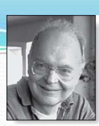
The Growth of Functions

Section 3.2

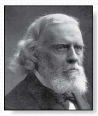
Section Summary



Donald E. Knuth
(Born 1938)

- Big-O Notation
- Big-O Estimates for Important Functions
- Big-Omega and Big-Theta Notation

Edmund Landau
(1877-1938)



Paul Gustav Heinrich Bachmann
(1837-1920)

The Growth of Functions

- In both computer science and in mathematics, there are many times when we care about how fast a function grows.
- In computer science, we want to understand how quickly an algorithm can solve a problem as the size of the input grows.
  - We can compare the efficiency of two different algorithms for solving the same problem.
  - We can also determine whether it is practical to use a particular algorithm as the input grows.
  - We'll study these questions in Section 3.3.
- Two of the areas of mathematics where questions about the growth of functions are studied are:
  - number theory (covered in Chapter 4)
  - combinatorics (covered in Chapters 6 and 8)

Big-*O* Notation

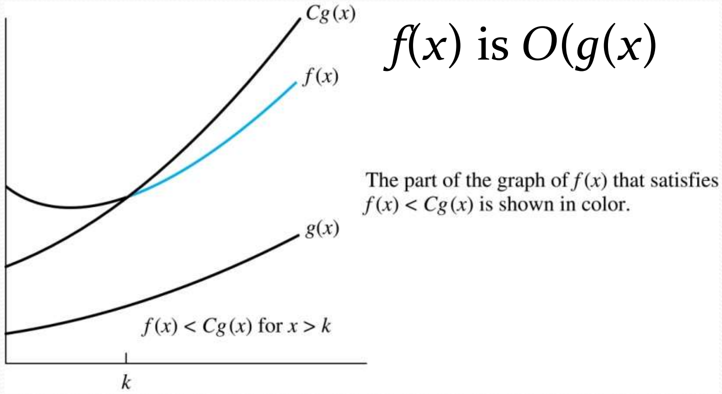
Definition: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. (illustration on next slide)

- This is read as "$f(x)$ is big-O of $g(x)$" or "g asymptotically dominates f."
- The constants C and k are called *witnesses*(凭证） to the relationship $f(x)$ is $O(g(x))$. Only one pair of witnesses is needed.

Illustration of Big-*O* Notation



$f(x)$ is $O(g(x))$

Some Important Points about Big-*O* Notation

- If one pair of witnesses is found, then there are infinitely many pairs. We can always make the k or the C larger and still maintain the inequality $|f(x)| \leq C|g(x)|$.
  - Any pair C' and k' where $C < C'$ and $k < k'$ is also a pair of witnesses since $|f(x)| \leq C|g(x)| \leq C'|g(x)|$ whenever $x > k' > k$.

You may see "$f(x) = O(g(x))$" instead of "$f(x)$ is $O(g(x))$."
  - But this is an abuse of the equals sign since the meaning is that there is an inequality relating the values of f and g, for sufficiently large values of x.
  - It is ok to write $f(x) \in O(g(x))$, because $O(g(x))$ represents the set of functions that are $O(g(x))$.
- Usually, we will drop the absolute value sign since we will always deal with functions that take on positive values.

Using the Definition of Big-*O* Notation

Example: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

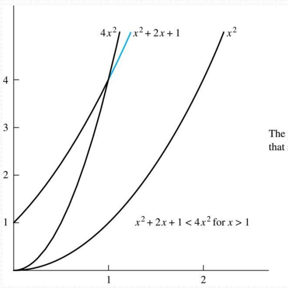
Solution: Since when $x > 1$, $x < x^2$ and $1 < x^2$

$$0 \leq x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2$$

- Can take $C = 4$ and $k = 1$ as witnesses to show that
  $f(x)$ is $O(x^2)$ (see graph on next slide)
- Alternatively, when $x > 2$, we have $2x \leq x^2$ and $1 < x^2$. Hence, $0 \leq x^2 + 2x + 1 \leq x^2 + x^2 + x^2 = 3x^2$ when $x > 2$.
  - Can take $C = 3$ and $k = 2$ as witnesses instead.

Illustration of Big-*O* Notation

$f(x) = x^2 + 2x + 1$ is $O(x^2)$



The part of the graph of $f(x) = x^2 + 2x + 1$ that satisfies $f(x) < 4x^2$ is shown in blue.

Big-O Notation

- Both $f(x) = x^2 + 2x + 1$ and $g(x) = x^2$ are such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.
  We say that the two functions are of the *same order*. (More on this later)
- If $f(x)$ is $O(g(x))$ and $h(x)$ is larger than $g(x)$ for all positive real numbers, then $f(x)$ is $O(h(x))$.
- Note that if $|f(x)| \leq C|g(x)|$ for $x > k$ and if $|h(x)| > |g(x)|$ for all x, then $|f(x)| \leq C|h(x)|$ if $x > k$. Hence, $f(x)$ is $O(h(x))$.
- For many applications, the goal is to select the function $g(x)$ in $O(g(x))$ as small as possible (up to multiplication by a constant, of course).

Using the Definition of Big-O Notation

Example: Show that $7x^2$ is $O(x^3)$.

Solution: When $x > 7$, $7x^2 < x^3$. Take $C = 1$ and $k = 7$ as witnesses to establish that $7x^2$ is $O(x^3)$.

(Would $C = 7$ and $k = 1$ work?)

Example: Show that n^2 is not $O(n)$.

Solution: Suppose there are constants C and k for which $n^2 \leq Cn$, whenever $n > k$. Then (by dividing both sides of $n^2 \leq Cn$) by n, then $n \leq C$ must hold for all $n > k$. A contradiction!

Big-O Estimates for Polynomials

Example: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.

Then $f(x)$ is $O(x^n)$.

Uses triangle inequality, an exercise in Section 1.8.

Proof: $|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_1|$

$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x^1 + |a_1|$

Assuming $x > 1$

$= x^n (|a_n| + |a_{n-1}| / x + \cdots + |a_1| / x^{n-1} + |a_1| / x^n)$

$\leq x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|)$

- Take $C = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|$ and $k = 1$. Then $f(x)$ is $O(x^n)$.
- The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-O Estimates for some Important Functions

Example: Use big-O notation to estimate the sum of the first n positive integers.

Solution: $1 + 2 + \cdots + n \leq n + n + \cdots n = n^2$

$1 + 2 + \ldots + n$ is $O(n^2)$ taking $C = 1$ and $k = 1$.

Example: Use big-O notation to estimate the factorial function $f(n) = n! = 1 \times 2 \times \cdots \times n$.

Solution:

$$n! = 1 \times 2 \times \cdots \times n \leq n \times n \times \cdots \times n = n^n$$

$$n! \text{ is } O(n^n) \text{ taking } C = 1 \text{ and } k = 1.$$

Continued →

Big-*O* Estimates for some Important Functions

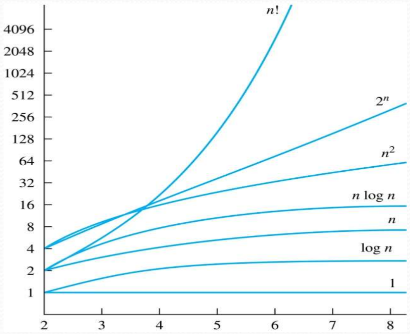
Example: Use big-*O* notation to estimate log n!

Solution: Given that $n! \leq n^n$ (previous slide)

then $\log(n!) \leq n \cdot \log(n)$.

Hence, $\log(n!)$ is $O(n \cdot \log(n))$ taking $C = 1$ and $k = 1$.

Display of Growth of Functions



Note the difference in behavior of functions as *n* gets larger

Useful Big-*O* Estimates Involving Logarithms, Powers, and Exponents

- If $d > c > 1$, then
  n^c is $O(n^d)$, but n^d is not $O(n^c)$.
- If $b > 1$ and c and d are positive, then
  $(\log_b n)^c$ is $O(n^d)$, but n^d is not $O((\log_b n)^c)$.
- If $b > 1$ and d is positive, then
  n^d is $O(b^n)$, but b^n is not $O(n^d)$.
- If $c > b > 1$, then
  b^n is $O(c^n)$, but c^n is not $O(b^n)$.

Combinations of Functions

- If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then
  $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.
  - See next slide for proof
- If $f_1(x)$ and $f_2(x)$ are both $O(g(x))$ then
  $(f_1 + f_2)(x)$ is $O(g(x))$.
  - See text for argument
- If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then
  $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x))$.
  - See text for argument

Combinations of Functions

- If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then
  $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.
  - By the definition of big-O notation, there are constants C_1, C_2, k_1, k_2 such that $|f_1(x)| \le C_1|g_1(x)|$ when $x > k_1$ and $f_2(x) \le C_2|g_2(x)|$ when $x > k_2$.
  - $|(f_1 + f_2)(x)| = |f_1(x) + f_2(x)|$
    $\le |f_1(x)| + |f_2(x)|$ by the triangle inequality $|a + b| \le |a| + |b|$
  - $|f_1(x)| + |f_2(x)| \le C_1|g_1(x)| + C_2|g_2(x)|$
    $\le C_1|g(x)| + C_2|g(x)|$ where $g(x) = \max(|g_1(x)|,|g_2(x)|)$
    $= (C_1 + C_2)\,|g(x)|$
    $= C|g(x)|$ where $C = C_1 + C_2$
  - Therefore $|(f_1 + f_2)(x)| \le C|g(x)|$ whenever $x > k$, where $k = \max(k_1,k_2)$.

Ordering Functions by Order of Growth

- Put the functions below in order so that each function is big-O of the next function on the list.
- $f_1(n) = (1.5)^n$
- $f_2(n) = 8n^3+17n^2 +111$
- $f_3(n) = (\log n)^2$
- $f_4(n) = 2^n$
- $f_5(n) = \log(\log n)$
- $f_6(n) = n^2(\log n)^3$
- $f_7(n) = 2^n(n^2 +1)$
- $f_8(n) = n^3+ n(\log n)^2$
- $f_9(n) = 10000$
- $f_{10}(n) = n!$

We solve this exercise by successively finding the function that grows slowest among all those left on the list.

- $f_9(n) = 10000$ (constant, does not increase with n)
- $f_5(n) = \log(\log n)$ (grows slowest of all the others)
- $f_3(n) = (\log n)^2$ (grows next slowest)
- $f_6(n) = n^2(\log n)^3$ (next largest, $(\log n)^3$ factor smaller than any power of n)
- $f_8(n) = n^3+ n(\log n)^2$ (tied with the one above)
- $f_2(n) = 8n^3+17n^2 +111$ (tied with the one below)
- $f_1(n) = (1.5)^n$ (next largest, an exponential function)
- $f_4(n) = 2^n$ (grows faster than one above since $2 > 1.5$)
- $f_7(n) = 2^n(n^2 +1)$ (grows faster than above because of the $n^2 +1$ factor)
- $f_{10}(n) = n!$ ($n!$ grows faster than c^n for every c)

Big-Omega Notation

Definition: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are constants C and k such that

$$|f(x)| \geq C|g(x)| \quad \text{when } x > k.$$

Ω is the upper case version of the lower case Greek letter ω.

- We say that "$f(x)$ is big-Omega of $g(x)$."
- Big-O gives an upper bound on the growth of a function, while Big-Omega gives a lower bound. Big-Omega tells us that a function grows at least as fast as another.
- $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$. This follows from the definitions. See the text for details.

Big-Omega Notation

Example: Show that $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(g(x))$ where $g(x) = x^3$.

Solution: $f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3$ for all positive real numbers x.

- Is it also the case that $g(x) = x^3$ is $O(8x^3 + 5x^2 + 7)$?

Big-Theta Notation

Θ is the upper case version of the lower case Greek letter θ.

- **Definition**: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. The function $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.
- We say that "f is big-Theta of $g(x)$" and also that "$f(x)$ is of *order* $g(x)$" and also that "$f(x)$ and $g(x)$ are of the *same order*."
- $f(x)$ is $\Theta(g(x))$ if and only if there exists constants C_1, C_2 and k such that $C_1 g(x) < f(x) < C_2 g(x)$ if $x > k$. This follows from the definitions of big-O and big-Omega.

Big Theta Notation

Example: Show that the sum of the first n positive integers is $\Theta(n^2)$.

Solution: Let $f(n) = 1 + 2 + \cdots + n$.

- We have already shown that $f(n)$ is $O(n^2)$.
- To show that $f(n)$ is $\Omega(n^2)$, we need a positive constant C such that $f(n) > Cn^2$ for sufficiently large n. Summing only the terms greater than $n/2$ we obtain the inequality

$$\begin{aligned} 1 + 2 + \cdots + n &\geq \lceil n/2 \rceil + (\lceil n/2 \rceil + 1) + \cdots + n \\ &\geq \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \\ &= (\mathrm{n} - \lceil n/2 \rceil + 1)\lceil n/2 \rceil \\ &\geq (\mathrm{n}/2)(\mathrm{n}/2) = n^2/4 \end{aligned}$$

- Taking $C = 1/4$, $f(n) > Cn^2$ for all positive integers n. Hence, $f(n)$ is $\Omega(n^2)$, and we can conclude that $f(n)$ is $\Theta(n^2)$.

Big-Theta Notation

Example: Show that $f(x) = 3x^2 + 8x \log x$ is $\Theta(x^2)$.

Solution:

- $3x^2 + 8x \log x \leq 11x^2$ for $x > 1$,
  since $0 \leq 8x \log x \leq 8x^2$.
  - Hence, $3x^2 + 8x \log x$ is $O(x^2)$.
- x^2 is clearly $O(3x^2 + 8x \log x)$
- Hence, $3x^2 + 8x \log x$ is $\Theta(x^2)$.

Big-Theta Notation

- When $f(x)$ is $\Theta(g(x))$ it must also be the case that
  $g(x)$ is $\Theta(f(x))$.
- Note that $f(x)$ is $\Theta(g(x))$ if and only if it is the case that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.
- Sometimes writers are careless and write as if big-O notation has the same meaning as big-Theta.

Big-Theta Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.
Then $f(x)$ is of order x^n (or $\Theta(x^n)$).
(The proof is an exercise.)

Example:

The polynomial $f(x) = 8x^5 + 5x^2 + 10$ is order of x^5 (or $\Theta(x^5)$).

The polynomial $f(x) = 8x^{199} + 7x^{100} + x^{99} + 5x^2 + 25$ is order of x^{199} (or $\Theta(x^{199})$).

Homework

Sec. 3.2 8(c), 26(a), 54, 56

Complexity of Algorithms

Section 3.3

Section Summary

- Time Complexity
- Worst-Case Complexity
- Algorithmic Paradigms
- Understanding the Complexity of Algorithms

The Complexity of Algorithms

- Given an algorithm, how efficient is this algorithm for solving a problem given input of a particular size? To answer this question, we ask:
  - How much time does this algorithm use to solve a problem?
  - How much computer memory does this algorithm use to solve a problem?
- When we analyze the time the algorithm uses to solve the problem given input of a particular size, we are studying the *time complexity* of the algorithm.
- When we analyze the computer memory the algorithm uses to solve the problem given input of a particular size, we are studying the *space complexity* of the algorithm.

The Complexity of Algorithms

- In this course, we focus on time complexity. The space complexity of algorithms is studied in later courses.
- We will measure time complexity in terms of the number of operations an algorithm uses and we will use big-O and big-Theta notation to estimate the time complexity.
- We can use this analysis to see whether it is practical to use this algorithm to solve problems with input of a particular size. We can also compare the efficiency of different algorithms for solving the same problem.
- We ignore implementation details (including the data structures used and both the hardware and software platforms) because it is extremely complicated to consider them.

Time Complexity

- To analyze the time complexity of algorithms, we determine the number of operations, such as comparisons and arithmetic operations (addition, multiplication, etc.). We can estimate the time a computer may actually use to solve a problem using the amount of time required to do basic operations.
- We ignore minor details, such as the "house keeping" aspects of the algorithm.
- We will focus on the *worst-case time* complexity of an algorithm. This provides an upper bound on the number of operations an algorithm uses to solve a problem with input of a particular size.
- It is usually much more difficult to determine the *average case time complexity* of an algorithm. This is the average number of operations an algorithm uses to solve a problem over all inputs of a particular size.

Complexity Analysis of Algorithms

Example: Describe the time complexity of the algorithm for finding the maximum element in a finite sequence.

procedure *max*($a_1, a_2, \ldots, a_n$: integers)
 $max := a_1$
 for $i := 2$ to n
  if $max < a_i$ then $max := a_i$
 return *max*{*max* is the largest element}

Solution: Count the number of comparisons.

- The $max < a_i$ comparison is made $n - 1$ times.
- Each time i is incremented, a test is made to see if $i \leq n$.
- One last comparison determines that $i > n$.
- Exactly $2(n - 1) + 1 = 2n - 1$ comparisons are made.

Hence, the time complexity of the algorithm is $\Theta(n)$.

Worst-Case Complexity of Linear Search

Example: Determine the time complexity of the linear search algorithm.

procedure *linear search*(x:integer, $a_1, a_2, \ldots, a_n$: distinct integers)
$i := 1$
while ($i \leq n$ and $x \neq a_i$)
  $i := i + 1$
if $i \leq n$ **then** *location* := i
else *location* := 0
return *location*{*location* is the subscript of the term that equals x, or is 0 if x is not found}

Solution: Count the number of comparisons.

- At each step two comparisons are made; $i \leq n$ and $x \neq a_i$.
- To end the loop, one comparison $i \leq n$ is made.
- After the loop, one more $i \leq n$ comparison is made.

If $x = a_i$, $2i + 1$ comparisons are used. If x is not on the list, $2n + 1$ comparisons are made and then an additional comparison is used to exit the loop. So, in the worst case $2n + 2$ comparisons are made. Hence, the complexity is $\Theta(n)$.

Average-Case Complexity of Linear Search

Example: Describe the average case performance of the linear search algorithm. (Although usually it is very difficult to determine average-case complexity, it is easy for linear search.)

Solution: Assume the element is in the list and that the possible positions are equally likely. By the argument on the previous slide, if $x = a_i$, the number of comparisons is $2i + 1$.

$$\frac{3+5+7+\ldots+(2n+1)}{n} = \frac{2(1+2+3+\ldots+n)+n}{n} = \frac{2[\frac{n(n+1)}{2}]}{n} + 1 = n + 2$$

Hence, the average-case complexity of linear search is $\Theta(n)$.

Worst-Case Complexity of Binary Search

Example: Describe the time complexity of binary search in terms of the number of comparisons used.

```
procedure binary search(x: integer, a1,a2,..., an: increasing integers)
 i := 1 {i is the left endpoint of interval}
 j := n {j is right endpoint of interval}
 while i < j
       m := ⌊(i + j)/2⌋
       if x > am then i := m + 1
       else j := m
 if x = ai then location := i
 else location := 0
 return location{location is the subscript i of the term ai equal to x, or 0 if x is not found}
```

Solution: Assume (for simplicity) $n = 2^k$ elements. Note that $k = \log n$.

- Two comparisons are made at each stage; $i < j$, and $x > a_m$.
- At the first iteration the size of the list is 2^k and after the first iteration it is 2^{k-1}. Then 2^{k-2} and so on until the size of the list is $2^1 = 2$.
- At the last step, a comparison tells us that the size of the list is the size is $2^0 = 1$ and the element is compared with the single remaining element.
- Hence, at most $2k + 2 = 2 \log n + 2$ comparisons are made.
- Therefore, the time complexity is $\Theta(\log n)$, better than linear search.

Worst-Case Complexity of Bubble Sort

Example: What is the worst-case complexity of bubble sort in terms of the number of comparisons made?

```
procedure bubblesort(a1,...,an: real numbers
                     with n ≥ 2)
  for i := 1 to n−1
    for j := 1 to n −i
      if aj >aj+1 then interchange aj and aj+1
{a1,..., an is now in increasing order}
```

Solution: A sequence of $n-1$ passes is made through the list. On each pass $n - i$ comparisons are made.

$$(n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2}$$

The worst-case complexity of bubble sort is $\Theta(n^2)$ since $\frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$.

Worst-Case Complexity of Insertion Sort

Example: What is the worst-case complexity of insertion sort in terms of the number of comparisons made?

Solution: The total number of comparisons are:

$$2 + 3 + \cdots + n = \frac{n(n-1)}{2} - 1$$

Therefore the complexity is $\Theta(n^2)$.

```
procedure insertion sort(a_1,...,a_n:
            real numbers with n ≥ 2)
  for j := 2 to n
     i := 1
     while a_j > a_i
        i := i + 1
     m := a_j
     for k := 0 to j − i − 1
        a_{j-k} := a_{j-k-1}
     a_i := m
```

Matrix Multiplication Algorithm

- The definition for matrix multiplication can be expressed as an algorithm; **C** = **A B** where **C** is an $m \times n$ matrix that is the product of the $m \times k$ matrix **A** and the $k \times n$ matrix **B**.
- This algorithm carries out matrix multiplication based on its definition.

```
procedure matrix multiplication(A,B: matrices)
  for i := 1 to m
     for j := 1 to n
        c_ij := 0
        for q := 1 to k
           c_ij := c_ij + a_iq b_qj
return C{C = [c_ij] is the product of A and B}
```

$\mathbf{A} = [a_{ij}]$ is a $m \times k$ matrix
$\mathbf{B} = [b_{ij}]$ is a $k \times n$ matrix

Complexity of Matrix Multiplication

Example: How many additions of integers and multiplications of integers are used by the matrix multiplication algorithm to multiply two $n \times n$ matrices.

Solution: There are n^2 entries in the product. Finding each entry requires n multiplications and $n - 1$ additions. Hence, n^3 multiplications and $n^2(n - 1)$ additions are used.

Hence, the complexity of matrix multiplication is $O(n^3)$.

Boolean Product Algorithm

- The definition of Boolean product of zero-one matrices can also be converted to an algorithm.

procedure *Boolean product*(**A**,**B**: zero-one matrices)
  for i := 1 to m
    for j := 1 to n
      c_{ij} := 0
      for q := 1 to k
        $c_{ij} := c_{ij} \vee (a_{iq} \wedge b_{qj})$
return C{**C** = $[c_{ij}]$ is the Boolean product of **A** and **B**}

Complexity of Boolean Product Algorithm

Example: How many bit operations are used to find $\mathbf{A} \odot \mathbf{B}$, where A and B are $n \times n$ zero-one matrices?

Solution: There are n^2 entries in the $\mathbf{A} \odot \mathbf{B}$. A total of n Ors and n ANDs are used to find each entry. Hence, each entry takes $2n$ bit operations. A total of $2n^3$ operations are used.

Therefore the complexity is $O(n^3)$

Matrix-Chain Multiplication

- How should the *matrix-chain* $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n$ be computed using the fewest multiplications of integers, where $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ are $m_1 \times m_2$, $m_2 \times m_3, \cdots m_n \times m_{n+1}$ integer matrices. Matrix multiplication is associative (exercise in Section 2.6).

Example: In which order should the integer matrices $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ - where $\mathbf{A}_1$ is 30×20 $\mathbf{A}_2$ 20×40, $\mathbf{A}_3$ 40×10 - be multiplied to use the least number of multiplications.

Solution: There are two possible ways to compute $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$.

- $\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3)$: $\mathbf{A}_2\mathbf{A}_3$ takes $20 \cdot 40 \cdot 10 = 8000$ multiplications. Then multiplying $\mathbf{A}_1$ by the 20×10 matrix $\mathbf{A}_2\mathbf{A}_3$ takes $30 \cdot 20 \cdot 10 = 6000$ multiplications. So the total number is $8000 + 6000 = 14{,}000$.
- $(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3$: $\mathbf{A}_1\mathbf{A}_2$ takes $30 \cdot 20 \cdot 40 = 24{,}000$ multiplications. Then multiplying the 30×40 matrix $\mathbf{A}_1\mathbf{A}_2$ by $\mathbf{A}_3$ takes $30 \cdot 40 \cdot 10 = 12{,}000$ multiplications. So the total number is $24{,}000 + 12{,}000 = 36{,}000$.

So the first method is best.

An efficient algorithm for finding the best order for matrix-chain multiplication can be based on the algorithmic paradigm known as *dynamic programming*. (see Ex. 57 in Section 8.1)

Algorithmic Paradigms

- An *algorithmic paradigm* is a general approach based on a particular concept for constructing algorithms to solve a variety of problems.
  - Greedy algorithms were introduced in Section 3.1.
  - We discuss brute-force algorithms in this section.
  - We will see divide-and-conquer algorithms (Chapter 8), dynamic programming (Chapter 8), backtracking (Chapter 11), and probabilistic algorithms (Chapter 7). There are many other paradigms that you may see in later courses.

Brute-Force Algorithms

- A *brute-force* algorithm is solved in the most straightforward manner, without taking advantage of any ideas that can make the algorithm more efficient.
- Brute-force algorithms we have previously seen are sequential search, bubble sort, and insertion sort.

Computing the Closest Pair of Points by Brute-Force

Example: Construct a brute-force algorithm for finding the closest pair of points in a set of n points in the plane and provide a worst-case estimate of the number of arithmetic operations.

Solution: Recall that the distance between (x_i,y_i) and (x_j, y_j) is $\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$. A brute-force algorithm simply computes the distance between all pairs of points and picks the pair with the smallest distance.

Note: There is no need to compute the square root, since the square of the distance between two points is smallest when the distance is smallest.

Continued →

Computing the Closest Pair of Points by Brute-Force

- Algorithm for finding the closest pair in a set of n points.

```
procedure closest pair((x1, y1), (x2, y2), … ,(xn, yn): xi, yi real numbers)
min = ∞
  for i := 1 to n
    for j := 1 to i
       if (xj − xi)² + (yj − yi)² < min
        then min := (xj − xi)² + (yj − yi)²
          closest pair := (xi, yi), (xj, yj)
return closest pair
```

- The algorithm loops through $n(n-1)/2$ pairs of points, computes the value $(x_j - x_i)^2 + (y_j - y_i)^2$ and compares it with the minimum, etc. So, the algorithm uses $\Theta(n^2)$ arithmetic and comparison operations.
- We will develop an algorithm with $O(n\log n)$ worst-case complexity in Section 8.3.

Understanding the Complexity of Algorithms

TABLE 1 Commonly Used Terminology for the Complexity of Algorithms.

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Understanding the Complexity of Algorithms

TABLE 2 The Computer Time Used by Algorithms.

Problem Size	*Bit Operations Used*					
n	$\log n$	n	$n \log n$	n^2	2^n	$n!$
10	3×10^{-11} s	10^{-10} s	3×10^{-10} s	10^{-9} s	10^{-8} s	3×10^{-7} s
10^2	7×10^{-11} s	10^{-9} s	7×10^{-9} s	10^{-7} s	4×10^{11} yr	*
10^3	1.0×10^{-10} s	10^{-8} s	1×10^{-7} s	10^{-5} s	*	*
10^4	1.3×10^{-10} s	10^{-7} s	1×10^{-6} s	10^{-3} s	*	*
10^5	1.7×10^{-10} s	10^{-6} s	2×10^{-5} s	0.1 s	*	*
10^6	2×10^{-10} s	10^{-5} s	2×10^{-4} s	0.17 min	*	*

Times of more than 10^{100} years are indicated with an *.

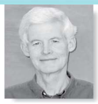
Example

- 以计算 nXn 阶行列式为例子。
- 如果我们用行列式定义去计算，其计算复杂度为 O(n•n!)
- 如果我们通过行消元将行列式化上三角行列计算，其计算复杂度为 $O(n^3)$
- 如果 n= 50,采用第一种方法所需要计算的乘法次数 $1.5X10^{65}$，第二种方法所需要的乘法次数为 $6X10^6$，如果用一台计算速度是每秒可执行1000亿次乘法的计算机计算，其计算时间分别为 $5x10^{46}$ 年及 $6x10^{-5}$ 秒

Complexity of Problems

- *Tractable*（易解） *Problem*: There exists a polynomial time algorithm to solve this problem. These problems are said to belong to the *Class P*.
- *Intractable* （难解） *Problem*: There does not exist a polynomial time algorithm to solve this problem
- *Unsolvable Problem* : No algorithm exists to solve this problem, e.g., halting problem.
- *Class NP*: Solution can be checked in polynomial time. But no polynomial time algorithm has been found for finding a solution to problems in this class.
- *NP Complete Class*: If you find a polynomial time algorithm for one member of the class, it can be used to solve all the problems in the class.

P Versus NP Problem



Stephen Cook
(Born 1939)

- The *P versus NP problem* asks whether the class P = NP? Are there problems whose solutions can be checked in polynomial time, but can not be solved in polynomial time?
  - Note that just because no one has found a polynomial time algorithm is different from showing that the problem can not be solved by a polynomial time algorithm.
- If a polynomial time algorithm for any of the problems in the NP complete class were found, then that algorithm could be used to obtain a polynomial time algorithm for every problem in the NP complete class.
  - Satisfiability (in Section 1.3) is an NP complete problem.
- It is generally believed that P≠NP since no one has been able to find a polynomial time algorithm for any of the problems in the NP complete class.
- The problem of P versus NP remains one of the most famous unsolved problems in mathematics (including theoretical computer science). The Clay Mathematics Institute has offered a prize of $1,000,000 for a solution.

Homework

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