Scalar Spherical Harmonics and Slepian Functions

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Contents

1 Plot a single spherical harmonic function

We will demonstrate plotting a spherical-harmonic on a sphere, in a standard Matlab plot, on a Mollweide projection, and on random points of a sphere.

First, designate a spherical-harmonic to be plotted:

For example,

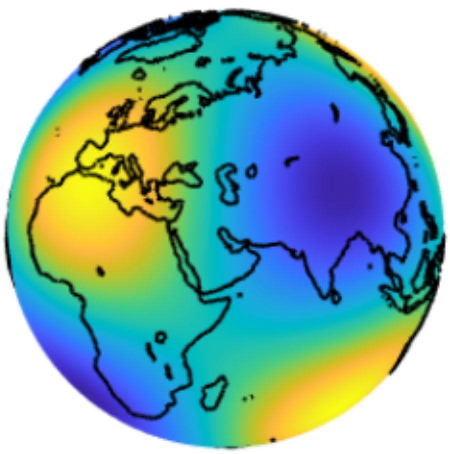
`l` = 3; `m` = -2;

0 `l` fixes the degree and `m` fixes the order.

1.1 Plot on sphere

Example:

```
l = 3; m = -2;
lon = 0:0.5:360;
lat = -90:0.5:90;
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon);
figure;
plotplm(Y, pi/180*lon, pi/180*lat,2)
```



1. Create a grid on the sphere

```
lon = 0:0.5:360; lat = -90:0.5:90;
```

This creates a coordinate point every half-degree.

2. Calculate the values of the function for coordinate points on the sphere

```
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon);
```

The function `slepian_alpha/ylm.m` evaluates the spherical harmonic function of degree `l` and order `m` at every point `pi/180*(90-lat)`, `pi/180*lon` on the grid. We name the vector of the spherical-harmonic values `Y`.

Note that `90-lat` is needed to convert latitude to colatitude and `pi/180` is needed to convert degrees to radians.

3. Plot

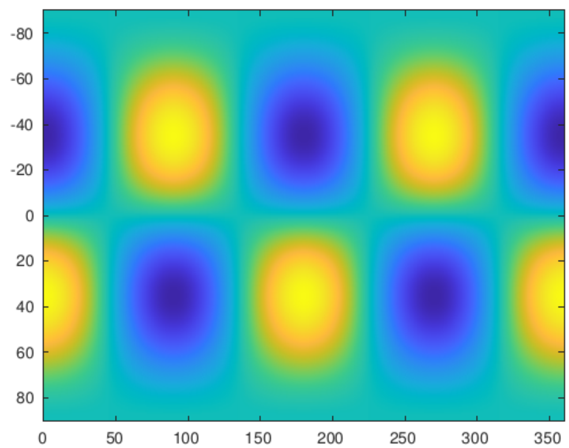
```
figure; plotplm(Y, pi/180*lon, pi/180*lat,2)
```

The function `slepian_alpha/plotplm.m` is here used to plot the vector `Y` using the grid specified by `lon` and `lat` in step 1. The input `2` dictates that the graph be on a sphere.

1.2 Plot in standard Matlab plot

Example:

```
l = 3; m = -2;
lon = 0:0.5:360;
lat = -90:0.5:90;
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon);
imagesc(lon, lat, Y)
```



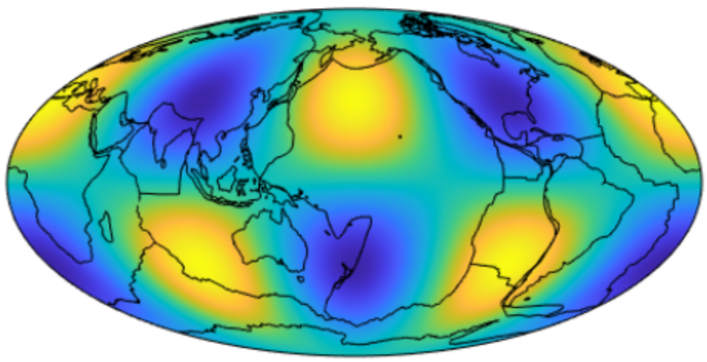
Do steps 1 and 2, and then run

```
imagesc(lon, lat, Y)
```

1.3 Plot on Mollweide projection

Example:

```
l = 3; m = -2;
lon = 0:0.5:360;
lat = -90:0.5:90;
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon);
figure;
plotplm(Y, pi/180*lon, pi/180*lat,1)
```



Do steps 1 and 2, and then run

```
figure;
```

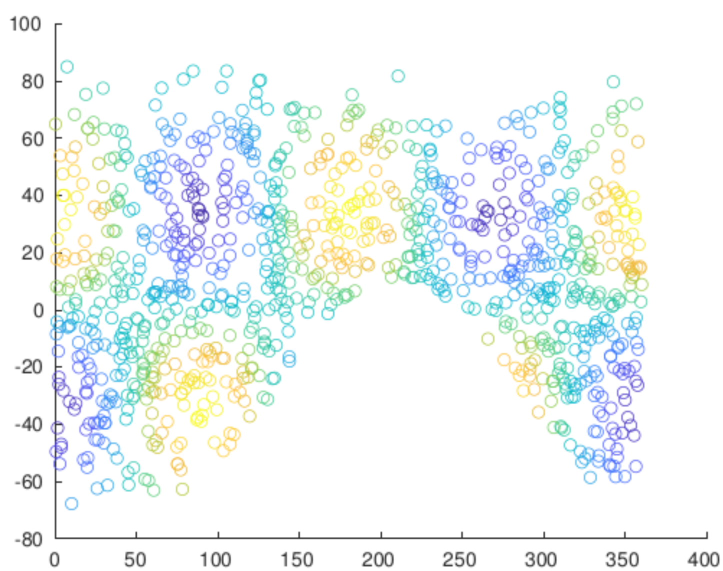
```
plotplm(Y, pi/180*lon, pi/180*lat,1)
```

The input `1` dictates that the graph be on the Mollweide projection.

1.4 Plot for random points on a sphere

Example:

```
l = 3; m = -2;
TH = 120; lon0 = 30; cola0 = 40; N=1000;
[lon, lat] = randpatch(N,TH,lon0,cola0);
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon,[],[],[],1);
scatter(lon, lat, [], Y)
```



1. Generate a subset of the sphere consisting of random points

In particular, we will create `N` randomly-generated coordinate points within a spherical cap of opening angle `TH` and centered at longitude `lon0` and colatitude `cola0`

For example,

```
TH = 120; lon0 = 30; cola0 = 40; N=1000;
```

```
[lon, lat] = randpatch(N,TH,lon0,cola0);
```

The function `slepian_alpha/randpatch.m` creates the set of random points within the spherical cap of the specified values. We name those coordinate points `[lon,lat]`.

2. Calculate the values of the spherical harmonic at those points

```
Y = ylm(l, m, pi/180*(90-lat), pi/180*lon,[],[],[],1);
```

`ylm.m` takes the arguments `l`, `m`, `pi/180*(90-lat)`, `pi/180*lon` as before. Run `help ylm` for information on all eight arguments.

3. Plot

If necessary, use the Matlab command

```
clf;
```

To clear existing figures, and then run the Matlab command

```
scatter(lon, lat, [], Y)
```

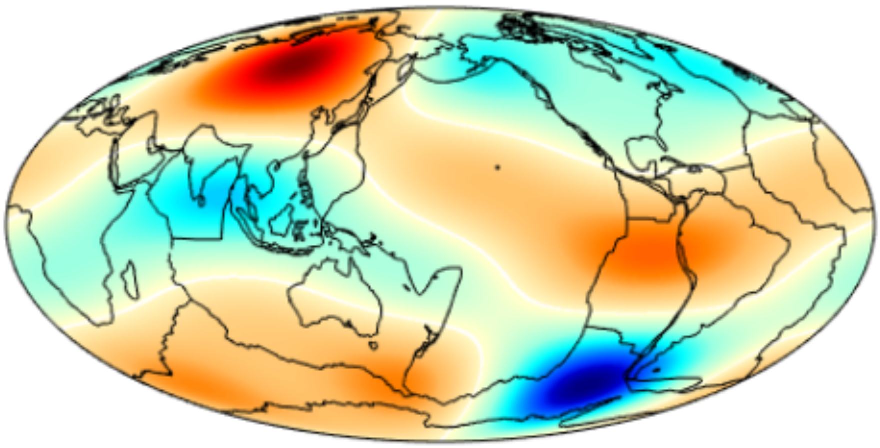
To create a scatter plot of circles having locations `[lon, lat]`. Here, `[]` indicates the default value for circle size and the vector of spherical-harmonic values `Y` is used to determine circle color.

Please see `Ch_01` in the `.edu` folder for more detailed information.

2 Plot a linear combination of spherical harmonics

Example:

```
lon = 0:0.5:360;
lat = -90:0.5:90;
Y1=ylm(3,1,pi/180*(90-lat),pi/180*lon);
Y2=ylm(1,-1,pi/180*(90-lat),pi/180*lon);
Y3=ylm(5,-2,pi/180*(90-lat),pi/180*lon);
Y4=4*Y1-0.2*Y2+2*Y3;
plotplm(Y4, pi/180*lon, pi/180*lat,1);
kelicol(1)
```



This task is a simple variation on the first.

Let us define three spherical harmonics:

`lon = 0:0.5:360;`

`lat = -90:0.5:90;`

`Y1=ylm(3,1,pi/180*(90-lat),pi/180*lon);`

`Y2=ylm(1,-1,pi/180*(90-lat),pi/180*lon);`

`Y3=ylm(5,-2,pi/180*(90-lat),pi/180*lon);`

Next, create a vector which is a linear combination of these three. For example,

`Y4=4*Y1-0.2*Y2+2*Y3;`

To plot the function, use `plotplm.m`. For example,

`plotplm(Y4, pi/180*lon, pi/180*lat,1)`

If you're interested in another color scheme, try out

`kelicol(1)`

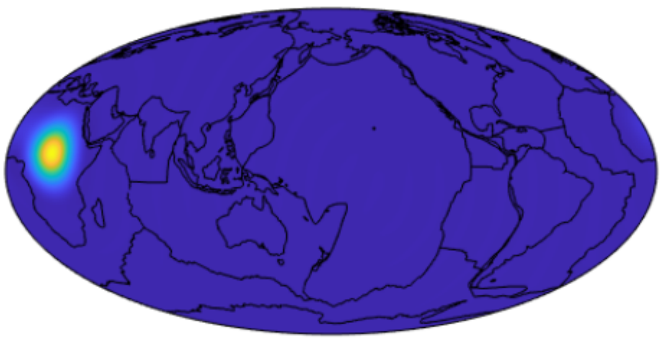
Please see `Ch_01` in the `.edu` folder for more detailed information.

3 Create and plot scalar Slepian functions

3.1 Named region and polar cap

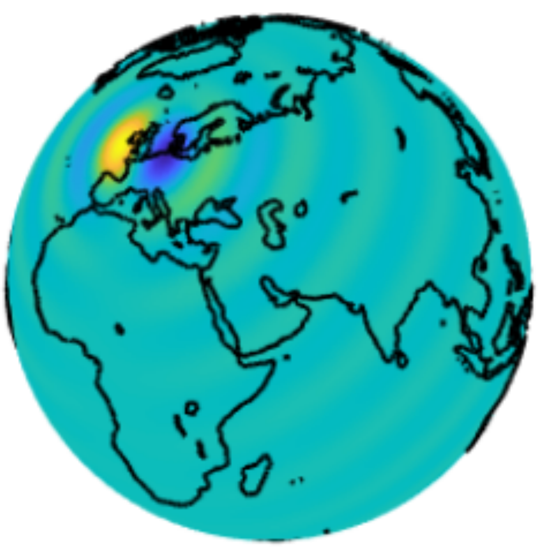
Named region example 1:

```
[G] = glmalpha('africa',20,[],0);
lmcs = coef2lmcosi(G(:,1),1);
data=plm2xyz(lmcs,0.5);
plotplm(data, [], [], 1, 0.5)
```



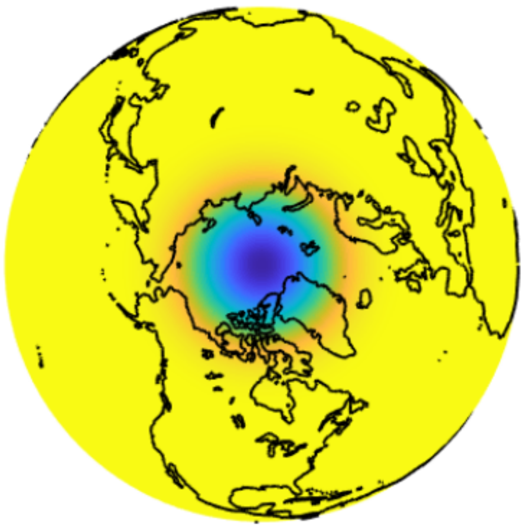
Named region example 2:

```
[G] = glmalpha('england',20,[],0);
lmcs = coef2lmcosi(G(:,3),1);
data=plm2xyz(lmcs,0.5);
plotplm(data, [], [], 2, 0.5)
```



Polar cap example:

```
[G] = glmalpha(40,20,1,0)
lmcs = coef2lmcosi(G(:,1),1);
data=plm2xyz(lmcs,0.5);
plotplm(data, [], [], 2, 0.5)
view(2)
```



To create the Slepian function to be plotted is to generate its spherical-harmonic coefficients. For this task we will use the function `glmalpha.m`, which computes the best spatially-concentrated Slepian functions given two constraints. Those constraints are the first input, `TH`, either a named region or the opening angle in degrees of a polar cap, and the second, `L`, the bandwidth. The named regions recognized by `glmalpha` are 'england', 'eurasia', 'namerica', 'australia', 'greenland', 'africa', 'samerica', 'amazon', 'orinoco', 'antarctica', 'contshelves', and 'alloceans'.

For example,

```
[G] = glmalpha('africa',20,[],0);
```

`[G]` is a matrix whose nth column holds the coefficients of the nth-best spatially-concentrated Slepian function. In order to plot the nth function we need to convert the nth column of `[G]` into the form recognized by the function `plotplm.m`, the so-called "lmcosi" format.

```
lmcs = coef2lmcosi(G(:,1),1);
```

Here we have chosen to use the first column `G(:,1)`. The second input `1` is necessary when the coefficients are calculated using `glmalpha`.

Now we input the (lmcosi-formatted) matrix `lmcs` and a resolution into the function `plm2xyz`. We choose the resolution to be 0.5 and name the ouput "data":

```
data=plm2xyz(lmcs,0.5);
```

Now to plot, run

```
plotplm(data, [], [], 1, 0.5)
```

The input `1` specifies Mollweide projection and the input `0.5` is just the resolution again.

The same sequence is used to plot a Slepian function on a polar cap, except for a change in the inputs to `glmalpha.m` Specifically, we will now let `TH` denote an opening angle in degrees.

For example, let `TH = 40`.

```
[G] = glmalpha(40,20,2,0)
```

The command

```
view(2)
```

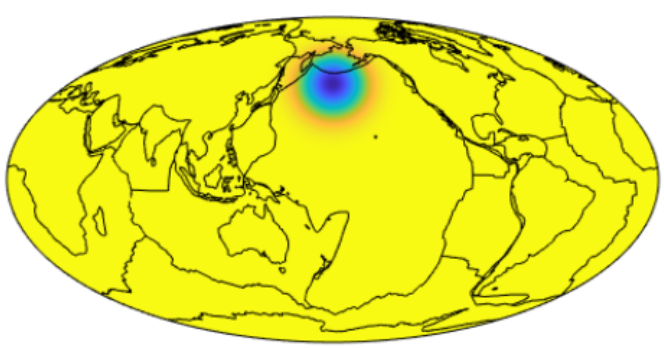
may be used to rotate the spherical figure so that the North pole is faced toward the viewer.

We suggest reading the `help` section for the relevant functions and/or Chapter 2, Section 1 in the folder `.edu` for a detailed discussion.

3.2 Rotated polar cap

Example:

```
[G] = glmalphaptoJ(40,20,180,45,0,10)
lmcs = coef2lmcosi(G(:,1),1);
data=plm2xyz(lmcs,0.5);
plotplm(data, [], [], 1, 0.5)
```



The general method of calculuating coefficients for a rotated polar cap is the same as above, but involves slightly different commands.

We will need to use the function `glmalphaptoJ` and provide the following inputs:

`TH`: opening angle

`L`: maximum spherical harmonic degree (bandwidth)

`phi`: longitude in degrees

`theta`: colatitude in degrees

`omega`: rotation of the region itself, if any

`J`: number of Slepian functions to be calculated

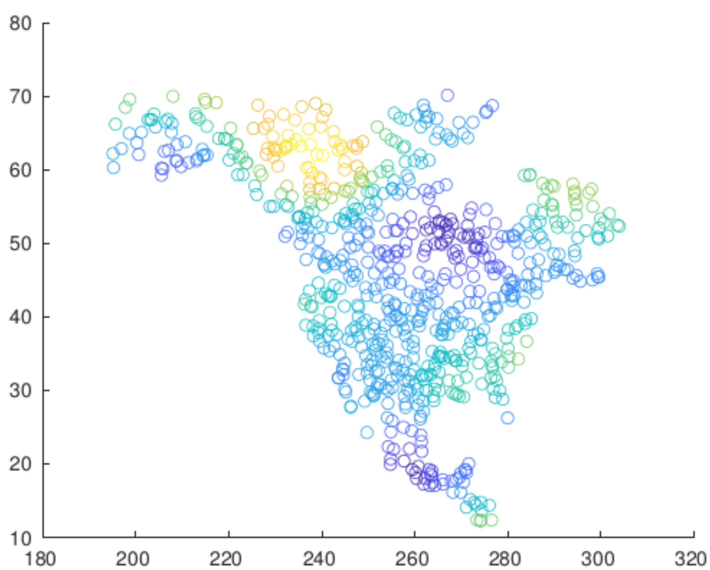
4 Compute spherical-harmonic coefficients from regional data

Slepian functions may be used to represent data obtained from regional observations. The goal of the current task is to calculate the spherical-harmonic coefficients of the Slepian function which best represents such a data set.

In a preliminary step we create the "observations" from random spherical-harmonic coefficients.

Example:

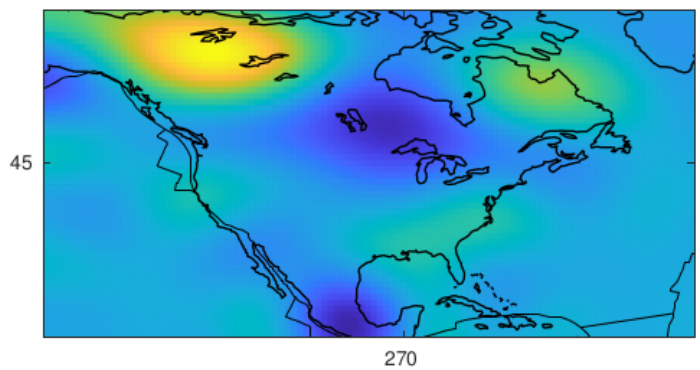
```
N=5000
dom='namerica'
[lon,lat]=randinpoly(dom,N);
Lmax=20
lmcosi=plm2rnd(Lmax,0)
data = plm2xyz(lmcosi,lat,lon);
subplot(2,1,1)
scatter(lon,lat,[],data)
```



The main task consists of obtaining the spherical-harmonic coefficients from the observations.

Example:

```
[G,V]=glmalpha(dom,Lmax);
Y=ylm([0 Lmax],[],(90-lat)*pi/180,lon*pi/180+pi,[],[],[],1)
J = min((Lmax+1)^2,round(1.5*(Lmax+1)^2*spharea(dom)));
Geval = G(:,1:J)'*Y;
gcoef = (Geval*Geval')\(Geval*data);
coef = G(:,1:J)*gcoef;
subplot(2,1,2)
plotplm(coef2lmcosi(coef,1),[],[],4,1)
```



Step 1.

First choose a region and create `N` random data locations within the region. For example,

```
N=5000

dom='namerica'

[lon,lat]=randinpoly(dom,N);
```

Next, create random spherical-harmonic coefficients using the function `plm2rnd.m`, which makes random coefficients for spherical harmonics up to degree `L` and stores them in an `lmcosi` matrix. For this example, let `Lmax=L=20`.

```
Lmax=20

lmcosi=plm2rnd(Lmax,0)
```

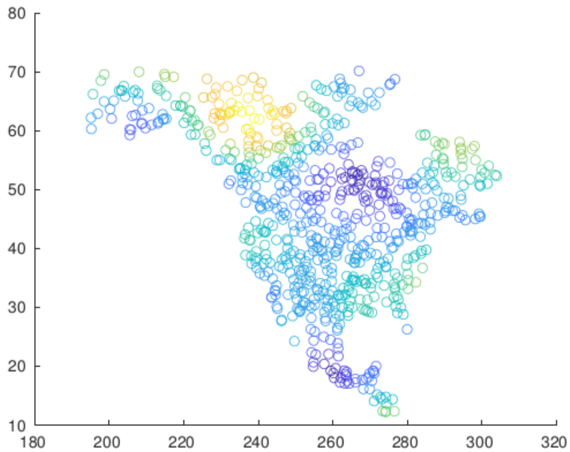
Now evaluate the Slepian function determined by these random coefficients at these locations:

```
data = plm2xyz(lmcosi,lat,lon);
```

To look at the data,

```
subplot(2,1,1)
```

```
scatter(lon,lat,[],data)
```



Step 2.

We will now treat data as a collection of observations and try to find the Slepian function which best fits the observations. We already know the function of best fit in this example; it is that whose spherical-harmonic coefficients are given in lmcosi and whose values are the entries of data. But be careful to note that data only represents the values of that function F over a subset of its domain, the randomly-generated [lon,lat] points in 'namerica'. The coefficients we will obtain in this Step will determine a Slepian function which matches F on [lon,lat], but which has no reason to coincide with it over the rest of the sphere.

First, we will use glmalpha to compute the coefficients of the best spatially-concentrated Slepian functions over our chosen region dom and chosen bandwidth Lmax. From this set of functions will be chosen the one which best fits data.

```
[G,V]=glmalpha(dom,Lmax);
```

Evaluate the spherical harmonics of degrees 0 to Lmax at the data points:

```
Y=ylm([0 Lmax],[],(90-lat)*pi/180,lon*pi/180+pi,[],[],[],1)
```

Next evaluate the Slepian functions at the data points. All of these evaluated Slepian functions are linear combinations of the above evaluated spherical harmonics; they differ by their coefficients, which are stored in the matrix [G]. Therefore, in order to evaluate a Slepain function whose coefficients come from the nth vector of [G], we would run something like

```
eval=G(:,n)'*Y
```

(Recall that `transpose(A)=A'` in Matlab.)

But we want to evaluate the functions for the first `J` vectors of `[G]` simultaneously, in order that we may compare them for best fit to `data`, so we run

```
J = min((Lmax+1)^2,round(1.5*(Lmax+1)^2*spharea(dom)));
```

```
Geval = G(:,1:J)'*Y;
```

See that we have replaced the column argument `n` with `1:J`.

(This choice of `J` should be sufficient in most situations. `(Lmax+1)^2` is the number of Slepian functions which exist up to a maximum degree `Lamx`, and `spharea(dom)` is the area of the domain relative to the whole sphere. The role of these quantities in choosing `J` was decided upon in order to optimize the solution to the linear system below relative to other factors like noise sensitivity. The scaling factor `1.5` is usually a safe choice but varies with the context.)

To determine which of these `J` functions best fits `data`, we must solve for a vector `gcoef` in

```
Geval*gcoef = data
```

This is an over-determined system, so we use the least-squares method to compute `gcoef`:

```
gcoef = (Geval*Geval')\(Geval*data);
```

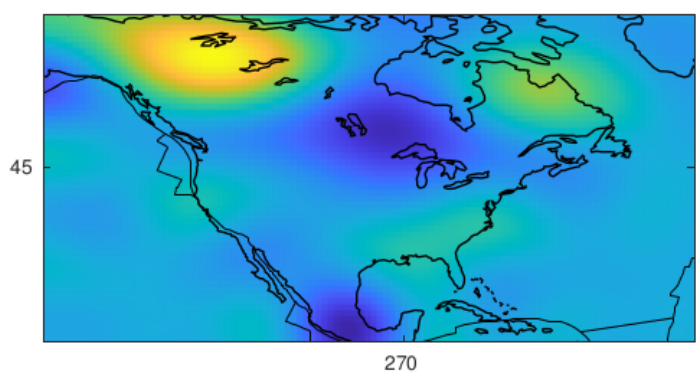
The vector of best-fit coefficients `gcoef` is currently in the Slepian basis. To translate it into the spherical-harmonic basis, run

```
coef = G(:,1:J)*gcoef;
```

To view the function whose coefficients are given in `coef`,

```
subplot(2,1,2)
```

```
plotplm(coef2lmcosi(coef,1),[],[],4,1)
```



For comparison, the Slepian function whose values are the entries in `data` (that is, whose coefficients are the entries in `lmcosi2coef(lmcosi,1)`) is graphed by

```
plotplm(lmcosi,[],[],4,1)
```