# CHAPTER 10 Graphs







## Graph Theory

**Graph theory is an old subject with many modern applications .**

**For example, graphs can be used to**

**- study the structure of the World Wide Web.**

**- determine whether a circuit can be implemented on a planar circuit board.**

**- solve problems such as finding the shortest path between two cities in a transportation network.**

**- to schedules exams, and so on.**

## 10.1 Graphs and Graph Models

**【Definition 1】 A *graph* $G=(V,E)$ consists of $V$, a nonempty set of *vertices* (or *nodes*) and $E$, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.**

**Remark:**

- ✓ *Unrelated to graphs of functions studied in Chapter 2*
- ✓ *All that matters is the connections made by the edges, not the particular geometry depicted.*
- ✓ *Infinite Graph, finite Graph*

**Types of *Undirected Graphs***

- **Simple graph**
- **Multigraph**
- **Pseudograph**



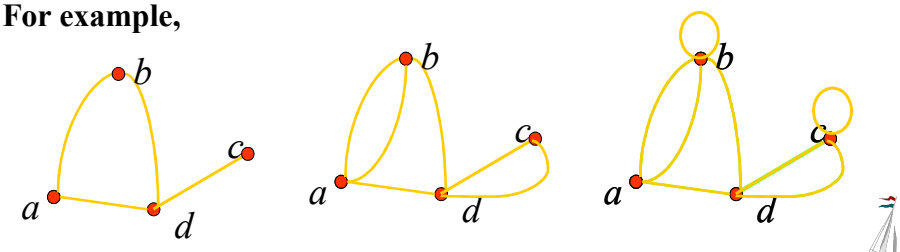


***Simple* graph: A graph in which each edge connects two different vertices *and* where no two edges connect the same pair of vertices.**

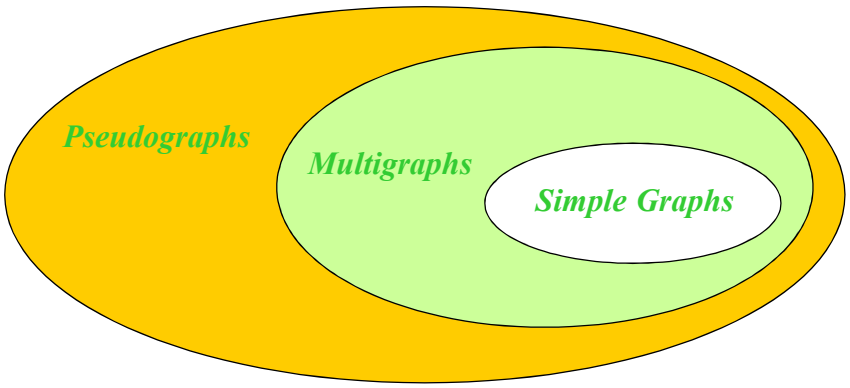
***Multigraph:* Graphs that may have multiple edges connecting the same vertices.**

***Pseudograph:* Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices.**

**For example,**




**The relations of different undirected graphs**







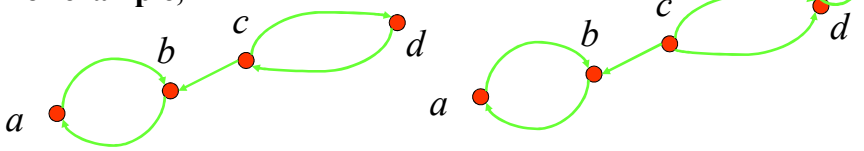

**【Definition 2】A *directed graph* (or *digraph*) *(V, E)* consists of a nonempty set of vertices *V* and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (*u*,*v*) is said to *start* at *u* and *end* at *v*.**

**Types of digraphs:**

***Simple directed graph:*** **a directed graph has no loops and has no multiple directed edges.**

***directed multigraph:*** **a directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex.**

**For example,**








## Graph Models

**Problems in almost every conceivable discipline can be solved using graph models.**

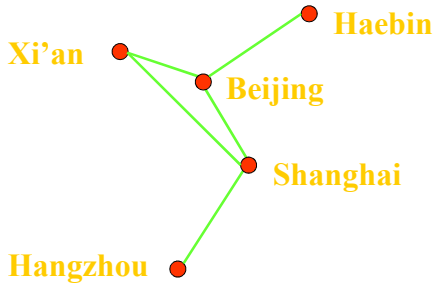
**For example,**

- ✓ **Niche overlap Graphs in Ecology**
- ✓ **Influence Graphs**
- ✓ **The Hollywood Graph**
- ✓ **Round-Robin Tournament**
- ✓ **The Web Graph**
- ✓ **……**

〖Example 1〗 How can we represent a network of (bi-directional) railways connecting a set of cities?

*Solution:*

We should use a simple graph with an edge $\{a, b\}$ indicating a direct train connection between cities $a$ and $b$.



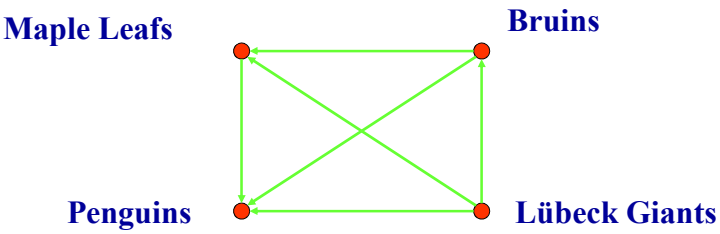





〖Example 2〗 In a round-robin tournament, each team plays against each other team exactly once. How can we represent the results of the tournament (which team beats which other team)?

*Solution:*

We should use a *directed graph* with an edge $(a, b)$ indicating that team $a$ beats team $b$.



# Other Applications of Graphs

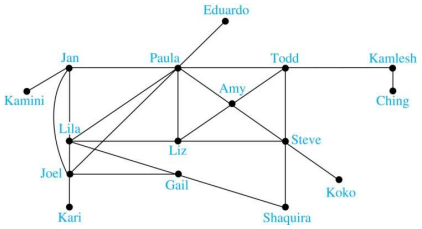
- We will illustrate how graph theory can be used in models of:
  - Social networks
  - Communications networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks
- It's a challenge to find a subject to which graph theory has not yet been applied. Can you find an area without applications of graph theory?

# Graph Models: Social Networks

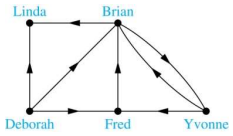
- Graphs can be used to model social structures based on different kinds of relationships between people or groups.
- In a *social network*, vertices represent individuals or organizations and edges represent relationships between them.
- Useful graph models of social networks include:
  - *friendship graphs* - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)
  - *collaboration graphs* - undirected graphs where two people are connected if they collaborate in a specific way
  - *influence graphs* - directed graphs where there is an edge from one person to another if the first person can influence the second person

# Graph Models: Social Networks (*continued*)

➢Example: A friendship graph where two people are connected if they are Facebook friends.




➢Example: An influence graph




➢*Next Slide: Collaboration Graphs*

# Examples of Collaboration Graphs

- The *Hollywood graph* models the collaboration of actors in films.
  - We represent actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie.
  - We will study the Hollywood Graph in Section 10.4 when we discuss Kevin Bacon numbers.
- An *academic collaboration graph* models the collaboration of researchers who have jointly written a paper in a particular subject.
  - We represent researchers in a particular academic discipline using vertices.
  - We connect the vertices representing two researchers in this discipline if they are coauthors of a paper.
  - We will study the academic collaboration graph for mathematicians when we discuss *Erdős numbers* in Section 10.4.

# Applications to Information Networks

- Graphs can be used to model different types of networks that link different types of information.
- In a *web graph*, web pages are represented by vertices and links are represented by directed edges.
  - A web graph models the web at a particular time.
  - We will explain how the web graph is used by search engines in Section 11.4.
- In a *citation network*:
  - Research papers in a particular discipline are represented by vertices.
  - When a paper cites a second paper as a reference, there is an edge from the vertex representing this paper to the vertex representing the second paper.

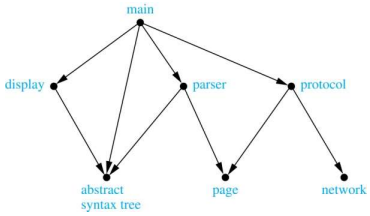
# Transportation Graphs

- Graph models are extensively used in the study of transportation networks.
- Airline networks can be modeled using directed multigraphs where
  - airports are represented by vertices
  - each flight is represented by a directed edge from the vertex representing the departure airport to the vertex representing the destination airport
- Road networks can be modeled using graphs where
  - vertices represent intersections and edges represent roads.
  - undirected edges represent two-way roads and directed edges represent one-way roads.

# Software Design Applications

- Graph models are extensively used in software design. We will introduce two such models here; one representing the dependency between the modules of a software application and the other representing restrictions in the execution of statements in computer programs.
- When a top-down approach is used to design software, the system is divided into modules, each performing a specific task.
- We use a *module dependency graph* to represent the dependency between these modules. These dependencies need to be understood before coding can be done.
  - In a module dependency graph vertices represent software modules and there is an edge from one module to another if the second module depends on the first.

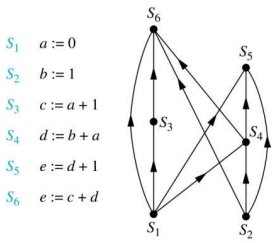
➢Example: The dependencies between the seven modules in the design of a web browser are represented by this module dependency graph.




# Software Design Applications (*continued*)

- We can use a directed graph called a *precedence graph* to represent which statements must have already been executed before we execute each statement.
  - Vertices represent statements in a computer program
  - There is a directed edge from a vertex to a second vertex if the second vertex cannot be executed before the first

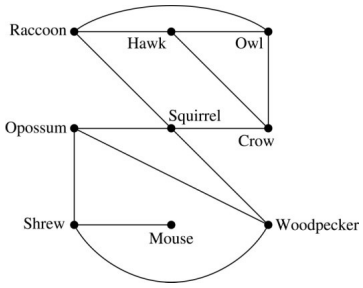
➢Example: This precedence graph shows which statements must already have been executed before we can execute each of the six statements in the program.




# Biological Applications

- Graph models are used extensively in many areas of the biological science. We will describe two such models, one to ecology and the other to molecular biology.
- *Niche overlap graphs* model competition between species in an ecosystem
  - Vertices represent species and an edge connects two vertices when they represent species who compete for food resources.

➢Example: This is the niche overlap graph for a forest ecosystem with nine species.




**Homework:**

**Sec. 10.1 1, 3-9**