Chapter 1: Introduction to Spherical Harmonics

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1 Initialization

Start Matlab or Octave and switch to the folder `Slepian`

To initialize the Slepian software in Matlab, run

```
initialize
```

To initialize the Slepian software in Octave, run

```
initialize_octave
```

The first time you run `initialize_octave`, it will download and install missing packages. This will take a few minutes. Make sure you have internet access.

1.1 Additional considerations for Octave

If you are running Octave and your figures further on in this tutorial look bad, then you may want to switch the graphics package Octave uses. Try for example

```
graphics_toolkit('fltk')
```

or

```
graphics_toolkit('gnuplot')
```

or

```
graphics_toolkit('qt')
```

When you are displaying a lot of output, Octave usually shows this step by step. This is a bit annoying. We'll turn this off by running

```
more off
```

You will need to run all of these initializations every time you start Octave.

2 Testing the software

To check if the software installed correctly you can run a few demos using the function `demos_slepian_golf`

```
help demos_slepian_golf
```

This will show you a description of what the program does. For example, try

```
demos_slepian_golf(6)
```

This will calculate and plot a single gradient vector Slepian function for Eurasia and maximum spherical-harmonic degree $L = 10$. You will learn later in this tutorial what the spherical-harmonic degree is. We leave the tutorial for gradient vector Slepian functions for a later time. If you are interested you can check out for example the article "Potential-field estimation using scalar and vector Slepian functions at satellite altitude" (doi: `10.1007/978-3-642-27793-1_64-2`) or the information including links on the website `www.alainplattner.net`.

3 Introduction to spherical harmonics

Spherical harmonics are a counterpart to monomials (building blocks of polynomials, the "wiggly" lines) but on the sphere. Remember that polynomials were always sums of monomials multiplied by some numbers. For example $3x^2 - 20.95x + 5.11113$ is a *polynomial* consisting of the *monomials* x^2, x, and 1 multiplied by the *coefficients* $3, -20.95, 5.1113$ and then summed up. Spherical harmonics do exactly the same. Each of them has a degree, we multiply them with coefficients and then sum them up. One thing that is different is that besides having a degree, spherical harmonics also have an order. This is because they live on a sphere which is a surface and therefore has two directions in which it can "wiggle".

Here we denote spherical harmonics with the capital letter Y followed by its degree l and its order m as indices. Both degree l and order m are integers. The degree l is 0 or greater whereas the order can be positive, zero, or negative. So we generally write $Y_{l\,m}$. For example for degree 4 and order 2 it will be $Y_{4\,2}$ (note the small gap between 4 and 2). The *degree* l of a spherical harmonic says, how many zero-crossing lines it has in total. The *order* m of a spherical harmonic says how many of the zero-crossings are perpendicular to the equator. The sign of the order says if one of the zero-crossings is along the prime-meridian (negative order m) or if the maximum value is on the prime meridian (positive order m). From this explanation it is clear that the order m must always be equal or smaller to the degree l (there can't be more zero-crossings perpendicular to the equator than there are zero-crossings altogether).

Let's look at a few examples. To make it easier to visualize the spherical harmonics, I am plotting them on a sphere and in Mollweide projection over Earth's map. $Y_{0\,0}$ is just a constant number over the entire planet. That's a bit boring so I won't plot it here.

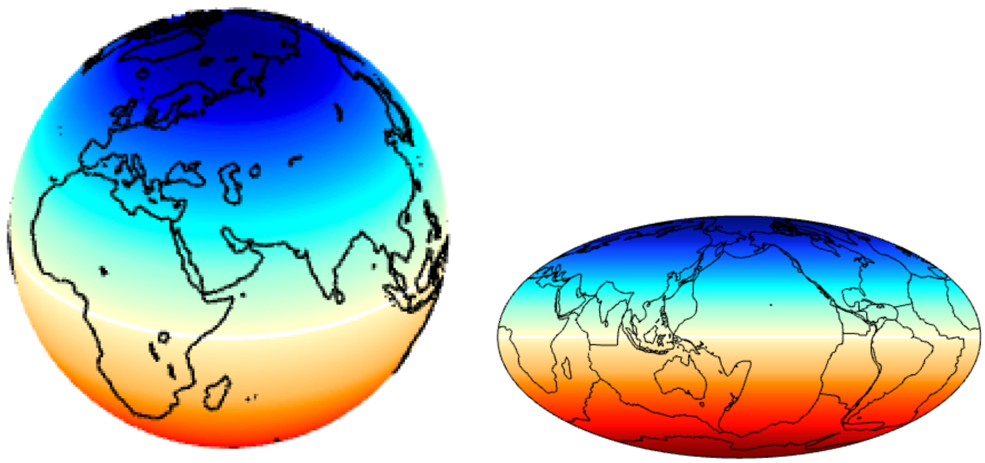

Figure 1: This is spherical harmonic $Y_{1\,0}$. It's a dipole with its north pole over the geographical North Pole and its south pole over the geographical South Pole. In the left panel I am plotting it over a sphere, in the right panel I am using Mollweide projection

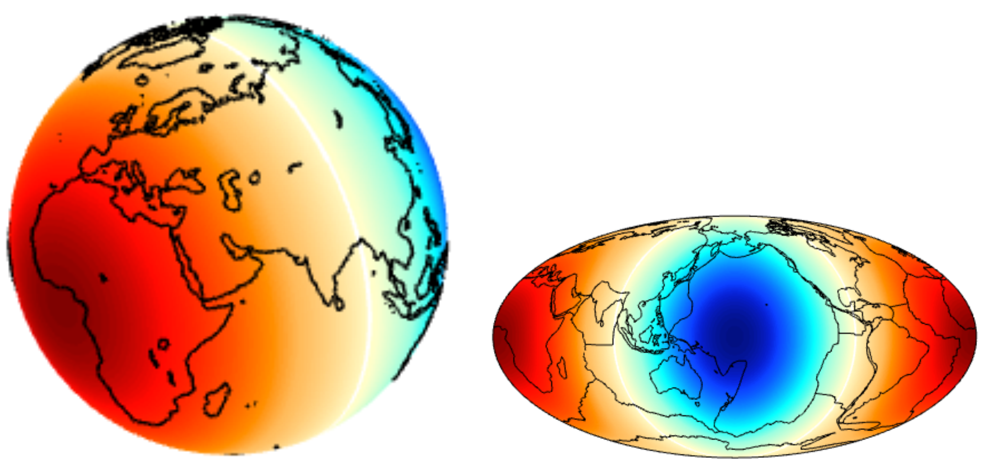

Figure 2: This is spherical harmonic $Y_{1\,-1}$. It's a dipole with its north pole over the Pacific and its south pole over Africa. In the left panel I am plotting it over a sphere, in the right panel I am using Mollweide projection.

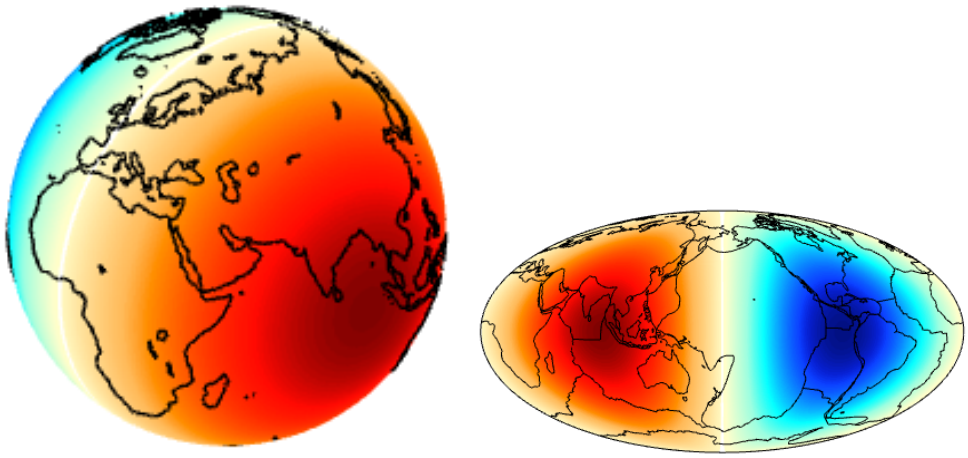

Figure 3: This is spherical harmonic $Y_{1\,1}$. It's a dipole with its north pole over the Indian Ocean and its south pole over Central America. In the left panel I am plotting it over a sphere, in the right panel I am using Mollweide projection.

Question: How can we make a dipole with its south pole over the geographical North Pole and its North pole over the geographical South Pole?

I made these plots using our codes that you just installed. Once you chose a degree l and an order m, you must select a plotting grid. Let's make a pixel every half degree. From your Octave tutorial you remember that

```
theta=-90:0.5:90;
```

Sets up such a vector for the latitudes and

```
phi=0:0.5:360;
```

for the longitudes. We will use the function `ylm.m` to calculate spherical-harmonic values on the sphere on the grid we just defined. Notice that the symbol `l` is the letter l. Run

```
l=3
```

and

```
m=-2
```

and then

```
Y3m2=ylm(l,m,pi/180*(90-theta),pi/180*phi);
```

This will calculate the values of Y_{lm} for your chosen l and m on the grid you defined with latitudes `theta` and longitudes `phi`. We need the factors $\pi/180$ because the function `ylm.m` requires the grid in radians and we need the $90-$ `theta` because the function `ylm.m` works with colatitudes instead of latitudes.

To plot the resulting evaluated spherical harmonic `Y3m2` on a Mollweide projection, run

```
plotplm(Y3m2,pi/180*phi,pi/180*theta,1)
```

Let's change the color to a bit a nicer color scheme, run

```
kelicol(1)
```

To plot the resulting evaluated spherical harmonic on a rotatable sphere, you need to first open a new figure or close the old one and then run

```
plotplm(Y3m2,pi/180*phi,pi/180*theta,2)
```

The Octave version on my computer had issues with using the `print` command to turn figures into pdfs. I needed to take screen shots to get these figures into the document. This bug will probably be fixed soon and might not occur on your computer.

Exercise: Reproduce figures 1, 2, and 3. Plot different spherical harmonics for different degrees and orders. Plot sums of spherical harmonics. Plot linear combinations of spherical harmonics. How does for example
$4Y_{3\,1} - 0.2Y_{1\,-1} + 2Y_{5\,-2}$ look like?

If it looks like in figure 4, then your programing was correct.

What we learn from this is that even though the individual spherical harmonics look very regular with their zero-crossings either parallel or perpendicular to the equator, linear combinations of spherical harmonics can look very arbitrary. In fact, every reasonably behaved function on the sphere (no infinite values and no sudden jumps, etc.) can be expressed as a linear combination of spherical harmonics. For some we might need spherical harmonics with very high degrees.

Exercise: Make your own linear combination of spherical harmonics. Play around with the degrees and coefficients. Can you make up some shape in your mind that you want to draw with spherical harmonics and then reproduce it?

Calculating spherical harmonics individually, multiplying them with coefficients, and then summing them up is cumbersome. Luckily, our software allows us to do this more efficiently. Instead of just calculating a single spherical harmonic we can calculate all of them up to a any degree L we choose. Run for example

```
L=10
```

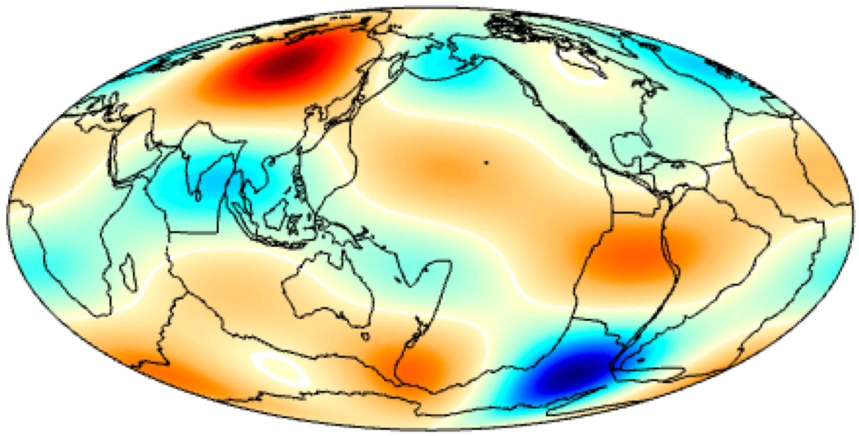

Figure 4: Linear combination of spherical harmonics $4Y_{3\,1} - 0.2Y_{1\,-1} + 2Y_{5\,-2}$ in Mollweide projection

and then

```
Y=ylm([0 L],[],pi/180*(90-theta),pi/180*phi);
```

This will calculate a matrix `Y` in which each row represents an evaluated spherical harmonic. the rows are ordered in what we call here the *addmout* format. This is an informal term used within the software suite and is not generally used in the community. Addmout means the following ordering for the (degree/order) pairs: (0/0), (1/-1), (1/0), (1/1), (2/-2), (2/-1), etc. This means that the spherical harmonic with degree 2 and order 1 is in the 8th row of the matrix `Y`.

Question: In which row is degree 3, order -2? Which degree/order pair do we find in row 20?

If we now have a vector of coefficients that we call `c`, then we can calculate the linear combination of the spherical harmonics evaluated in matrix `Y` with the coefficients in vector `c` using matrix-vector multiplication. For a maximum spherical harmonic degree L there are $(L+1)^2$ spherical harmonics including all degrees between 0 and L and all corresponding orders. We can calculate a random row vector with the correct length by running

```
c=randn(1,(L+1)^2);
```

and evaluate the linear combination of the spherical harmonics as

```
F=c*Y;
```

The linear combination `F` has the form of a vector. We need to put it into the right shape by running

```
Fp=reshape(F,length(theta),length(phi));
```

Plot this function with the command we have seen before

```
plotplm(Fp,pi/180*phi,pi/180*theta,1)
```

Question: Why are there $(L+1)^2$ spherical harmonics for degrees 0 to L and all coprresponding orders?

Question: Why does the matrix vector multiplication `F=c*Y;` calculate a linear combination of the spherical harmonics with the corresponding coefficients in `c`?

Exercise: If I give you a random list of data point values together with their locations, can you find a way to calculate best-fitting spherical-harmonic coefficients? Remember least-squares fitting.