Math 212 - Problem Set 6 SOLUTIONS

① (a) $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$

characteristic equation: $m^2 + 2m + 5 = 0$

roots: $m = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-x}\sin(2x) + c_2 e^{-x}\cos(2x)$$

particular solution: $y_p(x) = \alpha \sin 2x + \beta \cos 2x$

$$\Longrightarrow \quad y_p' = 2\alpha\cos 2x - 2\beta \sin 2x,$$

$$y_p'' = -4\alpha \sin 2x - 4\beta\cos 2x$$

substitute these into the nonhomogeneous ODE:

$$(-4\alpha\sin 2x - 4\beta\cos 2x) + (4\alpha\cos 2x - 4\beta\sin 2x)$$
$$+ (5\alpha\sin 2x + 5\beta\cos 2x) = 6\sin 2x + 7\cos 2x$$

$$\Longrightarrow \quad (\alpha - 4\beta)\sin 2x + (4\alpha + \beta)\cos 2x = 6\sin 2x + 7\cos 2x$$

$$\Longrightarrow \quad \begin{aligned} \alpha - 4\beta &= 6 \\ 4\alpha + \beta &= 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \alpha - 4\beta &= 6 \\ 16\alpha + 4\beta &= 28 \end{aligned}$$

$$17\alpha = 34 \Longrightarrow \alpha = 2$$

$$\alpha = 2 \Longrightarrow \beta = -1.$$

Thus, $y_p(x) = 2\sin 2x - \cos 2x$, and the general solution of the nonhomogeneous ODE is

$$y(x) = c_1 e^{-x}\sin 2x + c_2 e^{-x}\cos 2x + 2\sin 2x - \cos 2x .$$

(b) $y''' - 3y'' + 4y = 4e^{x} - 18e^{-x}$

characteristic equation: $m^3 - 3m^2 + 4 = 0$

$$\Longrightarrow (m+1)(m-2)^2 = 0$$

roots: $m = -1$, $m = 2$ (double root)

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-x} + (c_2 + c_3 x)e^{2x} .$$

particular solution: $y_p(x) = \alpha e^{x} + \beta x e^{-x}$

$$\Longrightarrow y_p' = \alpha e^{x} + \beta e^{-x} - \beta x e^{-x},$$

$$y_p'' = \alpha e^{x} - \beta e^{-x} - \beta e^{-x} + \beta x e^{-x} = \alpha e^{x} - 2\beta e^{-x} + \beta x e^{-x},$$

$$y_p''' = \alpha e^{x} + 2\beta e^{-x} + \beta e^{-x} - \beta x e^{-x} = \alpha e^{x} + 3\beta e^{-x} - \beta x e^{-x}$$

<u>Note</u>: The particular solution <u>cannot</u> be of the form $\alpha e^{x} + \beta e^{-x}$, since e^{-x} solves the homogeneous equation. That's the reason for multiplying e^{-x} by x.

Now substitute y_p, y_p'', y_p''' into the nonhomogeneous ODE ...

Get

$$\alpha e^{x} + 3\beta e^{-x} - \beta x e^{-x} - 3\alpha e^{x} + 6\beta e^{-x} - 3\beta x e^{-x}$$
$$+ 4\alpha e^{x} + 4\beta x e^{-x} = 4e^{x} - 18e^{-x}$$

$$\Longrightarrow \quad 2\alpha e^{x} + 9\beta e^{-x} = 4e^{x} - 18e^{-x}$$

$$\Longrightarrow \quad \alpha = 2, \ \beta = -2 \quad \Longrightarrow \quad y_p(x) = 2e^{x} - 2xe^{-x}.$$

The general solution of the ODE is thus

$$y(x) = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + 2e^{x} - 2xe^{-x}.$$

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② (a) $y'' + 4y' + 5y = e^{-2x} \sec x$

characteristic equation: $m^2 + 4m + 5 = 0$

$$\Longrightarrow \quad m = \frac{-4 \pm \sqrt{16-20}}{2} = \underline{\underline{-2 \pm i}}$$

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-2x} \sin x + c_2 e^{-2x} \cos x$$

particular solution via variation of parameters:

$$y_p(x) = u_1 e^{-2x} \sin x + u_2 e^{-2x} \cos x.$$

Note that $\frac{d}{dx}\left(e^{-2x} \sin x\right) = -2e^{-2x} \sin x + e^{-2x} \cos x,$

$$\frac{d}{dx}\left(e^{-2x} \cos x\right) = -2e^{-2x} \cos x - e^{-2x} \sin x.$$

To determine u_1 & u_2, we solve the system

$$\begin{cases} u_1' e^{-2x}\sin x + u_2' e^{-2x}\cos x = 0 \\ u_1'(-2e^{-2x}\sin x + e^{-2x}\cos x) + u_2'(-2e^{-2x}\cos x - e^{-2x}\sin x) = e^{-2x}\sec x \end{cases}$$

Cancel all of the exponentials to simplify:

$$\begin{cases} u_1' \sin x + u_2' \cos x = 0 \\ u_1'(-2\sin x + \cos x) + u_2'(-2\cos x - \sin x) = \sec x \end{cases}$$

Since $u_1' \sin x + u_2' \cos x = 0$, the 2nd equation simplifies further: $u_1'(-2\sin x + \cos x) + u_2'(-2\cos x - \sin x)$

$$= -2\underbrace{(u_1' \sin x + u_2' \cos x)}_{0} + u_1' \cos x - u_2' \sin x$$

$$= u_1' \cos x - u_2' \sin x.$$

Finally, we have the system

$$\begin{cases} u_1' \sin x + u_2' \cos x = 0 \\ u_1' \cos x - u_2' \sin x = \sec x \end{cases}$$

Multiply the top equation by $\sin x$ & the bottom by $\cos x$ to get

$$\begin{cases} u_1' \sin^2 x + u_2' \sin x \cos x = 0 \\ u_1' \cos^2 x - u_2' \sin x \cos x = 1 \end{cases}$$

Add: $u_1' = 1 \implies u_1 = x$ (don't need $+c$ for finding a particular solution!)

Now substitute $u_1 = x$ into the equation $u_1' \sin x + u_2' \cos x = 0$

to get $\sin x + u_2' \cos x = 0 \implies u_2' = \dfrac{-\sin x}{\cos x} = -\tan x$

$$\implies u_2 = \int \frac{-\sin x}{\cos x}\, dx = \ln|\cos x| \quad \left(\begin{matrix}\text{don't need}\\ +c\end{matrix}\right)$$

Thus, $y_p(x) = x e^{-2x} \sin x + \ln|\cos x| e^{-2x} \cos x$,

and the general solution of the nonhomogeneous ODE is

$$y(x) = c_1 e^{-2x} \sin x + c_2 e^{-2x} \cos x + x e^{-2x} \sin x + \ln|\cos x| e^{-2x} \cos x.$$

(b) $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

characteristic equation: $m^2 + 3m + 2 = 0$

$$\implies (m+1)(m+2) = 0 \implies m = -1, -2$$

general solution of homogeneous equation:

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$$

particular solution via variation of parameters:

$$y_p(x) = u_1 e^{-x} + u_2 e^{-2x},$$

where u_1 and u_2 solve the system

$$\begin{cases} u_1' e^{-x} + u_2' e^{-2x} = 0 \\ -u_1' e^{-x} - 2u_2' e^{-2x} = \dfrac{1}{1+e^x} \end{cases}$$

Adding these equations gives

$$-u_2' e^{-2x} = \frac{1}{1+e^x} \implies u_2' = \frac{-e^{2x}}{1+e^x}$$

$$\implies u_2 = -\int \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{e^x}{1+e^x} e^x\,dx$$

Using the substitution $w = e^x$, $dw = e^x dx$ and

$$\int \frac{e^x}{1+e^x} e^x dx = \int \frac{w}{1+w}\,dw = \int \frac{w+1-1}{1+w}\,dw$$

$$= \int \left(1 - \frac{1}{1+w}\right) dw$$

$$= w - \ln|1+w|$$

$$= e^x - \ln(1+e^x).$$

Thus, $u_2 = \ln(1+e^x) - e^x$.

Substituting $u_2' = \dfrac{-e^{2x}}{1+e^x}$ into the equation

$$u_1' e^{-x} + u_2' e^{-2x} = 0 \implies$$

$$u_1' e^{-x} + \left(\frac{-e^{2x}}{1+e^x} \right)\left(e^{-2x} \right) = 0$$

$$\Longrightarrow \quad u_1' e^{-x} - \frac{1}{1+e^x} = 0$$

$$\Longrightarrow \quad u_1' = \frac{e^x}{1+e^x} \Longrightarrow u_1 = \int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)$$

Thus, $y_p(x) = \ln(1+e^x) \cdot e^{-x} + \left(\ln(1+e^x) - e^x \right) e^{-2x}$

$$= e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x) - e^{-x}$$

and the general solution of the nonhomogeneous ODE is

This term solves the homogeneous ODE, so it gets absorbed in the $c_1 e^{-x}$ term.

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x)$$

(c) $x^2y'' - 6xy' + 10y = 3x^4 + 6x^3$

Homogeneous solutions: $y_1 = x^2$, $y_2 = x^5$

particular solution: $y_p = u_1 x^2 + u_2 x^5$, where

u_1 and u_2 solve the system

$$\begin{cases} u_1' \cdot x^2 + u_2' \cdot x^5 = 0 \\ u_1' \cdot 2x + u_2' \cdot 5x^4 = 3x^2 + 6x \end{cases}$$

Divide the right-hand side, $3x^4 + 6x^3$, by the coefficient of y'', x^2, to get this.

Multiply the top by 2 and the bottom by x:

$$\begin{cases} u_1' \cdot 2x^2 + u_2' \cdot 2x^5 = 0 \\ u_1' \cdot 2x^2 + u_2' \cdot 5x^5 = 3x^3 + 6x^2 \end{cases}$$

$$\Longrightarrow \quad u_2' \cdot 3x^5 = 3x^3 + 6x^2$$

$$\Longrightarrow \quad u_2' = \frac{1}{x^2} + \frac{2}{x^3} = x^{-2} + 2x^{-3}$$

$$\Longrightarrow \quad u_2 = -x^{-1} - x^{-2} = \underline{\underline{-\frac{1}{x} - \frac{1}{x^2}}}$$

Substitute $u_2' = \frac{1}{x^2} + \frac{2}{x^3}$ into the equation $u_1' \cdot x^2 + u_2' \cdot x^5 = 0$

to get $u_1' \cdot x^2 + \left(\frac{1}{x^2} + \frac{2}{x^3}\right)(x^5) = 0$

$$\Longrightarrow \quad u_1' \cdot x^2 + x^3 + 2x^2 = 0$$

$$\Rightarrow \quad u_1' + x + 2 = 0$$

$$\Rightarrow \quad u_1' = -x - 2 \Rightarrow u_1 = -\frac{1}{2}x^2 - 2x$$

Thus, $y_p = \left(-\frac{1}{2}x^2 - 2x\right)(x^2) + \left(-\frac{1}{x} - \frac{1}{x^2}\right)(x^5)$

$$= -\frac{1}{2}x^4 - 2x^3 - x^4 - x^3$$

$$= -\frac{3}{2}x^4 - 3x^3$$, and the general solution is

$$y(x) = c_1 x^2 + c_2 x^5 - \frac{3}{2}x^4 - 3x^3.$$

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③ (a) $(2x+1)y'' - 4(x+1)y' + 4y = 0$; $y(x) = e^{2x}$ is one solution

2^{nd} solution via reduction of order:

$$y(x) = u(x)e^{2x}$$

$$\Rightarrow y' = u'e^{2x} + 2ue^{2x}, \quad y'' = u''e^{2x} + 2u'e^{2x} + 2u'e^{2x} + 4ue^{2x}$$

$$\Rightarrow \quad y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}.$$

Substitute these into the ODE; for simplicity, go ahead and eliminate all of the e^{2x} factors!

$$(2x+1)(u''+4u'+4u) - 4(x+1)(u'+2u) + 4u = 0$$

$$\Longrightarrow \quad (2x+1)u'' + [8x+4-4x-4]u' + [8x+4-8x-8+4]u = 0$$

$$\Longrightarrow \quad (2x+1)u'' + 4x\,u' = 0$$

This equation is 1^{st} order in u'.

$$\Longrightarrow \quad u'' = \frac{-4x}{2x+1}u'$$

$$\Longrightarrow \quad \frac{u''}{u'} = \frac{-4x}{2x+1} = \frac{-4x-2+2}{2x+1} = -2 + \frac{2}{2x+1}$$

$$\Longrightarrow \quad \frac{u''}{u'} = -2 + \frac{2}{2x+1}$$

Now integrate!

$$\ln|u'| = -2x + \ln|2x+1|$$

(don't need to here — incorporate arbitrary constants later)

$$\Longrightarrow u' = e^{-2x} \cdot e^{\ln|2x+1|}$$

$$= (2x+1)e^{-2x}$$

integration by parts

$$\Longrightarrow u = \int (2x+1)e^{-2x}\,dx = -\frac{1}{2}(2x+1)e^{-2x} + \int e^{-2x}\,dx$$

$$v = 2x+1 \qquad dw = e^{-2x}\,dx$$

$$dv = 2\,dx \qquad w = -\frac{1}{2}e^{-2x}$$

$$= -\frac{1}{2}(2x+1)e^{-2x} - \frac{1}{2}e^{-2x}$$

Thus, a second solution is

$$y(x) = \left(-\frac{1}{2}(2x+1)e^{-2x} - \frac{1}{2}e^{-2x}\right)\left(e^{2x}\right)$$

$$= -\frac{1}{2}(2x+1) - \frac{1}{2} = \underline{\underline{-x-1}}.$$

(A lot of work for such a simple solution!)

(b) $(x^3-x^2)y'' - (x^3+2x^2-2x)y' + (2x^2+2x-2)y = 0$;

$y(x) = x^2$ is one solution.

solution via reduction of order:

$$y = x^2 u$$

$$\Rightarrow \quad y' = 2xu + x^2u',$$

$$y'' = 2u + 2xu' + 2xu' + x^2u''$$

$$= 2u + 4xu' + x^2u''.$$

Substitute these into the ODE:

$$(x^3-x^2)(2u+4xu'+x^2u'') - (x^3+2x^2-2x)(2xu+x^2u')$$

$$+ (2x^2+2x-2)(x^2u) = 0$$

$\Longrightarrow$

$$\left[x^5 - x^4\right]u'' + \left[4x^4 - 4x^3 - x^5 - 2x^4 + 2x^3\right]u'$$

$$+ \left[-2x^4 - 4x^3 + 4x^2 + 2x^3 - 2x^2 + 2x^4 + 2x^3 - 2x^2\right]u = 0$$

$$\Longrightarrow (x^5 - x^4)u'' + (-x^5 + 2x^4 - 2x^3)u' = 0$$

$$\Longrightarrow (x^2 - x)u'' + (-x^2 + 2x - 2)u' = 0 \longleftarrow$$ 1^{st}-order in u'.

$$\Longrightarrow u'' = \frac{(x^2 - 2x + 2)}{x^2 - x}u'$$

$$\Longrightarrow \frac{u''}{u'} = \frac{x^2 - 2x + 2}{x^2 - x} = \frac{x^2 - x - x + 2}{x^2 - x} = 1 - \frac{x - 2}{x^2 - x}$$

Note that $\dfrac{x-2}{x^2 - x} = \dfrac{2}{x} - \dfrac{1}{x-1}$, so we have

$$\frac{u''}{u'} = 1 - \frac{2}{x} + \frac{1}{x-1}.$$

Integrate: $\ln|u'| = x - 2\ln x + \ln|x-1|$

$$= x + \ln\left|\frac{x-1}{x^2}\right|$$

$$\Longrightarrow u' = e^x \cdot \left(\frac{x-1}{x^2}\right) = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$\Longrightarrow u' = e^x\left(\frac{1}{x}\right) + e^x\left(-\frac{1}{x^2}\right).$$

Note that $\frac{d}{dx}\left(e^x \cdot \frac{1}{x}\right) = e^x \cdot \frac{1}{x} + e^x\left(-\frac{1}{x^2}\right)$, (product rule)

so $u' = e^x \cdot \frac{1}{x} + e^x\left(-\frac{1}{x^2}\right) \Longrightarrow u = \frac{e^x}{x}$.

Thus, a second solution is

$$y(x) = \frac{e^x}{x}(x^2) = \underline{\underline{x e^x}}.$$