Laplace transform

Definition:

$$f(t) \longmapsto F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Use: solve initial-value problems using table, properties

Example

$f(t) = t$:

$$F(s) = \int_0^\infty \underbrace{t}_{u}\, \underbrace{e^{-st}\, dt}_{dv}$$

$$= \cancel{t \cdot \frac{e^{-st}}{-s} \Big|_0^\infty} + \int_0^\infty \frac{e^{-st}}{s}\, dt =$$

$$= \left. \frac{e^{-st}}{-s^2} \right|_0^\infty = \frac{1}{s^2}, \quad s > 0$$

$$f(t) = t \longmapsto F(s) = \frac{1}{s^2}$$

$$t^n \longmapsto \frac{n!}{s^{n+1}}, \quad s > 0$$

Properties / facts:

① $$\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{at} f(t) e^{-st} \, dt$$

$$= \int_0^\infty f(t) e^{-(s-a)t} \, dt$$

$$= F(s-a).$$

② $$\mathcal{L}(f') = \int_0^\infty \underbrace{f'(t)}_{dv} \underbrace{e^{-st}}_{u} \, dt = e^{-st} f(t) \Big|_0^\infty + \int_0^\infty f(t) \, s e^{-st} \, dt$$

Then $\mathcal{L}(f') = -f(0) + sF(s)$.

and $\mathcal{L}(f'') = -f'(0) - sf(0)$
$+ s^2 F(s)$.

<u>example</u>: $y'' - 4y' + 4y = 0$

$y(0) = 1, \; y'(0) = 1$

Apply $\mathcal{L}$:

$$-y'(0) - sy(0) + s^2 Y - 4(-y(0) + sY) + 4Y = 0$$

$$-1 - s + s^2 Y + 4 - 4sY + 4Y = 0$$

$$(s^2 - 4s + 4)Y = s - 3$$

$$Y(s) = \frac{s-3}{(s-2)^2} = \frac{(s-2)-1}{(s-2)^2}$$

$$= \frac{1}{s-2} - \frac{1}{(s-2)^2} \Longrightarrow$$

$$Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

$$y(t) = e^{2t} - e^{2t} \cdot t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \cdot 1$$

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = \underline{\underline{e^{2t} \cdot t}}$$

$$F = \frac{1}{s^2}, \; f = t$$

$$\mathcal{L}(e^{at} f) = F(s-a)$$

Next fact:

$$\mathcal{L}\big(\underline{H(t-c)}\,f(t-c)\big) = \int_0^\infty H(t-c)f(t-c)\,e^{-st}\,dt$$

Heaviside:

$$H(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

$$= \int_c^\infty f(t-c)\,e^{-st}\,dt$$

$z = t-c,\ dz = dt$

$$= \int_0^\infty f(z)\,e^{-s(z+c)}\,dz$$

$$= e^{-cs}\int_0^\infty f(z)\,e^{-sz}\,dz$$

$$= \underline{e^{-cs}\,F(s)}.$$

In other notation:

$$\mathcal{L}\left(\underbrace{u_c(t)}\, f(t-c)\right) = \underline{e^{-cs}\, F(s)}$$

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

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$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases} = -2u_3 + 4u_5 - u_7$$

Find the transform of

$$f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases} = u_1(t)\left((t-1)^2 + 1\right)$$

$$= u_1(t)(t-1)^2 + u_1(t)$$

$$\mathcal{L}(f) = \mathcal{L}\left(u_1(t-1)^2 + u_1 \cdot 1\right) = e^{-s} \cdot \frac{2}{s^3} + e^{-s} \cdot \frac{1}{s}$$

$$= e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

$$\mathcal{L}(u_c \cdot f(t-c)) = e^{-cs} F(s)$$

#14 $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

inverse will include $u_2(t)\, f(t-2)$; $\mathcal{L}(f) = \dfrac{1}{s^2+s-2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s-2}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+2)(s-1)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right)$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\mathcal{L}(F) = u_2(t)\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right].$$