OPTIMIZING BASIS FUNCTIONS

OPTIMIZING BASIS FUNCTIONS

GOALS FOR TODAY

A step towards deep learning and neural networks

How to optimize the basis function

General formulization

OPTIMIZING BASIS FUNCTIONS

OVERVIEW

Choose heuristic basis functions and do pretty good

If the fit is not good enough, we generate more features

— create more bins, etc

OPTIMIZING BASIS FUNCTIONS

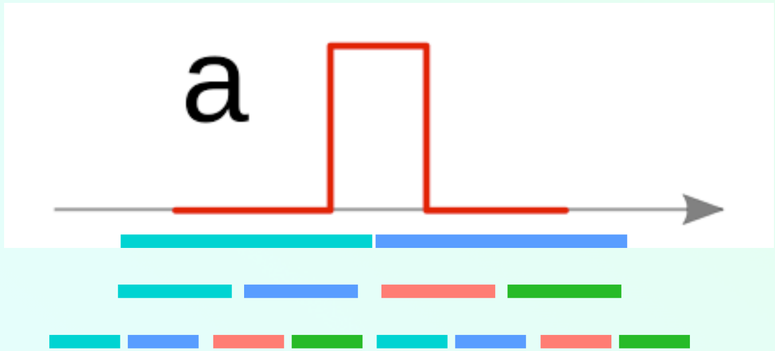
OVERVIEW

Choose heuristic basis functions and do pretty good

If the fit is not good enough, we generate more features

— create more bins, etc

Can lead to a huge number of features to get small error



OPTIMIZING BASIS FUNCTIONS

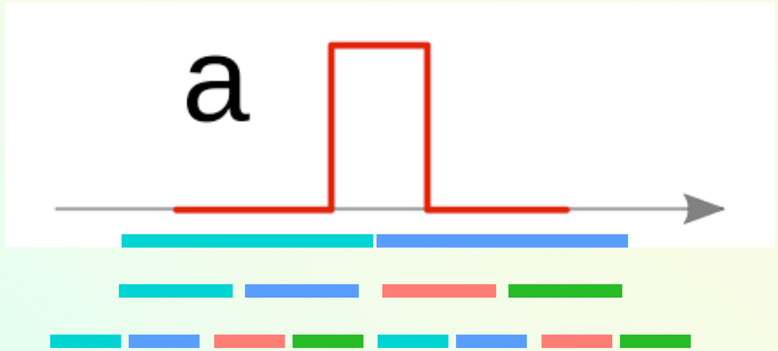
OVERVIEW

Choose heuristic basis functions and do pretty good

If the fit is not good enough, we generate more features

— create more bins, etc

Can lead to a huge number of features to get small error



Features might just need a little bit of adjustment

Solution: Optimize a fixed set of features so they can best approximate the function.

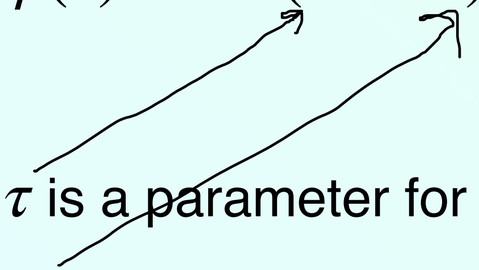
— move the center or scale a little bit to decrease the loss

PARAMETERIZING BASIS FUNCTIONS

MECHANICS

$$f(x, w) = w\phi(x)$$

$$\phi(x) \doteq \sin(\tau x + \kappa)$$



τ is a parameter for the frequency

κ is a parameter for the offset

PARAMETERIZING BASIS FUNCTIONS

MECHANICS

$$f(x, w) = w\phi(x, \tau, \kappa)$$

$$\phi(x, \tau, \kappa) \doteq \sin(\tau x + \kappa)$$

PARAMETERIZING BASIS FUNCTIONS

MECHANICS

$$f(x, w, \tau, \kappa) = w\phi(x, \tau, \kappa)$$

$$\phi(x, \tau, \kappa) \doteq \sin(\tau x + \kappa)$$

PARAMETERIZING BASIS FUNCTIONS

MECHANICS

$$\beta = \begin{bmatrix} \tau \\ \kappa \end{bmatrix}$$

$$f(x, w, \beta) = w\phi(x, \beta)$$

$$\phi(x, \beta) \doteq \sin(\beta_1 x + \beta_2)$$

PARAMETERIZING BASIS FUNCTIONS

MECHANICS

$$\beta = \begin{bmatrix} \tau \\ \kappa \end{bmatrix}$$ ≒ Frequency (β_1)
≒ offset (β_0) or β_2

$$f(x, w, \beta) = w\phi(x, \beta)$$

$$\phi(x, \beta) \doteq \sin(\beta_1 x + \beta_2)$$

$$l(w, \beta) \doteq \mathbf{E}\left[\left(f(X, w, \beta) - Y\right)^2\right]$$

QUIZ

OPTIMIZING BASIS FUNCTIONS

MECHANICS

$$
\begin{aligned}
\nabla l(w,\beta) = & \left[\frac{\partial l(w,\beta)}{\partial w}, \frac{\partial l(w,\beta)}{\partial \beta}\right] \\
= & \mathbf{E}\left[\left(f(X,w,\beta) - Y\right)\left[\frac{\partial f(X,w,\beta)}{\partial w}, \frac{\partial f(X,w,\beta)}{\partial \beta}\right]\right] \\
= & \mathbf{E}\left[\left(f(X,w,\beta) - Y\right)\left[\phi(X,\beta), \frac{\partial f(X,w,\beta)}{\partial \beta}\right]\right]
\end{aligned}
$$

OPTIMIZING BASIS

MECHANICS

$$\frac{\partial f(X, w, \beta)}{\partial \beta} = \frac{\partial}{\partial \beta} w \phi(X, \beta)$$

$$= w \frac{\partial}{\partial \beta} \phi(X, \beta)$$

OPTIMIZING BASIS

MECHANICS

$$\phi(x, \beta) = \sin(\beta_1 x + \beta_2)$$

$$\begin{aligned}
\frac{\partial \phi(x, \beta)}{\partial \beta} &= \frac{\partial}{\partial \beta} \sin(\beta_1 x + \beta_2) \\
&= \cos(\beta_1 x + \beta_2) \frac{\partial}{\partial \beta} \left(\beta_1 x + \beta_2\right) \\
&= \cos(\beta_1 x + \beta_2) \begin{bmatrix} x \\ 1 \end{bmatrix}
\end{aligned}$$

OPTIMIZING BASIS

MECHANICS

$$\frac{\partial f(X, w, \beta)}{\partial \beta} = w \frac{\partial}{\partial \beta} \phi(X, \beta)$$

$$\frac{\partial \phi(x, \beta)}{\partial \beta} = \cos(\beta_1 x + \beta_2) \begin{bmatrix} x \\ 1 \end{bmatrix}$$

$$\frac{\partial f(x, w, \beta)}{\partial \beta} = w \cos(\beta_1 x + \beta_2) \begin{bmatrix} x \\ 1 \end{bmatrix}$$

OPTIMIZING BASIS

MECHANICS

$$\begin{aligned}\nabla l(w,\beta) = &\ \mathbf{E}\left[\left(f(X,w,\beta)-Y\right)\left[\phi(X,\beta),\frac{\partial f(X,w,\beta)}{\partial\beta}\right]\right]\\ = &\ \mathbf{E}\left[\left(f(X,w,\beta)-Y\right)\left[\phi(X,\beta),w\cos(\beta_1 X+\beta_2)\begin{bmatrix}x\\1\end{bmatrix}\right]\right]\end{aligned}$$

CHAIN RULE DERIVATIVES

MECHANICS

$$\frac{\partial l(X, Y, w, \beta)}{\partial w} = \frac{\partial l(X, Y, w, \beta)}{\partial f(X, w, \beta)} \frac{\partial f(X, w, \beta)}{\partial w}$$

$$\frac{\partial l(X, Y, w, \beta)}{\partial \beta} = \frac{\partial l(X, Y, w, \beta)}{\partial f(X, w, \beta)} \frac{\partial f(X, w, \beta)}{\partial \phi(X, \beta)} \frac{\partial \phi(X, \beta)}{\partial \beta}$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE

$$f(x, w, \beta) = w\phi(x, \beta)$$

$$\phi(x, \beta) \doteq \sin(\beta_1 x + \beta_2)$$

$$Y = w_* \sin\left(2X + \frac{1}{2}\right) + \xi$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE

$$f(x, w, \beta) = w\phi(x, \beta)$$

$$\phi(x, \beta) \doteq \sin(\beta_1 x + \beta_2)$$

$$Y = w_* \sin\left(2X + \frac{1}{2}\right) + \xi$$

$$l(w, \beta) = \mathbf{E}\left[\left(f(X, w, \beta) - Y\right)^2\right]$$

$$\arg\min_{w,\beta} l(w, \beta)$$

Use gradient descent to optimize $l(w, \beta)$

OPTIMIZING BASIS FUNCTIONS

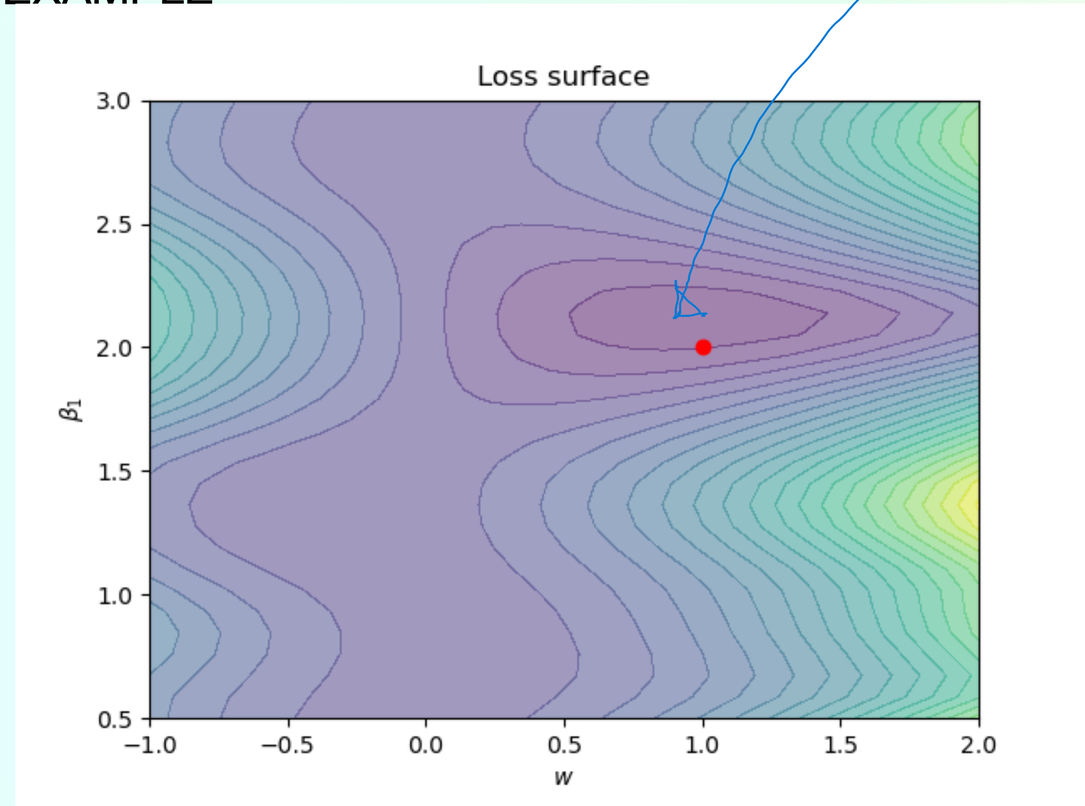
EXAMPLE

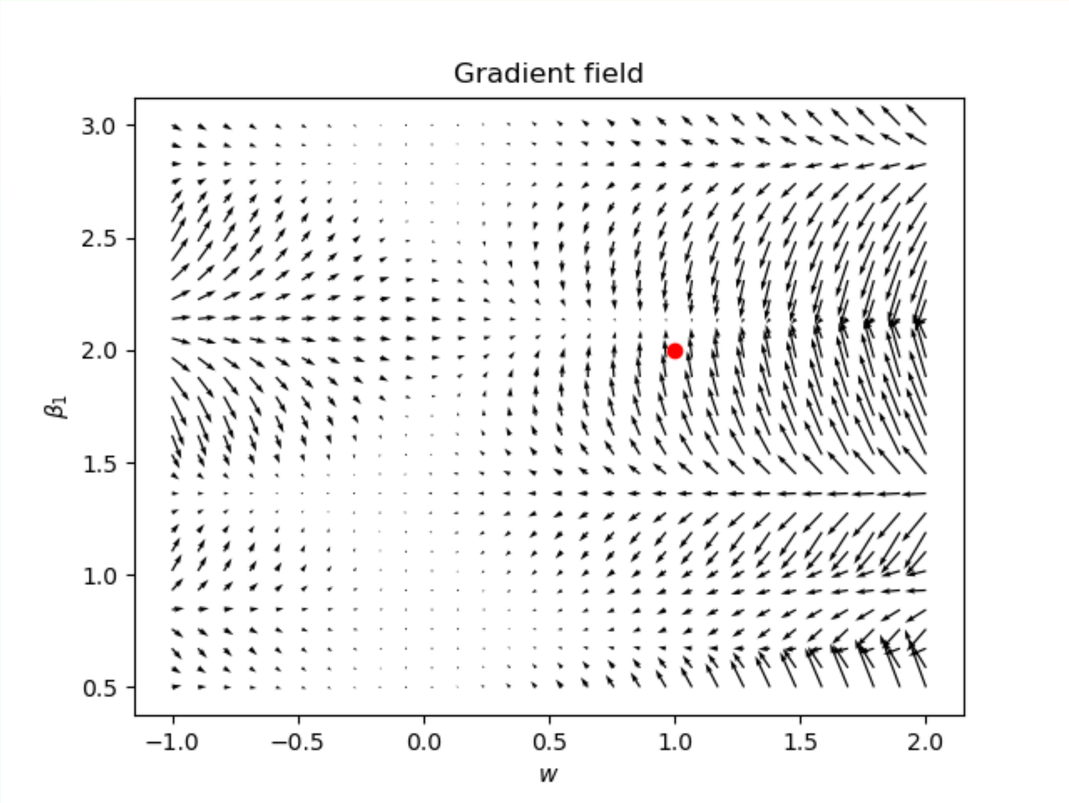
$$\begin{bmatrix} w^{k+1} \\ \beta^{k+1} \end{bmatrix} = \begin{bmatrix} w^{k} \\ \beta^{k} \end{bmatrix} - \eta \nabla l(w, \beta)$$

Is this going to be successful?

What does $l(w, \beta)$ and $\nabla l(w, \beta)$ look like?

EXAMPLE

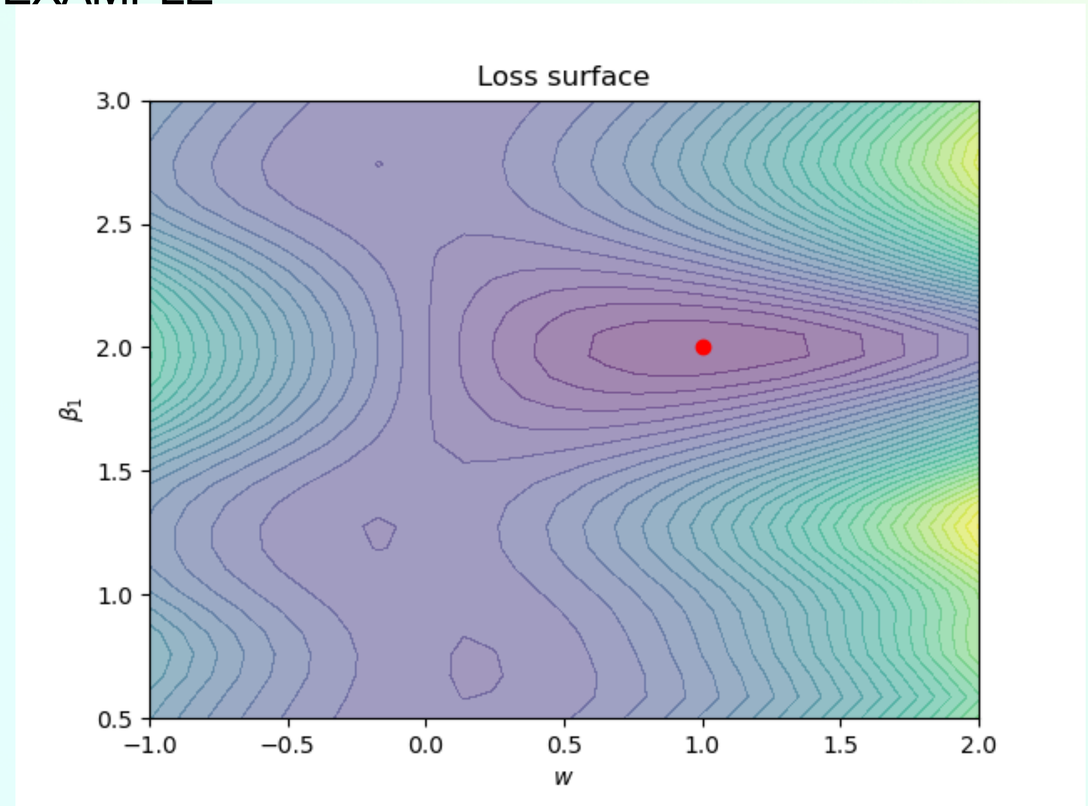


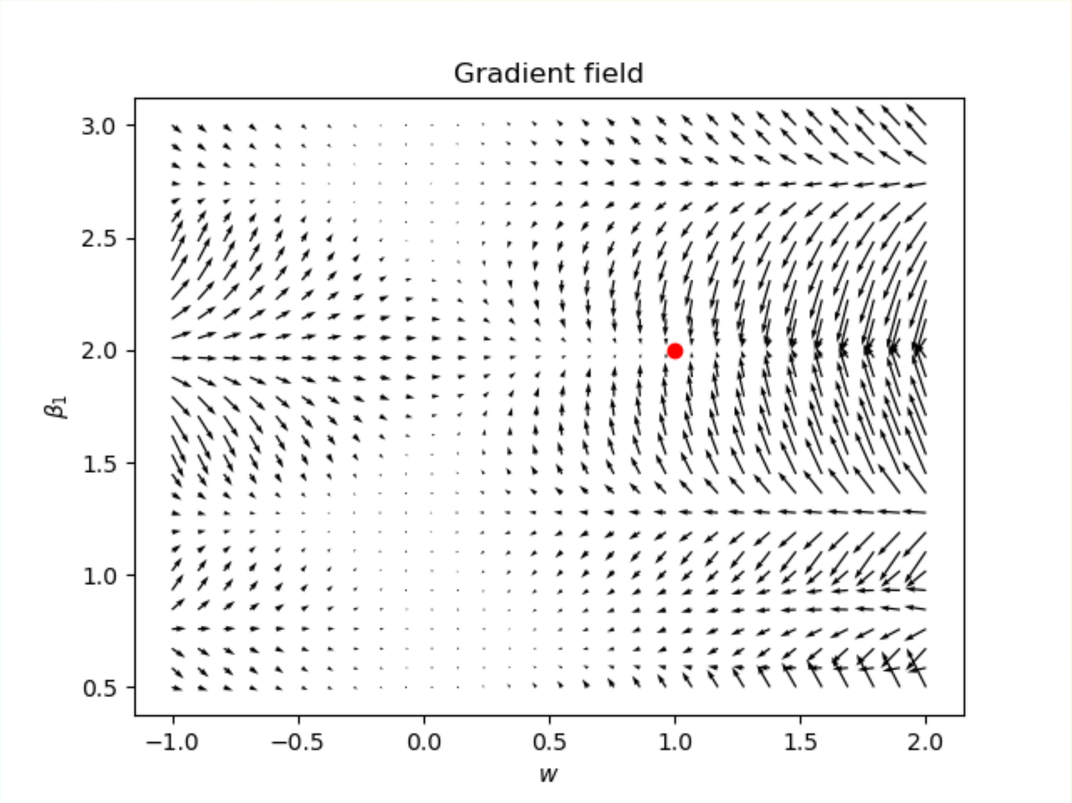


$$\beta_2 = 0$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE

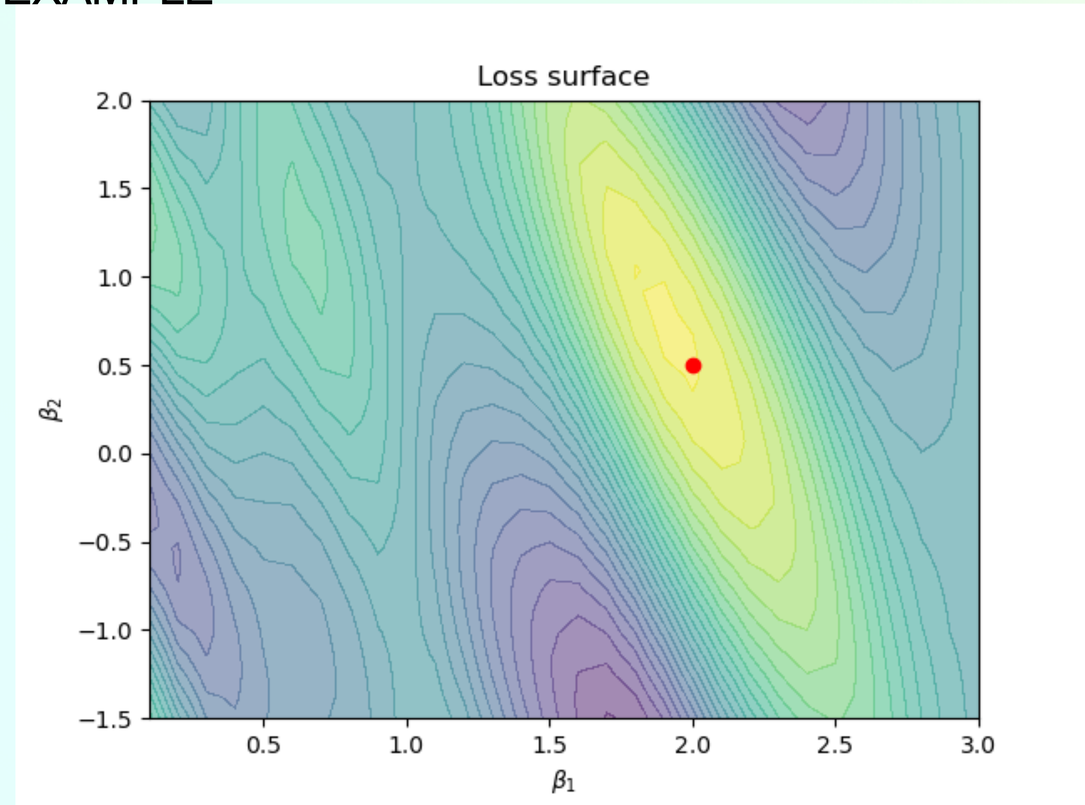


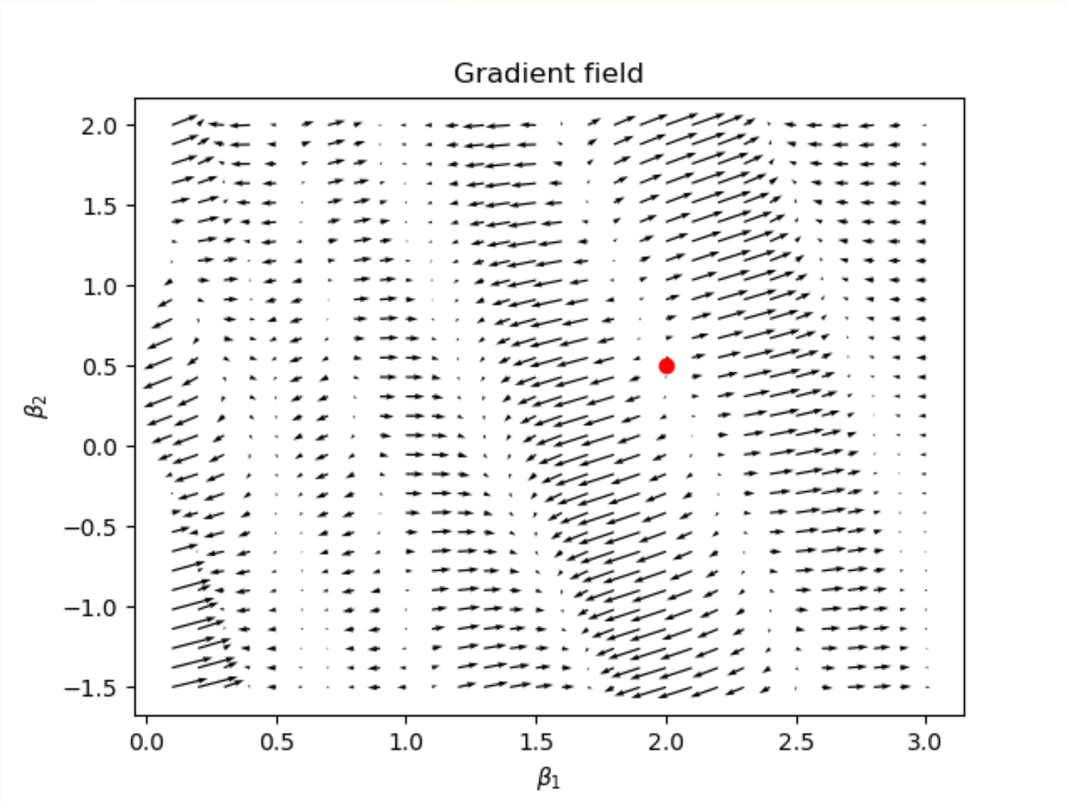


$$\beta_2 = \frac{1}{2}$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE

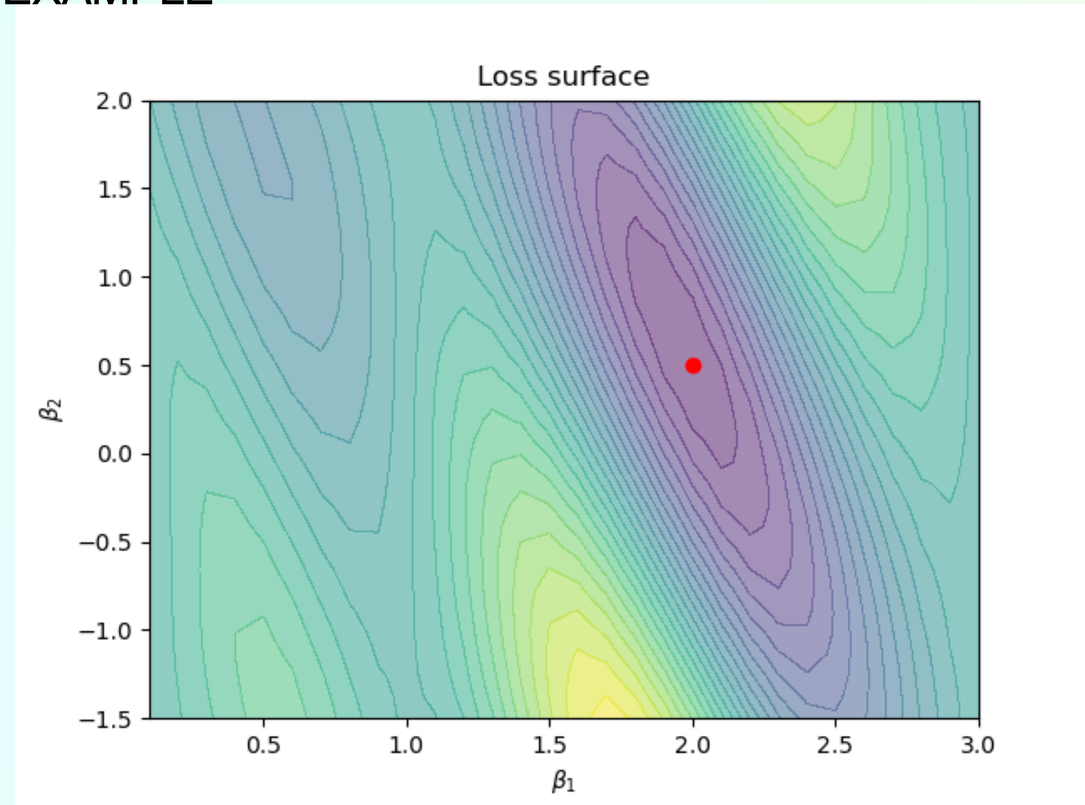


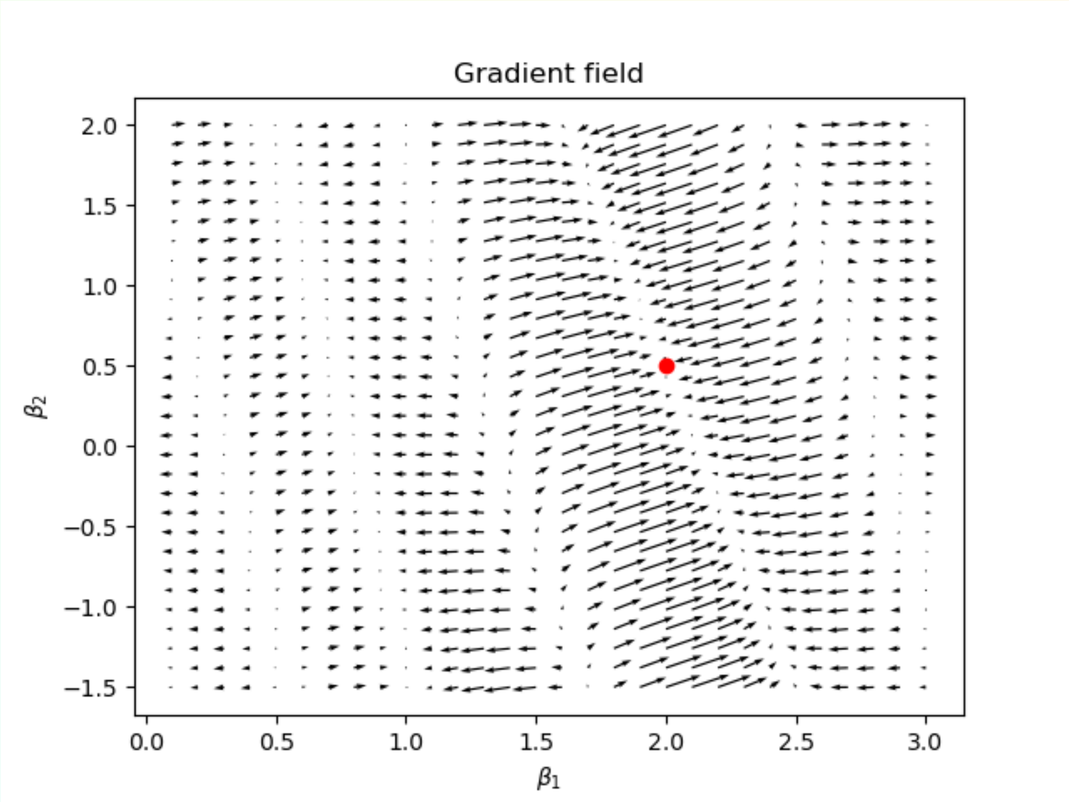


$$w = -1$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE





$$w = 0.1$$

OPTIMIZING BASIS FUNCTIONS

EXAMPLE

What does $l(w, \beta)$ and $\nabla l(w, \beta)$ look like?

- Many local minima
- Many plateaus/saddle points where the gradient is near zero in all or most dimensions
- Wrong value of w can cause β to move away from the optimum
- A small range of values to initialize w^0, β^0 that leads to correct values
  - With $\xi = 0$ there is no approximation error (perfect fit to the data)

Disappointing — gradient descent is not a good optimization method for this function.

OPTIMIZING BASIS FUNCTIONS

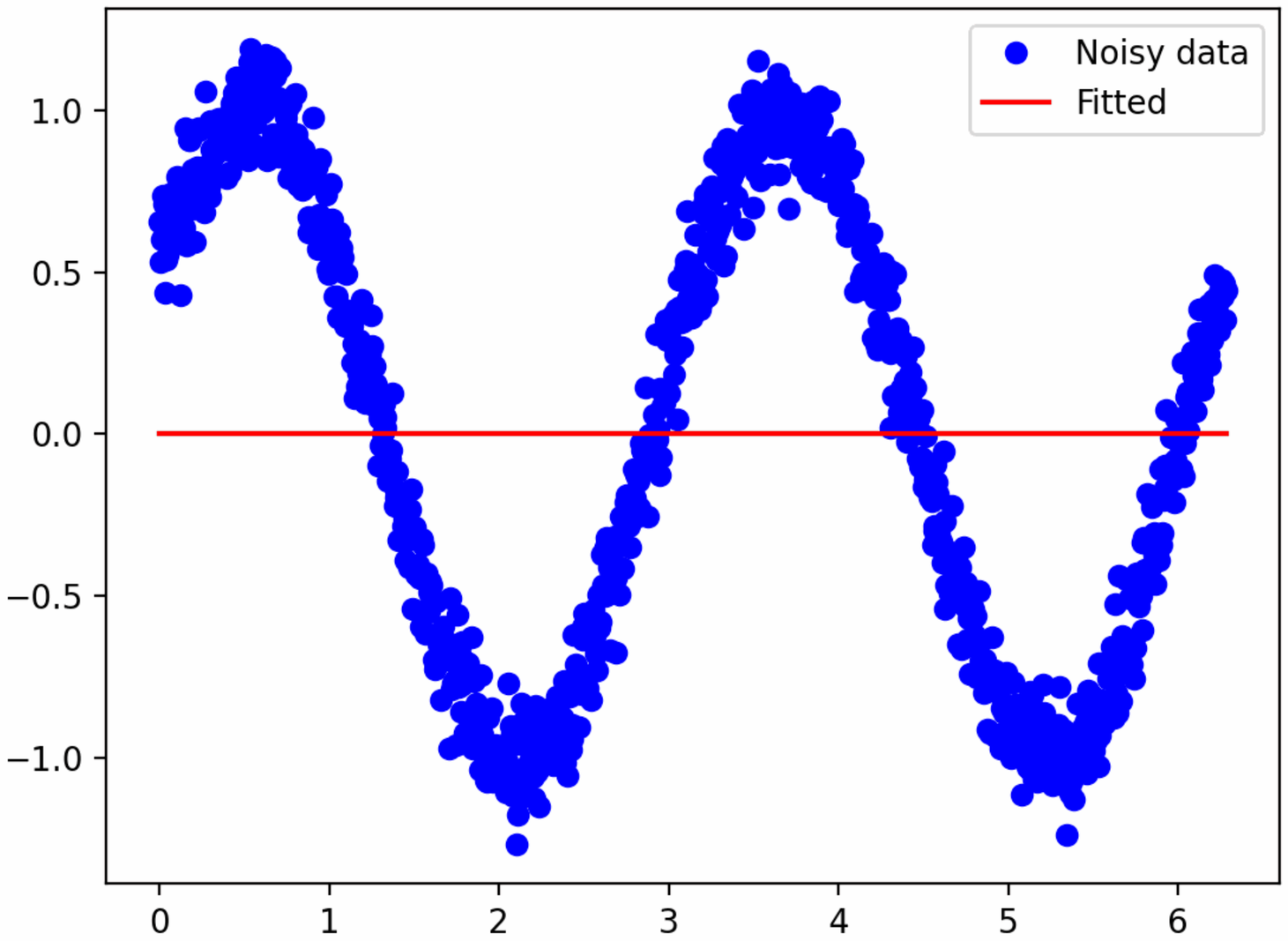
WHAT IF THERE ARE MANY FEATURES?

$$\phi(x, \beta) = \begin{bmatrix} \sin(\beta_{1,1}x + \beta_{1,2}) \\ \sin(\beta_{2,1}x + \beta_{2,2}) \\ \sin(\beta_{3,1}x + \beta_{3,2}) \\ \sin(\beta_{4,1}x + \beta_{4,2}) \\ \sin(\beta_{5,1}x + \beta_{5,2}) \end{bmatrix}$$

$$f(x, w, \beta) = w^\top \phi(x, \beta)$$

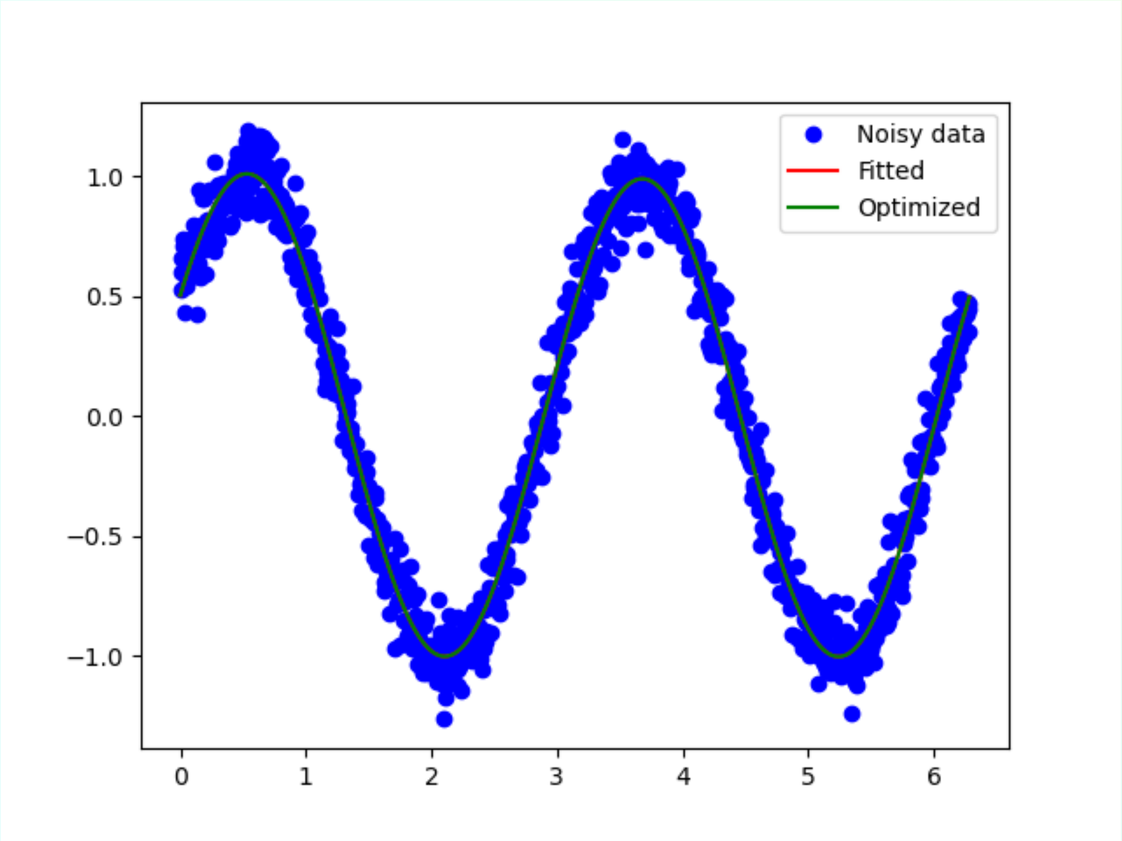
OPT

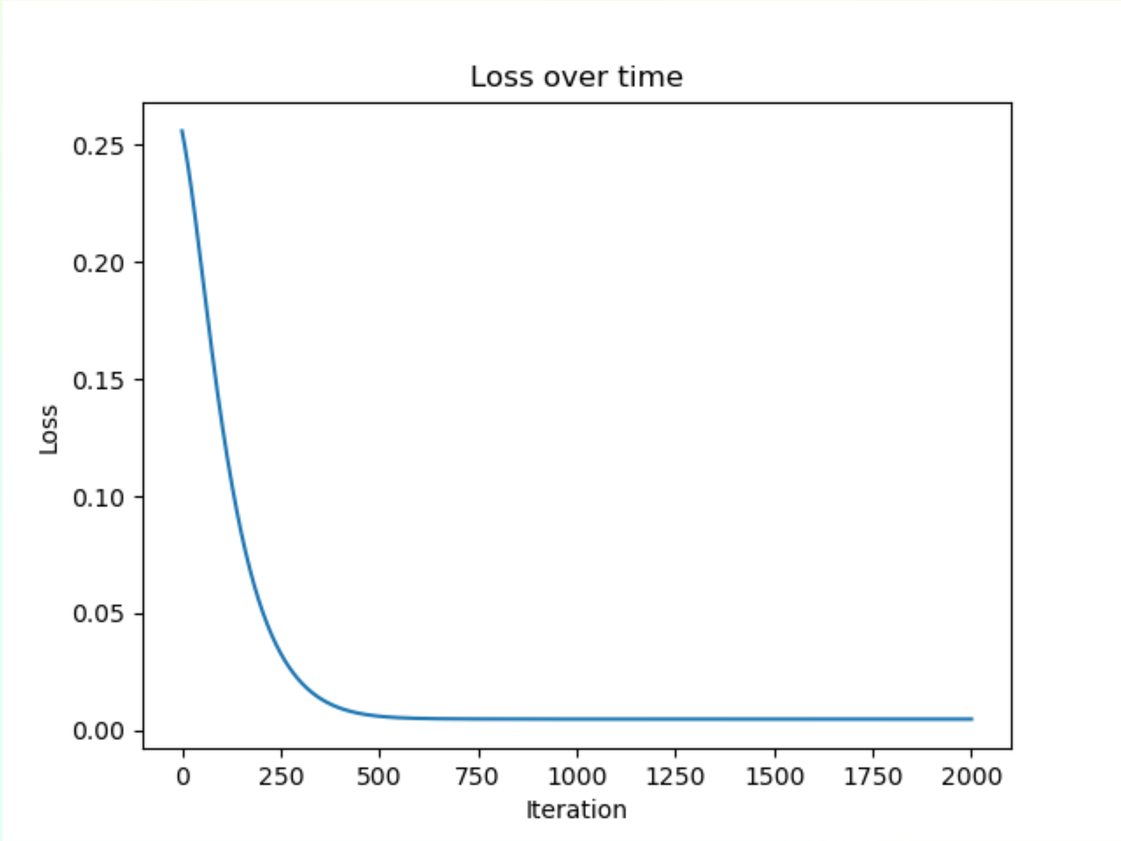
WHAT IF THE



OPTIMIZING BASIS FUNCTIONS

WHAT IF THERE ARE MANY FEATURES?



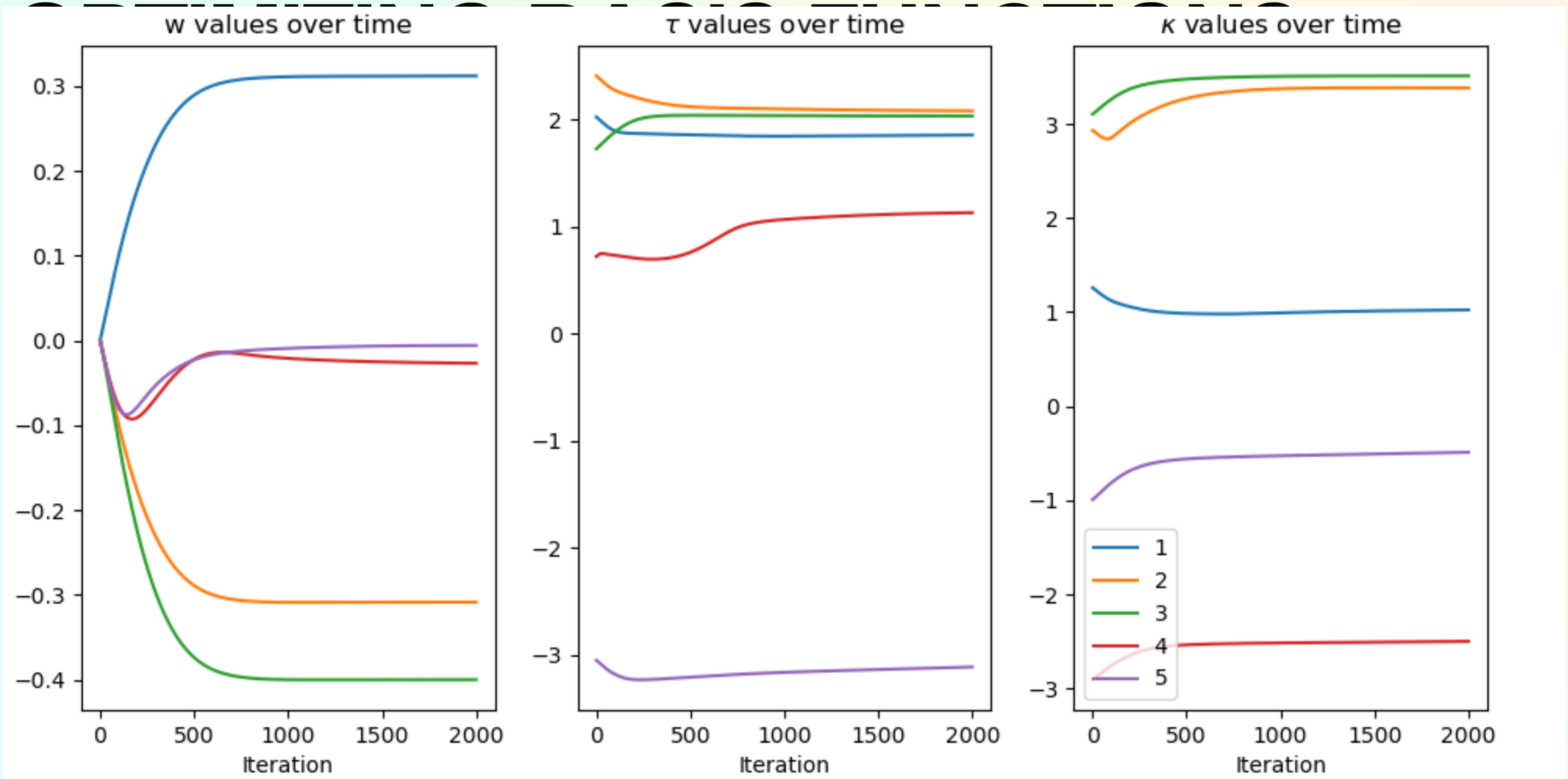


OPTIMIZING BASIS FUNCTIONS

WHAT IF THERE ARE MANY FEATURES?

Does it identify the correct sinewave?

$$f_*(x) = \sin(2x + 0.5)$$

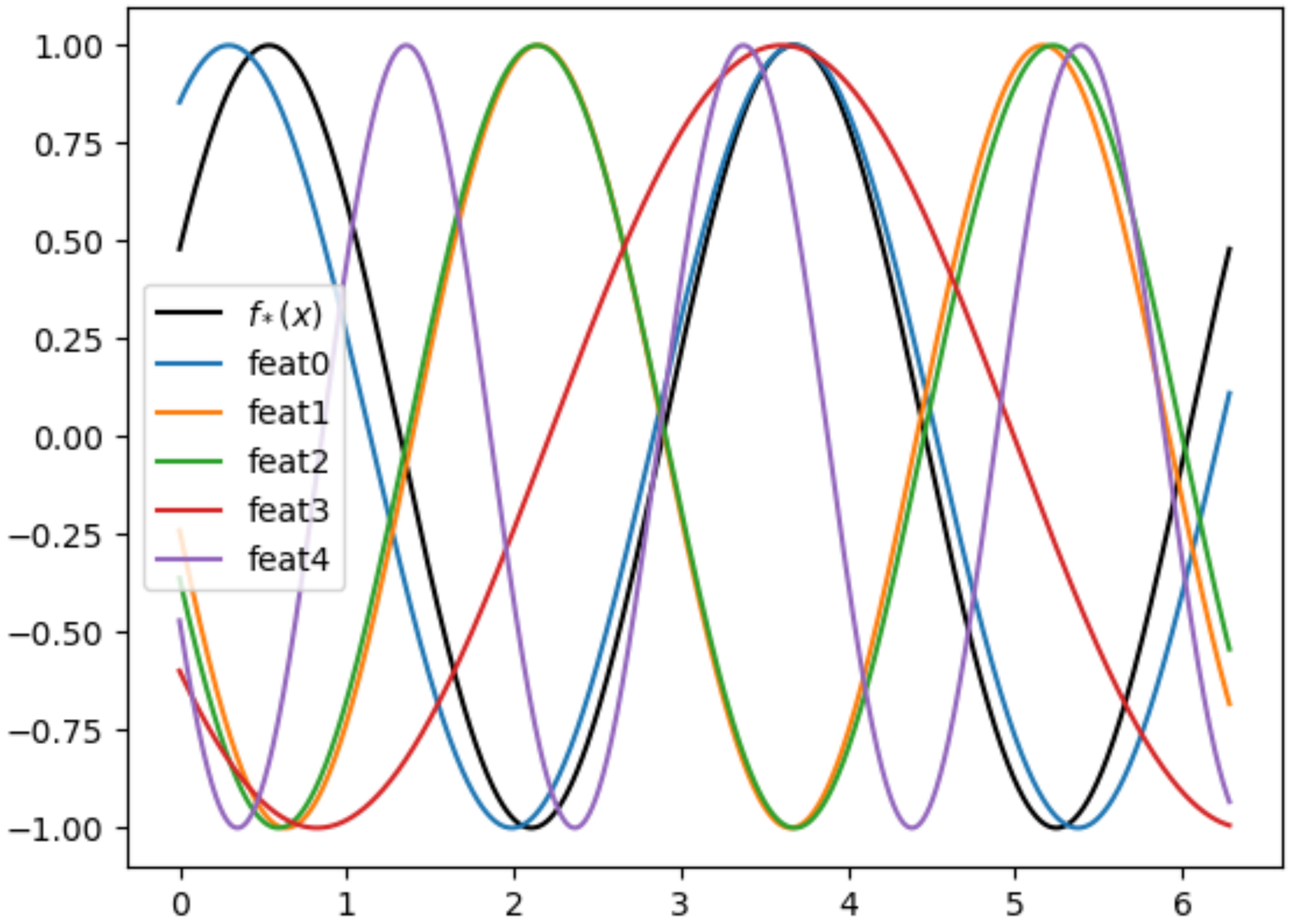

w values over time
τ values over time
κ values over time
Iteration
Iteration
Iteration
1
2
3
4
5

OPTI

WHAT IF THER

Does it identi

$$f_*(x) = \sin$$



OPTIMIZING BASIS FUNCTIONS

WHAT IF THERE ARE MANY FEATURES?

Does it identify the correct sinewave?

$$f_*(x) = \sin(2x + 0.5)$$

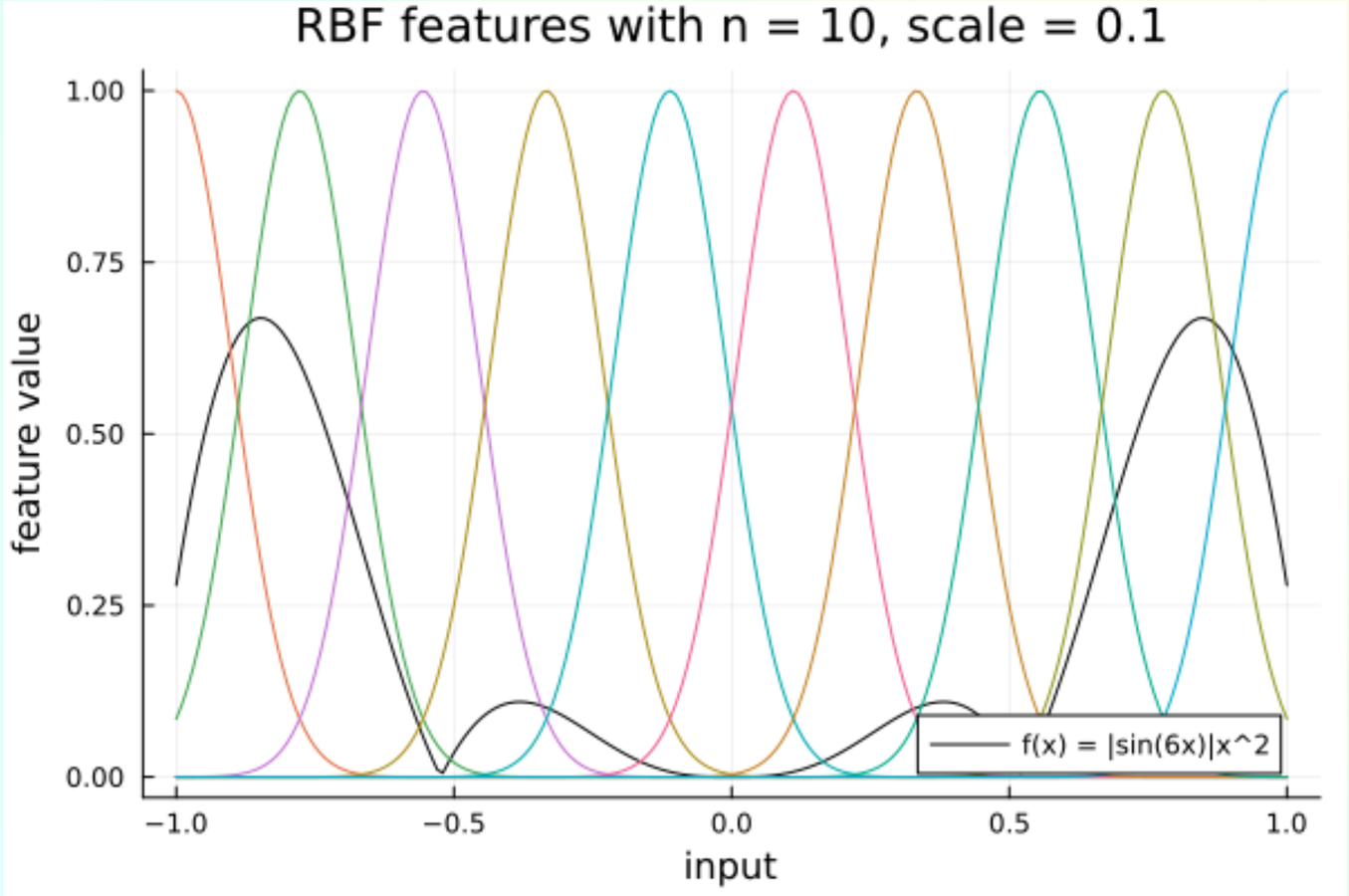
No, but it got close.

The gradient is close to zero

The model does not need to find the exact function, only a close one.

No guarantee that $f \to f_*$ in the same parameterization

DIFFERENTIABLE BINNING



RADIAL BASIS FUNCTIONS

$$\phi_i(x) = e^{-\frac{1}{2}\frac{(a-\mu_i)^2}{\sigma_i^2}}$$

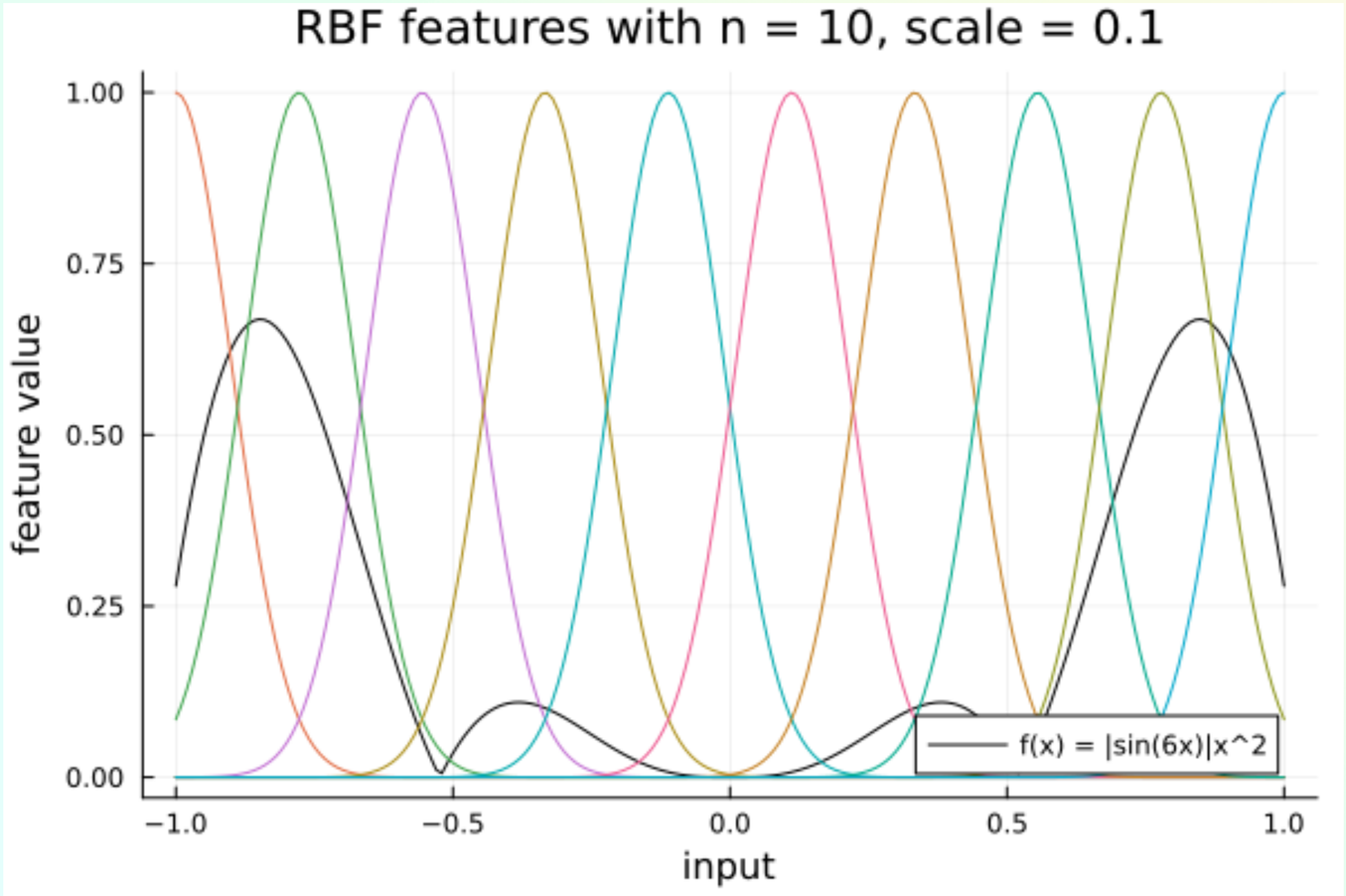
μ_i is the center

σ_i is the scale and controls the “width” of the feature

$\phi_i(\mu) = 1$

$\phi_i(\mu + 3\sigma) \approx 0$

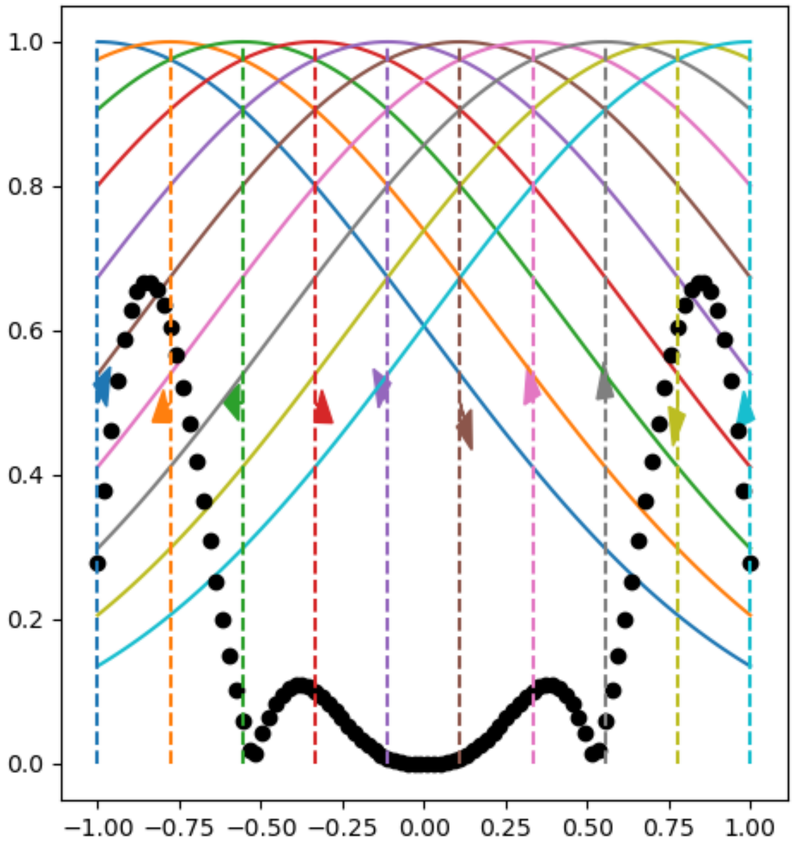
OPTIMIZING RBFS

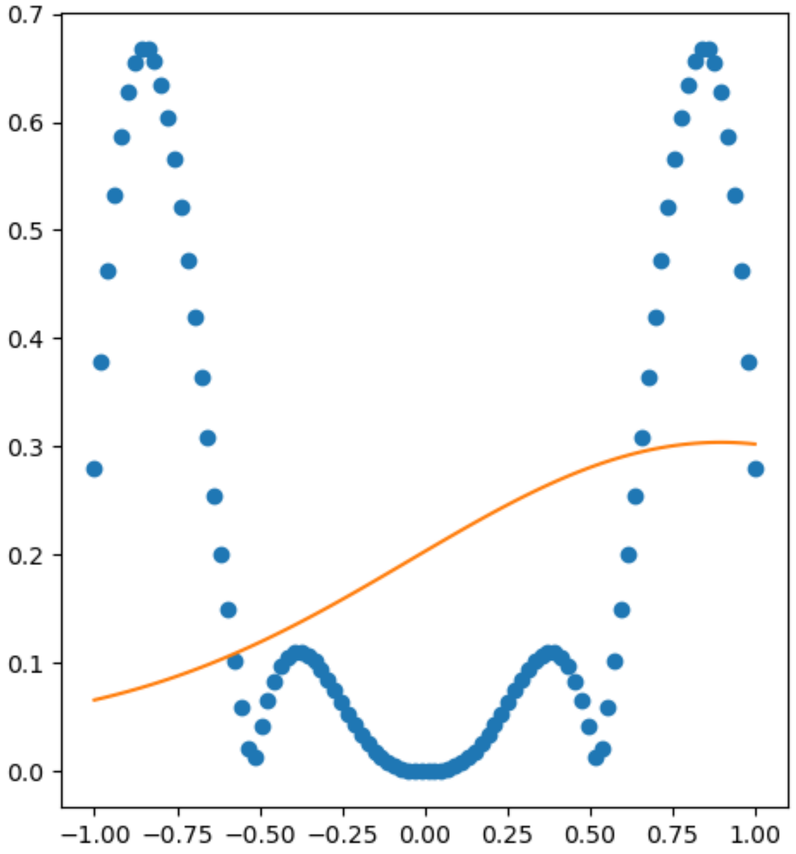


Unlikely local peaks will align on a nice grid.

OPTIMIZING RBFS

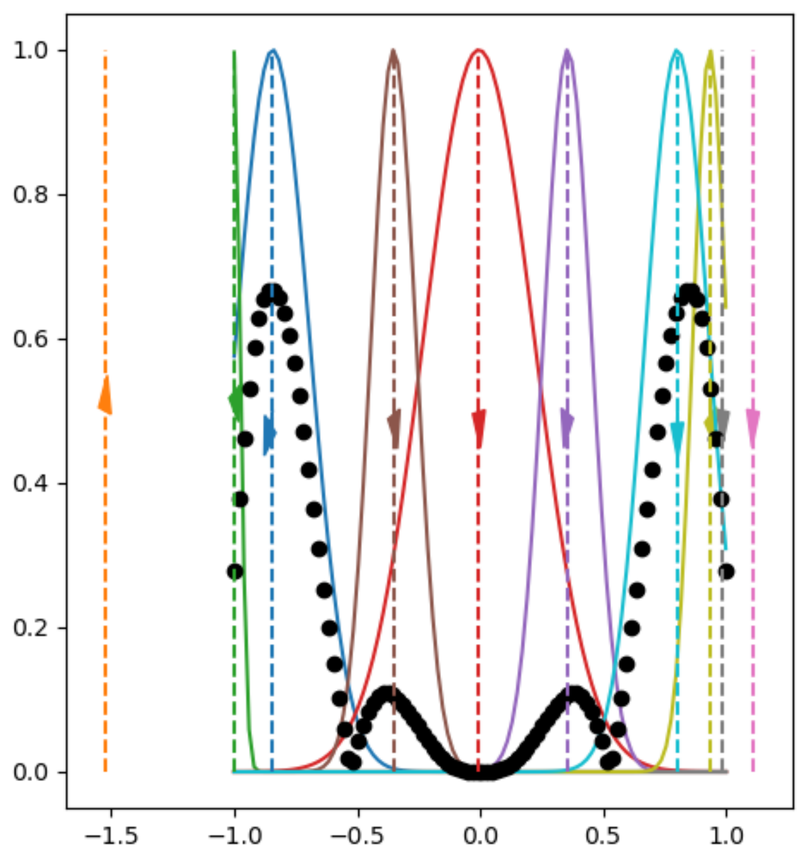
EXAM

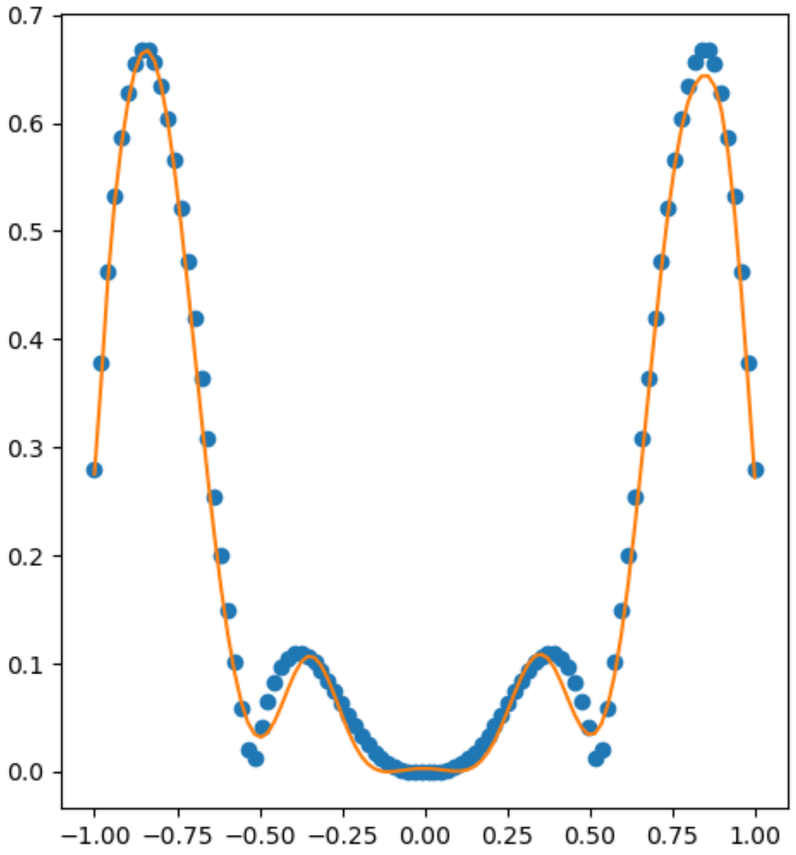




OPTIMIZING RBFS

EXAMPLE





OPTIMIZING RBFS

EXAMPLE

- Centers and scales adapted to fit the function’s peaks and valleys
- Useless features were removed from the representation (no guarantee)

SCALING TO HIGHER DIMENSIONS

CURSE OF DIMENSIONALITY

In low-dimension feature vectors x (1 to 5)

- Basis functions work great!

What about higher dimensions?

- The number of features generated grows exponentially with the size of the input.
- Can’t consider all possible combinations of features

COMPOSING BASIS FUNCTIONS

EXAMPLE

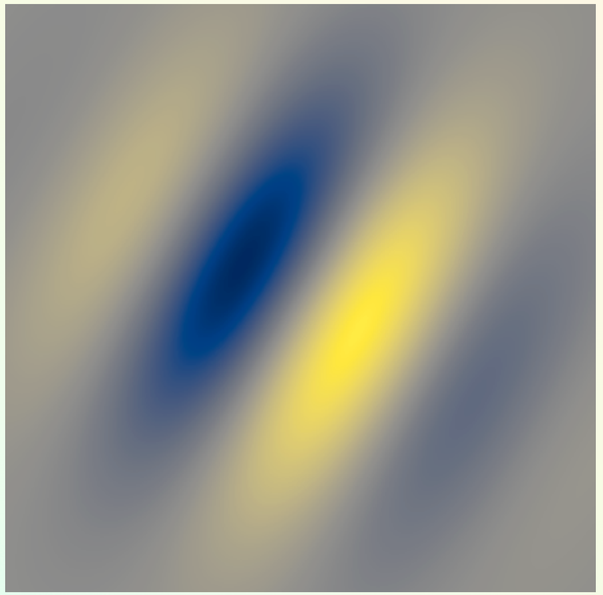
- Need to represent images to classify cat/dog
- Images can represented by lines, then groups of lines, groups of groups

COMPOSING BASIS FUNCTIONS

EXAMPLE

Line filter (a feature that activates if there is a line in a specific orientation)

Evidence that this is how neurons in early layers of the visual cortex represent images



COMPOSING BASIS FUNCTIONS

GENERAL PROCESS

$$f(x, w, \beta^1, \beta^2) = w^\top \phi^2 \left(\phi^1(x, \beta^1), \beta^2\right)$$

$$h^1 = \phi^1(x, \beta^1)$$

$$h^2 = \phi^2(h^1, \beta^2)$$

$$\hat{y} = w^\top h^2$$

COMPOSING BASIS FUNCTIONS

GENERAL PROCESS

$$f(x, w, \beta^1, \beta^2) = w^\top \phi^2 \left(\phi^1(x, \beta^1), \beta^2\right)$$

$$h^1 = \phi^1(x, \beta^1)$$

$$h^2 = \phi^2(h^1, \beta^2)$$

$$\hat{y} = w^\top h^2$$

NEXT CLASS

Next Class — Neural Networks!