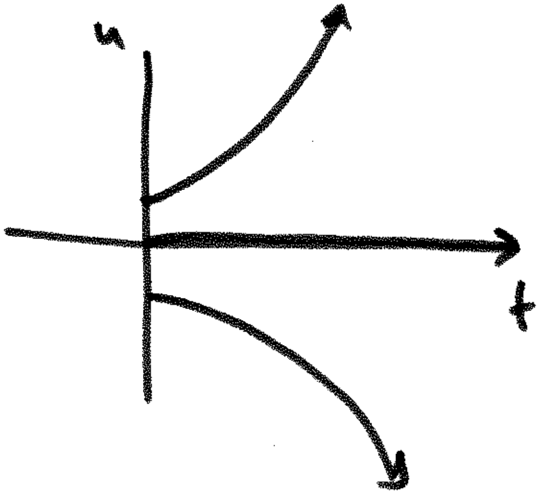
$$u'' - 3u' + 2u = 0$$

$$u = e^{rt} \Longrightarrow r^2 - 3r + 2 = 0$$

$$r = 1, 2$$

$$\Longrightarrow u(t) = c_1 e^{t} + c_2 e^{2t}$$



$x = u$, $y = u'$ (position & momentum)

$$\Longrightarrow x' = y, \quad y' = u'' = 3u' - 2u = 3y - 2x$$

$$\begin{cases} x' = y \\ y' = 3y - 2x \end{cases}$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Ansatz: $\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^{\lambda t} \vec{v}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \lambda e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Longrightarrow \lambda e^{\lambda t}\vec{v} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} e^{\lambda t}\vec{v}$$

$$\longrightarrow \lambda \vec{v} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\vec{v} \Longrightarrow \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -2v_1 + 3v_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \lambda v_1 = v_2 \\ \lambda v_2 = -2v_1 + 3v_2 \end{cases}$$

$$\Longrightarrow \lambda^2 v_1 = -2v_1 + 3\lambda v_1$$

$$\Longrightarrow (\lambda^2 - 3\lambda + 2)v_1 = 0$$

$\times$ $v_1 = 0 \Longrightarrow v_2 = 0$ $\ddot{\frown}$

$v_1 \neq 0 \Longrightarrow \underline{\underline{\lambda = 1, \lambda = 2}}$

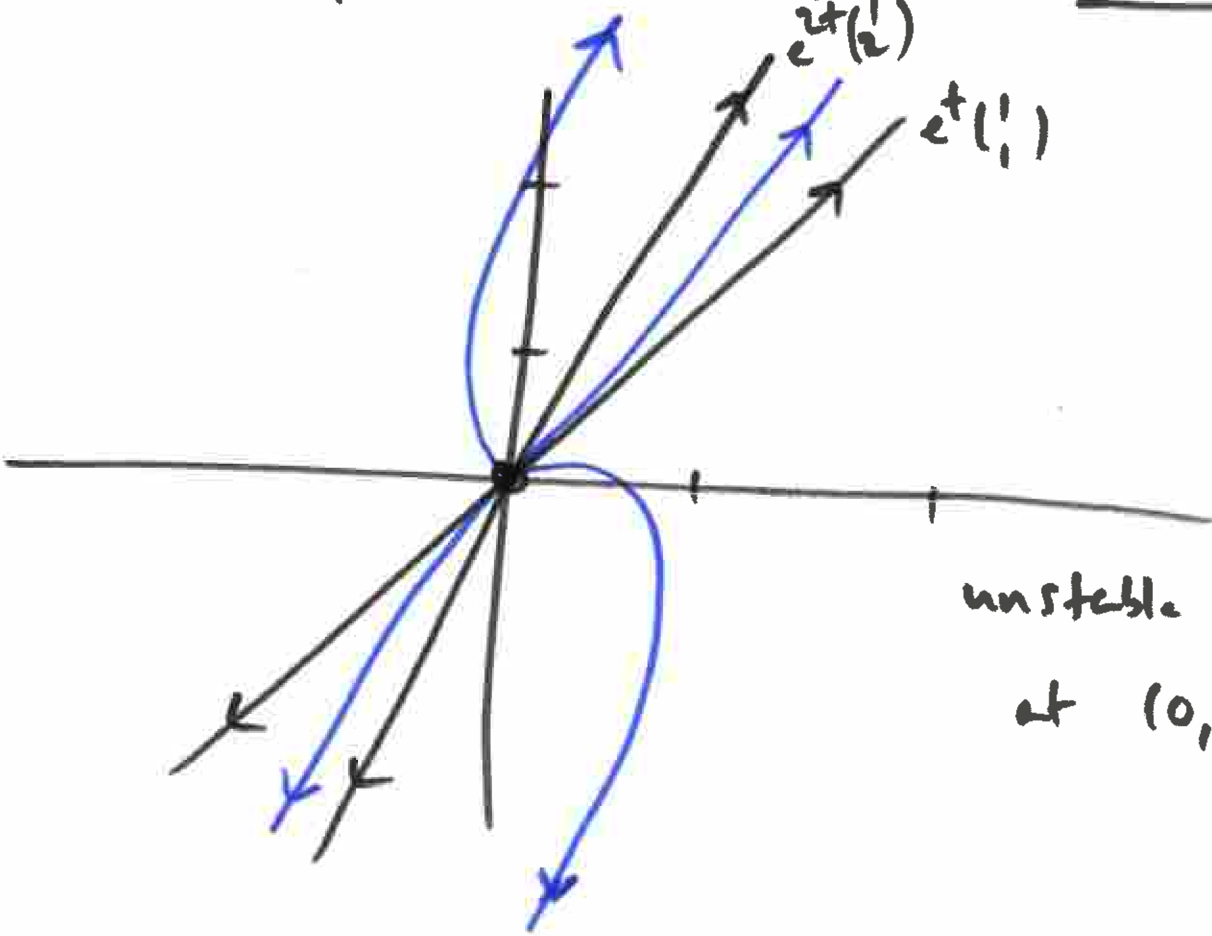
(eigenvalues)

$\underline{\lambda=1}$: $v_1 = v_2 \Longrightarrow v = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \Longrightarrow v = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \underline{\underline{v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

$\underline{\lambda=2}$: $2v_1 = v_2 \Longrightarrow v = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow \underline{\underline{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$

$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ General <u>solution</u>

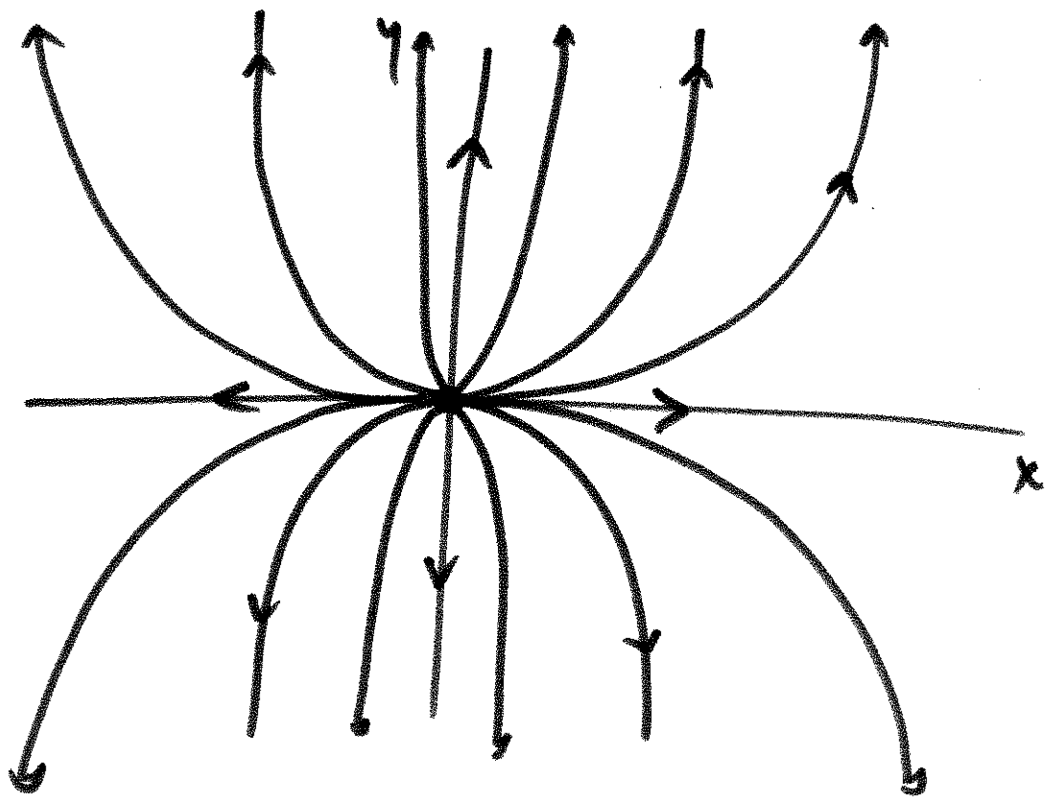
phase <u>plane</u>



unstable <u>node</u> (source) at $(0,0)$

$$\begin{cases} x' = x \\ y' = 2y \end{cases} \longrightarrow \begin{array}{l} x = c_1 e^{t} \\ y = c_2 e^{2t} \end{array} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \underline{\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} + c_2 e^{2t} \underline{\underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

$\underline{y \propto x^2}$



$\underline{1}$ $\quad \vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x} \qquad$ i.e., $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

ansatz: $\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \lambda e^{\lambda t} \vec{v}$

$$\Longrightarrow \lambda e^{\lambda t} \vec{v} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} e^{\lambda t} \vec{v} \Longrightarrow \lambda \vec{v} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{v}$$

$$\Longrightarrow \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 - 2v_2 \\ 2v_1 - 2v_2 \end{pmatrix} \Longrightarrow \begin{cases} \lambda v_1 = 3v_1 - 2v_2 \\ \lambda v_2 = 2v_1 - 2v_2 \end{cases}$$

$$\Longrightarrow \begin{array}{l} (\lambda - 3)v_1 + 2v_2 = 0 \\ -2v_1 + (\lambda+2)v_2 = 0 \\ \hline \end{array} \Longrightarrow \begin{array}{l} 2(\lambda - 3)v_1 + 4v_2 = 0 \\ -2(\lambda - 3)v_1 + (\lambda+2)(\lambda-3)v_2 = 0 \\ \hline \end{array}$$

$$\Longrightarrow \Big((\lambda+2)(\lambda-3) + 4 \Big) v_2 = 0$$

$$v_2 \neq 0 \Longrightarrow (\lambda+2)(\lambda-3) + 4 = 0$$

$$\Longrightarrow \lambda^2 - \lambda - 2 = 0$$

$$\Longrightarrow (\lambda - 2)(\lambda + 1) = 0 \Longrightarrow \underline{\underline{\lambda = 2}}, \ \underline{\underline{\lambda = -1}}$$

$\underline{\lambda = 2}$: $\quad -v_1 + 2v_2 = 0 \Longrightarrow v_1 = 2v_2 \Longrightarrow v = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \underline{\underline{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$

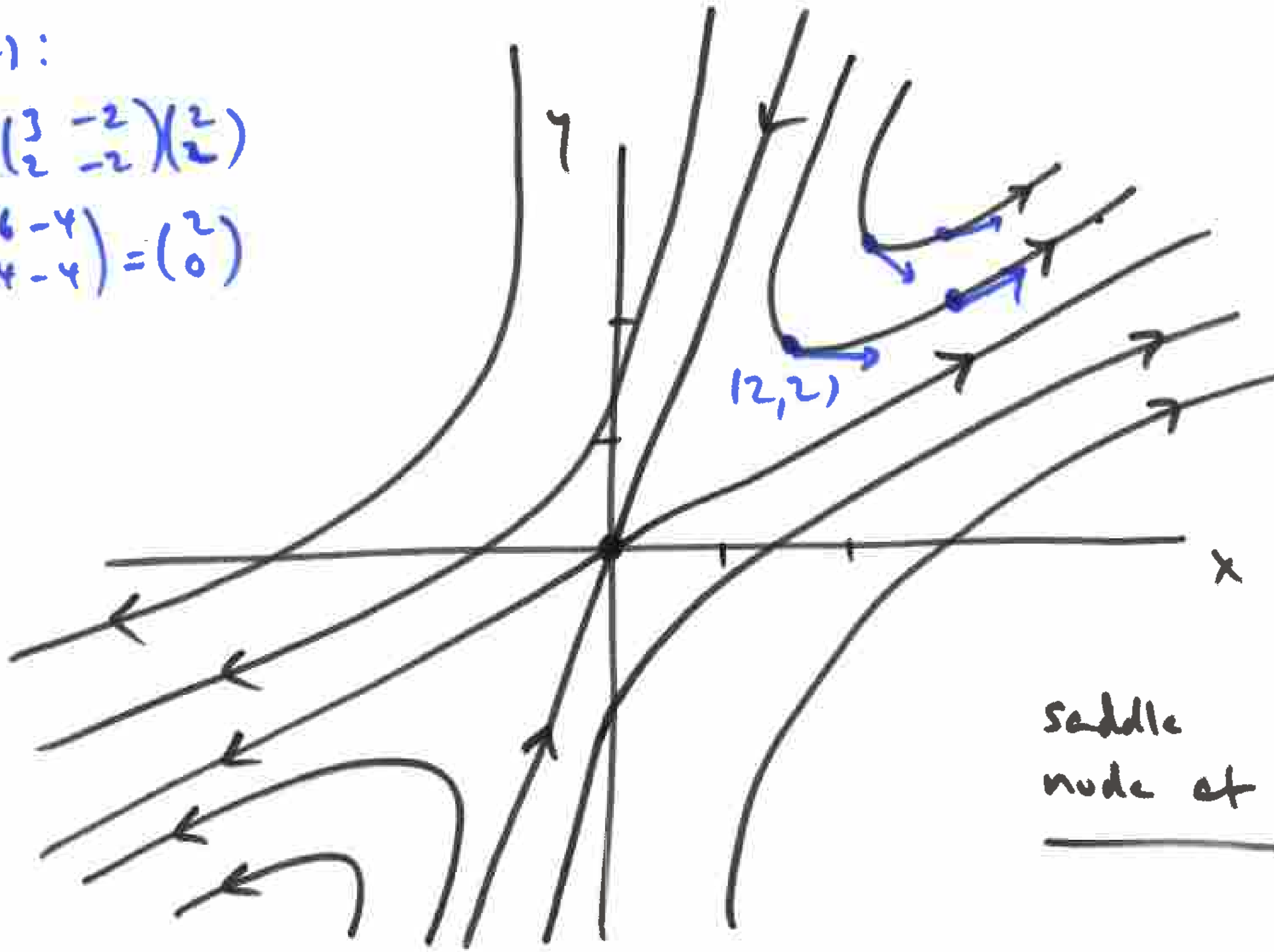
$\underline{\lambda = -1}$: $\quad -4v_1 + 2v_2 = 0 \Longrightarrow 4v_1 = 2v_2 \Longrightarrow v = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$2v_1 = v_2$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

At $(2,2)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6-4 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



Saddle node at $(0,0)$

<u>11</u> $$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}} , \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \lambda e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} e^{\lambda t}\vec{v} \Longrightarrow \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + v_2 \\ -5v_1 + 4v_2 \end{pmatrix}$$

$$\Longrightarrow \left.\begin{aligned} \lambda v_1 &= -2v_1 + v_2 \\ \lambda v_2 &= -5v_1 + 4v_2 \end{aligned}\right\}$$

$$\Longrightarrow \left\{\begin{aligned} (\lambda+2)v_1 - v_2 &= 0 \\ 5v_1 + (\lambda-4)v_2 &= 0 \end{aligned}\right. \Longrightarrow \begin{bmatrix} \lambda+2 & -1 \\ 5 & \lambda-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Need $(\lambda+2)(\lambda-4) - (-1)(5) = 0$

$$(\lambda+2)(\lambda-4)+5=0 \Longrightarrow \lambda^2-2\lambda-3=0$$

$$\Longrightarrow (\lambda-3)(\lambda+1)=0$$

$$\Longrightarrow \underline{\underline{\lambda=3}}, \ \underline{\underline{\lambda=-1}}$$

$\underline{\lambda=3}$: $\quad 3v_1 = -2v_1+v_2 \longrightarrow v_2 = 5v_1 \Longrightarrow v=\begin{pmatrix}1\\5\end{pmatrix}$

$\underline{\lambda=-1}$: $\quad -v_1 = -2v_1+v_2 \Longrightarrow v_2 = v_1 \Longrightarrow v=\begin{pmatrix}1\\1\end{pmatrix}$

$$\Longrightarrow \begin{pmatrix}x\\y\end{pmatrix} = c_1 e^{3t}\begin{pmatrix}1\\5\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$$

$x(0)=1$

$y(0)=3$ $\Longrightarrow \begin{pmatrix}1\\3\end{pmatrix} = c_1\begin{pmatrix}1\\5\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 5c_1 + c_2 \end{pmatrix}$$

$$\Longrightarrow \begin{matrix} c_1 + c_2 = 1 \\ 5c_1 + c_2 = 3 \end{matrix} \Longrightarrow 4c_1 = 2 \Longrightarrow c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$$

Solution satisfying condition is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} ax+by=0 \\ cx+dy=0 \end{cases} \Longrightarrow \begin{aligned} adx + bdy &= 0 \\ bcx + bdy &= 0 \end{aligned}$$

$$(ad-bc)\,x = 0$$

$$x \neq 0 \Longrightarrow \underline{\underline{ad-bc=0}}$$

$$\Big\updownarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Need $\underline{ad-bc=0}$

i.e., $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$