Illusions and anamorphosis

Terrien Tanguy, Ahmed M'Sabou, Fabien Da Costa Barros

Contents

Introduction

An illusion is a distortion of senses, which can show how the humain brain naturally interprets sensory stimulation. Although illusions modify our perception of reality, they can often be experienced by most people. Illusions may act on almost all of human senses. For example, there are auditory illusions : the Shepard tone, a recording where the notes seem to rise endlessly and still rise when it restarts, the recording where some people ear Yanny while some hear Laurel...
Here we are going to mainly focus on optical illusions, which act on our vision and play on the behaviour of our brain when facing atypical or unusual objects. Indeed, visual illusions are the best known among all the illusions. They also are the most interesting to talk about, because of the prevalence of vision over the four other senses. Some illusions are based on general assumptions that the brain makes during perception. Other illusions occur because of the biological structures of the human body or its physical conditions.
When thinking about eyes, the first modern object that comes to mind is a camera. However, there is a lot of features that our optical system has that a camera does not have. This optical system corresponds to the work done side by side by the eyes and the brain. There is a hidden purpose in comparing eyes to a camera, it is to think that eyes provide image directly to the brain. The eye transform the image into neural activity so that the brain could understand this flow of information. The neural activity is a representation of this object and the brain interpret those signals as the object.
The purpose of the presentation is to understand how the optical system can be tricked. For this, we need a better understanding of how it works. The point is to understand how one perceives a "world of object". We first need to clarify the underlying process of perceptions to find out when does this system fail to work properly. Perception involves a lot of mechanisms going beyond the immediate perception made by the senses. This perception is assessed on many grounds and generally, the brain makes the easiest interpretation and see things more or less correctly. Indeed, since the eyes do not give to the brain directly the world how it is, the neuronal system have to work on it.

Chapter 1

Eyes physical mechanisms

1.1 Light

The composition of light has not always been obvious wave or matter where two confronted assumptions. During the 17th century, we had on one hand Isaac Newton was thinking that light was a chain of small pieces of matter. On the other hand Christiaan Huygens was claiming that light was waves travelling through a transparent matter, "that he called aether" according to R.L. Gregory [1]. Nowadays we know that it is travelling through matter as waves and composed by quanta of energy called photons. During the same century another interrogation about light was discussed that revolved around the speed of light, is it instantaneous or does it have a speed ? And, if it has speed, what is it? As far as the science knows the speed of light is approximately 300,000 km per second. Studies have shown us that when it travels through different types of matter it can be slowed down depending on the matter and this difference of speed bends and refracts light as shown by the well-known glass prism experience. This is called refraction and it is important to mention it before going further into details.

1.2 Eyes

Firstly, the signal emitted by light can be processed in different ways to obtain, what we call an image. One way is to place a little hole in between the source of light and a screen, so that only the ray of light aligned the hole could reach the screen. we can think of it as being bijective in mathematical terms : one source, one point on the screen. This example can be demonstrated in a dark room and it represents quite accurately the human ocular system. We can compare the iris with a little hole and the retina with the screen but with only those elements the ocular system is incomplete and we often obtain a blurred image. we will detail further the role of crystalline lens and the importance of refracting the light to obtain a clear image, it is comparable to the focus of a camera. Secondly, another important aspect of vision is the movement of the eyes which is very useful to perceive our environment. Briefly, the eyes are able to move thanks to six muscles who work together to rotate and hold the eyes in their orbit. It allows us to keep track of moving objects in sight or even simply to see objects that are not directly in front of our eyes. It is obvious but important to mention that the movement of the eyes is team work, they converge together for close objects and always look at the same place to have a clear image.
Here is an explanation of the role of the different parts of human ocular system.

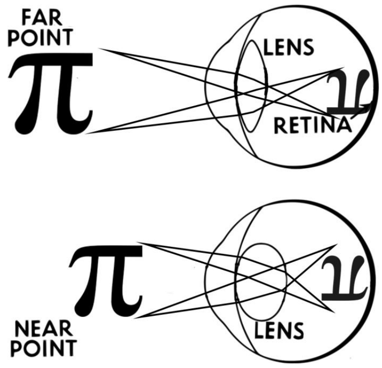


Figure 1.1: Illustration of eye accommodation to distance through lens shape changes

The crystalline lens

It is often thought that the lenses serve to bend the incoming light to form a clear image. In fact, the region where the bending is the most effective in the human eye is not the lens, but the front surface of the cornea. This occurs because the difference between the refractive index of the cornea and the one of air is higher than the one between the cornea and the liquid just behind it. The liquid behind the lens is nearly as dense as the lens, and as such the refractive power of the lens is not as efficient as commonly thought. Although the role of the lens is not essential for forming an image in the human eye, it is important for accommodation. The lens is composed of a membrane, the zonula whose curvature changes depending on the contraction of the ciliary muscle. The curvature of the lens is higher for near vision, the lens becoming increasingly convergent and so adding more precision to the primary bending accomplished by the cornea (see 1.2). This curvature improvement is due to the contraction of the ciliary muscle who push toward the lens and makes it more convex. Conversely, the lens need to be thiner to accommodate for vision of distant objects and it is done by a relaxation of the ciliary muscle.

The iris

The only use of the iris color is his opacity so that he can stop the light and create the hole described before. This could hardly be its primary function however, comparing to the occupied area the changes of the aperture count only for a ratio of about 16%. The goal of pupil contraction is to focus the rays on the more efficient part of the lens. This use is restraint in the dark and the pupil open to have a best sensation of our surroundings. It also closes for closer vision, and this increases the depth of field for closer objects. It is the hole described before which protects our eye from too much light, as it can burn the retina, or not enough light which would make us blind.

The retina

The retina is a thin sheet of interconnected nerve cells, including the light-sensitive rod and cone cells which convert light into electrical pulses, the language of the nervous system. "The retina has been described as 'an outgrowth of the brain' " according to G.L. Gregory [1]. It is a region of the surface of the brain which has come out and become sensitive to light, but it is still made of brain cells. The nerve cells are situated between the receptors and the optic nerve but represent the front layers of the retina which greatly modify the electrical activity from the receptors themselves. Some of the data processing to compute an image takes place in the eyes, which makes it even more a part of the brain. As said before, there are two different types of receptor cells, the rods and the cones. The cones are most effective during daylight conditions, and gives us the ability to perceive colours. The rods works at a lower light level, and allow us to see shades of grey. Daylight vision, using the cones of the retina, is referred to as photopic while the grey world given by the rods in low illumination is called scotopic.

Perception of distances

When looking at close objects, our eyes have to converge towards the center of our face (we're nearly cross-eyed), but when looking at objects that are further away, our eyes are likely to be at a distance from one another symmetrical to our nose. If we drew lines between our eyes and the object, they would be likely be almost parallel. The convergence of eyes is used to process distance by the brain, using a convergent or divergent lens, we can trick our brain into thinking that the object is respectively closer or further away than in real life. If the angle of light ray is modify enough to change the convergence of the eyes we will think that the object is in a place where it is not in reality. Depth perception is done partially by this mechanism but it is completed by another computation based on the difference between the two images given by the two eyes.
The brain evaluates a distance using the difference between the two images received by both eyes. This difference is called disparity. Special cells are involved in this combination of images using disparity which are called a "binocular cell". They are triggered by a stimulation of two related points of the two retinas. We believe that the firing of binocular cell is responsible for the "same" stimulation on two retinal points and are strongly involved in the disparity computation. These two mechanisms work together with the convergence of the eyes to adjust the disparity. The disparity between the images of both eyes when looking at two distant objects is higher than when looking at two closer objects.

Chapter 2

The brain

Scientists agree to say that the brain is the most complex part of human body.
We roughly understand the basic mechanisms which are pretty well understood nowadays. First, it is composed of two different substances which we call grey matter and white matter. To explain what the difference between grey and white matter is, we need to examine the key component of a nerve cell. In a nerve cell, we can distinguish three important components ; the cell body and its many thin dendrites, both composed the grey matter, and the axon which is a long fibre that conduct a message to the muscles. The grey matter is situated in the outer part of the brain whereas the white matter is situated in the inner part of the brain.

2.1 How is a signal transmitted across the brain?

The neural signals are transmitted by electrical oscillation which is produced by the alteration of ions in the cell's membrane. At rest, the potential inside the axon is negative with respect to the surface. The signals are caused by a depolarisation, meaning that the potential increases in the axon (it goes positive). At the end of the signal occurs repolarisation, the electrical potential going lower than at rest, in the negatives. These signals propagate in the entire axon during these steps. In other terms, during the firing of a signal there is an inversion and the potential inside the axon becomes greater than the potential at the surface (depolarisation) then the potential goes negative (repolarisation) and stabilizes at rest potential again.
We can illustrate the importance our brain gives to different parts of our body using a homunculus figure, which gives us a good graphical explanation. We can clearly see that the two regions that stand out are the face and the hand.
The face is the area where all, where the five traditional senses, that excludes touch, are concentrated. The mouth stands out clearly in this image meaning it is incredibly sensible of its surrounding environment. About our main subject, the eyes, we can see that they need a non-negligible surface of the brain to work. We can identify three different sub-areas the deeper we go in the brain, those are the cortex, the area stratie and Superior colliculus.

2.2 Optical part of the brain

There has been different experiments over time that have tried to distinguish the different sections of the brain responsible for the different parts of vision. The stimulation of the cortex (external part of the brain) produces a flash. The area stratie is the optical region of the brain situated on the cortex who is in charge of pairing eye movement. It has been found that there is a second area, the Superior colliculus responsible of retinal sensorial mapping and eye movement.

Figure 2.1: Picture of a sculpted Homunculus

Processing

The primary visual cortex is the place where spatial informations are computed. The neural signal is transmitted to the Superior Colliculus which translates this activity into eye movement. We discovered the importance and role of this part of the brain when an accident occured on people or animals in this region.
For example, experiments on monkeys have shown that lesions in this section can cause irregular eye movement. Decades before, another experiment was conducted on cats lead by American physiologists, Hubel and Wiesel describe in *Eye and brain : The psychology of seeing* [1]. The purpose of the experiment was to record neuronal activity while showing cats bars projected on screen. Results showed that the region where the brain activity is higher depends on the orientation of the bar shown. Finally the conclusion is that the first role of the area striate is dedicated to patterns recognition and each part of it recognizes a stimulation of a specific region of the retina. Those results highlight the importance of the brain making the right selection of inputs created by the stimulation of the retina. Pattern stimulation of the retina is coded by coordinate neuron firing and inhibiting themselves to give the right sensation. The visual cortex organisation is described as different layers, which are in turn organised in columns.. A stimulation of a special part in this three dimensional space is linked with a particular line orientation projection (formed by the received stimulus) on the retina.
Moreover, the brain cells related to vision is responding to two different points from the two retinae. It's mainly thanks to the link between two points from different both retinae that allows us to recognize patterns, colours and extract a contour from an image.

Chapter 3

Kinds and classes of visual illusions

Introduction

To be better understood and explained, illusions may be classified. The most commonly accepted classification of the types and kinds of illusions was thought up by Richard Langton Gregory, a psychologist and neuropsychologist from Bristol. He was one of the first to develop the idea of cognitive perception, and by extension the main creator of the different classes of illusions. He has written numerous books on the psychology of vision and perception.
In his article "Putting illusions in their place" [4], he explains the basis of his classification, composed of 3 conceptual classes.

3.1 Physical illusions

Physical illusions are due to explainable physical phenomena. For example, when a stick is in water, it gives the impression of being divided into 2 non-parallel parts. This is caused by the phenomenon called refraction (this is the change of direction of light waves from one environment to another).

Figure 3.1: This stick seems to be broken in two parts.

3.2 Physiological illusions

Definition

Physiological illusions are created by excessive stimulation of the eyes and the brain or recurrent interaction with the same stimuli. Linked to neuronal systems and sensitive receptors, they often come from high luminosity or strong colours observed over a long period of time.

Figure 3.2: Stare for a minute at the flag then look at a white spot. The french flag will appear.

Explanation of the image above

The eye has colour sensors. They are not used to face highly saturated images. This is why after a certain period of time, the sensors of a color X perceived at a place of the eye get tired. Then, looking at white (composed of all the colors) directly afterwards, the eye will see all the colors except the color X. The color additive system will make the brain interpret the information received as the opposite of the color the sensor is tired of. By looking at white, one thus sees the saturated image in reverse.

3.3 Cognitive illusions

Cognitive illusions are the most complex kind. They are indeed deeply linked to the brain, especially to the interpretation of the brain : therefore, effects of cognitive illusions sometime vary person-to-person. Moreover, human understanding of these effects are still evolving. This is why they have been subdivided into 4 types by Richard Gregory.

3.3.1 Ambiguity

Ambiguous illusions are images that can be interpreted in many different ways. In general, with a short training or habit, anyone can switch their way to see the picture.

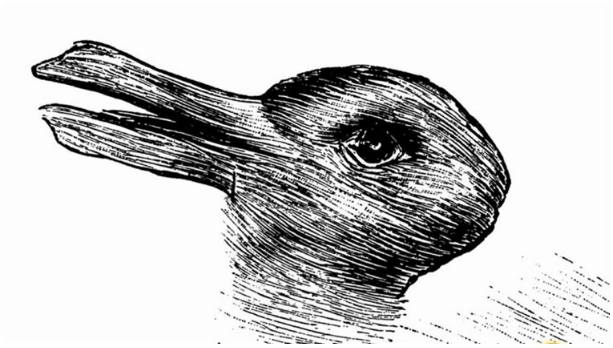


Figure 3.3: Is it a duck or a rabbit ?

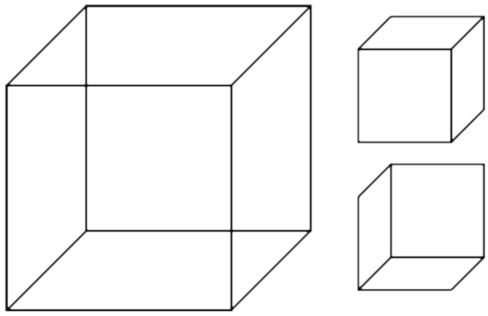


Figure 3.4: The Necker cube

The duck-rabbit illusion was used in "Fact and Fable in Psychology" of Joseph Jastrow, an american psychologist. In his work, he reproduced this illusion to illustrate the correlation between brain and cultur in visual perception. This illusion really widespread through the book "Philosophical investigations", where Ludwig Wittgenstein use it to describe the opposition between "seeing something" and "seeing something as something".
The Necker cube, published for the first time by Swiss crystallographer Louis-Albert Necker, is a two-dimensional drawing of a cube that can be interpreted from 2 ways (front side top right or front side down left).

3.3.2 Distortions

These illusions are based on the geometric distortion of objects. They are often deliberately invented to deceive the viewer about the shape and size of certain images.

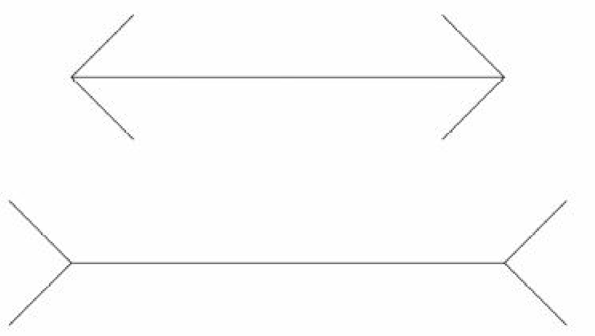


Figure 3.5: The classical Muller-Lyer illusion.

Named after German sociologist and psychologist Franz Müller-Lyer, this illusion tricks the observer into thinking that top segment is longer than the bottom one. This impression is given by the inward/outward pointing arrows at both ends.

3.3.3 Paradoxes

These illusions are created by two-dimensional images representing objects that cannot physically exist in real life. Paradoxical three-dimensional objects also exist, they are called impossible objects. The illusions they generate usually only appear from a certain perspective.

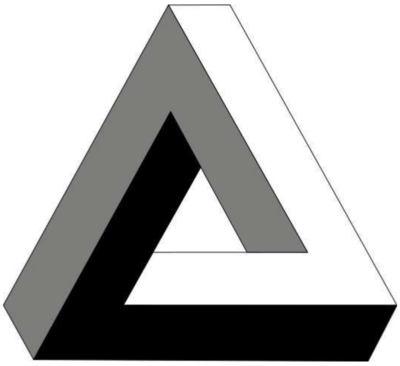


Figure 3.6: Penrose triangle

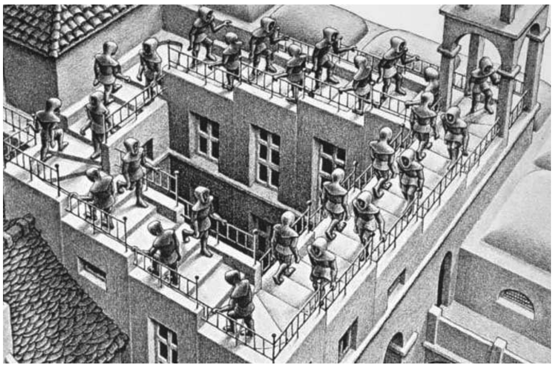


Figure 3.7: Penrose and Escher stairs

Penrose triangle is an impossible object described for the first time in 1934 by Oscar Reutersvärd, a swedish artist, and rediscovered by mathematician Roger Penrose in 1958. At the same time, his father Lionel designed a drawing showing endless ascending stairs. The two Penrose envisioned many drawings of impossible objects, that they published in the British Journal of Psychology in 1958.
Penrose staircase has been reused many times, and the great Maurits Escher variations of it are now very well known (for example, the never-ending fountain).

3.3.4 Fictions

These illusions occur when we see something that does not exist in the image. They are often due to an interpretation of the brain.

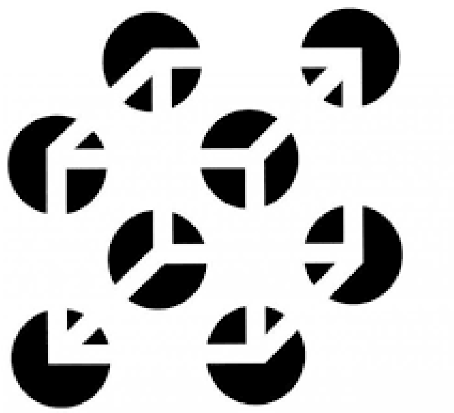


Figure 3.8: Incomplete black disks are enough to make one see a (Necker) cube.

When perceiving an illusory figure, we unconsciously interpret it based on our previous experiences. In this example, the black disks are intentionally crossed by white lines to make the human brain see a cube that does not exist.

Chapter 4

A particular type of illusion : Anamorphosis

4.1 An unconventional way of seeing

The word "anamorphosis" is derived from the Greek prefix ana, meaning again, and the word morphe, meaning shape. An anamorphosis can be defined as a deformed picture whose true shape is revealed when seen in a particular way. An anamorphosis is a distorted image, often a projection of an image on a curved surface. When seen from a certain position, or reflected from a curved mirror, the illusion effect arises. There are two typical forms of anamorphoses. The most common one is oblique anamorphosis, which must be seen from a position that is far from the conventional in-front position from which we normally expect pictures to be viewed at. The other known kind of anamorphosis is the "mathematically distorted image", which generally uses the reflection of a mirror for the illusion to emerge.

4.1.1 The Ambassadors

In 1533, German painter Hans Holbein made a painting called "The Ambassadors". It is preserved in the National Gallery in London. This work of art contains an excellent example of oblique anamorphosis.

Figure 4.1: The Ambassadors by Hans Holbein (1533)

At first sight, wealth and power show up in this painting. However, looking at the work in more details, one can understand that it is not that joyful. The first step towards deciphering the picture's meaning is to look at it sideways. Indeed, the illusion comes into focus : a human skull emerges.

Figure 4.2: Apparition of the effect of the anamorphosis

Visual part

The skull has its own special lighting. Initially, the moral seems obvious : Death enters and the world and its vanities simply disappear. Neither wealth, power nor even knowledge can save anybody! However, the barely visible figure of Christ in the top left-hand corner reminds the viewer that there is a second viewpoint, which wipes death itself from the picture. Since one can view his picture from different angles, Holbein is using the physical movement of the spectator to make him aware of the different possible approaches of this painting. We then must move into the picture,immersing ourselves in its details.

Historical part

The painting shows two young ambassadors (their ages, 29 and 25, are respectively discreetly marked on a dagger and a book). They are both French and share the same vision of diplomacy, a vision symbolised on the shelves. "The Ambassadors" holds three strong diplomatic cards. The first one, symbolised by the hymn, is the division of Christianity between Catholics and Protestants : by supporting the Protestant princes and towns, they can weaken the Empire,which is already fragmented into thousands of states. The second one is their contact with the Musmlim world, symbolised by the red carpet. When Holbein painted his picture, the Ottomans was reaching the gates of Vienna and were threatening Christianity. The third one is the love-life of the King of England, symbolised by this tiled floor,which replicates that in London's Westminister Abbey.France has backed the King's decision to repudiate his first wife,who has links with the Empire,and marry Anne Boleyn. This is its way of forestalling a possible alliance between England and the Empire.

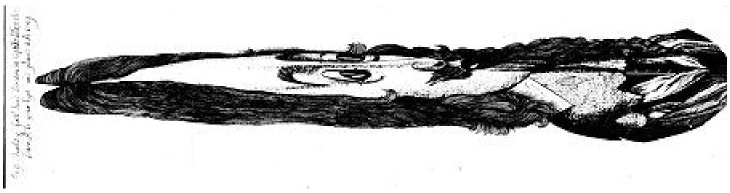
4.1.2 Anamorphosis in the streets

Road markings is without doubt the most popular use of anamorphic designs. Indeed, as a passenger, one can often see very long marks on the ground. In fact, these marks are deliberately stretched out: it uses the principle of foreshortening (the apparent shrinking of objects as they move further away) so that from their point of view, drivers can see marks with normal proportions.

Figure 4.3: The same spot from two different points of view.

Viewed from the front and from a distance, the letters seems to be as high as wide. Viewed from the side or from a close position, the letters are actually twice as high as wide.

History of anamorphosis



Anamorphic designs were used as "secret images" in past centuries, occasionally to hide erotic drawings, sometimes pictures of a politically-sensitive nature, for example this very well-known portrait of the British King Charles the First.which circulated amongst Royalist sympathisers after the King's execution in 1649:

Films and Series that used Anamorphosis

The concept of anamorphosis has been used in many movies as for example The Graduate (1967), McCabe and Mrs Miller (1971), Death in Venice (1971)... In 2013 and 2014, Angenieux released a new series of high end anamorphic zooms. All these movies used what one may call forced perspective. It is the use of optical illusion to make an object appear farther away, closer, larger or smaller than it actually is. It is usually achieved by cleverly placing a miniature in the frame with the life-sized objects in order to pull off special effect. For example, in the well-known movie series "Lord Of The Rings", anamorphoses system are used to make normal-sized humans appears as "hobbits".

Chapter 5

Illusions on a daily basis

5.1 Art

5.1.1 Pareidolia

Pareidolia is a psychological phenomenon where one believes to see a face in an image, while there is actually not. This concept of illusion is regularly used in the field of art. A good example of this use is the painting "All is Vanity", made by American illustrator C. Allan Gilbert in 1892 and published for the first time in a magazine named "Life" in 1902.

Figure 5.1: "All is vanity", Charles Allan Gilbert, 1892

Explanation

In the foreground, we can see a woman looking at herself in the mirror, and behind, we have this mirror and the reflection of the woman. This painting creates a perspective effect: at first sight, one can see a sitting woman and his reflect, but stepping back on it, the elements are all coming together to form a human skull. The mirror and the table form the skull, while the head of the woman, duplicated by the mirror, makes the eyes of the skull appear. In addition, the flasks take the place of teeth and the candle forms the nose.

5.1.2 Bela Borsodi

Bela Borsodi, an austrian photographer and graphic designer, created a fascinating piece of architecture.

Figure 5.2: The same work of art from two different points of view

The two pictures represent exactly the same thing. However, the first one is taken facing the work and the second one is taken from a different angle. It is important to note that no object moved.

Explanation

This work of art is an optical illusion that uses the principle of perspective. Indeed, seeing only the left picture, one may think that it is actually four pictures of different places sticked together. At the sight of the right picture, everyone understands that it is genuinely a single one : the illusion occurs when viewed from the front, and is totally broken if you see it from another point view. To establish this feeling of separation, Bela Borsodi uses many differents objects of various colors. The placing of the elements and the sorting of colors trick the spectators into seeing four separated pictures, while there is, in reality, only one.

5.2 Architecture

Introduction

The architectural field is another creative use of optical illusions. By their conception, at first glance these works defy the logic that passes through every person's brain. Moreover, the human scale size of these illusions strenghtened the effects created on spectators.

5.2.1 The Parthenon

Introduction

A well-known exemple is the temple named the Parthenon. It is a former greek temple located on the Athenian Acropolis. Its construction began in 438 BC, and was completed 9 years later, which is an astonishing speed of execution.

Description

Few people are aware of the subtlety of its achievement. The optical illusion occuring in this monument is more of a reversed illusion. Indeed, the Parthenon is on the top of the Acropolis of Athens. Hence, if it had been built in a normal way, with straight lines, we would see it curved. The temple has been designed accordingly : the center of the steps around the structure is 7 centimeters higher than the end of them. Similarly, the center of the edges forming the roof and the end of them are different from 12 centimeters high. The columns are also slightly curved : their bend radius is about 1,5 kilometers.

Illusion

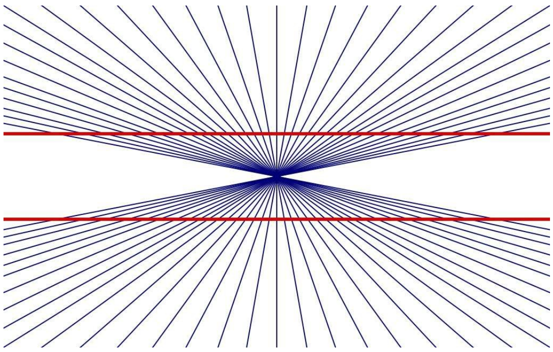


Figure 5.3: The Hering illusion.

In the Hering illusion, designed by German psychologist Ewald Hering in 1861, all the lines in different directions tricks our brain into seeing both red lines curved even though they are straight and parallel. A feeling of depth is therefore induced by the background.

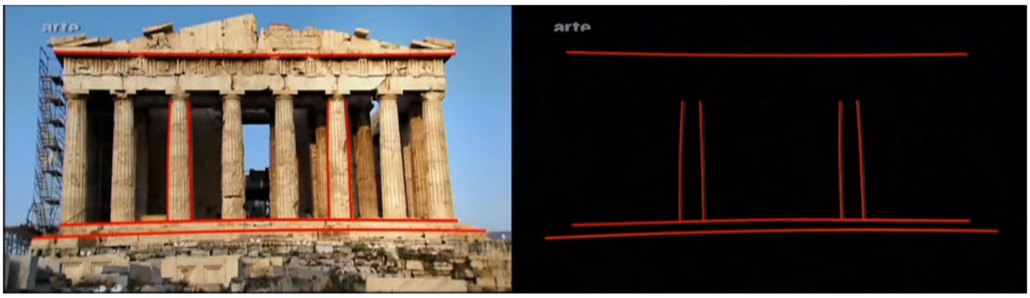


Figure 5.4: Front side of the Parthenon.[5]

Explanation

As one can see, lines of the Parthenon are inwardly curved. The curvature cancels the effects of a Hering illusion that appears here. This leads the human brain to interpret the monument as if it was only composed of straight lines, and therefore see a perfectly dimensioned structure, where in reality it is curved. Note that the Parthenon is not the first temple to use this trick : the temple of Demeter, located on an island next to Greece, was built a century earlier with curved lines.

5.2.2 Ames Room

Introduction

An Ames room, invented by the american scientist Adelbert Ames in 1946, is a distorted room that creates a visual illusion. Almost everyone knows this concept, although the name "Ames room" does not sound familiar.

Figure 5.5: Working of the illusion.[6]

Description

Everybody sees a normal room, but actually, looking at it from a different point of view, one can clearly see that the room has distorted proportion. For the illusion to work, an Ames room has to be viewed with one eye through a peephole, or from a camera if it is put in the right place. Forcing the observer to use one eye to look into the room prevents him from getting any information about the real shape of the room from stereopsis, the perception of depth and 3-dimensional structure, which obviously requires two eyes.

Explanation

In reality, the Ames room is trapezoid-shaped. Multiples sources of information of the shape of the room are manually modified by its creator. First of all, a strategic use of lighting makes the far corner as bright as the near corner. Elements on the floor such as windows or paintings are distorted. Patterns on the floor are also modified to appear straight and horizontal.

Power of the illusion

If an Ames room is well-built, the illusion will be strong enough to overcome other information such as true location and forms of objects in the room. Although everyone knows that a human cannot change size, our brain interprets what we see as a person growing and shrinking only by

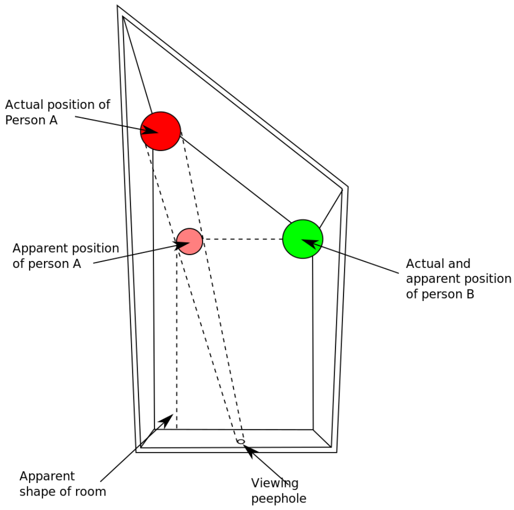


Figure 5.6: Example of an Ames room.

walking. The power of the effects created by this illusion comes from the fact that it combines differents kinds of illusions. Indeed, an Ames room is at the same time a physical illusion, and a cognitive illusion (in this case, it is of the distorted class).

Ames rooms in arts

Ames room are often used in movies, instead of using special effects (for example, giants in Harry Potter or hobbits in The Lord Of The Rings).

Chapter 6

Transform an image into a cylindric anamorphosis

Algorithmic transformation of an image

Overall, the main purpose of the algorithm is to map every pixel of an image to a nicely distorted image. The goal of the distortion is to see the initial image in a cylindric mirror. We split the work in two parts :

- The projection of every pixel on a different spot of $\mathbb{R}^2$.
- The pixelisation of a set of points in $\mathbb{R}^2$. In other words, get a value for each pixel (tuple of integers).

We will use image following the uncompress PGM format (also called PGM P2), which defines images in given value of shade of grey (all our images are in 255 shades of grey).
The code is visible and executable on our GitHub Project :
https://github.com/YargonIV/CylindricAnamorphosis .

6.1 Mathematical part[7]

An important part of the algorithm is purely mathematical. Indeed, before obtaining the final distorted image, it is required to compute, for each point of the initial image, its position after transformation, i.e. its projection on the xy-plan, corresponding to the white sheet.

6.1.1 The setup

We imagine that the original image is standing in the yz-plan in the cylinder (which is centered on the z-axis). We choose an arbitrary point V for the position of the viewer. However, the final illusion will better work if the point V is on the xz-plan (in front of the center of the image). We arbitrarily set the radius of the cylinder as 0,6 times the image width.

6.1.2 Transformation of a point P in the original image

Position on the cylinder

We imagine a line drawn between the point V, which is representing the viewer, and the point P, the point that we want to obtain the projection of. This line has for equation :

$$L(t) = P + t(V - P) \tag{6.1}$$

with t ranging from 0 to 1.
We then want to compute the point of intersection I of this line and the cylinder (i.e. the location of the cylinder where the spectator will see the reflection of the point P). By putting the expressions of L(t) for the x and y coordinates into the following equation of a cylinder

$$r^2 = x^2 + y^2 \tag{6.2}$$

we obtain a quadratic equation of the form

$$r^2 = at^2 + bt + c \tag{6.3}$$

which can be solved since we know the radius of the cylinder. We then compute the point L(t) corresponding to the value of t that we found by solving the previous quadratic equation : this is the point of intersection of the cylinder and the line and the PV segment.

Reflection of P on the xy-plan

We now want to compute the coordinates of the reflection of the point P on the xy-plan, where the final image will be projected. First, we consider the cylinder-normal vector, which has the same coordinates as the point I, with the exception that since it is cylinder-normal, its coordinate on the z-axis is 0. Then the reflection vector r is calculated with the equation

$$r = \vec{PV} + 2\vec{VU} \tag{6.4}$$

where U is the orthogonal projection of V on the cylinder-normal vector that we computed before. We now have to compute the intersection between the line drawn by the reflection vector and the xy-plan. Then, in a similar way as to find the intersection point of the segment PV and the cylinder, we find the equation of the line corresponding to the reflection vector: this is the reflection line. Its representative equation is :

$$R(t) = (I_x, I_y, I_z) + t(r_x, r_y, r_z) \tag{6.5}$$

We then compute the value t by focusing on the Z coordinates of R(t). We want to find u such that the point R(u) belongs at the intersection of the reflection line and the xy-plan. That is a value such that the z-coordinate of R(t) is 0. We have :

$$I_z + ur_z = 0 \iff u = -\frac{I_z}{R_z} \tag{6.6}$$

We replace this value of u in the equation (6.5) and we obtain the following point :

$$(I_x - \frac{I_z}{R_z} r_x; I_y - \frac{I_z}{R_z} r_y; 0) \tag{6.7}$$

This point is therefore the projection of point P of the fictive original image on the xy-plan. In other words, it is the reflection of the point P on the cylinder from the point V representing the viewer.

6.1.3 Transformation of the whole image

Finally, we repeat this operation for all the points of the image that we choose to transform: we obtain the distorted image, projected on the xy-plan. However, since the image after transformation is stretched, we cannot simply make each pixel on the target image correspond to a pixel on the original image.

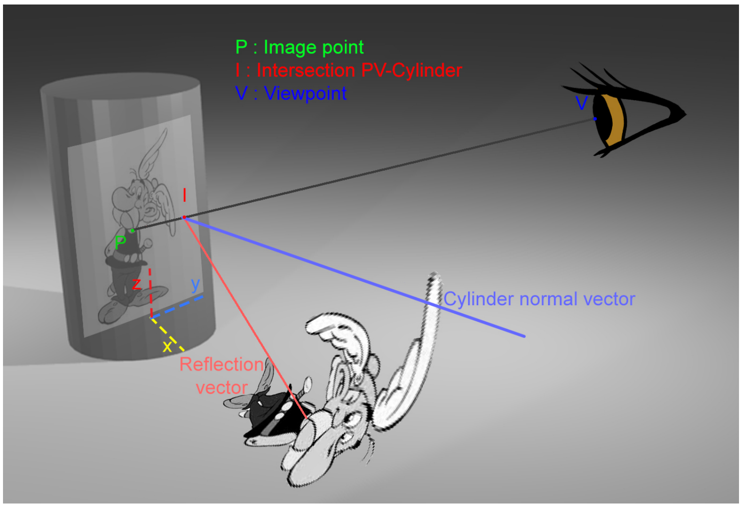


Figure 6.1: Graphical representation of the situation

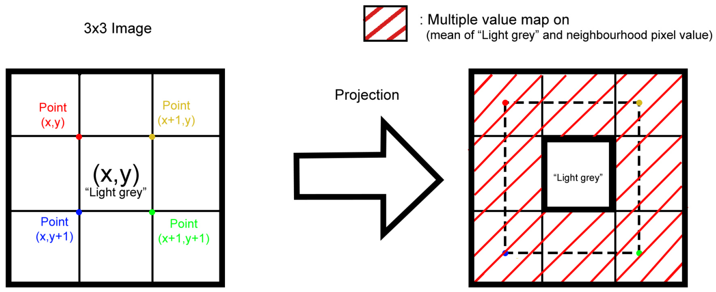
6.2 Pixelisation

In this section, we refer to W as the width and H as the height of the image that we want to transform. The goal is to map each $(x, y) \in \{(x, y)\mathbb{R}^2|(x, y)$ *is the projection of a point calculated before* $\}$ to some $(n_x, n_y) \in \mathbb{N}^2$.

For this we should compute the projection of each pixel (x,y) where $0 \leq x < W$ and $0 \leq y < H$. Instead, we will compute the projection of each pixel (x,y) where $0 \leq x \leq W$ and $0 \leq y \leq H$ (notice that the $<$ become a second $\leq$) to have a grid (see 6.2) and project the area occupied by the pixel instead of the pixel himself. We store the projection of each point (x,y) in a table at index (x,y) to access it easily and calculate it one time. Here the index (x,y) is equal $x + y * (L + 1)$ in a one dimension table and L+1 represent the length of rows as $0 \leq x \leq W$. As illustrated below we define a correspondence between a pixel and his four corners and the projection of a pixel. On the distorted image, we gives the color of the origin pixel to each pixel in the smallest rectangle containing the projection of the four corners.

There still are two problems we will have integers but they can be negative. Moreover some $(n_x, n_y) \in \mathbb{N}^2$ would match multiple value because the smallest rectangle containing all points will overlap with those of his neighbourhood pixel.

I will define four new constants : $min_x, min_y, max_x, max_y$ *respectively equal to the minimum in the x-coordinate of the projection, the minimum in the y-coordinate of the projection and reciprocally for the maximum*



ig:pixelProj

Figure 6.2: Illustration of pixel mapping

The easiest problem to solve is to avoid negative integers as we define our image as grid of pixel with non-negative coordinates. We need to move each pixel as follows :

$$translation : \mathbb{R}^2 \mapsto \mathbb{R}^2$$
$$(x, y) \mapsto (x - min_x, y - min_y)$$

so that now, $0 \leq x \leq max_x - min_x$ and $0 \leq y \leq max_y - min_y$

Now, to fix the overlapping problem, we create a table containing a list of pixel representing our distorted image. At each time that we project a pixel, we do not overwrite the ones that were computed before, if there are some : we add them to the list at the same index (x,y). The, we compute the final value of each pixel by computing the mean of the list for each pixel. We finally write the image in the PGM format and we can observe the result.

References - Sources

Gr66 [1] R.L. Gregory, *Eye and brain : the psychology of seeing*, Vol. 3, McGraw-Hill Paperbacks, 1966, Third edition 1978. ↑3, 4, 7

Ho11 [2] R. John Leigh Anja K.E. Horn, *Handbook of Clinical Neurology*, 2011. Chapter 2 - The anatomy and physiology of the ocular motor system. ↑

[3] *Visual cortex*, `https://en.wikipedia.org/wiki/Visual_cortex`. ↑

Classes [4] Richard L. Gregory, *Putting Illusions in their place*, Even Odder Perceptions (02/01/1991), 1–4. Volume 20. ↑8

Arte [5] Gary Glassman and Christine Le Goff, *Les secrets du Parthénon* (2008), Film documentary, Arte France ↑17

Ames [6] Andrew Marmery, *Tales from the Prep Room: The Ames Room* (12/21/2011), The Royal Institution ↑18

SU [7] Dr. Bergner Dr. Spickler, *Cylinder reflections*: presented during **Science Night @ SU - The Mathematics Behind Anamorphic Art**, Salisbury University (11/12/2011), `https://facultyfp.salisbury.edu/despickler/personal/ManualResources.html`, see pdf ↑20

[8] Phillip Kent, *What is Anamorphosis?*, `https://www.anamorphosis.com/what-is.html`. ↑