CHAPTER 3

PARABOLIC EQUATIONS

SECTION 1. THE HEAT EQUATION

Suppose a long, thin rod of length l is situated on the interval $(0, l)$ along the x-axis. We shall assume that the material of the rod is homogeneous. Heat may be put into or removed from the rod, and we assume that the temperature u at any point in the rod is a function only of x, the location of a particular cross section, and of t, the time. We write $u = u(x, t)$. Under certain assumptions on the physical properties of the rod, the differential equation governing the flow of heat (in appropriate units) in the rod is given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t).$$

The function f is the rate of heat removal in the bar. The temperature function $u(x, t)$ satisfies a maximum principle somewhat different from the one which was established for elliptic equations and inequalities.

Suppose $u(x, t)$ satisfies the strict inequality

$$L[u] \equiv \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} > 0$$

in a region E of the x, t-plane (Fig. 1). It is clear that u cannot have a (local) maximum at any interior point. For at such a point

$$\frac{\partial^2 u}{\partial x^2} \leq 0 \text{ and } \frac{\partial u}{\partial t} = 0,$$

thereby violating $L[u] > 0$. We shall not only extend this statement to solutions u of $L[u] \geq 0$, but we shall also show that for operators of this type, the maximum principle takes a stronger form.

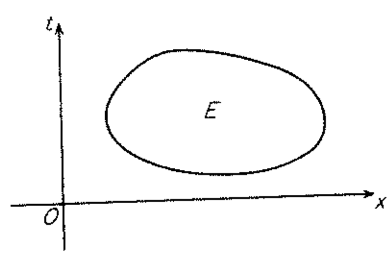

FIGURE 1

To illustrate a typical problem,

we shall suppose that the rod described above has its temperature prescribed initially (i.e., at time $t = 0$) and that the temperatures at the ends of the rod are known functions of time. The *principle of causality* states that the temperature distribution at any fixed time T is unaffected by any changes in the rod which happen at a time $t > T$. Thus it is natural to consider the rectangular region

$$E: \{0 < x < l, 0 < t \leq T\} \tag{1}$$

in the x, t-plane. We suppose that the temperature $u(x, t)$ is known on three sides of E:

$$S_1: \{x = 0, 0 \leq t \leq T\}, \qquad S_2: \{0 \leq x \leq l, t = 0\},$$
$$S_3: \{x = l, 0 \leq t \leq T\}.$$

On physical grounds, we expect that this information and the fact that the temperature u satisfies the equation

$$L[u] \equiv \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

in E suffice to determine the temperature uniquely throughout E (Fig. 2).

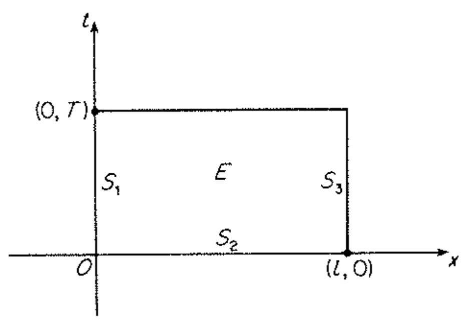

FIGURE 2

The uniqueness of the solution is easily established as a corollary of the following maximum principle.

THEOREM 1. Suppose $u(x, t)$ satisfies the inequality

$$L[u] \equiv \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \geq 0 \tag{2}$$

in the rectangular region E given by (1). Then the maximum of u on the closure $E \cup \partial E$ must occur on one of the three sides S_1, S_2 or S_3 (Fig. 2).

Proof. Suppose that M is the maximum of the values of u which occur on S_1, S_2, and S_3. We shall assume there is a point $P(x_0, t_0)$ of E where u has a value $M_1 > M$ and establish a contradiction. We define the auxiliary function

$$w(x) = \frac{M_1 - M}{2l^2}(x - x_0)^2.$$

Then, since $u \leq M$ on S_1, S_2, and S_3, we have

$$v(x, t) \equiv u(x, t) + w(x) \leq M + \frac{M_1 - M}{2} < M_1 \tag{3}$$

for all points of S_1, S_2, and S_3. Furthermore,

$$v(x_0, t_0) = u(x_0, t_0) = M_1 \tag{4}$$

and

$$L[v] \equiv L[u] + L[w] = L[u] + \frac{M_1 - M}{l^2} > 0 \tag{5}$$

throughout E. Conditions (3) and (4) show that v must assume its maximum either at an interior point of E or along the open interval

$$S_4: \{0 < x < l, t = T\}.$$

The inequality (5) shows that v cannot have an interior maximum. At a maximum along S_4, we have $\partial^2 v/\partial x^2 \leq 0$, implying that $\partial v/\partial t$ is strictly negative. Thus v must be larger at an earlier time so that the maximum in E cannot be on S_4. We see in this way that the assumption $u(x_0, t_0) > M$ leads to a contradiction.

Remarks. (i) The theorem states that the maximum not only cannot occur at an interior point of E, but also cannot occur at the "latest" time, except possibly at the ends of the rod.

(ii) The maximum principle of Theorem 1 is not one of the strong form, since this theorem permits the maximum of u to occur both on the boundary and at interior points. Later we shall see that if the maximum occurs in E, then the solution must be constant in a certain region, a result which contains Theorem 1 as a special case.

(iii) For solutions of $L[u] = 0$ we obtain an associated minimum principle when we replace u by $-u$. The uniqueness theorem alluded to earlier then follows easily.

(iv) For elliptic differential inequalities the maximum of a solution could occur anywhere on the boundary. In the case of the heat equation, we have a stronger result—namely, the maximum can occur only on a specified portion of the boundary. This fact is true both for more general equations of which the heat equation is the prototype, and for more general domains.

The equation of heat propagation in a three-dimensional, homogeneous object D is

$$L[u] \equiv \Delta u - \frac{\partial u}{\partial t} = f(x, y, z, t),$$

EXERCISES

1. Show that the problem

$$u_{xx} - tu_t + 2u = 0$$
$$u(x, 0) = 0, \qquad 0 \leq x \leq \pi$$
$$u(0, t) = u(\pi, t) = 0, \qquad t \geq 0$$

has the solutions $u = at \sin x$ for all values of the constant a. Why are the uniqueness theorems not applicable?

2. Show that the problem (polar coordinates in the plane)

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} - u_t = 0, \qquad 0 < r < 1, \qquad t > 0$$
$$u(r, \theta, 0) = 0, \qquad 0 \leq r \leq 1, \qquad 0 \leq \theta \leq 2\pi$$
$$u_\theta(1, \theta, t) = 0, \qquad 0 \leq \theta \leq 2\pi, \qquad t > 0$$

has more than one solution. Why doesn't Theorem 8 apply?

SECTION 5. A THREE-CURVES THEOREM

The Hadamard three-circles theorem for subharmonic functions (Chapter 2, Section 12) does not have an exact analogue for functions which satisfy parabolic inequalities. However, by means of the maximum principle, it is possible to obtain a three-curves theorem which is similar to the Hadamard inequality. The result given here has a number of applications. As an example, we use it to establish the uniqueness of the solution of an initial value problem for the heat equation.

Let t_0 be a fixed positive constant, and consider the one-parameter family of parabolas

$$\frac{x^2}{t_0 - t} = \rho,$$

where the constant ρ takes on all positive values (see Fig. 12). Except for points on the t-axis, each point in the strip $\{0 < t < t_0, -\infty < x < \infty\}$ lies on exactly one member of this family.

We may consider ρ as a function of x and t. A computation yields

$$\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial \rho}{\partial t} = \frac{2(t_0 - t) - x^2}{(t_0 - t)^2}.$$

We seek a function of ρ alone, say $\sigma(\rho)$, which is a solution of the heat equation. To determine σ, we write

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{\partial \sigma}{\partial t} \equiv \sigma''(\rho)\rho_x^2 + \sigma'(\rho)(\rho_{xx} - \rho_t).$$

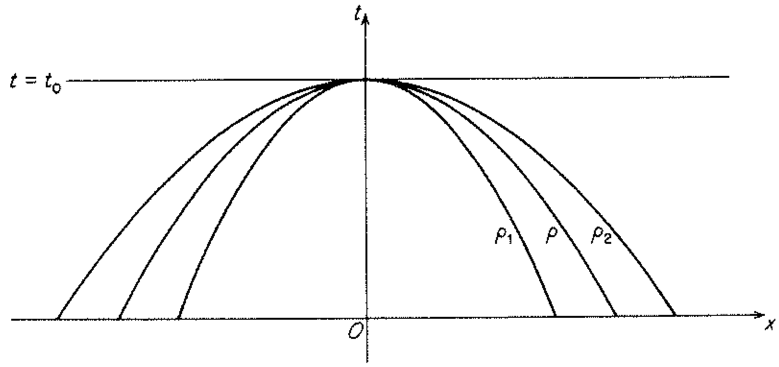

FIGURE 12

Thus σ must satisfy

$$\frac{\sigma''}{\sigma'} = -\frac{\rho_{xx} - \rho_t}{\rho_x^2} = \frac{1}{4} - \frac{1}{2\rho}$$

and, upon integration,

$$\log \sigma' = \tfrac{1}{4}\rho - \tfrac{1}{2}\log \rho.$$

Therefore,

$$\sigma' = \frac{1}{\sqrt{\rho}} e^{\rho/4},$$

and a second integration gives

$$\sigma(\rho) = \int_0^\rho \frac{1}{\sqrt{\rho_1}} e^{\rho_1/4} d\rho_1. \tag{1}$$

The function $\sigma(\rho)$ as given by (1) satisfies the heat equation for $t < t_0$.

We consider a function $u(x, t)$ which satisfies

$$u_{xx} - u_t \geq 0$$

in a region D described as follows: D is bounded from below by the line $t = 0$ and from above by the line $t = \bar{t}$, where $\bar{t} < t_0$; D is bounded on its sides by the arcs of the parabolas $\rho = \rho_1$ and $\rho = \rho_2$ situated in the first quadrant (see Fig. 13). For $\rho_1 \leq \rho \leq \rho_2$, we define the functions

$$M_1(\rho) = \max_{\substack{x^2=\rho(t_0-t)\\ 0\leq t\leq \bar{t}}} u(x, t), \qquad M_2 = \max_{\sqrt{\rho_1 t_0}\leq x\leq\sqrt{\rho_2 t_0}} u(x, 0),$$

and

$$M(\rho) = \max\,(M_1(\rho), M_2).$$

The function

$$\varphi(\rho) = a + b\sigma(\rho)$$

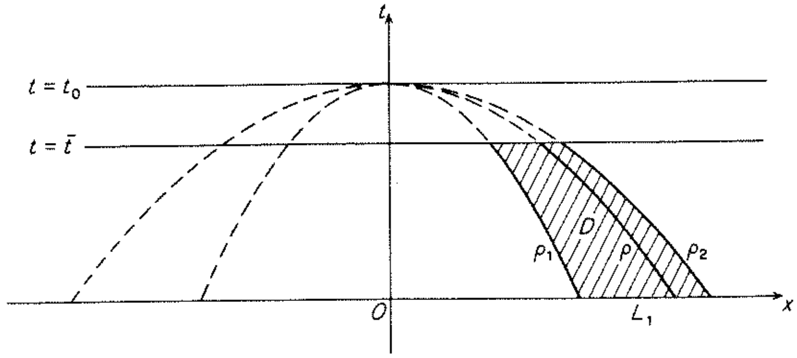

FIGURE 13

satisfies the heat equation. We now determine a and b by the relations

$$a + b\sigma(\rho_1) = M(\rho_1),$$
$$a + b\sigma(\rho_2) = M(\rho_2),$$

and we find

$$\varphi(\rho) = \frac{M(\rho_1)[\sigma(\rho_2) - \sigma(\rho)] + M(\rho_2)[\sigma(\rho) - \sigma(\rho_1)]}{\sigma(\rho_2) - \sigma(\rho_1)}.$$

The function

$$v = u - \varphi(\rho)$$

satisfies the inequality

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} \geq 0 \text{ in } D.$$

Furthermore, since

$$u(x, 0) \leq M_2$$

and

$$u(x, t) \leq M(\rho_1) \text{ for } x^2 = \rho_1(t_0 - t),$$
$$u(x, t) \leq M(\rho_2) \text{ for } x^2 = \rho_2(t_0 - t),$$

we find that $v \leq 0$ on the entire boundary of D below the line $t = \bar{t}$. Applying the maximum principle to v, we derive the formula

$$M(\rho) \leq \frac{M(\rho_1)[\sigma(\rho_2) - \sigma(\rho)] + M(\rho_2)[\sigma(\rho) - \sigma(\rho_1)]}{\sigma(\rho_2) - \sigma(\rho_1)}. \tag{2}$$

That is, *$M(\rho)$ is a convex function of $\sigma(\rho)$*. With the aid of (2) we can establish the following uniqueness result, which was first proved by A. N. Tikhonov [1].

THEOREM 9. Let $u(x, t)$ and $v(x, t)$ be solutions of $u_{xx} - u_t = f(x, t)$ in the strip $D: \{-\infty < x < \infty, 0 < t < T\}$, and suppose that u and v are continuous on $D \cup \partial D$. If $u(x, 0) = v(x, 0) = g(x)$, where g is a prescribed function, and if there are constants A and c such that

$$|u(x, t)|, |v(x, t)| \leq Ae^{cx^2} \tag{3}$$

uniformly in t for $0 \leq t \leq T$, then $u(x, t) \equiv v(x, t)$ in D.

Proof. We employ the convexity inequality (2). We select $t_0 < 1/4c$ and consider the function $w(x, t) \equiv u(x, t) - v(x, t)$ in the domain D_1: $\{-\infty < x < \infty, 0 \leq t \leq t_0/2\}$. Then w satisfies the heat equation and $w(x, 0) \equiv 0$. We apply inequality (2) to w and let $\rho_2 \to \infty$. Since w is bounded by a multiple of e^{cx^2} and $\sigma(\rho)$ grows as rapidly as $e^{x^2/4t_0}$, we find that $M(\rho_2)/\sigma(\rho_2) \to 0$ as $\rho_2 \to \infty$. Therefore

$$M(\rho) \leq M(\rho_1).$$

Letting $\rho_1 \to 0$, we see that w takes on its maximum in the half-strip $\{x \geq 0, 0 \leq t \leq \frac{1}{2}t_0\}$ at $x = 0$. Applying the same reasoning for negative values of x, we conclude that w takes on its maximum in D_1 at $x = 0$. Then by Theorem 2 this maximum must be zero, and so $w \leq 0$ in the whole strip. Applying the same reasoning to $(-w)$, we find that $w \equiv 0$ in D_1. We may repeat the entire process using the line $t = \frac{1}{2}t_0$ as initial line and find that $w \equiv 0$ in D_2: $\{-\infty < x < \infty, t_0/2 \leq t \leq t_0\}$. After a finite number of steps, we conclude that $w \equiv 0$ in D.

Remarks. (i) In n dimensions, we can use the same method to derive an inequality such as (2) applicable to functions which satisfy $\Delta u - u_t \geq 0$ with the maximum taken over paraboloids of the form $(x_1^2 + \cdots + x_n^2)/(t_0 - t) = \rho$. Then we obtain a uniqueness theorem for solutions of the heat equation which grow no more rapidly than $Ae^{c(x_1^2 + x_2^2 + \cdots + x_n^2)}$.

(ii) The growth condition (3) is needed, since examples can be found of nonuniqueness of solutions if growth more rapid than (3) is allowed. See Tikhonov [1], Täcklind [1], Widder [1], Rosenbloom and Widder [1].

(iii) As in the case of elliptic equations, a general three-surfaces theorem can be obtained which is applicable to linear parabolic equations. A uniqueness theorem for parabolic equations with conditions as in Theorem 9 can also be obtained. However, it is also possible to derive a uniqueness theorem for unbounded domains by a direct application of the maximum principle. At the same time a Phragmèn-Lindelöf principle for parabolic equations may be established. We shall develop this method in Section 6.