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P real polynomial degree $n \geq 1$ such that

$P(0), P(1), P(4), P(9), \ldots, P(n^2)$ are in $\mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}, P(a^2) \in \mathbb{Z}$.

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Lemma. Let $Q(x)$ be a real polynomial of degree d such that

$$Q(0), Q(1), Q(2), \ldots, Q(d) \in \mathbb{Z}$$

Then $Q(a) \in \mathbb{Z}, \forall a \in \mathbb{Z}$.

Before proving the lemma, let us see why it solves the given question.

Let $P(x)$ be a real polynomial of degree n such that

$$P(0), P(1), P(4), P(9), \ldots, P(n^2) \in \mathbb{Z}$$

Define $Q(x) := P((x-n)^2)$. Then $Q(x)$ is a real polynomial of degree $2n$ such that

$$Q(0), Q(1), Q(2), \ldots, Q(2n) \in \mathbb{Z}$$

Hence by the lemma above, $Q(a) \in \mathbb{Z}, \forall a \in \mathbb{Z} \Rightarrow P(a^2) = Q(a+n) \in \mathbb{Z}, \forall a \in \mathbb{Z}$.

Proof the lemma: We will prove this by induction on d.

Base case: $d = 1$: Let $Q(x) = rx + s$. Then $Q(0) = s \in \mathbb{Z}$ and $Q(1) = r + s \in \mathbb{Z}$

This implies that both r and s are integers. So, $\forall a \in \mathbb{Z}, Q(a) = ar + s \in \mathbb{Z}$

This proves the base case. Assume now that the result is true for all polynomials of degree $\leq d$ (where $d \geq 1$) satisfying the given conditions.

Induction step: Let degree of $Q(x)$ be $d + 1$ such that

$$Q(0), Q(1), Q(2), \dots, Q(d+1) \in \mathbb{Z}$$

Define $R(x) := Q(x+1) - Q(x)$. Then $R(x)$ is a polynomial of degree d such that

$$R(0), R(1), R(2), \dots, R(d) \in \mathbb{Z}$$

Thus by induction hypothesis, $R(k) \in \mathbb{Z}, \forall k \in \mathbb{Z}; \quad (*)$

If $a > d + 1$, then $Q(a) = Q(d+1) + \sum_{k=d+1}^{a-1} R(k) \in \mathbb{Z}$ using $(*)$

If $a < 0$, then $Q(a) = Q(0) - \sum_{k=a}^{-1} R(k) \in \mathbb{Z}$ using $(*)$ again

$$\Rightarrow Q(a) \in \mathbb{Z}, \forall a \in \mathbb{Z}$$

This completes the proof of the lemma.

Note by editor:

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