Introduction to RBC models

TA Session 3

Alaïs Martin-Baillon

SciencesPo

February 2018

Objective of the third TA session

At the end of this session you should be able to :

- Understand the Taylor's theorem
- Do a Taylor approximation of a function in a given point
- Log-linearize any equations

Taylor's theorem :
Let $n \geqslant 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be (n+1) times differentiable at the point $a \in \mathbb{R}$. Then,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n)$$

So,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n)$$

With the Taylor serie at order n :

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

And $f = o(g)$ at a that means that $\frac{f(x)}{g(x)} \to 0$ when $x \to a$

What does it mean with words ?

By doing a Taylor expansion we are doing an approximation. Two functions are equal with a given error that we wish to be locally small ie that it tends toward zero.

Moreover, we want to know at wich rate this error tends toward zero, $(x-a)^n$ is a scale of comparaison, we know here that the error tends toward zero quicker that $(x-a)^n$.

Finally, the rate at which $(x-a)^n$ tends toward zero increases with n.

For instance, at the first order we are doing a linear approximation of the function and we approach $f(x)$ at a with $f(a) + f'(a)(x - a)$.

For instance, one can take the first-order Taylor approximation at the point zero of $f(x) = e^x$:

$$f(0) + f'(0)(x - 0) = e^0 + e^0(x - 0) = 1 + x$$

And do the same operation with $f(x) = \ln(1 + x)$

$$f(0) + f'(0)(x - 0) = ln(1 + 0) + \frac{1}{1 + 0}(x - 0) = x$$

Preliminary step

Definition

With X_t a variable and X it's value at its non-stochastic steady-state. Let's define :

$$\hat{X}_t \equiv ln(X_t) - ln(X)$$

$\hat{X}_t$ is the log-deviation of the value of the variable from its value in the non-stochastic steady state.

Remark : $X_t = Xe^{\hat{X}_t}$

Let's log-linearize these equations

Step 1 : when necessary, rewrite the equation to simplifiy the process

$$C_t^{-\gamma} = \mathbb{E}_t \left[\beta C_{t+1}^{-\gamma} (r_{t+1} + 1 - \delta) \right]$$
$$\iff C_t^{-\gamma} = \mathbb{E}_t \left[\beta C_{t+1}^{-\gamma} R_{t+1} \right]$$

With

$$R_{t+1} \equiv r_{t+1} + 1 - \delta$$

Let's log-linearize these equations

Step 2 : Express the endogenous variables in terms of log-deviations from the non-stochastic steady

$$
\begin{aligned}
(Ce^{\hat{C}_t})^{-\gamma} &= \mathbb{E}_t\left[\beta(Ce^{\hat{C}_t})^{-\gamma}Re^{\hat{R}_{t+1}}\right] \\
\iff \; C^{-\gamma}e^{-\gamma\hat{C}_t} &= R\beta C^{-\gamma}\,\mathbb{E}_t\left[e^{-\gamma\hat{C_{t+1}}+\hat{R}_{t+1}}\right]
\end{aligned}
$$

Step 3 : Use the fact that the equation also holds in the non-stochastic steady state

$$
C^{-\gamma} = \beta R C^{-\gamma}
$$

Then,

$$
e^{-\gamma\hat{C}_t} = \mathbb{E}_t\left[e^{-\gamma\hat{C}_{t+1}+\hat{R}_{t+1}}\right]
$$

Let's log-linearize these equations

Step 4 : Do a first-order Taylor approximation of the function on the left-hand side at the point zero and do a first-order Taylor approximation of the function on the right-hand side at the point zero. If there is an expectation operator in the equation, do the Taylor approximation of the function inside the expectation operator.

$$1 - \gamma \hat{C}_t = \mathbb{E}_t \left[-\gamma \hat{C}_{t+1} + \hat{R}_{t+1} \right]$$

Step 5 : Simplify the equation

$$\hat{C}_t = \mathbb{E}_t \left[\hat{C}_{t+1} - \frac{1}{\gamma} \hat{R}_{t+1} \right]$$