# TA Sessions : Business Cycle
Method of undetermined coefficient

Alaïs Martin-Baillon

Consumption Euler equation

$$c_t = \mathbb{E}_t \left[ -\frac{1}{\gamma}(r_t - \pi_{t+1}) + c_{t+1} \right]$$

New Keynesian Phillips curve

$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi(c_t - c_t^f) + \beta\,\mathbb{E}_t(\pi_{t+1})$$

Monetary policy rule

$$r_t = \phi_\pi \pi_t + v_t$$

Output gap

$$c_t - c_t^f = c_t - \frac{\psi+1}{\psi+\gamma}a_t$$

## 1 Monetary policy shock

Let us assume that the exogenous component of the interest rate, $v_t$, follows an AR(1) process

$$v_t = \rho v_{t-1} + \varepsilon_t^v$$

With $\rho \in [0,1]$. And $\varepsilon_t^v$ is IID white noise with mean zero.

A positive (negative) realization of $\varepsilon_t^v$ should be interpreted as a contractionary (expansionary) monetary policy shock, leading to a rise (decline) in the nominal interest rate, *given* inflation, and the output gap.

### Step 1 : Guess

The method on undetermined coefficient consists in guessing a functional form for the solution. We know from the simple monetary model that a solution of the model is to express the endogenous variables as a function of the structural shocks.

Guess that the solution takes the form $c_t = \psi_c \times v_t$ and $\pi_t = \psi_\pi \times v_t$

## Step 2 : Verify

### Step 2.1 : Substitute in the guess

$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi\psi_c v_t + \beta\,\mathbb{E}_t(\psi_\pi v_{t+1})$$

$$c_t = \mathbb{E}_t\left[-\frac{1}{\gamma}(\phi_\pi\psi_\pi v_t + v_t - \psi_\pi v_{t+1}) + \psi_c v_{t+1}\right]$$

### Step 2.2 : Use the law of motion for the monetary policy shock

If $v_t = \rho v_{t-1} + \varepsilon_t^v$ then $\mathbb{E}_t(v_{t+1}) = \rho v_t$

$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi\psi_c v_t + \beta\psi_\pi\rho v_t$$

$$c_t = \frac{-1}{\gamma}(\phi_\pi\psi_\pi v_t + v_t - \psi_\pi\rho v_t) + \psi_c\rho v_t$$

### Step 2.3 : Factorize

$$\pi_t = \left[\frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi\psi_c + \beta\psi_\pi\rho\right]v_t$$

$$c_t = \left[\frac{-1}{\gamma}(\phi_\pi\psi_\pi + 1 - \psi_\pi\rho) + \psi_c\rho\right]v_t$$

### Step 2.4 : Match the coefficients

$$\psi_\pi = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi\psi_c + \beta\psi_\pi\rho$$
$$\psi_c = \frac{-1}{\gamma}(\phi_\pi\psi_\pi + 1 - \psi_\pi\rho) + \psi_c\rho$$

### Step 2.5 : result, the guess is correct if and only if

$$\psi_\pi(1-\beta\rho) = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi\psi_c$$
$$\psi_c(1-\rho) = \frac{-1}{\gamma}(\phi_\pi\psi_\pi + 1 - \psi_\pi\rho)$$

$$\iff \psi_\pi = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda(1-\beta\rho)}\xi\psi_c$$
$$\psi_c = \frac{-1}{\gamma(1-\rho)}(\phi_\pi\ \psi_\pi + 1 - \psi_\pi\rho)$$

Using, $\kappa \equiv \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi$

$$\iff \psi_\pi = \frac{\kappa}{(1-\beta\rho)}\psi_c$$
$$\psi_c = \frac{-1}{\gamma(1-\rho)}(\phi_\pi\psi_\pi + 1 - \psi_\pi\rho)$$

**Step 2.5 : Solve**

$$\psi_c = \frac{-1}{\gamma(1-\rho)}(\frac{\kappa}{(1-\beta\rho)}\psi_c(\phi_\pi - \rho) + 1)$$
$$\iff \psi_c(1 + \frac{1}{\gamma(1-\rho)}(\frac{\kappa}{(1-\beta\rho)}(\phi_\pi - \rho)) = \frac{-1}{\gamma(1-\rho)}$$
$$\iff \psi_c = \frac{-1}{\gamma(1-\rho)}\frac{\gamma(1-\rho)(1-\beta\rho)}{\gamma(1-\rho)(1-\beta\rho) + \kappa(\phi_\pi - \rho)}$$
$$\iff \psi_c = -\frac{(1-\beta\rho)}{\gamma(1-\rho)(1-\beta\rho) + \kappa(\phi_\pi - \rho)}$$
$$\iff \psi_c = -\frac{1}{\gamma(1-\rho) + \frac{\kappa}{(1-\beta\rho)}(\phi_\pi - \rho)}$$

$$\psi_\pi = \frac{\kappa}{(1-\beta\rho)}\psi_c$$
$$\iff \psi_\pi = -\frac{\kappa}{(1-\beta\rho)}\frac{1}{\gamma(1-\rho) + \frac{\kappa}{(1-\beta\rho)}(\phi_\pi - \rho)}$$

A contractionary monetary policy shock causes a reduction in consumption and inflation. The larger $\phi_\pi$ the smaller the effect on output and inflation. Because, the central bank is counteracting the shock.

This result is intuitive: a contractionary monetary policy shock triggers a deflation at the cost of a recession (a decrease of output below potential). Conversely, a monetary policy expansion will increase output below potential at the cost of inflation.

The size of this "trade-off" between inflation/deflation and expansion/recession is given by the slope of the Phillips curve $\kappa$ : for one percentage increase in the output gap, inflation will increase by $\kappa$ percent. Both responses are increasing (in absolute value) with the elasticity of intertemporal substitution - since interest rates 'work' through this channel.

A contractionary monetary policy shock leads to a fall in consumption today by intertemporal substitution. By the labor supply decision, the household will accept a lower wage to work the same hours. This fall in demand has an asymmetric effect on the firms, of whom we have two types. Those who can set prices will reduce their prices, causing deflation (a fall in inflation), and by the Phillips curve a fall in average marginal cost. Deflation will imply a fall in the real interest rates, so we expect demand to expand partially from this source. Those firms who cannot adjust prices will instead demand less labor. The economy establishes at a new equilibrium whereby both labor and real wages have fallen, there is deflation, and a fall in output (gap) and consumption.

# 2 Productivity shock

Let us assume that the exogenous labor productivity, $a_t$, follows an AR(1) process

$$a_t = \rho a_{t-1} + \varepsilon_t^a$$

With $\rho \in [0, 1]$. And $\varepsilon_t^a$ is IID white noise with mean zero.

## Step 1 : Guess

Guess that the solution takes the form $c_t = \psi_c \times a_t$ and $\pi_t = \psi_\pi \times a_t$

## Step 2 : Verify

### Step 2.1 : Substitute in the guess

$$c_t = \mathbb{E}_t \left[ -\frac{1}{\gamma}(\phi_\pi \psi_\pi a_t - \psi_\pi a_{t+1}) + \psi_c a_{t+1} \right]$$
$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi(\psi_c a_t - \frac{\psi+1}{\psi+\gamma}a_t) + \beta\,\mathbb{E}_t(\psi_\pi a_{t+1})$$

### Step 2.2 : Use the law of motion for the productivity shock

If $a_t = \rho a_{t-1} + \varepsilon_t^a$ then $\mathbb{E}_t(a_{t+1}) = \rho a_t$

$$c_t = \mathbb{E}_t \left[ -\frac{1}{\gamma}(\phi_\pi \psi_\pi a_t - \psi_\pi \rho a_t) + \psi_c \rho a_t \right]$$
$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}\xi(\psi_c a_t - \frac{\psi+1}{\psi+\gamma}a_t) + \beta\,\mathbb{E}_t(\psi_\pi \rho a_t)$$

### Step 2.3 : Factorize

$$c_t = \left[ -\frac{1}{\gamma}\psi_\pi(\phi_\pi - \rho) + \psi_c \rho \right] \times a_t$$
$$\pi_t = \left[ \kappa(\psi_c - \frac{\psi+1}{\psi+\gamma}) + \beta\psi_\pi\rho \right] a_t$$

### Step 2.4 : Match the coefficients

$$\psi_c = -\frac{1}{\gamma}\psi_\pi(\phi_\pi - \rho) + \psi_c \rho$$
$$\psi_\pi = \kappa(\psi_c - \frac{\psi+1}{\psi+\gamma}) + \beta\psi_\pi\rho$$

### Step 2.5 : result, the guess is correct if and only if

$$\psi_c(1-\rho) = -\frac{1}{\gamma}\psi_\pi(\phi_\pi - \rho)$$
$$\iff \psi_c = -\frac{1}{\gamma(1-\rho)}\psi_\pi(\phi_\pi - \rho)$$

And,

$$\psi_\pi(1-\beta\rho) = \kappa(\psi_c - \frac{\psi+1}{\psi+\gamma})$$
$$\iff \psi_\pi = \frac{\kappa}{(1-\beta\rho)}(\psi_c - \frac{\psi+1}{\psi+\gamma})$$

**Step 2.5 : Solve**

$$\psi_\pi = \frac{\kappa}{(1-\beta\rho)}(\psi_c - \frac{\psi+1}{\psi+\gamma})$$
$$\psi_\pi = \frac{\kappa}{(1-\beta\rho)}(-\frac{1}{\gamma(1-\rho)}\psi_\pi(\phi_\pi-\rho) - \frac{\psi+1}{\psi+\gamma})$$
$$\psi_\pi\left(1+\frac{\kappa(\phi_\pi-\rho)}{(1-\beta\rho)\gamma(1-\rho)}\right) = \frac{-\kappa}{(1-\beta\rho)}\frac{\psi+1}{\psi+\gamma})$$
$$\psi_\pi = \frac{-\kappa}{(1-\beta\rho)}\frac{\psi+1}{\psi+\gamma} \times \frac{(1-\beta\rho)\gamma(1-\rho)}{(1-\beta\rho)\gamma(1-\rho)+\kappa(\phi_\pi-\rho)}$$
$$\psi_\pi = -\kappa\times\gamma(1-\rho)\times\frac{\psi+1}{\psi+\gamma}\times\frac{1}{(1-\beta\rho)\gamma(1-\rho)+\kappa(\phi_\pi-\rho)} \quad < 0$$

With $\frac{1}{(1-\beta\rho)\gamma(1-\rho)+\kappa(\phi_\pi-\rho)} = \Omega$

$$\psi_c = -\frac{1}{\gamma(1-\rho)}\psi_\pi(\phi_\pi-\rho)$$
$$\psi_c = -\frac{(\phi_\pi-\rho)}{\gamma(1-\rho)} - \kappa\gamma(1-\rho)\frac{\psi+1}{\psi+\gamma}\Omega$$
$$\psi_c = \frac{(\phi_\pi-\rho)}{\gamma(1-\rho)}\kappa\gamma(1-\rho)\frac{\psi+1}{\psi+\gamma}\Omega$$
$$\psi_c = (\phi_\pi-\rho)\kappa\frac{\psi+1}{\psi+\gamma}\Omega \quad > 0$$

The response of consumption to a productivity shock is positive and is increasing in $\kappa$ and $\phi_\pi$. $\kappa$ measures the degree of price stickiness ($\kappa$ is a decreasing function of price stickiness, a low $\kappa$ means a high degree of price stickiness). The response of inflation to a productivity shock is negative, the magnitude of the response is increasing in $\kappa$ and decreasing in $\phi_\pi$. When $\kappa$ tends to zero (price are very sticky), inflation does not react to productivity shock.

The increase in productivity implies a fall in marginal cost what implies that the labor demand will fall. Firms who can adjust prices will cut them. This will lead to deflation, therefore a fall in real interest rates and an increase in consumption today relative to tomorrow. It will imply a leftward shift in labor supply compared that obtained in the flexprice equilibrium (consumption increases by less). This explains why the output gap will be negative. Therefore, a positive technology shock generates deflation and a fall in the output gap. Note, however, that the actual level of output can increase, since potential output increases with technology.

After a technology shock, the equilibrium allocation no longer equals the first-best allocation. Namely, after a positive technology shock, output does not increase enough and thus the output gap is negative (due to price stickiness the real interest rate does not fall enough after a positive technology shock and therefore consumption does not increase enough after a positive technology shock). A higher coefficient on inflation in the monetary policy rule ($\psi_\pi$) yields a stronger response of output to a technology shock.

The fall in hours in response to a productivity shock has been the subject of a lively debate in the profession over the past few years.