Generalized Permutations and Combinations

Section 6.5

Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition. ◀

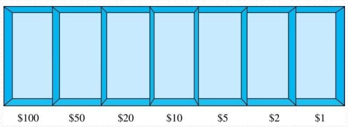
Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

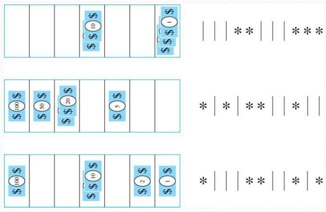
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

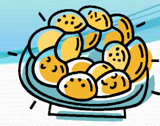
Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof: Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star for each time the ith element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. ◄



Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Combinations with Repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 12}{1\cdot 2} = 78$$

solutions.

〖Example 〗How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $x_i (i = 1,2,3,4)$ is nonnegative integer?

Solution:

Since a solution of this equation corresponds to a way of selecting 16 items from a set with four elements so that x_1 items of type one, x_2 items of type two, x_3 items of type three, x_4 items of type four are chosen.

Hence the number of solutions is

$$C(4\text{-}1+16, 16) = C(19, 16)$$
$$= C(19, 3)$$

9

6.5 Generalized Permutations and Combinations

〖Example 〗How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

Where $x_i (i = 1,2,3,4)$ is nonnegative integer?

Question:

(1) $x_i > 1$, for $i = 1,2,3,4$ ⇒ $x_i \geq 2$

$C(4\text{-}1+8, 8) = C(11, 8) = C(11, 3)$

(2) $x_1 + x_2 + x_3 + x_4 \leq 16$

We can introduce an auxiliary variable x_5 so that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

$C(5\text{-}1+16, 16) = C(20, 16) = C(20, 4)$

10

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$ ◀

〖Example 〗There are 50 students in a class.

(1) How many ways to select 7 students to construct a leading group?

C(50,7)

(2) If two of the leading group are e[…]itor and a vice monitor, then how many ar[…]

$$=\frac{P(50,7)}{1!1!5!}$$

(3) If these 7 students are elected to have different tasks, then how many are there?

P(50,7)

14

〖Example 〗

(1) How many bit strings of length 10?

2^{10}

(2) How many bit strings of length 10 are there that contain exactly two 0s, eight 1s?

$10!/(2!8!)$

〖Example 3〗 How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

Solution:

$A = \{ 1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$

$$\frac{11!}{4!4!2!}$$

15

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
  - (*See Exercises* 47 *and* 48 *for two different proofs.*)
  - Example: There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.

Distributing Objects into Boxes

- *Indistinguishable objects* and *distinguishable boxes*.
  - There are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10)$ = C(17,10) = 19,448 ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - Example: There are 14 ways to put four employees into three indistinguishable offices

〖Example 〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution:

We represent the four employees by A,B,C,D.

(1) All four are put into one office: 1 ways {A,B,C,D}

(2) Three are put into one office and a fourth is put into a second office:

4 ways {{A,B,C},{D}}; {{A,B,D}, {C}}; {{A,C,D}, {B}}; {{B,C,D}, {A}}

(3) Two are put into one office and two put into a second office:

3 ways {{A,B},{C,D}}; {{A,D},{B,C}}; {{A,C},{B,D}}

(4) Two are put into one office, and one each put into the other two office:

6 ways

{{A,B},{C},{D}}; {{A,C},{B},{D}}; {{A,D},{B},{C}}; {{B,C},{A},{D}}; {{B,D},{A},{C}}; {{C,D},{A},{B}}

20

〖Example 〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution:

There are 1+4+3+6=14 ways to put four different employees into three indistinguishable offices.

Another way:

Look at the number of offices into which we put employees.

1) There are 6 ways to put four different employees into three indistinguishable offices so that no office is empty.
2) There are 7 ways to put four different employees into two indistinguishable offices so that no office is empty.
3) There are 1 ways to put four different employees into one offices so that it is not empty.

21

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.

Distributing Objects into Boxes

- *Indistinguishable objects* and *indistinguishable boxes*.

〖Example 〗How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution:

We can enumerate all ways to pack the books. For each ways to pack the books, we will list the number of books in the box with the largest of books, followed by the number of books in each box containing at least one book, in order of decreasing number of books in a box.

The ways we can pack the books are

6	5,1	
4,2	4,1,1	
3,3	3,2,1	
3,1,1,1	2,2,2	2,2,1,1

Distributing Objects into Boxes

- *Indistinguishable objects* and *indistinguishable boxes*.
  - The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
  - No simple closed formula exists for this number.

Homework

第八版 Sec. 6.5 10, 16, 28, 34, 48, 52, 56, 63

Generating Permutations and Combinations

Section 6.6

Section Summary

- Generating Permutations
- Generating Combinations

Generating Permutations

Problem:

List the permutations of any set of n elements.

Solution:

- Any set with n elements can be placed in one-to-one correspondence with the set {1, 2, …, n}
- Generate the permutation of the n smallest positive integers, and then replace these integers with the corresponding elements.

Generating Permutations

The lexicographic ordering of the set of permutations of {1, 2, …, n}

The permutation $a_1a_2\ldots a_n$ precedes the permutation of $b_1b_2\ldots b_n$, if for some k, with $1\leq k\leq n$, $a_1=b_1$, $a_2=b_2$, …, $a_{k-1}=b_{k-1}$, and $a_k<b_k$,

For example:

123465 124635

Generating Permutations

Algorithm of producing the n! permutations of the integers 1, 2, …, n

- ❖ Begin with the smallest permutation in lexicographic order, namely $1234\ldots n$.
- ❖ Produce the next larger permutation.
- ❖ Continue until all $n!$ permutations have been found.

Generating Permutations

Given permutation $a_1a_2\ldots a_n$, find the next larger permutation in increasing order:

(1) Find the integers

a_j, a_{j+1} *with* $a_j < a_{j+1}$ *and* $a_{j+1} > a_{j+2} > \cdots > a_n$

(2) Put in the *j*th position the least integer among

$a_{j+1}, a_{j+2}, \cdots, a_n$ that is greater than a_j

(3) List in increasing order the rest of the integers

$a_j, a_{j+1}, \cdots, a_n$

Generating Permutations

Example 1:

What is the next larger permutation in lexicographic order after 124653?

Solution:

- The next largest permutation of 124653 in lexicographic order is 125346

Generating Combinations

Problem 1:

Generate all combinations of the elements of a finite set .

Solution:

- A combination is just a subset. ⇒ We need to list all subsets of the finite set.
- Use bit strings of length n to represent a subset of a set with n elements. ⇒ We need to list all bit strings of length n.
- The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions.

Generating Combinations

Algorithm of producing all bit strings

- ❖ Start with the bit string 000…00,with *n* zeros.
- ❖ Then, successively find the next larger expansion until the bit string 111…11 is obtained.

The method to find the next larger binary expansion:

Locate the first position from the right that is not a 1,
then changing all the 1s to the right of this position to 0s
and making this first 0 a 1.

For example:

1000110011 → 1000110100

Generating Combinations

Problem 1:

Generate all r-combinations of the set {1, 2, …, n}

Solution:

The algorithm for generating the r-combination of the set $\{1, 2, \ldots, n\}$

(1) $S_1 = \{1, 2, \ldots, r\}$

(2) If $S_i = \{a_1, a_2, \cdots, a_r\}, 1 \le i \le C_n^r - 1$ has found, then the next combination can be obtained using the following rules.

First, locate the last element a_i in the sequence such that $a_i \ne n - r + i$. Then replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$.

Generating Permutations

Example 2:

S_i =$\{2,3,5,6,9,10\}$ is given from the set $\{1,2,3,4,5,6,7,8,9,10\}$. Find S_{i+1}.

Solution:

S_{i+1} =$\{2,3,5,7,8,9\}$

Homework

Sec. 6.6 6(f), 7, 9