

# Partial Orderings

Section 9.6

## Section Summary

- Partial Orderings (偏序) and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices (格)
- Topological Sorting

## Partial Orderings

**Definition 1:** A relation  $R$  on a set  $S$  is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering  $R$  is called a *partially ordered set*, or *poset*, and is denoted by  $(S, R)$ . Members of  $S$  are called *elements* of the poset.

## Partial Orderings (*continued*)

**Example 1:** Show that the “greater than or equal” relation ( $\geq$ ) is a partial ordering on the set of integers.

- *Reflexivity:*  $a \geq a$  for every integer  $a$ .
- *Antisymmetry:* If  $a \geq b$  and  $b \geq a$ , then  $a = b$ .
- *Transitivity:* If  $a \geq b$  and  $b \geq c$ , then  $a \geq c$ .

These properties all follow from the order axioms for the integers.  
(See Appendix 1).

## Partial Orderings (*continued*)

**Example 2:** Show that the divisibility relation ( $|$ ) is a partial ordering on the set of integers.

- *Reflexivity:*  $a | a$  for all integers  $a$ . (see Example 9 in Section 9.1)
- *Antisymmetry:* If  $a$  and  $b$  are positive integers with  $a | b$  and  $b | a$ , then  $a = b$ . (see Example 12 in Section 9.1)
- *Transitivity:* Suppose that  $a$  divides  $b$  and  $b$  divides  $c$ . Then there are positive integers  $k$  and  $l$  such that  $b = ak$  and  $c = bl$ . Hence,  $c = a(kl)$ , so  $a$  divides  $c$ . Therefore, the relation is transitive.
- $(\mathbb{Z}^+, |)$  is a poset.

## Partial Orderings (*continued*)

**Example 3:** Show that the inclusion relation ( $\subseteq$ ) is a partial ordering on the power set of a set  $S$ .

- *Reflexivity:*  $A \subseteq A$  whenever  $A$  is a subset of  $S$ .
- *Antisymmetry:* If  $A$  and  $B$  are positive integers with  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .
- *Transitivity:* If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

The properties all follow from the definition of set inclusion.

## Comparability

**Definition 2:** The elements  $a$  and  $b$  of a poset  $(S, \leq)$  are *comparable* if either  $a \leq b$  or  $b \leq a$ . When  $a$  and  $b$  are elements of  $S$  so that neither  $a \leq b$  nor  $b \leq a$ , then  $a$  and  $b$  are called *incomparable*.

The symbol  $\leq$  is used to denote the relation in any poset.

**Definition 3:** If  $(S, \leq)$  is a poset and every two elements of  $S$  are comparable,  $S$  is called a *totally ordered* or *linearly ordered set*, and  $\leq$  is called a *total order* (全序) or a *linear order* (线序). A totally ordered set is also called a *chain* (链).

**Definition 4:**  $(S, \leq)$  is a well-ordered (良序) set if it is a poset such that  $\leq$  is a total ordering and every nonempty subset of  $S$  has a least element.

## Lexicographic Order

**Definition:** Given two posets  $(A_1, \leq_1)$  and  $(A_2, \leq_2)$ , the *lexicographic ordering* on  $A_1 \times A_2$  is defined by specifying that  $(a_1, a_2)$  is less than  $(b_1, b_2)$ , that is,

$$(a_1, a_2) < (b_1, b_2),$$

either if  $a_1 <_1 b_1$  or if  $a_1 = b_1$  and  $a_2 <_2 b_2$ .

- This definition can be easily extended to a lexicographic ordering on strings (see text).

**Example:** Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- $discreet < discrete$ , because these strings differ in the seventh position and  $e < t$ .
- $discreet < discreteness$ , because the first eight letters agree, but the second string is longer.

[[Example ]] Let  $A_1 = A_2 = \mathbb{Z}^+$  and  $R_1 = R_2 = \mid$ . Then

(1)  $(2, 4) < (2, 8)$  since  $x_1 = x_2$  and  $y_1 R_2 y_2$

(2)  $(2, 4)$  is not related under  $R$  to  $(2, 6)$  since  $x_1 = x_2$  but 4 does not divide 6.

(3)  $(2, 4) < (4, 5)$  since  $x_1 R_1 x_2$

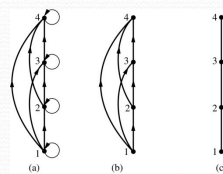
$(2, 3, 4, 5) < (2, 3, 8, 2)$  since  $x_1 = x_2$  and  $y_1 = y_2$  and 4 divides 8.

$(2, 3, 4, 5)$  is not related to  $(3, 6, 8, 10)$  since 2 does not divide 3.

29

## Hasse Diagrams

**Definition:** A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



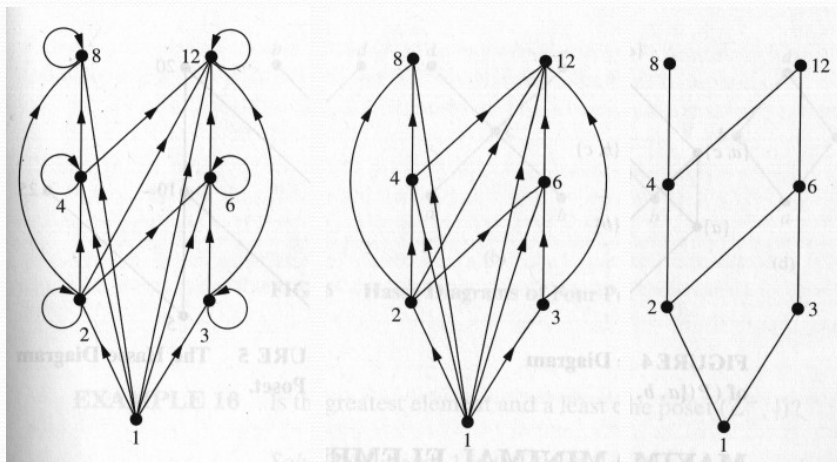
A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).



## Procedure for Constructing a Hasse Diagram

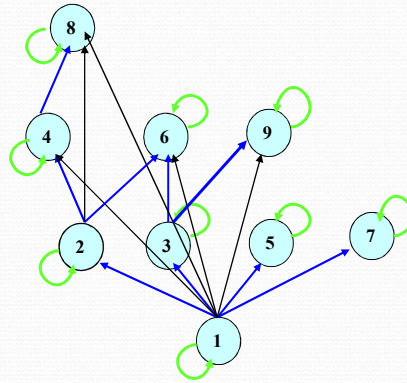
- To represent a finite poset  $(S, \preceq)$  using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops  $(a, a)$  present at every vertex due to the reflexive property.
  - Remove all edges  $(x, y)$  for which there is an element  $z \in S$  such that  $x < z$  and  $z < y$ . These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

For the poset  $(\{1, 2, 3, 4, 6, 8, 12\}, |)$



## 9.6 Partial Orderings

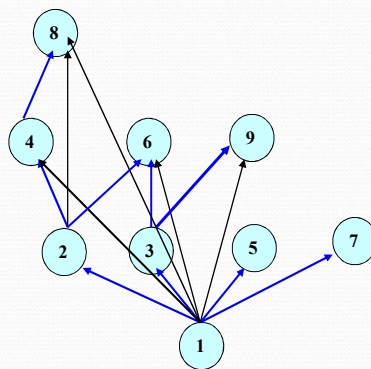
[[Example]]  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$   $R = \{(a, b) \mid a \mid b, a, b \in A\}$



33

## 9.6 Partial Orderings

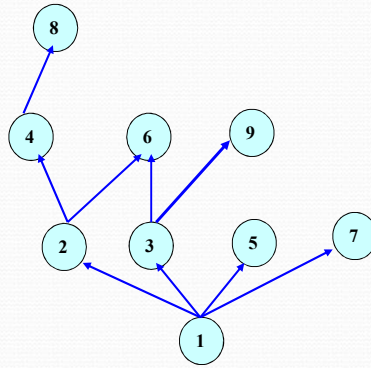
[[Example]]  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$   $R = \{(a, b) \mid a \mid b, a, b \in A\}$



34

## 9.6 Partial Orderings

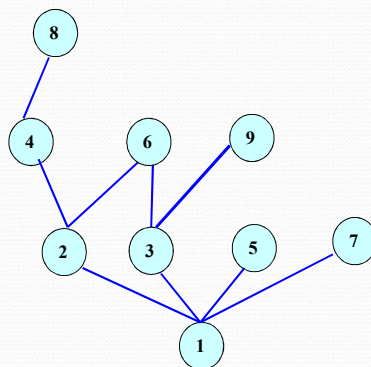
【Example】  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$      $R = \{(a, b) \mid a \mid b, a, b \in A\}$



35

## 9.6 Partial Orderings

【Example】  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$      $R = \{(a, b) \mid a \mid b, a, b \in A\}$

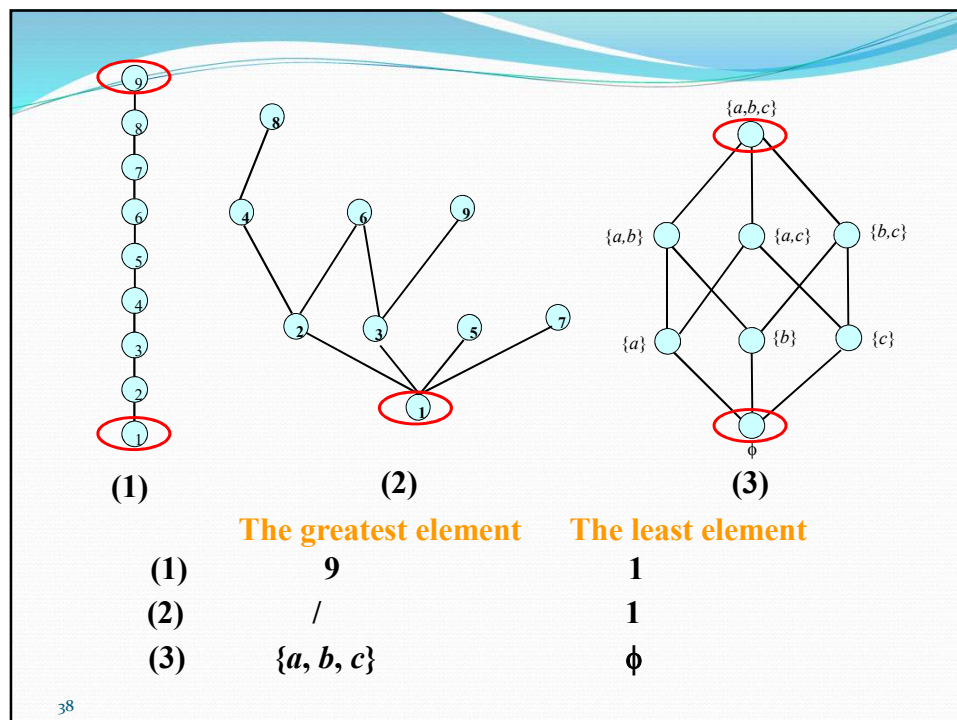


36



## Hasse Diagram Terminology

- Let  $(S, \preceq)$  be a poset.
- $a$  is *maximal* (极大元) in  $(S, \preceq)$  if there is no  $b \in S$  such that  $a \preceq b$ . (**top** of the Hasse diagram)
- $a$  is *minimal* (极小元) in  $(S, \preceq)$  if there is no  $b \in S$  such that  $b \preceq a$ . (**bottom** of the Hasse diagram)
- $a$  is the *greatest element* (最大元) of  $(S, \preceq)$  if  $b \preceq a$  for all  $b \in S$ .
- $a$  is the *least element* (最小元) of  $(S, \preceq)$  if  $a \preceq b$  for all  $b \in S$ .



**【 Theorem 】** The greatest and least element of the poset  $(A, \leq)$  are unique when they exist.

*Proof:*

**Suppose that  $a_1$  is a greatest element in  $A$ . It follows that**

**$x \leq a_1$  for every  $x$  in  $A$ .**

**Suppose that  $a_2$  is a greatest element in  $A$ . It follows that**

**$x \leq a_2$  for every  $x$  in  $A$ .**

**It implies that  $a_2 \leq a_1$  and  $a_1 \leq a_2$**

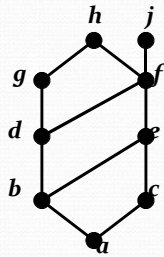
**That is  $a_1 = a_2$**

39

## Hasse Diagram Terminology (Cont ..)

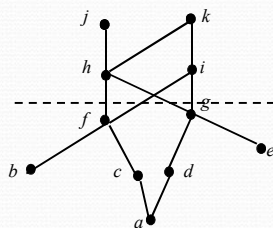
- Let  $A$  be a subset of  $(S, \preceq)$ .
- If  $u \in S$  such that  $a \preceq u$  for all  $a \in A$ , then  $u$  is called an *upper bound* of  $A$ .
- If  $l \in S$  such that  $l \preceq a$  for all  $a \in A$ , then  $l$  is called an *lower bound* of  $A$ .
- If  $x$  is an upper bound of  $A$  and  $x \preceq z$  whenever  $z$  is an upper bound of  $A$ , then  $x$  is called the *least upper bound* of  $A$ .
- If  $y$  is a lower bound of  $A$  and  $z \preceq y$  whenever  $z$  is a lower bound of  $A$ , then  $y$  is called the *greatest lower bound* of  $A$ .

## Example



Maximal elements:  $h, j$   
 Minimal elements:  $a$   
 Greatest element: none  
 Least element:  $a$   
 Upper bound of  $\{a, b, c\}$ :  $\{e, f, j, h\}$   
 Least upper bound of  $\{a, b, c\}$ :  $e$   
 Lower bound of  $\{a, b, c\}$ :  $\{a\}$   
 Greatest lower bound of  $\{a, b, c\}$ :  $a$

【Example】  $S = \{a, b, c, d, e, f, g, h, i, j, k\}$   
 $A = \{a, b, c, d, e, f, g\}, A' = \{h, i, j, k\}$



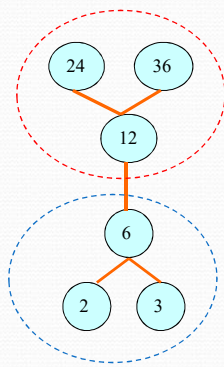
the upper bounds of set  $A$ :  $h, i, k, j$

the lower bounds of set  $A$ : /

the upper bounds of set  $A'$ : /

the lower bounds of set  $A'$ :  $f, g, a, b, c, d, e$

【Example】  $A = \{2, 3, 6, 12, 24, 36\}$ ,  $B_1 = \{2, 3, 6\}$ ,  $B_2 = \{12, 24, 36\}$ ,  $R : |$   
Determine the maximal elements, minimal elements, greatest element, least element of set  $A$ , the upper bounds, lower bounds, least upper bound, greatest lower bound of set  $B_1, B_2$ .



$A$ :

maximal elements: 24, 36    minimal elements: 2, 3

the greatest element: /    the least element: /

$B_1$ :

upper bounds: 6, 12, 24, 36    lower bounds: /

The least upper bound: 6    The greatest lower bound: /

$B_2$ :

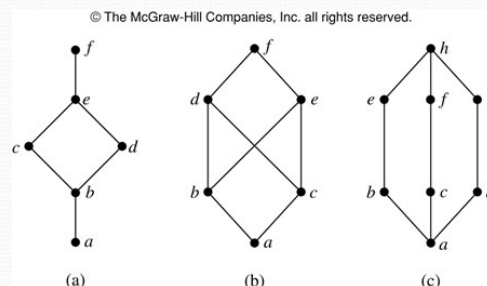
upper bounds: /    lower bounds: 12, 6, 2, 3

The least upper bound: /    The greatest lower bound: 12

43

## Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and greatest lower bound is called a *lattice*.



**FIGURE 8** Hasse Diagrams of Three Posets.

## 9.6 Partial Orderings

(2)  $(\mathbb{Z}, \leq)$  ?

**lub:** the larger of the two elements,

**glb:** the smaller of the two elements.

Hence, the poset  $(\mathbb{Z}, \leq)$  is a lattice.

**Note:** Every totally ordered set is a lattice.

(3)  $(\mathbb{Z}^+, |)$  ?

$\forall a, b \in \mathbb{Z}^+$

**lub:** the least common multiple,

**glb:** the greatest common divisor

Hence, the poset  $(\mathbb{Z}^+, |)$  is a lattice.

(4)  $(P(s), \subseteq)$  ?

$\forall s_1, s_2 \in P(s)$  , **lub:**  $s_1 \cup s_2$     **glb:**  $s_1 \cap s_2$

Hence, the poset  $(P(s), \subseteq)$  is a lattice.

45

## Topological Sorting

- A total ordering  $\preceq$  is said to be compatible with the partial ordering  $R$  if  $a \preceq b$  whenever  $a R b$ .
  - Topological sorting: constructing a compatible total ordering from a partial ordering
- Lemma 1: Every finite nonempty poset  $(S, \preceq)$  has at least one minimal element.

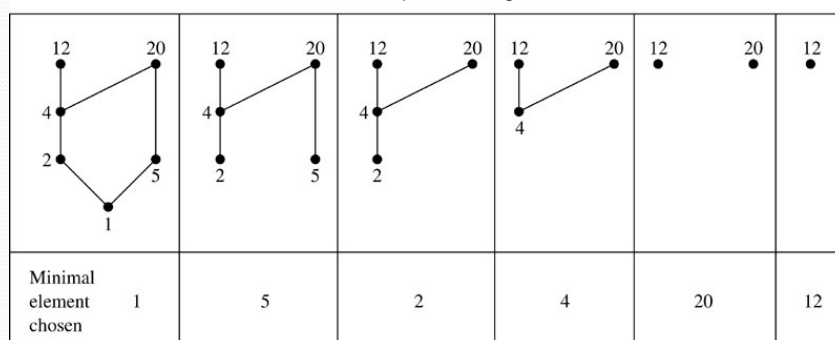


## Topological Sorting

- Algorithm 1: topological sorting
  - Procedure topological sort( $S, \preceq$  : finite poset)
    - $k := 1$
    - while  $S \neq \emptyset$
    - begin
      - $a_k :=$  a minimal element of  $S$
      - $S := S - \{a_k\}$
      - $k := k + 1$
    - end
  - ( See Fig. 9, Fig. 10, 11)

### FIGURE 9 (9.6,p629)

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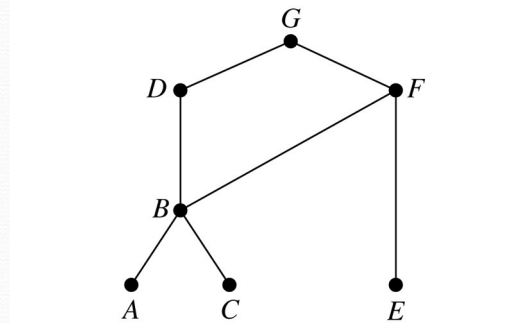


**FIGURE 9** A Topological Sort of  $(\{1, 2, 4, 5, 12, 20\}, \preceq)$ .

P. 577

## FIGURE 10 (9.6)

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**FIGURE 10** The Hasse Diagram for Seven Tasks.

## FIGURE 11 (9.6)

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Minimal element chosen     A	C	B	E	F	D	G

**FIGURE 11** A Topological Sort of the Tasks.

# Homework

Sec. 9.6 5, 10, 23(a),(c), 32, 44, 66