7.6 #3 $\quad x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$

$x = e^{\lambda t}\vec{v} \Longrightarrow$ Need $\det \begin{pmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 + 2\lambda + 2 = 0$

$\Longrightarrow \lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = \underline{\underline{-1 \pm i}}$

$\underline{\lambda = -1+i}$:

$$\begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow (2-i)v_1 - v_2 = 0 \Longrightarrow v_2 = (2-i)v_1$, $\underline{\underline{v = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}}}$

$\Longrightarrow$ solution : $e^{\lambda t}\vec{v} = e^{(-1+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} = e^{-t}e^{it}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$$\Longrightarrow e^{\lambda t}\vec{v} = e^{-t}(\cos t + i\sin t)\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}(\cos t + i\sin t) \\ e^{-t}(\cos t + i\sin t)(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos t + i(e^{-t}\sin t) \\ e^{-t}(2\cos t + 2i\sin t - i\cos t + \sin t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos t \\ e^{-t}(2\cos t + \sin t) \end{pmatrix} + i\begin{pmatrix} e^{-t}\sin t \\ e^{-t}(2\sin t - \cos t) \end{pmatrix}$$

$$= \underbrace{e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}} + i\underbrace{e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}}$$

These are the fundamental solutions...

$\Longrightarrow$ general solution : $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$

<u>#4</u> $x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x$

$x = e^{\lambda t}\vec{v} \Longrightarrow \det\begin{pmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 + 9 = 0$

$\Longrightarrow \underline{\lambda = \pm 3i}$

$\underline{\lambda = 3i}$: $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow \left.\begin{aligned} (1-3i)v_1 + 2v_2 &= 0 \\ -5v_1 + (-1-3i)v_2 &= 0 \end{aligned}\right\}$$

We can use <u>either</u> equation — they lead to solutions that appear different!

1st equation: $(1-3i)v_1 + 2v_2 = 0 \Longrightarrow v_2 = \left(\frac{3i-1}{2}\right)v_1$

$$\Longrightarrow v = \begin{pmatrix} 2 \\ 3i-1 \end{pmatrix}$$

2nd equation: $-5v_1 + (-1-3i)v_2 = 0 \Longrightarrow v_1 = \frac{-1-3i}{5} v_2$

$$\Longrightarrow v = \begin{pmatrix} -1-3i \\ 5 \end{pmatrix}.$$

These eigenvectors might <u>appear</u> different, but they lead to <u>equivalent</u> solutions!!

Here's one...

$$e^{\lambda t}\vec{v} = e^{3it}\begin{pmatrix}-1-3i\\5\end{pmatrix} = (\cos 3t + i\sin 3t)\begin{pmatrix}-1-3i\\5\end{pmatrix}$$

$$= \begin{pmatrix}-\cos 3t + 3\sin 3t - i\sin 3t - 3i\cos 3t\\ 5\cos 3t + 5i\sin 3t\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}-\cos 3t + 3\sin 3t\\ 5\cos 3t\end{pmatrix}} + i\underbrace{\begin{pmatrix}-\sin 3t - 3\cos 3t\\ 5\sin 3t\end{pmatrix}}$$

These are the fundamental sol'ns.

$\Longrightarrow$ general solution: $x(t) = c_1\begin{pmatrix}-\cos 3t + 3\sin 3t\\ 5\cos 3t\end{pmatrix} + c_2\begin{pmatrix}-\sin 3t - 3\cos 3t\\ 5\sin 3t\end{pmatrix}$

#7

$$x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x \Longrightarrow \det \begin{pmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow \lambda^2 + 2\lambda + 2 = 0 \Longrightarrow \underline{\underline{\lambda = -1 \pm i}} \quad \left(\begin{matrix} \text{just} \\ \text{like} \\ \#3 \end{matrix}\right)$$

$\underline{\lambda = -1+i}$:
$$\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow v_1 - (2+i)v_2 = 0 \Longrightarrow \underline{\underline{v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}}}$$

$$e^{\lambda t}\vec{v} = e^{(-1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{-t}(\cos t + i \sin t) \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$= \underbrace{e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}} + i\underbrace{e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}}$$

Fundamental sol'ns

$\Longrightarrow$ general solution: $\quad x(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{aligned} 2c_1 + c_2 &= 1 \\ c_1 &= 1 \end{aligned} \Longrightarrow c_1 = 1,\ c_2 = -1,$$

solution is $\quad x(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

#1 $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x$

At $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$: $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$

$x = e^{\lambda t}\vec{v} \Longrightarrow \det\begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)^2 = 0 \Longrightarrow \underline{\underline{\lambda = 1}}$

v: $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow v_1 - 2v_2 = 0 \Longrightarrow \underline{\underline{v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$

$\Longrightarrow$ <u>one</u> solution: $e^t\vec{v} = \underline{\underline{e^t\begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$

<u>second solution</u>: $\vec{x} = e^{\lambda t}\vec{v} + te^{\lambda t}\vec{u}$ with $\lambda = 1$

$\Longrightarrow \underline{\underline{x = e^t\vec{v} + te^t\vec{u}}}$

$$x' = e^t\vec{v} + e^t\vec{u} + te^t\vec{u} = A\left(e^t\vec{v} + te^t\vec{u}\right)$$

$$\implies v + u + \underline{t\vec{u}} = A\vec{v} + \underline{tA\vec{u}}$$

$$\implies Au = u \implies \underline{u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$\implies Av = v + u \implies (A - I)v = u$$

$$\implies (A - I)v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\implies v_1 - 2v_2 = 1 \implies \underline{v_1 = 2v_2 + 1}$$

$$v = \begin{pmatrix} 2v_2 + 1 \\ v_2 \end{pmatrix} = v_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

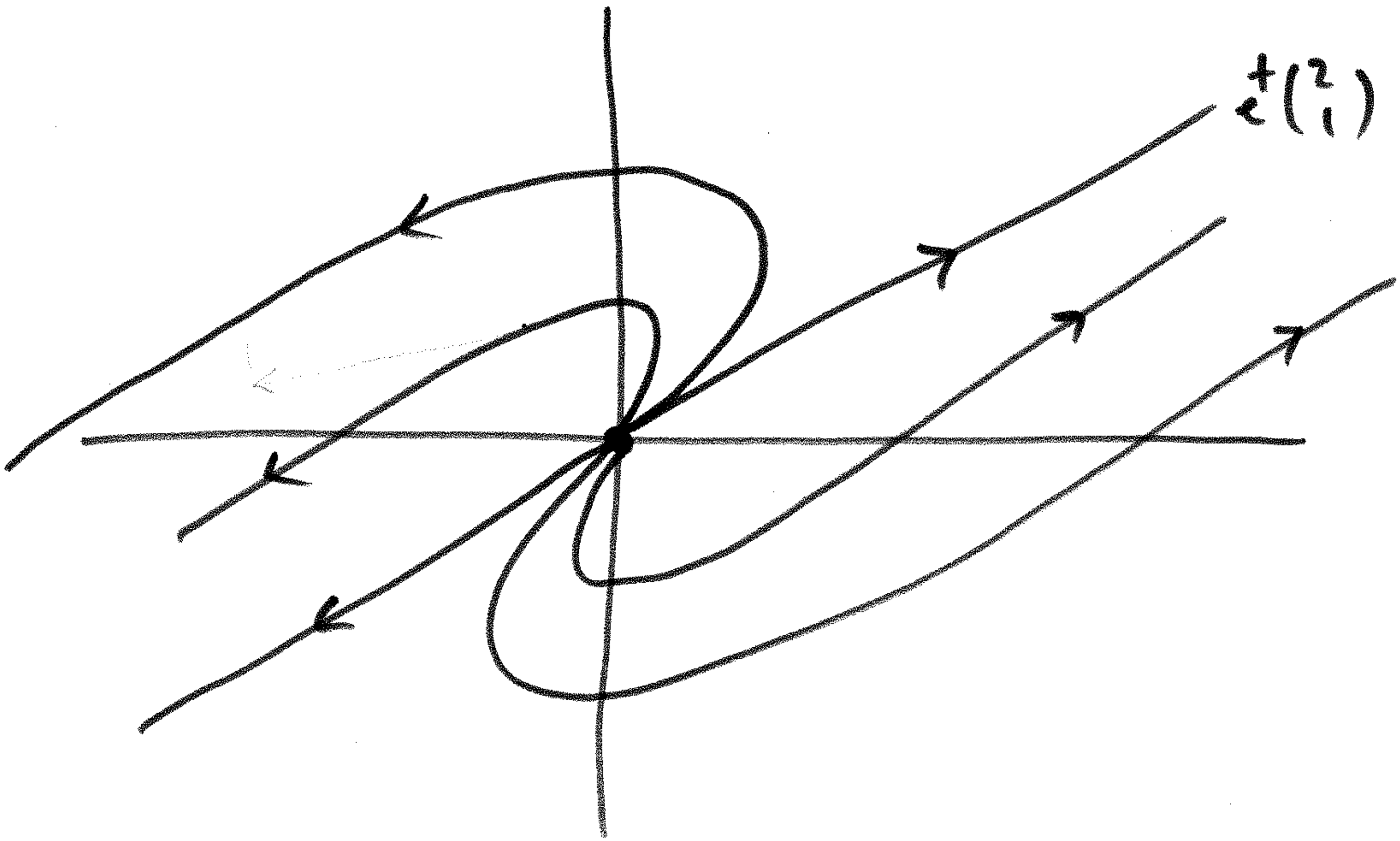
<u>other sol'n:</u>

$$e^t\left[\cancel{v_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] + te^t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\longrightarrow$ other sol'n is

$$\underline{te^t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t}$$

$$\Longrightarrow \vec{x} = c_1 e^{t}\begin{pmatrix}2\\1\end{pmatrix} + c_2\left(e^{t}\begin{pmatrix}1\\0\end{pmatrix} + te^{t}\begin{pmatrix}2\\1\end{pmatrix}\right)$$ General solution



#2 $x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x$

$\Longrightarrow \det\begin{pmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 = 0 \Longrightarrow \lambda = 0$

$\Longrightarrow v$: $\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow 4v_1 - 2v_2 = 0$

$\Longrightarrow 2v_1 = v_2, \; v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

one sol'n: $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(= e^{\lambda t} u + t e^{\lambda t} v$ with $\lambda = 0)$

other sol'n: $x = u + tv \Longrightarrow x' = v$

$$\Longrightarrow v = A(u+tv) \Longrightarrow v = Au + tAv$$

$$\Longrightarrow Av = 0 \Longrightarrow v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow v = Au \quad \text{, i.e.,}$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow 4u_1 - 2u_2 = 1 \Longrightarrow 2u_2 = 4u_1 - 1$$

$$\Longrightarrow u_2 = 2u_1 - \frac{1}{2}$$

$$x = u + tv \Longrightarrow x = \begin{pmatrix} u_1 \\ 2u_1 - \frac{1}{2} \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= u_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow \; x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

General sol'n