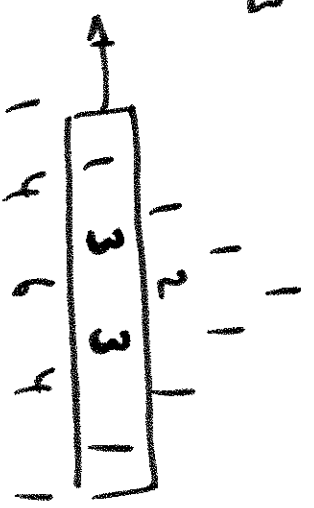


31b)

$$(a+b)^3 = a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$



etc.

$$n \in \mathbb{N} \implies n = 3k$$

or

$$n = 3k+1$$

or

$$n = 3k+2$$

$$(3k)^3 = 27k^3$$

$$= 9m$$

$$m = 3k^3$$

$$(3k+1)^3 =$$

$$27k^3 + 27k^2 +$$

$$9k + 1$$

$$= 9m+1,$$

$$m = 3k^3 + 3k^2 + k$$

$$(3k+2)^3 =$$

$$27k^3 + 54k^2 +$$

$$36k + 8$$

$$= 9(3k^3 + 6k^2 + 4k)$$

$$+ 8$$

$$n \in \mathbb{N} \implies n = 3k$$

or

$$n = 3k+1$$

or

$$n = 3k-1$$

$$n^3 \equiv 0 \pmod{9}$$

$$n^3 \equiv 1 \pmod{9}$$

$$n^3 \equiv -1 \pmod{9}$$

Diophantine equations

$$\boxed{2x + 3y = 17} \quad \text{yes!}$$

$$\boxed{4x + 6y = 25} \quad \text{No!}$$

$$\boxed{4x + 6y = 24} \quad \text{yes!}$$

$$\boxed{5x + 11y = 173} \quad \text{yes!}$$

$$(x=28, y=3)$$

Are there integer

solutions?

$$(x, y \in \mathbb{Z})$$

$$\underline{\underline{ax + by = c}}, \quad a, b, c \in \mathbb{Z}$$
$$\underline{\underline{a \in \mathbb{N}}}$$

$a, b \in \mathbb{N}$

Defn a is a divisor of b iff $b = ac$ for some ~~number~~ $c \in \mathbb{N}$. Notation: $a|b$.

Note: $a \nmid b$ means that a does not divide b .

Defn $a, b \in \mathbb{N}$. The greatest common divisor of a and b , denoted $\gcd(a, b)$, is an ~~int~~ integer $d \in \mathbb{N}$ satisfying

(1) $d|a$ and $d|b$, and

(2) $c \leq d$ for any divisor c of a and b .

n	Divisors of n
12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
15	1, 3, 5, 15
18	1, 2, 3, 6, 9, 18
48	1, 2, 3, 4, 6, 8, 12, 16, 24, 48

$$3 = (-1) \cdot 12 + 1 \cdot 15$$

$$\swarrow \quad \text{gcd}(12, 15) = 3$$

$$\text{gcd}(12, 18) = 6 \rightarrow 6 = (-1) \cdot 12 + (1) \cdot 18$$

$$\text{gcd}(18, 48) = 6$$

$$\swarrow \quad 6 = (3) \cdot 18 + (-1) \cdot 48$$

Theorem If $d = \gcd(a, b)$, then $\exists x, y \in \mathbb{Z}$

$$\text{s.t. } d = ax + by.$$

pf: Define $S = \{ax + by \mid x, y \in \mathbb{Z}, ax + by > 0\}$.

S nonempty since $a \cdot \left(\frac{a}{|a|}\right) > 0$.

Thus, by the W.O.P., S has a minimal element, d .

Then $d = ax + by$ for some integers x, y .

To see that $d = \gcd(a, b)$, apply the Division Lemma:

$$a = qd + r, \text{ with } 0 \leq r < d.$$

$$\begin{aligned} \Rightarrow r &= a - qd = a - q(ax + by) \\ &= a - qxa - qyb \end{aligned}$$

$$= (1 - qx)a + (-qy)b \in S \text{ if } r > 0,$$

can't happen!

Thus, $r=0$ & $a=qd$, i.e., $d|a$.

Similarly, $d|b$.

If c is any common divisor of a and b , then $c|a+br$ for any $u, v \in \mathbb{Z}$.

Thus, if c is positive, $c|a+br$, i.e., $c|d$, and $c \leq d$.

□