<u>Arithmetic functions</u>

$\tau(n)$ = number of divisors of n, $n \in \mathbb{N}$

$\sigma(n)$ = sum of divisors of n

n	divisors	$\tau(n)$	$\sigma(n)$
1	1	1	1
2	1, 2	2	3
3	1, 3	2	4
4	1, 2, 4	3	7
5	1, 5	2	6
6	1, 2, 3, 6	4	12

* if n is prime, then $\tau(n) = 2$ and $\sigma(n) = n+1$.

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* $\sigma(6) = 12$, so its proper divisors add up to 6. 6 is a <u>perfect number</u>.

$$\tau(n) = \sum_{d \mid n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d \mid n} d .$$

$n>1$

$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ for primes $p_1, \dots, p_r$ and exponents $k_1, \dots, k_r$. $(k_i \geq 1)$

Divisors of n: $d = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}$ for $0 \leq j_i \leq k_i$.

Then $\tau(n) = (k_1+1)(k_2+1) \cdot \ldots \cdot (k_r+1) = \prod_{i=1}^{r} (k_i+1)$

example: $n = 60 = 2^2 \cdot 3 \cdot 5$

$\tau(n) = 3 \cdot 2 \cdot 2 = 12$

$$\sigma(n) = (1 + p_1 + p_1^2 + \dots + p_1^{k_1})(1 + p_2 + p_2^2 + \dots + p_2^{k_2}) \times \dots$$

$$\times (1 + p_r + p_r^2 + \dots + p_r^{k_r})$$

$$= \left(\frac{p_1^{k_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{k_2+1} - 1}{p_2 - 1}\right) \times \dots \times \left(\frac{p_r^{k_r+1} - 1}{p_r - 1}\right)$$

$$= \prod_{i=1}^{r} \left(\frac{p_i^{k_i+1} - 1}{p_i - 1}\right).$$

$$1 + p + p^2 + \dots + p^k = S$$

$$p + p^2 + \dots + p^k + p^{k+1} = pS$$

$$p^{k+1} - 1 = (p-1)S$$

$$\frac{p^{k+1} - 1}{p - 1} = S$$

example: $n = 60$

$= 2^2 \cdot 3 \cdot 5$

$$\sigma(60) = \left(\frac{2^3 - 1}{2 - 1}\right)\left(\frac{3^2 - 1}{3 - 1}\right)\left(\frac{5^2 - 1}{5 - 1}\right)$$

$$= (7)(4)(6)$$

$$= 168.$$

$n = 4, \quad m = 6, \quad mn = 24 = \underline{2^2 \cdot 2 \cdot 3} = \underline{2^3 \cdot 3}$

$\tau(4) = 3, \quad \tau(6) = 4$

$\tau(24) = 4 \cdot 2 = 8 \neq \tau(4) \cdot \tau(6)$

$\sigma(4) = 7, \quad \sigma(6) = 12, \quad \sigma(24) = \left(\frac{2^4 - 1}{2-1}\right)\left(\frac{3^2-1}{3-1}\right) = 15 \cdot 4 = 60$

$$\sigma(24) \neq \sigma(4) \cdot \sigma(6).$$

A function $f : \mathbb{N} \to \mathbb{N}$ is <u>multiplicative</u> if

$f(mn) = f(m) \cdot f(n)$, when m, n are relatively prime.

If $\gcd(m,n) = 1$, $\quad m = p_1^{k_1} \cdots p_r^{k_r}, \quad n = q_1^{j_1} \cdots q_s^{j_s}$

then $mn = p_1^{k_1} \cdots p_r^{k_r} \cdot q_1^{j_1} \cdots q_s^{j_s}$.

__Theorem__ τ and σ are multiplicative.

__pf.__: m, n are relatively prime, with
$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $n = q_1^{j_1} \cdots q_s^{j_s}$.

Then
$$\tau(mn) = \tau\left(p_1^{k_1} \cdots p_r^{k_r} \cdot q_1^{j_1} \cdots q_s^{j_s}\right)$$
$$= \underbrace{(k_1+1)\cdots(k_r+1)} \cdot \underbrace{(j_1+1)\cdots(j_s+1)}$$
$$= \tau(m) \cdot \tau(n).$$

and
$$\sigma(mn) = \underbrace{\left(\frac{p_1^{k_1+1}-1}{p_1-1}\right)\cdot \cdots \cdot \left(\frac{p_r^{k_r+1}-1}{p_r-1}\right)}\underbrace{\left(\frac{q_1^{j_1+1}-1}{q_1-1}\right)\cdots\left(\frac{q_s^{j_s+1}-1}{q_s-1}\right)}$$
$$= \sigma(m) \cdot \sigma(n).$$

∎

<u>FACT</u>: If f is multiplicative and

$$F(n) = \sum_{d \mid n} f(d),$$

then F is multiplicative.

<u>pf.</u>: Let m, n be relatively prime. Then

$$F(mn) = \sum_{d \mid mn} f(d) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$

$$= \left(\sum_{d_1 \mid m} f(d_1) \right) \left(\sum_{d_2 \mid n} f(d_2) \right)$$

$$= F(m) F(n). \quad \boxtimes$$

$$\Big(\prod_{d_1 | n} d_1\Big)\Big(\prod_{d_2 | n} d_2\Big) = n^{\tau(n)} \implies \Big(\prod_{d | n} d\Big) = n^{\frac{\tau(n)}{2}}$$

the same!

Arithmetic functions, cont'd

The following are multiplicative:

(1) $f(n) = 1$: $f(mn) = 1 = 1 \cdot 1 = f(m) f(n)$

(2) $f(n) = n$: $f(mn) = mn = f(m) f(n)$

(3) for $s \in \mathbb{R}$, $f(n) = n^s$: $f(mn) = (mn)^s = m^s n^s = f(m) f(n)$

Theorem If f is multiplicative, then so is

$$F(n) := \sum_{d \mid n} f(d).$$

pf: m, n s.t. $\gcd(m,n) = 1$. Then

$$F(mn) = \sum_{d \mid mn} f(d) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$

$$= \sum f(d_1) \sum f(m_2) = F(m)F(n)$$

Recall: $\tau(n) = \#$ divisors of n $= \sum_{d \mid n} 1 \implies \tau$ is multiplicative.

$\sigma(n) =$ sum of divisors $= \sum_{d \mid n} d \implies \sigma$ is multiplicative.

Möbius function $\mu(n) := \begin{cases} 1, & n = 1 \\ 0, & \text{if } p^2 \mid n \text{ for any prime } p \\ (-1)^r, & \text{if } n = p_1 p_2 \cdot \ldots \cdot p_r \\ & (r \text{ distinct primes}) \end{cases}$

FACT: μ is multiplicative: $m, n > 1$, $\gcd(m,n) = 1$

if either m or n is <u>not</u> square-free, $\mu(mn) = 0 = \mu(m)\mu(n)$

otherwise, $m = p_1 \cdots p_r$ & $n = q_1 \cdots q_s$ and

$\mu(mn) = (-1)^{r+s} = (-1)^r (-1)^s = \mu(m)\mu(n)$. $\boxtimes$

$\sum_{d|n} \mu(d)$: when $n = p^k$, have

$$\sum_{d|p^k} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \dots + \mu(p^k)$$

$$= 1 + (-1) + 0 + \dots + 0$$

$(n>1)$

$$= 0.$$

in general, $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and

$$\mu(p_1^{k_1} \cdots p_r^{k_r}) = \mu(p_1^{k_1}) \cdot \dots \cdot \mu(p_r^{k_r}) = 0.$$

$\underline{n=1}$: $\sum_{d|1} \mu(d) = \mu(1) = 1.$

Möbius inversion formula

If $F(n) := \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d}) \overset{\text{automatically}}{=} \sum_{d|n} \mu(\frac{n}{d}) F(d)$.

pf.:

$$\sum_{d|n} \mu(d) F(\tfrac{n}{d}) = \sum_{d|n} \mu(d) \Big(\sum_{c|\frac{n}{d}} f(c) \Big)$$

$$= \sum_{c|n} \sum_{d|\frac{n}{c}} \mu(d) f(c)$$

$$= \sum_{c|n} f(c) \sum_{d|\frac{n}{c}} \mu(d)$$

$\neq 0$ iff $c = n$

$$= f(n). \quad \boxtimes$$

NOTE:

$d|n$ & $c|\frac{n}{d}$ iff $c|n$ & $d|\frac{n}{c}$:

$n = dr$, $cs = \frac{n}{d} \Longleftrightarrow \underline{cds = n} \Longleftrightarrow c|n$

~~$cds = dr$~~

$ds = \frac{n}{c}$

Theorem Suppose that $F(n) = \sum_{d \mid n} f(d)$. $\boxed{\text{grok this!}}$

Then F is multiplicative iff f is multiplicative.

pf.: Remains to show that F mult. $\Longrightarrow$ f mult.

$$f(mn) = \sum_{d \mid mn} \mu(d) F\left(\frac{mn}{d}\right) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} \mu(d_1 d_2) F\left(\frac{mn}{d_1 d_2}\right)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} \mu(d_1)\mu(d_2) F\left(\frac{m}{d_1}\right) F\left(\frac{n}{d_2}\right)$$

$$= \left(\sum_{d_1 \mid m} \mu(d_1) F\left(\frac{m}{d_1}\right)\right)\left(\sum_{d_2 \mid n} \mu(d_2) F\left(\frac{n}{d_2}\right)\right)$$

$$= f(m) f(n). \quad \boxtimes$$

20 $\omega(n) = \#$ distinct prime divisors of n

(a) $2^{\omega(n)}$ is multiplicative:

$$n = p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_r^{k_r} \Longrightarrow \omega(n) = r$$

$$m = q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_s^{j_s} \Longrightarrow \omega(m) = s$$

$$\omega(mn) = r + s \qquad (\omega \text{ is additive})$$

$$2^{\omega(mn)} = 2^{r+s} = 2^r 2^s = 2^{\omega(n)} 2^{\omega(m)}. \quad \blacksquare$$

(b) $\tau(n^2) = \sum_{d|n} 2^{\omega(d)}$:

$$n = p^k \Longrightarrow \tau(n^2) = \tau(p^{2k}) = 2k+1 \text{ and}$$

$$\sum_{d|n} 2^{\omega(d)} = 2^{\omega(1)} + 2^{\omega(p)} + \ldots + 2^{\omega(p^k)}$$

$$= 2^0 + 2^1 + \ldots + 2^1 = 2k+1.$$

Then $n = p_1^{k_1} \cdots p_r^{k_r} \Longrightarrow \tau(p_1^{2k_1} \cdots p_r^{2k_r}) = \tau(p_1^{2k_1}) \cdots \tau(p_r^{2k_r}) = \sum_{d|n} 2^{\omega(d)}.$

Greatest Integer Function

$\lfloor x \rfloor :=$ largest integer $\leq x$

$\lfloor \pi \rfloor = 3$, $\lfloor e \rfloor = 2$, $\lfloor 7 \rfloor = 7$, $\lfloor 4.2 \rfloor = 4$

FACT Let n be an integer, let p be some prime, $p \leq n$. Then the power of p that divides $n!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

$p, 2p, 3p, \ldots, tp \leq n$, where $t = \left\lfloor \frac{n}{p} \right\rfloor$ ex: $n=9, p=2$ $\left\lfloor \frac{9}{2} \right\rfloor = 4$

$p^2, 2p^2, \ldots, tp^2 \leq n$, $t = \left\lfloor \frac{n}{p^2} \right\rfloor$

etc.

ex: Number of zeroes ~~in~~ at end 20! :

$$\sum_{k=1}^{\infty} \left\lfloor \frac{20}{2^k} \right\rfloor = \left\lfloor \frac{20}{2} \right\rfloor + \left\lfloor \frac{20}{4} \right\rfloor + \left\lfloor \frac{20}{8} \right\rfloor + \left\lfloor \frac{20}{16} \right\rfloor$$

$$= 10 + 5 + 2 + 1 = \underline{\underline{18}} \implies 2^{18} \mid 20!$$

$$\sum_{k=1}^{\infty} \left\lfloor \frac{20}{p^k} \right\rfloor = \left\lfloor \frac{20}{5} \right\rfloor$$

$$= 4 . \implies 5^4 \mid 20!$$

20! ~~has~~ ends in 4 zeroes.

$$\boxed{n! = \prod_{\text{primes } p} p^{\sum_{1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor}}$$