3.6 #1, continued from last time

Fundamental solutions: $y_1 = e^{2t}$, $y_2 = e^{3t}$

System to be solved:
$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= 2e^t \end{aligned}\right\}$$

$$\Longrightarrow \left.\begin{aligned} u_1' e^{2t} + u_2' e^{3t} &= 0 \\ 2u_1' e^{2t} + 3u_2' e^{3t} &= 2e^t \end{aligned}\right\}$$

$$\Longrightarrow \begin{aligned} 2u_1' e^{2t} + 2u_2' e^{3t} &= 0 \\ 2u_1' e^{2t} + 3u_2' e^{3t} &= 2e^t \end{aligned} \Longrightarrow u_2' e^{3t} = 2e^t$$

$$u_2' = 2e^{-2t}$$

integrate: $u_2 = -e^{-2t}$

$u_1 = 2e^{-t}$

Then: $u_1' e^{2t} + (2e^{-2t}) e^{3t} = 0$

$u_1' e^{2t} + 2e^t = 0 \Longrightarrow u_1' e^{2t} = -2e^t \Rightarrow u_1' = -2e^{-t}$

Thus: $y_p = 2e^{-t} \cdot e^{2t} + (-e^{-2t})(e^{3t})$

$\Longrightarrow$ $y_p = e^t$

$\Rightarrow$ General solution : $y(t) = \underbrace{c_1 e^{2t} + c_2 e^{3t}}_{\text{Homogeneous sol'n}} + \underbrace{e^t}_{\text{particular sol'n}}$

#4 $y'' + y = \tan t, \quad 0 < t < \frac{\pi}{2}$

Step 1: Solve $y'' + y = 0$

$$r^2 + 1 = 0 \Longrightarrow r = \pm i \Longrightarrow \underline{\underline{y_1 = \cos t}}, \quad \underline{\underline{y_2 = \sin t}}$$

Step 2: apply variation of parameters; find u_1 & u_2 so that $y_p = u_1 y_1 + u_2 y_2$ solves $y'' + y = \tan t$.

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \tan t \end{cases}$$

$$\Longrightarrow \left.\begin{aligned} u_1' \cos t + u_2' \sin t &= 0 \\ -u_1' \sin t + u_2' \cos t &= \tan t \end{aligned}\right\}$$

$$\Longrightarrow \quad \left.\begin{aligned} u_1' \sin t \cos t + u_2' \sin^2 t &= 0 \\ - u_1' \sin t \cos t + u_2' \cos^2 t &= \sin t \end{aligned}\right\}$$

$$u_2' (\sin^2 t + \cos^2 t) = \sin t$$

$$\Longrightarrow \quad \underline{\underline{u_2' = \sin t}} \quad \Longrightarrow \quad \underline{\underline{u_2 = -\cos t}}$$

Then $u_1' \cos t + \sin^2 t = 0$

$$\Longrightarrow \quad u_1' \cos t = -\sin^2 t$$

$$\Longrightarrow \quad u_1' = -\frac{\sin^2 t}{\cos t} \quad \Longrightarrow \quad u_1 = \int -\frac{\sin^2 t}{\cos t}\, dt$$

option #1: $u_1 = \int - \sin t \cdot \tan t \, dt \quad \ldots \qquad \left(\int u\,dv = uv - \int v\,du \right)$

option #2: $u_1 = \int \frac{\cos^2 t - 1}{\cos t}\, dt = \int (\cos t - \sec t)\, dt$

$$\Longrightarrow u_1 = \int (\cos t - \sec t)\,dt$$

$$u_1 = \underline{\underline{\sin t - \ln|\sec t + \tan t|}}$$

<u>Thus</u>: $y(t) = c_1 \cos t + c_2 \sin t + \left(\sin t - \ln|\sec t + \tan t|\right)\cos t$

$$+ (-\cos t)\sin t$$

$$\Longrightarrow \underline{\underline{y(t) = c_1 \cos t + c_2 \sin t - (\cos t)\ln|\sec t + \tan t|}}$$

#8 $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

$\int \frac{-t}{1+t^2}\,dt$

$u = 1+t^2 \ldots$

<u>step 1</u>: $y_1 = e^t, \; y_2 = te^t$

<u>step 2</u>: $y_p = u_1 y_1 + u_2 y_2 = \underline{-\frac{1}{2}\ln(1+t^2)e^t + (\arctan t)te^t}$

$$\begin{cases} u_1' e^t + u_2' te^t = 0 \\ u_1' e^t + u_2'(e^t + te^t) = \dfrac{e^t}{1+t^2} \end{cases}$$

$$\Longrightarrow \begin{cases} u_1' + t u_2' = 0 \\ u_1' + (1+t)u_2' = \dfrac{1}{1+t^2} \end{cases} \Longrightarrow u_2' = \frac{1}{1+t^2}$$

$\underline{u_2 = \arctan t}$

Then: $u_1' + \dfrac{t}{1+t^2} = 0 \Longrightarrow u_1' = \dfrac{-t}{1+t^2} \Longrightarrow u_1 = -\frac{1}{2}\ln(1+t^2)$