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**If $a,b>0$ then:**

$$\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)tan^{-1}\left(\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab}(a^2+b^2)}\right)+\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)tan^{-1}\left(\frac{(a-b)^2}{2+2ab}\right)\geq 0$$

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*Solution by Florică Anastase-Romania*

Let $a,b\geq 1$

$$\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab}(a^2+b^2)}\geq\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab(a^2+b^2)}}=\frac{\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}}{1+\sqrt{ab}\sqrt{\frac{a^2+b^2}{2}}}$$

$$\frac{(a-b)^2}{2+2ab}\geq\frac{(a-b)^2}{2(1+ab)(a+b)}=\frac{\frac{a+b}{2}-\frac{2ab}{a+b}}{1+\frac{a+b}{2}\cdot\frac{2ab}{a+b}}$$

$$\frac{2ab}{a+b}+\sqrt{ab}\leq\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}$$

$$\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}\leq\frac{a+b}{2}-\frac{2ab}{a+b}$$

$$\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)tan^{-1}\left(\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab}(a^2+b^2)}\right)+\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)tan^{-1}\left(\frac{(a-b)^2}{2+2ab}\right)\geq 0;(1)$$

$$\Leftrightarrow \frac{tan^{-1}\left(\frac{(a-b)^2}{2+2ab}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}-\frac{tan^{-1}\left(\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab}\left(a^2+b^2\right)}\right)}{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\geq 0;(2)$$

$$\frac{tan^{-1}\left(\frac{(a-b)^2}{2+2ab}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}\geq\frac{tan^{-1}\left(\frac{a+b}{2}\right)-tan^{-1}\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2}-\frac{2ab}{a+b}}\overset{MVT}{=}f'(\xi_1),\xi_1\in\left(\frac{2ab}{a+b},\frac{a+b}{2}\right)$$

$$\frac{tan^{-1}\left(\frac{\sqrt{2ab}-\sqrt{a^2+b^2}}{\sqrt{2}+\sqrt{ab}\left(a^2+b^2\right)}\right)}{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\geq\frac{tan^{-1}\left(\sqrt{\frac{a^2+b^2}{2}}\right)-tan^{-1}\left(\sqrt{ab}\right)}{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\overset{MVT}{=}f'(\xi_2),$$

$$\xi_2\in\left(\sqrt{ab},\sqrt{\frac{a^2+b^2}{2}}\right),f:\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right),f(t)=tan^{-1}t$$

$f'(t)=\frac{1}{1+t^2}>0,f''(t)=-\frac{2t}{(1+t^2)^2}>0,\forall t>0\Rightarrow f$ −increasing and $f'$ −decrasing function.

$$\begin{cases}\frac{2ab}{a+b}<\xi_1<\frac{a+b}{2}\\ \sqrt{ab}<\xi_2<\sqrt{\frac{a^2+b^2}{2}}\\ f'-decreasing\end{cases}\Rightarrow\begin{cases}f'\left(\frac{2ab}{a+b}\right)>f'(\xi_1)>f'\left(\frac{a+b}{2}\right)\\ f'\left(\sqrt{ab}\right)>f'(\xi_2)>f'\left(\sqrt{\frac{a^2+b^2}{2}}\right)\end{cases}$$

Hence,

$$f'(\xi_1)-f'(\xi_2)>f'\left(\frac{a+b}{2}\right)-f'\left(\sqrt{\frac{a^2+b^2}{2}}\right)\geq 0\Rightarrow(2)\Rightarrow(1)true.$$

Similarly, $0<a,b<1$