must diverge like the log of the log, which Euler writes [E4] as

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots=\log(\log\infty). \tag{2}$$

It is not clear exactly what Euler understood this equation to mean—if indeed he understood it as anything other than a mnemonic—but an obvious interpretation of it would be

$$\sum_{p<x} 1/p \sim \log(\log x) \qquad (x\to\infty), \tag{2'}$$

where the left side denotes the sum of $1/p$ over all primes p less than x and where the sign $\sim$ means that the relative error is arbitrarily small for x sufficiently large or, what is the same, that the ratio of the two sides approaches one as $x\to\infty$. Now

$$\log(\log x)=\int_1^{\log x}\frac{du}{u}=\int_e^x\frac{dv}{v\log v}$$

so (2′) says that the integral of $1/v$ relative to the measure $dv/\log v$ diverges in the same way as the integral of $1/v$ relative to the point measure which assigns weight 1 to primes and weight 0 to all other points. In this sense (2′) can be regarded as saying that the density of primes is roughly $1/\log v$. However, there is no evidence that Euler thought about the density of primes, and his methods were not adequate to prove the formulation (2′) of his statement (2).

Gauss states† in a letter [G2] written in 1849 that he had observed as early as 1792 or 1793 that the density of prime numbers appears on the average to be $1/\log x$ and he says that each new tabulation of primes which was published in the ensuing years had tended to confirm his belief in the accuracy of this approximation. However, he does not mention Euler's formula (2) and he gives no analytical basis for the approximation, which he presents solely as an empirical observation. He gives, in particular, Table I.

TABLE I[a]

x	Count of primes $< x$	$\int \frac{dn}{\log n}$	Difference
500,000	41,556	41,606.4	50.4
1,000,000	78,501	78,627.5	126.5
1,500,000	114,112	114,263.1	151.1
2,000,000	148,883	149,054.8	171.8
2,500,000	183,016	183,245.0	229.0
3,000,000	216,745	216,970.6	225.6

[a]From Gauss [G2].

†For some corroboration of Gauss's claim see his collected works [G3].

Gauss does not say exactly what he means by the symbol $\int(dn/\log n)$, but the data given in Table II, taken from D.N. Lehmer [L9], would indicate that he means n to be a continuous variable integrated from 2 to x, that is, $\int_2^x (dt/\log t)$. Note that Lehmer's count† of primes, which can safely be assumed to be accurate, differs from Gauss's information and that the difference is in *favor* of Gauss's estimate for the larger values of x.

TABLE II[a]

x	Count of primes $< x$	$\int_2^x \frac{dt}{\log t}$	Difference
500,000	41,538	41,606	68
1,000,000	78,498	78,628	130
1,500,000	114,155	114,263	108
2,000,000	148,933	149,055	122
2,500,000	183,072	183,245	173
3,000,000	216,816	216,971	155

[a]Data from Lehmer [L9].

Around 1800 Legendre published in his *Theorie des Nombres* [L11] an empirical formula for the number of primes less than a given value which amounted more or less to the same statement, namely, that the density of primes is $1/\log x$. Although Legendre made some slight attempt to prove his formula, his argument amounts to nothing more than the statement that if the density of primes is assumed to have the form

$$1/(A_1x^{m_1}+A_2x^{m_2}+\cdots)$$

where $m_1>m_2>\ldots$, then m_1 cannot be positive [because then the sum (1) would converge]; hence m_1 must be "infinitely small" and the density must be of the form

$$1/(A\log x+B).$$

He then determines A and B empirically. Legendre's formula was well known in the mathematical world and was mentioned prominently by Abel [A2], Dirichlet [D3], and Chebyshev [C2] during the period 1800–1850.

The first significant results beyond Euler's were obtained by Chebyshev around 1850. Chebyshev proved that the relative error in the approximation

$$\pi(x)\sim\int_2^x\frac{dt}{\log t}, \tag{3}$$

†Lehmer insists on counting 1 as a prime. To conform to common usage his counts have therefore been reduced by one in Table II.