Thus, if $(\mu M)_y$ is to equal $(\mu N)_x$, it is necessary that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{27}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (27), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (23) has an integrating factor depending only on y; see Problem 23.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{19}$$

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{28}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{29}$$

Hence

$$\mu(x) = x. \tag{30}$$

Multiplying Eq. (19) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{31}$$

The latter equation is exact, and its solutions are given implicitly by

$$x^3y + \tfrac{1}{2}x^2y^2 = c. \tag{32}$$

Solutions may also be found in explicit form since Eq. (32) is quadratic in y.

You may also verify that a second integrating factor for Eq. (19) is

$$\mu(x,y) = \frac{1}{xy(2x + y)},$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. $(2x + 3) + (2y - 2)y' = 0$
2. $(2x + 4y) + (2x - 2y)y' = 0$
3. $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$
4. $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$

5. $\dfrac{dy}{dx} = -\dfrac{ax + by}{bx + cy}$ 6. $\dfrac{dy}{dx} = -\dfrac{ax - by}{bx - cy}$

7. $(e^x \sin y - 2y \sin x)\, dx + (e^x \cos y + 2 \cos x)\, dy = 0$

8. $(e^x \sin y + 3y)\, dx - (3x - e^x \sin y)\, dy = 0$

9. $(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x)\, dx + (xe^{xy} \cos 2x - 3)\, dy = 0$

10. $(y/x + 6x)\, dx + (\ln x - 2)\, dy = 0, \qquad x > 0$

11. $(x \ln y + xy)\, dx + (y \ln x + xy)\, dy = 0; \qquad x > 0, \quad y > 0$

12. $\dfrac{x\, dx}{(x^2 + y^2)^{3/2}} + \dfrac{y\, dy}{(x^2 + y^2)^{3/2}} = 0$

In each of Problems 13 and 14 solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x - y)\, dx + (2y - x)\, dy = 0, \qquad y(1) = 3$

14. $(9x^2 + y - 1)\, dx - (4y - x)\, dy = 0, \qquad y(1) = 0$

In each of Problems 15 and 16 find the value of b for which the given equation is exact, and then solve it using that value of b.

15. $(xy^2 + bx^2 y)\, dx + (x + y)x^2\, dy = 0$

16. $(ye^{2xy} + x)\, dx + bxe^{2xy}\, dy = 0$

17. Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x, y)$ is

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)\, ds + \int_{y_0}^{y} N(x, t)\, dt,$$

where (x_0, y_0) is a point in R.

18. Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

In each of Problems 19 through 22 show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

19. $x^2 y^3 + x(1 + y^2)y' = 0, \qquad \mu(x, y) = 1/xy^3$

20. $\left(\dfrac{\sin y}{y} - 2e^{-x} \sin x\right) dx + \left(\dfrac{\cos y + 2e^{-x} \cos x}{y}\right) dy = 0, \qquad \mu(x, y) = ye^x$

21. $y\, dx + (2x - ye^y)\, dy = 0, \qquad \mu(x, y) = y$

22. $(x + 2) \sin y\ dx + x \cos y\ dy = 0, \qquad \mu(x, y) = xe^x$

23. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)\, dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.