) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5})\right)$$

$$\Longrightarrow \quad 0 = 2\cos(\sqrt{5}) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5})$$

$$\Longrightarrow \quad \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}) = -2\cos(\sqrt{5})$$

$$\Longrightarrow \quad \alpha+2 = (-2\sqrt{5})\left(\frac{\cos(\sqrt{5})}{\sin(\sqrt{5})}\right) = 3.50877..$$

$$\Longrightarrow \quad \underline{\underline{\alpha = 1.50877...}}$$

(c) smallest t for which $y = 0$:

$$0 = e^{-t}\left(2\cos(\sqrt{5}\,t) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}\,t)\right)$$

$$\Longrightarrow \quad \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}\,t) = -2\cos(\sqrt{5}\,t)$$

$$\Longrightarrow \quad \frac{\sin(\sqrt{5}\,t)}{\cos(\sqrt{5}\,t)} = \frac{-2\sqrt{5}}{\alpha+2} \Longrightarrow \tan(\sqrt{5}\,t) = \frac{-2\sqrt{5}}{\alpha+2}$$

Since $\alpha \geq 0$, this says that $\tan(\sqrt{5}\,t) \leq 0 \implies \sqrt{5}\,t$ is between $\frac{\pi}{2}$ and π (smallest t for which $y = 0$),

so that
$$\tan(\sqrt{5}\,t) = \frac{-2\sqrt{5}}{\alpha+2} \implies \sqrt{5}\,t = \pi - \underbrace{\arctan\left(\frac{+2\sqrt{5}}{\alpha+2}\right)}_{\text{between } +\pi/2 \text{ and } 0}$$

Thus,
$$t = \frac{\pi - \arctan\left(\frac{-2\sqrt{5}}{\alpha+2}\right)}{\sqrt{5}}.$$

~~(d) lim $\alpha \to 0$~~

$$\text{(d)} \lim_{\alpha \to \infty} \left(\frac{\pi - \arctan\left(\frac{-2\sqrt{5}}{\alpha+2}\right)}{\sqrt{5}} \right) = \frac{\pi}{\sqrt{5}}$$

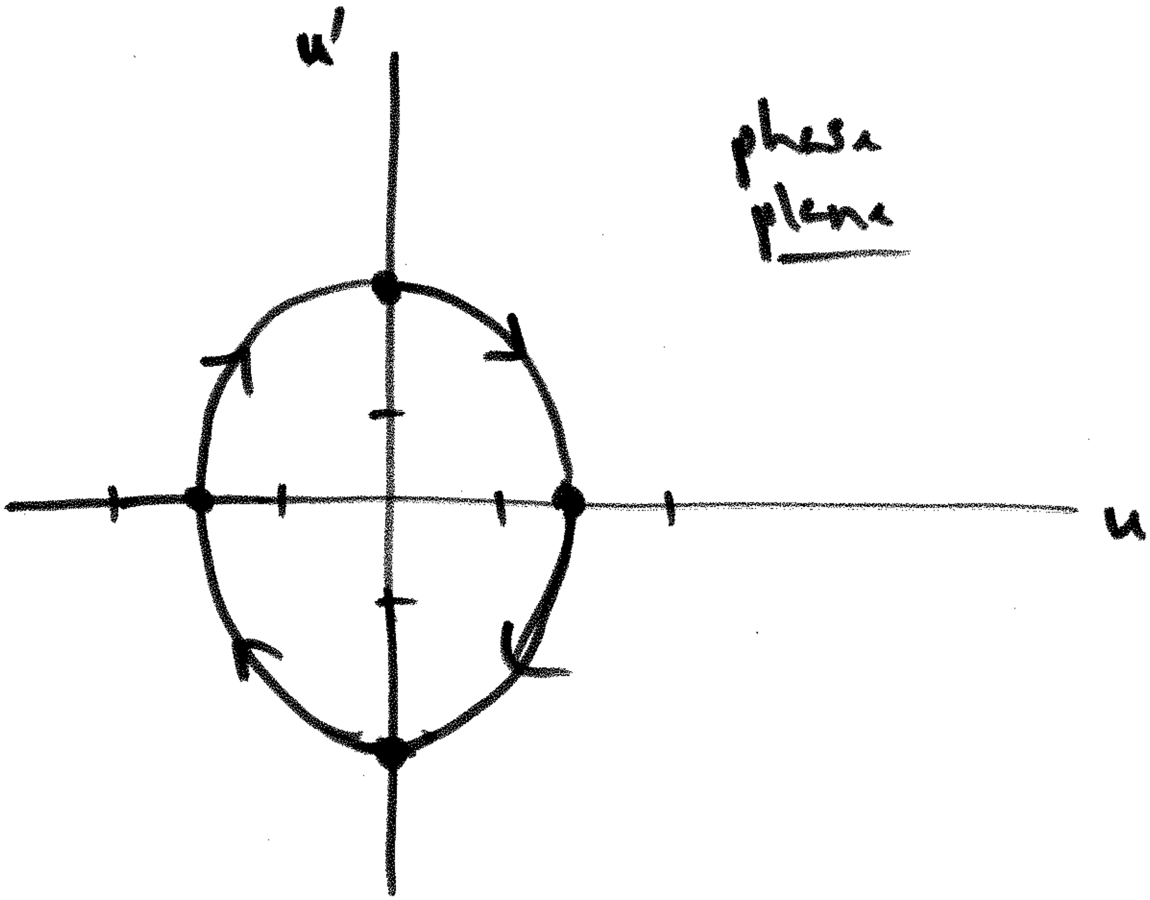
since $\arctan(0) = 0$.

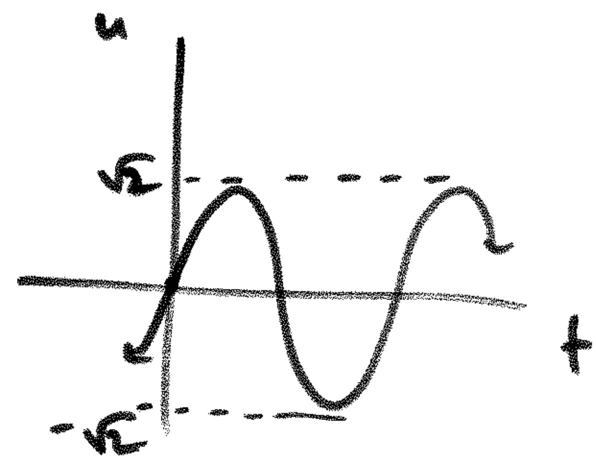
#20 $\quad u'' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2$

$$\Longrightarrow \left.\begin{aligned} u(t) &= \sqrt{2}\,\sin(\sqrt{2}\,t) \\ u'(t) &= 2\cos(\sqrt{2}\,t) \end{aligned}\right\} \quad \begin{aligned} (u)^2 &= 2\sin^2(\sqrt{2}\,t) \\ (u')^2 &= 4\cos^2(\sqrt{2}\,t) \end{aligned}$$

$$\Longrightarrow \underline{\underline{2(u^2) + (u')^2 = 4}}$$

phase plane



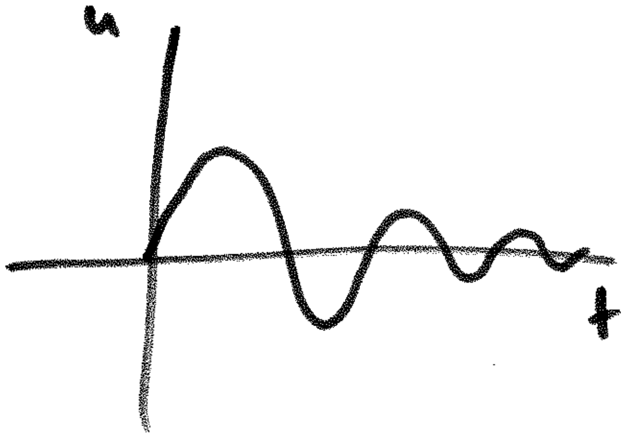


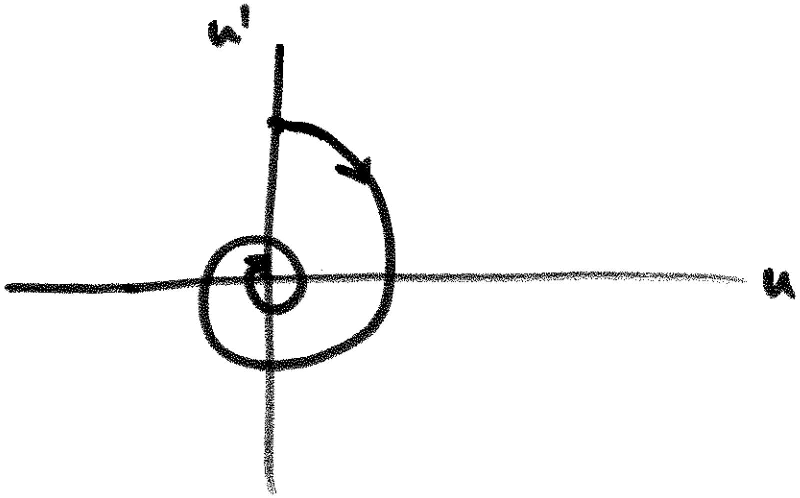
#21 $u'' + \frac{1}{4}u' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2$

$$\Longrightarrow r^2 + \frac{1}{4}r + 2 = 0 \Longrightarrow r = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 8}}{2}$$

$$= \frac{-\frac{1}{4} \pm \alpha i}{2}$$

$$= -\frac{1}{8} \pm \beta i$$





#10 $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

$$y = e^{rt} \Longrightarrow r^2 - 6r + 9 = 0 \Longrightarrow (r-3)^2 = 0$$

$$r = 3 \qquad y_1 = e^{3t}$$

reduction of order : $y_2 = u y_1 = ue^{3t}$

$$y_2' = u'e^{3t} + 3ue^{3t}$$

$$y_2'' = u''e^{3t} + 3u'e^{3t} + 3u'e^{3t} + 9ue^{3t}$$

$$= u''e^{3t} + 6u'e^{3t} + 9ue^{3t}$$

$$\Longrightarrow \underbrace{u'' + \cancel{6u'} + \cancel{9u}}_{y_2''} \underbrace{- \cancel{6u'} - \cancel{18u}}_{-6y'} \underbrace{+ \cancel{9u}}_{+9y} = 0$$

$$\Longrightarrow u'' = 0 \Longrightarrow u(t) = a + bt \Longrightarrow y_2 = te^{3t}$$

General sol'n : $y(t) = c_1 e^{3t} + c_2 te^{3t}$

$y(0) = 0 \Longrightarrow c_1 = 0$

$y = cte^{3t}$

$y' = ce^{3t} + 3cte^{3t}$

$y' = 2 \Longrightarrow c = 2$

$$\boxed{y(t) = 2te^{3t}}$$

3.6 #1 $y'' - 5y' + 6y = 2e^t$

step 1: Solve the homogeneous ODE,

$$y'' - 5y' + 6y = 0$$

$$\Longrightarrow \quad r^2 - 5r + 6 = 0 \Longrightarrow r = 2, 3$$

$$y(t) = c_1 e^{2t} + c_2 e^{3t}$$

step 2: find a particular solution of the nonhomogeneous ODE, $y'' - 5y' + 6y = 2e^t$

$$y_p = u_1 y_1 + u_2 y_2 \qquad \text{Ansatz}$$

$$y_p' = \cancel{u_1' y_1} + u_1 y_1' + \cancel{u_2' y_2} + u_2 y_2' \qquad \boxed{u_1' y_1 + u_2' y_2 = 0}$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' - 5u_1y_1' - 5u_2y_2' +$$
$$+ 6u_1y_1 + 6u_2y_2 = 2e^t$$

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{\cancel{(y_1'' - 5y_1' + 6y_1)}}_{0} + u_2\underbrace{\cancel{(y_2'' - 5y_2' + 6y_2)}}_{0}$$
$$= 2e^t$$

$$\Longrightarrow \left.\begin{aligned} u_1'y_1' + u_2'y_2' &= 2e^t \\ \& \quad u_1'y_1 + u_2'y_2 &= 0 \end{aligned}\right\}$$