CLASSIFICATION & GRADIENT DESCENT

GOALS FOR TODAY

AND WHAT YOU SHOULD LEARN

1. Know the classification problem
2. What is gradient gradient descent
3. What is logistic regression

FUNCTION APPROXIMATION

RECAP

For any function f_*, e.g., $f_*(x) = ax^2 + bx + c$, we want an approximation $f(x)$

$$\forall x,\ |f(x) - f_*(x)| < \epsilon$$

LINEAR FUNCTION APPROXIMATION

RECAP

Represent as a linear function with a basis function

$\phi : \mathbb{R} \to \mathbb{R}^n$ is the basis function

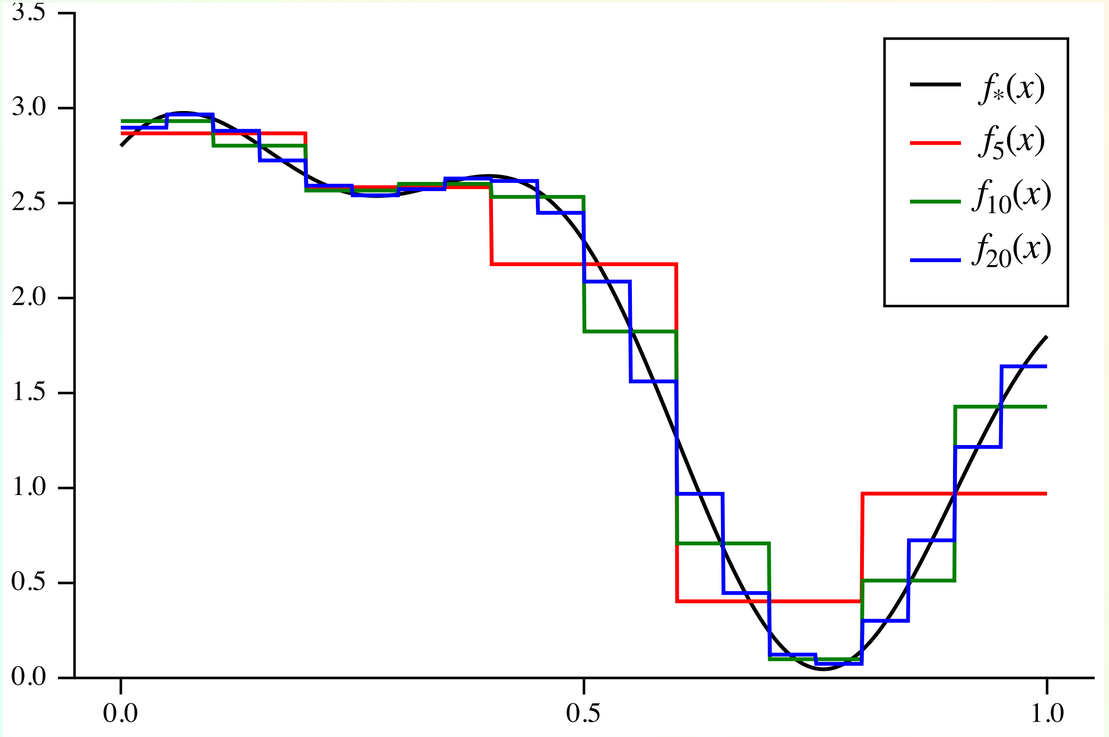
$\phi_i(x)$ is the output of i^{th} the basis function

For discrete approximation

$$\phi_i(x) = \begin{cases} 1 & x \in \text{bin}_i \\ 0 & \text{otherwise} \end{cases}$$

$\phi(x) = [0,0,1,0,0]^\top$ — one-hot vector

$$f(x, w) = w^\top \phi(x) = \sum_{i=1}^{n} w_i \phi_i(x)$$



LINEAR FUNCTION APPROXIMATION

RECAP

$$\phi(x) = \left[1, x, x^2, \cos(3\pi x), x \in [0.5, 0.6], \ldots\right]^\top$$

REGRESSION

RECAP

Let X be a random variable representing an a draw of x from $\mathcal{X}$

Let $Y = f_*(X) + \xi$

Assume: $\mathbf{E}[Y \mid X = x] = f_*(x) + \underbrace{\mathbf{E}[\xi \mid X = x]}_{=0} = f_*(x)$

REGRESSION

RECAP

Loss function (mean squared error):

$$l(w) \doteq \mathbf{E}\left[\left(f(X, w) - Y\right)^2\right]$$

“On average, how far away is the estimate from the samples Y”

LEAST SQUARES

RECAP

Find optimal weights:

$$w^* \in \arg\min_w l(w)$$

Solution:

$$\begin{aligned} w^* &= \underbrace{\mathbf{E}\left[\phi(X)\phi(X)^\top\right]^{-1}}_{=A} \underbrace{\mathbf{E}\left[Y\phi(X)\right]}_{=b} \\ &= A^{-1}b \end{aligned}$$

FINDING THE BEST FIT

GRADIENT DESCENT

Assume we have a data set of inputs and outputs:

$$x_1, x_2, \ldots, x_m,\ y_1 = f(x_1), y_2 = f(x_2), \ldots, y_m = f(x_m)$$

Loss function for weights w (how bad the approximation is)

$$l(w) = \frac{1}{m} \sum_{i=1}^{m} \left(f(x_i, w) - y_i \right)^2$$

GRADIENT DESCENT ON $l(w)$

STOCHASTIC GRADIENT DESCENT

$$l(w) = \frac{1}{2}\frac{1}{m}\sum_{i=1}^{m}\left(f(x_i, w) - y_i\right)^2$$

$$\nabla l(w) = \frac{\partial}{\partial w}\frac{1}{2}\frac{1}{m}\sum_{i=1}^{m}\left(y_i - f(x_i, w)\right)^2$$

$$= \frac{1}{2}\frac{1}{m}\sum_{i=1}^{m}\frac{\partial}{\partial w}\left(f(x_i, w) - y_i\right)^2 = \frac{1}{m}\sum_{i=1}^{m}\left(f(x_i, w) - y_i\right)\frac{\partial f(x_i, w)}{\partial w}$$

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Idea: move in the direction of the negative gradient

$$w \leftarrow w - \eta \nabla l(w)$$

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Idea: move in the direction of the negative gradient

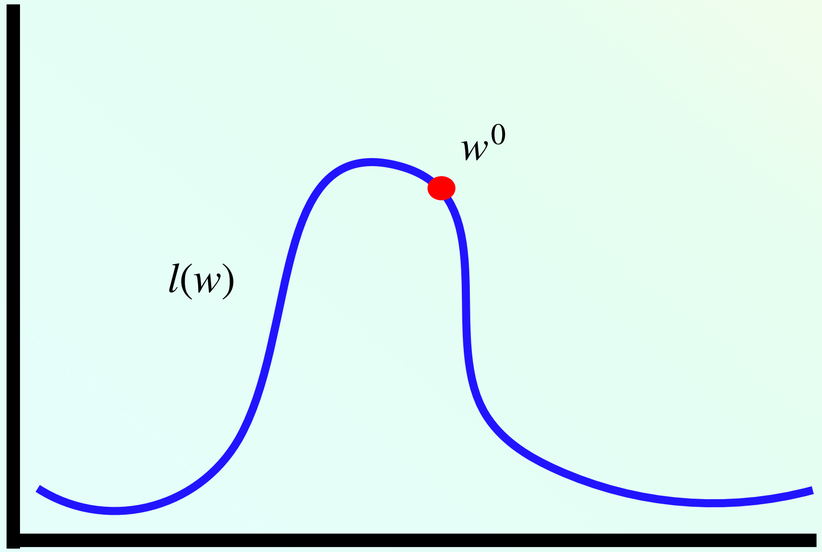
$$w \leftarrow w - \eta \nabla l(w)$$

η is called the *step size*
Some people refer to this as a *learning rate*

GRADIENT DESCENT

PROCESS

$$w^{k+1} = w^k - \eta \nabla l(w^k)$$



GRADIENT DESCENT

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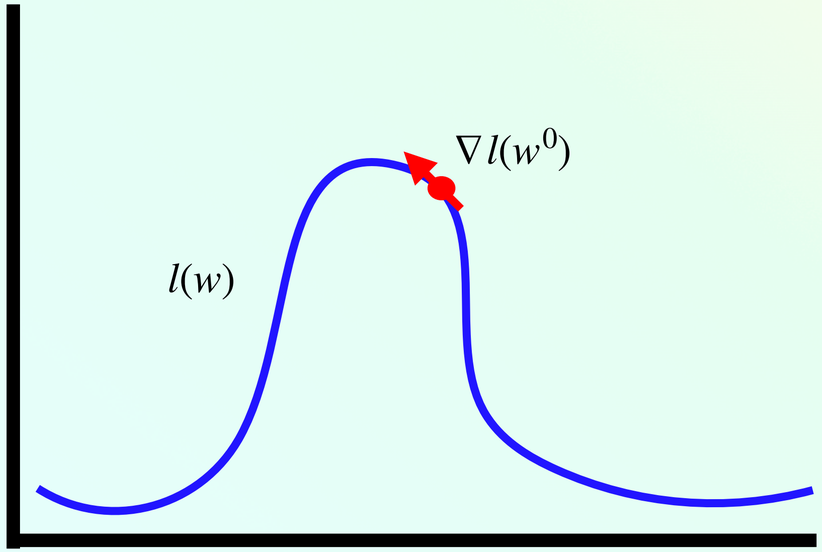

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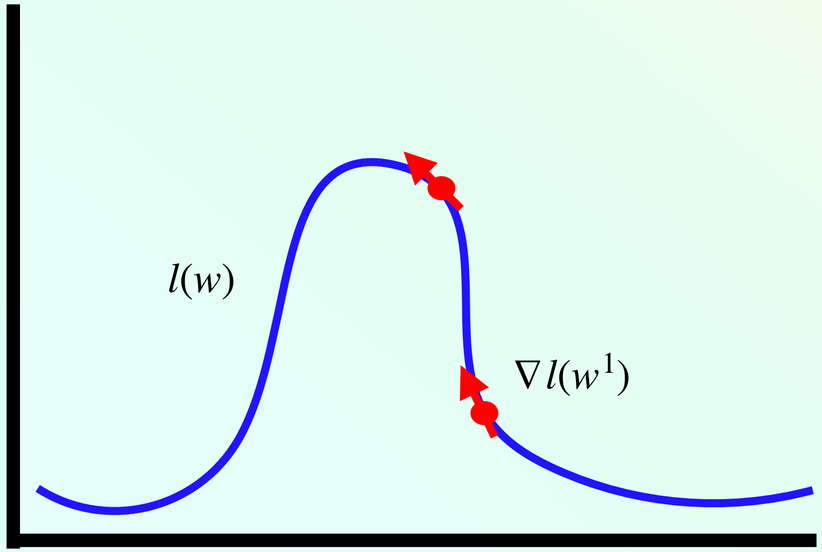

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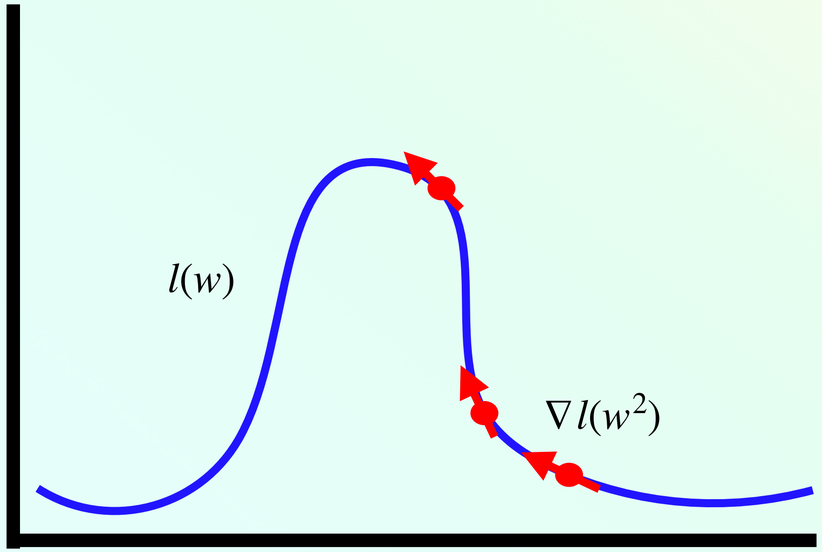

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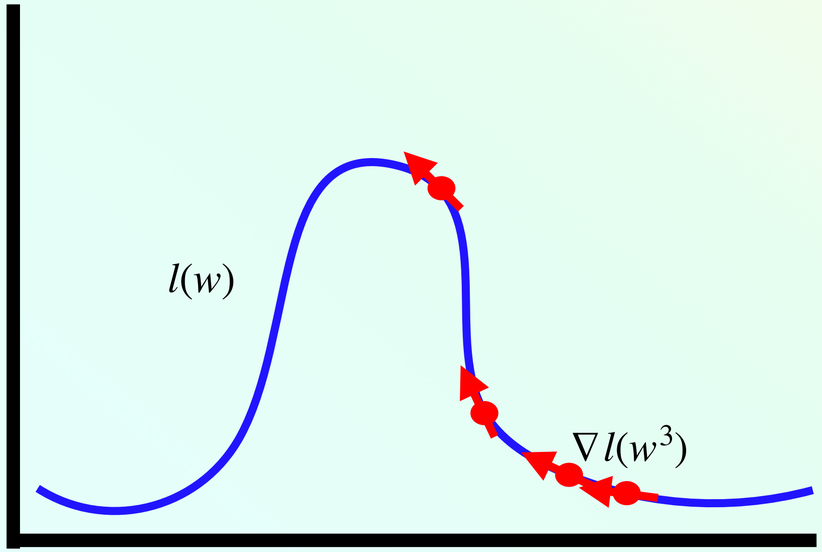

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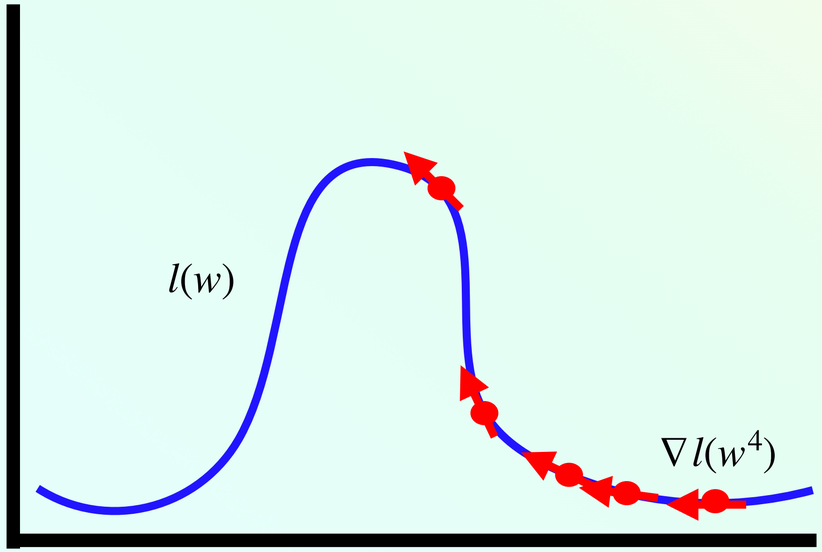

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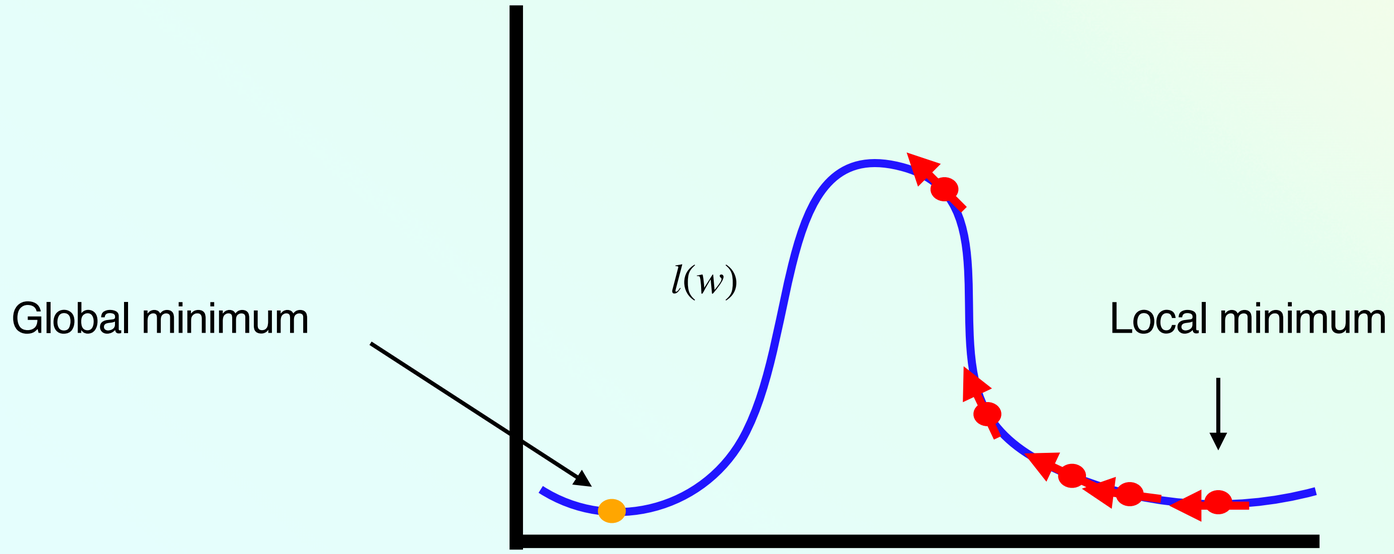


GRADIENT DESCENT

PROPERTIES

- Only guaranteed to find a **local minimum**
  - Some l only have one minimum (e.g., $l(w)$ is convex for all w)

GRADIENT DESCENT

PROPERTIES

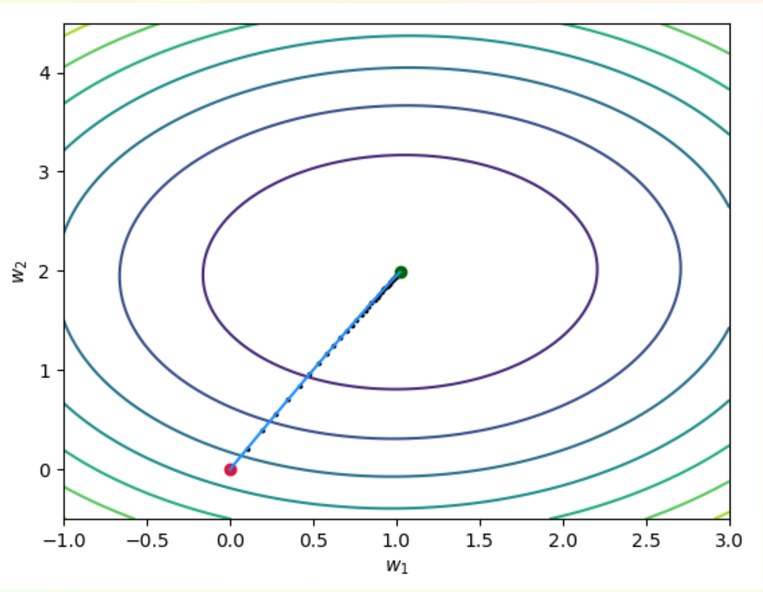
- Only guaranteed to find a **local minimum**
  - Some l only have one minimum (e.g., $l(w)$ is convex for all w)
- l must be differentiable at least once
- Works best when $\nabla l(w)$ is smooth
  - $\forall w, w' \ \frac{\|\nabla l(w) - \nabla l(w')\|_2}{\|w - w'\|_2} \leq L$
  - Small change in w implies small change in $\nabla l(w)$
  - No discrete jumps in $l(w)$

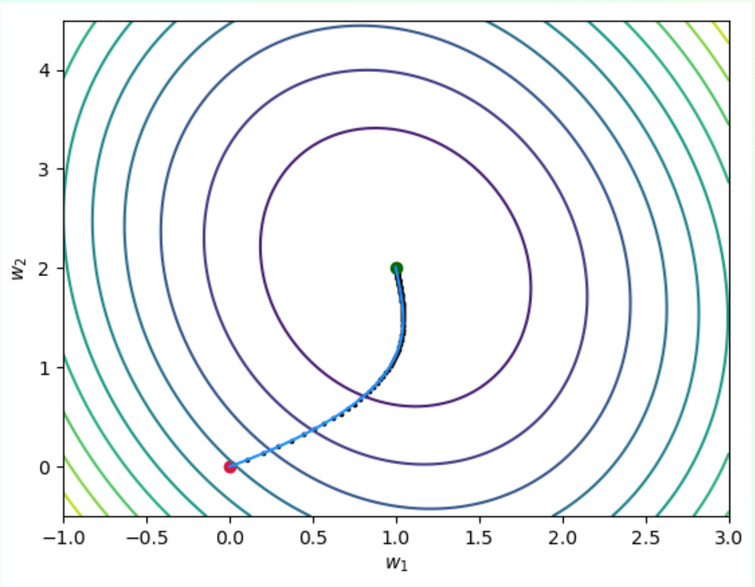
$$\|x\|_2 \doteq \sqrt{\sum_{i=1}^{n} x_i^2}$$

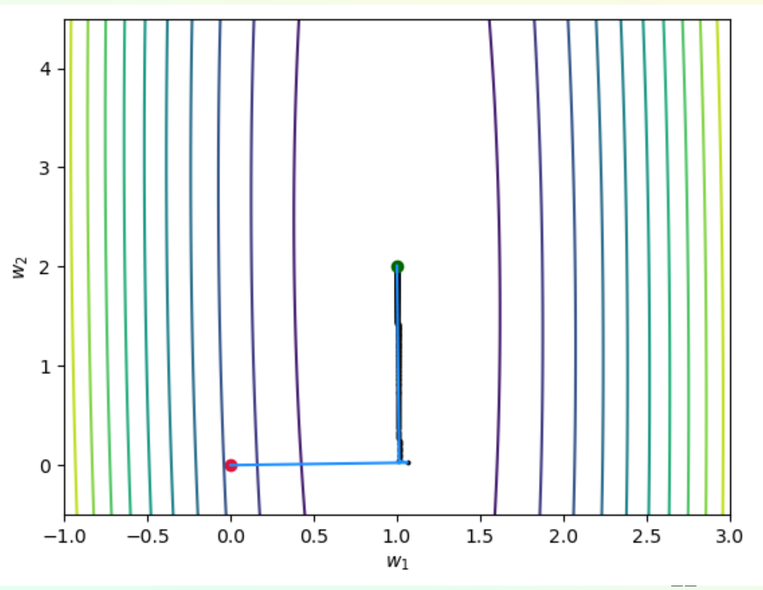
LINEAR REGRESSION

LOSS FUNCTION HAS ONE MINIMIZER

- The loss function looks like a bowl
- It can be squashed or stretched in different directions







QUIZ

CLASSIFICATION

OVERVIEW

f_*

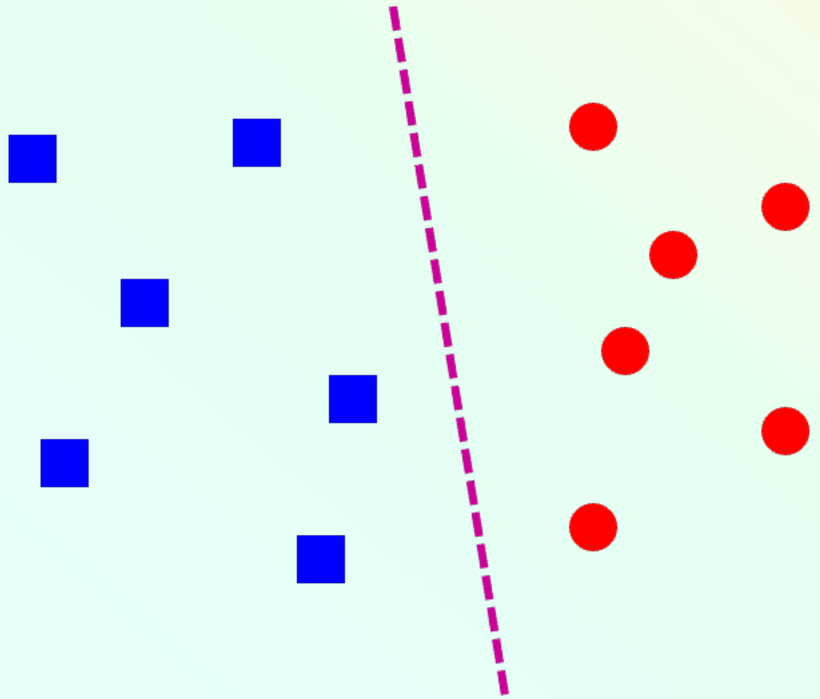
$f_*(X_{cat}) = \text{Cat}$

f_*

$f_*(X_{dog}) = \text{Dog}$

LINEAR CLASSIFICATION

OVERVIEW



LINEAR CLASSIFICATION

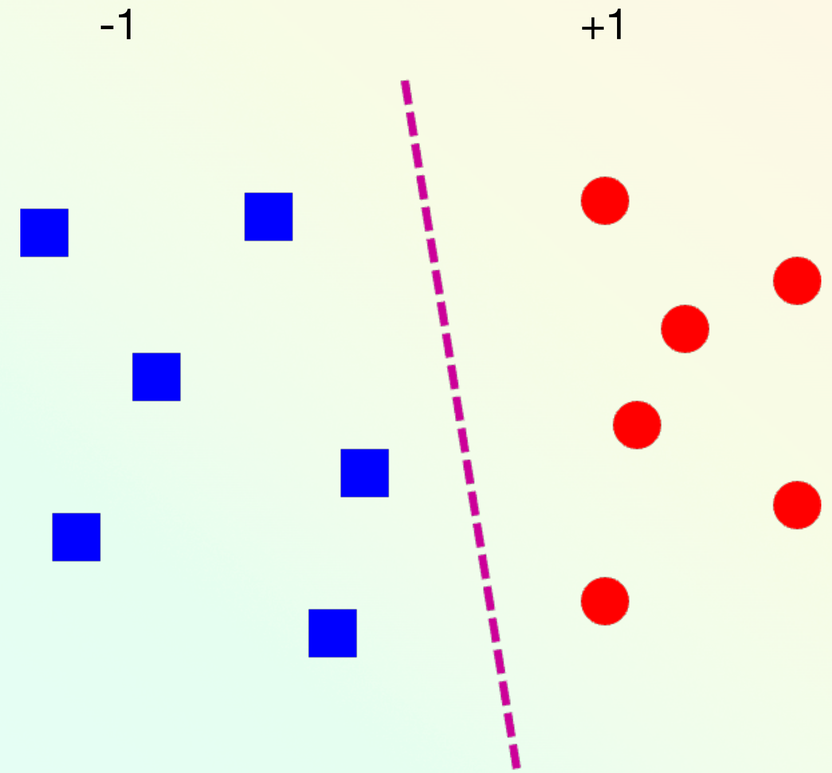
OVERVIEW

$$\mathcal{X} = \mathbb{R}^2$$

$$\mathcal{Y} = \{-1, +1\}$$

$$f\colon \mathcal{X} \times \mathbb{R}^2 \to \mathcal{Y}$$

$$f(x, w) = \text{sgn}(w^\top x)$$



LINEAR CLASSIFICATION

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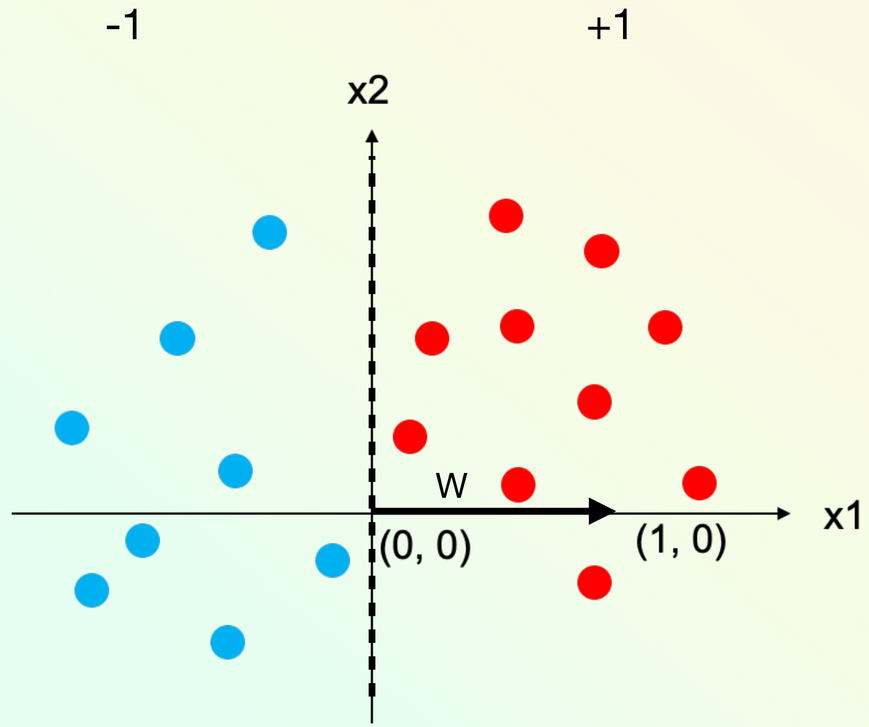

$$f(x, w) = \text{sgn}(w^\top x) = \text{sgn}(w_1 x_1 + w_2 x_2)$$

What should the weights w be?

LINEAR CLASSIFICATION

OVERVIEW

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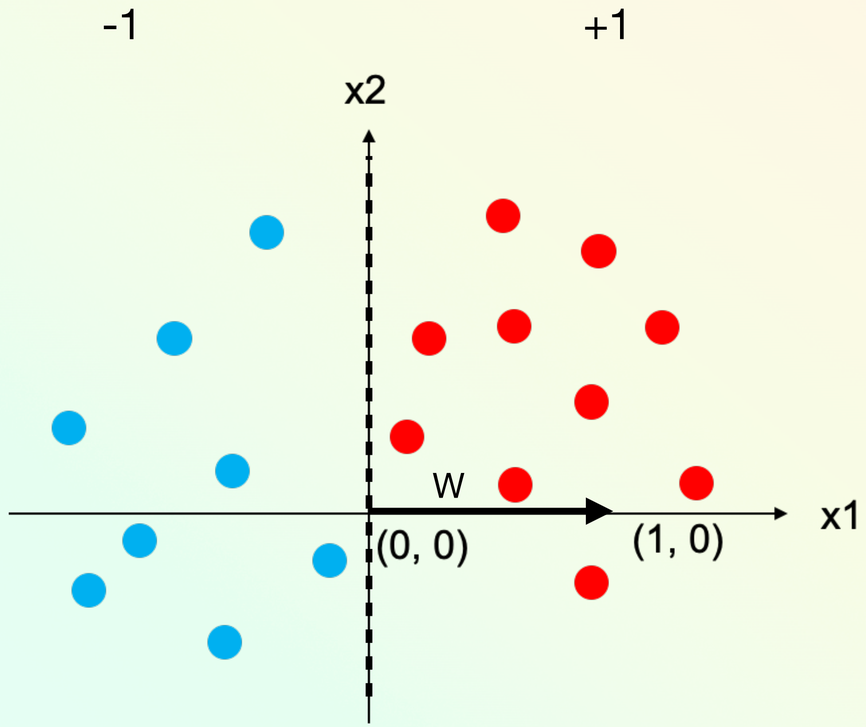
What should the weights w be?

$$w = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \text{ for any } \alpha > 0$$



LINEAR CLASSIFICATION

OVERVIEW

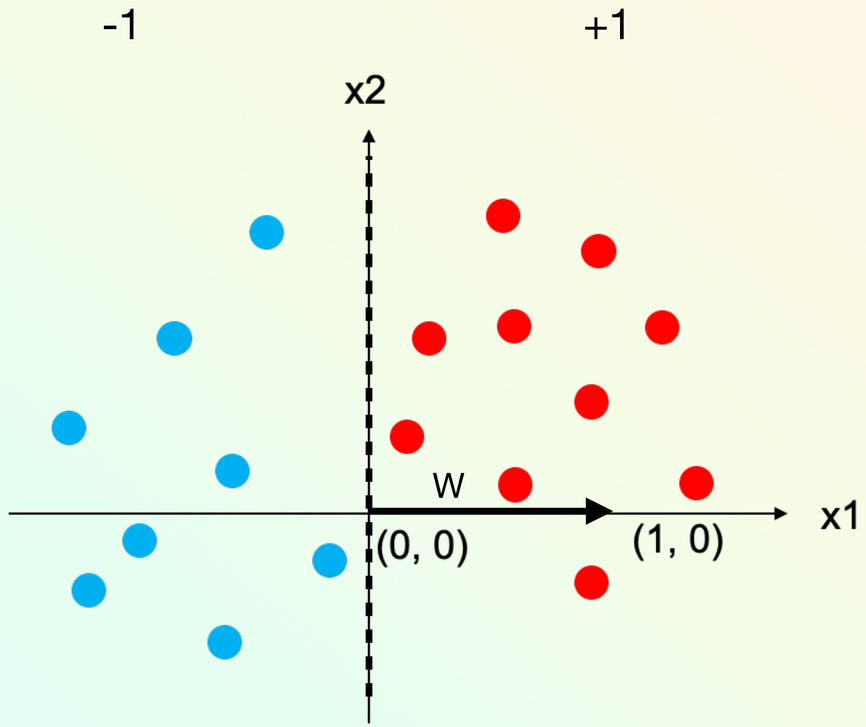
$$f(x, w) = \text{sgn}(w^\top x) = \text{sgn}(w_1 x_1 + w_2 x_2)$$

What should the weights w be?

$$w = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \text{ for any } \alpha > 0$$

$$w^\top \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha > 0 \qquad \text{+1 class}$$

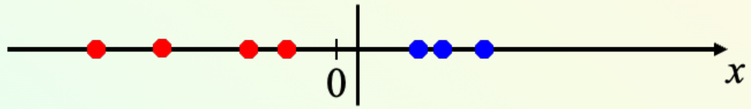
$$w^\top \begin{bmatrix} -0.1 \\ 100 \end{bmatrix} = -0.1\alpha < 0 \qquad \text{-1 class}$$



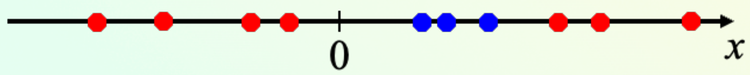
THE IMPORTANCE OF BASIS FUNCTIONS

BASIS EXPANSION (MAPPING TO HIGHER DIMINSIONS)

Some data is linearly separable:

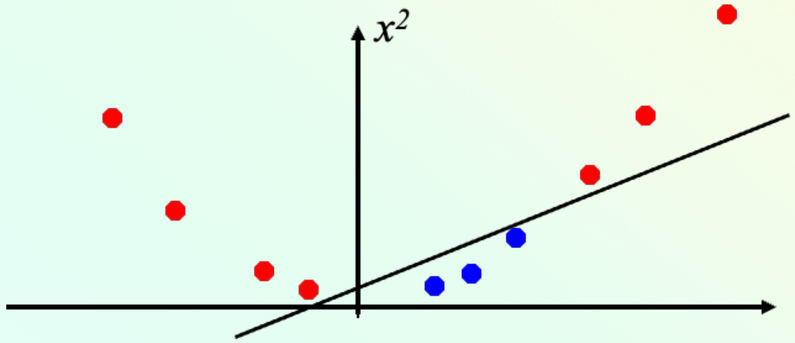


Some is not:



Data becomes separable in higher dimensions

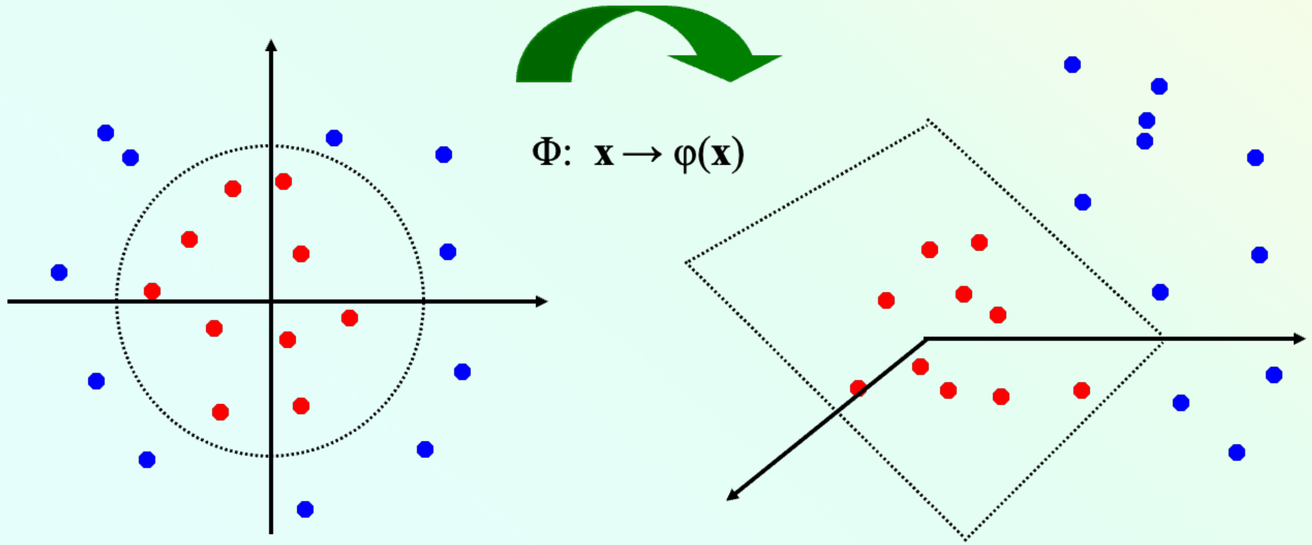
$$\phi(x) = \begin{bmatrix} x \\ x^2 \end{bmatrix}$$



THE IMPORTANCE OF BASIS FUNCTIONS

BASIS EXPANSION (MAPPING TO HIGHER DIMINSIONS)

General idea: the original input space can always be mapped to some higher-deminsional feature space where the training set is separable.

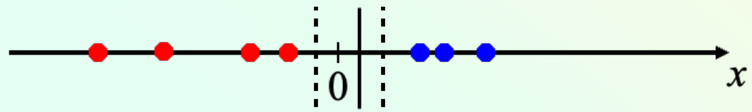


WHICH LINES

MAXIMUM MARGIN

Many lines that separete the data

Which one do we want?



Heuristic: One that separates the data with the most distance between classes.

This is line is called the maximum margin line

LINEAR CLASSIFICATION

FINDING THE WEIGHTS

$D = \{(\mathbf{x}_i, y_i)\}_{i=1}^{m}$

$\mathbf{x}_i$ are the features for the i^{th} data point

$y_i \in \{-1, 1\}$ is the class label for the i^{th} data point

$$\text{Minimize } \frac{1}{2} w^\top w$$

$$\text{Subject to } \forall i, y_i(w^\top \mathbf{x}_i + b) \geq 1$$

Classifier is: $\text{sgn}(w^\top \mathbf{x} + b - 1)$

LINEAR CLASSIFICATION

FINDING THE WEIGHTS

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Only consider the w that correctly classify all data points

Classifier is: $\text{sgn}(w^\top \mathbf{x} + b - 1)$

LINEAR CLASSIFICATION

FINDING THE WEIGHTS

$D = \{(\mathbf{x}_i, y_i)\}_{i=1}^{m}$

$\mathbf{x}_i$ are the features for the i^{th} data point

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Finds the line with the smallest weights —> most distance between classes

$$\text{Minimize } \frac{1}{2} w^\top w$$

$$\text{Subject to } \forall i, y_i(w^\top \mathbf{x}_i + b) \geq 1$$

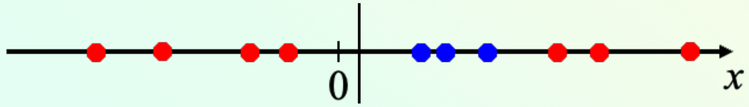
Only consider the w that correctly classify all data points

Classifier is: $\text{sgn}(w^\top \mathbf{x} + b - 1)$

LINEAR CLASSIFICATION

FINDING THE WEIGHTS

Sometimes it is not possible to separate all data points



Need to define a penalty for being misclassified

- Soft-margin Support Vector Machines — not covered in this course
- Create optimization problem to maximize margin and minimize classification penalty

CLASSIFICATION

DIFFERENT APPROACH

Deep learning approach:

1. Define a loss function $l(w)$ to calculate the error in classification
2. Find a local minimum w^* of $l(w)$ using (stochastic) gradient descent

CLASSIFICATION

CLASSIFICATION LOSS

Idea 1: +1 error for each miss classified data point

$$l(w) = -\sum_{i=1}^{m} \text{sgn}(w^\top \mathbf{x}_i) y_i$$

CLASSIFICATION

CLASSIFICATION LOSS

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$$l(w) = -\sum_{i=1}^{m} \text{sgn}(w^\top \text{x}_i) y_i$$

What is $\nabla l(w)$?

What is $\frac{d}{dx}\text{sgn}(x)$?

CLASSIFICATION

CLASSIFICATION LOSS

Idea 1: +1 error for each miss classified data point

$$l(w) = -\sum_{i=1}^{m} \text{sgn}(w^\top \text{x}_i) y_i$$

What is $\nabla l(w)$?

$$\frac{d}{dx}\text{sgn}(x) = 0 \text{ for } x \neq 0$$

Not differentiable everywhere and the derivative has no direction to maximize function

CLASSIFICATION

CLASSIFICATION LOSS

Idea #2:

- treat labels as random variables, $Y \in \{-1, +1\}$
- $\hat{Y}$ random variable representing the prediction from the model f
- we want to know $\Pr(\hat{Y} = +1 \,|\, X = x)$ for each x

  $$\Pr(\hat{Y} = -1 \,|\, X = x) = 1 - \Pr(\hat{Y} = +1 \,|\, X = x)$$

- Classification error is based on how unlikely the label is
  - $\Pr(\hat{Y} = y_i \,|\, X = x_i)$ If the probability is low under our model estimate, then that is bad

MODELING PROBABILITIES

FUNCTION REPRESENTATION

$$f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$$

$$f(x, w) \in [0,1]$$

$$f(x, w) = \sigma(w^\top x)$$

$$w^\top x \to \mathbb{R}$$

$$\sigma\colon \mathbb{R} \to [0,1]$$

MODELING PROBABILITIES

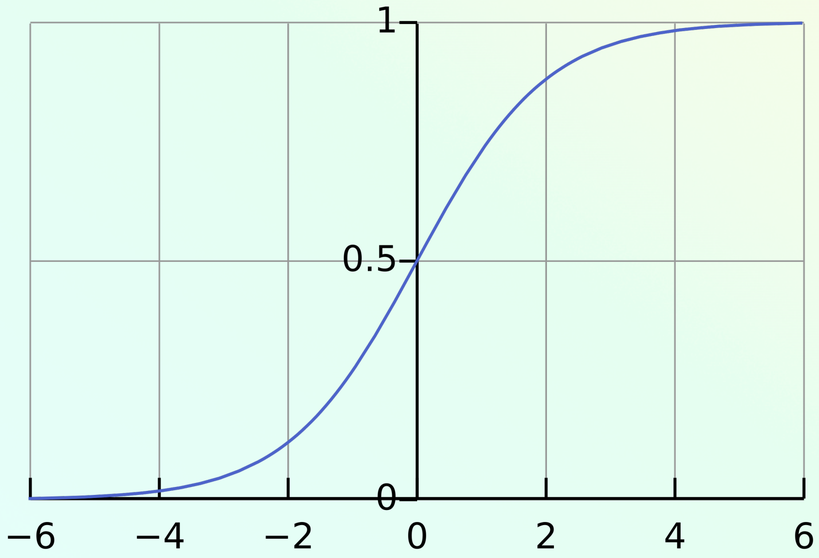
FUNCTION REPRESENTATION

$$f(x, w) = \sigma(w^\top x)$$

Sigmoid function:

$$\sigma(x) \doteq \frac{1}{1 + e^{-x}}$$

$$\sigma(x) \in (0,1)$$



MODELING PROBABILITIES

FUNCTION REPRESENTATION

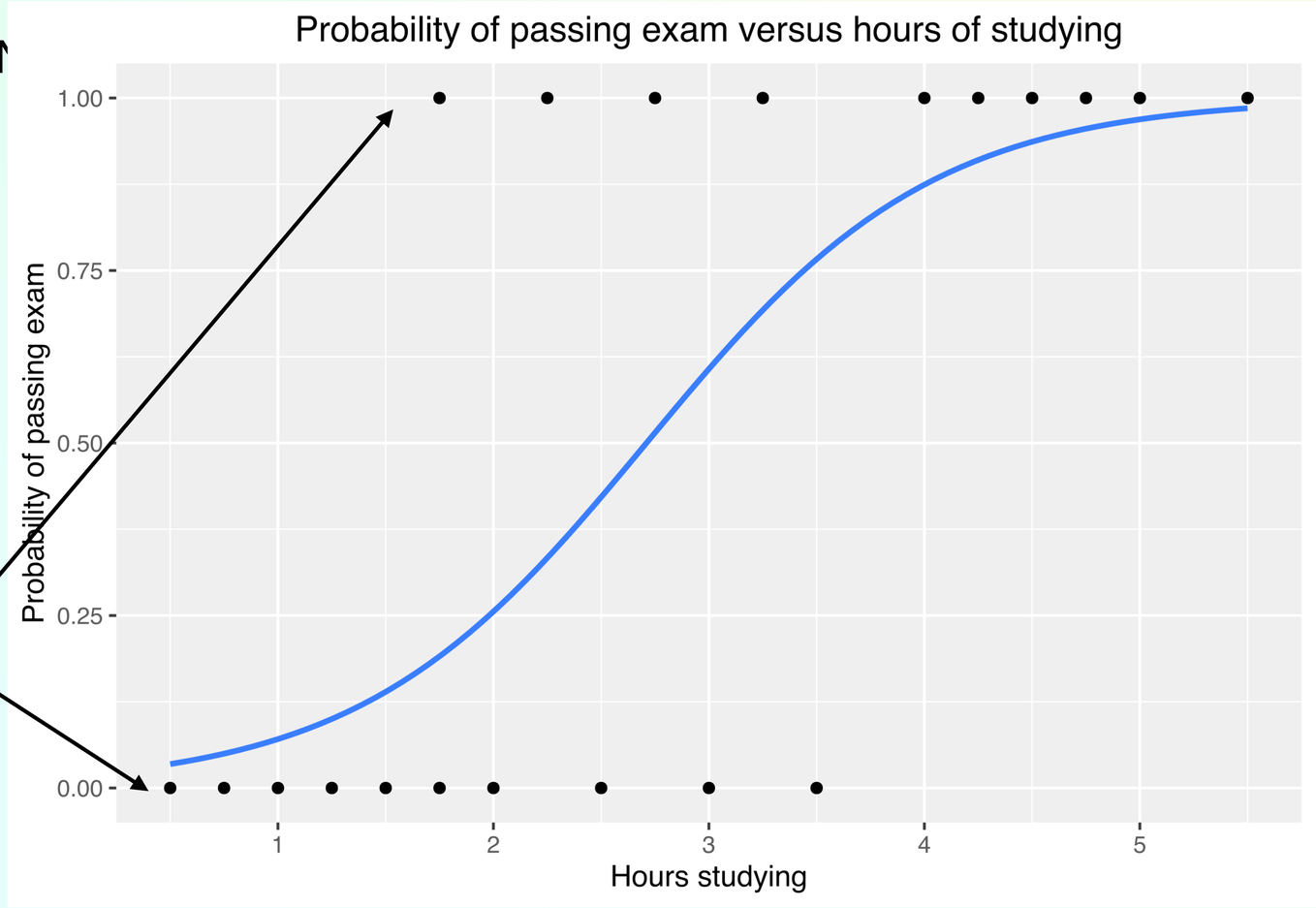
$$f(x, w) = \sigma(w^\top x) = \frac{1}{1 - e^{-w^\top x}}$$

$$\Pr(\hat{Y} = +1 \,|\, X = x) \doteq f(x, w)$$

This is the probability under the model, not the actual probability $\Pr(Y = +1 \,|\, X = x)$

MODELING PROBABILITIES

FUNCTION REPRESEN



Y labels are $\{0,1\}$
Instead of $\{-1,1\}$

MODELING PROBABILITIES

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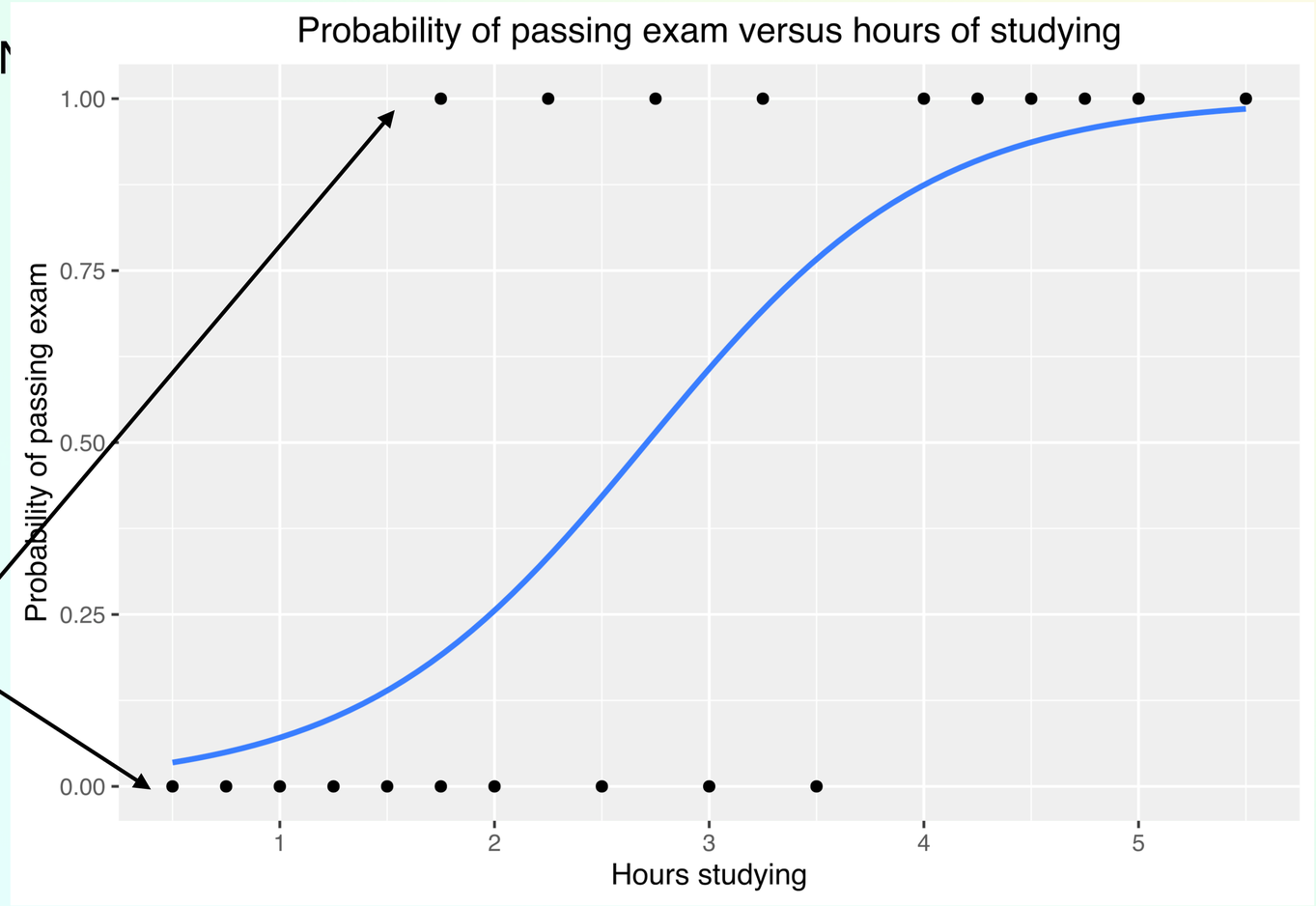

Y labels are $\{0,1\}$
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Can view $f(x, w)$ as a probability

More accurate to call $f(x, w)$ model confidence

$f(x, w) \approx 1$ is very confident that $Y = +1$

$f(x, w) \approx 0$ is very confident that $Y = 0$

CLASSIFICATION

LOSS FUNCTION

Want to maximize the probability of all data being correctly classified

$$\Pr(\hat{Y} = y_1 \,|\, X = x_1) \Pr(\hat{Y} = y_2 \,|\, X = x_2) \ldots \Pr(\hat{Y} = y_m \,|\, X = x_m) = \prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

The product of all probabilities is in (0,1)

Maximum is reached when all probabilities are 1

$$l(w) \doteq - \prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

CLASSIFICATION

LOSS FUNCTION

Minimize a loss function:

$$l(w) \doteq -\prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

Minimize the negative product of probabilities

Equivalent to maximizing product of probabilities

CLASSIFICATION

LOSS FUNCTION

Gradient descent:

We need to compute $\nabla l(w)$

Calculus — Product rule

$$\nabla l(w) = -\frac{\partial}{\partial w} \prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

CLASSIFICATION

LOSS FUNCTION

$$\nabla l(w) = -\frac{\partial}{\partial w}\prod_{i=1}^{m}\Pr(\hat{Y}=y_i\,|\,X=x_i)$$

$$= -\left(\prod_{j=1,j\neq 1}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_1\,|\,X=x_1) - \left(\prod_{j=1,j\neq 2}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_2\,|\,X=x_2) - \ldots$$

CLASSIFICATION

LOSS FUNCTION

$$
\begin{aligned}
\nabla l(w) = & -\frac{\partial}{\partial w}\prod_{i=1}^{m}\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
= & -\left(\prod_{j=1,j\neq 1}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_1\,|\,X=x_1) - \left(\prod_{j=1,j\neq 2}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_2\,|\,X=x_2) - \ldots \\
= & -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_i\,|\,X=x_i)
\end{aligned}
$$

CLASSIFICATION

LOSS FUNCTION

For some function $p(x) \in (0,1)$

$$\frac{\partial}{\partial x} \ln p(x) = \frac{1}{p(x)} \frac{\partial}{\partial x} p(x)$$ logorithm rule –> $$\frac{\partial}{\partial x} \ln x = \frac{1}{x}$$ plus the chain rule

Rewrite this as:

$$\frac{\partial}{\partial x} p(x) = p(x) \frac{\partial}{\partial x} \ln p(x)$$

CLASSIFICATION

LOSS FUNCTION

$$\nabla l(w) = -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_i\,|\,X=x_i)$$

$$= -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\Pr(\hat{Y}=y_i\,|\,X=x_i)\frac{\partial}{\partial w}\ln\Pr(\hat{Y}=y_i\,|\,X=x_i)$$

CLASSIFICATION

LOSS FUNCTION

$$
\begin{aligned}
\nabla l(w) = & -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
= & -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\Pr(\hat{Y}=y_i\,|\,X=x_i)\frac{\partial}{\partial w}\ln\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
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\end{aligned}
$$

CLASSIFICATION

LOSS FUNCTION

$$
\begin{aligned}
\nabla l(w) = & -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
= & -\sum_{i=1}^{m}\left(\prod_{j=1,j\neq i}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\Pr(\hat{Y}=y_i\,|\,X=x_i)\frac{\partial}{\partial w}\ln\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
= & -\sum_{i=1}^{m}\left(\prod_{j=1}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\frac{\partial}{\partial w}\ln\Pr(\hat{Y}=y_i\,|\,X=x_i) \\
= & -\left(\prod_{j=1}^{m}\Pr(\hat{Y}=y_j\,|\,X=x_j)\right)\sum_{i=1}^{m}\frac{\partial}{\partial w}\ln\Pr(\hat{Y}=y_i\,|\,X=x_i)
\end{aligned}
$$

QUIZ

CLASSIFICATION

LOSS FUNCTION

$$\nabla l(w) = \underbrace{-\left(\prod_{j=1}^{m} \Pr(\hat{Y} = y_j \,|\, X = x_j)\right)}_{(a)} \sum_{i=1}^{m} \frac{\partial}{\partial w} \ln \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

For any w, $(a) > 0$. We can interpret this as rescaling the gradient, but not changing the direction, i.e., it does not really matter.

CLASSIFICATION

LOSS FUNCTION

Could instead define a different loss function l' that measure the log-likelihood

$$l'(w) \doteq -\ln \prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

This is called the **negative log-likelihood** (NLL) and is a common loss function in machine learning

We do not change the optimal point because we are just rescaling the loss l, e.g.,

$$l'(w) = -\ln\left(-l(w)\right)$$

CLASSIFICATION

LOSS FUNCTION

Logs turns products into summations!

$$l'(w) = -\ln \prod_{i=1}^{m} \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

$$= -\sum_{i=1}^{m} \ln \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

CLASSIFICATION

LOSS FUNCTION

Simpler derivative

$$\nabla l'(w) = -\frac{\partial}{\partial w} \sum_{i=1}^{m} \ln \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

$$= -\sum_{i=1}^{m} \frac{\partial}{\partial w} \ln \Pr(\hat{Y} = y_i \,|\, X = x_i)$$

CLASSIFICATION

LOSS FUNCTION

Same direction as $\nabla l(w)$

$$\nabla l(w) = -(a) \sum_{i=1}^{m} \frac{\partial}{\partial w} \ln \Pr(\hat{Y} = y_i \,|\, X = x_i)$$
$$= (a) \nabla l'(w)$$

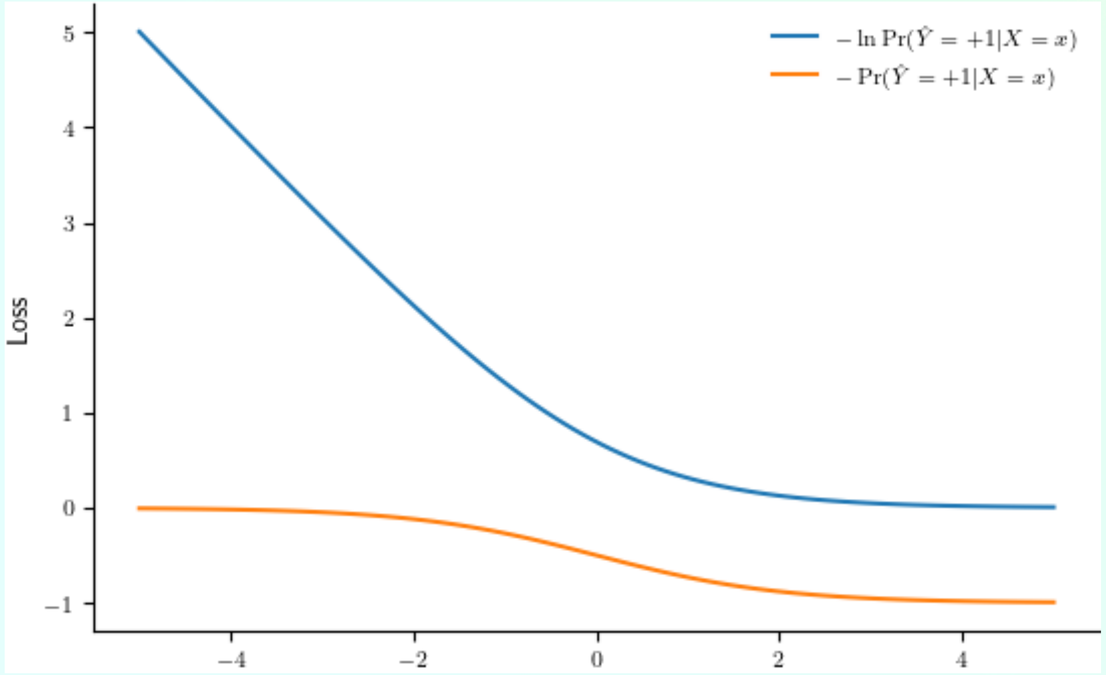
$\nabla l(w)$ points in the exact same direction as $\nabla l'(w)$, they just have different lengths.

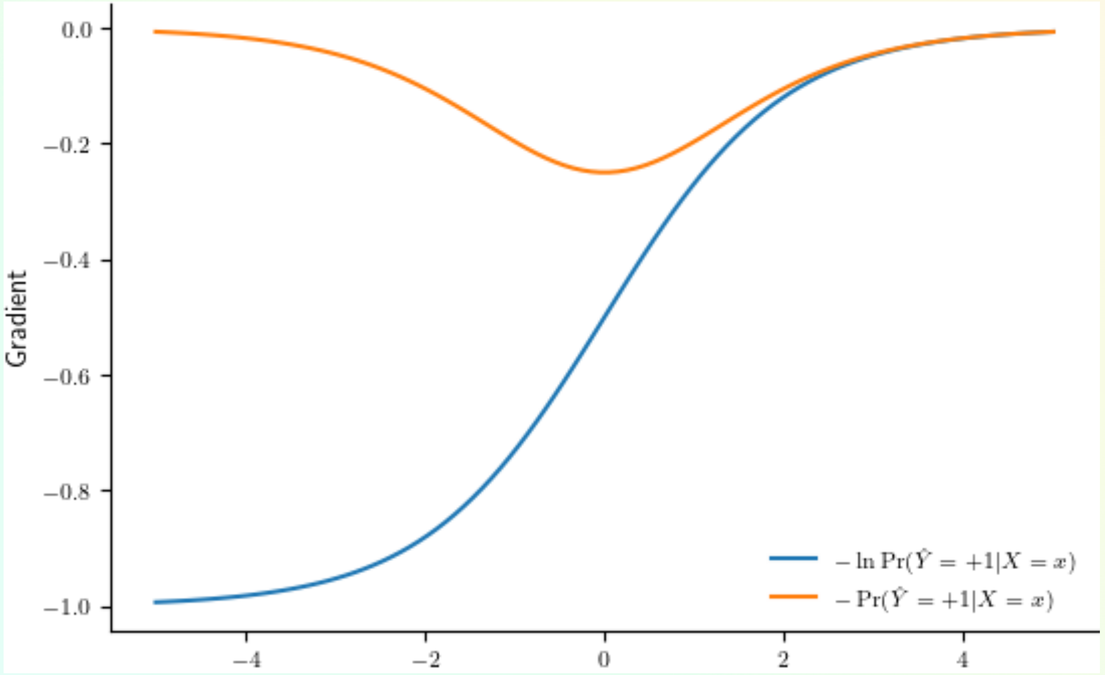
This means we can use NLL objectives, which are often easier to work with.

NLL also has a better gradient

CLASSIFICATION

LOSS FUNCTION



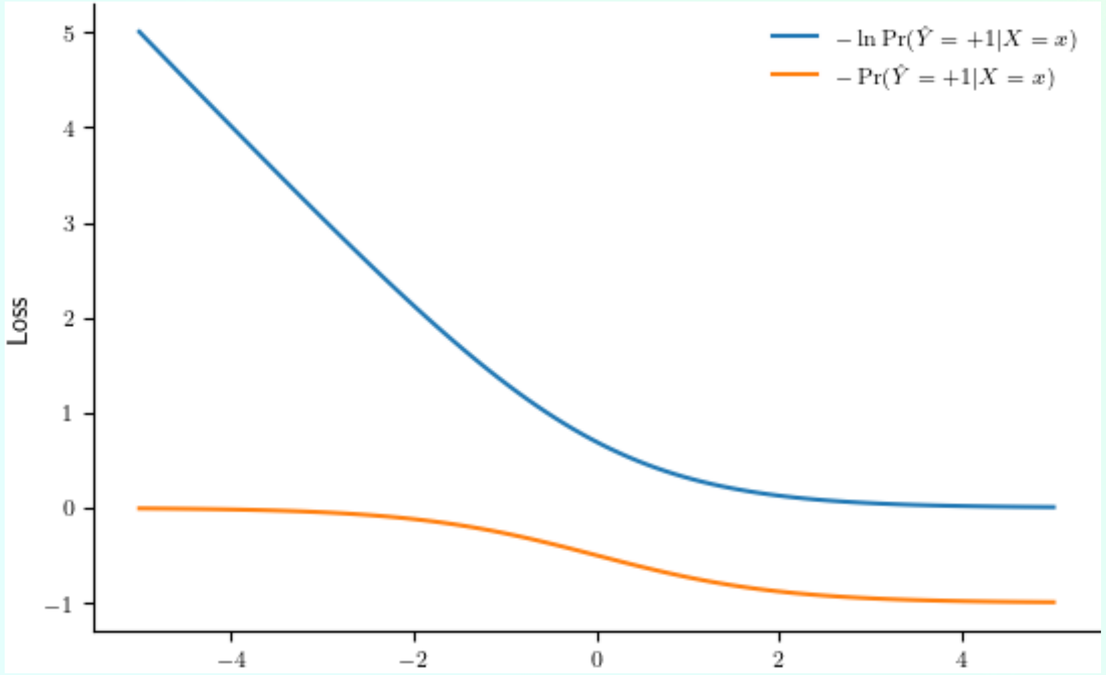


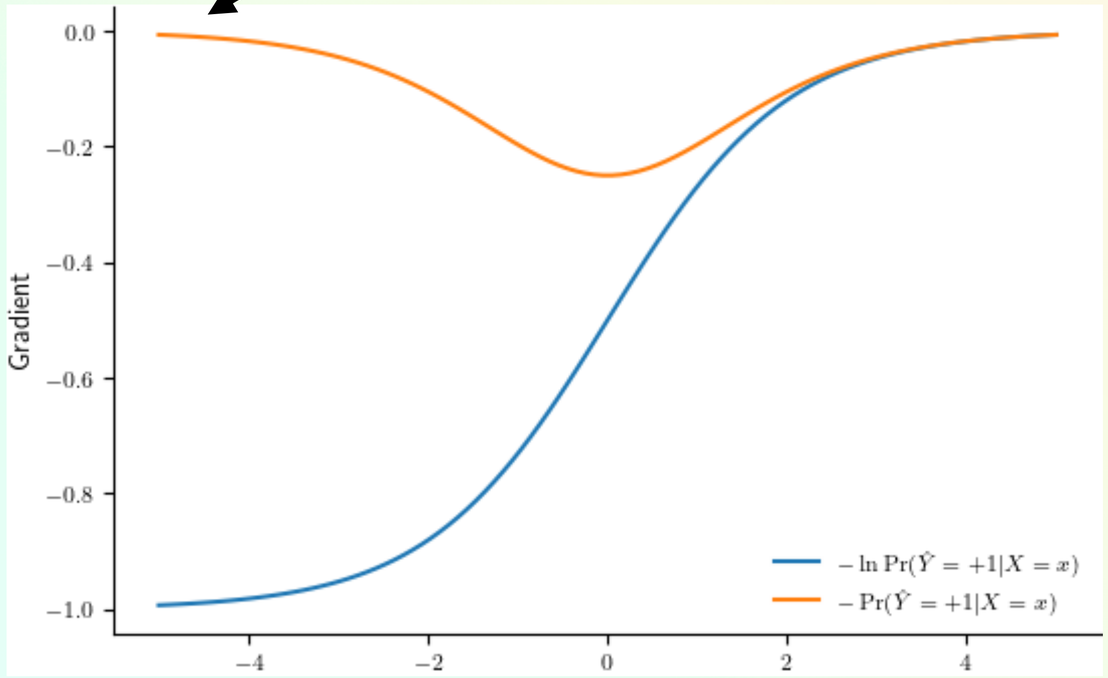
CLASSIFICATION

LOSS FUNCTION

A flat gradient (near zero) means parameters should not change much.

But predictions were really bad and we want them to change a lot





CLASSIFICATION

LOSS FUNCTION

We need an expression for $\Pr(\hat{Y} = y_i \,|\, X = x_i)$ in terms $f(x_i, w)$ so that we can differentiate

$$\Pr(\hat{Y} = +1 \,|\, X = x_i) = f(x_i, w)$$

$$\Pr(\hat{Y} = 0 \,|\, X = x_i) = 1 - \Pr(\hat{Y} = +1 \,|\, X = x_i) = 1 - f(x_i, w)$$

CLASSIFICATION

LOSS FUNCTION

We need an expression for $\Pr(\hat{Y} = y_i \,|\, X = x_i)$ in terms $f(x_i, w)$ so that we can differentiate

$$\Pr(\hat{Y} = +1 \,|\, X = x_i) = f(x_i, w)$$

$$\Pr(\hat{Y} = 0 \,|\, X = x_i) = 1 - \Pr(\hat{Y} = +1 \,|\, X = x_i) = 1 - f(x_i, w)$$

$$\Pr(\hat{Y} = y_i \,|\, X = x_i) = \mathbf{1}_{y_i=+1} \Pr(\hat{Y} = +1 \,|\, X = x_i) + \mathbf{1}_{y_i=0} \Pr(Y = 0 \,|\, X = x_i)$$

$\mathbf{1}_A = 1$ if A is true and $\mathbf{1}_A = 0$ otherwise (called an indicator function)

CLASSIFICATION

LOSS FUNCTION

$$\begin{aligned}\Pr(\hat{Y} = y_i \,|\, X = x_i) = &\ \mathbf{1}_{y_i=+1} \Pr(\hat{Y} = +1 \,|\, X = x_i) + \mathbf{1}_{y_i=0} \Pr(\hat{Y} = 0 \,|\, X = x_i) \\ = &\ \mathbf{1}_{y_i=+1} f(x_i, w) + \mathbf{1}_{y_i=0} \Big(1 - f(x_i, w)\Big)\end{aligned}$$

Simplify the computation by replacing $\mathbf{1}_A$

$$\Pr(\hat{Y} = y_i \,|\, X = x_i) = y_i f(x_i, w) + (1 - y_i)\Big(1 - f(x_i, w)\Big)$$

CLASSIFICATION

LOSS FUNCTION

$$\begin{aligned}\Pr(\hat{Y} = y_i \,|\, X = x_i) &= \mathbf{1}_{y_i=+1} \Pr(\hat{Y} = +1 \,|\, X = x_i) + \mathbf{1}_{y_i=0} \Pr(\hat{Y} = 0 \,|\, X = x_i) \\ &= \mathbf{1}_{y_i=+1} f(x_i, w) + \mathbf{1}_{y_i=0} \left(1 - f(x_i, w)\right)\end{aligned}$$

Simplify the computation by replacing $\mathbf{1}_A$

$$\Pr(\hat{Y} = y_i \,|\, X = x_i) = y_i f(x_i, w) + (1 - y_i)\left(1 - f(x_i, w)\right)$$

Similarly, we have

$$\ln \Pr(\hat{Y} = y_i \,|\, X = x_i) = y_i \ln f(x_i, w) + (1 - y_i) \ln \left(1 - f(x_i, w)\right)$$

CLASSIFICATION

LOSS FUNCTION

NLL loss function for classification with $y_i \in \{0,1\}$

$$l(w) = -\sum_{i=1}^{m} \ln \Pr\left(\hat{Y} = y_i \,|\, X = x_i\right)$$

$$= -\sum_{i=1}^{m} y_i \ln\left(f(x_i, w)\right) + (1 - y_i)\ln\left(1 - f(x_i, w)\right)$$

CLASSIFICATION

LOSS FUNCTION

Take-home questions:

1. $f(x, w) = \frac{1}{1 + e^{-w^\top x}}$, what is $\frac{\partial f(x, w)}{\partial w}$?
2. What is $\nabla l(w)$ for the NLL loss?

We will go over these answers next time.

NEXT CLASS

Next Class — Stochastic Gradient Descent