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**Solve for real numbers:**

$$\left|\left|\frac{tan^2x-1}{tan^4x}\right|+\frac{1}{sin^2x}-2cot^3x\right|+2(cotx+cot^3x)=4$$

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*Solution by Khaled Abd Imouti-Damascus-Syria*

$$4-2(cotx+cot^3x)\geq 0\Leftrightarrow 2-(cotx+cot^3x)\geq 0$$

$$\Leftrightarrow cotx+cot^3x\leq 2\Leftrightarrow \frac{cosx}{sin^3x}\leq 2$$

Let $f: D-\{k\pi \mid k\in\mathbb{Z}\}\to\mathbb{R}, f(x)=cotx(1+cot^2x)$

$$f(-x)=-f(x), x\in[0,\pi]$$

$$\lim_{x\to 0} f(x)=+\infty;\ \lim_{x\to\pi} f(x)=-\infty$$

$$f'(x)=\frac{-sin^4x-3sin^2xcos^2x}{sin^6x}<0\Rightarrow f\searrow[0,\pi]$$

Let $g(x)=4-2(cotx+cot^3x)\geq 0$

$$\left|\frac{tan^2x-1}{tan^4x}\right|=-(1+cot^2x)+2cot^3x+4-2cotx(1+cot^2x)$$

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I. $cot^2x - cot^4x = -1 - cot^2x + 2cot^3x + 4 - 2cotx - 2cot^3x \Leftrightarrow$

$$-cot^4x + 2cot^2x + 2cotx - 3 = 0$$

$$cotx = y \Rightarrow -y^4 - 2y^2 - 2y + 3 = 0 \Leftrightarrow (y-1)(y^3 + y^2 - y - 3) = 0$$

$$y - 1 = 0 \Rightarrow cotx = 1 \Rightarrow x = \frac{\pi}{4}$$

$$y^3 + y^2 - y - 3 = 0$$

Let $h(y) = y^3 + y^2 - y - 3$

$$h'(y) = 3y^2 + 2y - 1; h'(y) = 0 \Leftrightarrow y_1 = -1, y_2 = \frac{1}{3}$$

$$\Rightarrow y^3 + y^2 - y - 3 < 0, \forall x \leq 1$$

II. $cot^4x - cot^2x = -1 - cot^2x + 2cot^3x + 4 - 2cotx - 2cot^3x \Leftrightarrow$

$$cot^4x + 2cotx - 3 = 0; y = cotx \Rightarrow$$

$$y^4 + 2y - 3 = 0 \Leftrightarrow (y-1)(y^3 + y^2 + y + 3) = 0$$

$$y = 1 \Rightarrow cotx = 1 \Rightarrow x = \frac{\pi}{4}$$

$$s = \left\{ \frac{\pi}{4} + k\pi \;\middle|\; k \in \mathbb{Z} \right\}$$

**Note by editor:**

**Many thanks to Florică Anastase-Romania for typed solution.**