Handy relations for making bounding proofs

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The following relations are true and can be very useful when making upper bound and lower bound arguments. For the sake of simplicity, and without lost of generality, I maintained a variable n in all the statements. All the relations are still true if n is replaced with an arbitrary function $p(n)$. Besides n, I also used the constants a, b, c. These constants can also be replaced with functions if the outputs of such functions adhere to the constraints on the constants as defined in the relations. If not stated, $n \geq 0$ since it denotes the size of an input.

In using these relations, always substitute the expressions on the LHS with their corresponding appropriate expression on the RHS. Also always keep in mind the bound you are seeking to prove. Finally, this list is only a short guide to get you started. Practice regularly and do search for other bounding propositions too.

1 Proving upper bounds

Given two functions $f(n)$ and $g(n)$, if $f(n) \leq g(n)$, $g(n)$ is said to be an upper bound of $f(n)$.

1. $n + c \leq 2n$ where $c \geq 1$ and $n \geq c$. E.g, $n + 3 \leq 2n \quad \forall n \geq 3$
2. $n + c \leq cn$ where $c \geq 2$. E.g, $n + 3 \leq 3n$
3. $n^a \leq n^{a+c}$ where $a, c \geq 1$. E.g, $n^2 \leq n^{2+1} \leq n^3$
4. $cn^a \leq n^{a+1}$ where a and c are positive integers and $n \geq c$. E.g $1000n^2 \leq n^3 \quad \forall n \geq 1000$.
5. $n^a - cn^b \leq n^a$ where $a, b, c \geq 1$ and $a \geq b$. E.g, $n^2 - 100n \leq n^2$
6. $\frac{n}{c} \leq n$ where $c \geq 1$. E.g, $\frac{n^3}{7} \leq n^3$
7. $c^{n-a} \leq c^n$ where $c, a \geq 1$ E.g, $2^{n-1} \leq 2^n$.
8. $c^n \leq (c + a)^n$ where $a \geq 1$. E.g, $2^n \leq (2 + 1)^n \leq 3^n$
9. $(n - a)! \leq n!$ where $0 \leq a \leq n$. E.g, $(n - 2)! < n!$.
10. $n \leq c^n$ where $c \geq 2$. E.g, $n \leq 2^n$

2 Proving lower bounds

Given two functions $f'(n)$ and $g'(n)$, if $f'(n) \geq g'(n)$, $g'(n)$ is said to be a lower bound of $f'(n)$. To minimize repeating myself, I will not include relations which can be deduced through a simple inversion of the relations for the upper bound arguments.

1. $n + c \geq n$ where $c \geq 0$. E.g $n \log n + 100n \geq n \log n$.
2. $n^a \geq n^{a-b}$ where $a, b \geq 1$ and $b \leq a$. E.g, $n^3 \geq n^{3-1} \geq n^2$.
3. $n - h(n) \geq n - c * h(n)$ where $h(n) \geq 1$ and $c \geq 1$. E.g, $n^3 - n \geq n^3 - 10n$
4. $c^n \geq (c - a)^n$ where c and a are positive integers and $c > a$. E.g $4^n \leq 3^n$.
5. $(n + a)! \geq n!$ where $a \geq a$. E.g, $(n + 100)! \geq n!$.

***Note. All the relations above still apply even if they are arguments to a monotonically increasing function. For example, because the logarithmic function is always increasing, $\log(n + c) \leq \log(2n) \quad \forall n \geq c$. E.g, $\log(n^2 + 100) \leq \log(2n^2) \quad \forall n \geq 100$. Also, $\log(n^2) \leq \log(n^3)$, $\quad \log(n^2) \leq \log(n^3)$. And so on.....