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Let be $(a_n)_{n\geq1},(b_n)_{n\geq1}, a_n,b_n\in\mathbb{R}_+^*=(0,\infty)$ such that $\lim\limits_{n\to\infty}a_n=a\in\mathbb{R}_+^*$

and $(b_n)_{n\geq1}$ is a bounded sequence. If $(x_n)_{n\geq1}, x_n=\prod_{k=1}^{n}(ka_k+b_k)$ find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{x_{n+1}}-\sqrt[n]{x_n}\right)$$

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Solution by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}\left(\sqrt[n+1]{x_{n+1}}-\sqrt[n]{x_n}\right)=\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}\cdot n\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}-1\right);\ \ (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\overset{C-D}{=}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{x_{n+1}}{(n+1)x_n}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{(n+1)a_{n+1}+b_{n+1}}{n+1}=\frac{a}{e};\ \ (2)$$

Because $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$ and $\lim\limits_{n\to\infty}\frac{(n+1)a_{n+1}+b_{n+1}}{n+1}=\lim\limits_{n\to\infty}\left(a_{n+1}+\frac{b_{n+1}}{n+1}\right)=a, b_n$ –is

bounded.

$$\lim_{n\to\infty}n\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}-1\right)=\lim_{n\to\infty}n\left(e^{\log\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)}-1\right)=$$

$$=\lim_{n\to\infty}\left(\frac{e^{log\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)}-1}{log\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)}\right)nlog\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)=$$

$$=\lim_{n\to\infty}log\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)^n=log\left(\lim_{n\to\infty}\frac{x_{n+1}}{x_n}\cdot\frac{1}{\sqrt[n+1]{x_{n+1}}}\right)=$$

$$=log\left(\lim_{n\to\infty}\frac{(n+1)a_{n+1}+b_{n+1}}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\right)=$$

$$=log\left(a\cdot\frac{e}{a}\right)=loge=1;(3)$$

From $(1),(2),(3)$ we get: $L=\frac{a}{e}$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.