#11 $(3-x^2)y'' - 3xy' - y = 0$

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^n} \Longrightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\left[3(2)(1) a_2 - a_0\right]}_{n=0} + \underbrace{\left[3(3)(2) a_3 - 3a_1 - a_1\right]}_{n=1} x^1$$

$$+ \sum_{n=2}^{\infty} \left[3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n a_n - a_n\right] x^n = 0$$

$\underline{n=0}$: $\boxed{a_2 = \frac{a_0}{6}}$

$\underline{n=1}$: $18a_3 - 4a_1 = 0$

$\boxed{a_3 = \frac{2}{9}a_1}$

$\underline{n \geq 2}$: $3(n+2)(n+1)a_{n+2} - [n(n-1) + 3n + 1]a_n = 0$

$n^2 - n + 3n + 1$

$n^2 + 2n + 1$

$(n+1)^2$

$$\Longrightarrow \quad a_{n+2} = \frac{(n+1)^2 a_n}{3(n+2)(n+1)} = \frac{(n+1)a_n}{3(n+2)}, \quad n \geq 2$$

$$\boxed{a_{n+2} = \frac{(n+1)a_n}{3(n+2)}, \quad n \geq 0}$$

$$a_2 = \frac{a_0}{6}$$

$$a_4 = \frac{3a_2}{3(4)} = \frac{a_2}{4} = \frac{1}{24}a_0$$

$$a_6 = \frac{5a_4}{3(6)} = \frac{5}{18}a_4 = \frac{5}{18(24)}a_0$$

etc.

$$a_3 = \frac{2}{9}a_1$$

$$a_5 = \frac{4a_3}{3(5)} = \frac{4}{15}\cdot\frac{2}{9}a_1$$

$$= \frac{8}{135}a_1$$

$$a_7 = \frac{6a_5}{3(7)} = \frac{2}{7}\cdot\frac{8}{135}a_1$$

etc.

THEN

$$a_0\left(1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \frac{5}{18(24)}x^6 + \dots\right)$$

$$y(x) = \left(a_0 + \frac{a_0}{6}x^2 + \frac{1}{24}a_0x^4 + \frac{5}{18(24)}a_0x^6 + \dots\right) +$$

$$\left(a_1x + \frac{2}{9}a_1x^3 + \frac{8}{135}a_1x^5 + \frac{16}{7\cdot 135}a_1x^7 + \dots\right)$$

$$a_1\left(x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \frac{16}{7\cdot 135}x^7 + \dots\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x - 2y \end{pmatrix}$$

$$\begin{cases} x' = 2x - y \\ y' = 3x - 2y \end{cases} \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} v = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda e^{\lambda t} v = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} e^{\lambda t} v$$

$$\Longrightarrow \lambda v = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} v$$

$$\underbrace{\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$(2-\lambda)(-2-\lambda) - (-3) = 0$$

$$\lambda^2 - 4 + 3 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

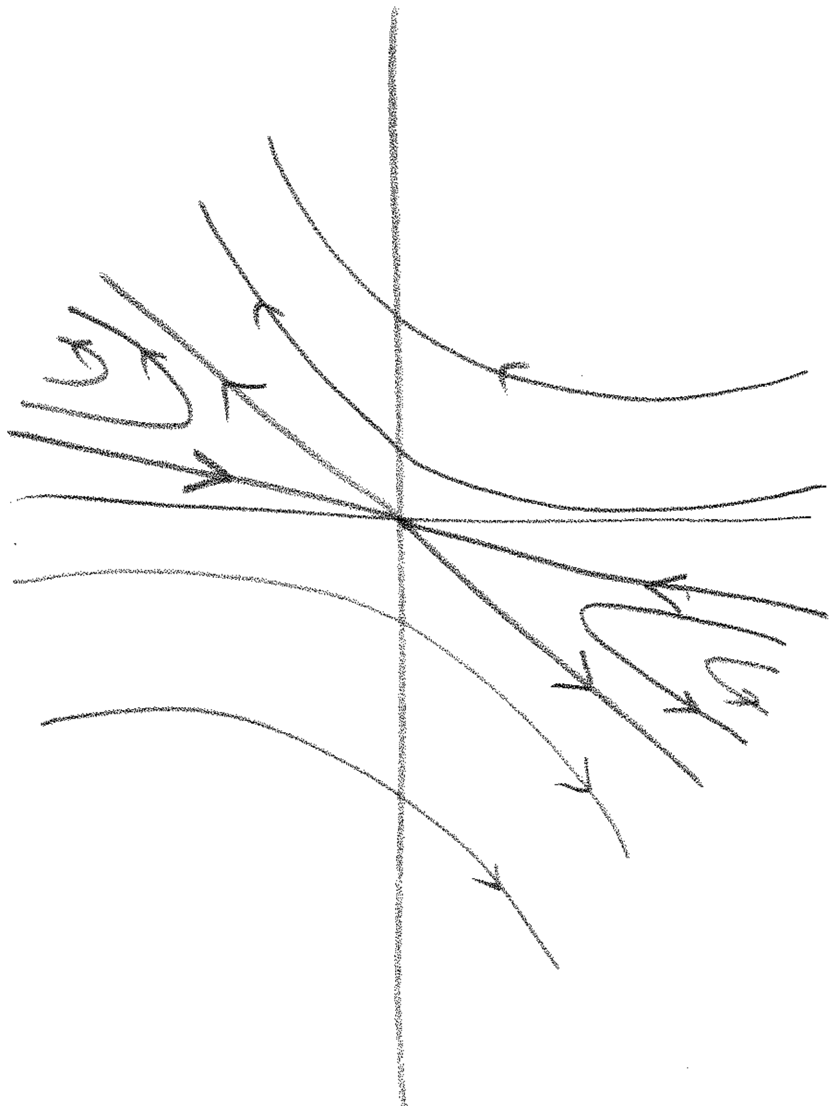
$\lambda = 1$: $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$v_1 - v_2 = 0$$

$$v_1 = v_2 \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$: $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$$3v_1 = v_2 \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



$$x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x$$

$$\det \begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0$$

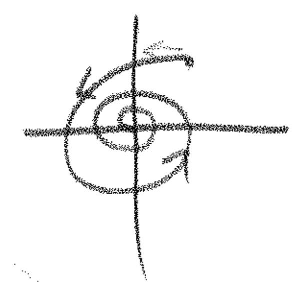
$$(-1-\lambda)(-1-\lambda) + 4 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$



$\lambda = -1+2i$:

$$\begin{pmatrix} -1+1-2i & -4 \\ 1 & -1+1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

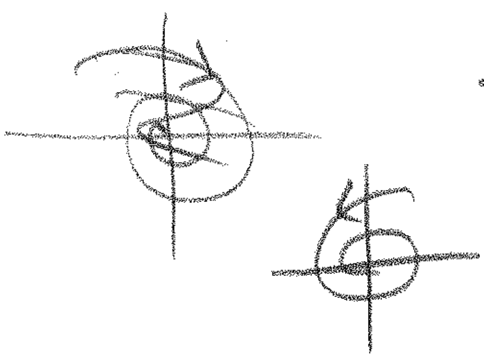
$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_1 - 2iv_2 = 0$$

$$v_1 = 2iv_2$$

$$v = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} = e^{-t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}\cos(2t)\, i - 2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) + i\sin(2t)e^{-t} \end{pmatrix}$$



$$= \begin{pmatrix} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{pmatrix} + i\begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix}$$

$$x(t) = c_1\begin{pmatrix} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{pmatrix} + c_2\begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix}$$