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If $0 < m \leq a_i \leq M, i \in \overline{1,n}, n \in \mathbb{N}, n > 2$ then:

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j\neq i}}^{n} a_j^2}} \leq \frac{n}{2\sqrt{n-1}}\left(\frac{m}{M}+\frac{M}{m}\right)$$

Proposed by Seyran Ibrahimov-Masilli-Azerbaijan

Solution 1 by Michael Sterghiou-Greece, Solution 2 by Florică Anastase-Romania

Solution 1 by Michael Sterghiou-Greece

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j\neq i}}^{n} a_j^2}} \leq \frac{n}{2\sqrt{n-1}}\left(\frac{m}{M}+\frac{M}{m}\right); \quad (1)$$

$f(t) = \sqrt{t}$ **is concave on** $(0,\infty)$

$$\sum_{i=1}^{n} \sqrt{\frac{a_1^2}{x-a_i^2}} \leq n \cdot \sqrt{\frac{\left(\frac{p}{n}\right)^2}{x-\left(\frac{p}{n}\right)^2}}; \quad (2)$$

Where $x = \sum_{i=1}^{n} a_i^2$ and $p = \sum_{i=1}^{n} a_i$ with $x \geq \frac{p^2}{n}$.

$$(2) \Rightarrow LHS_{(1)} \overset{Jensen}{\leq} n \cdot \sqrt{\frac{p^2}{n^2 x - p^2}} \leq n \cdot \sqrt{\frac{p^2}{n^2 \cdot \frac{p^2}{n} - p^2}} = \frac{n}{\sqrt{n-1}}; \quad (3)$$

But $\frac{m}{M} + \frac{M}{m} \overset{AM-GM}{\geq} 2 \Rightarrow \frac{1}{2}\left(\frac{m}{M} + \frac{M}{m}\right) \geq 1$ and from (3)

Since $LHS_{(1)} \leq \frac{n}{\sqrt{n-1}}$ it follows that:

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j\neq i}}^{n} a_j^2}} \leq \frac{n}{2\sqrt{n-1}}\left(\frac{m}{M} + \frac{M}{m}\right)$$

Solution 2 by Florică Anastase-Romania

Let $S = \sum_{i=1}^{n} a_i$

Fom Cauchy-Schwartz Inequality we have:

$$\begin{cases} a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \geq \dfrac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n-1} \\ \cdots \\ a_2^2 + a_3^2 + \cdots + a_n^2 \geq \dfrac{(a_2 + a_3 + \cdots + a_n)^2}{n-1} \end{cases} \Rightarrow$$

$$\begin{cases} \dfrac{a_n}{\sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}} \leq \dfrac{a_n\sqrt{n-1}}{S - a_n} \\ \cdots \\ \dfrac{a_1}{\sqrt{a_2^2 + a_3^2 + \cdots + a_n^2}} \leq \dfrac{a_1\sqrt{n-1}}{S - a_1} \end{cases}$$

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j\neq i}}^{n} a_j^2}} \leq \sqrt{n-1}\left(\frac{a_1}{S - a_1} + \frac{a_2}{S - a_2} + \cdots + \frac{a_n}{S - a_n}\right) \overset{(1)}{\leq} \frac{n}{\sqrt{n-1}}$$

$$n\left(\frac{a_1}{S-a_1}+\frac{a_2}{S-a_2}+\cdots+\frac{a_n}{S-a_n}\right)\le\frac{S}{S-a_1}+\frac{S}{S-a_2}+\cdots+\frac{S}{S-a_n}\Leftrightarrow$$

$$(n-1)\left(\frac{a_1}{S-a_1}+\frac{a_2}{S-a_2}+\cdots+\frac{a_n}{S-a_n}\right)\le\frac{S-a_1}{S-a_1}+\frac{S-a_2}{S-a_2}+\cdots+\frac{S-a_n}{S-a_n}\Leftrightarrow$$

$$\frac{a_1}{S-a_1}+\frac{a_2}{S-a_2}+\cdots+\frac{a_n}{S-a_n}\le\frac{n}{n-1}\Rightarrow(1)\ true.$$

But $\frac{m}{M}+\frac{M}{m}\overset{AM-GM}{\ge}2\Rightarrow\frac{1}{2}\left(\frac{m}{M}+\frac{M}{m}\right)\ge 1;\ \ (2)$

From $(1),(2)$ it follows that:

$$\sum_{i=1}^{n}\frac{a_i}{\sqrt{\sum_{\substack{j=1\\ j\ne i}}^{n}a_j^2}}\le\frac{n}{2\sqrt{n-1}}\left(\frac{m}{M}+\frac{M}{m}\right)$$