Quiz 6

Name:

1. Which of the following vector spaces are isomorphic? Group them accordingly and provide your rationale.

$$\mathbb{R}^8 \qquad \mathcal{M}_{3\times 3} \qquad \mathcal{P}_8 \qquad \mathbb{R}^7 \qquad \mathcal{M}_{4\times 2} \qquad \mathbb{R}^9 \qquad \mathcal{P}_7$$

$\underbrace{\mathbb{R}^8, M_{4\times 2}, P_7}_{\text{each has dimension } 8}$ $\qquad$ $\underbrace{M_{3\times 3}, P_8, \mathbb{R}^9}_{\text{each has dimension } 9}$ $\qquad$ $\underbrace{\mathbb{R}^7}_{\text{dimension } 7}$ $\quad$ 3

2. Define the map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Verify that T is a homomorphism:

$$T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right). \qquad \blacksquare$$

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(b) Why is T not an isomorphism?

T is not 1-1; for example, $T\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right)$, while $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. $\quad$ 2

(c) Identify one easy way to change T to make it an isomorphism.

<u>Option 1</u>: redefine as

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>Option 2</u>: redefine as

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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