$$\begin{cases} ax+by=0 \\ cx+\underline{dy}=0 \end{cases} \Longrightarrow dy=-cx \Longrightarrow y=-\frac{cx}{d}$$

$$\Longrightarrow ax+b\left(-\frac{cx}{d}\right)=0$$

$$\left(a-\frac{bc}{d}\right)x=0$$

$$\left(\frac{ad-bc}{d}\right)x=0$$

$$\Longrightarrow \text{Nontrivial solution iff}$$

$$\underline{ad-bc=0}, \quad \text{i.e.,}$$

Equivalent to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}=0$$

$$\Longrightarrow \det(A)=0$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{x' = Ax} \qquad \text{ansatz: } x(t) = \underline{e^{\lambda t} v} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Downarrow \qquad \begin{cases} \lambda : \text{eigenvalues} \\ v : \text{eigenvectors} \end{cases}$$

$$\lambda e^{\lambda t} v = A(e^{\lambda t} v) \implies \lambda v = Av$$

$$\implies Av - \lambda v = 0$$

$$\implies Av - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} v = 0 \implies \underline{(a-\lambda)(d-\lambda) - bc = 0}$$

#2 $x' = \underbrace{\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}} x$

$x(t) = e^{\lambda t} v, \quad \lambda: \quad \det\begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} = 0$

$$(1-\lambda)(-4-\lambda) - (-6) = 0$$

$$\lambda^2 + 3\lambda - 4 + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\underline{\lambda = -1, \quad \lambda = -2}$$

General Sol'n:

$$x(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\underline{\lambda = -1}$: $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

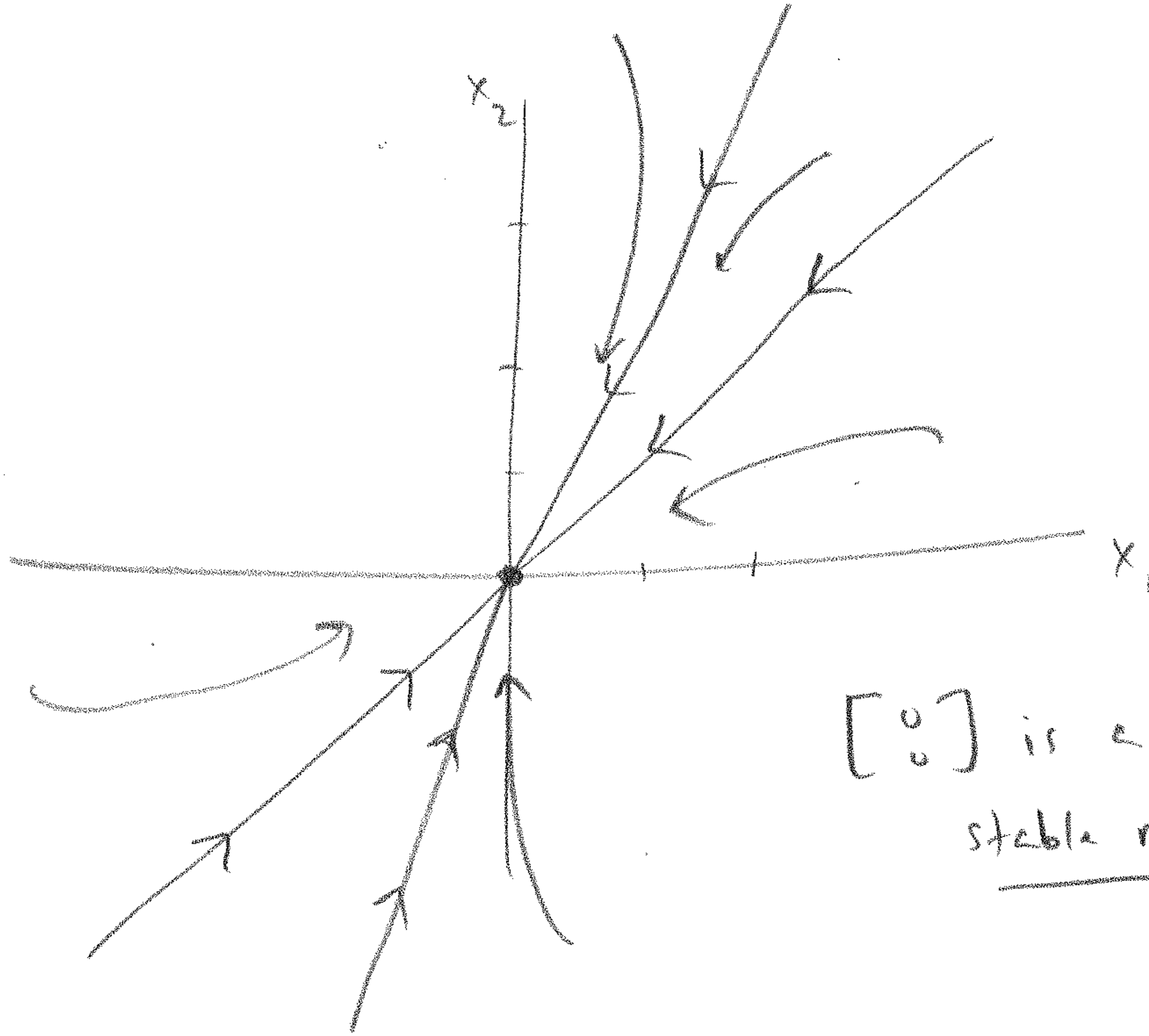
$2v_1 - 2v_2 = 0$

$\underline{v_1 = v_2} \implies v = \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\underline{\lambda = -2}$: $\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3v_1 - 2v_2 = 0 \implies v_2 = \frac{3}{2} v_1$

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a <u>stable node</u>

#4 $\quad x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x$

$$x = e^{\lambda t}\vec{v} \Longrightarrow \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = 0$$

$$\Longrightarrow (1-\lambda)(-2-\lambda) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0 \Longrightarrow \lambda = -3,\ \lambda = +2$$

$\lambda = -3$:

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 + v_2 = 0$$

$$v_2 = -4v_1$$

$$v = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$\lambda = 2$:

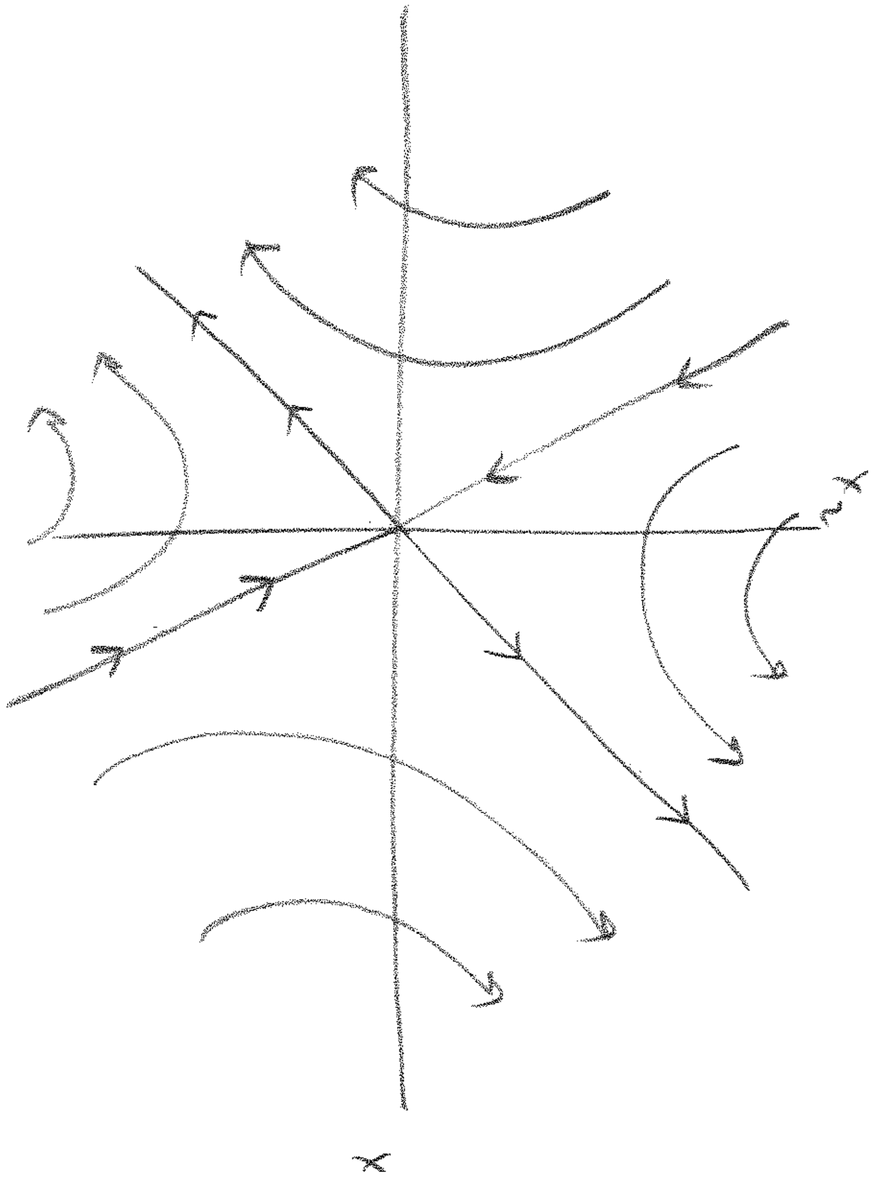
$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-v_1 + v_2 = 0$$

$$v_2 = v_1$$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General sol'n: $x(t) = c_1 e^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$x' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} x$$

$$x(t) = e^{\lambda t} v \Longrightarrow \det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = \underline{1 \pm 2i}$$

$\underline{\lambda = 1+2i}$; $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$e^{(1+2i)t}\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$\left.\begin{aligned} (2-2i)v_1 - 2v_2 &= 0 \\ 2v_2 &= (2-2i)v_1 \\ \underline{v_2} &\underline{= (1-i)v_1} \end{aligned}\right\} \quad \underline{v = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}}$$

$$= e^t(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$= \begin{bmatrix} e^t(\cos(2t) + i\sin(2t)) \\ e^t\cos(2t) + e^t\sin(2t) + i(e^t\sin(2t) - e^t\cos) \end{bmatrix}$$

$\underline{\lambda = 1-2i}$: $\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$a+bi$

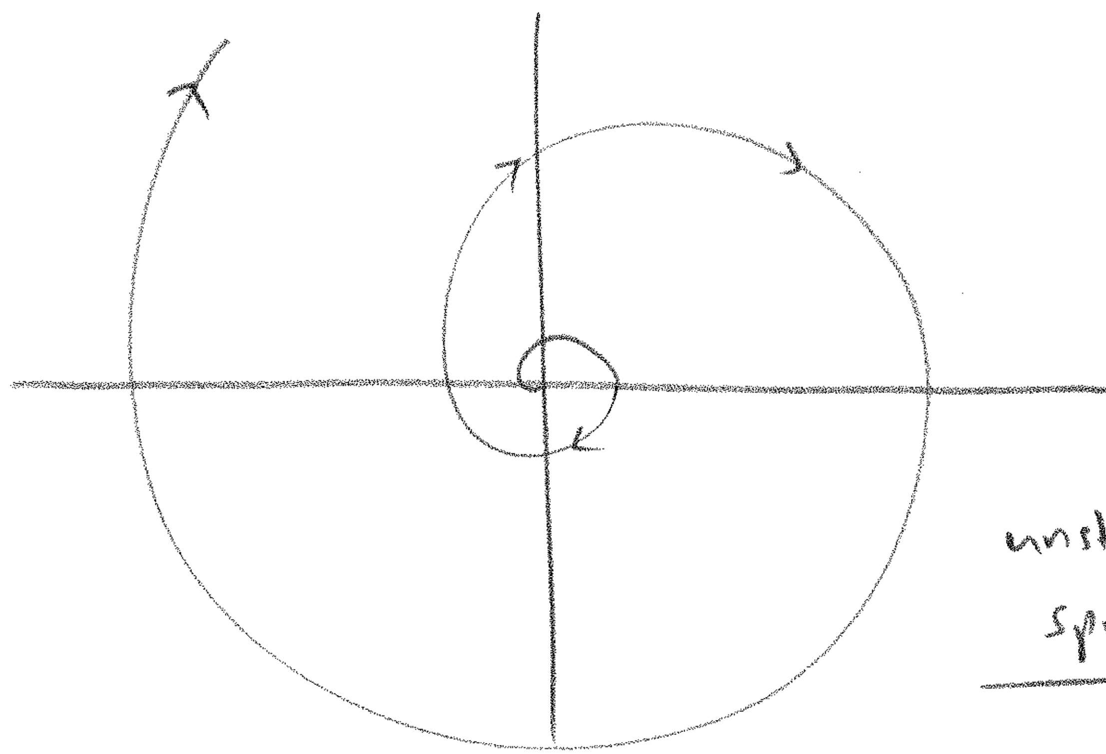
$a-bi$

$(2+2i)v_1 - 2v_2 = 0$

$$\left.\begin{aligned} 2v_2 &= (2+2i)v_1 \\ v_2 &= (1+i)v_1 \end{aligned}\right\} \quad v = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Then :

$$x(t) = c_1 e^t \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix}$$



unstable
spiral

#3 $\quad x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$

$$(2-\lambda)(-2-\lambda) + 5 = 0$$

$$\underline{\lambda^2 + 1 = 0} \implies \lambda = \pm i$$

$\underline{\lambda = i}$: $\quad \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(2-i)v_1 - 5v_2 = 0$$

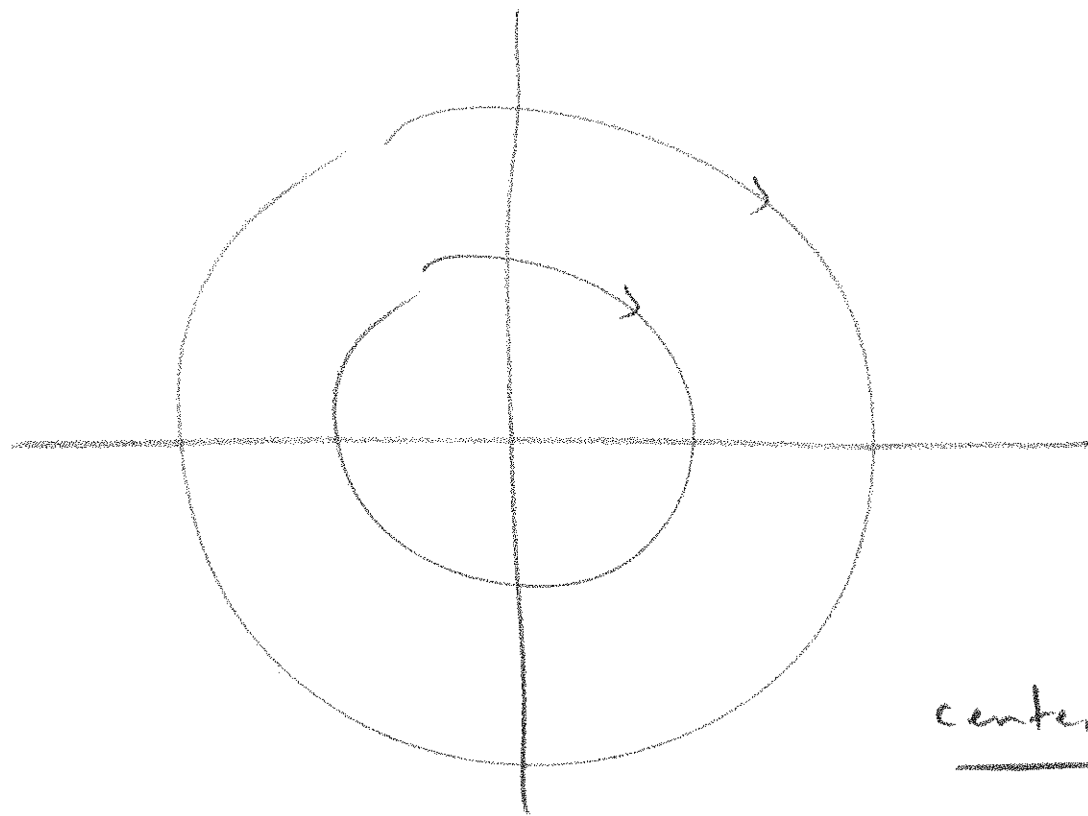
$$v_2 = \frac{2-i}{5} v_1$$

$$v = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos t + 5i\sin t \\ (2\cos t + \sin t) + i(2\sin t - \cos t) \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$$



<u>center</u>