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If $f, f': (0,\infty) \to (0,\infty)$, f −differentiable, $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{(f(x)+f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}\,dxdy \le \log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right)^{f^2(b)-f^2(a)}$$

Proposed by Daniel Sitaru-Romania

Solution by Remus Florin Stanca-Romania

Let's prove that:

$$\frac{a+b}{\sqrt{1+ab}} \le \frac{a}{\sqrt{1+b^2}} + \frac{b}{\sqrt{1+a^2}}, \forall a,b > 0$$

The inequality can be written as:

$$\frac{1}{\sqrt{1+ab}} \le \frac{a}{a+b} \cdot \frac{1}{\sqrt{1+b^2}} + \frac{b}{a+b} \cdot \frac{1}{\sqrt{1+a^2}}$$

Let $g: (0,\infty) \to (0,\infty), g(x) = \frac{1}{\sqrt{x+1}}; \frac{\partial g}{\partial x} = -\frac{1}{2}(x+1)^{-\frac{3}{2}}; \frac{\partial^2 g}{\partial x^2} = \frac{3}{2} \cdot \frac{1}{2}(x+1)^{-\frac{5}{2}} \ge 0$

$\Rightarrow g$ **−convexe, then for any** $t_1, t_2 \in (0,1), t_1 + t_2 = 1$ **and for any** $x_1, x_2 \in I$ **we have:**

$$t_1 f(x_1) + t_2 f(x_2) \ge f(t_1 x_1 + t_2 x_2)$$

Let $t_1 = \frac{a}{a+b}; t_2 = \frac{b}{a+b}$ **and** $x_1 = b^2, x_2 = a^2$

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$$\frac{a}{a+b}\cdot\frac{1}{\sqrt{1+b^2}}+\frac{b}{a+b}\cdot\frac{1}{\sqrt{1+a^2}}\geq\frac{1}{\sqrt{1+\frac{a^2b+ab^2}{a+b}}}=\frac{1}{\sqrt{1+\frac{ab(a+b)}{a+b}}}=\frac{1}{\sqrt{1+ab}}$$

$$\Rightarrow\frac{a+b}{\sqrt{1+ab}}\leq\frac{a}{\sqrt{1+b^2}}+\frac{b}{\sqrt{1+a^2}},\forall a,b>0$$

$$\frac{f(x)+f(y)}{\sqrt{1+f(x)f(y)}}\leq\frac{f(x)}{\sqrt{1+f^2(y)}}+\frac{f(y)}{\sqrt{1+f^2(x)}}$$

$$\frac{(f(x)+f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}\leq\frac{f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}}+\frac{f(y)f'(x)f'(y)}{\sqrt{1+f^2(x)}};\ (3)$$

$$\frac{1}{2}\int_a^b\int_a^b\frac{2f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}}dxdy=\frac{1}{2}\int_a^b\frac{f^2(x)f'(y)}{\sqrt{1+f^2(y)}}\Bigg|_a^b dy=$$

$$=\frac{1}{2}\left(f^2(b)-f^2(a)\right)\left(\log\left(f(b)+\sqrt{1+f^2(b)}\right)-\log\left(f(a)+\sqrt{1+f^2(a)}\right)\right)=$$

$$=\frac{1}{2}\left(f^2(b)-f^2(a)\right)\log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right);\ (1)$$

$$\frac{1}{2}\int_a^b\int_a^b\frac{2f(y)f'(y)f'(x)}{\sqrt{1+f^2(x)}}dxdy=\frac{1}{2}\int_a^b\frac{f^2(y)f'(x)}{\sqrt{1+f^2(x)}}\Bigg|_a^b dy=$$

$$=\frac{1}{2}\left(f^2(b)-f^2(a)\right)\left(\log\left(f(b)+\sqrt{1+f^2(b)}\right)-\log\left(f(a)+\sqrt{1+f^2(a)}\right)\right)=$$

$$=\frac{1}{2}\left(f^2(b)-f^2(a)\right)\log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right);\ (2)$$

From $(1),(2),(3)$ it follows that:

$$\int_a^b\int_a^b\frac{(f(x)+f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}dxdy\leq\log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right)^{f^2(b)-f^2(a)}$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.