The History Representation of Heterogeneous Agent Models

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Introduction

- Heterogeneous agent models are becoming the standard tool in macroeconomics (Bewely-Aiyagari-Huggett models: HANK, Kaplan, Moll, Violante 2018; Auclert 2017 among many others)
- Hard to study with aggregate shocks (Krusell and Smith 1998; Den Haan 2010; Reiter 2009)
- Hard to think about optimal monetary policy Aiyagari 1995; Aiyagari McGrattan 1995; Acikgoz 2013; Nuno and Thomas 2017)
- Very hard to solve for optimal policy and aggregate shocks (Bhandari, Evans, Golosov, and Sargent 2017)

What we do

- Provide a representation (History Representation) to solve optimal policy in heterogeneous agent models with aggregate shocks
- Additionally, improve on current projection methods i. e Reiter 2009 algorithm.
- Embbed reduced history representations Challe and Ragot, 2016; Challe, Matheron, Ragot, Rubio-Ramirez 2017; Le Grand Ragot 2017
  - Apply to two examples
  - Standard KS model (to benchmark), the optimal provision of public goods in the business cycle.

The main idea

- Provide a finite state-space representation using a time-invariant partition in the space of history
  1. Explicit partition : truncation of idiosyncratic histories
  2. Implicit partition : using the distribution of wealth (as Reiter)
- Can Apply Lagrangian Approach (Marcet and Marimon 2011) to solve for optimal policies with aggregate shocks
- Can use perturbation methods (DYNARE) to solve the model

Literature

Models with Incomplete insurance markets: A new benchmark

("Bewley-Aiyagari-Huggett", "Heterogeneous agents", "liquidity constrained").

- Empirically relevant (Zeldes, 1989; Carroll 2001; Chodorow-Reich and Karabarbounis 2014; Heathcote, Storesletten, Violante 2009; Relative success for wealth Benhabib and Bisin 2016, among many others).
- Now solved with many additional frictions (Krusell, Mukuyama, Sahin 2010; McKay, Reis 2013; Ravn, Sterck 2015; Challe, Matheron, Ragot, Rubio-Ramirez 2015; Heathcote 2005; Kaplan Violante 2014; HANK, Kaplan, Moll, Violante 2015; Den Haan, Pontus, Riegler 2016; Farhi, Werning 2017)
- Hard to solve with aggregate shocks (Krusell and Smith 1998; Reiter 2009; Den Haan and coauthors 2010; Winberry 2017).
- Hard/Impossible to solve for optimal policies with aggregate shocks (Aiyagari, McGrattan 1998; Shin 2006; Davila, Hong, Krusell, Rios-Rul 2012; Acikgoz 2013; Nuno, Moll 2015; Ragot 2015; McKay, Reis 2016; Bhandari, Evans, Golosov, Sargent 2015, 2016; Challe 2017; Bilbiie, Ragot 2017).

Outline of the presentation

1. The Economic Problem
2. History Representation
3. Ramsey
4. Numerical example

1 - The Economic Problem

- Unit mass of agents, discount factor β, face uninsurable (time-varying) unemployment risk:

$$M_t = \begin{bmatrix} 1 - f_t & f_t \\ s_t & 1 - s_t \end{bmatrix} \quad (1)$$

- $e_t = 0, 1$. History denoted as $e^t = \{e_0, \ldots, e_t\} \in \mathcal{E}^{t+1}$. Solve

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^i\right) + \chi v(G_t) \right)$$

$$c_t^i + a_t^i = (1 + r_t) a_{t-1}^i + \left((1 - \tau_t) e_t^i + \mu (1 - e_t^i) \right) w_t$$

$$a_t^i \geq -\bar{a}$$

Production

The production function

$$Y_t = \quad A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}$$

Competition implies the factor pricing

$$\begin{aligned} \tilde{w}_t &= (1-\alpha)\, A_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha} \\ r_t &= \alpha A_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha-1} - \delta \end{aligned}$$

where $\tilde{r}_t$ is the before-tax return of capital.

$$w_t = (1 - \tau_t^L)\tilde{w}_t$$

Government

- The budget constraint of the government is

$$\tau_t = \frac{1 - L_t}{L_t}\mu$$

such that $L_t \left(1 - \tau_t\right) w_t + \left(1 - L_t\right) \mu w_t = w_t L_t$

- Public good financing

$$G_t \quad = \quad \tau_t^L \tilde{w}_t L_t$$

Markets

- The financial market equilibrium

$$\int a_t^i di = K_t$$

- Goods market equilibirum

$$\int c_t^i di + G_t + K_t = Y_t + K_{t-1}$$

Questions

Two simple questions to present the methodology

1. Dynamics of the economy after TFP shocks?
2. What is the optimal provision of the public good in the business cycle?

Recursive formulation

A the steady state

$$\begin{aligned} V(a,e) &= u(c) + v\left(G\right) + \mathbb{E}\beta\tilde{V}(a',e') \\ c + a' &= w\left(e\right) + (1+r)a \\ & \quad c,\, l \geq 0,\, a' \geq -\bar{a}, \end{aligned}$$

Foc

$$u'(c) = \beta(1+r)\mathbb{E}u'(c') + \nu$$

- Density $d\Gamma\left(a,e\right)$,
- Policy rules $c = g_c\left(a,e\right)$ and $a' = g_a\left(a,e\right)$
- Lagrange multipliers $\nu\left(a,e\right)$ where $a \in \mathbb{R}$ and $e = 0, 1$.

2 - History Representation

Denote as $\mathcal{P}$ a finite partition of the space of histories. At each date, any agent has one and exactly one history in $\mathcal{P}$. Examples :

1. $\mathcal{P} = (e, u)$ (Ravn and Sterck, 2015)
2. $\mathcal{P} = (ee, eu, ue, uu)$ (Challe and Ragot, 2016)
3. $\mathcal{P} = (e, eu, euu, euuu, uuuu)$ (Challe, Matheron, Ragot, Rubio-Ramirez 2017)
4. Truncation of length N : $h = (e_{-N+1}, \ldots, e_0)$: 2^N elements (Le Grand and Ragot, 2017).
5. Implicit partition (Reiter, 2009, see below.)

Transitions $\mathcal{P}$

The transition matrix derived from labor market transitions. For instance

- $\Pi_{t,\{e\},\{eu\}} = s_t$
- $\Pi_{t,\{eu\},\{e\}} = \Pi_{t,\{uu\},\{e\}} = f_t$,

Values by history

From the solution of the Bewley model

- Distribution of wealth $d\Gamma(a,e)$, $e \in \{0,1\}$
- Policy rules for saving and consumption $g_{a'}(a,e)$ and $g_c(a,e)$.

Deduce the conditional distribution of wealth : $d\Gamma^{\mathcal{P}}(a,h)$.

$$
\begin{aligned}
S_h &= \int_a d\Gamma^{\mathcal{P}}(a,h) \\
c_h &= \frac{\int_a g_c(a,h)\, d\Gamma^{\mathcal{P}}(a,h)}{S_h} \\
\tilde{a}_h &= \frac{\int_a a d\Gamma^{\mathcal{P}}(a,h)}{S_h} \\
a'_h &= \frac{\int_a g_a(a,h)\, d\Gamma^{\mathcal{P}}(a,h)}{S_h} \\
\nu_h &= \frac{\int_a \nu(a,h)\, d\Gamma^{\mathcal{P}}(a,h)}{S_h}
\end{aligned}
$$

Aggregation by history

- budget constraint $c_h + a'_h = (1+r)\,\tilde{a}_h + w\,(h)$

$$\tilde{a}'_{h'} = \sum_{h' \succeq h} \Pi_{h,h'} a'_h \frac{S_h}{S'_{h'}}$$

- Size by history $S'_{h'} = \sum_{h \preceq h'} \Pi_{h,h'} S_h$
- Euler equations. Can find ξ_h, $h \in \mathcal{P}$:

$$\xi_h u'\,(c_h) \quad = \quad \beta(1+r) \left(\sum_{h' \succeq h} \Pi_{h,h'} \xi_{h'} u'(c_{h'}) \right) + \nu_h$$

- Welfare $W = \sum_{h \in \mathcal{P}} S_h \eta_h u\,(c_h) + \beta W'$ where

$$\eta_h \equiv \frac{\int_{-\bar{a}}^{\infty} u\,(g_c\,(a,h))\, d\Gamma^{\mathcal{P}}\,(a,h)}{u\,(c_h)} \frac{1}{S_h}$$

Aggregation by history

$$
\begin{aligned}
c_h^b + a_h^{b\prime} &= (1+r)\,\tilde{a}_h^b + w(h) \\
\xi_h^b u'\left(c_h^b\right) &= \beta(1+r)\mathbb{E}_{h'}\left(\xi_{h'}^b u'\left(c_{h'}^b\right)\right) + \nu_h^b \\
\tilde{a}_h^{b\prime} &= \sum_{h'\succeq g} \Pi_{g,h} a_g^{b\prime} \frac{S_g}{S_h'} \\
K &= \sum_{h\in\mathcal{P}} S_h^b a_h^b \\
W &= \sum_{h\in\mathcal{P}} S_h \eta_h^b u'\left(c_h^b\right) + \beta W'
\end{aligned}
$$

Key idea

- We approximate the model by a finite number of histories
- We follow histories with relevant adjustment ξ_h and η_h.
- Easy to solve with aggregate shocks (in Dynare)

3 - Ramsey

$$\max_{\left(r_t,w_t,G_t,(a_{t,h},c_{t,h})_{h\in\mathcal{P}}\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\sum_{h\in\mathcal{P}}S_{t,h}\eta_{t,h}u(c_{t,h})+v\left(G_t\right)\right)\right],$$

$$G_t+K_{t-1}r_t+L_tw_t\leq K_{t-1}^{\alpha}L_t^{1-\alpha}-\delta K_{t-1}$$

for all $h\in\mathcal{P}$:

$$c_{t,h}+a_{t,h}\leq(1+r_t)\tilde{a}_{t-1,h}+w_t\left(h\right)$$

$$\tilde{a}_{t,h}=\sum_{h'\in\mathcal{P}}\frac{S_{t-1,h'}}{S_{t,h}}\Pi_{t,h',h}a_{t-1,h'}.$$

$$\xi_{h,t}u'\left(c_{h,t}\right)=\beta\mathbb{E}_t\left[(1+r_{t+1})\left(\sum_{h'\succeq h}\Pi_{h,h'}\xi_{h',t+1}u'(c_{h',t+1})\right)\right]+\nu_{t,h}$$

$$K_t=\sum_{h\in\mathcal{P}}S_{t,h}a_{t,h},\ L_t=\left(1-s_t\right)L_{t-1}+f_t\left(1-L_t\right),$$

A reformulation of the Ramsey

Using Marcet and Marimon techniques

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\sum_{h \in \mathcal{P}} S_{t,h} \left[\eta_{t,h} u(c_{t,h}) + \xi_{t,h} \Big((1+r_t)\Lambda_{t,h} - \lambda_{t,h} \Big) u'(c_{t,h}) \right] + v\left(G_t\right) \right.$$
$$\left. + \mu_t \left(F(A_{t-1}, L_t, s_t) - G - r_t K_{t-1} - w_t L_t \right) \right)$$

subject to

$$c_{t,h} + a_{t,h} \leq (1+r_t)\tilde{a}_{t-1,h} + w_t\left(h\right)$$
$$\tilde{a}_{t,h} = \sum_{h' \in \mathcal{P}} \frac{S_{t-1,h'}}{S_{t,h}} \Pi_{t,h',h} a_{t-1,h'}.$$
$$K_t = \sum_{h \in \mathcal{P}} S_{t,h} a_{t,h},\ L_t = \left(1 - s_t\right) L_{t-1} + f_t\left(1 - L_t\right),$$

First order conditions

Define the liquidity value

$$\psi_{t,h} \equiv \eta_{t,h} u'(c_{t,h}) + \xi_{t,h} \Big((1+r_t)\Lambda_{t,h} - \lambda_{t,h}\Big) u''(c_{t,h})$$

with

$$\Lambda_{t,h} \equiv \sum_{\hat{h} \in \mathcal{P}} \frac{S_{t-1,\hat{h}}}{S_{t,h}} \Pi_{t,\hat{h},h} \lambda_{t-1,\hat{h}}$$

The first order conditions are

$$\begin{aligned} \mu_t &= v'\left(G_t\right) \\ \mu_t L_{t-1} &= \sum_{h \in \mathcal{P}} S_{t,h} \tilde{w}_{t,h} \psi_{t,h} \\ \psi_{t,h} &= \beta \mathbb{E}_t (1 + r_{t+1}) \sum_{h' \in \mathcal{P}} \Pi_{t,h,h'} \psi_{t+1,h'}, \text{for } h \neq h_{cc} \end{aligned}$$

Implicit partition

Connection with Reiter algorithm. Consider a partition B_h in the space of wealth

$$\begin{cases} \cup_{h=1,\dots,H} B_h = [-\bar{a}, a^{max}], & \\ B_h \cap B_{h'} = \emptyset & \text{for all } h, h' \in 1, \dots, H. \end{cases}$$

- It is an implicit partition in the space of histories.
- From the steady-state solution we can deduce $\Pi_{h,h'}$

Time-varying idiosyncratic risk

$$
\begin{aligned}
F_{t,h,h} &= \quad \phi_h S_{t,h} M_t\,(0,0) && \text{if } h = 1,\ldots,H \\
F_{t,h,h-1} &= (1-\phi_h)\, S_{t,h} M_t\,(0,0) && \text{if } h = 2,\ldots,H \\
F_{t,h,h+H} &= \quad \phi_h S_{t,h} M_t\,(0,1) && \text{if } h = 1,\ldots,H \\
F_{t,h,h-1+H} &= (1-\phi_h)\, S_{t,h} M_t\,(0,1) && \text{if } h = 1,\ldots,H \\
F_{t,h,h} &= \quad \phi_h S_{t,h} M_t\,(1,1) && \text{if } h = H+1,\ldots,2H \\
F_{t,h,h+1} &= (1-\phi_h)\, S_{t,h} M_t\,(1,1) && \text{if } h = H+1,\ldots,2H-1 \\
F_{t,h,h-H} &= \quad \phi_h S_{t,h} M_t\,(1,0) && \text{if } h = H+1,\ldots,2H \\
F_{t,h,h+1-H} &= (1-\phi_h)\, S_{t,h} M_t\,(1,0) && \text{if } h = H+1,\ldots,2H-1
\end{aligned}
$$

Discussion

Advantage compared to Reiter's algorithm

1. **Small number of agents.** Using the quantities $(\xi_h)_{h=1,\ldots,H}$ and $(\eta_h)_{h=1,\ldots,H}$
2. **Computation of optimal policies.** : new range of problems
3. **Time-varying idiosyncratic risk.** : easy to compute

4 - Numerical example

From Den Haan 2010

Table 1 : Parameter values

Parameter	Description	Value
β	Discount Factor	0.96
α	Capital share	0.36
δ	Depreciation rate	0.1
μ	UI replacement rate	0.1
π_{01}	U to E probability	0.5
π_{10}	E to U probability	0.038
ρ_A	TFP persistence	0.0859
σ_A	Sd. TFP innov.	0.014
χ	Pref. pub good	0

Partition

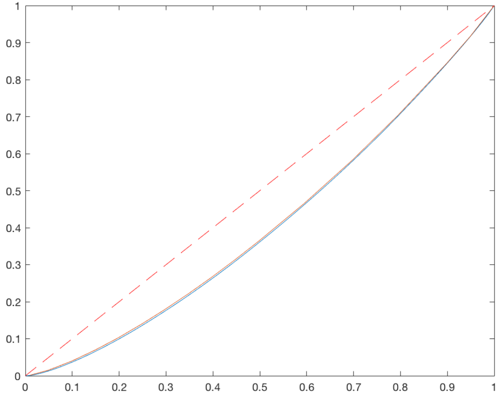
2H	6	22	34	64	200
$sd\,(\xi)(\%)$	5.72	2.43	1.54	0.95	0.79
$sd(\Gamma)(\%)$	8.48	3.93	2.89	1.73	0.68
Gini	15	19	19	19	19

Benchamrk : 34 agents (17 employed and 17 unemployed)

[8.10^{-4},5,10,15,20,25,30,35,40,45,50,60,70,80,90,95];

The initial number is the fraction of credit constrained agents.

Lorenz curve



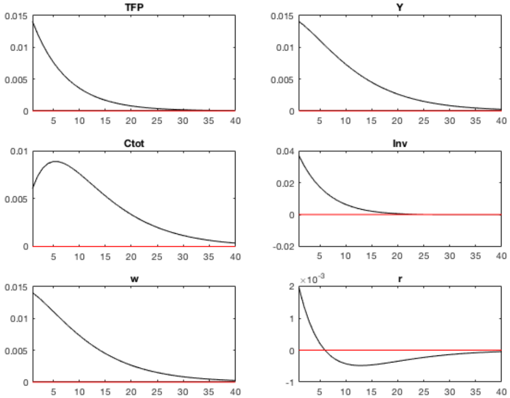
Gini index is 0.19 in both case.

Dynamics

Table 2 : Business Cycle Statistics and Comparison

Variable	Sd (relative to Sd(Y))			Correlation with Y		
	Model	KS	Reiter	Model	KS	Reiter
output (%, base)	1.32	1.32	1.32	1	1	1
Consumption	0.49	0.49	0.50	0.91	0.91	0.92
Investment	2.64	2.67	2.64	0.98	0.98	0.98
Real wage	1	1	1	1	1	1
Real interest rate	0.15	0.15	0.15	0.90	0.90	0.90

IRF



Undistinguishable from Reiter/Winberry.

Provision of a public good in the business cycle

Now $\chi = 0.082$ and log case, we find

- $\tau_L = 10\%$
- $G/GDP = 4\%$

Steady State

Table : Steady-state convergence

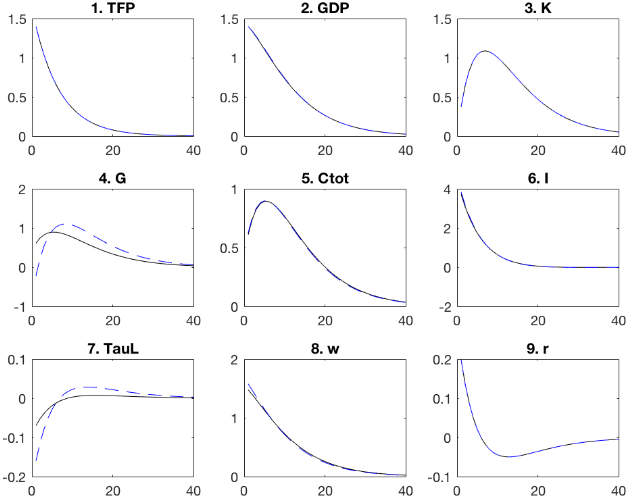
H	358	394	976
G	0.088	0.091	0.091
τ^L	0.0965	0.100	0.100
Sd(ξ)	0.015	0.014	0.014
sd(η)	0.003	0.003	0.003

Dynamics

Table : Business Cycle Statistics of Ramsey allocation

Variable	Sd (relative to Sd(Y))			Correlation with Y		
H	CM	4	22	CM	4	22
output (%, base)	1.32	1.33	1.33	1	1	1
Consumption	0.50	0.50	0.50	0.91	0.92	0.92
Investment	2.67	2.74	2.79	0.98	0.97	0.97
Public spend. G	0.05	0.05	0.05	0.91	-0.18	-0.12
Real wage	1.05	1.12	1.12	1	1	1
Int. rate (bef. tax)	0.15	0.15	0.15	0.90	0.89	0.89
Labor tax	0.05	0.12	0.12	-0.94	-0.91	-0.91

IRF after a TFP shock



Conclusion

- We construct a History-Representation of heterogeneous-agent models
- Increase the efficiency of the algorithm to solve models with aggregate shocks and perturbation methods
- Solve for optimal policies : fiscal, monetary...
- Bayesian Estimation techniques