$\underline{y'' + y = 0}$, $y = y(t)$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = \underline{a_0 + a_1 t + a_2 t^2 + \dots}$$

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + \dots = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = 2a_2 + 6a_3 t + 12a_4 t^2 + \dots$$

Then get $\underbrace{\left(2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + \dots\right)}_{y''} + \left(a_0 + a_1 t + a_2 t^2 + \dots\right) = 0$

$$\Longrightarrow \underbrace{(2a_2 + a_0)} + \underbrace{(6a_3 + a_1)}t + \underbrace{(12a_4 + a_2)}t^2 + \underbrace{(20a_5 + a_3)}t^3 + \dots = 0$$

$$\Longrightarrow 2a_2 + a_0 = 0, \quad 6a_3 + a_1 = 0, \quad 12a_4 + a_2 = 0, \quad 20a_5 + a_3 = 0, \dots$$

$$\Longrightarrow a_2 = \frac{-a_0}{2}, \quad a_3 = \frac{-a_1}{6}, \quad a_4 = \frac{-a_2}{12} = \frac{-\left(\frac{-a_0}{2}\right)}{12} = \frac{+a_0}{24}$$

$$a_5 = \frac{-a_3}{20} = \frac{-\left(-\frac{a_1}{6}\right)}{20} = \frac{a_1}{120}, \ \ldots$$

$$\underline{a_2 = -\frac{1}{2}a_0}, \quad \underline{a_3 = -\frac{1}{6}a_1}, \quad \underline{a_4 = \frac{1}{24}a_0}, \quad \underline{a_5 = \frac{1}{120}a_1}, \ \ldots$$

$$y(t) = \left(a_0 + a_2t^2 + a_4t^4 + \ldots\right) + \left(a_1t + a_3t^3 + a_5t^5 + \ldots\right)$$

$$= \left(a_0 - \frac{1}{2}a_0t^2 + \frac{1}{24}a_0t^4 + \ldots\right) + \left(a_1t - \frac{1}{6}a_1t^3 + \frac{1}{120}a_1t^5 + \ldots\right)$$

$$= a_0\underbrace{\left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \ldots\right)} + a_1\underbrace{\left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \ldots\right)}$$

$$= a_0 \cos t + a_1 \sin t$$

$$\Longrightarrow \underline{y(t) = a_0 \cos t + a_1 \sin t}$$

$$y'' + y = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}}_{y''} + \underbrace{\sum_{n=0}^{\infty} a_n t^n}_{y} = 0$$

$$m = n-2 \Longrightarrow n = m+2$$
$$n-1 = m+1$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} \left[\underline{(n+2)(n+1) a_{n+2} + a_n} \right] t^n = 0$$

$$\Longrightarrow \boxed{\begin{array}{c} (n+2)(n+1) a_{n+2} + a_n = 0, \quad n = 0, 1, 2, 3, \ldots \\ \text{recurrence relation} \end{array}}$$

$$\Longrightarrow \quad \underbrace{a_{n+2} = \frac{-a_n}{(n+2)(n+1)}}, \quad n = 0, 1, 2, 3, \ldots$$

$$\Longrightarrow \quad a_n = \frac{(-1)^n}{n!}$$

#3 $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = ?$

$$\sum_{n=0}^{\infty} ar^n = a + ar + \dots = \frac{a}{1-r}, \quad |r| < 1$$

Ratio test:

$$\lim_{n\to\infty} \left(\underbrace{\frac{|x|^{2(n+1)}}{(n+1)!}}_{(n+1)^{st} \text{ term}} \cdot \underbrace{\frac{n!}{|x|^{2n}}}_{n^{th} \text{ term}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{|x|^{2n+2}}{(n+1)!} \cdot \frac{n!}{|x|^{2n}} \right)$$

$$= \lim_{n\to\infty} \frac{|x|^2}{n+1} = |x|^2 \left(\lim_{n\to\infty} \frac{1}{n+1} \right) = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \text{ converges for all } x.$$

#5 $\displaystyle\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n}$

$$\lim_{n\to\infty}\left(\frac{|x-x_0|^{n+1}}{n+1}\right)\left(\frac{n}{|x-x_0|^n}\right) = \lim_{n\to\infty}\left(\frac{|x-x_0|\cdot n}{n+1}\right)$$

$$= |x-x_0| \underbrace{\lim_{n\to\infty}\left|\frac{n}{n+1}\right|}_{1}$$

$$= |x-x_0|$$

$\Longrightarrow \displaystyle\sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n}$ converges when $|x-x_0|<1$,

diverges when $|x-x_0|>1$.

? $|x-x_0|=1$? $x-x_0=1 \Longrightarrow$ diverges

$x-x_0=-1 \Longrightarrow$ converges

#7 $\underline{\sin x}$, $x_0 = 0$:

$$\sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$x = 0 \Longrightarrow \underline{a_0 = 0}$$

Differentiate:

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$x = 0 \Longrightarrow \underline{1 = a_1}$$

$\underline{\text{Diff.}}$:

$$-\sin x = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots$$

$$x = 0 \Longrightarrow a_2 = 0$$

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

$\underline{\text{Diff}}$:

$$-\cos x = 6a_3 + 24a_4 x + \dots$$

$$x = 0 \Longrightarrow -1 = 6a_3 \Longrightarrow a_3 = \frac{-1}{6} = \frac{-1}{3!}$$