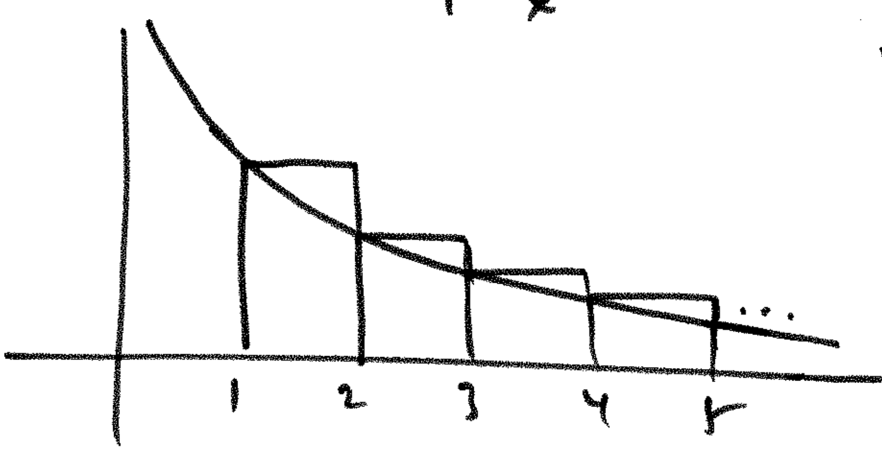
__Harmonic series__ : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

Divergence: __Method #1__

$$1 + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{\geq \frac{1}{2}} + \dots = \infty$$

__Method #2__

$y = \frac{1}{x}$



$$\sum_{n=1}^{N} \frac{1}{n} > \int_{1}^{N} \frac{1}{x}\,dx = \ln(N)$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n} = \infty = \ln(\infty)$$

__Also__:

Euler's constant

$$\lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \int_{1}^{N} \frac{1}{x}\,dx \right) = \gamma$$

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$\frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots$$

$$\left(1 - \frac{1}{2}\right) S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \dots$$

$$\frac{1}{3}\left(1 - \frac{1}{2}\right) S = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \dots$$

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right) S = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

$$\Longrightarrow \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right) S = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

$$\Longrightarrow \left(\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)\right) S = 1 \Longrightarrow$$

Euler product formula

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{\substack{p \\ \text{prime}}} \left(1 - \frac{1}{p}\right)^{-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{\substack{p \\ \text{prime}}} \left(1 - \frac{1}{p}\right)^{-1}$$

$$\Longrightarrow \ln\left(\sum_{1}^{\infty} \frac{1}{n}\right) = \ln\left(\prod_{p} \left(1 - \frac{1}{p}\right)^{-1}\right)$$

$$= \sum_{p} \left(\ln\left(\left(1 - \frac{1}{p}\right)^{-1} \right) \right)$$

$$= -\sum_{\substack{p \\ \text{prime}}} \ln\left(1 - \frac{1}{p}\right)$$

$$= \sum_{p \text{ prime}} \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \dots\right)$$

$$= \sum_{p} \frac{1}{p} + \sum_{p} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \dots\right)$$

$$\ln(1+x) = \cancel{a_0} + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$x = 0 \Longrightarrow \underline{a_0 = 0}$$

$$(1+x)^{-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$x = 0 \Longrightarrow a_1 = 1$$

$$-(1+x)^{-2} = 2a_2 + 6a_3 x + \dots$$

$$x = 0 \Longrightarrow a_2 = -\frac{1}{2}$$

$$2(1+x)^{-3} = 6a_3 + 24a_4 x + \dots$$

$$x = 0 \Longrightarrow a_3 = \frac{1}{3}$$

$$\vdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots$$

$$\ln\left(\sum_{1}^{\infty} \frac{1}{n}\right) = \sum_{\substack{p \\ \text{prime}}} \frac{1}{p} + \underbrace{\sum_{\substack{p \\ \text{prime}}} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \dots \right)}$$

$< \frac{1}{2}$

Thus:

$$\ln\left(\sum_{1}^{n} \frac{1}{n}\right) \approx \sum_{p} \frac{1}{p}$$

$$\boxed{\begin{array}{l} \text{so} \quad \displaystyle\sum_{\substack{p \\ \text{prime}}} \frac{1}{p} = \infty \quad \text{(diverges)} \\[2em] \text{and} \quad \displaystyle\sum_{\substack{p \\ \text{prime}}} \frac{1}{p} = \ln(\ln(\infty)) \end{array}}$$

Also:

Mertens

$$\lim_{p \to \infty} \left(\sum_{p=2}^{p} \frac{1}{p} - \ln(\ln(p)) \right) = \text{const.}$$

$$< \sum_{p} \left(\frac{1}{2p^2} + \frac{1}{2p^3} + \dots \right)$$

$$= \sum_{p} \left(\frac{1}{2p^2} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots \right) \right)$$

$$= \sum_{p} \left(\frac{1}{2p^2} \cdot \frac{1}{1 - \frac{1}{p}} \right)$$

$$= \sum_{p} \left(\frac{1}{2p^2} \cdot \frac{p}{p-1} \right)$$

$$= \sum_{p} \left(\frac{1}{2} \cdot \frac{1}{p(p-1)} \right) < \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

$$\frac{1}{2} \sum \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$= \frac{1}{2}$

<u>Linear congruences</u>: given $a, b, n \in \mathbb{N}$, want $x \in \mathbb{N}$ ($x \in \mathbb{Z}$) such that

$$ax \equiv b \pmod{n}.$$

<u>Note</u>: $ax \equiv b \pmod{n} \iff ax - b = ny$ for some $y \in \mathbb{Z}$

$\iff \underbrace{ax - ny = b}$, a linear Diophantine equation.

solvable iff $d = \gcd(a,n)$ divides b.

if $d \mid b$, then all solutions given by

$$x = x_0 + \frac{n}{d}t, \quad y = y_0 + \frac{a}{d}t \quad \text{for } t \in \mathbb{Z},$$

where x_0, y_0 is <u>a</u> solution.

<u>FACT</u>: There are exactly d solutions mod n.

<u>pf</u>: consider $x = x_0 + \frac{n}{d}t,\ y = y_0 + \frac{a}{d}t$ for $t = 0, 1, \dots, d-1$.