# LAPLACE TRANSFORM

given $f(t)$, its Laplace transform is

$$\mathcal{L}[f] \equiv F(s) = \int_0^\infty e^{-st} f(t)\,dt .$$

example: $\boxed{f(t) = e^{at} \Longrightarrow F(s) = \frac{1}{s-a} \text{ if } s > a}$

$$\Longrightarrow F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)}\Bigg|_{t=0}^{t=\infty} = \underbrace{0}_{s>a} + \frac{1}{s-a}$$

$$\mathcal{L}(f') = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)dt}_{dv}$$

$u = e^{-st}$ $dv = f'(t)dt$

$du = -se^{-st}$ $v = f(t)$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= \underbrace{0}_{s>0} - f(0) + s\mathcal{L}[f]$$

$$\boxed{\mathcal{L}(f') = -f(0) + s\mathcal{L}[f]}$$

$$\mathcal{L}(f'') = \int_0^\infty e^{-st} f''(t)\,dt \qquad u = e^{-st} \quad dv = f'' \qquad du = -se^{-st}\,dt \quad v = f'$$

$$= e^{-st} f'(t)\Big|_{t=0}^{t=\infty} + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= 0 - f'(0) + s\underbrace{\mathcal{L}(f')}_{\cancel{f'(0)}}$$

$$\boxed{-f(0) + s\mathcal{L}(f) = \mathcal{L}(f')}$$

$$\boxed{\mathcal{L}(f'') = -f'(0) - sf(0) + s^2\mathcal{L}(f)}$$

• $y'' - y = 0, \quad y(0) = 1, \quad y'(0) = -1 \Longrightarrow \underline{y(t) = e^{-t}}$

Apply the Laplace transform:

$$\mathcal{L}(y'') - \mathcal{L}(y) = 0$$

$$\Longrightarrow -y'(0) - s\,y(0) + s^2\mathcal{L}(y) - \mathcal{L}(y) = 0$$

$$\Longrightarrow 1 - s + s^2 Y(s) - Y(s) = 0 \qquad (Y = \mathcal{L}(y))$$

$$(s^2 - 1)Y(s) = s - 1$$

$$Y(s) = \frac{s-1}{s^2-1} = \frac{1}{s+1} \Longrightarrow \boxed{y(t) = e^{-t}}$$

$$\mathcal{L}(t) = \int_0^\infty e^{-st}\, t\, dt \qquad u = t \quad dv = e^{-st}dt$$

$$du = dt \quad v = -\frac{e^{-st}}{s}$$

$$= -\frac{e^{-st}\,t}{s}\Bigg|_{t=0}^{t=\infty} + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= 0 + -\frac{e^{-st}}{s^2}\Bigg|_0^\infty = \frac{1}{s^2} \qquad (s>0)$$

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$$\mathcal{L}(t^2) = \int_0^\infty e^{-st}\, t^2\, dt$$

$$= \underline{\underline{\frac{2}{s^3}}}$$

$$\begin{array}{ccc} t^2 & & e^{-st} \\ & + & \\ 2t & & -\dfrac{e^{-st}}{s} \\ & - & \\ 2 & & \dfrac{e^{-st}}{s^2} \\ & + & \\ 0 & & -\dfrac{e^{-st}}{s^3} \end{array}$$

$$\mathcal{L}(t^3) = \int_0^\infty e^{-st} t^3 \, dt$$

$$= \frac{6}{s^4}$$

In general:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad (s>0)$$

$$\begin{array}{ccc}
t^3 & & e^{-st} \\
 & \oplus & \\
3t^2 & & -\dfrac{e^{-st}}{s} \\
 & \ominus & \\
6t & & \dfrac{e^{-st}}{s^2} \\
 & \oplus & \\
6 & & -\dfrac{e^{-st}}{s^3} \\
 & \ominus & \\
0 & & \dfrac{e^{-st}}{s^4}
\end{array}$$

$$\mathcal{L}(\sin(at)) = \int_0^\infty e^{-st}\sin(at)\,dt$$

$u = \sin(at)$ $\quad dv = e^{-st}dt$

$du = a\cos(at)$ $\quad v = -\frac{e^{-st}}{s}$

$$= -\frac{e^{-st}\sin(at)}{s}\Bigg|_{t=0}^{t=\infty} + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt$$

$$= \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt$$

$u = \cos(at)$ $\quad dv = e^{-st}dt$

$du = -a\sin(at)$ $\quad v = -\frac{e^{-st}}{s}$

$$= \frac{a}{s}\left[-\frac{e^{-st}\cos(at)}{s}\Bigg|_0^\infty - \int_0^\infty \frac{a}{s}\sin(at)e^{-st}\,dt\right]$$

$$= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\mathcal{L}(\sin(at))\right]$$

$$= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}(\sin(at))$$

$$\overbrace{\left(1+\frac{a^2}{s^2}\right)}^{\frac{a^2+s^2}{s^2}} \mathcal{L}(\sin(at)) = \frac{a}{s^2}$$

$$\Longrightarrow \mathcal{L}(\sin(at)) = \frac{a}{s^2} \cdot \frac{s^2}{a^2+s^2} = \frac{a}{a^2+s^2}$$

**#8** $y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Apply $\mathcal{L}$:

$$\underbrace{-y'(0) - sy(0) + s^2 Y}\ \underbrace{+\,y(0) - sY}\ \underbrace{-\,6Y} = 0$$

$$1 - s + 1 + (s^2 - s - 6)Y = 0$$

$$(s^2 - s - 6)Y = s - 2$$

$$\boxed{\begin{aligned} &s-2 = A(s+2) + B(s-3) \\ &s=3: \quad A = \tfrac{1}{5} \\ &s=-2: \quad B = \tfrac{4}{5} \end{aligned}}$$

$$Y = \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s-3)(s+2)}$$

$$= \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2) + B(s-3)}{s^2 - s - 6}$$

Thus:

$$Y = \frac{\frac{1}{5}}{s-3} + \frac{\frac{4}{5}}{s+2}$$

$$\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}$$

**#10** $y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

$$-1 + s^2 Y - 2sY + 2Y = 0$$

$$(s^2 - 2s + 2)Y = 1$$

$$Y(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}$$

$$\xrightarrow[\text{table}]{\text{via}} \underline{\underline{y(t) = e^t \sin t}}$$