

# Chapter 3: Introduction to altitude-cognizant scalar Slepian functions

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## 1 Introduction

Now that we understand the classical scalar Slepian functions, we can consider the altitude of a satellite from which the data was taken. Altitude-cognizant scalar Slepian functions can optimize the linear combination of spherical harmonics by either a downward continuation from the satellite radius to the planet's surface or a upward continuation from the planet's surface to the satellite radius. The altitude-cognizant scalar Slepian functions (AC-SSF) are constructed through the maximum spherical harmonic degree  $L$ , region of concentration, planet radius, and the average satellite radius. Since there is 'negligible' potential field in the area between the planet's surface and the satellite altitude, we optimize the potential from the planet to create a model. The AC-SSF optimizes the linear combination of scalar spherical harmonics to incorporate the satellite altitude at which the data was taken. Basically, we are able to take the field data collected by a satellite at some altitude and, through an optimal linear combination, we are able to plot the field on the planet's surface.

\*See Plattner and Simons (2017) for an in depth explanation.

## 2 Application

The following will include the methods for which we put Slepian functions to use.

## 2.1 Using glmalphapotup.m

To acquire the Slepian functions, given a data set, we may use the function `glmalphapotup`. This function requires the following inputs: TH (or dom for domain), L,

rnew – average satellite altitude

rold – planet radius

The outputs of this function are G, the matrix of spherical harmonic coefficients on the planet's surface, and V, the eigenvalues or the suitability factors. For all options for this function, see `help glmalphapotup`.

In our example, we will use a familiar domain, Africa. Set the parameters:

```
dom = 'africa';  
L = 20;  
rnew = 6371+400;  
rold = 6371;
```

As mentioned in the previous chapter, we must know whether our spherical harmonic coefficients are in *addmout* or *addmon* ordering. By using the `help` function, it is clear that the output is in *addmout* ordering; this will be useful information in a moment. So, we can run:

```
[G,V] = glmalphapotup(dom,L,rnew,rold);
```

## 2.2 Plotting

However, to plot the Slepian function at satellite altitude, we must upward continue the functions by using the function `potup`. This requires the inputs: G, radius to the satellite, planet radius, and L.

### 2.2.1 Example One:

For this example, we will use the spherical harmonic coefficient for the optimal Slepian function, which would be the first column of  $G$ . Let's first plot the Slepian functions on the surface of the planet. We will first need to convert our  $G$  into  $lmcosi$ , since  $G$  is in *addmout*, so run:

```
lmcosi = coef2lmcosi(G(:,1),1);
```

To plot the Slepian functions on the surface of the planet, we may use the same functions as in the previous chapter, for the classical scalar Slepian functions: `plm2xyz` and `plotplm`. Now set the resolution:

```
res = 1;
```

and create a spacial field from the spherical harmonic coefficients by running:

```
data = plm2xyz(lmcosi,res);
```

Finally, to plot the figure on a flat surface, run:

```
plotplm(data,[],[],4,res)
```

Let's add a colorbar and make the color scale nicer by running:

```
kelicol(1)
colorbar
caxis([-1,1]*max(abs(caxis)))
```

To plot the figure on a 3-D sphere, run:

```
plotplm(lmcosi,[],[],2,res)
```

For the results, see Figure 1.

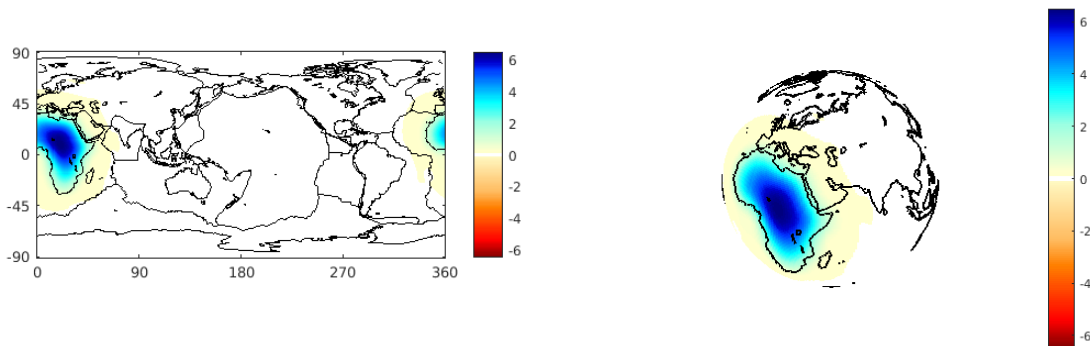


Figure 1: Optimal Slepian function,  $G(:,1)$ , plotted on the surface of the planet within the region Africa and spherical harmonic degree  $L=20$ .

Let's now use the function `potup` to upward continue the Slepian functions to satellite altitude. We can now run:

```
coef = potup(G(:,1),rnew,rold,L,0);
```

providing the output of the spherical harmonic coefficient vector for radial field at the new altitude. We need to convert this to `lmcosi` format:

```
lmcosi = coef2lmcosi(coef,1);
```

We can plot the functions at the satellite altitude. To plot on a flat surface, run:

```
data = plm2xyz(lmcosi,res);
plotplm(data,[],[],4,res)
caxis([-1,1]*max(abs(caxis)))
kelicol(1)
```

To plot on a sphere, run:

```
plotplm(lmcosi,[],[],2,res)
```

For the results, see figure 2.

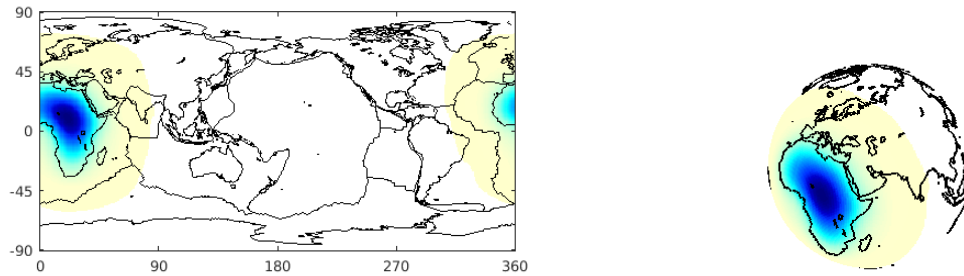


Figure 2: Upward continued optimal Slepian function,  $G(:,1)$ , in the region Africa with  $L=20$ . We can see this is the optimal Slepian function since it describes much of the region.

**Exercise:** Let's try another region, say a spherical cap:

```
dom = [16]
```

Run every step above again and plot on a sphere. What is the result? Compare your result to figure 3.

**Exercise:** We can try another region, say a spherical ring:

```
dom = [100,25]
```

What is the result? Compare to figure 3.



Figure 3: Optimal Slepian function upward continued at satellite radius in the spherical cap with opening angle  $\text{dom}=16$  (on the left) and in the spherical ring between the angles  $\text{dom}=[100,25]$  (on the right).

See the `help` function to see what other regions we can choose for `glmalphapotup`.

**Exercise:** Let's try to rotate a region. We will walk through this together. In order to rotate our region, we must use the function, `glmalphapotuptoJp`. This function requires an angular extent of a spherical cap or angles between two spherical caps, where we want to look at the ring between them. See `help glmalphapotuptoJp` for more information on the required parameters. Let's set our TH and other parameters as follows:

```
TH = 16;  
phi = 90;  
theta = 20;  
omega = 25;  
J = 50;
```

Now let's get our spherical harmonic coefficients:

```
[Grot,V] = glmalphapotuptoJp(TH,L,rnew,rold,phi,theta,omega,J);
```

We can now upward continue these and plot them:

```
coef = potup(Grot(:,1),rnew,rold,L,0);
lmcosi = coef2lmcosi(coef,1);
data = plm2xyz(lmcosi,res);
plotplm(data,[],[],4,res)
```

Plotting on the 3D sphere:

```
plotplm(lmcosi,[],[],2,res)
```

The rotated spherical cap is shown in figure 4.

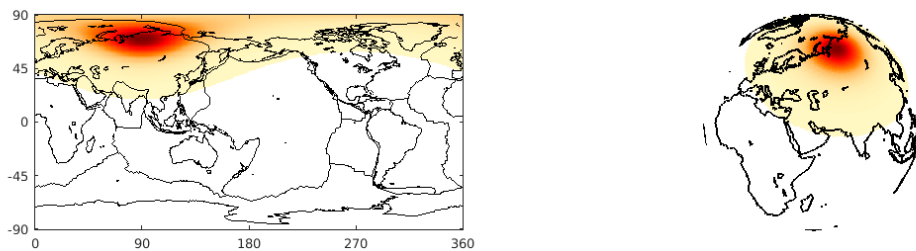


Figure 4: Optimal Slepian function to describe the rotated spherical cap of  $TH = 16$  upward continued with rotations  $\phi=90$ ,  $\theta=20$ , and  $\omega=25$  and eigenfunctions up to  $J=50$ . Compare this with figure 3, which is the same spherical cap not rotated.

**Exercise:** Let's try a rotated ring. We will need to change only the  $TH$  so we have a ring. So set:

```
TH = [100,25]
```

Now we can obtain the spherical harmonic coefficients:

```
[Grot,V] = glmalphapotuptoJp(TH,L,rnew,rold,phi,theta,omega,J);
```

Upward continue and plot:

```
coef = potup(Grot(:,1),rnew,rold,L,0);
lmcosi = coef2lmcosi(coef,1);
data = plm2xyz(lmcosi,res);
plotplm(data,[],[],4,res)
```

The rotated rotated ring is shown in figure 5.

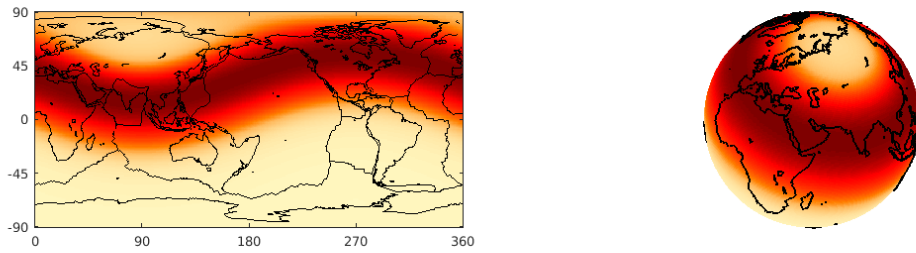


Figure 5: Optimal Slepian function to describe the rotated spherical ring of  $TH=[100,25]$  upward continued with rotations  $\phi=90$ ,  $\theta=20$ , and  $\omega=25$  and eigenfunctions up to  $J=50$ . Compare this with figure 3, which is the same spherical ring not rotated.

**Exercise:** Try the second best Slepian function, now in Africa:

```
dom = 'africa';
[G,V] = glmalphapotup(dom,L,rnew,rold);
coef = potup(G(:,2),rnew,rold,L,1);
lmcosi = coef2lmcosi(coef,1);
plotplm(lmcosi,[],[],2,res)
kelicol(1)
```

This results in figure 6. Compare with figure 2. What do you notice?

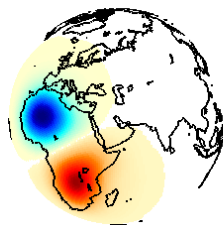


Figure 6: Upward continued second best Slepian function,  $G(:,2)$ , to describe the region, Africa. Notice this looks like two separate regions on the sphere. These are formed due to the nature of spherical harmonics. See chapter 1 for more information.

## 2.3 Linear combinations of Slepian functions

Now that we know how to create and plot the Slepian functions at satellite altitude with different regions, we can create a linear combination of the altitude-cognizant Slepian functions. What does the linear combination

```
h = G(:,1)+G(:,2);
```

look like? You will need to input this linear combination into your command line and run the same process, with `h` instead of `G(:,1)` in the functions:

```
coef = potup(h,rnew,rold,L,1);  
lmcosi = coef2lmcosi(coef,1);  
plotplm(lmcosi,[],[],2,res)  
caxis([-1,1]*max(abs(caxis)))  
kelicol(1)
```

This should result in figure 7. Compare this figure with figures 2 and 4. What do you notice?

**Exercise:** Try some of your own linear combinations and examine the plots.

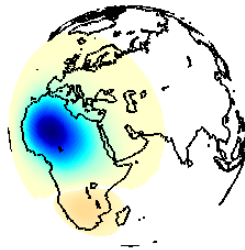


Figure 7: Linear combination of Slepian functions,  $h = G(:,1)+G(:,2)$ . Notice the upper portion dominates the plot since  $G(:,1)$  is the most optimal Slepian function, while  $G(:,2)$  is the second best Slepian function to describe the region. See figures 2 and 6 to notice how both of these Slepian functions plot.



### 2.3.1 Example Two:

Now, let's try a set of linear combinations including all of the spherical harmonic coefficients in our case. The length of our matrix,  $G$ , is  $(L + 1)^2$ . Let:

```
K = length(G)
```

We need to determine a proper value for  $J$  to plot the optimal linear combination of Slepian functions. We will first need to plot our index  $j$  versus  $V$ :

```
plot(1:K,V)
```

Now we can analyze the plot to determine which value of  $J$  such that most of the eigenvalues are concentrated within our region. Our plot will look like figure 8, as an exponential decay curve.

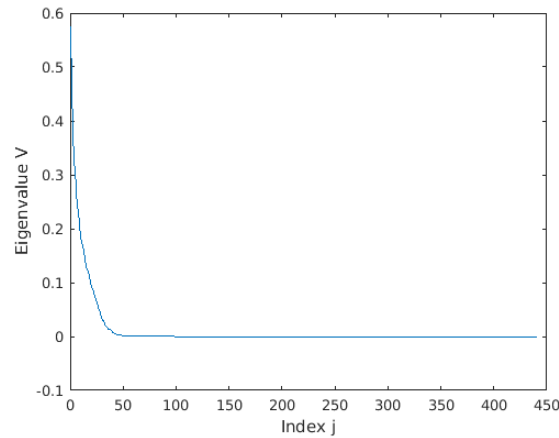


Figure 8: Eigenvalues  $V$  versus index  $j$  (column number in the vector  $V$ ). From this plot, we can see that the eigenvalues approach zero near  $j=50$ , which we will then set as our maximum  $J$  value.

By analyzing the graph, it looks like the eigenvalues near zero at about  $J = 50$ . You will have to manually choose your  $J$  value, so run:

```
J = 50;
```

We want to choose some random values to create our linear combination:

```
u = randn(K,1);
```

and we will need to create a matrix,  $f$ , which is the same size as our coefficients. Run:

```
f = zeros(size(G(:,1)));
```

Now we can make a for loop to create a linear combination of the coefficients by running:

```
for j=1:J
    f = f + u(j)*G(:,j);
end
```

This will create a matrix, `f`, containing the linear combinations of spherical harmonics. Now we can sort them into the correct format and plot:

```
[F,lon,lat] = plm2xyz(coef2lmcosi(f,1),1);
plotplm(F,lon,lat,4,res)
```

The plot of `F` is shown in figure 9. We can alternatively directly plot this by:

```
plotplm(coef2lmcosi(f,1),[],[],4,res)
```

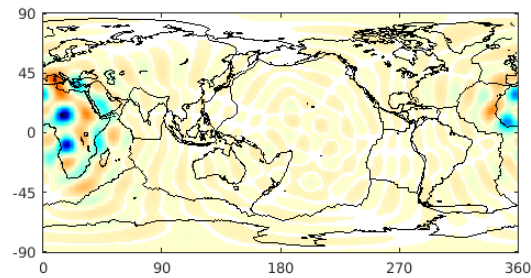


Figure 9: Linear combination of Slepian functions with random coefficients for eigenvalues up to  $J = 50$ , plotted on the surface. The linear combination of the Slepian functions gives a plot which describes the region, Africa.

Now, let's use all of the Slepian functions, rather than just up to our value  $J$ . To do this, let's run a similar code with eigenvalues  $1:K$ .

```
f2 = zeros(size(G(:,1)));
for j=1:K
    f2 = f2 + u(j)*G(:,j)
end
```

Plot:

```
plotplm(coef2lmcosi(f2,1),[],[],4,res)
```

Figure 10 shows the plot of all of the eigenvalues.

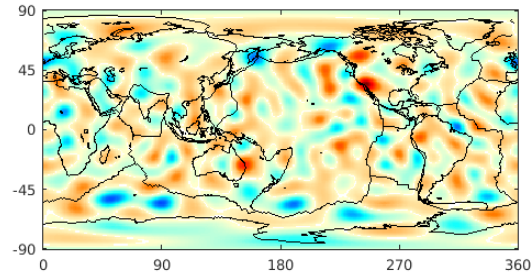


Figure 10: Linear combination of Slepian functions with random coefficients for all eigenvalues plotted on the surface. This linear combination does not describe the region, Africa, so this plot would not be useful.

Now that we understand the building of the linear combinations, we can use `potup` to upward continue the coefficients. We want to use our original `f`, since it would be the best linear combination of Slepian functions. We can do this in one step:

```
fup = potup(f,rnew,rold,L,1);
```

Now we can plot the upward continued coefficients:

```
plotplm(coef2lmcosi(fup,1),[],[],4,res)
```

The best fit Slepian functions, upward continued, are shown in figure 11.

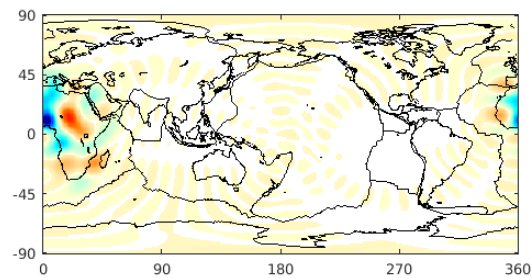


Figure 11: Upward continued linear combination of Slepian functions with random coefficients up to  $J = 50$ .

**Question:** Choose a lower  $J$  value and compare the plots. What do you notice?

This tutorial is currently under construction. Please check back later for more by keeping your software updated.