#24 $\quad y' = ry - ky^2 \qquad r, k > 0$

$$y' - ry = -ky^2 \quad \begin{cases} p = -r \\ q = -k \\ n = 2 \end{cases}$$

$$v = y^{-1} \Rightarrow v' = -y^{-2} y'$$

$$v\underline{y^2} = y \ , \quad v'\underline{(-y^2)} = y'$$

$$-v'y^2 - rvy^2 = -ky^2$$

$$\Longrightarrow -v' - rv = -k$$

$$\Longrightarrow \underline{v' + rv = k} \qquad (\mu = e^{rt})$$

$$\Longrightarrow (ve^{rt})' = ke^{rt}$$

$$\Longrightarrow ve^{rt} = \frac{k}{r}e^{rt} + c$$

$$\Longrightarrow v = \frac{k}{r} + ce^{-rt}$$

$$y' + p(t)y = q(t)y^n$$

$$\underline{v = y^{1-n}}$$

$$v = \frac{k}{r} + ce^{-rt}$$

$$\Longrightarrow y^{-1} = \frac{k}{r} + ce^{-rt}$$

$$\Longrightarrow y(t) = \frac{1}{\frac{k}{r} + ce^{-rt}}$$

or

$$y(t) = \frac{e^{rt}}{\frac{k}{r}e^{rt} + c}$$

AUTONOMOUS POPULATION MODELS

$$\boxed{y' = f(y)}$$

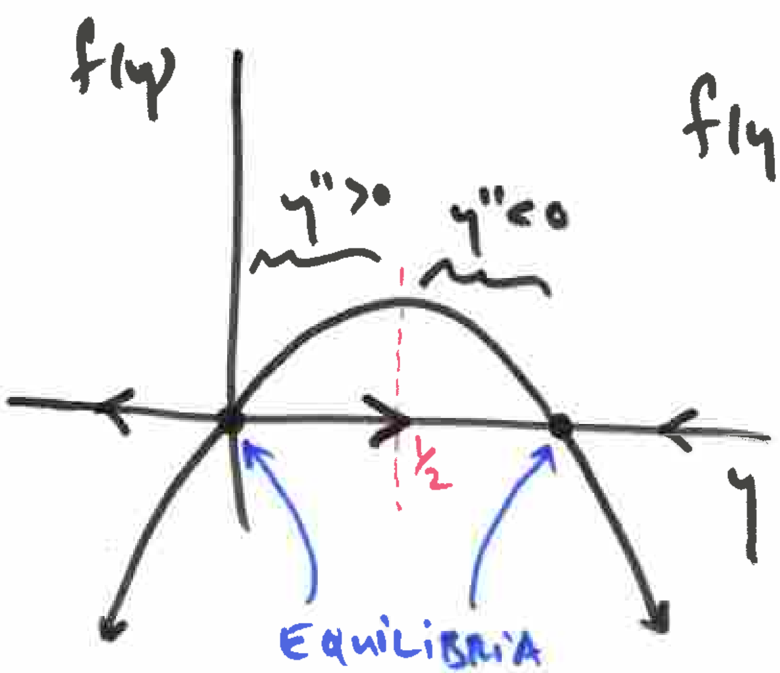
e.g. $f(y) = y(1-y)$ LOGISTIC

or $f(y) = y \ln\left(\frac{k}{y}\right)$ GOMPERTZ

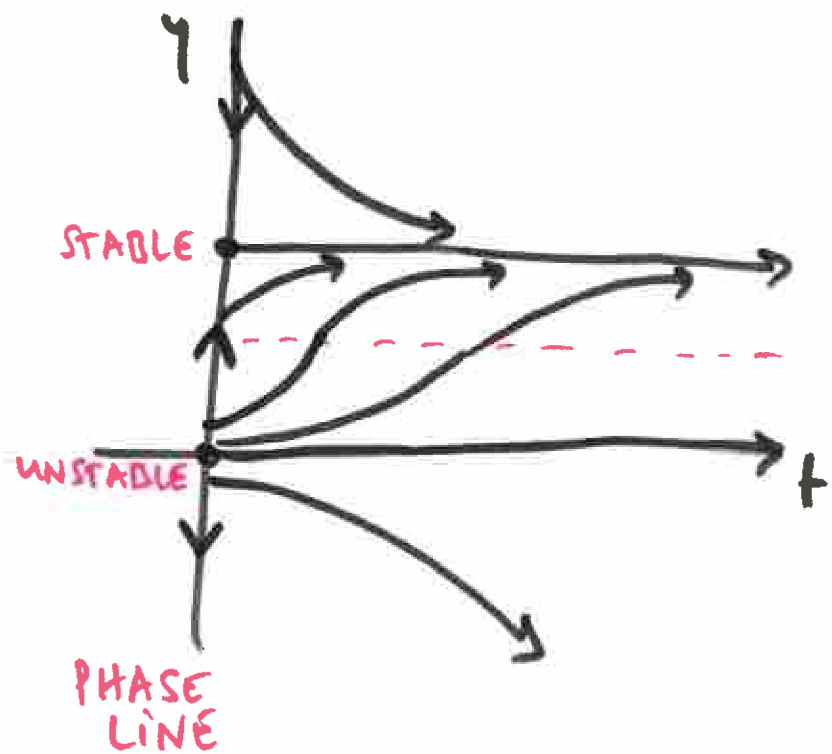
or $f(y) = y(1-y)(y-\frac{1}{2})$

$\Rightarrow \quad y'' = f'(y)y'$

$\Rightarrow \quad \boxed{y'' = f'(y)f(y)}$

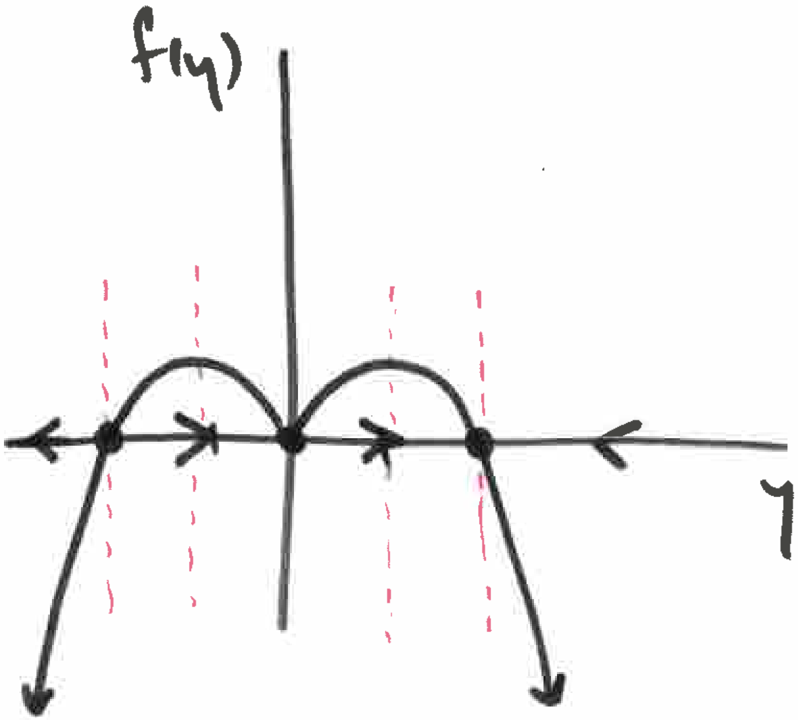


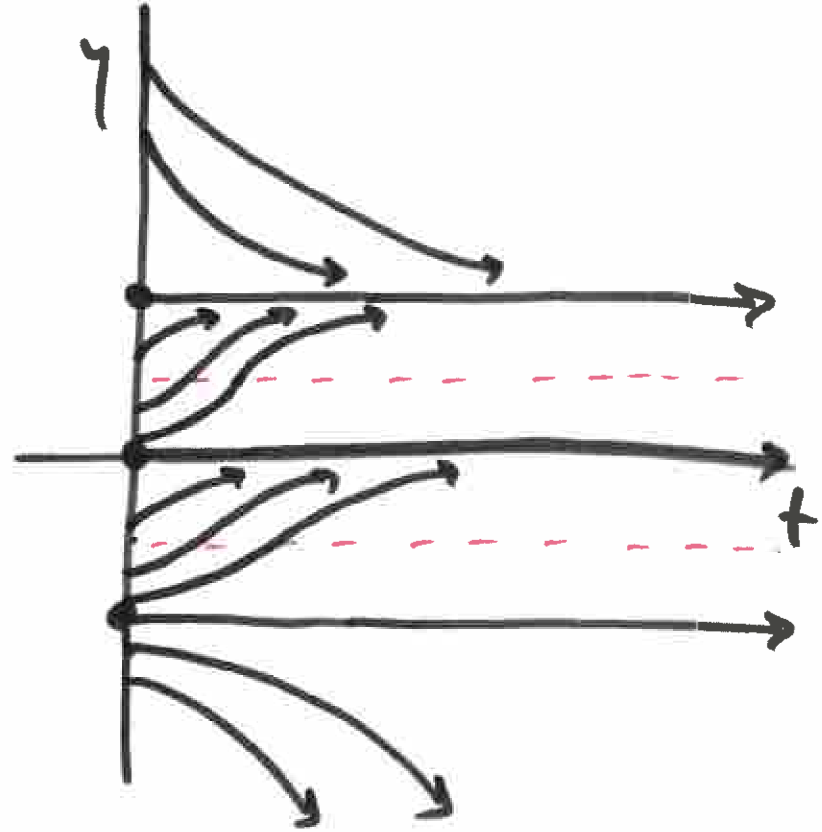
$f(y) = y(1-y)$



#8 $y' = \underbrace{y^2(4-y^2)}_{f(y)}$

$y'' = f'(y)f(y)$





$$y' = y^2(4-y^2)$$

$$\frac{y'}{y^2(4-y^2)} = 1$$

Integrate:

$$\int \frac{dy}{y^2(4-y^2)} = t + c$$

$$\vdots$$

$$y' = +ry - ky^2$$

$$y' = ry\left(1 - \frac{k}{r}y\right)$$

$$\frac{y'}{y\left(1-\frac{k}{r}y\right)} = r$$

Integrate:

$$\int \frac{dy}{y\left(1-\frac{k}{r}y\right)} = rt + c$$

$$\vdots$$

3.1 #1 $y'' + 2y' - 3y = 0$

Ansatz : $y(t) = e^{rt} \Longrightarrow y' = re^{rt}, \; y'' = r^2 e^{rt}$

$\Longrightarrow r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$

$\Longrightarrow r^2 + 2r - 3 = 0$ characteristic equation

$\Longrightarrow (r+3)(r-1) = 0 \Longrightarrow$ characteristic values/roots are $r = -3$, $r = 1$

This means that $y_1(t) = e^{-3t}$ & $y_2(t) = e^{t}$ solve the ODE, and the general solution is

$$\boxed{y(t) = c_1 e^{-3t} + c_2 e^{t}}$$

#8 $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$

$y(t) = e^{rt} \implies y' = re^{rt}, \quad y'' = r^2 e^{rt}$

$\implies r^2 + 4r + 3 = 0$ characteristic equation

$\implies r = -1, -3$

$\implies y_1(t) = e^{-t}, \quad y_2(t) = e^{-3t}$

General sol'n: $y(t) = c_1 e^{-t} + c_2 e^{-3t} \implies y(0) = c_1 + c_2$

$y' = -c_1 e^{-t} - 3c_2 e^{-3t} \implies y'(0) = -c_1 - 3c_2$

$$\implies \left.\begin{aligned} c_1 + c_2 &= 2 \\ -c_1 - 3c_2 &= -1 \end{aligned}\right\} \implies \begin{aligned} -2c_2 = 1 &\implies c_2 = \tfrac{-1}{2} \\ c_1 &= \tfrac{5}{2} \end{aligned}$$

Sol'n of IVP: $\underline{\underline{y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}}}$