have to choose only one, not both at a time.

2.4. Implication: contrapositive, converse and inverse

Implication is a binary operation on propositions which corresponds to the phrase "if...then...". This operation is denoted using the symbol "⇒". So P ⇒ Q is read "if P then Q", or "P implies Q", "P is sufficient for Q", "Q if P", or "Q is necessary for P". P is called the **antecedent** (or premise), and Q is called the consequent (or conclusion). In particular when an implication p ⇒ q is identified as a theorem, it is customary to refer to p as the **hypothesis** and q as the **conclusion**.
The implication is *false only when the antecedent is true and the consequent is false*. It is true in all remaining cases. So the truth table of the implication is the following.

	P	Q	P ⇒ Q
Case 1	T	T	T
Case 2	T	F	**F**
Case 3	F	T	T
Case 4	F	F	T

Example: x is a real number. Consider the predicates P(x): "$x \geq 3$" and Q(x): "$x^2 \geq 9$". Then the predicate P(x) ⇒ Q(x) is defined by "$x \geq 3 \Rightarrow x^2 \geq 9$".
P(4) is true, Q(4) is true, P(4) ⇒ Q(4) is true.
P(2) is false, Q(2) is false, P(2) ⇒ Q(2) is true.
P(-4) is false, Q(-4) is true, P(-4) ⇒ Q(-4) is true.

Note:
The **converse** of an implication P⇒Q is the implication Q⇒P.
The **inverse** of an implication P⇒Q is the implication ⌉ P⇒⌉ Q.
The **contrapositive** of an implication P⇒Q is the implication ⌉ Q⇒⌉ P.

2.5. Equivalence

Equivalence is a binary operation on propositions such that P is equivalent to Q means that P is true whenever Q is true, and vice versa. This operation is denoted using the symbol "⇔". So P ⇔ Q is read "P is equivalent to Q", or "P iff (if and only if) Q", or "A is necessary and sufficient for B".
The equivalence P ⇔ Q is true when P and Q have the same truth value (either true or false); it is false in all other cases. So the truth table of the implication is the following.

	P	Q	P ⇔ Q
Case 1	T	T	T
Case 2	T	F	F
Case 3	F	T	F
Case 4	F	F	T

Note 1: The equivalence P ⇔ Q can also be defined in terms of conjunction and implication as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
Note 2: Two propositions are **logically equivalent** when they have the same truth tables.
Example: Using a truth table, it is easy to see that both an implication and its contrapositive are logically equivalent.
Exercise: Is p ⇒ q logically equivalent to q ⇒ p ?

2.6. Properties of logical connectors

a) Logical laws or tautologies

A compound predicate R which is true irrespective of the truth values of the composing

predicates is called a tautology.
Example:

	P	⌉ P	P ∨ ⌉ P
Case 1	T	F	T
Case 2	F	T	T

Exercise: By constructing their truth tables, show that the following propositions are logical laws:

- $(A \land B) \Rightarrow (A \lor B)$
- Commutative law of disjunction $(A \lor B) \Leftrightarrow (B \lor A)$
- Associative law of disjunction $[(A \lor B) \lor C] \Leftrightarrow [A \lor (B \lor C)]$
- First distributive law $[A \land (B \lor C)] \Leftrightarrow [(A \land B) \lor (A \land C)]$
- Second distributive law $[A \lor (B \land C)] \Leftrightarrow [(A \lor B) \land (A \lor C)]$
- $(A \Rightarrow B) \Leftrightarrow (\neg A \lor B)$
- De Morgan's first law $\neg(A \lor B) \Leftrightarrow (\neg A \land \neg B)$
- De Morgan's second law $\neg(A \land B) \Leftrightarrow (\neg A \lor \neg B)$

b) Incompatibles predicates

Two (compound) propositions are incompatible if their conjunction is false irrespective of the truth values of the composing predicates.
Example:

	P	⌉ P	P ∧ ⌉ P
Case 1	T	F	F
Case 2	F	T	F

Exercise: Show that the predicates "$x \leq 1$" and "$x \geq 2$" are incompatibles.

3- Mathematical quantifiers

3.1. Simple quantifiers

Consider the predicate P(x): "$x^2 + 2x - 1 = 0$". When we know the value of the variable x, the predicate becomes an assertion.
Example: P(0) is true and P(1) is true.
The predicate can also become an assertion if we add special phrases called **quantifiers**.

- **Universal quantifier**

Let us change the above predicate as "For every x, $x^2 - 2x + 1 = 0$" (by default x is a real number). That is an assertion since it is false. In symbolic form, we write

$$\text{“}\forall\, x, x^2 - 2x + 1 = 0\text{”}.$$

The symbol ∀ is the universal quantifier read "For every…," "For all…," "For each…,".

- **Existential quantifier**

Let us change the predicate as "There exists an x, such that $x^2 - 2x + 1 = 0$". That is an assertion since it is true (for $x = 1$). In symbolic form, we write

$$\text{“}\exists\, x \mid x^2 - 2x + 1 = 0\text{”}.$$

The symbol ∃ is the existential quantifier read "There exists…," "There is at least one...," or anything equivalent. The symbol | (or : or sometimes ∋) means "such as".

In the statement "$\exists\, x \mid x^2 - 2x + 1 = 0$", x=1 yields a true, but this does not prevent other

values of x to yield a true outcome (e.g. x=-1). If and only if a unique value of x satisfies that, we can use the symbol "∃! (there exists a unique…).
Example: "$\exists\ x \mid 2x - 1 = 0$" is true (the unique value is $1/2$).

Note: Sometimes the quantifier is not explicitly written down. In such a case, if the variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.

Example: "If x is greater than 1, then x^2 is greater than 1" can be written as "$\forall\ x$, if $x > 1$, then $x^2 > 1$".

3.2. Nested/multiple quantifiers

One can nest two or more quantifiers together. Consider a two-variable predicate P(x, y) where x and y are elements of sets A and B, respectively. Then the statement

$$\text{“}\forall\ y \in B, P(x, y)\text{”}$$

is again a predicate since its truth value is still dependent on the variable x. In contrast, the statement

$$\text{“}\forall\ x \in A, \forall\ y \in B, P(x, y)\text{”}$$

is an assertion. It is true when ALL elements of A and B satisfy P(x,y). Note that the quantified assertion

$$\text{“}\exists\ x \in A, \exists\ y \in B \mid P(x, y)\text{”}$$

is true when at least an element of A and at least an element of B satisfy P(x,y).

Examples:
1. The assertion "$\forall\ x \in R_+, \forall\ n \in N, (1 + x)^n \geq 1$" is true.
2. The assertion "$\exists\ x \in R, \exists\ y \in R \mid x + y = 4$" is true.

Exercise: Rewrite each statement using ∃, or ∀ as appropriate.
(a) Any non-negative real number has a non-negative square.
(b) For every positive number M, there is a positive number N such that
N < 1/M.
(c) No positive number x satisfies the equation f (x) = 5.

- **Rules on the use of nested quantifiers**

- We can permute two identical quantifiers
- We cannot permute two different quantifiers

Examples:
1. The following assertions are identical:
"$\forall\ x \in R_+, \forall\ n \in N, (1 + x)^n \geq 1$" and "$\forall\ n \in N, \forall\ x \in R_+, (1 + x)^n \geq 1$".
"$\exists\ x \in R, \exists\ y \in R \mid x + y = 4$" and "$\exists\ y \in R, \exists\ x \in R \mid x + y = 4$"
2. The following assertions are different:
"$\forall\ y \in R, \exists\ x \in R \mid x \leq y$" is true.
"$\exists\ x \in R \mid \forall\ y \in R, x \leq y$" is false (because R is not bounded from below).

3.3 Negation of quantified statements

Consider the statement

"Everyone in the class is awake".

Symbolically, let P(x): "x is awake" (x is anyone in the class). Then we can rewrite the statement as

$$“\forall\, x, P(x)”.$$

What is the condition for the statement to be false? At least a person should be asleep. Hence "Someone in the class is asleep" is the logical opposite of the first statement. Symbolically,

$$“\exists\, x \mid \urcorner\, P(x)”.$$

Hence we get

$$\urcorner\, [\forall\, x, P(x)] \Leftrightarrow [\exists\, x \mid \urcorner\, P(x)].$$

Or equivalently,

$$\urcorner\, [\exists\, x \mid \urcorner\, P(x)] \Leftrightarrow [\forall\, x, P(x)].$$

More generally, we negate a complex predicate with an initial quantifier by first changing ∀ to ∃ and ∃ to ∀ then negating the predicate.

Example (with nested quantifiers):

$\neg[(\forall x \in X)(\exists y \in Y)(\forall z \in Z)\, P(x, y, z)] \Leftrightarrow [(\exists x \in X)(\forall y \in Y)(\exists z \in Z)\, \neg P(x, y, z)]$.

Exercise: Rewrite each statement using ∃, or ∀ as appropriate.

(a) There exists a positive number x such that $x^2 = 5$.

(b) For every positive number M, there is a positive number N such that $N < 1/M$.

(c) No positive number x satisfies the equation $f(x) = 5$.

Exercise: Rewrite each of the following statements using ∃, or ∀ as appropriate then negate it.

(a) For every x in A, $f(x) > 5$.

(b) There exists a positive number y such that $0 < g(y) \leqslant 1$.

Exercise: Negate the following statements:

(a) $\forall x \in Z_+)\ (\exists y \in Z_+)\ (2y > x)$

(b) $(\exists x \in Z_+)\ (\forall y \in Z_+)\ (2y \leqslant x)$

4- Proof techniques

We discuss here a few methods of proof.

4.1. Direct proof

The direct proof assumes the hypotheses that are given and try to argue directly to let the conclusion follow. This is a well-suited approach when the hypotheses can be translated into algebraic expressions (equations or inequalities) that can be manipulated to get other algebraic expressions, which are useful in verifying the conclusion.

Example: For all integers m and n, prove that if m and n are odd, then m + n is even.

Proof structure: We are proving the implication $P \Rightarrow Q$.

- *Hypotheses.* P: "m is an odd integer", "n is an odd integer"

- *Conclusion.* Q: "m + n is an even integer".

- *Familiar definitions* (translate information into mathematical expressions): an integer n is even if $n = 2k$ for some integer k; an integer n is odd if $n = 2k + 1$ for some integer k.

- *Structure:* Assume the hypotheses held for arbitrarily m and n, and then write down equations that follow from the definition to get to the conclusion.

Proof. Assume m and n are arbitrary odd integers. Then m and n can be written in the form $m = 2a + 1$ and $n = 2b + 1$, where a and b are also integers.

Then $m + n = (2a + 1) + (2b + 1)$ (substitution)

$= 2a + 2b + 2$ (associative and commutative laws of addition)

$= 2(a + b + 1)$ (distributive law)

Since m+ n is twice another integer, namely, a + b +1, m+ n is an even integer.

Exercise: Prove the statement "For all integers m and n, if m is odd and n is even, then m + n is odd."

4.2. Contrapositive proof

If a direct proof of an assertion appears problematic, the next most natural strategy to try is a proof of the contrapositive. The **contrapositive proof** (or proof by contrapositive) is based on the equivalence

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

To prove that $P \Rightarrow Q$ is true, we simply assume $\neg Q$ and then prove that $\neg Q \Rightarrow \neg P$ is true.
Example: For all integers m and n, if mn is odd then so are m and n.
Proof structure. We have to prove that $(\forall m, n \in Z_+)$

$$(mn \text{ is odd}) \Rightarrow [(m \text{ is odd}) \wedge (n \text{ is odd})],$$

which is the same as to prove that

$$[(m \text{ is even}) \vee (n \text{ is even})] \Rightarrow (mn \text{ is even}).$$

The latter is evident.
Proof. Suppose that m and n are arbitrary odd integers. Then $m = 2a + 1$ and $n = 2b + 1$; where a and b are integers. Then

$$\begin{aligned} mn &= (2a + 1)(2b + 1) \text{ (substitution)} \\ &= 4ab + 2a + 2b + 1 \text{ (associative, commutative, and distributive laws)} \\ &= 2(2ab + a + b) + 1 \text{ (distributive law)} \end{aligned}$$

Since mn is twice an integer (namely, $2ab + a + b$) plus 1, mn is odd.
Exercise. For every integer n, if n^2 is even, then n is even.

4.3. Proof by contradiction

To prove by contradiction that a proposition is true, it suffices to show that it is not false, while to show that a proposition is false, it suffices to show that it is not true.
The proof by contradiction is based on the equivalence

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q).$$

To prove that $P \Rightarrow Q$ is true, it suffices to prove that $\neg P \vee Q$ is true or else that $P \wedge \neg Q$ is false.
Example:
Let $x, y \in R$ be positive. If $x^2 + y^2 = 25$ and $x \neq 3$, then $y \neq 4$.
Proof. In order to prove by contradiction we assume that $[(x^2+ y^2= 25) \wedge (x \neq 3)] \wedge (y = 4)$. Then $x^2+ y^2= x^2+ 16= 25$. Hence $(x = 3) \wedge (x \neq 3)$ which is a contradiction.
Exercise. Prove each of the statements is true:

- Let x and y be real numbers. If $5x + 25y = 1723$, then x or y is not an integer.
- For all real numbers x and y, if $35x + 14y = 253$, then x is not an integer or y is not an integer.
- For all positive real numbers a, b, and c, if $ab = c$, then $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.

4.4. Indirect proof

This is a particular case of proof by contradiction. It is also called **reductio ad absurdum** argument or reduction to absurdity, and is used to disprove a statement by showing that it would inevitably leads to a ridiculous result. To show something must be false, assume first that it is true, and show that this implies something which you know to be false. So at some point in the proof we must assume a hypothesis which later turns out to be false.
Example: Suppose that x be a positive number such that $\sin(x)=1$.Then $x \geqslant \pi /2$.
Proof. Suppose for sake of contradiction that $x<\pi/2$. Since x is positive, we thus have $0<x<\pi/2$. Since $\sin(x)$ is increasing for $0 <x<\pi/2$, and $\sin(0) = 0$ and $\sin(\pi/2) = 1$, we thus have $0<\sin(x)$

<1. But this contradicts the hypothesis that sin(x)=1. Hence $x \geqslant \pi/2$.

4.5. Proof by induction

Mathematical Induction is often used when one needs to prove statements of the form

$$(\forall n \in N)\ P(n)$$

or similar types of statements.
The principle of Mathematical Induction is as follows. If for a statement P (n)
(i) P (1) is true,
(ii) [P (n) ⇒ P (n + 1)] is true,
then $(\forall n \in N)$ P (n) is true.
Part (i) is called the base case; (ii) is called the induction step.

Example. Prove that $\forall n \in N$:
$1^2+ 2^2+ 3^2+ \cdots + n^2=n(n + 1)(2n + 1)/6$.
Solution.
Base case: n = 1, we get $1^2=1\cdot 2\cdot 3/6$ which is true.
Induction step: Suppose that the statement is true for n = k (k > 1). We have to prove that it is true for n = k + 1. So our assumption is
$1^2+ 2^2+ 3^2+ \cdots + k^2=k(k + 1)(2k + 1)/6$.
Therefore we have
$1^2+ 2^2+ 3^2+ \cdots + k^2+(k + 1)^2=k(k + 1)(2k + 1)/6 +(k + 1)^2=(k + 1)(k + 2)(2k + 3)/6$, which proves the statement for n = k + 1. By the principle of mathematical induction the statement is true for all $n \in N$.
Exercise. Let P(n) be the proposition
$1+ 2+ 3+ \cdots + n=n(n + 1)/2$.
Using induction proof, show that the proposition is true.

References

- T. Tao, Analysis 1, 3rd Ed, Springer and Industan book Agency, New Delhi (2015)
- S. Balac and F. Sturm, Analyse et algebre, 2nd Ed, Presses Polytechniques et Universitaires Romandes, Lausanne (2009)
- Vitali Liskevich, Analysis 1 : Lecture Notes 2013/2014, University of Bristol, UK

Recommended Textbooks

Steven R. Lay, Analysis with an Introduction to Proof, 5th Ed, Pearson Education Limited, Boston (2014)

Tutorial sheet:

Exercise 1.0.
Negate the following predicate: « E is a set containing an infinite number of elements »

Exercise 1.1. What is the negation of the statement "either X is true, or Y is true, but not both"?

Exercise 1.2. What is the negation of the statement "X is true if and only if Y is true"? (There may be multiple ways to phrase this negation).

Exercise 1.3. Suppose that you have shown that whenever X is true, then Y is true, and whenever X is false, then Y is false. Have you now demonstrated that X and Y are logically equivalent? Explain.

Exercise 1.4. Suppose that you have shown that whenever X is true, then Y is true, and whenever Y is false, then X is false. Have you now demonstrated that X is true if and only if Y is true? Explain.

Exercise 1.5. Suppose you know that X is true if and only if Y is true, and you know that Y is true if and only if Z is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

Exercise 1.6. Suppose you know that whenever X is true, then Y is true; that whenever Y is true, then Z is true; and whenever Z is true, then X is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

Exercise 1.7
x is a real number. Consider the predicates P(x): "x > 1" and Q(x): "x < 2". Then fill in the following truth tables:

	P	Q	$P \wedge Q$
$x \in]-\infty, 1]$			
$x \in]1,2[$			
$x \in [2, \infty]$			
$x \in]-\infty, 1] \cup [2, \infty]$			

	P	Q	$P \vee Q$
$x \in]-\infty, 1]$			
$x \in]1,2[$			
$x \in [2, \infty]$			
$x \in]-\infty, 1] \cup [2, \infty]$			

Exercise 1.8
Give the truth value of the following statement: "$\forall\, x, \sqrt{x^2} = x$".

Exercises 1-31 pages 35-37, Steven R. Lay, Analysis with an Introduction to Proof, 5th Ed, Pearson Education Limited, Boston (2014)