2.2 #13 $y' = xy^3(1+x^2)^{-1/2}$, $y(0)=1$

Separate $\Longrightarrow$ $y^{-3}y' = \dfrac{x}{\sqrt{1+x^2}}$

Integrate $\Longrightarrow$ $-\frac{1}{2}y^{-2} = \int \dfrac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{2}u^{-\frac{1}{2}}du = u^{\frac{1}{2}}+c$

$= \sqrt{1+x^2} + c$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\Longrightarrow$ $-\frac{1}{2}y^{-2} = \sqrt{1+x^2} + c$ $\qquad$ $y(0)=1 \Longrightarrow -\frac{1}{2} = 1+c$

$\Longrightarrow c = -\frac{3}{2}$

$\Longrightarrow$ $-\frac{1}{2}y^{-2} = \sqrt{1+x^2} - \frac{3}{2}$

$\Longrightarrow$ $y^{-2} = -2\sqrt{1+x^2} + 3$

$\Longrightarrow$ $y^2 = \dfrac{1}{3-2\sqrt{1+x^2}}$ $\longrightarrow$ $y(x) = \sqrt{\dfrac{1}{3-2\sqrt{1+x^2}}}$

#13, continued

$y(x) = \sqrt{\dfrac{1}{3-2\sqrt{1+x^2}}}$ is defined as long as $3-2\sqrt{1+x^2}$

remains <u>positive</u> $\Longrightarrow$ need $3-2\sqrt{1+x^2} > 0$

$$\Longrightarrow \quad 2\sqrt{1+x^2} < 3$$

$$\Longrightarrow \quad \sqrt{1+x^2} < \frac{3}{2}$$

$$\Longrightarrow \quad 1+x^2 < \frac{9}{4}$$

$$\Longrightarrow \quad x^2 < \frac{5}{4}$$

$$\Longrightarrow \quad -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}, \text{ i.e., } \underline{\underline{|x| < \frac{\sqrt{5}}{2}}}.$$

1^{st}-order ODEs :

^ linear!

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

$$\Longrightarrow \mu y' + \mu p y = \mu g$$

$$(\mu y)' = \mu y' + \mu' y$$

$$(\mu y)' = \mu g$$

Integrate: $\mu y = \int \mu g \, dt$

$$\Longrightarrow y(t) = \frac{1}{\mu} \int \mu g \, dt$$

$$\boxed{y(t) = e^{-\int p\,dt} \int \mu g \, dt}$$

Need $\mu' = \mu p$

$$\frac{\mu'}{\mu} = p$$

Integrate: $\ln|\mu| = \int p \, dt$

$$\mu(t) = e^{\int p\,dt}.$$

THE POINT OF THIS IS NOT THE FORMULA — THE TAKEAWAY IS THAT LINEAR 1^{st} order ODEs ARE "EASY" TO SOLVE.

Nonlinear ODEs

Example #1: $y' = y^2 \Longrightarrow \frac{1}{y^2}\frac{dy}{dx} = 1$

Integrate: $-\frac{1}{y} = x + c$

$\longrightarrow \frac{1}{y} = -x + c$

$\longrightarrow y = \frac{1}{-x+c}$ Finite-time blow-up!

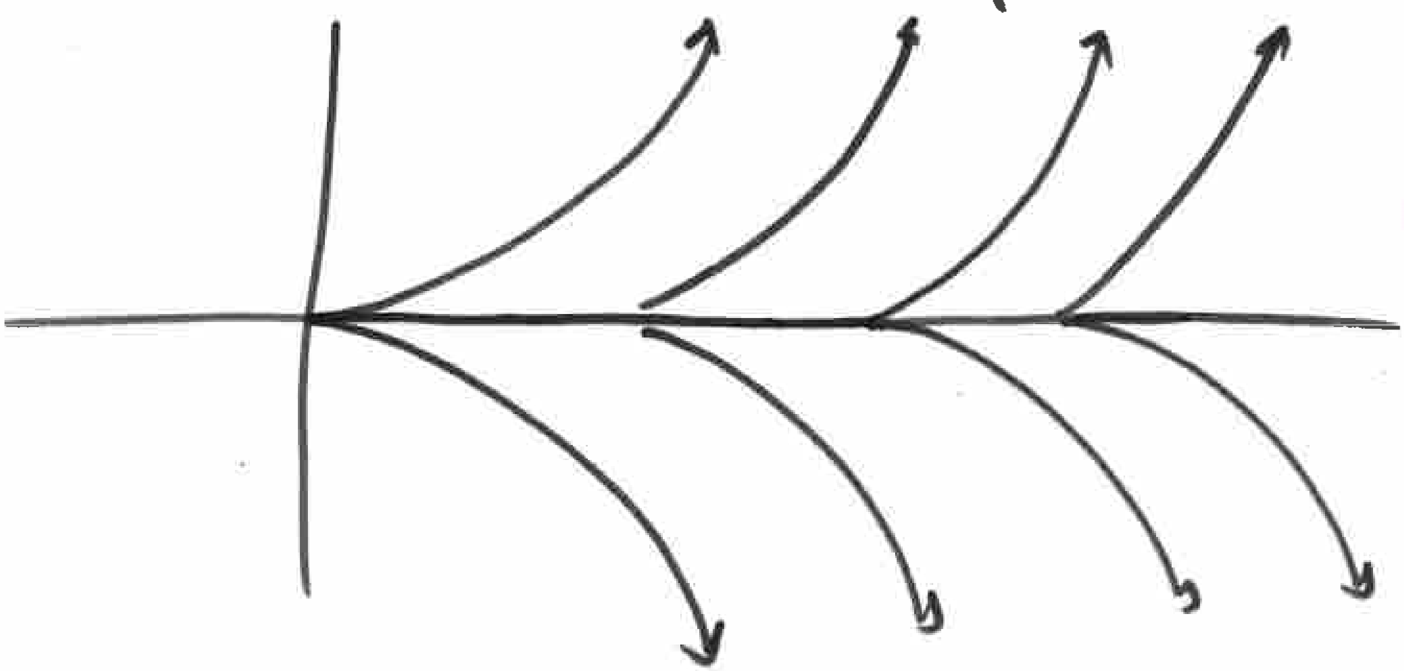
Example #2 $\quad y' = y^{\frac{1}{3}} = \sqrt[3]{y} \quad , \; y = y(t) \; ; \; y(0) = 0$

$$\Longrightarrow y^{-\frac{1}{3}} y' = 1$$

Integrate: $\frac{3}{2} y^{\frac{2}{3}} = t + c \Longrightarrow c = 0$

$$\frac{3}{2} y^{2/3} = t \Longrightarrow y^{2/3} = \frac{2}{3} t$$

$$\Longrightarrow y(t) = \pm \left(\frac{2}{3} t \right)^{3/2}$$



NONUNIQUENESS OF SOLUTIONS!

Bernoulli equations $\quad y' + py = qy^n \quad , n \neq 0, 1$

$v = y^{1-n}$
$vy^n = y$ $\Longleftarrow$ Use $v = y^{1-n}$: $v' = (1-n)y^{-n}y'$

$\Longrightarrow y' = \frac{1}{1-n} v' y^n$

$\Longrightarrow \frac{1}{1-n} v' y^n + pvy^n = qy^n$

See problem #23, Section 2.4

$\Longrightarrow \frac{1}{1-n} v' + pv = q$

$\Longrightarrow \boxed{v' + (1-n)pv = (1-n)q}$

Linear 1st order ODE for v !

#24 $\quad y' = ry - ky^2 \Longrightarrow y' - ry = -ky^2$

$p(t) = -r, \ q(t) = -k, \ n = 2$

$\Longrightarrow v' + rv = k$