A Special Integral: $\psi(m, a)$

1 Introduction

In this paper, the Mellin transform of $\frac{e^{-x^2}}{(1+x^2)^a}$ is studied. Thus, if we let

$$\psi(m, a) = \int_0^\infty \frac{e^{-x^2} x^{m-1}}{(1+x^2)^a} dx$$

then $\psi(m, a)$ represents the Mellin transform, where $m > 0$ and $a \in \mathbb{R}$. This integral can be used in solving various infinite series, improper integrals and differential equations.

2 Prerequisites

In order to find specific values and properties of $\psi(m, a)$, we should be familiar with the following known integrals, namely,
Error Function:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Complimentary Error Function:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

Generalized Exponential Integral:

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$$

Confluent Hypergeometric Function:

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (t+1)^{b-a-1} dt$$

Modified Bessel function of the second kind:

$$K_\alpha(x) = \int_0^\infty e^{-x cosh(t)} cosh(\alpha t) dt$$

3 Properties

1. $$\psi(m,0) = \frac{\Gamma(m/2)}{2}$$

2. If $m-1 \geq 2a$, then
$$\psi(m,a) \leq \frac{\Gamma(\frac{m}{2}-a)}{2}$$

3. $$\psi(m,a) = \frac{2}{m}\psi(m+2,a) + \frac{2a}{m}\psi(m+2,a+1)$$

4. If $n \in \mathbb{N}$, then
$$\psi(m,a) = \sum_{k=0}^{n} \binom{n}{k} \psi(m+2k, a+n)$$

5. If $a \in \mathbb{N}$, then
$$\psi(2,a) = \frac{1}{2}\sum_{k=1}^{a-1} \frac{(-1)^{k-1}}{(a-1)_k} - \frac{(-1)^a e E_1(1)}{2\Gamma(a)}$$

6. $$2\psi(m,a)\psi(n,a) = \sum_{k=0}^{\infty} \frac{(-1)^k a^{(k)} B(\frac{m}{2}+k, \frac{n}{2}+k)}{k!} \psi(m+n+4k, k+a)$$

4 Proof of properties

Proof of 3.1: It can be easily proved using the definition of gamma function.
Proof of 3.2: Since $(1+x^2)^a \geq x^{2a}$ $\forall x \in \mathbb{R}$. Hence, if $m-1 \geq 2a$, then

$$\psi(m,a) \leq \int_0^\infty e^{-x^2} x^{m-2a-1} dx = \frac{1}{2}\int_0^\infty e^{-x} x^{\frac{m}{2}-a-1} dx = \frac{\Gamma(\frac{m}{2}-a)}{2}$$

Proof of 3.3: Integrating $\psi(m,a)$ by parts we obtain the desired result.
Proof of 3.4: Observe that,

$$\psi(m,a) = \int_0^\infty \frac{e^{-x^2} x^{m-1}}{(1+x^2)^a} dx = \int_0^\infty \frac{e^{-x^2} x^{m-1}(1+x^2)^n}{(1+x^2)^{a+n}} dx$$

Expand $(1+x^2)^n$ using binomial theorem to obtain the desired result.
Proof of 3.5: Observe that,

$$\psi(2,a)=\int_0^\infty \frac{e^{-x^2}x}{(1+x^2)^a}dx$$

substituting $1+x^2=t$, we get,

$$\psi(2,a)=\frac{1}{2}\int_1^\infty \frac{e^{-(t-1)}}{t^a}dt=\frac{e}{2}\int_1^\infty \frac{e^{-t}}{t^a}dt$$

thus,

$$\psi(2,a)=\frac{e}{2}E_a(1)$$

Now if we integrate $E_a(1)$ by parts, we obtain the following relation,

$$E_a(1)=\frac{e^{-1}}{a-1}-\frac{1}{a-1}E_{a-1}(1)$$

Thus,

$$E_a(1)=e^{-1}\sum_{k=1}^{a-1}\frac{(-1)^{k-1}}{(a-1)_k}-\frac{(-1)^a E_1(1)}{\Gamma(a)}$$

hence after multiplying the above equation by $e/2$ we obtain the desired result.
Proof of 3.6: Multiplying $\psi(m,a)$ by $\psi(n,a)$ and changing cartesian coordinates to polar coordinates, we get

$$\psi(m,a)\psi(n,a)=\int_0^{\frac{\pi}{2}} cos^{m-1}(\theta)sin^{n-1}(\theta)\int_0^\infty \frac{e^{-r^2}r^{m+n-1}}{(1+r^2+r^4sin^2(\theta)cos^2(\theta))^a}drd\theta$$

Taking $(1+r^2)^a$ common from the denominator and then expanding the denominator using binomial theorem and finally inverting the order of integration and summation, we have,

$$\psi(m,a)\psi(n,a)=\sum_{k=0}^\infty \frac{(-1)^k a^{(k)}}{k!}\int_0^\infty \frac{e^{-r^2}r^{4k+m+n-1}}{(1+r^2)^{k+a}}dr\int_0^{\frac{\pi}{2}} cos^{m+2k-1}(\theta)sin^{n+2k-1}(\theta)d\theta$$

Simplifying it further using beta function, we obtain the desired result.

5 Special values

1.
$$\psi(1,1) = \frac{e\pi erfc(1)}{2}$$

2.
$$\psi(2,1) = \frac{eE_1(1)}{2}$$

3.
$$\psi(1,2) = \frac{\sqrt{\pi}}{2} - \frac{e\pi erfc(1)}{4}$$

4.
$$\psi(2,3) = \frac{eE_1(1)}{4}$$

5.
$$\psi(2,1/2) = \frac{e\sqrt{\pi} erfc(1)}{2}$$

6.
$$\psi(1,1/2) = \frac{1}{2}\sqrt{e}K_0\left(\frac{1}{2}\right)$$

6 Proof of specific values

Proof of 5.1: Let,

$$I(n) = \int_0^\infty \frac{e^{-nx^2}}{1+x^2}dx$$

then, it is easy to show that,

$$I^{'} - I = -\frac{1}{2}\sqrt{\frac{\pi}{n}}$$

hence

$$I(n) = \frac{1}{2}\pi e^n erfc(\sqrt{n})$$

put $n = 1$ to complete the proof.
Proof of 5.2: Put $a = 1$ in property 3.5.
Proof of 5.3: Observe that, using property 3.3

$$\psi(3,2) = \frac{\psi(1,1)}{2} - \psi(3,1)$$

and using property 3.4($n = 1$)

$$\psi(1,1) = \psi(1,2) + \psi(3,2)$$

we complete the proof.
Proof of 5.4: Put $a = 3$ in property 3.5.
Proof of 5.5: Integrating $\frac{d}{dx}erf((1+x^2)^{1/2})$ from 0 to ∞, we obtain the desired result.
Proof of 5.6: Substitute $x = sinh(t)$ in $\psi(1,1/2)$ and then using the definition of modified Bessel function of the second kind and the identity $cosh(2t) - 2sinh^2(t) = 1$, we obtain the result.

7 Corollaries

1. $$erfc(1) < 1/e$$

2. $\psi(m,1)$ and $\psi(2m,a)$ can be evaluated $\forall m \in \mathbb{N}$ and $\forall a \in \mathbb{N}$.

3. $$\left(\frac{e\pi erfc(1)}{2}\right)^2 = \frac{\pi}{2}\psi(2,1) - \frac{\pi}{16}\psi(6,2) + \frac{3\pi}{256}\psi(10,3)...$$

4. $$\psi(m,a) = \frac{\Gamma(\frac{m}{2})U\Big(\frac{m}{2}, \frac{m}{2} - a + 1, 1\Big)}{2}$$

8 Proof of corollaries

Proof of 7.1: Put $m = 2$ and $a = 1/2$ in property 3.2.
Proof of 7.2: Using property 3.3 and 3.4($n = 1$) and specific values 5.1 and 5.2, all the values of $\psi(m,1)$ can be evaluated, similarly using property 3.5, one can calculate $\psi(2,a)$ and repeatedly using property 3.4($n = 1$), $\psi(2m,a)$ can be evaluated.
Proof of 7.3: Put $m = n = a = 1$ in property 3.6.
Proof of 7.4: Using the definition of confluent hypergeometric function and with some manipulations, last corollary can be proved.

9 References

[1] Lokenath Debnath, Dambaru Bhatta, Integral Transforms and Their Applications, Second Edition

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