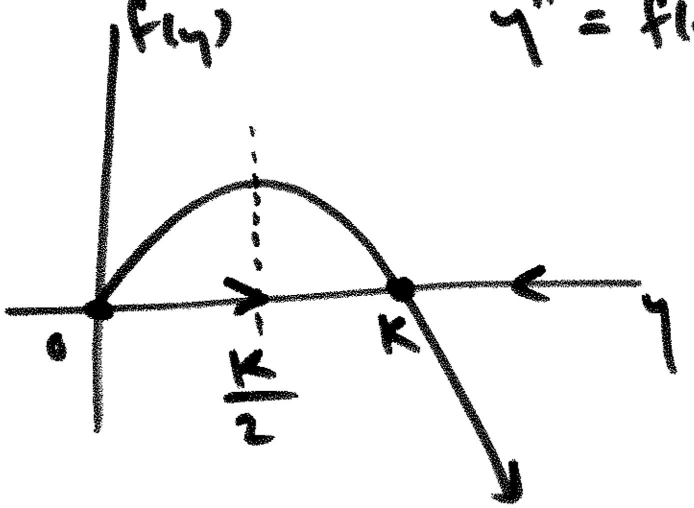
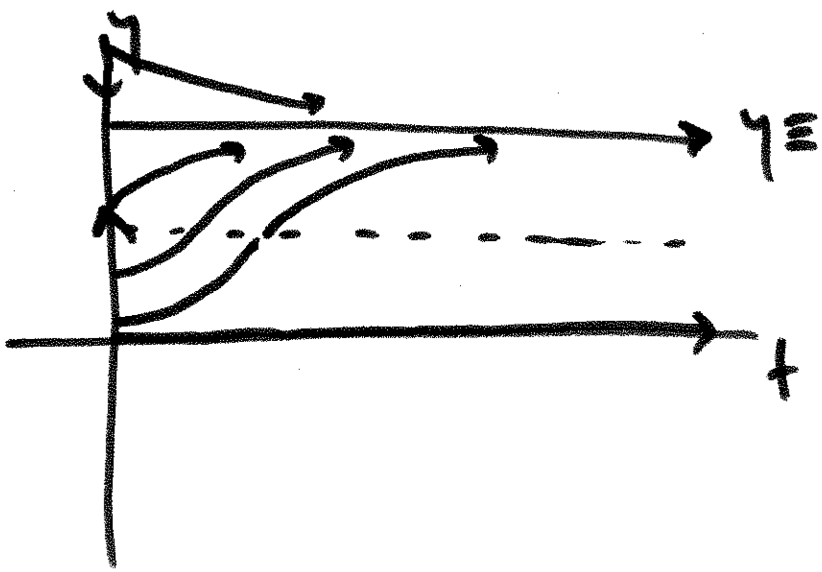
Gompertz $\quad y' = \underbrace{ry\ln\left(\frac{K}{y}\right)}_{f(y)}$ $\quad$ vs. $\quad$ Logistic $\quad y' = ry(1-y)$

$$y'' = f(y)\,f'(y)$$





$$\lim_{y\to 0^+} f(y) = \lim_{y\to 0^+}\left(ry\ln\left(\frac{K}{y}\right)\right)$$

$$= \lim_{y\to 0^+}\left(\frac{r\ln\left(\frac{K}{y}\right)}{\frac{1}{y}}\right)$$

L'H

$$= \lim_{y\to 0^+}\left(\frac{r\cdot\frac{y}{K}\cdot\frac{-K}{y^2}}{-\frac{1}{y^2}}\right)$$

$$= \lim_{y\to 0^+}(ry) = 0$$

$$y' = ry\ln\left(\frac{K}{y}\right)$$

$$\Longrightarrow \quad \frac{y'}{y\ln\left(\frac{K}{y}\right)} = r$$

Integrate: $$\int \frac{dy}{y\ln\left(\frac{K}{y}\right)} = rt + c$$

$$\left.\begin{aligned} u &= \ln\left(\frac{K}{y}\right) = \ln K - \ln y \\ du &= -\frac{1}{y}\,dy \\ -du &= \frac{1}{y}\,dy \end{aligned}\right\}$$

$$\int -\frac{du}{u} = rt + c$$

$$-\ln|u| = ce^{rt}$$

$$\ln|u| = ce^{rt}$$

$$\underline{\underline{u = e^{ce^{rt}}}}$$

2.5 #19 Schaefer model: $\frac{dy}{dt} = r(1-\frac{y}{K})y - Ey$

(a) Equilibria: $f(y) = r(1-\frac{y}{K})y - Ey = 0$

$$\iff y\left(r(1-\frac{y}{K}) - E \right) = 0$$

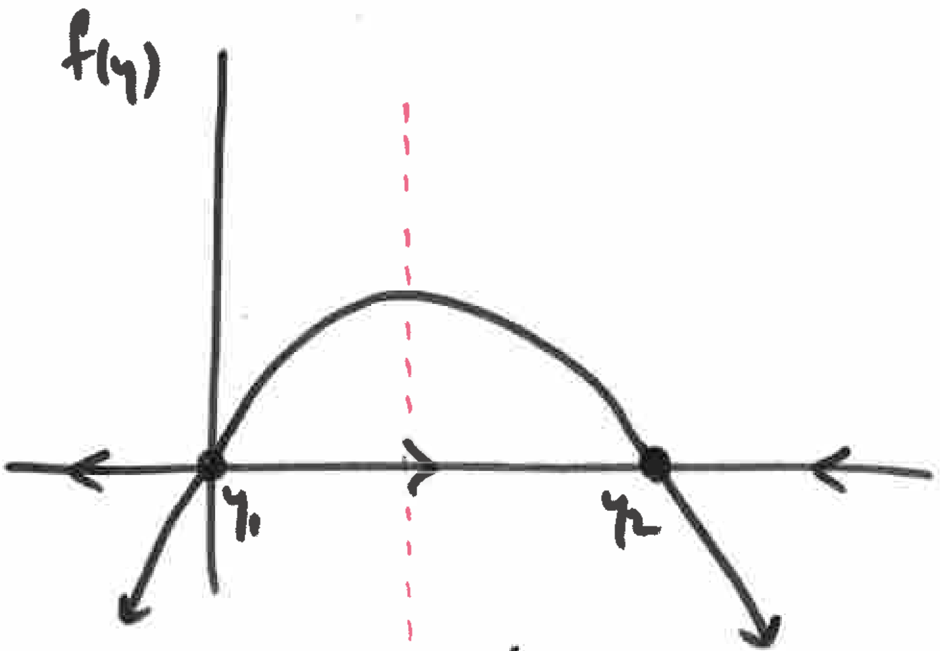
$$\iff \underline{y=0} \quad \text{or} \quad \underbrace{r(1-\frac{y}{K}) = E}$$

$$\iff r - \frac{r}{K}y = E \iff \underline{y = \frac{K(r-E)}{r}}$$

As a population model, only non-negative equilibria make sense. If $r > E$, there are 2 non-negative equilibria: $\underbrace{y \equiv 0}_{y_1}$ and $\underbrace{y \equiv \frac{K(r-E)}{r}}_{y_2}$.

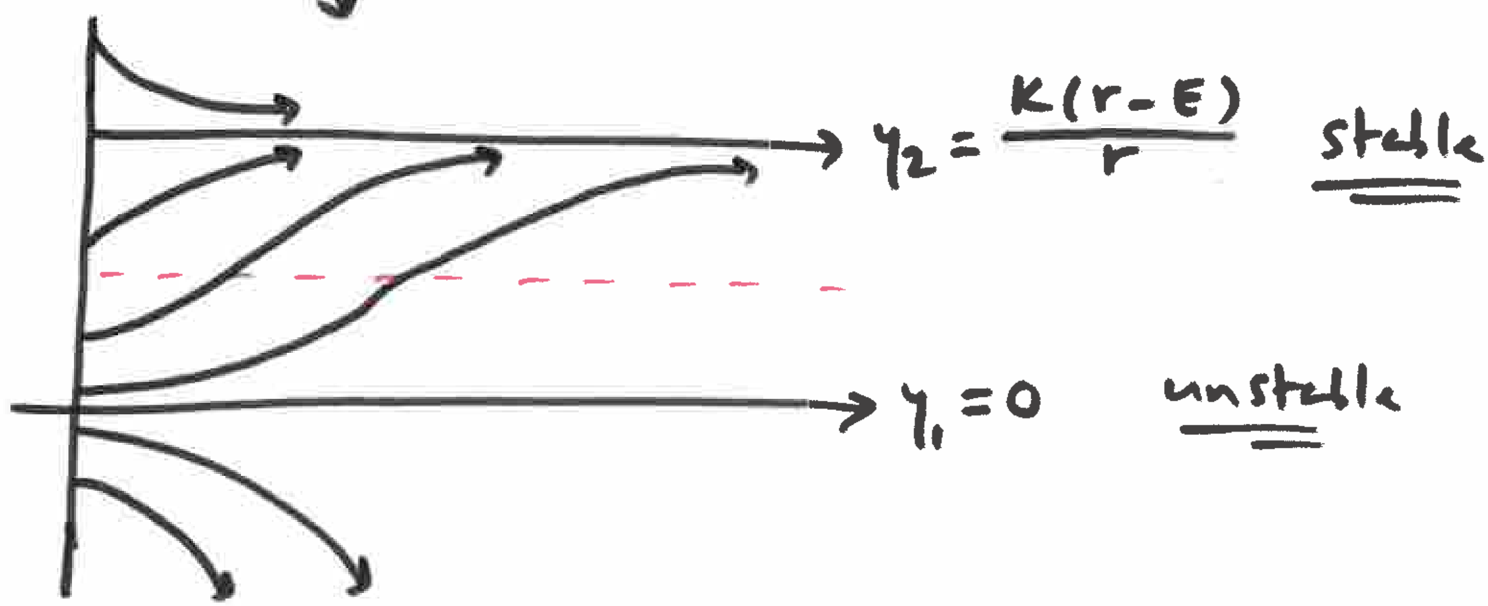
(b) <u>stability</u>

$$f(y) = r\left(1-\frac{y}{K}\right)y - Ey = (r-E)y - \frac{r}{K}y^2$$
$$= y\left((r-E) - \frac{r}{K}y\right).$$



<u>Equilibria</u>: $y_1 = 0$,
$$y_2 = \frac{K(r-E)}{r}$$
(Assuming $E < r$)

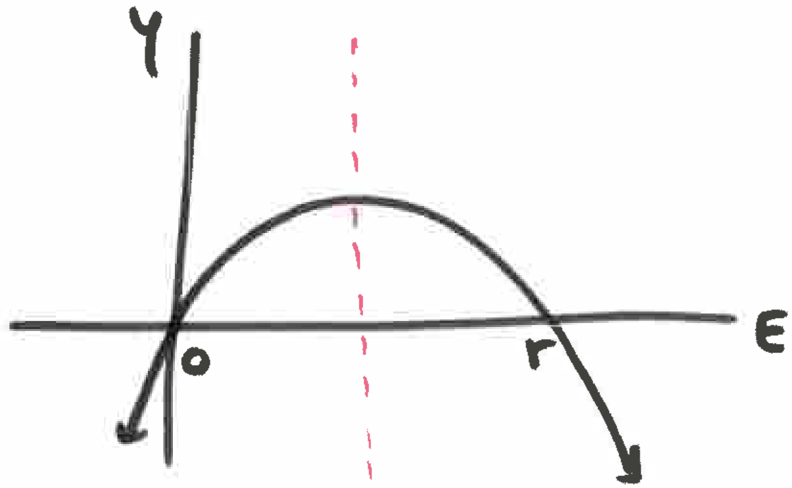
$\Longrightarrow$



(c) yield-effort curve :

yield $Y = Ey_2 = E\left(\frac{K(r-E)}{r}\right) = KE - \frac{K}{r}E^2$

Y vs. E :



(d) maximum sustainable yield :

$Y' = K - \frac{2K}{r}E = 0$ when $E = \frac{K}{\frac{2K}{r}} = \underline{\underline{\frac{r}{2}}}$; max since

$Y'' = -\frac{2K}{r} < 0$ (see plot above!)

$\Longrightarrow$ max of Y is $K \cdot \frac{r}{2} - \frac{K}{r} \cdot \frac{r^2}{4} = \underline{\underline{\frac{Kr}{4}}}$

#7 $y'' + 2y' + 2y = 0$

$\boxed{i^2 = -1}$

$y = e^{rt} \Longrightarrow$ characteristic equation $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$\Longrightarrow \underline{\underline{r = -1 \pm i}}$$

$$y_1 = e^{(-1+i)t}, \quad y_2 = e^{(-1-i)t}$$

$$\Longrightarrow y_1 = e^{-t} e^{it}, \quad y_2 = e^{-t} e^{-it}$$

$a+bi,\ a-bi$

$\boxed{e^{it} = ??}$

$\boxed{e^{-t}\cos t,\ e^{-t}\sin t}$

$$e^{it} = \cos t + i\sin t \Longrightarrow \begin{aligned} y_1 &= e^{-t}(\cos t + i\sin t) \\ y_2 &= e^{-t}(\cos t - i\sin t) \end{aligned}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

with

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$n! = n(n-1)\cdots(2)(1)$$

$$\underline{\underline{i^2 = -1}}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$$

$$= 1 + \frac{it}{1} + \frac{(it)^2}{2} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \dots$$

$$= 1 + it - \frac{1}{2}t^2 - \frac{1}{3!}it^3 + \frac{1}{4!}t^4 + \frac{t^5}{5!}i - \frac{t^6}{6!} + \dots$$

$$= \left(1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots\right) + i\left(t - \frac{1}{3!}t^3 + \frac{t^5}{5!} - \frac{t^7}{7} + \dots\right)$$

Thus: $\boxed{\begin{array}{c} e^{it} = \cos t + i \sin t \\ \text{Euler's formula} \end{array}}$

ex: $e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$\underline{\underline{e^{i\pi} = -1}}$$

$$e^{rt} \Longrightarrow r = \lambda \pm i\mu$$

$$\Longrightarrow y_1 = e^{\lambda t}\cos(\mu t), \quad y_2 = e^{\lambda t}\sin(\mu t)$$