Math 212 - Problem Set 7 SOLUTIONS

① (a) $y'' + 8xy' - 4y = 0$

ansatz: $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (Maclaurin series)

$$\Longrightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these into the ODE:

$$\underline{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}} + \sum_{n=1}^{\infty} 8n a_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0$$

shift index to write this as $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$,

then combine all 3 series together; get

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 8n a_n - 4a_n \right] x^n = 0$$

The recurrence relation is thus

$$(n+2)(n+1) a_{n+2} + (8n-4) a_n = 0$$

$$\Longrightarrow \quad a_{n+2} = \frac{(4-8n) a_n}{(n+2)(n+1)}, \quad n \geq 0$$

Note that a_0 will determine a_n for all even n, while a_1 will determine a_n for all odd n. In particular:

$\underline{n=0}$: $\quad a_2 = \dfrac{4a_0}{2} = 2a_0$;

$\underline{n=2}$: $\quad a_4 = \dfrac{-12a_2}{12} = -a_2 = -2a_0$;

$\underline{n=4}$: $\quad a_6 = \dfrac{-28a_4}{30} = \dfrac{-14}{15}a_4 = \dfrac{28}{15}a_0$; etc.

As for the odd-powered terms:

$\underline{n=1}$: $\quad a_3 = \dfrac{-4a_1}{6} = -\dfrac{2}{3}a_1$;

$\underline{n=3}$: $\quad a_5 = \dfrac{-20a_3}{20} = -a_3 = \dfrac{2}{3}a_1$;

$\underline{n=5}$: $\quad a_7 = \dfrac{-36a_5}{42} = \dfrac{-6}{7}a_5 = -\dfrac{4}{7}a_1$; etc.

Thus, we get

$$y(x) = a_0 + a_1x + 2a_0x^2 - \frac{2}{3}a_1x^3 - 2a_0x^4 + \frac{2}{3}a_1x^5 + \frac{28}{15}a_0x^6 - \frac{4}{7}a_1x^7 + \dots$$

$$= a_0\underbrace{\left(1 + 2x^2 - 2x^4 + \frac{28}{15}x^6 + \dots\right)}_{y_1(x)} + a_1\underbrace{\left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5 - \frac{4}{7}x^7 + \dots\right)}_{y_2(x)}$$

(b) $y'' - xy' + (3x-2)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Longrightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{same as } \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n} - \sum_{n=1}^{\infty} n a_n x^n + \underbrace{\sum_{n=0}^{\infty} 3 a_n x^{n+1}}_{\text{same as } \sum_{n=1}^{\infty} 3 a_{n-1} x^n} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Then get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 3 a_{n-1} x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\Longrightarrow [2a_2 - 2a_0] + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n + 3a_{n-1} - 2a_n \right] x^n = 0$$

Thus, $\underline{\underline{a_2 = a_0}}$ and, for $n \geq 1$, we have the recurrence relation

$$(n+2)(n+1) a_{n+2} - n a_n + 3a_{n-1} - 2a_n = 0$$

$$\Longrightarrow (n+2)(n+1) a_{n+2} = -3a_{n-1} + (n+2) a_n$$

$$\Longrightarrow \underline{\underline{a_{n+2} = \frac{(n+2) a_n - 3a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.}}$$

$\underline{n=1}$: $\quad a_3 = \dfrac{3a_1 - 3a_0}{6} = -\frac{1}{2}a_0 + \frac{1}{2}a_1$

$\underline{n=2}$: $\quad a_4 = \dfrac{4a_2 - 3a_1}{12} = \frac{1}{3}a_2 - \frac{1}{4}a_1 = \frac{1}{3}a_0 - \frac{1}{4}a_1$

(since $a_2 = a_0$)

$\underline{n=3}$: $\quad a_5 = \dfrac{5a_3 - 3a_2}{20} = \frac{1}{4}a_3 - \frac{3}{20}a_2$

$$= \frac{1}{4}\left(-\frac{1}{2}a_0 + \frac{1}{2}a_1\right) - \frac{3}{20}a_0$$

$$= -\frac{1}{8}a_0 - \frac{3}{20}a_0 + \frac{1}{8}a_1$$

$$= -\frac{11}{40}a_0 + \frac{1}{8}a_1$$

Thus,

$$y(x) = a_0 + a_1x + a_0x^2 - \frac{1}{2}a_0x^3 + \frac{1}{2}a_1x^3 + \frac{1}{3}a_0x^4 - \frac{1}{4}a_1x^4 - \frac{11}{40}a_0x^5 + \frac{1}{8}a_1x^5 + \dots$$

$$= a_0\left(1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{11}{40}x^5 + \dots\right) + a_1\left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \dots\right)$$

(c) $(x+3)y'' + (x+2)y' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \implies y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Then

$$(x+3)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2}$$

$$= \underbrace{\sum_{n=0}^{\infty}}_{\text{can start @ } n=0} (n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n ,$$

and

$$(x+2)y' = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^{n-1}$$

$$= \underbrace{\sum_{n=0}^{\infty}}_{\text{can start @ } n=0} n a_n x^n + \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n ,$$

so we get

$$\sum_{n=0}^{\infty} (n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n$$

$$+ \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0 , \quad \text{i.e.,}$$

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+1)n a_{n+1} + 3(n+2)(n+1) a_{n+2} + n a_n + 2(n+1) a_{n+1} + a_n \right]}_{\text{This coefficient must be zero, so}} x^n = 0$$

$$\underbrace{\left[(n+1)n + 2(n+1) \right]}_{(n+2)(n+1)} a_{n+1} + 3(n+2)(n+1) a_{n+2} + (n+1) a_n = 0 \quad \text{for } n \geq 0$$

$\implies$

Thus,

$$3(n+2)\cancel{(n+1)}a_{n+2} = -\cancel{(n+1)}a_n - \cancel{(n+1)}(n+2)a_{n+1}$$

$$\Longrightarrow \quad 3(n+2)a_{n+2} = -a_n - (n+2)a_{n+1}$$

$$\Longrightarrow \quad a_{n+2} = \frac{-a_n - (n+2)a_{n+1}}{3(n+2)}, \quad n \geq 0$$

$\underline{n=0}$:

$$a_2 = \frac{-a_0 - 2a_1}{6} = -\frac{1}{6}a_0 - \frac{1}{3}a_1$$

$\underline{n=1}$:

$$a_3 = \frac{-a_1 - 3a_2}{9} = -\frac{1}{9}a_1 - \frac{1}{3}a_2$$

$$= -\frac{1}{9}a_1 + \frac{1}{18}a_0 + \frac{1}{9}a_1 = \frac{1}{18}a_0$$

$\underline{n=2}$:

$$a_4 = \frac{-a_2 - 4a_3}{12} = -\frac{1}{12}a_2 - \frac{1}{3}a_3$$

$$= \frac{1}{72}a_0 + \frac{1}{36}a_1 - \frac{1}{54}a_0$$

$$= -\frac{1}{216}a_0 + \frac{1}{36}a_1$$

etc. etc.

$$\Longrightarrow$$

Thus,

$$y(x) = a_0 + a_1 x + \left(-\frac{1}{6}a_0 - \frac{1}{3}a_1\right)x^2 + \frac{1}{18}a_0 x^3 + \left(\frac{-1}{216}a_0 + \frac{1}{36}a_1\right)x^4 + \cdots$$

$$= a_0 \overbrace{\left[1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 + \cdots\right]}^{y_1(x)} + a_1 \underbrace{\left[x - \frac{1}{3}x^2 + \frac{1}{36}x^4 + \cdots\right]}_{y_2(x)}.$$

---

(2) (a) $2x^2y'' + xy' + (2x^2 - 3)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} \Longrightarrow y' = \sum_{n=0}^{\infty} (n+m)a_n x^{n+m-1},$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1)a_n x^{n+m-2}$$

Substitute these into the ODE:

$$\sum_{n=0}^{\infty} 2(n+m)(n+m-1)a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m)a_n x^{n+m} + \sum_{n=0}^{\infty} 2a_n x^{n+m+2} - \sum_{n=0}^{\infty} 3a_n x^{n+m} = 0$$

This is the same as $\sum_{n=2}^{\infty} 2a_{n-2}x^{n+m}$; then get $\Longrightarrow$

$$\left[2m(m-1)+m-3\right]a_0 x^m + \left[2m(m+1)+(m+1)-3\right]a_1 x^{m+1}$$

$$+\sum_{n=2}^{\infty}\left[2(n+m)(n+m-1)a_n+(n+m)a_n+2a_{n-2}-3a_n\right]x^{n+m}=0$$

The x^m term must vanish; this yields the indicial equation,

$$2m^2-2m+m-3=0$$

$$\Longrightarrow \quad 2m^2-m-3=0$$

$$\Longrightarrow \quad m=\frac{1\pm\sqrt{1+24}}{4}=\underline{\underline{-1,\ \frac{3}{2}}}$$

Since the x^{m+1} term must vanish, $\underline{\underline{a_1=0}}$.

Finally, the coefficient of x^{n+m} must be zero, so

$$\left(2(n+m)(n+m-1)+(n+m)-3\right)a_n+2a_{n-2}=0$$

$$\Longrightarrow \quad \underline{\underline{a_n=\frac{-2a_{n-2}}{(n+m)(2n+2m-1)-3}}}, \quad n\geq 2.$$

$$\Longrightarrow$$

$m = -1$ The recurrence relation is

$$a_n = \frac{-2a_{n-2}}{(n-1)(2n-3)-3}$$

$$\Longrightarrow \quad a_n = \frac{-2a_{n-2}}{2n^2 - 5n}, \quad n \geq 2$$

$n=2$: $\quad a_2 = \dfrac{-2a_0}{-2} = a_0$

$n=4$: $\quad a_4 = \dfrac{-2a_2}{12} = -\dfrac{1}{6}a_2 = -\dfrac{1}{6}a_0$

$n=6$: $\quad a_6 = \dfrac{-2a_4}{42} = -\dfrac{a_4}{21} = \dfrac{1}{126}a_0$ etc.

Note: $a_1 = 0 \Longrightarrow a_n = 0$ for all odd n.

Thus, one solution is

$$y_1(x) = x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 + \dots\right)$$

(One way to get this is to let $a_0 = 1$.)

$\underline{\underline{m = 3/2}}$: The recurrence relation is

$$a_n = \frac{-2a_{n-2}}{(n+3/2)(2n+2) - 3}$$

$$\Longrightarrow \quad a_n = \frac{-2a_{n-2}}{2n^2 + 5n}, \quad n \geq 2.$$

$\underline{n=2}$: $$a_2 = \frac{-2a_0}{18} = -\frac{1}{9}a_0$$

$\underline{n=4}$: $$a_4 = \frac{-2a_2}{52} = -\frac{1}{26}a_2 = \frac{1}{234}a_0$$

etc. $\underline{\text{Note}}$: $a_1 = 0 \Longrightarrow a_n = 0$ for all odd n.

Thus, a second solution is

$$y_2(x) = x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 + \cdots\right).$$

(Take $a_0 = 1$.)

The general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

(b) $2xy'' + y' + 2y = 0$

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$\Longrightarrow \quad y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1},$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

Substitute these into the ODE:

$$\sum_{n=0}^{\infty} 2(n+m)(n+m-1) a_n x^{n+m-1} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ \sum_{n=0}^{\infty} 2a_n x^{n+m} = 0$$

same as $\sum_{n=1}^{\infty} 2a_{n-1} x^{n+m-1}$; use this

to get

$$\sum_{n=0}^{\infty} 2(n+m)(n+m-1) a_n x^{n+m-1} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ \sum_{n=1}^{\infty} 2a_{n-1} x^{n+m-1} = 0$$

$$\Longrightarrow \left[2m(m-1) + m\right] a_0 x^{m-1} + \sum_{n=1}^{\infty} \Big[2(n+m)(n+m-1) a_n + (n+m) a_n + 2a_{n-1} \Big] x^{n+m-1} = 0$$

indicial equation: $2m^2 - 2m + m = 0$

$$\Longrightarrow \quad 2m^2 - m = 0$$

$$\Longrightarrow \quad m(2m-1) = 0$$

$$\Longrightarrow \quad \underline{\underline{m=0}}, \quad \underline{\underline{m=\tfrac{1}{2}}}.$$

recurrence relation:

$$2(n+m)(n+m-1)a_n + (n+m)a_n + 2a_{n-1} = 0$$

$$\Longrightarrow \quad (n+m)(2n+2m-1)a_n + 2a_{n-1} = 0$$

$$\Longrightarrow \quad a_n = \frac{-2a_{n-1}}{(n+m)(2n+2m-1)}, \quad n \geq 1$$

$\underline{\underline{m=0}}$: The recurrence relation is

$$a_n = \frac{-2a_{n-1}}{n(2n-1)}, \quad n \geq 1$$

$\underline{n=1}$: $a_1 = \dfrac{-2a_0}{1\cdot 1} = -2a_0$

$\underline{n=2}$: $a_2 = \dfrac{-2a_1}{6} = -\dfrac{a_1}{3} = \dfrac{2}{3}a_0$

$\underline{n=3}$: $a_3 = \dfrac{-2a_2}{15} = \dfrac{-4}{45}a_0 \Longrightarrow$

Thus, one solution (taking $a_0 = 1$) is

$$y_1(x) = 1 - 2x + \frac{2}{3}x^2 - \frac{4}{45}x^3 + \cdots$$

$\underline{\underline{n = \frac{1}{2}}}$: The recurrence relation is

$$a_n = \frac{-2a_{n-1}}{(n+\frac{1}{2})(2n)} = \frac{-2a_{n-1}}{2n^2 + n}$$

$\underline{n=1}$: $a_1 = \frac{-2a_0}{3} = -\frac{2}{3}a_0$

$\underline{n=2}$: $a_2 = \frac{-2a_1}{10} = -\frac{1}{5}a_1 = \frac{2}{15}a_0$

$\underline{n=3}$: $a_3 = \frac{-2a_2}{21} = \frac{-2}{21} \cdot \frac{2}{15}a_0 = \frac{-4}{315}a_0$

A second solution (taking $a_0 = 1$) is thus

$$y_2(x) = x^{1/2}\left(1 - \frac{2}{3}x + \frac{2}{15}x^2 - \frac{4}{315}x^3 + \cdots\right)$$