Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$, where $c_1, c_2, \ldots., c_k$ are real numbers, and $c_k \neq 0$

• it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
• it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
• the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1$, $a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

〖Example 〗

(1) $a_n = (1.02)a_{n-1}$

linear; constant coefficients; homogeneous; degree 1

(2) $a_n = (1.02)\ a_{n-1} + 2^{n-1}$

linear; constant coefficients; nonhomogeneous; degree 1

(3) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

linear; constant coefficients; nonhomogeneous; degree 3

(4) $a_n = n\ a_{n-1} + n^2\ a_{n-2} + a_{n-1}\ a_{n-2}$

nonlinear; coefficients are not constants; nonhomogeneous; degree 2

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Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_k\ a_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_k\ r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:
  $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

【Theorem 1】 Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. Then the Sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0,1,2,\ldots$, where α_1, α_2 are constants.

Proof:

- **Show that if r_1, r_2 are the roots of the characteristic equation, and α_1, α_2 are constant, then the sequence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation**

$$r_1^2 = c_1 r_1 + c_2$$

$$r_2^2 = c_1 r_2 + c_2$$

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$$c_1a_{n-1} + c_2a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$
$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$
$$= \alpha_1 r_1^{n} + \alpha_2 r_2^{n}$$
$$= a_n$$

- **Show that if $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constant α_1, α_2 .**

Suppose that $\{a_n\}$ is a solution, and the initial condition $a_0 = C_0, a_1 = C_1$ hold.

$$a_0 = C_0 = \alpha_1 + \alpha_2$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$$

$$\Rightarrow \quad \alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2} \qquad \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$.

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Using Theorem 1

Example: What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$a_0 = 2 = \alpha_1 + \alpha_2$ and $a_1 = 7 = \alpha_1 2 + \alpha_2(-1)$.

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3\cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 .$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$a_0 = 1 = \alpha_1$ and $a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$.

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence,

$a_n = 3^n + n3^n$.

Example: What is the solution to the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$?

Solution: $a_n = 3^n - \frac{2}{3} n3^n$.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

〖Example 〗 What is the solution of the recurrence relation

$$a_{n+2} = 3a_{n+1}, a_0 = 4$$

Solution:

(1) The Characteristic equation of the recurrence relation is $r - 3 = 0$.

(2) Find the root of the characteristic equation: $r_1 = 3$

(3) Compute the general solution: $a_n = c3^n$

(4) Find the constants based on the initial conditions:

$$a_0 = c3^n = 4$$

(5) Produce the specific solution: $a_n = 4 \cdot 3^n$

Example

- Recurrence relation:
  - $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with a_0=2, a_1=5, a_2=15
- Characteristic equation: $r^3-6r^2+11r-6=0$
- The distinct roots are 1,2,3 hence
  - $a_n = l_1 1^n + l_2 2^n + l_3 3^n$
- Use the initial condition we find that
  - l_1=1, l_2=-1, l_3=2. $a_n = 1 - 2^n + 2 \bullet 3^n$

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

The General Case with Repeated Roots Allowed

$$a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1}\alpha_{i,j}n^j\right)r_i^n$$

E.g., Roots of C.E. are 2, 2, 2, 5, 5 and 9.

Solution:

$(\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)*2^n + (\alpha_{2,0} + \alpha_{2,1}n)*5^n + \alpha_{3,0}9^n$

Example

- Find the solution to the recurrence relation
- $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ *with initial condition*
- $a_0=1,\ a_1 = -2,\ a_2 = -1$
- *Solution:* the characteristic equation is *:*
- $r^3+3r^2+3r+1=0$, -1 is the single root of multiplicity 3. hence
- $a_n = (l_{1,0} + l_{1,1}n + l_{1,2}n^2)(-1)^n$

Cont..

- Use the initial conditions, we obtain
- $a_0=1=l_{1,0}$
- $a_1=-2=-l_{1,0}-l_{1,1}-l_{1,2}$
- $a_2=-1= l_{1,0}+2l_{1,1}+4l_{1,2}$ solving the equation we get $l_{1,0}=1$, $l_{1,1}=3$ and $l_{1,2}=-2$
- Hence $a_n=(1+3n-2n^2)(-1)^n$

More Example

- the recurrence relation
- $a_n = 3a_{n-1}+6a_{n-2}-28a_{n-3}+24\,a_{n-4}$
- *with initial condition* $a_0=-2$, $a_1 = 12$, $a_2 = 22$, $a_3 = 222$
- *Solution:* the characteristic equation is *:*
- $r^4-3r^3-6r^2+28r-24=(r-2)^3(r+3)=0$,

  2 is the root of multiplicity 3; and −3 is a root of multiplicity 1, then

Cont.

- $a_n = (l_{1,0} + l_{1,1}\, n + l_{1,2}\, n^2)2^n + l_{2,0}\,(-3)^n$
- Use the initial conditions, we obtain
- $a_0 = l_{1,0} + l_{2,0} = -2$
- $a_1 = 2l_{1,0} + 2\, l_{1,1} + 2l_{1,2} - 3\, l_{2,0} = 12$
- $a_2 = 4l_{1,0} + 8\, l_{1,1} + 16l_{1,2} + 9\, l_{2,0} = 22$
- $a_3 = 8l_{1,0} + 24l_{1,1} + 72l_{1,2} - 27\, l_{2,0} = 222$
- solving the equation we get $l_{1,0}=0$, $l_{1,1}=1$ and $l_{1,2}=2$, $l_{2,0}= -2$ Hence $a_n=(n+2n^2)2^n-2(-3)^n$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k} + F(n),$$

where c_1, c_2, ….,c_k are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k},$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_k\, a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_k\, a_{n-k}.$$

Proof:

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$$

Suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)$$

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)})$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.
Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.
Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.
Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Step of solution

- A. find the general form of the solution of *associated homogeneous recurrence relation,say g(n)*
- B. find a particular solution of linear non-homogeneous recurrence with constant coefficients,say f(n)
- C. use the initial condition to solve out the parameters in f(n)+g(n)

The key problem

- The key problem is how to find a particular solution to a linear non-homogeneous recurrence with constant coefficients. We cannot always "guess" the particular solution, but if f(n) is the product of a polynomial in n and the nth power of a constant, we know exactly what form a particular solution is.

Theorem 6

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$
- $F(n)=(b_t n^t+b_{t-1}n^{t-1}+\ldots+b_1 n+b_0)s^n$
- When s is not a root of the characteristic equation of the associated linear homogeneous recurrence solution, this is a particular solution of the form $f(n)=(p_t n^t+p_{t-1}n^{t-1}+\ldots+p_1 n+p_0)s^n$
- When s is a root of the characteristic equation and its multiplicity is m, this is a particular solution of the form $f(n)=n^m(p_t n^t+p_{t-1}n^{t-1}+\ldots+p_1 n+p_0)s^n$

Cont…

- ***Example*** $a_n=6a_{n-1}-9a_{n-2}+F(n)$ what form does a particular solution has?

  (a) $F(n)=3^n$,(b)$F(n)=n^2 3^n$,(c)$F(n)=(n^2+1)3^n$ (d)$F(n)=n^2 2^n$
- **Solution**: 3 is root of characteristic equation with multiplicity 2.
- (a) $F(n)=3^n$, $f(n)=p_0 n^2 3^n$
- (b)$F(n)=n^2 3^n$, $f(n)=n^2(p_2 n^2+p_1 n+p_0)3^n$
- (c)$F(n)=(n^2+1)3^n$, $f(n)=n^2(p_2 n^2+p_1 n+p_0)3^n$
- (d)$F(n)=n^2 2^n$, $f(n)= (p_2 n^2+p_1 n+p_0)2^n$

Cont…

- **Example** $a_n=a_{n-1}+n$, $a_1=1$, the general solution of its associated linear homogeneous recurrence solution is $g(n)=c$
- Since $F(n) = n = n\cdot 1^n$ and 1 is also a root of degree one of the characteristic equation, from theorem 6 there is a particular solution of the form $f(n) = n\cdot (p_1n+p_0)\cdot 1^n = p_1n^2+p_0n$ substituting it to original equation we obtain:
- $p_1n^2+p_0n = p_1(n-1)^2+p_0(n-1) + n$
- $p_1= p_0=1/2$ at last $a_n=n(n+1)/2$

More example

- **Exercise** find all the solutions of recurrence relation

$a_n=5a_{n-1}- 6a_{n-2} +2^n+3n$, $a_0=0$, $a_1=1$,

It can be seen two recurrence relations

$a_n=5a_{n-1}- 6a_{n-2} + 2^n$ and

$a_n=5a_{n-1}- 6a_{n-2} + 3n$

the particular solutions of this two recurrence relations are: p_0n2^n, p_1n+p_2 respectively,

Since 2 and 3 are single roots of characteristic equation, the general solution of its associated linear homogeneous recurrence solution is $g(n) = c_12^n+c_23^n$

- So the all solutions are:
- $a_n= c_12^n+c_23^n + p_0n2^n+p_1n+p_2$

More example

- **Exercise** solve the simultaneous recurrence relations:
- $a_n = 3a_{n-1} + 2b_{n-1}$
- $b_n = a_{n-1} + 2b_{n-1}$ with $a_0 = 1$ and $b_0 = 2$
- Solution:

$$b_n = a_n - 2a_{n-1}$$

$$a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2}$$

- $a_1 = 3a_0 + 2b_0 = 7.$

$$a_n = -1 + 2 \cdot 4^n \qquad b_n = 1 + 4^n$$

Homework

Sec. 8.2 2,4(g), 20, 30, 35