FORMALIZING THE RIEMANN HYPOTHESIS IN THE LEAN INTERACTIVE THEOREM PROVER

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Abstract. Abstract

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The simplest form of the Riemann Hypothesis we could construct is the following:

$$\forall\,(s : \mathbb{C}),\ 0 < \sigma \to \eta(s) = 0 \to \sigma = 2^{-1}$$

where $\sigma := \mathfrak{Re}(s)$ and η is the Dirichlet Eta function typically defined as follows:

$$\eta(s) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}$$

Before proving that this series is well-defined, we want to define the Riemann Zeta function on $\mathbb{R}$:

$$\zeta(\sigma) := \sum_{n \geq 1} n^{-\sigma}$$

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To prove that this is a Cauchy sequence, we use the Cauchy-Schlömilch Condensation test so that we are comparing against the condensed sequence:

$$\sum_{n\geq 1} 2^n (2^n)^{-\sigma}$$

Simplifying each term, we get instead a geometric series in $2^{1-\sigma}$:

$$2^n(2^n)^{-\sigma} = (2^n)^{1-\sigma} = (2^{1-\sigma})^n$$

For this ratio to be less than 1 we need that $\sigma > 1$ which gives us our domain of convergence.

Now to prove that the Eta function converges, we collect terms in odd-even pairs as follows:

$$\eta(s) := \left(\frac{1}{1^s} - \frac{1}{2^s}\right) + \left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \cdots$$

For the nth term indexing from zero, we have,

$$\eta_n(s) := (2n+1)^{-s} - (2n+2)^{-s}$$

To prove that the partial sums of this sequence are a Cauchy sequence, we compare is against the terms of the Zeta function evaluated at $1+\sigma$,

$$\left|(2n+1)^{-s} - (2n+2)^{-s}\right| \leq C \cdot (n+1)^{-(1+\sigma)}$$

for some constant C to be determined. Rewriting the left hand side, we get

$$\left|\frac{1-(1-\frac{1}{2n+2})^s}{(2n+1)^s}\right| \leq C \cdot (n+1)^{-(1+\sigma)}$$

Since the absolute value of a power keeps only the real part of the exponent, we can cancel a factor of $(2n+1)^{-\sigma}$ from both sides,

$$\left|1-(1-\tfrac{1}{2n+2})^s\right| \leq C \cdot \frac{1}{n+1}$$

We can sharpen the right side to $(2n+2)^{-1}$ to match the term on the left hand side, and we are left with the following inequality:

$$\left|1-(1-\tfrac{1}{2n+2})^s\right| \leq C \cdot \frac{1}{2n+2}$$

Since this must be true for all n and all of the functions are continuous as a function of n, we will assume the inequality holds for all positive real $x \leq 1/2$ and find the constant which makes this true.

Opening up the power, we have

$$(1-x)^s := \exp(\log(1-x) \cdot s)$$

Since $x \leq 2^{-1}$ we have the inequality $|\log(1-x)| \leq 2|x|$. We also have the following inequality for exp,

$$\forall z \forall s,\ |\exp(zs) - (1 + zs)| \leq \exp(|s|)|z|^2$$

We begin again at the target inequality and proceed as follows:

$$\begin{aligned}
|1 - (1-x)^s| &= |1 - \exp(\log(1-x) \cdot s)| \\
&\leq |1 - (1 + \log(1-x) \cdot s)| \\
&+ |(1 + \log(1-x) \cdot s) - \exp(\log(1-x) \cdot s)| \\
&= |\log(1-x)| \cdot |s| \\
&+ |\exp(\log(1-x) \cdot s) - (1 + \log(1-x) \cdot s)|
\end{aligned}$$

Applying the exp inequality we get,

$$|1 - (1-x)^s| \leq |\log(1-x)| \cdot |s| + \exp(|s|) \cdot |\log(1-x)|^2$$

Applying the log inequality we get,

$$|1 - (1-x)^s| \leq 2|x| \cdot |s| + 4\exp(|s|)|x|^2$$

We weaken the right hand side factor of $|x|^2$ to $|x|$ since $|x| < 1$ and we have,

$$|1 - (1-x)^s| \leq (2|s| + 4e^{|s|})|x|$$

so we have found our constant and the inequality is proved. From this fact we deduce that the Dirichlet Eta function converges for $\mathfrak{Re}(s) > 0$.

3. Lean Details

Implementing this construction in the Lean Theorem Prover requires that we provide proofs of the Condensation Test, geometric series convergence, and the Comparison Test.

3.1. **Comparison Test.**

3.2. **Condensation Test.**

3.3. **Geometric Series Convergence.**

References

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