CHAPTER 10 Graphs

【Definition】 A graph is called *planar* if it can be drawn in the plane without any edges crossing .

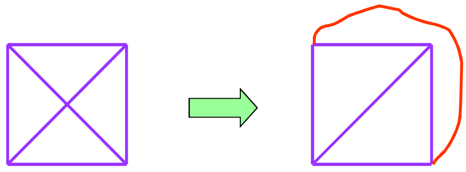
Such a drawing is called a *planar representation* of the graph.

Understanding planar graph is important:

- Any graph representation of maps is planar.
- Electronic circuits usually represented by planar graphs.

〖Example 1〗 Are the following graphs planar?

(1)



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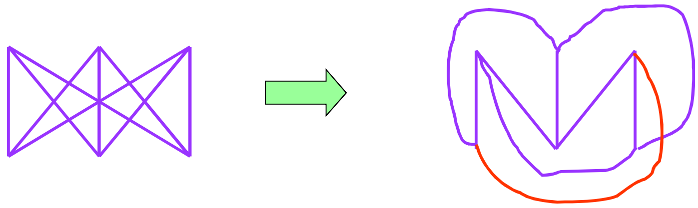

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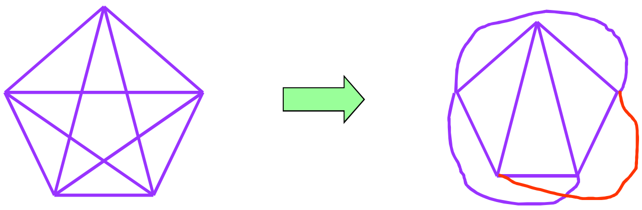


Note:

To prove that a graph is planar amounts to redrawing the edges in a way that no edges will cross. May need to move vertices around and the edges may have to be drawn in a very indirect fashion.

1. Euler's Formula

Region

-- A region is a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.

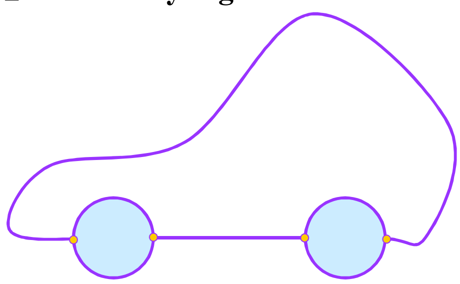
- Bounded region
- Unbounded region

Note:

There is one unbounded region in a planar graph.

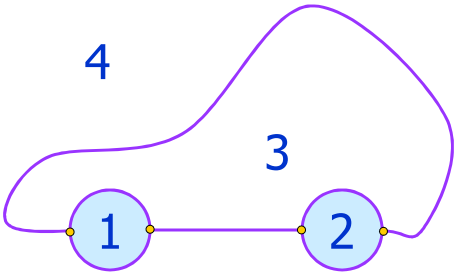
〖Example 2〗 How many regions does the following graph have?

(1)



〖Example 2〗 How many regions does the following graph have?

(1)



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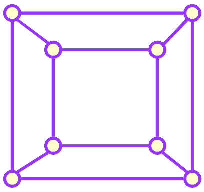
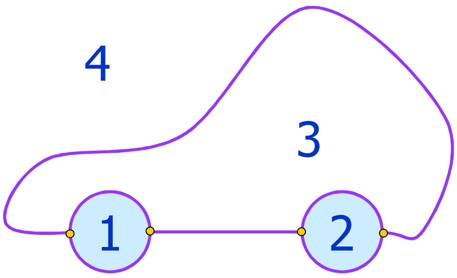


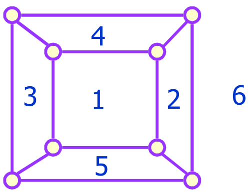
10.7 Planar Graphs

〖Example 2〗 How many regions does the following graph have?

(1)



(2)



【 Theorem 1】 Euler's formula

Let G be a *connected planar simple* graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r=e-v+2$.

Proof:

First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs $G_1, G_2, \cdots, G_e = G$, successively adding an edge at each stage.

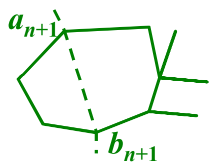
The constructing method: Arbitrarily pick one edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already in G_{n-1}, adding the other vertex incident with this edge if it is not already in G_{n-1}

Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G, respectively.

(1) The relationship $r_1= e_1 - v_1 +2$ is true for G_1, since e_1=1, v_1 =2,and r_1= 1.

(2) Now assume that $r_n= e_n - v_n +2$ .Let $\{a_{n+1}, b_{n+1}\}$be the edge that is added to G_n to obtain G_{n+1}.

◆ Both a_{n+1} and b_{n+1} are already in G_n.



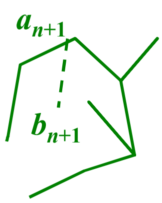
These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (and G_{n+1} is planar).

The addition of this new edge splits R into two regions.

Consequently, $r_{n+1}= r_n +1$, $e_{n+1}= e_n +1$,and $v_{n+1}= v_n$ .Thus, $r_{n+1}= e_{n+1}- v_{n+1} +2$.

- One of the two vertices of the new edge is not already in G_n .

Suppose that a_{n+1} is in G_n but that b_{n+1} is not.



Adding this new edge does not produce any new regions, since b_{n+1} must be in a region that has a_{n+1} on its boundary.

Consequently, $r_{n+1}= r_n$. Moreover, $e_{n+1}= e_n +1$ and $v_{n+1}= v_n +1$.

Hence, $r_{n+1}= e_{n+1}+1- v_{n+1}-1 +2$.

Note:

1) The Euler's formula is a *necessary condition*.

2) How about unconnected simple planar graph?

〖Example 3〗 Suppose that a connected planar simple graph has 20 vertices, each of degree 3. How many regions does this planar graph have?

Solution:

By handshaking theorem,

$$3v = 2e \Rightarrow e = 30$$

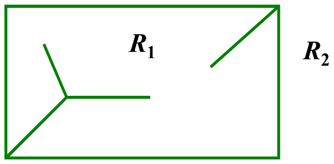
From Euler's formula, the number of regions is

$$r = e-v+2 = 30-20+2 = 12$$

【Definition】 **Suppose R is a region of a connected planar simple graph, the number of the edges on the boundary of R is called the *Degree of R*.**

Notation: Deg(R)

For example,



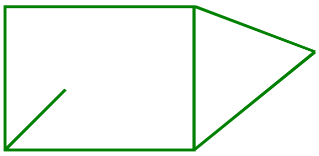
Deg(R_1) = 12
Deg(R_2) = 4

【 Corollary 1 】 If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v-6$

Proof:

Suppose that a connected planar simple graph divides the plane into r regions, the degree of each region is at least 3.

Since $2e = \sum \deg(R_i) \geq 3r$,



it imply $r \leq (2/3)e$

Using Euler's formula $e-v+2=r$, we obtain

$e-v+2 \leq (2/3)e$, this shows that

$e \leq 3v-6$

Note:

- **The equality holds if and only if every region has exactly three edges.**
- **For unconnected planar simple graph,** $e \le 3v - 6$ **also holds.**

Since for a component, $e_i \le 3v_i - 6$

$$e = \sum e_i \le \sum (3v_i - 6) < 3\sum v_i - 6 = 3v - 6$$

【Corollary 2】 If *G* is a connected planar simple graph, then *G* has *a* vertex of degree not exceeding five.

Proof:

(1) *G* has one or two vertices

(2) G has at least three vertices

By Corollary 1 , we know that $e \le 3v-6$ **, so** $2e \le 6v-12$

If the degree of every vertex were at least six, then

$2e \ge 6v$

【 Corollary 3 】 If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3,then $e \leq 2v-4$.

Generally, if every region of a connected planar simple graph has at least k edges, then

$$e \leq \frac{(v-2)k}{k-2}$$

r=e-v+2

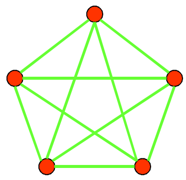
Kr<=2e

r<=2e/k

〖Example 4〗 Show that k_5, $k_{3,3}$ are nonplanar.

Proof:

(1)

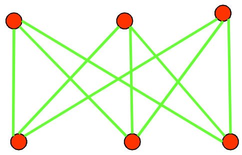


The graph k_5 has 5 vertices and 10 edges.

However, the inequality $e \leqq 3v-6$ is not satisfied for this graph since e=10 and $3v$-6=9.

Therefore, k_5 is not planar.

(2)



$k_{3,3}$ has 6 vertices and 9 edges.

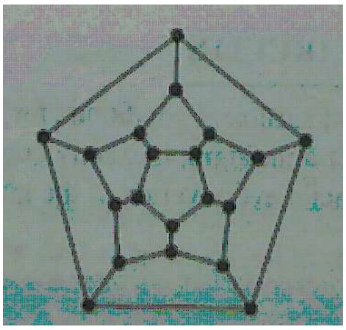
Since $k_{3,3}$ has no circuits of length 3 (this is easy to see since it is bipartite), Corollary 3 can be used .

Since e=9 and 2v-4=8, corollary 3 shows that $k_{3,3}$ is nonplanar.

〖Example 5〗 The construction of Dodecahedron .

Solution:

Since the degree of every vertex is 3 and the degree of every region is 5. Then

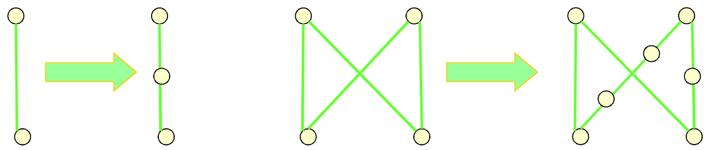


$$\begin{cases} 2e = 3v \\ 2e = 5r \\ r = e - v + 2 \end{cases}$$

It follows that v=20, e=30 and r=12.

2. KURATOWSKI'S THEOREM

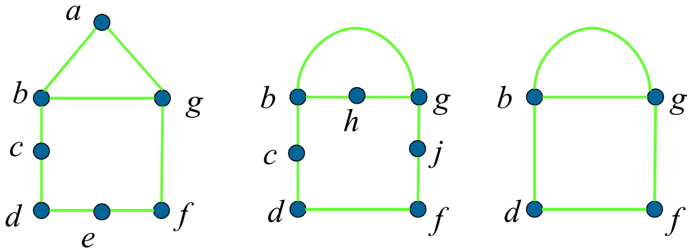
Elementary subdivision



Homeomorphic

-- **The graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.**

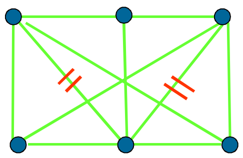
For example,

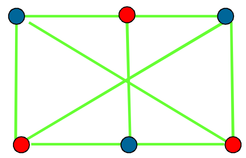


【Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

〖Example 6〗 Determine whether the following graphs are planar.

(1)





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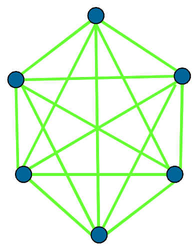

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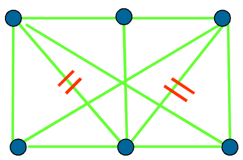

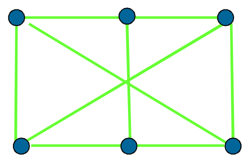

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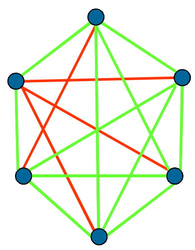

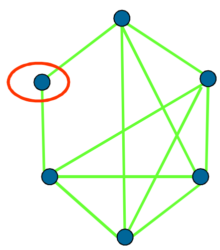

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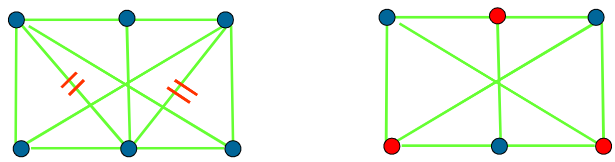
(1)



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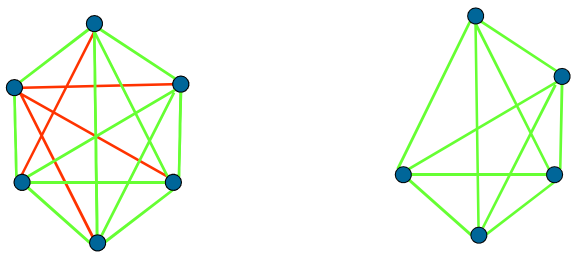
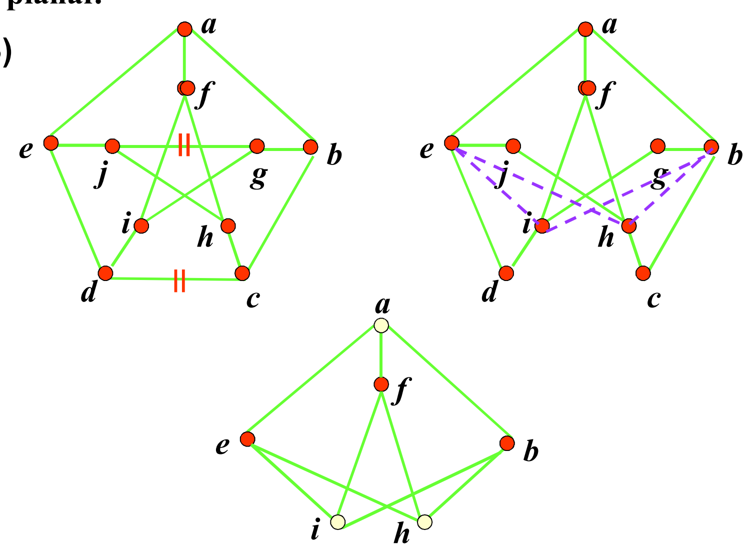


〖Example 6〗 Determine whether the following graphs are planar.

(3)



Homework:

Sec. 10.7 7, 20, 22, 23, 25