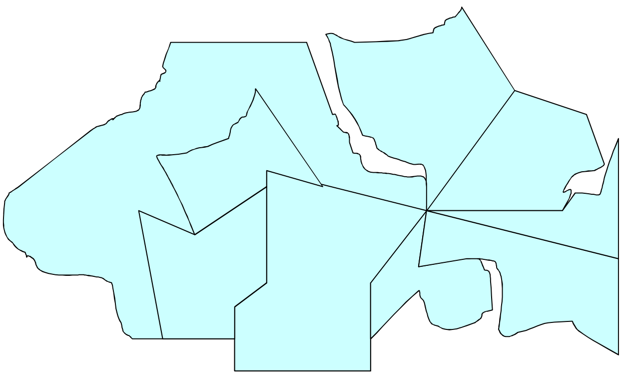
CHAPTER 10 Graphs

The problem related to the coloring of maps:

Determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.



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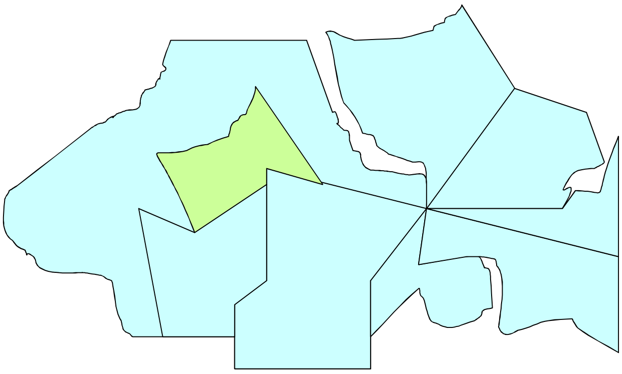

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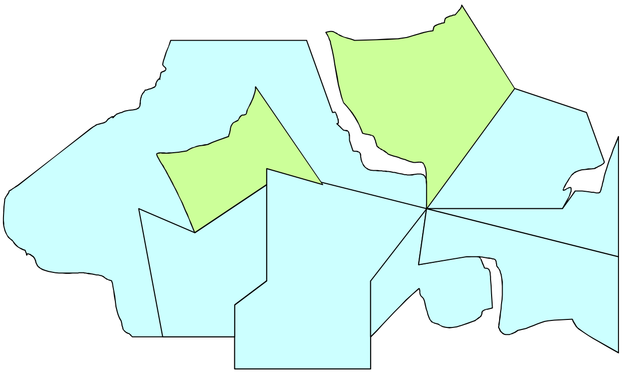

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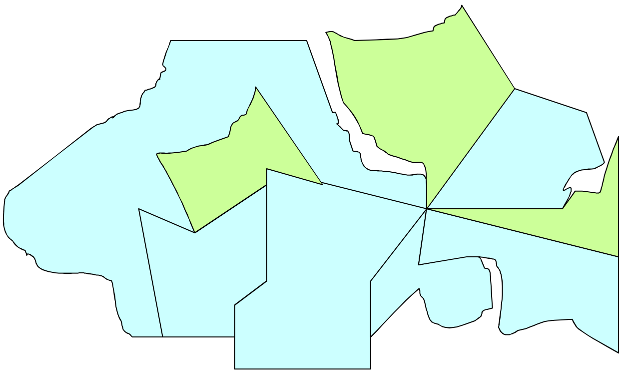

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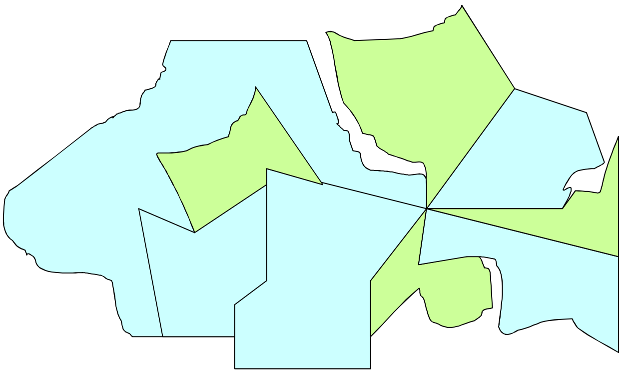

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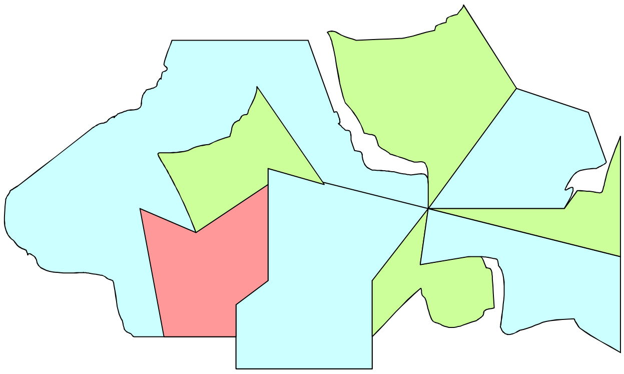

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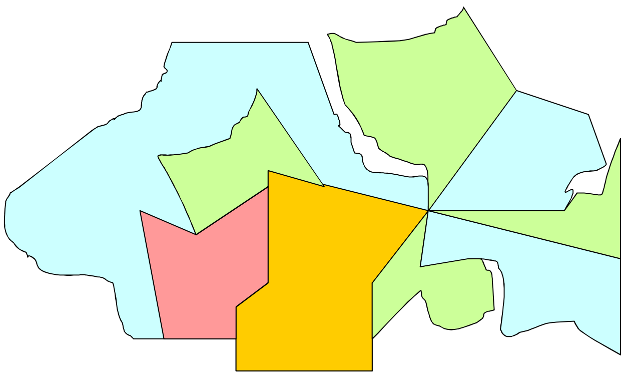
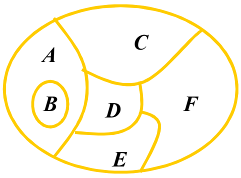


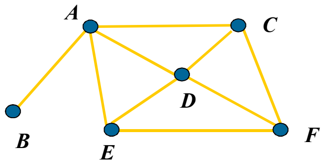
The problem of coloring a map, can be reduced to a graph-theoretic problem.

Each map in the plane can be represented by a graph, namely *the dual graph of the map*.

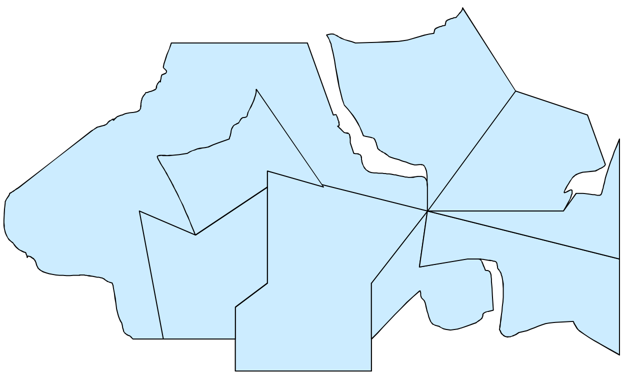
- **Each region of the map is represented by a vertex.**
- **An edge connect two vertices if the regions represented by these vertices have a common border.**
- **Two regions that touch at only one point are not considered adjacent.**

For example,

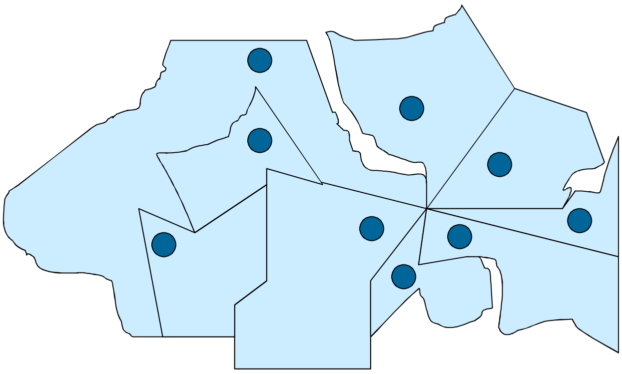




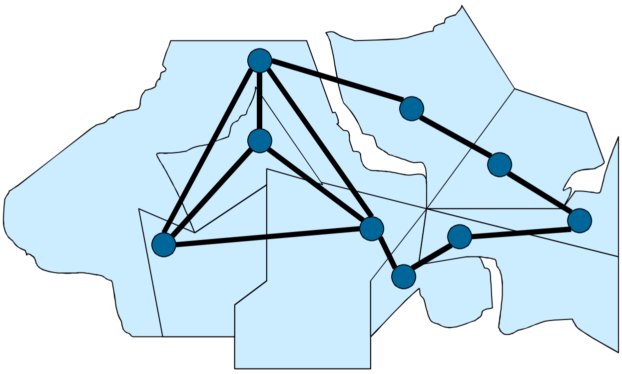
Coloring a map is equivalent to coloring the vertices of its dual graph so that no two adjacent vertices in this graph have the same color.



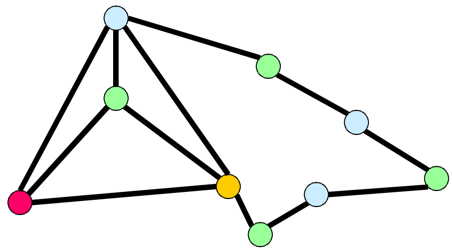
For each region introduce a vertex.



For each pair of regions with a positive-length common border introduce an edge.



Coloring regions of a map is equivalent to coloring vertices of its dual graph.



Terminologies:

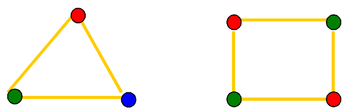
coloring

-- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The chromatic number of a graph

-- is the least number of colors needed for a coloring of this graph, denoted by $x(G)$

For example,



【Theorem 1】 The Four Color Theorem

The chromatic number of a planar graph is no greater than four.

- Any planar map of regions can be depicted using 4 colors so that no two regions with a common border have the same color.
- The four color theorem was originally proposed as a conjecture in 1850s.
- Proof by Haken and Appel used exhaustive computer search in 1976.
- The four color theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.

1. The chromatic numbers of some simple graphs

Show that the chromatic numbers of a graph is n.

- Show that the graph can be colored with n colors.

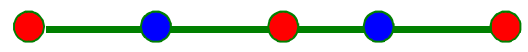
  Method: constructing such a coloring.
- Show that the graph cannot be colored using fewer than n colors.

The chromatic numbers of some simple graphs:

(1) The graph G contains only some isolated vertices.

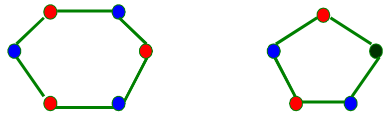
$$x(G) = 1$$

(2) The graph *G* is a path containing no circuit.



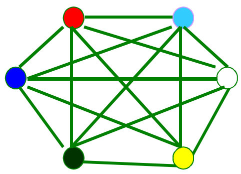
$x(G) = 2$

(3) C_n



$$\begin{cases} x(G) = 2 & \textbf{if } \boldsymbol{n} \textbf{ is even} \\ x(G) = 3 & \textbf{if } \boldsymbol{n} \textbf{ is odd} \end{cases}$$

(4) K_n



$x(G) = n$

(5) A simple graph with a chromatic number of 2 is bipartite.

A connected bipartite graph has a chromatic number of 2.

2. Applications of graph colorings

(1) Scheduling Exams

How can the exams at a university be scheduled so that no student has two exams at the same time?

Solution:

This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent.

Each time slot for a final exam is represented by a different color.

A scheduling of the exams corresponds to a coloring of the associated graph.

For example, Suppose we want to schedule some final exams for CS courses with following code numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are no common students in the following pairs of courses because of prerequisites:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

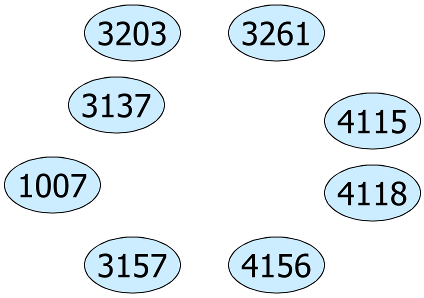
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

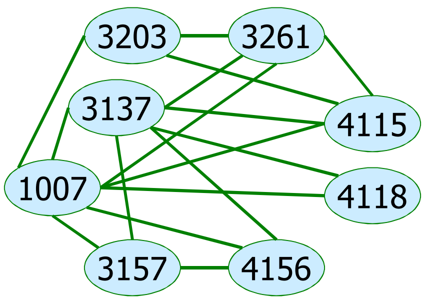
How many exam slots are necessary to schedule exams?

Turn this into a graph coloring problem.

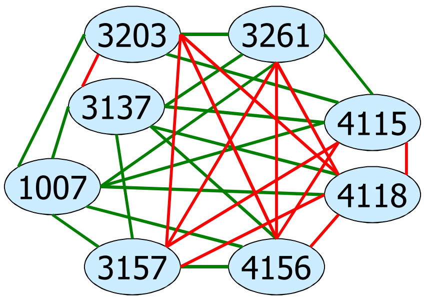
-- Vertices are courses, and edges connect courses which *cannot* be scheduled simultaneously because of possible common students in both courses



One way to do this is to put edges down where students mutually excluded…



…and then compute the complementary graph:



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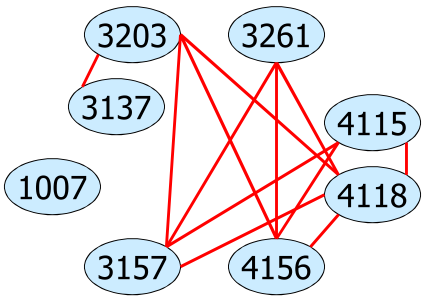
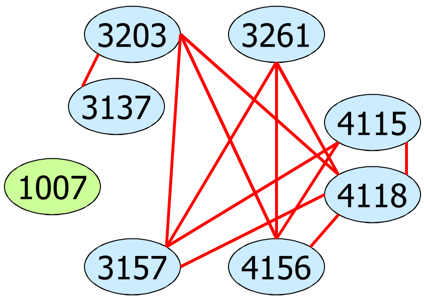


Coloring …



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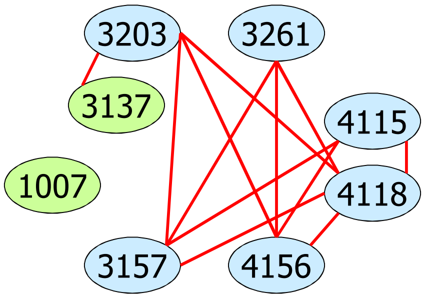

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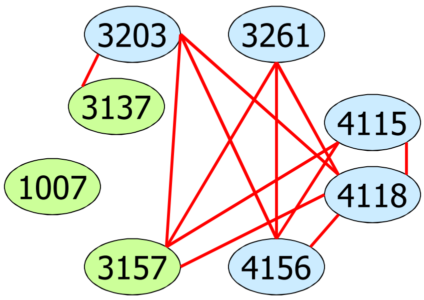

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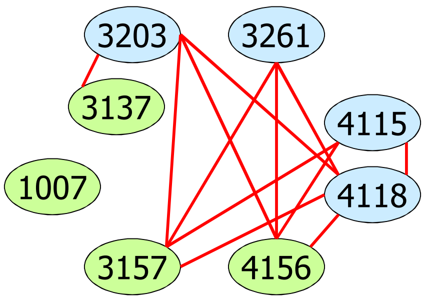

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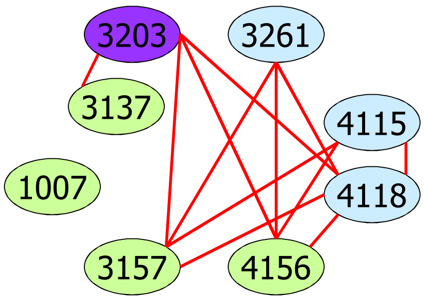

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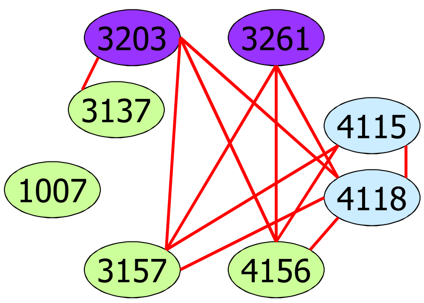

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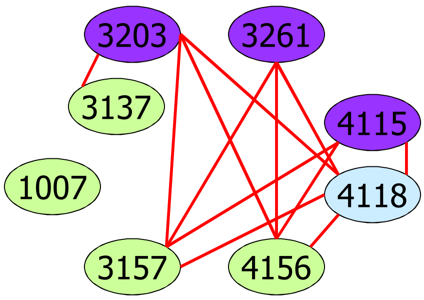

2. Applications of graph colorings

(2) Set up natural habitats of animals in a zoo

Solution:

Let the vertices of a graph represent the animals.

Draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits.

A coloring of this graph gives an assignment of habitats.

Homework:

Sec. 10.8 3, 8, 9, 10, 17