Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- 【Definition】Two quantifiers are nested if one is within the scope of the other.
- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \exists y(x + y = 0)$

  where the domains of x and y are the real numbers.
- We can also think of nested propositional functions:

  $\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x\, \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\, \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y\ \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\ P(x,y)$
   Answer: True
4. $\exists x\ \exists\ y\ P(x,y)$
   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the positive real numbers,
Define $P(x,y) : x / y = 1$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\ P(x,y)$
   Answer: False
4. $\exists x\ \exists\ y\ P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	***When True?***	***When False?***
$\forall x\ \forall y\ P(x, y)$ $\forall y\ \forall x\ P(x, y)$	**$P(x, y)$ is true for every pair x, y.**	**There is a pair x, y for which $P(x, y)$ is false.**
$\forall x\ \exists y\ P(x, y)$	**For every x there is a y for which $P(x, y)$ is true.**	**There is an x for which $P(x, y)$ is false for every y.**
$\exists x\ \forall y\ P(x, y)$	**There is an x for which $P(x, y)$ is true for every y.**	**For every x there is a y for which $P(x, y)$ is false.**
$\exists x\ \exists y\ P(x, y)$ $\exists y\ \exists x\ P(x, y)$	**There is a pair x, y for which $P(x, y)$ is true.**	**$P(x, y)$ is false for every pair x, y.**

Translating from Nested Quantifiers into English

- Examples

1. Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$ into English, where $C(x)$ is "x has a computer," $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students at ZJU.

Solution:

For every student x at ZJU, x has a computer or there is a student y such that y has a computer and x and y are friends.

$\Updownarrow$

Every student at ZJU has a computer or has a friend who has a computer.

Translating from Nested Quantifiers into English

- Examples

2. Translate the following statement into English, where $F(x,y)$ means x and y are friends and the domain for x, y, and z consists of all students at ZJU. $\exists x \forall y \forall z(((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$

Solution:

There is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends.

$\Updownarrow$

There is a student none of whose friends are also friends with each other.

Translating English into Logical Expressions

- Examples

1. Express the statement "Everyone has exactly one best friend" as a logical expression with a domain consisting of all people.

Solution:
Rewrite the original statement as

"For every person x , $\boxed{x \text{ has exactly one best friend.}}$"

⇩

"There is a person y who is the best friend of x, and furthermore, that for every person z, if z is not y, then z is not the best friend of x."

Let $B(x, y)$ be the statement "y is the best friend of x."

$$\forall x \exists y \forall z (B(x, y) \wedge ((z \neq y) \rightarrow \neg B(x, z)))$$

Translating English into Logical Expressions

- Examples

2. Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression with a domain consisting of all people.

Solution:
Rewrite the original statement as

"For every person x, if x is female and x is a parent, then there exists a person y such that x is the mother of y."

Let $F(x)$: "x is female,"

$P(x)$: "x is a parent,"

$M(x, y)$: "x is the mother of y."

$$\forall x ((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$$

$$\forall x \exists y ((F(x) \wedge P(x)) \rightarrow M(x, y))$$

Quantifiers with Restricted Domains

- Example: What do the statements $\forall x<0\ (x^2>0)$, $\exists y>0\ (y^2=2)$ mean, where the domain in each case consists of the real numbers?

Solution:

$\forall x<0\ (x^2>0)$

For every real number x with $x<0$, $x^2>0$

$\forall x\ (x<0 \rightarrow x^2>0)$

$\exists y>0\ (y^2=2)$

There exists a real number y with $y>0$ such that $y^2=2$

$\exists y(y>0 \land y^2=2)$

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Translating English into Logical Expressions

- Examples

3. Express the definition of a limit using quantifiers.

Solution:

The definition of the limit $\lim_{x\to a} f(x) = L$ is:

"For every real number ε>0 there exists a real number δ>0 such that $|f(x)-L|<\varepsilon$ *wherever* $|x-a|<\delta$ "

Assume the domain for the variables ε, δ, and x consist of all real numbers, we have

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x (0 < |x-a| < \delta \rightarrow |f(x)-L| < \varepsilon)$$

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, $x + y$ is positive."

3. The result is:

   $\forall x \, \forall \, y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

   $\exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$

Calculus in Logic (*optional*)

Example: Use quantifiers to express the definition of the limit of *a* real-valued function *f(x)* of a real variable *x* at a point *a* in its domain.

Solution: Recall the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is "For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$."

Using quantifiers:

$$\forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \rightarrow | f(x) - L | < \epsilon)$$

Where the domain for the variables ε and δ consists of all positive real numbers and the domain for *x* consists of all real numbers.

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x \, \forall y \, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \, \exists y \, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \, \forall x \, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \, \exists y \, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x \, L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$
2. $\forall w \neg \forall a \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Negating Nested Quantifiers

- Examples:

1. Express the negation of the statement $\forall x \exists y\ (xy=1)$ so that no negation precedes a quantifiers.

Solution:

$$\neg(\forall x \exists y\ (xy=1))$$
$$\equiv \exists x(\neg \exists y\ (xy=1))$$
$$\equiv \exists x \forall y\ (\neg\ (xy=1))$$
$$\equiv \exists x \forall y\ (xy \neq 1)$$

Return to Calculus and Logic (*Opt*)

Example : Recall the logical expression developed in the calculus example three slides back. Use quantifiers and predicates to express that $\lim_{x \to a} f(x)$ does not exist.

1. We need to say that for all real numbers L, $\lim_{x \to a} f(x) \neq L$
2. The result from the previous example can be negated to yield:
   $$\neg \forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$
3. Now we can repeatedly apply the rules for negating quantified expressions:

$$\begin{aligned} &\neg \forall \epsilon \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon) \\ &\equiv \exists \epsilon \neg \exists \delta \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon) \\ &\equiv \exists \epsilon \forall \delta \neg \forall x (0 < | x - a | < \delta \to | f(x) - L | < \epsilon) \\ &\equiv \exists \epsilon \forall \delta \exists x \neg (0 < | x - a | < \delta \to | f(x) - L | < \epsilon) \\ &\equiv \exists \epsilon \forall \delta \exists x \neg (0 < | x - a | < \delta \wedge | f(x) - L | \geq \epsilon) \end{aligned}$$

The last step uses the equivalence $\neg(p \to q) \equiv p \wedge \neg q$

Calculus in Predicate Logic (*optional*)

4. Therefore, to say that $\lim_{x \to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x \to a} f(x) \neq L$ can be expressed as:
   $$\forall L \exists \epsilon \forall \delta \exists x \neg (0 < | x - a | < \delta \wedge | f(x) - L | \geq \epsilon)$$
   Remember that ε and δ range over all positive real numbers and *x* over all real numbers.
5. Translating back into English we have, for every real number L, there is a real number ε > 0, such that for every real number δ > 0, there exists a real number x such that $0 < | x - a | < \delta$ and $|f(x) - L| \geq \varepsilon$.

Some Questions about Quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

    Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

    Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.
- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.
  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

Homework

- 第8版：Sec. 1.5 6(e, f), 12(d, h, k, n), 14(c, d, e, f), 24(a, d), 34，32(d), 38(b, d), 42