Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permuation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permuatations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \le r \le n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}\ .$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$

〖Example 〗 A soccer club has 8 female and 7 male members. For today's match, how many possible configurations are there?

(1) The coach wants to have 6 female and 5 male players on the grass.

(2) The coach wants to have 11 players with at most 5 male players on the grass.

Solution:

(1) C(8, 6) • C(7, 5)
= 8!/(6!·2!) • 7!/(5!·2!)
= 28 • 21
= 588

(2) C(8，6)C(7，5)+C(8，7)C(7，4)+C(8，8) C(7，3)

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Homework

Sec 6.3: 20, 44, 46

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}. \quad ◀$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is $\sum_{k=0}^{n} \binom{n}{k}.$

Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad ◀$$

【 **Corollary 2** 】 **Let n be a positive integer. Then**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

Proof:

Using the Binomial Theorem with $x = 1$ and $y = -1$.

Remark:

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

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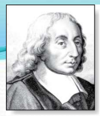
Blaise Pascal
(1623-1662)



Pascal's Identity

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

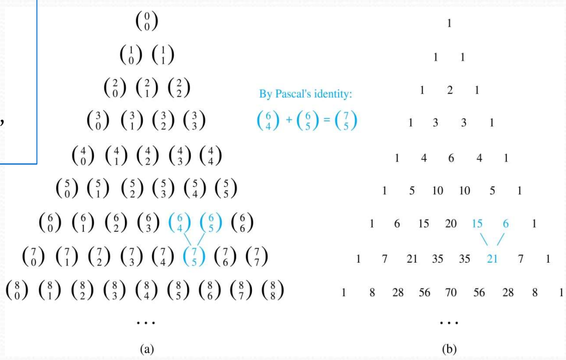
Hence,

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}. \blacktriangleleft$$

See Exercise 19 for an algebraic proof.

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.



By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

【 Theorem 3 】 ***Vandermonde's Identity***

Let m, n and r be nonnegative integer with r not exceeding either m or n. Then $$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof:

A and B are two disjoint sets. $|A|=m$, $|B|=n$,

$C(m+n, r)$ ---- the number of ways to pick r elements from $A \cup B$

Another way to pick r element from $A \cup B$ is to pick r-k elements from A and then k elements from B, where $0 \le k \le r$, which can be done in $C(m, r-k)\ C(n, r)$

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【 Corollary 4 】If n is a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof:

We use Vandermonde's Identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

【 Theorem 4 】Let n and r be nonnegative integer with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof:

The left-hand side counts the bit strings of length $n+1$ containing $r+1$ 1s.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Homework

Sec.6.4: 18, 26, 30, 34