MONASH University

MONASH BUSINESS SCHOOL

ETC3580: Advanced Statistical Modelling

Week 1: Visualizing linear models

Outline

Linear Models

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

- Response: y
- Predictors: $x_1, \ldots, x_p$
- Error: $\varepsilon \sim$ IID

Let $\boldsymbol{y} = (y_1, \ldots, y_n)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)'$, $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_p)'$ and

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \ldots & x_{p,1} \\ 1 & x_{1,2} & x_{2,2} & \ldots & x_{p,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,n} & x_{2,n} & \ldots & x_{p,n} \end{bmatrix}$$

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Then

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}.$$

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Least squares estimation

Minimize: $(\boldsymbol{y} - \boldsymbol{X}\beta)'(\boldsymbol{y} - \boldsymbol{X}\beta)$

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Differentiate wrt β gives

The "normal" equations

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$

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So MLE = OLS.

Outline

R modelling notation

```
fit <- lm(y ~ x1 + x2 + x3,
  data=tibble)
```

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2)$$

```
## # A tibble: 200 x 4
##         y    x1      x2    x3
##     <dbl> <dbl>   <dbl> <dbl>
##   1  78.6 51.3    47.8    2.8
##   2  19.4 76.6    37.8    1.05
##   3  97.6 26.1    24.2    4.63
##   4  81.5 21.6   -19.3    1.9
```

Useful helper functions

Base functions

- `summary`
- `coef`
- `fitted`
- `predict`
- `residuals`

`broom` functions

- `tidy`
- `augment`
- `glance`

R formulas

Categorical predictors:

- R will create the required dummy variables from a categorical *factor*.
- The first level is used as the reference category.
- Use `relevel` to change the reference category

Expression	Description
`y ~ x`	Simple regression
`y ~ 1 + x`	Explicit intercept
`y ~ -1 + x`	Through the origin
`y ~ x + I(x^2)`	Quadratic regression
`y ~ x1 + x2 + x3`	Multiple regression
`sqrt(y) ~ x + I(x^2)`	Transformed
`y ~ . -x1`	All variables except `x1`

Outline

Partial residuals

$$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{X}\hat{\beta}$$

denote residuals for a given model fit.

Then the partial residuals for variable j are given by

$$\boldsymbol{r}_j = \boldsymbol{y} - \boldsymbol{X}_{-j}\hat{\beta}_{-j}$$

where the $-j$ subscript indicates the removal of the jth column/element.

- Equivalent to y adjusted for all variables other than x_j.
- Plotting $\boldsymbol{r}_j$ vs $\boldsymbol{x}_j$ shows the relationship of $\boldsymbol{y}$ vs $\boldsymbol{x}_j$ after adjustment.

Conditional plots

- Slope of regression of $\boldsymbol{r}_j$ on $\boldsymbol{x}_j$ is β_j.
- Conditional plots show $\boldsymbol{r}_j + \boldsymbol{x}_{-j|m}\beta_{-j}$ vs $\boldsymbol{x}_j$, where $\boldsymbol{x}_{-j|m}$ corresponds to median of numeric variables and mode for factors.
- Let $\boldsymbol{x}^{*\prime}$ denote row of design matrix constructed from $x_j = x$ and $\boldsymbol{x}_{-j|m}$. Then equation of line is $\boldsymbol{x}^{*\prime}\hat{\beta}$ and standard error at x is

$$\text{se}(x) = \sqrt{\boldsymbol{x}^{*\prime}\text{Var}(\hat{\beta})\boldsymbol{x}^{*}}.$$

- Construct confidence interval using

$$\boldsymbol{x}^{*\prime}\hat{\beta} \pm t_{n-p,1-\alpha/2}\text{se}(x)$$

Visualization using visreg

- `visreg(ls_object)`
- `visreg(ls_object, "xvar", gg=TRUE)`

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- Plot r_j vs x_j for different values of x_k
- Plot r_k vs x_k for different values of x_j

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Much harder to see higher-order interactions

Interactions

Interactions:

- One possible type of interaction is obtained by multiplying the relevant columns of the model matrix.
- Use `a:b` for the interaction between `a` and `b`.
- Use `a*b` to mean `a + b + a:b`
- Need to specify explicit functions for other types of interaction. e.g., `I(a/b)`

Limited order interactions:

- Interactions up to 2nd order can be specified using the ^ operator.
- `(a+b+c)^2` is identical to `(a+b+c)*(a+b+c)`

Nested factors:

- `a + b %in% a` expands to `a + a:b`

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- Each coefficient gives effect of one unit increase of predictor on response variable, *holding all other variables constant*.
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Visualization using visreg

- `visreg2d(ls_object, "xvar1", "xvar2")`
- `visreg2d(ls_object, "xvar1", "xvar2", plot.type='persp')`
- `visreg2d(ls_object, "xvar1", "xvar2", plot.type='rgl')`
- `visreg2d(ls_object, "xvar1", "xvar2", plot.type='gg')`

Outline

Hypothesis testing

- Use F-tests between models:
  - Model 1: p_1 parameters.
  - Model 2 (nested within Model 1): p_2 parameters

$$F = \frac{(\text{RSS}_2 - \text{RSS}_1)/(p_1 - p_2)}{\text{RSS}_1/(n - p_1)} \sim F_{p_1 - p_2, n - p_1}$$

- Helper functions: `anova`, `drop1`
- If one term dropped, this is equivalent to a t-test on coefficient.

Hypothesis testing

`anova(model)`

- provides sequential testing of terms (conditional on all previous terms)
- Order of terms will usually affect the p-values.
- Uses “Type 1” SS

`anova(model1, model2)`

- Tests two nested models.
- Avoids ordering problems

`drop1(model, test="F")`

- Equivalent to series of `anova(model1, model2)` calls where `model2` drops one variable at a time.
- Equivalent to “Type 3” SS

Outline

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$$-2\log L = n\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}(\boldsymbol{y} - \boldsymbol{X}\beta)'(\boldsymbol{y} - \boldsymbol{X}\beta)$$

$$= c + \frac{\text{SSE}}{2\sigma^2}$$

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`extractAIC()` used by `step()`
handles c and q differently from `AIC()`

Variable selection

`step()` will minimize AIC using backwards selection

Best model with only main effects

```
mod1 <- lm(y ~ x1 + x2 + x3, data=data) %>%
  step()
```

Best model with up to 2-way interactions

```
mod2 <- lm(y ~ (x1 + x2 + x3)^2, data=data) %>%
  step()
```

Best model with up to 3-way interactions

```
mod3 <- lm(y ~ (x1 + x2 + x3)^3, data=data) %>%
  step()
```

Variable selection and inference

- Do not use coefficient t-tests for variable selection.
- Beware of *any* statistical tests after variable selection.
- Confidence intervals after variable selection are too narrow.
- Variable selection is most useful for prediction. If you are only interested in inference, don't do it!