Inclusion-Exclusion

Section 8.5

Section Summary

- The Principle of Inclusion-Exclusion
- Examples

Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

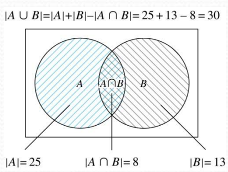
- We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

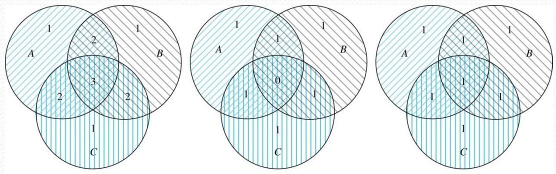
Solution: $|A \cup B| = |A| + |B| - |A \cap B|$

$= 25 + 13 - 8 = 30$



Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Three Finite Sets Continued

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

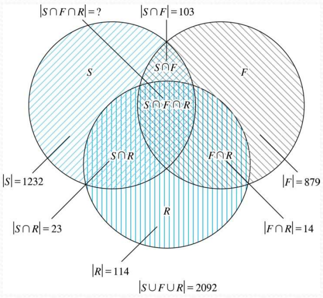
Using the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

Illustration of Three Finite Set Example



The Principle of Inclusion-Exclusion

Theorem 1. The Principle of Inclusion-Exclusion:
Let $A_1, A_2, \ldots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

$$\sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

The Principle of Inclusion-Exclusion (*continued*)

Proof: An element in the union is counted exactly once in the right-hand side of the equation. Consider an element a that is a member of r of the sets $A_1, \ldots, A_n$ where $1 \leq r \leq n$.

- It is counted $C(r,1)$ times by $\sum |A_i|$
- It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r,m)$ times by the summation of m of the sets A_i.

The Principle of Inclusion-Exclusion (cont)

- Thus the element is counted exactly
  $C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1} C(r,r)$
  times by the right hand side of the equation.
- By Corollary 2 of Section 6.4, we have
  $C(r,0) - C(r,1) + C(r,2) - \cdots + (-1)^{r} C(r,r) = 0.$
- Hence,
  $1 = C(r,0) = C(r,1) - C(r,2) + \cdots + (-1)^{r+1} C(r,r).$

〖Example 〗How many positive integers not exceeding 1000 that are not divisible by 5, 6 or 8?

Solution:

U: the set of positive integers not exceeding 1000

A: the set of positive integers not exceeding 1000 that are divisible by 5,

B: the set of positive integers not exceeding 1000 that are divisible by 6,

C: the set of positive integers not exceeding 1000 that are divisible by 8.

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 1000 - \left(\left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{6}\right\rfloor + \left\lfloor\frac{1000}{8}\right\rfloor - \left\lfloor\frac{1000}{5\times 6}\right\rfloor - \left\lfloor\frac{1000}{2\times 3\times 4}\right\rfloor - \left\lfloor\frac{1000}{5\times 8}\right\rfloor + \left\lfloor\frac{1000}{5\times 2\times 3\times 4}\right\rfloor\right)$$

$$= 600$$

Homework

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Applications of Inclusion-Exclusion

Section 8.6

Section Summary

- An alternative form of inclusion-exclusion
- Counting Onto-Functions
- Derangements

8.6 Applications of Inclusion-Exclusion

1. An alternative form of inclusion-exclusion

Problems that ask for the number of elements in a set that have none of *n* properties

$$P_1, P_2, \cdots, P_n\ .$$

Let A_i **be the subset containing the elements that have property** P_i **.**

$N(P_1'P_2'\cdots P_n')$

----The number of elements with none of the properties $P_1, P_2, \cdots, P_n$**.**

From the inclusion-exclusion principle, we see that

$$N(P_1'P_2'\cdots P_n') = N - |A_1 \cup A_2 \cdots \cup A_n| = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \cdots + (-1)^n N(P_1P_2\cdots P_n)$$

8.6 Applications of Inclusion-Exclusion

〖Example 1〗 How many solutions does $x_1 + x_2 + x_3 = 13$ **have, where** x_i **are nonnegative integers with** $x_i < 6, i = 1,2,3$ **?**

Solution:

Let a solution have property P_1 **is** $x_1 \ge 6$**, property** P_2 **is** $x_2 \ge 6$ **, property** P_3 **is** $x_3 \ge 6$ **.**

The number of solutions is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

$C(3-1+13,13)$

$N(P_i) = C(3-1+7,7)$

$N(P_iP_j) = C(3-1+1,1)$

$N(P_1P_2P_3) = 0$

2. The sieve of Eratosthenes

〖Example 2〗 Find the number of primes not exceeding a specified positive integer.

Take 100 for example.

Solution:

- ✧ **A composite integer is divisible by a prime not exceeding its square root.**
- ✧ **Composite integer not exceeding 100 must have a prime factor not exceeding 10.**
- ✧ **Since the only primes less than 10 are 2,3,5,7, the primes not exceeding 100 are these four primes and the positive integers greater than 1 and not exceeding 100 that are divisible by none of 2,3,5,7.**

P_1: the property that an integer is divisible by 2
P_2: the property that an integer is divisible by 3
P_3: the property that an integer is divisible by 5
P_4: the property that an integer is divisible by 7

The number of primes not exceeding positive integer 100 is

$$
\begin{aligned}
&4+N(P_1'P_2'P_3'P_4')\\
&=4+N-N(P_1)-N(P_2)-N(P_3)-N(P_4)+N(P_1P_2)+N(P_1P_3)+N(P_1P_4)\\
&+N(P_2P_3)+N(P_2P_4)+N(P_3P_4)-N(P_1P_2P_3)-N(P_1P_2P_4)-N(P_1P_3P_4)-N(P_2P_3P_4)+N(P_1P_2P_3P_4)\\
&=25
\end{aligned}
$$

99

$\lfloor 100/2 \rfloor$

$\lfloor 100/(2\times 3) \rfloor$

$\lfloor 100/(2\times 3 \times 5) \rfloor$

$\lfloor 100/(2\times 3 \times 5 \times 7) \rfloor$

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

8.6 Applications of Inclusion-Exclusion

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

8.6 Applications of Inclusion-Exclusion

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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8.6 Applications of Inclusion-Exclusion

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	**25**	26	27	28	29	30
31	32	33	34	**35**	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	**55**	56	57	58	59	60
61	62	63	64	**65**	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	**85**	86	87	88	89	90
91	92	93	94	**95**	96	97	98	99	100

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8.6 Applications of Inclusion-Exclusion

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	**25**	26	27	28	29	30
31	32	33	34	**35**	36	37	38	39	40
41	42	43	44	45	46	47	48	**49**	50
51	52	53	54	**55**	56	57	58	59	60
61	62	63	64	**65**	66	67	68	69	70
71	72	73	74	75	76	**77**	78	79	80
81	82	83	**84**	**85**	86	87	88	89	90
91	92	93	94	**95**	96	97	**98**	99	100

The Number of Onto Functions

Example: How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. The function is onto if none of the properties P_1, P_2, and P_3 hold.

By the inclusion-exclusion principle the number of onto functions from a set with six elements to a set with three elements is

$$N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3)$$

- Here the total number of functions from a set with six elements to one with three elements is $N = 3^6$.
- The number of functions that do not have in the range is $N(P_1) = 2^6$. Similarly, $N(P_2) = N(P_1) = 2^6$.
- Note that $N(P_1P_2) = N(P_1P_3) = N(P_2P_3) = 1$ and $N(P_1P_2P_3)= 0$.

Hence, the number of onto functions from a set with six elements to a set with three elements is:

$$3^6 - 3\cdot 2^6 + 3 = 729 - 192 + 3 = 540$$

The Number of Onto Functions (continued)

Theorem 1: Let m and n be positive integers with $m \geq n$. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

Proof follows from the principle of inclusion-exclusion (*see Exercise* 27).

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements (continued)

Theorem 2: The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof follows from the principle of inclusion-exclusion (*see text*).

Derangements (continued)

The Hatcheck Problem: A new employee checks the hats of n people at restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat.

Solution: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Remark: It can be shown that the probability of a derangement approaches $1/e$ as n grows without bound.

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

Proof:

Let a permutation have property P_i if it fixes element *i*.

The number of derangements is the number of permutations having none of the properties P_i for i=1, 2, …, n.

$$D_n = N(P_1'P_2'\cdots P_n')$$
$$= N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \cdots + (-1)^n N(P_1P_2\cdots P_n)$$
$$= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots + (-1)^n \times C(n,n)(n-n)!$$
$$= n! - \frac{n!}{1!(n-1)!}\times(n-1)! + \frac{n!}{2!(n-2)!}\times(n-2)! - \frac{n!}{3!(n-3)!}\times(n-3)! + \ldots + (-1)^n \frac{n!}{n!(n-n)!}\times(n-n)!$$
$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

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Homework

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