$$f(x) \longmapsto F(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

$$e^{ax} \longmapsto \frac{1}{s-a}, \quad s > a$$

$$\sin(ax) \longmapsto \frac{a}{a^2+s^2}, \quad s > 0$$

$$\cos(ax) \longmapsto \frac{s}{a^2+s^2}, \quad s > 0$$

$$e^{ax} f(x) \longmapsto F(s-a)$$

$$x^n \longmapsto \frac{n!}{s^{n+1}}$$

$$u_a(x) f(x-a) \longmapsto e^{-as} F(s)$$

$$u_a(x) = \begin{cases} 1, & x > a \\ 0, & x < a \end{cases}$$

Math 212 – Problem Set 8   SOLUTIONS

① $y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Transform:

$$-y'(0) - sy(0) + s^2 L[y] - \left(-y(0) + L[y]\right) - 6L[y] = 0$$

$$\Longrightarrow 1 - s + 1 + (s^2 - s - 6)L[y] = 0$$

$$\Longrightarrow L[y] = \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s-3)(s+2)}$$

$$\Longrightarrow L[y] = \frac{1/5}{s-3} + \frac{4/5}{s+2} \quad \text{(partial fractions)}$$

$$\Longrightarrow y(x) = \frac{1}{5}e^{3x} + \frac{4}{5}e^{-2x}$$

---

② $y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -1$

$$\Longrightarrow -y'(0) - sy(0) + s^2 L[y] - 2y(0) + 2sL[y] + 5L[y] = 0$$

$$\Longrightarrow 1 - 2s - 4 + (s^2 + 2s + 5)L[y] = 0$$

$$\Longrightarrow L[y] = \frac{2s+3}{s^2 + 2s + 5} = \frac{2(s+1) + 1}{(s+1)^2 + 4}$$

$$\Longrightarrow \mathcal{L}[y] = \frac{2(s+1)}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$$

$$\Longrightarrow y(x) = e^{-x}(2\cos(2x)) + e^{-x}\left(\tfrac{1}{2}\sin(2x)\right)$$

$$= 2e^{-x}\cos(2x) + \tfrac{1}{2}e^{-x}\sin(2x).$$

(3) $y'' + 2y' + y = 4e^{-x}, \quad y(0) = 2, \quad y'(0) = -1$

$$\Longrightarrow -y'(0) - sy(0) + s^2\mathcal{L}[y] - 2y(0) + 2s\mathcal{L}[y] + \mathcal{L}[y] = \frac{4}{s+1}$$

$$\Longrightarrow 1 - 2s - 4 + (s^2+2s+1)\mathcal{L}[y] = \frac{4}{s+1}$$

$$\Longrightarrow (s^2+2s+1)\mathcal{L}[y] = \frac{4}{s+1} + 2s + 3$$

$$\Longrightarrow (s+1)^2\mathcal{L}[y] = \frac{4}{s+1} + 1 + 2(s+1)$$

$$\Longrightarrow \mathcal{L}[y] = \frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}$$

$$\Longrightarrow y(x) = 2e^{-x}\cdot x^2 + e^{-x}\cdot x + 2e^{-x}$$

$$\Longrightarrow y(x) = (2x^2 + x + 2)e^{-x}.$$

④ $y'' + y = h(x)$, $h(x) = \begin{cases} 1, & 0 \le x < 3\pi \\ 0, & 3\pi \le x < \infty \end{cases}$

$y(0) = 0$, $y'(0) = 1$

Laplace transform of h:

$$L[h] = \int_0^\infty e^{-sx} h(x)\,dx = \int_0^{3\pi} e^{-sx}\,dx = \frac{e^{-sx}}{-s}\Bigg|_{x=0}^{x=3\pi}$$

$$= \frac{1 - e^{-3\pi s}}{s}$$

Transform the ODE:

$$-y'(0) - sy(0) + (s^2+1)L[y] = \frac{1 - e^{-3\pi s}}{s}$$

$$\Longrightarrow (s^2+1)L[y] = \frac{1 - e^{-3\pi s}}{s} + 1$$

$$\Longrightarrow L[y] = \frac{1 - e^{-3\pi s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$= \frac{1}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

by partial fractions: $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$$= \frac{1}{s} - \frac{s}{s^2+1} - e^{-3\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$$

Thus, $$L[y] = \frac{1}{s} - \frac{s}{s^2+1} - \frac{e^{-3\pi s}}{s} + \frac{e^{-3\pi s} \cdot s}{s^2+1} + \frac{1}{s^2+1}$$

Invert the transforms, using the table and the fact that $L[u_a(x)f(x-a)] = e^{-as}F(s)$;
doing so yields

$$y(x) = 1 - \cos x - u_{3\pi}(x)\cdot 1 + u_{3\pi}(x)\cos(x-7\pi) + \sin x$$

$$= 1 - \cos x + \sin x + u_{3\pi}(x)\Big(\cos(x-7\pi) - 1\Big).$$

(5) $y'' + 3y' + 2y = h(x)$, $\quad h(x) = \begin{cases} 1, & 0 \le x < 10 \\ 0, & x \geq 10 \end{cases}$

$y(0) = 0$, $y'(0) = 0$

Transform of $h(x)$:

$$L[h] = \int_0^\infty e^{-sx} h(x)\,dx = \int_0^{10} e^{-sx}\,dx = \frac{1-e^{-10s}}{s}$$

Transform the ODE: since $y(0) = y'(0) = 0$, get

$$(s^2+3s+2)L[y] = \frac{1-e^{-10s}}{s}$$

$$\Longrightarrow L[y] = \frac{1}{s(s+1)(s+2)} - \frac{e^{-10s}}{s(s+1)(s+2)}$$

$$= \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

$$+ e^{-10s}\left(\frac{-\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}\right)$$

Invert as in #4:

$$y(x) = \frac{1}{2} - e^{-x} + \frac{1}{2}e^{-2x} + u_{10}(x)\left(-\frac{1}{2} + e^{-(x-10)} - \frac{1}{2}e^{-2(x-10)}\right).$$