Math 212 - Problem Set #2 SOLUTIONS

(1) $x\sin y\,dx + (x^2+1)\cos y\,dy = 0, \quad y(1) = \frac{\pi}{2}$

Rearrange to recognize this as a separable ODE:

$$\frac{x}{x^2+1}\,dx + \frac{\cos y}{\sin y}\,dy = 0$$

Now integrate:

$$\frac{1}{2}\ln|x^2+1| + \ln|\sin y| = c$$

$$\Longrightarrow \quad \ln\left|\sqrt{x^2+1}\cdot\sin y\right| = c$$

$$\Longrightarrow \quad \sqrt{x^2+1}\cdot\sin y = c$$ This is the general solution.

For the given condition, $y(1) = \frac{\pi}{2}$:

$$\sqrt{1+1}\cdot\sin\left(\frac{\pi}{2}\right) = c \Longrightarrow c = \sqrt{2}.$$

The particular solution is $\sqrt{x^2+1}\cdot\sin y = \sqrt{2}$.

② $(y\sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$

check for exactness:

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(y\sec^2 x + \sec x\tan x\right) &= \sec^2 x \\ \frac{\partial}{\partial x}\left(\tan x + 2y\right) &= \sec^2 x \end{aligned}\right\} \ \underline{\text{exact!}}$$

Now find $f(x,y)$:

$$\frac{\partial f}{\partial x} = y\sec^2 x + \sec x \tan x$$

$$\Longrightarrow f(x,y) = y\tan x + \sec x + R(y)$$

$$\Longrightarrow \frac{\partial f}{\partial y} = \tan x + R'(y) = \tan x + 2y$$

$$R'(y) = 2y \Rightarrow R(y) = y^2$$

The general solution is thus

$$\underline{\underline{y\tan x + \sec x + y^2 = c}}\ .$$

③ $\left(\frac{3-y}{x^2}\right)dx + \left(\frac{y^2-2x}{xy^2}\right)dy = 0$, $y(-1)=2$.

check for exactness:

$$\left.\begin{aligned} &\frac{\partial}{\partial y}\left(\frac{3-y}{x^2}\right) = -\frac{1}{x^2} \, , \\ &\frac{\partial}{\partial x}\left(\frac{y^2-2x}{xy^2}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x} - \frac{2}{y^2}\right) = -\frac{1}{x^2} \end{aligned}\right\} \underline{\text{exact!}}$$

Now find $f(x,y)$:

$$\frac{\partial f}{\partial x} = \left(\frac{3-y}{x^2}\right) \Longrightarrow f(x,y) = \frac{y-3}{x} + R(y)$$

$$\Longrightarrow \frac{\partial f}{\partial y} = \frac{1}{x} + R'(y)$$, which has to equal

$$\frac{y^2}{xy^2} - \frac{2x}{xy^2} = \frac{1}{x} - \frac{2}{y^2}.$$

Thus, $R'(y) = \frac{-2}{y^2} \Rightarrow R(y) = \frac{2}{y}$.

The general solution is $\underline{\underline{\frac{y-3}{x} + \frac{2}{y} = c}}$.

$y(-1)=2 \Rightarrow \frac{2-3}{-1} + \frac{2}{2} = c \Longrightarrow c = 2$, so the

particular solution is $\frac{y-3}{x} + \frac{2}{y} = 2$.

④ $(x+4)(y^2+1)\,dx + y(x^2+3x+2)\,dy = 0$

Rearrange to recognize this as a separable ODE:

$$\frac{x+4}{x^2+3x+2}\,dx + \frac{y}{y^2+1}\,dy = 0$$

Now integrate:

$$\int \frac{x+4}{x^2+3x+2}\,dx = \int \left(\frac{3}{x+1} + \frac{-2}{x+2}\right)dx = 3\ln|x+1| - 2\ln|x+2|$$

partial fraction decomposition:

$$\frac{x+4}{x^2+3x+2} = \frac{x+4}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$x+4 = A(x+2) + B(x+1) \implies A = 3,\ B = -2$$

$$\int \frac{y}{y^2+1}\,dy = \frac{1}{2}\ln|y^2+1|$$

Thus, the general solution is

$$3\ln|x+1| - 2\ln|x+2| + \frac{1}{2}\ln|y^2+1| = c.$$

$$\implies \ln\left|(x+1)^3(x+2)^{-2}(y^2+1)^{\frac{1}{2}}\right| = c$$

$$\implies \frac{(x+1)^3\sqrt{y^2+1}}{(x+2)^2} = c$$

$$\implies \underline{\underline{(x+1)^3\sqrt{y^2+1} = c(x+2)^2}}.$$ (other forms are possible!)

(5) $8\cos^2 y\,dx + \csc^2 x\,dy = 0, \quad y(\frac{\pi}{12}) = \frac{\pi}{4}$

Rearrange as a separable equation:

$$8\sin^2 x\,dx + \sec^2 y\,dy = 0$$

Now integrate:

$$\int 8\sin^2 x\,dx = \int 4(1-\cos 2x)\,dx$$

since $\cos 2x = 1 - 2\sin^2 x$,
so $2\sin^2 x = 1 - \cos 2x$

$$= \int (4 - 4\cos 2x)\,dx$$

$$= 4x - 2\sin 2x$$

$$\int \sec^2 y\,dy = \tan y$$

Thus, the general solution is

$$\underline{\underline{4x - 2\sin 2x + \tan y = c}}$$

Given the condition $y(\frac{\pi}{12}) = \frac{\pi}{4}$:

$$4(\tfrac{\pi}{12}) - 2\sin(\tfrac{\pi}{6}) + \tan(\tfrac{\pi}{4}) = c$$

$$\Longrightarrow \frac{\pi}{3} - 2(\tfrac{1}{2}) + 1 = c \Longrightarrow c = \frac{\pi}{3}$$

The particular solution is thus

$$\underline{\underline{4x - 2\sin 2x + \tan y = \frac{\pi}{3}}}.$$

⑥ $(2xy+1)\,dx + (x^2+4y)\,dy = 0$

check for exactness:

$$\left.\begin{aligned} \frac{\partial}{\partial y}(2xy+1) &= 2x \\ \frac{\partial}{\partial x}(x^2+4y) &= 2x \end{aligned}\right\} \text{ exact!}$$

Find $f(x,y)$:

$$\frac{\partial f}{\partial x} = 2xy+1 \Longrightarrow f(x,y) = x^2y + x + R(y)$$

$$\Longrightarrow \frac{\partial f}{\partial y} = x^2 + R'(y), \quad R' = 4y$$

$$\Longrightarrow R(y) = 2y^2$$

Thus, the general solution is

$$\underline{\underline{x^2y + x + 2y^2 = c}}$$