Quiz 9

Name: SOLUTIONS

1. Define the homomorphisms $g\colon \mathbb{R}^3 \longrightarrow \mathcal{P}_1$ and $h\colon \mathcal{M}_{2\times 2} \longrightarrow \mathbb{R}^3$ as follows:

$$g\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (2a+b)\,x + (b-c) \qquad \text{and} \qquad h\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + 3\delta \\ \beta - \gamma \\ 2\delta - \beta \end{pmatrix}$$

(a) Compute the matrix representations of g and h using the standard bases of $\mathbb{R}^3$, $\mathcal{P}_1$, and $\mathcal{M}_{2\times 2}$.

(b) Use the matrix representations from part (a) to compute the matrix representation of the composition of g and h. (There is only one way to compose these two homomorphisms!)

(c) Compute the matrix representations of g and h using the standard basis for $\mathcal{M}_{2\times 2}$ and the following bases for $\mathbb{R}^3$ and $\mathcal{P}_1$.

$$\mathbb{R}^3 : \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{P}_1 : \{\, 1-x, 1+x \,\}$$

(d) Use the matrix representations from part (c) to compute the matrix representation of the composition of g and h with respect to the given bases.

(a) $g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2x = 0 \cdot 1 + 2 \cdot x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

$g\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x + 1 = 1 \cdot 1 + 1 \cdot x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

$g\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1 = -1 \cdot 1 + 0 \cdot x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Thus, $[g] = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$.

$$h\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$h\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} = 0\cdot e_1 + 1\cdot e_2 + (-1)\cdot e_3 = \begin{bmatrix}0\\1\\-1\end{bmatrix},$$

$$h\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix} = 0\cdot e_1 + (-1)e_2 + 0\cdot e_3 = \begin{bmatrix}0\\-1\\0\end{bmatrix},$$

and $h\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix} = 3e_1 + 0e_2 + 2e_3 = \begin{bmatrix}3\\0\\2\end{bmatrix}.$

Thus, $[h] = \begin{bmatrix}1 & 0 & 0 & 3\\ 0 & 1 & -1 & 0\\ 0 & -1 & 0 & 2\end{bmatrix}.$

(b) $g\circ h : M_{2\times 2} \longrightarrow \mathcal{P}_1$; $h\circ g$ is $\underline{\underline{\text{not}}}$ defined.

The matrix representation of $g\circ h$ is

$$[g\circ h] = [g][h] = \begin{bmatrix}0 & 1 & -1\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 3\\ 0 & 1 & -1 & 0\\ 0 & -1 & 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 2 & -1 & -2\\ 2 & 1 & -1 & 6\end{bmatrix}$$

(c) $g\begin{pmatrix}1\\1\\0\end{pmatrix} = 3x+1 = -1(1-x)+2(1+x) = \begin{bmatrix}-1\\2\end{bmatrix}$,

$g\begin{pmatrix}0\\1\\1\end{pmatrix} = x = -\frac{1}{2}(1-x)+\frac{1}{2}(1+x) = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$,

and $g\begin{pmatrix}1\\0\\1\end{pmatrix} = 2x-1 = -\frac{3}{2}(1-x)+\frac{1}{2}(1+x) = \begin{bmatrix}-\frac{3}{2}\\\frac{1}{2}\end{bmatrix}$

Thus, $[g] = \begin{bmatrix}-1 & -\frac{1}{2} & -\frac{3}{2}\\ 2 & +\frac{1}{2} & \frac{1}{2}\end{bmatrix}$.

$h\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$

$h\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\1\end{pmatrix} + (-1)\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$h\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \left(-\frac{1}{2}\right)\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$

and

$h\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{5}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{5}{2}\end{bmatrix}$

Thus, $[h] = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & \frac{5}{2} \end{bmatrix}.$

(d) The matrix representation for $g \circ h$ is

$$[g][h] = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{3}{2} \\ 2 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} & 0 & -4 \\ 1 & \frac{3}{2} & -1 & 2 \end{bmatrix}.$$