4.3 | 3 The last 2 digits of 9^{9^9} :

First, compute modulo 10 : $\quad 9^2 \equiv 1 \pmod{10} \implies 9^8 \equiv 1 \pmod{10}$

$$\implies 9^9 \equiv 9 \pmod{10}.$$

Thus, as indicated in the hint, $9^9 = 10k + 9$, some $k \in \mathbb{N}$.

Next, compute modulo 100 : $\quad 9^2 \equiv 81 \pmod{100}$

$$\implies 9^4 \equiv 61 \pmod{100}$$

$$\implies 9^8 \equiv 21 \pmod{100}$$

$$\implies 9^9 \equiv 89 \pmod{100}$$

$$\implies 9^{10} \equiv 1 \pmod{100}$$

$$\implies 9^{10k} \equiv 1 \pmod{100}$$

$$\implies 9^{9^9} \equiv \underline{\underline{89}} \pmod{100}.$$

8 For any integer a, $a^2 - a + 7$ ends in 3, 7, or 9.

pf. To prove this, we show that $a^2 - a$ ends in 0, 2, or 6.

Just compute mod 10:

Final digit of a	$a^2 - a \pmod{10}$
0	0
1	0
2	2
3	6
4	2
5	0
6	0
7	2
8	6
9	2

■

__9__ Compute the remainder when $4444^{''''}$ is divided by 9.

__sol'n:__ Note/compute that $4444 \equiv 7 \equiv -2 \pmod{9}$

$$\Longrightarrow 4444^2 \equiv 4 \pmod{9},$$

$$4444^3 \equiv -8 \equiv 1 \pmod{9}$$

$$\Longrightarrow (4444)^{3 \cdot 1481} \equiv 1 \pmod{9}, \text{ i.e.,}$$

$$4444^{4443} \equiv 1 \pmod{9}$$

$$\Longrightarrow 4444^{4444} \equiv 7 \pmod{9}$$

the desired remainder!

12 The last 3 digits of 7^{999} : $\underline{\underline{143}}$

Note that $7^4 = 2401 \equiv 401 \pmod{1000}$

$\Longrightarrow 7^8 \equiv 801 \pmod{1000}$

$\Longrightarrow 7^{16} \equiv 601 \pmod{1000}$

$\Longrightarrow 7^{32} \equiv 201 \pmod{1000}$

$\Longrightarrow 7^{64} \equiv 401 \pmod{1000}$

$\Longrightarrow 7^{128} \equiv 801 \pmod{1000}$

$\Longrightarrow 7^{256} \equiv 601 \pmod{1000}$

$\Longrightarrow 7^{512} \equiv 201 \pmod{1000}$.

Also, note that

$$999 = 512 + 256 + 128 + 64 + 32 + 4 + 2 + 1,$$

so

$$\begin{aligned} 7^{999} &\equiv 201 \times 601 \times 801 \times 401 \times 201 \times 401 \times 49 \times 7 \\ &\equiv 801 \times 801 \times 401 \times 201 \times 401 \times 49 \times 7 \\ &\equiv 601 \times 401 \times 201 \times 401 \times 49 \times 7 \\ &\equiv 201 \times 401 \times 49 \times 7 \\ &\equiv 601 \times 49 \times 7 \\ &\equiv 143 \pmod{1000}. \end{aligned}$$

25 For any prime $p > 3$, $13 \mid 10^{2p} - 10^p + 1$.

pf.: Since p is prime, $p = 3m+1$ or $p = 3m+2$.

if $p = 3m+1$, m is even; if $p = 3m+2$, m is odd.

Now observe that $10^3 \equiv -1 \pmod{13}$.

$\underline{p = 3m+1}$: $10^p = 10^{3m+1}$, and $10^{3m} \equiv (-1)^m = 1 \pmod{13}$

$\Longrightarrow 10^{3m+1} \equiv 10 \pmod{13}$

$10^{2p} = (10^p)^2$, $(10^p)^2 \equiv 100 \equiv 9 \pmod{13}$

Then $10^{2p} - 10^p \equiv 9 - 10 = -1 \pmod{13}$

and $10^{2p} - 10^p + 1 \equiv 0 \pmod{13}$

$\underline{p = 3m+2}$: $10^{3m} \equiv (-1)^m = -1 \pmod{13}$

$\Longrightarrow 10^{3m+2} \equiv -100 \pmod{13} \Longrightarrow 10^{3m+2} \equiv 4 \pmod{13}$

$(10^p)^2 \equiv 16 \equiv 3 \pmod{13}$, $10^{2p} - 10^p + 1 \equiv 0 \pmod{13}$.

∎

4.4

1(a) $25x \equiv 15 \pmod{29}$

$\gcd(25, 29) = 1$, and $7 \cdot 25 - 6 \cdot 29 = 1$ (Euclidean algorithm)

so $(15 \cdot 7) \cdot 25 - (6 \cdot 15) \cdot 29 = 15$.

Thus, $x_0 = 15 \cdot 7 = 105$ is one solution; $105 \equiv 18 \pmod{29}$,

so $\underline{\underline{18}}$ is the unique solution modulo 29.

1(b) $5x \equiv 2 \pmod{26}$

$\gcd(5, 26) = 1$, and $-5 \cdot 5 + 26 = 1$; thus,

$(-10) \cdot 5 + 2 \cdot 26 = 2$ and $x_0 = -10$ solves the congruence.

Any solution is of the form $x = -10 + 26t$, $t \in \mathbb{Z}$, so the

unique solution (in the interval $[0, 26)$) modulo 26 is $x = -10 + 26 = \underline{\underline{16}}$.

11(c) $6x \equiv 15 \pmod{21}$

$\gcd(6,21)$: $\begin{aligned} 21 &= 3\cdot 6 + 3 \\ 6 &= 2\cdot 3 \end{aligned}$ $\Longrightarrow \gcd(6,21) = 3$, and $3 = -3\cdot 6 + 21$.

Thus, $(-15)\cdot 6 + 5\cdot 21 = 15$, and solutions of the congruence are of the form $x = -15 + 7t$, $t \in \mathbb{Z}$

$\Longrightarrow$ $\underline{\underline{x = 6, 13, 20}}$ are the 3 solutions mod 21.

11(d) $36x \equiv 8 \pmod{102}$

$\gcd(36,102)$: $\begin{aligned} 102 &= 2\cdot 36 + 30 \\ 36 &= 30 + 6 \\ 30 &= 5\cdot 6 \end{aligned}$ $\Longrightarrow \gcd(36,102) = 6$.

Since $6 \nmid 8$, this congruence has $\underline{\text{No}}$ solution. :(

11e) $34x \equiv 60 \pmod{98}$

$\gcd(34, 98)$:

$$\left.\begin{aligned} 98 &= 2\cdot 34 + 30 \\ 34 &= 30 + 4 \\ 30 &= 7\cdot 4 + \underline{\underline{2}} \\ 4 &= 2\cdot 2 \end{aligned}\right\}$$

$\gcd(34, 98) = 2$, and

$$\begin{aligned} 2 &= 30 - 7\cdot 4 \\ &= 30 - 7(34 - 30) \\ &= 8\cdot 30 - 7\cdot 34 \\ &= 8(98 - 2\cdot 34) - 7\cdot 34 \\ &= \underline{\underline{-23\cdot 34 + 8\cdot 98}} \end{aligned}$$

Since $-23\cdot 34 + 8\cdot 98 = 2$,

$$\underbrace{-(30\cdot 23)}_{-690}\cdot 34 + (30\cdot 8)\cdot 98 = 60,$$

and solutions of the congruence are of the form

$$x = -690 + 49t, \quad t \in \mathbb{Z}.$$

$\Longrightarrow$ The 2 solutions modulo 98 are $\underline{x = 45}$ $(t = 15)$

and $\underline{x = 94}$ $(t = 16)$.

11f) $140x \equiv 133 \pmod{301}$

$$\gcd(140, 301): \left.\begin{aligned} 301 &= 2\cdot 140 + 21 \\ 140 &= 6\cdot 21 + 14 \\ 21 &= 14 + \underline{\underline{7}} \\ 14 &= 2\cdot 7 \end{aligned}\right\} \quad \begin{aligned} &\gcd(140, 301) = 7, \text{ and} \\ &7 = 21 - 14 = 21 - (140 - 6\cdot 21) \\ &\quad = -140 + 7(301 - 2\cdot 140) \\ &\quad = -15\cdot 140 + 7\cdot 301 . \end{aligned}$$

Since $133 = 7\cdot 19$, we have

$$\underbrace{(-15\cdot 19)}_{x_0 = -285}\cdot 140 + (7\cdot 19)\cdot 301 = 133$$

and every solution of the congruence is of the form

$$x = -285 + 43t, \quad t \in \mathbb{Z} .$$

The 7 solutions modulo 301 are therefore

$$16, 59, 102, 145, 188, 231, \text{ and } 274 .$$

$\underline{t = 7}$

2(a) $4x + 51y = 9$

Note that $\gcd(4, 51) = 1$ and that $13 \cdot 4 - 51 = 1$.

Equivalent congruences:

$4x \equiv 9 \pmod{51}$ and $51y \equiv 9 \pmod{4}$

Have $(9 \cdot 13) \cdot 4 - 9 \cdot 51 = 9$,

so $x = 117 - 51t$, $t \in \mathbb{Z}$

$\Longrightarrow$ unique sol'n mod 51 is $x = 15$; all sol'ns of form $x = 15 + 51t$, $t \in \mathbb{Z}$.

Have $-9 \cdot 51 + (9 \cdot 13) \cdot 4 = 9$,

so $y = -9 + 4t$, $t \in \mathbb{Z}$

$\Longrightarrow$ unique sol'n mod 4 is $y = 3$; all sol'ns of form $y = 3 + 4s$, $s \in \mathbb{Z}$.

Need $s + t = 0$, so get

$$x = 15 + 51t, \quad y = 3 - 4t, \quad t \in \mathbb{Z}.$$

2(b) $12x + 25y = 331$

Note that $\gcd(12, 25) = 1$, and $-2\cdot 12 + 25 = 1$.

Thus, $(-662)\cdot 12 + 331\cdot 25 = 331$.

Equivalent congruences:

$12x \equiv 331 \pmod{25}$, $25y \equiv 331 \pmod{12}$

$x = -662 + 25t$, $t \in \mathbb{Z}$ | $y = 331 + 12t$, $t \in \mathbb{Z}$

$\Longrightarrow x = 13 + 25t$, $t \in \mathbb{Z}$ | $\Longrightarrow y = 7 + 12t$, $t \in \mathbb{Z}$

since 13 is the unique sol'n mod 25. | since 7 is the unique sol'n mod 12.

$$x = 13 + 25t, \quad y = 7 - 12t, \quad t \in \mathbb{Z}.$$

2(e) $5x - 53y = 17$

Note that $\gcd(5, 53) = 1$ and that $32 \cdot 5 - 3 \cdot 53 = 1$. (Euclidean algorithm)

Thus, $\underbrace{(32 \cdot 17)}_{544} \cdot 5 - \underbrace{(3 \times 17)}_{51} \cdot 53 = 17$.

Equivalent congruences:

$5x \equiv 17 \pmod{53}$, $53y \equiv 17 \pmod{5}$

$x = 544 + 53t, \ t \in \mathbb{Z}$ | $y = 51 + 5t, \ t \in \mathbb{Z}$

$\Longrightarrow x = 14 + 53t, \ t \in \mathbb{Z}$ | $\Longrightarrow y = 1 + 5t, \ t \in \mathbb{Z}$.

$$\underbrace{\qquad\qquad}$$

$$x = 14 + 53t, \quad y = 1 + 5t, \quad t \in \mathbb{Z}.$$

4(a) $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$:

$$\left.\begin{aligned} 35x \equiv 1 \pmod 3 &\Longrightarrow x = 2 \\ 21x \equiv 1 \pmod 5 &\Longrightarrow x = 1 \\ 15x \equiv 1 \pmod 7 &\Longrightarrow x = 1 \end{aligned}\right\} \quad \begin{aligned} x &= 70 + 21\cdot 2 + 15\cdot 3 \\ &= 70 + 42 + 45 \\ &= 157 \equiv \underline{\underline{52}} \bmod 105. \end{aligned}$$

Solution: $\underline{\underline{x = 52}}$

4(b) $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 15 \pmod{31}$

$$\left.\begin{aligned} 899x \equiv 1 \pmod{11} &\Longrightarrow x = 7 \\ 341x \equiv 1 \pmod{29} &\Longrightarrow x = 4 \\ 319x \equiv 1 \pmod{31} &\Longrightarrow x = 7 \end{aligned}\right\} \quad \begin{aligned} x &= 899\cdot 7\cdot 5 + 341\cdot 4\cdot 14 + 319\cdot 7\cdot 15 \\ &= 31465 + 19096 + 33495 \\ &= 84056 \equiv \underline{\underline{4944}} \bmod 11\cdot 29\cdot 31 \end{aligned}$$

All of these were found using the Euclidean algorithm.

4(c) $x \equiv 5 \pmod 6, \quad x \equiv 4 \pmod{11}, \quad x \equiv 3 \pmod{17}$

$$\left.\begin{aligned} 187x &\equiv 1 \pmod 6 \implies x = 1 \\ 102x &\equiv 1 \pmod{11} \implies x = 4 \\ 66x &\equiv 1 \pmod{17} \implies x = 8 \end{aligned}\right\} \quad \begin{aligned} \bar{x} &= 187 \cdot 5 + 4 \cdot 102 \cdot 4 + 8 \cdot 66 \cdot 3 \\ &= 935 + 1632 + 1584 \\ &= 4151 \equiv 785 \pmod{6 \cdot 11 \cdot 17} \end{aligned}$$

$$\begin{aligned} 66 &= 3 \cdot 17 + 15 \\ 17 &= 15 + 2 \\ 15 &= 7 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 \end{aligned} \implies \begin{aligned} 1 &= 15 - 7 \cdot 2 \\ &= 15 - 7 \cdot (17 - 15) \\ &= -7 \cdot 17 + 8 \cdot (66 - 3 \cdot 17) \\ &= 8 \cdot 66 - 31 \cdot 17 \end{aligned}$$

$$x = 8$$

$$\boxed{\bar{x} = 785}$$

4(d) $2x \equiv 1 \pmod 5$, $3x \equiv 9 \pmod 6$, $4x \equiv 1 \pmod 7$, $5x \equiv 9 \pmod{11}$

$2x \equiv 1 \pmod 5 \implies 6x \equiv 3 \pmod 5 \implies x \equiv 3 \pmod 5$

$3x \equiv 9 \pmod 6 \implies x \equiv 3 \pmod 2$

$4x \equiv 1 \pmod 7 \implies 8x \equiv 2 \pmod 7$, $x \equiv 2 \pmod 7$

$5x \equiv 9 \pmod{11} \implies 10x \equiv 18 \equiv 7 \pmod{11}$
$10x \equiv 7 \pmod{11}$
$-x \equiv 7 \pmod{11}$
$x \equiv 4 \pmod{11}$

New system: $x \equiv 3 \pmod 5$, $x \equiv 3 \pmod 2$, $x \equiv 2 \pmod 7$, $x \equiv 4 \pmod{11}$

$154x \equiv 1 \pmod 5 \implies x = 4$

$385x \equiv 1 \pmod 2 \implies x = 1$

$110x \equiv 1 \pmod 7 \implies x = 3$

$70x \equiv 1 \pmod{11} \implies x = 3$

$\bar{x} = 4 \cdot 154 \cdot 3 + 385 \cdot 3 + 3 \cdot 110 \cdot 2 + 3 \cdot 70 \cdot 4$
$= 4503 \equiv 653 \pmod{770}$

$\boxed{\bar{x} = 653}$