Math 212 – Problem Set 3 SOLUTIONS

① $y' + 3y = 3x^2 e^{-3x}$

Find an integrating factor:

$$\underbrace{\mu y' + 3\mu y}\ = \mu(3x^2 e^{-3x})$$

$\frac{d}{dx}(\mu y) = \mu y' + \mu' y$, so need $\mu' = 3\mu \Longrightarrow \mu(x) = e^{3x}$

with $\mu = e^{3x}$, we get

$$\frac{d}{dx}(e^{3x} y) = 3x^2$$

Integrate:

$$e^{3x} y = x^3 + c \Longrightarrow y(x) = (x^3 + c)e^{-3x}$$

② $2x \frac{dy}{dx} + y = \frac{2x^2}{y^3}$, $y(1) = 2$

Divide through by $2x$, then multiply through by y^3:

$$\frac{dy}{dx} + \frac{1}{2x} y = \frac{x}{y^3} \qquad \text{(Bernoulli ODE)}$$

$$\Longrightarrow y^3 \frac{dy}{dx} + \frac{1}{2x} y^4 = x$$

To make the leftmost term a derivative, multiply through by 4:

$$\underbrace{4y^3 \frac{dy}{dx}} + \frac{2}{x} y^4 = 4x$$

This is now $\frac{d}{dx}(y^4)$, so the ODE is

$$\frac{d}{dx}(y^4) + \frac{2}{x} y^4 = 4x.$$

Find an integrating factor:

$$\underbrace{\mu \frac{d}{dx}(y^4) + \mu\left(\frac{2}{x}\right) y^4} = 4\mu x$$

$\frac{d}{dx}(\mu y^4) = \mu \frac{d}{dx}(y^4) + \mu' y^4$, so we need

$$\mu' = \mu \cdot \frac{2}{x} \implies \frac{\mu'}{\mu} = \frac{2}{x}$$

$$\implies \ln|\mu| = 2\ln|x| = \ln|x^2|$$

$$\implies \mu = x^2.$$

Using $\mu = x^2$, we get

$$\frac{d}{dx}(x^2 y^4) = 4x^3 \quad ; \text{ integrate} \ldots$$

$$\Longrightarrow \quad x^2 y^4 = x^4 + c$$

$$\Longrightarrow \quad y^4 = x^2 + \frac{c}{x^2}$$

$$\Longrightarrow \quad y(x) = \left(x^2 + \frac{c}{x^2}\right)^{1/4}$$

To find the particular sol'n with $y(1) = 2$:

$$2 = (1+c)^{1/4} \Longrightarrow 16 = 1 + c \Longrightarrow \underline{c = 15}.$$

The particular solution is thus $\underline{y(x) = \left(x^2 + \frac{15}{x^2}\right)^{1/4}}$.

(3) $(5xy + 4y^2 + 1)\,dx + (x^2 + 2xy)\,dy = 0$

check for exactness:

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = (5x + 8y) - (2x + 2y)$$

$$= 3x + 6y$$

$$= 3(x + 2y). \qquad \underline{\text{Not exact!}}$$

Find an integrating factor:

note that $\dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \dfrac{3(x+2y)}{x(x+2y)} = \dfrac{3}{x}$, so we can solve

$$\frac{h'}{h} = \frac{3}{x} \implies \ln|h| = 3\ln|x| = \ln|x^3|$$

$$\implies \underline{\underline{h(x) = x^3}}$$

to get an integrating factor. Using $h(x) = x^3$, we get

$$x^3(5xy + 4y^2 + 1)\,dx + x^3(x^2 + 2xy)\,dy = 0$$

$$\implies \underbrace{(5x^4y + 4x^3y^2 + x^3)}_{P}\,dx + \underbrace{(x^5 + 2x^4y)}_{Q}\,dy = 0$$

check for exactness:

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= 5x^4 + 8x^3y \\ \frac{\partial Q}{\partial x} &= 5x^4 + 8x^3y \end{aligned}\right\} \underline{\text{exact!}}$$

Find $f(x,y)$:

$$\frac{\partial f}{\partial x} = 5x^4y + 4x^3y^2 + x^3$$

$$\implies f(x,y) = x^5y + x^4y^2 + \frac{1}{4}x^4 + R(y)$$

$$\implies \frac{\partial f}{\partial y} = x^5 + 2x^4y + R'(y), \text{ so } R' = 0 \to \underline{\underline{R = 0}}.$$

The general solution of the original ODE is thus

$$x^5 y + x^4 y^2 + \frac{1}{4} x^4 = c.$$

④ $y' + 3x^2 y = x^2, \quad y(0) = 2$

Find an integrating factor:

$$\underbrace{\mu y' + 3x^2 \mu y}_{} = \mu x^2$$

$$\frac{d}{dx}(\mu y) = \mu y' + \mu' y \Longrightarrow \text{need } \mu' = 3x^2 \mu$$

$$\Longrightarrow \frac{\mu'}{\mu} = 3x^2 \Longrightarrow \ln|\mu| = x^3$$

$$\Longrightarrow \mu = e^{x^3}.$$

with $\mu = e^{x^3}$, we get

$$\frac{d}{dx}\left(e^{x^3} y\right) = x^2 e^{x^3}.$$

Integrate:

$$e^{x^3} y = \frac{1}{3} e^{x^3} + c$$

$$\Longrightarrow \quad y(x) = \frac{1}{3} + c e^{-x^3}.$$

To find the solution with $y(0) = 2$:

$$2 = \frac{1}{3} + c \implies c = \frac{5}{3}.$$

The particular solution is thus

$$y(x) = \frac{1}{3} + \frac{5}{3} e^{-x^3}.$$

(5) $$\frac{dy}{dx} - \frac{y}{x} = \frac{-y^2}{x}$$

Divide through by y^2:

$$\underbrace{y^{-2} \frac{dy}{dx}} - y^{-1} x^{-1} = -x^{-1}$$

Turn this into $\frac{d}{dx}(y^{-1})$ by multiplying through by -1:

$$-y^{-2} \frac{dy}{dx} + y^{-1} x^{-1} = x^{-1}$$

i.e.,

$$\frac{d}{dx}(y^{-1}) + x^{-1} y^{-1} = x^{-1}.$$

Find an integrating factor:

$$\underbrace{\mu \frac{d}{dx}(y^{-1}) + \mu x^{-1} y^{-1}}_{\frac{d}{dx}(\mu y^{-1}) = \mu \frac{d}{dx}(y^{-1}) + \mu' y^{-1}} = \mu x^{-1}$$

Thus, need $\mu' = \mu x^{-1}$

$$\Longrightarrow \frac{\mu'}{\mu} = \frac{1}{x} \Longrightarrow \underline{\underline{\mu = x}}.$$

Using $\mu = x$, we get

$$\frac{d}{dx}(x y^{-1}) = 1$$

Integrate:

$$x y^{-1} = x + c$$

$$\Longrightarrow y^{-1} = 1 + \frac{c}{x}$$

$$\Longrightarrow y(x) = \frac{1}{1 + \frac{c}{x}} = \underline{\underline{\frac{x}{x+c}}}.$$

(6) $(2x + \tan y)\,dx + (x - x^2 \tan y)\,dy = 0$

check for exactness:

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = (\sec^2 y) - (1 - 2x\tan y)$$

$$= \underbrace{(\sec^2 y - 1)} + 2x\tan y$$

$$= \tan^2 y + 2x\tan y$$

$$= \tan y\,(\tan y + 2x).$$

The original ODE is <u>not</u> exact, but

$$\frac{\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}}{P} = \frac{-\tan y\,(\tan y + 2x)}{\tan y + 2x} = -\tan y,$$

so $h(y)$ will be an integrating factor as long as

$$\frac{h'}{h} = -\tan y = \frac{-\sin y}{\cos y}$$

$$\Longrightarrow \ln|h| = \ln|\cos y| \Longrightarrow \underline{\underline{h(y) = \cos y}}.$$

using $h(y) = \cos y$, we get

$$\cos y\,(2x + \tan y)\,dx + \cos y\,(x - x^2\tan y)\,dy = 0$$

$$\Longrightarrow \underbrace{(2x\cos y + \sin y)}_{P}\,dx + \underbrace{(x\cos y - x^2\sin y)}_{Q}\,dy = 0$$

check for exactness:

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= -2x\sin y + \cos y \\ \frac{\partial Q}{\partial x} &= \cos y - 2x\sin y \end{aligned}\right\}\ \underline{\text{exact!}}$$

Now find $f(x,y)$:

$$\frac{\partial f}{\partial x} = 2x\cos y + \sin y$$

$$\Longrightarrow f(x,y) = x^2\cos y + x\sin y + R(y)$$

$$\Longrightarrow \frac{\partial f}{\partial y} = -x^2\sin y + x\cos y + R' \Longrightarrow R' = 0$$

$$\Longrightarrow R = 0.$$

Thus, the general solution of the original ODE is

$$\underline{\underline{x^2\cos y + x\sin y = c}},$$