Rules of Inference

Section 1.6

Section Summary

- Valid Arguments（有效论证）
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Revisiting the Socrates Example

- We have the two premises:
  - "All men are mortal."
  - "Socrates is a man."
- And the conclusion:
  - "Socrates is mortal."
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore \quad Mortal(Socrates)$$

- We will see shortly that this is a valid argument.

Valid Arguments

- We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.
  1. Propositional Logic
     Inference Rules
  2. Predicate Logic
     Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion. An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p_1, p_2, ..., p_n$ and the conclusion is q then

  $(p_1 \land p_2 \land ... \land p_n) \to q$ is a tautology.
- Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

Valid Arguments

An *argument* is a sequence of statements that end with a conclusion

Proofs in mathematics are *valid arguments*

By *valid*, we mean the conclusion must follow from the truth of the preceding statements (premises)

We use *rules of inference* to construct valid arguments

Rules of Inference for Propositional Logic: Modus Ponens（假言推理）

$$p \rightarrow q$$
$$p$$
$$\overline{}$$
$$\therefore q$$

Corresponding Tautology:
$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:
Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore, I will study discrete math."

Modus Tollens（取拒式）

$$p \to q$$
$$\neg q$$
$$\overline{\therefore \neg p}$$

Corresponding Tautology:
$(\neg q \land (p \to q)) \to \neg p$

Example:
Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism（假言三段论）

$$p \to q$$
$$q \to r$$
$$\overline{\therefore p \to r}$$

Corresponding Tautology:
$((p \to q) \land (q \to r)) \to (p \to r)$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism（析取三段论）

$$p \vee q$$
$$\frac{\neg p}{\therefore q}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition（附加律）

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:
$p \rightarrow (p \vee q)$

Example:
Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification（化简律）

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:
$(p \wedge q) \rightarrow p$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction（合取律）

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution（消解律）

Resolution plays an important role in AI and is used in Prolog.

$$\neg p \lor r$$
$$\underline{p \lor q}$$
$$\therefore q \lor r$$

Corresponding Tautology:
$((\neg p \lor r) \land (p \lor q)) \rightarrow (q \lor r)$

Example:
Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$S_1$$
$$S_2$$
$$.$$
$$.$$
$$.$$
$$S_n$$

$$\therefore C$$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \to q)$$

Show that q is a conclusion.

Solution:

Step	Reason
1. $p \wedge (p \to q)$	Premise
2. p	Simplification using (1)
3. $p \to q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:
  - "It is not sunny this afternoon and it is colder than yesterday."
  - "We will go swimming only if it is sunny."
  - "If we do not go swimming, then we will take a canoe trip."
  - "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the conclusion:
  - "We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon." r : "We will go swimming." t : "We will be home by sunset."
   q : "It is colder than yesterday." s : "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q, \ r \to p, \ \neg r \to s, \ s \to t$$
$$\text{Conclusion: } t$$

Valid Arguments

3. Construct the Valid Argument

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)
（全称实例）

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)
（全称引入）

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)
（存在实例）

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)
（存在引入）

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid
   argument to show that
   "John Smith has two legs"
   is a consequence of the premises:
   "Every man has two legs." "John Smith is a man."
Solution: Let $M(x)$ denote "x is a man" and $L(x)$ "x has two legs"
   and let John Smith be a member of the domain.
   Valid Argument:

Step	Reason
1. $\forall x(M(x) \to L(x))$	Premise
2. $M(J) \to L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument
   showing that the conclusion
   "Someone who passed the first exam has not read the book."
   follows from the premises
   "A student in this class has not read the book."
   "Everyone in this class passed the first exam."
Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read
   the book," and $P(x)$ denote "x passed the first exam."
   First we translate the
   premises and conclusion
   into symbolic form.

$$\exists x(C(x) \wedge \neg B(x))$$
$$\underline{\forall x(C(x) \to P(x))}$$
$$\therefore \ \exists x(P(x) \wedge \neg B(x))$$

Continued on next slide →

Using Rules of Inference

Valid Argument:

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore \quad Mortal(Socrates)$$

Solution for Socrates Example

Valid Argument

Step	Reason
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens （全称假言推理）

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\forall x(P(x) \rightarrow Q(x))$$
$$P(a), \text{where } a \text{ is a particular}$$
$$\text{element in the domain}$$
$$\overline{\therefore Q(a)}$$

This rule could be used in the Socrates example.

Exercise

- Sec. 1.6 12, 14(d), 18, 24, 29, 34(a)