Relations

Chapter 9

Chapter Summary

- Relations and Their Properties
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Social Relationships

- There are many kinds of relationships in the world:
- Relative: Relationship by blood or by a common ancestor.
- Friendship: boyfriend and girlfriend
- Relations between Teachers and students
- Relations between bosses and employees

Social Relationships

- Relations between war and peace
- Relations between city and village
- Relations between God and mankind
- Relations between mankind and their environment
- Relations between obama and osama (bin laden)
- And so on...

Abstract Relationships

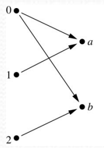
- The question is how to represent relationship in mathematical methods
- N-ary relationships (complex): relationships among many objects.
- But most of the relationship can be formalized in the idea of binary relation.
- Binary relation is the simplest relation, it is what we will study in this course.

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are
  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4, and R_6: $(1,2)$ is in R_1 and R_6: $(2,1)$ is in R_2, R_5, and R_6: $(1, -1)$ is in R_2, R_3, and R_6 : $(2,2)$ is in R_1, R_3, and R_4.

Reflexive（自反） Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in$ A. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \longrightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),
$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition:A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$

- **Example**: The following relations on the integers are antisymmetric:

$R_1 = \{(a,b) \mid a \leq b\}$, ⟵ For any integer, if a $a \leq b$ and $a \leq b$, then $a = b$.
$R_2 = \{(a,b) \mid a > b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$.

The following relations are not antisymmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \wedge (y,z) \in R \longrightarrow (x,z) \in R]$$

- **Example**: The following relations on the integers are transitive:

$R_1 = \{(a,b) \mid a \leq b\}$, ← For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Question:

Symmetric, transitive $\Rightarrow$ reflexive ?

$$\left.\begin{matrix}(a,b) \in R \\ R \text{ is symmetric}\end{matrix}\right\} \Rightarrow \left.\begin{matrix}(b,a) \in R \\ R \text{ is transitive}\end{matrix}\right\} \Rightarrow (a,a) \in R$$

This argument makes an assumption that $\forall a \exists b (a,b) \in R$

Therefore, symmetry and transitivity are not enough to infer reflexivity

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

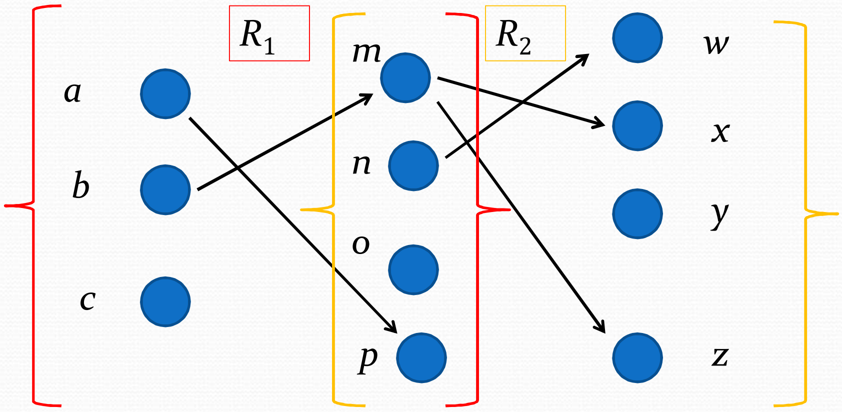
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$

relational composition

- Let M be the relation "is mother of"
- Let F be the relation "is father of"
- What is $M \circ F$?
  - If $(a,b) \in F$, then a is the father of b
  - If $(b,c) \in M$, then b is the mother of c
  - Thus, $M \circ F$ denotes the relation "maternal grandfather"（外公）
- What is $F \circ M$?
  - If $(a,b) \in M$, then a is the mother of b
  - If $(b,c) \in F$, then b is the father of c
  - Thus, $F \circ M$ denotes the relation "paternal grandmother"（奶奶）
- What is $M \circ M$?
  - If $(a,b) \in M$, then a is the mother of b
  - If $(b,c) \in M$, then b is the mother of c
  - Thus, $M \circ M$ denotes the relation "maternal grandmother"（外婆）
- Note that M and F are not transitive relations!!!

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

- Example: $R=\{(1,1),(2,1),(3,2),(4,3)\}$

  Find R^2, R^3, and R^4

  $R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$

  $R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

  $R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

Example

- R={(a,b), a is parent of b or vice versa}
- R^2= {(a,b), a is grandparent of b or vice versa}
- N-generations blood relationship: if $(a,b) \in R^n$, we say a and b have n-generations blood relationship

Theorem 1

- Then relation R on a set A is transitive if and only if $R^n \subseteq R$.(n=1,2,3,...)
- If part: $R^n \subseteq R$, $R^2 \subseteq R$. *if* $(a,b) \in R$ and $(b,c) \in R$ for any a,b,c $\in A$, then (a,c) $\in R^2$, *hence,* (a,c) $\in R$, R is transitive.
- Only if part: if R is transitive, (a,c) $\in R^2$, then there exist $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R$. Hence (a,c) $\in R$
- *This implies that* $R^2 \subseteq R$

Cont...

- Further more, $R^3=R^2 \circ R \subseteq R \circ R = R^2 \subseteq R$
- Then for any n=1,2,3,...
- $R^n=R^{n-1} \circ R \subseteq \ldots\ldots \subseteq R \circ R = R^2 \subseteq R$

- *Inverse Relation*: Let R be a relation from set A to set B, the inverse of R is a relation from B to A such that :
- $R^{-1} = \{(a,b) | (b,a) \in R\}$

Homework

- 第八版 Sec. 9.1 7(a,c,h), 26, 32, 49, 53

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1,a_2, a_3\}$ and $B = \{b_1,b_2, b_3,b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

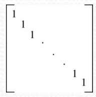
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i,b_j) with $m_{ij} = 1$, it follows that:

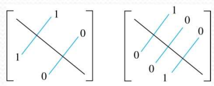
$R = \{(a_1, b_2), (a_2, b_1),(a_2, b_3), (a_2, b_4),(a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

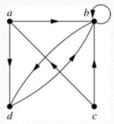
Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

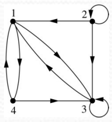
- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?

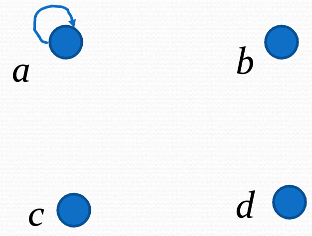


Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Determining which Properties a Relation has from its Digraph

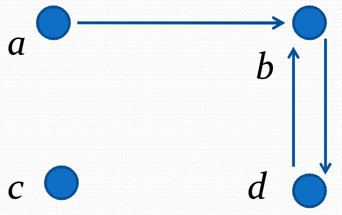
- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

Determining which Properties a Relation has from its Digraph – Example 1



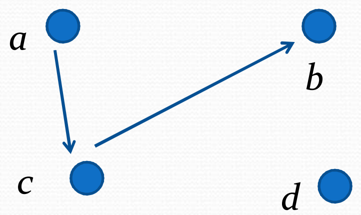
- *Reflexive?* No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



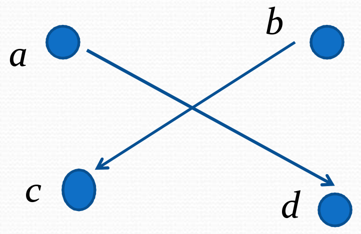
- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from a to b, but not from b to a
- *Antisymmetric?* No, there is an edge from d to b and b to d
- *Transitive?* No, there are edges from a to c and from c to b, but there is no edge from a to d

Determining which Properties a Relation has from its Digraph – Example 3



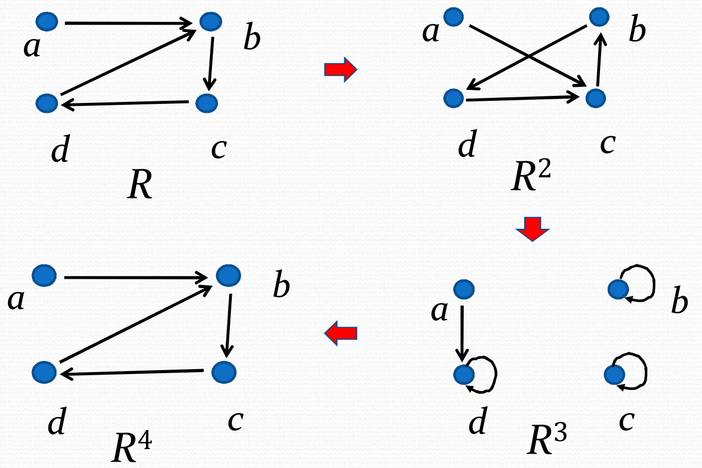
Reflexive? No, there are no loops
Symmetric? No, for example, there is no edge from c to a
Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back
Transitive? No, there is no edge from a to b

Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive?* No, there are no loops
- *Symmetric?* No, for example, there is no edge from d to a
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

Inverse relation

$R = \{(a,b) \mid a \in A, b \in B, aRb\}$

The *inverse relation* from *B* to *A*: $R^{-1}(R^c)$

$$\{(b,a) \mid (a,b) \in R, a \in A, b \in B\}$$

Question:

How to get R^{-1} **?**

(1) Using the definition directly

For example, $R = \{(a,b) \mid a \mid b, a,b \in Z^+\}$

$R^{-1} = \{(a,b) \mid b \mid a, a,b \in Z^+\}$

(2) Reverse all the arcs in the digraph representation of *R*

(3) Take the transpose $M_R{}^T$ of the connection matrix M_R of *R*.

41

The properties of relation operations

Suppose that *R*, *S* are the relations from *A* to *B*, *T* is the relation from *B* to *C*, *P* is the relation from *C* to *D*, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof:

$\forall (x,y) \in (R \cup S)^{-1}$

$\Leftrightarrow (y,x) \in R \cup S$

$\Leftrightarrow (y,x) \in R \quad or \quad (y,x) \in S$

$\Leftrightarrow (x,y) \in R^{-1} \quad or \quad (x,y) \in S^{-1}$

$\Leftrightarrow (x,y) \in R^{-1} \cup S^{-1}$

The properties of relation operations

Suppose that R, S are the relations from A to B, T is the relation from B to C, P is the relation from C to D, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(2) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(3) $(\overline{R})^{-1} = \overline{R^{-1}}$

(4) $(R - S)^{-1} = R^{-1} - S^{-1}$

(5) $(A \times B)^{-1} = B \times A$

Proof:

$$\forall (x, y) \in (A \times B)^{-1}$$

$$\Leftrightarrow \quad (y, x) \in A \times B$$

$$\Leftrightarrow \quad (x, y) \in B \times A$$

The properties of relation operations

Suppose that R, S are the relations from A to B, T is the relation from B to C, P is the relation from C to D, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(2) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(3) $(\overline{R})^{-1} = \overline{R^{-1}}$

(4) $(R - S)^{-1} = R^{-1} - S^{-1}$

(5) $(A \times B)^{-1} = B \times A$

(6) $\overline{R} = A \times B - R$

(7) $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$

(8) $(R \circ T) \circ P = R \circ (T \circ P)$

(9) $(R \cup S) \circ T = R \circ T \cup S \circ T$

Homework

Sec. 9.3 13,14,31

Closures of Relations

Section 9.4

Definition of Closure

- The *closure* of a relation R with respect to property **P** is the relation obtained by adding the minimum number of ordered pairs to R to obtain property **P**.

Reflexive Closure

- In terms of the digraph representation of R:
  - Add loops to all vertices to find the reflexive closure
- In terms of the 0-1 matrix representation:
  - Put 1's on the diagonal to find the reflexive closure
- $r(R)=R\cup\Delta$ where $\Delta=\{(a,a)|a\in A\}$

Symmetric Closure

- In terms of the digraph representation of R:
  - Add arcs in the opposite direction to find the symmetric closure

  $S(R)= R \cup R^{-1}$
- In terms of the 0-1 matrix representation:
  - Add 1's to the pairs across the diagonals that differ in value

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \Rightarrow \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 1 & 0 \end{matrix}$$

Transitive Closure

- It is very easy to find the reflexive closure and the symmetric closure, but it is difficult to find the transitive closure
- In terms of the digraph representation of R:
  - To find the transitive closure, if there is a path from a to b, add an arc from a to b (can be complicated)

Transitive Closure

- R={(1,3),(1,4),(2,1),(3,2)} first adding the pairs (1,2),(2,3),(2,4)(3,1) to R obtain R'={(1,3),(1,4),(2,1),(3,2), (1,2),(2,3),(2,4) (3,1) } is not transitive either.
- A path from *a* to *b* in the digraph *G* is a sequence of one or more edges (x_0,x_1), (x_1,x_2), …, (x_{n-1},x_n) in *G* where $x_0=a$ and $x_n=b$. if a=b, the path is called circuit or cycle.

Transitive Closure (Cont.)

- This path is denoted by $x_0,x_1,x_2, \ldots, x_n$ and has length n. the path is called a cycle if it starts and ends at the same vertex.
- Theorem 1: Let R be a relation on a set A, there is a path of length n from a to b if and only if $(a,b) \in R^n$

Proof:

① **Inductive basis**

An edge from a to b is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

② **Inductive step**

There is a path of length n+1 from a to b if and only if there is an x in A such that there is a path of length 1 from a to x and a path of length n from x to b.

From the Induction Hypothesis,

$$(a,x) \in R \qquad (x,b) \in R^n$$

$$(a,b) \in R^n \circ R = R^{n+1}$$

53

Transitive Closure (Cont.)

- $R^* = \cup_1^\infty R^n$, is called the connectivity relation of R,which consists of the (a,b) such that there is path from a to b.
- Theorem 2: the transitive closure of a relation R (denoted by t(R)) equals the connectivity R*
- $R^* = \cup_1^\infty R^i = t(R)$.

Proof

- To prove R* is transitive closure we must prove:
- (1) R*⊇R. It is obvious by definition
- (2)R* is transitive. If (a,b)∈R*,(b,c)∈R*, it implies there is a path from a to b and a path from b to c, hence there is a path from a to c through b.
- (3)R* is minimum. If S is also a transitive relation containing R, then S ⊇R*. It is obvious that S*=S. since S ⊇R, then S*⊇R*, hence S ⊇R*.

Lemma 1

- A is a set containing n elements. R is relation on A. if there is a path from a to b, then there is such path with length not exceeding n. if a≠b, there is such path with length not exceeding n-1.
- From this lemma, t(R)= $\cup_1^n R^i$

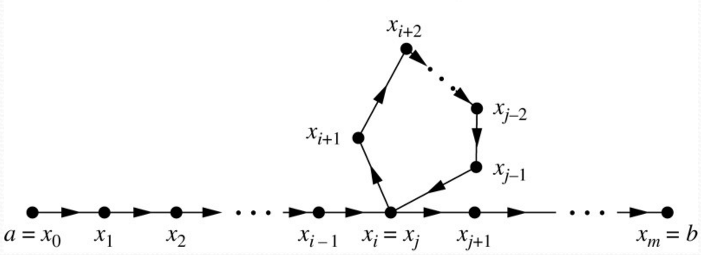


FIGURE 2 Producing a Path with Length Not Exceeding *n*.

Transitive Closure (Cont.)

- Theorem 3 $M_{R^*} = M_R \vee M_R^2 \vee M_R^3 \vee \dots \vee M_R^n$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad M_R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = {M_R}^3 = M_{R^*}$$

Cont…

Algorithm 1 A procedure for computing the transitive closure

procedure *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)

```
A := M_R
B := A
for i := 2 to n
begin
    A := A ⊙ M_R
    B := B ∨ A
end { B is the zero-one matrix for R* }
```

Transitive Closure (Cont.)

- Warshall's algorithm an efficient method for computing the transitive closure of a relation.
- Interior vertices of a path: a,x_1,x_2, …, x_{m-1}, b. x_1,x_2, …, x_{m-1} are interior vertices
- Matrices: $M_R = W_0, W_1, W_2, \ldots, W_n = M_{R^*}$
- Named after Stephen Warshall in 1960
  - $2n^3$ bit operation
  - Also called Roy-Warshall algorithm, Bernard Roy in 1959
  - Previous algorithm 1 using $2n^3$ (n-1) bit operation

$$n^2(2n-1)(n-1)+(n-1)n^2 = 2n^3(n-1) = O(n^4)$$

FIGURE 3 (9.4,p604)

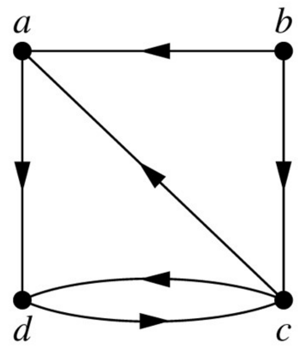


FIGURE 3 The Directed Graph of the Relations *R*.

Warshall's algorithm

- Observation: we can compute W_k directly from W_{k-1}
  - Two cases (Fig. 4)
    (a) There is a path from v_i to v_j with its interior vertices among the first k-1 vertices
      - $w_{ij}^{(k-1)}=1$

    (b) There are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first k-1 vertices
      - $w_{ik}^{(k-1)}=1$ and $w_{kj}^{(k-1)}=1$

FIGURE 4 (9.4)

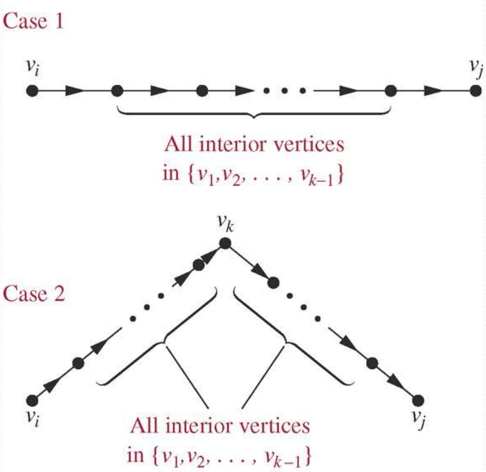


FIGURE 4 p605 Adding v_k to the Set of Allowable Interior Vertices.

Warshall's algorithm

- Lemma 2: Let $Wk=w_{ij}^{(k)}$ be the zero-one matrix that has a 1 in its (i,j)th position iff there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then $w_{ij}^{(k)}=w_{ij}^{(k-1)} \vee (w_{ik}^{(k-1)} \wedge w_{kj}^{(k-1)})$,
  whenever I, j, and k are positive integers not exceeding n.

Transitive Closure (Cont.)

- Algorithm 2 warshall algorithm
- Procedure warshall(M_R:n × n zero-one matrix)
- W= M_R
- For k=1 to n
- Begin
-   For I=1 to n
-     Begin
-       For j=1 to n
-       $W_{ij}=w_{ij} \vee (w_{ik} \wedge w_{kj})$
-     End
- End

【Example】 Let $A=\{1,2,3,4,5\}, R=\{(1,1),(1,2),(2,4),(3,5),(4,2)\}, t(R)=?$

Solution:

$$M=\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=1}\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=2}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$$\xrightarrow{k=3}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=4}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=5}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$W_{ij}=w_{ij}\vee(w_{ik}\wedge w_{kj})$

If $(w_{ik}==1)$

$W_{ij}=w_{ij}\vee w_{kj}$

65

Homework

Sec.9.4 2, 6, 9(6), 11(6), 20, 28(a), 29

Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations（等价关系）
- Equivalence Classes（等价类）
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

  Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s|(a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative （代表元）of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a-2m, a-m, a+2m, a+2m, \dots\}$. For example,

$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$ $[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$

$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$ $[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$

Proof: We show that (i) implies (ii). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(1) aRb
(2) $[a]=[b]$
(3) $[a]\cap[b]\neq\phi$

- **Show that (2) implies (3)**

$$\left.\begin{array}{l}[a]=[b] \\ \textbf{\textit{R} is reflexive} \Rightarrow \textbf{[\textit{a}] is nonempty}\end{array}\right\} \Rightarrow [a]\cap[b]\neq\phi$$

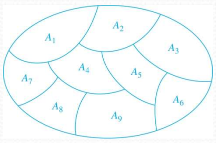
- **Show that (3) implies (1)**

$$[a]\cap[b]\neq\phi \quad \Rightarrow \exists x\in[a]\cap[b]$$
$$\Rightarrow (a,x)\in R,(b,x)\in R$$
$$\Rightarrow (a,x)\in R,(x,b)\in R$$
$$\Rightarrow (a,b)\in R$$

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j=\emptyset$ when $i \neq j$,
- and $\bigcup_{i\in I} A_i = S.$



Notation: $pr(A)=\{A_i \mid i\in I\}$

A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

.

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

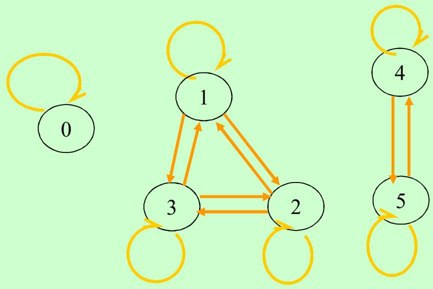
- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

〖Example〗 **Find the partition of the set A from R.**

$A = \{0,1,2,3,4,5\}$,

$R = \{(0,0),(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,4),(4,5),(5,4),(5,5)\}$

Solution:



[0]={0}

[1]={1,2,3}

[4]={4,5}

***pr*(*A*)={ [0], [1], [4]}**

Question:

Congruence Modulo *m*

$$R = \{(a,b) \mid a \equiv b (\mathrm{mod}\, m), a,b \in Z\}, pr(Z) = ?$$

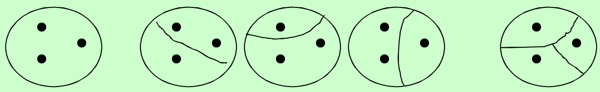
$$pr(Z) = \{[0]_m, [1]_m, \cdots, [m-1]_m\}$$

Question:

|*A*|=3. How many different equivalence relations on the set *A* are there?

Solution:

an equivalence relation on a set A $\leftrightarrow$ a partition of A



The combining of the Equivalence Relations

- Let R and S be equivalence relations on A, how about $R \cap S$, $R \cup S$?
- The answer for $R \cap S$ is yes.
- (1) Is it reflexive?:
- Since $(a,a) \in R$ $(a,a) \in S$,then $(a,a) \in R \cap S$ for every $a \in A$
- (2) Is it symmetric?
- Since $R^{-1}=R$ $S^{-1}=S$, then $(R \cap S)^{-1} = R^{-1} \cap S^{-1} = R \cap S$

The combining of the Equivalence Relations

- Is it transitive?
- Since R and S are transitive, then $R^2 \subseteq R$, $S^2 \subseteq S$
- Then $(R \cap S)^2 = R^2 \cap RoS \cap SoR \cap S^2 \subseteq R^2 \cap S^2 = R \cap S$
- So $R \cap S$ is equivalence relation.
- But the answer for $R \cup S$ is No!

【 Theorem 】 If R_1, R_2 are equivalence relations on *A*, then $R_1 \cup R_2$ is a reflexive and symmetric relation on A.

Proof:

(1) reflexive

$$\forall a \in A \quad \because (a,a) \in R_1, (a,a) \in R_2 \quad \therefore (a,a) \in R_1 \cup R_2$$

(2) symmetric

$$(a,b) \in R_1 \cup R_2 \quad \Rightarrow (a,b) \in R_1 \text{ or } (a,b) \in R_2$$
$$\Rightarrow (b,a) \in R_1 \text{ or } (b,a) \in R_2 \quad \Rightarrow (b,a) \in R_1 \cup R_2$$

Question: transitive?

〖**Example** 〗 $A = \{a,b,c\}$,

$R_1 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$

$R_2 = \{(a,a),(b,b),(c,c),(b,c),(c,b)\}$

Is $R_1 \cup R_2$ a transitive relation ?

Solution:

$$R_1 \cup R_2 = \{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$$

【 Theorem 】 If R_1, R_2 are equivalence relations on A, then $(R_1 \cup R_2)^*$ is an equivalence relation on A.

Proof:

(1) reflexive

(2) symmetric

(3) transitive

Homework

- Sec. 9.5 3, 10, 16, 36(b), 39, 41