Reiter's Projection and Perturbation Algorithm Applied to a Search-and-Matching Model

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05/03/2018

Sciences Po and LIEPP

Road map

Outline

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- A typical Heterogeneous agents model
- The Krusell-Smith algorithm
- Reiter's projection and perturbation approach
- Application to a dynamic search-and-matching model

A typical heterogeneous agents model

Model

Households with idiosyncratic labor income shocks:

- continuum of utility-maximizing households indexed by $j \in [0,1]$:

$$\max_{\{c_{jt}\}_{t=0}^{\infty}} \Big(\mathbb{E} \sum_{t=0}^{+\infty} \beta^t \frac{c_{jt}^{1-\sigma} - 1}{1-\sigma} \Big)$$

  such that:

$$c_{jt} + k_{jt+1} - (1-\delta) k_{jt} = y_{jt}$$

- inelastic labor supply l, shock ε_{jt} independent across households, follows 2-state Markov process within households:

$$\varepsilon_{jt} \in \{\varepsilon_0 = 0, \varepsilon_1 = 1\}$$

- labor earnings:

$$y_{jt} = \begin{cases} w_t l \text{ if } \varepsilon_{jt} = 1 \text{ (employed)} \\ 0 \text{ if } \varepsilon_{jt} = 0 \text{ (unemployed)} \end{cases}$$

Model

Representative Firm:

- Production function:

$$Y_t = e^{z_t} K_t^{\alpha} L^{1-\alpha}$$

  z_t aggregate productivity shock, K_t aggregate capital shock, L aggregate labor supply, α capital share.

- AR(1) process for the TFP:

$$z_{t+1} = \rho_z z_t + \sigma_z \omega_{t+1}$$

  with $\omega_{t+1} \sim \mathcal{N}(0, 1)$

Model

Incomplete Market Arrangement:

- Factor prices:

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L^{1-\alpha} - \delta$$

$$w_t = (1-\alpha) e^{z_t} K_t^{\alpha} L^{-\alpha}$$

- Borrowing constraint:

$$k \in [0, \infty)$$

Model

Recursive competitive equilibrium

- Aggregate state (Γ, z) with Γ the measure of consumers over holdings of capital and employment status
- Law of motion $\Gamma' = H(\Gamma, z, z')$
- State variables $(k, \varepsilon; \Gamma, z)$

Households optimization problem:

$$v(k, \varepsilon; \Gamma, z) = \max_{c} \left(U(c) + \beta \, \mathbb{E} \left[v(k', \varepsilon'; \Gamma', z') | z, \varepsilon \right] \right)$$

such that

$$c + k' = r(K, L, z)k + w(K, L, z)l\varepsilon + (1 - \delta)k$$

$$\Gamma' = H(\Gamma, z, z')$$

$$k' \geq 0$$

Recursive competitive equilibrium

A recursive competitive equilibrium is a law of motion H, a pair of policy rule (v, f) with $k' = f(k, \varepsilon; \Gamma, z)$, and a pricing function such that

- (v, f) solves the consumer problem
- r and w are competitive
- H is generated by f

Krusell-Smith Algorithm

Krusell-Smith Algorithm

Step 1: dimension reduction

- only the first N moments of Γ are relevant for the pricing decision $\mathbf{m} = (\mathbf{m_1}, \mathbf{m_2}, ...)$
- Approximate law of motion $m' = H(\mathbf{m}, z, z'|\theta_N)$
- Approximate policy rule $k' = f(k, \varepsilon; \Gamma, z|\theta_N)$

Step 2: Monte-Carlo over H

Algorithm

1. Solve for the households optimization problem, holding $H(\mathbf{m}, z, z'|\theta_N^{i-1})$ fixed
2. Simulate an economy for large number of period using the policy rule $f(k, \varepsilon; \Gamma, z|\theta_N^{i-1})$
3. Use the simulated data to update $H(\mathbf{m}, z, z'|\theta_N^{i})$
4. Compare $d(\theta_N^{i-1}, \theta_N^{i})$ and accuracy check for the forecasting rule.

Example Krusell-Smith Algorithm

- $\mathbf{m}' = H(\mathbf{m}, z, z' | \theta_N^i)$ with $N = 1$ (the mean of Γ is enough for making accurate forecasts)

$$\log(\bar{k}') = \begin{cases} a_{0i} + a_{1i} \log(\bar{k}) \text{ if } z = z_g \\ b_{0i} + b_{1i} \log(\bar{k}) \text{ if } z = z_b \end{cases}$$

- Convergence reached in 2 hours on my laptop

Reiter's projection and perturbation method

General Idea

- Circumvent the need for finding H by Monte-Carlo
- **Projection**: finite representation of the infinite dimensional problem by using an histogram to approximate Γ
- **Perturbation**: solve for a steady-state of the finite model and use perturbation method around the steady-state

Reiter (2009) is Krusell and Smith (1998) with 2 modifications:

- **stochastic tax rate** following an AR(1) process:

$$\tau_{t+1} - \tau^* = \rho_t(\tau_t - \tau^*) + \varepsilon_{\tau,t+1}$$

  The government taxes end-of-the period capital k_{t-1}. **Lump sum** redistribution to households at the beginning of period t. **Balanced budget** at every period.

- **Continuous** distribution of idiosyncratic shocks ξ_{it}, **i.i.d**, with pdf $f_\xi(.)$ and cdf $F_\xi(.)$. Normalization:

$$\mathbb{E}_t[\xi_{it}] = \mathbb{E}_{t-1}[\xi_{it}] = 1$$

Equilibrium

State variables

$$\Omega_t = (Z_t, \tau_t, \Psi_{k,t-1}(.))$$

with

- Z_t the aggregate productivity variable in period t
- τ_t the tax rate in period t
- $\Psi_{k,t-1}(.)$ the cross-sectional distribution of capital holdings inherited from period $t-1$

Equilibrium

Definition An equilibrium consists in:

- a consumption function $C(\chi, \Omega_t)$ with χ after-transfer disposable income in period t

$$\chi_{it} = (1 + r_t)k_{t-1} + w_t \xi_{it} + T_t$$

- a stochastic process of cross-sectional distribution $\Psi_{k,t}(.)$
- a process of lump sum transfers T_t

such that:

- $C(\chi, \Omega_t)$ satisfies the Euler equation
- $\Psi_{k,t}(.)$ is consistent with the dynamic equation implied by the Euler equation
- Transfers satisfy the balanced budget condition

(Reiter, 2009) 3-step approach

1. Provide a finite representation of the model

Replace infinite-dimensional objects by discrete counterparts to represent the dynamic system as

$$F(\mathbf{X_t}, \mathbf{X_{t-1}}, \eta_\mathbf{t}, \varepsilon_\mathbf{t}) = \mathbf{0}$$

with $\dim(\mathbf{X_t}) = \dim(\mathbf{X_{t-1}}) = (n_X \times 1)$, $\dim(\eta_\mathbf{t}) = (n_\eta \times 1))$, $\dim(\varepsilon_\mathbf{t}) = (n_\varepsilon \times 1)$

2. Solve for a steady-state of the discrete model

$$F(\mathbf{X}^\star, \mathbf{X}^\star, \mathbf{0}, \mathbf{0}) = \mathbf{0}$$

Define the steady-state as the state of the system when there is no aggregate uncertainty ($\varepsilon_\mathbf{t} = 0$) and no expectation errors ($\eta_\mathbf{t} = 0$)

3. Linearize F around its steady-state and use a rational expectation solver to solve for $\eta_\mathbf{t}$

$$F_1(\mathbf{X_t} - \mathbf{X_{ss}}) + F_2(\mathbf{X_{t-1}} - \mathbf{X_{ss}}) + F_3\eta_\mathbf{t} + F_4\varepsilon_\mathbf{t} = \mathbf{0}$$

$F_1 = \frac{\partial F}{\partial \mathbf{X_t}}|\mathbf{X_{ss}}$, $F_2 = \frac{\partial F}{\partial \mathbf{X_{t-1}}}|\mathbf{X_{ss}}$, $F_3 = \frac{\partial F}{\partial \eta_\mathbf{t}}|\mathbf{X_{ss}}$, $F_4 = \frac{\partial F}{\partial \varepsilon_\mathbf{t}}|\mathbf{X_{ss}}$

with $\dim(F_1) = \dim(F_2) = (n_X \times n_X)$, $\dim(F_3) = (n_X \times n_\eta)$, $\dim(F_4) = (n_X \times n_\varepsilon)$

Discretezing the model

Savings function $K(\chi, \Omega_t) = \chi - C(\chi, \Omega_t))$

- Approximate $K(\chi, \Omega_t))$ by $n_p + 1$ points collected in $\mathbf{s_t}$.
- Use the collocation method: the Euler equation has to be exactly satisfied at the n_p knots points $[\chi_{t,1}, ..., \chi_{t,n_p}]$.
- Knots points are chosen such that the borrowing constraint is not binding (Euler equation holds with equality)

Discretezing the model

Euler equation: For $i = 0, 1, ..., n_p$:

$$U'(\hat{C}(\chi_{t-1,i}; \mathbf{s_{t-1}})) = \beta \sum_{j=1}^{n_\zeta} w_j^\zeta \left[(1 + r(\mathbf{p_{t-1}}, Z_t) U'(\hat{C}(\hat{X}_{ij}, \mathbf{s_t})\right] + \eta_{i,j}^c \tag{1}$$

with the approximate disposable income for an individual i facing an idiosyncratic shock j

$$\hat{X}_{ij} = (1 + r(\mathbf{p_{t-1}}, Z_t))(K(\chi_{t-1,i}; \mathbf{s_{t-1}})) + w(\mathbf{p_{t-1}}, Z_t) + T_t \tag{2}$$

- ω_j^ζ Gaussian quadrature weights, to approximate the expectation with respect to idiosyncratic risks
- The vector $\mathbf{s_t}$ is a function of the aggregate state vector Ω_t. The dependency of $\mathbf{s_t}$ on Ω_t is solved at the perturbation stage
- Directly solving for $\mathbf{s_t}(\Omega_t)$ would require knowing the law of motion for Ω_t (back to the Krusell-Smith algorithm)

Discretezing the model

Off-knots values are calculated by interpolation:

$$\hat{K}(\chi_{t,i}, \mathbf{s_t}) = \begin{cases} \underline{k} & \text{for } \chi \leq \mathcal{X}_t \\ CSI(\chi_{t,i}, \mathbf{s_t}) & \text{for } \mathcal{X}_t < \chi \leq \chi_{t,n_p} \\ k_{t,n_p} + CSI'(\chi_{t,i})(\chi_{t,i} - \chi_{t,n_p}) & \text{for } \chi > \chi_{t,n_p} \end{cases} \tag{3}$$

- $\underline{k}$ the borrowing constraint ($= 0$ in Krusell and Smith (1998))
- $\mathcal{X}_t$ the value for which the borrowing constraint starts to be binding
- $n_p + 1$ time-dependent knot points $[\chi_{t,0}, \chi_{t,1}, ..., \chi_{t,n_p}]$ $\chi_{t,0} = \mathcal{X}_t$ and $\chi_{t,0} = \mathcal{X}_t + X_i,\ i = 1, ..., n_p$
- "CSI" stands for a cubic spline interpolation using the points $\mathbf{s_t}$
- k_{t,n_p} is the capital invested when disposable income is $\chi_{t,n-p}$

Discretezing the model

Wealth Distribution: approximate the cross-sectional distribution of capital $\Psi_{k,t}(.)$

Summarize the cdf using a vector $\mathbf{p_t}$ of n_d points with

$$p_t^i = \Psi_t(\kappa_i) - \Psi_t(\kappa_i - 1)$$

, $i = 1, ..., n_d$

Assume constant density within each interval $[\kappa_{i-1}, \kappa_i]$

Linear dynamic equation:

$$\mathbf{p_t} = \Pi(\hat{\mathbf{\Omega}}_\mathbf{t})\mathbf{p_{t-1}}$$

with $\hat{\mathbf{\Omega}}_\mathbf{t} = (\mathbf{p_{t-1}}, Z_t, \tau_t)$

Discretezing the model

Expectation errors

Expectation errors result from the aggregate shock (Gaussian quadrature for idiosyncratic shock)

Replace the expectation operator $\mathbb{E}$ by defining $\mathbb{E}_{\hat{\mathbf{\Omega}}_{\mathbf{t}}}[x_t] = x_t + \eta_t$

$n_p + 1$ expectation errors (Euler equation solved by collocation on the grid with $n_p + 1$ points)

Solving for a steady-state

Set $\eta^* = \mathbf{0}$, $\varepsilon^* = \mathbf{0}$; Solving a **one-dimensional** fixed-point:

1. Guess an aggregate capital $K^* \rightarrow$ determines r^*, w^* and T^* $\rightarrow$ solve for $\mathbf{s}^*$ using the Euler equation For $i = 0, 1, ..., n_p$:

$$U'(\hat{C}(\chi_i^*; \mathbf{s}^*)) = \beta \sum_{j=1}^{n_\zeta} w_j^\zeta \left[(1 + r(K^*, Z^*) U'(\hat{C}(\hat{X}_{ij}^*, \mathbf{s}^*) \right] \quad (4)$$

2. Given r^*, T^* and $\mathbf{s}^*$ find $\mathbf{p}^*$:

$$\mathbf{p}^* = \Pi^*(r^*, T^*, s^*)\mathbf{p}^*$$

3. Check whether the guess K^* is consistent with the one implied by $\mathbf{p}^*$

Linearization and solving for rational expectation errors

Define the column vector $\mathbf{X_t} = (\mathbf{s_t}, \mathbf{p_t}, Z_t, \tau_t, T_t)'$ with $(n_p + 1) + n_d + 3 = n_p + n_d + 4$ elements.

Numerical differentiation of $F(\mathbf{X_t}, \mathbf{X_{t-1}}, \eta_\mathbf{t}, \varepsilon_\mathbf{t})$ around its non-stochastic steady-state to obtain F_1, F_2, F_3, F_4 evaluated at $\mathbf{X_t} = \mathbf{X_{t-1}} = \mathbf{X}^*$ $\eta^* = \mathbf{0}$ and $\varepsilon^* = \mathbf{0}$.

The linearized system can be written into **Sims (2002)** canonical form:

$$\Gamma_0 \mathbf{y_t} = \Gamma_1 \mathbf{y_{t-1}} + \mathbf{C} + \Psi \mathbf{z_t} + \Pi \eta_\mathbf{t}$$

with $\mathbf{y_t} = \mathbf{X_t} - \mathbf{X_{ss}}$, $\Gamma_0 = -F_1$, $\Gamma_1 = -F_2$, $\mathbf{C} = \mathbf{0}, \Psi = F_3, \Phi = F_4$

Linearization and solving for rational expectation errors

Outcome of Sims (2002) gensys solver: matrix A and B such that:

$$\mathbf{y_t} = A\mathbf{y_{t-1}} + B\varepsilon_\mathbf{t}$$

Projection and Perturbation in a Search-and-Matching Model

Countercyclical Left Skewness of Income Shocks

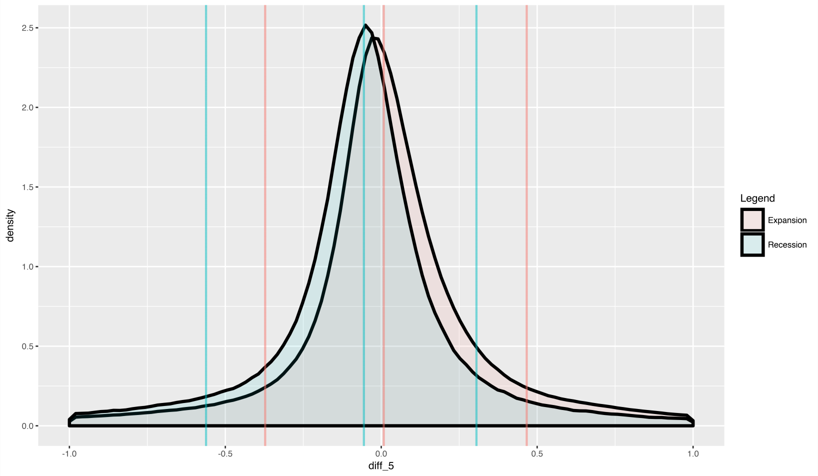

Figure 1: Distribution of income shocks: recessions versus expansions

Countercyclical Left Skewness of Income Shocks

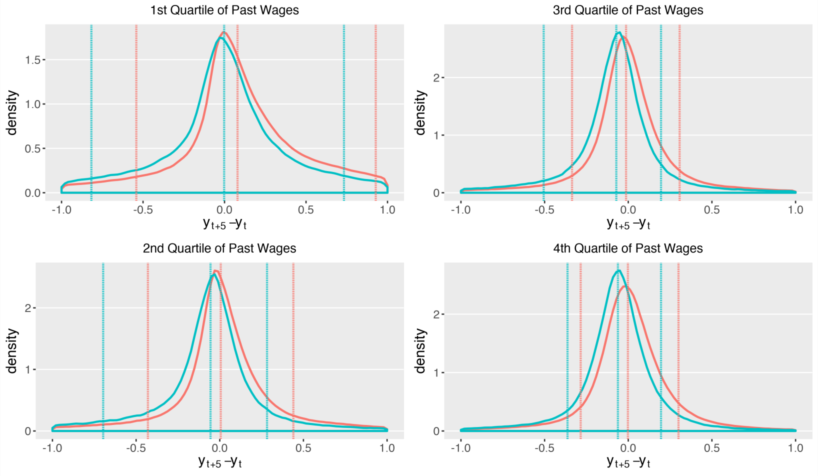

Figure 2: Distribution of income shocks: recessions versus expansions

Our Contribution

Main Question

What are the impacts of payroll taxation on labor income risks along the business cycle?

Contribution

- Build a dynamic $\underbrace{\text{history-dependent}}_{\text{Technical contribution}}$ model with non-linear taxes on wages, frictional unemployment and heterogeneous workers
- Counter-factual: "flat-tax"

Literature

Empirical Literature

- Guvenen, Ozkan, and Song (2014) Busch, Domeij, Guvenen, and Madera (2015)

Costs of Business Cycle

- Lucas Jr (2003) Gali, Gertler, and Lopez-Salido (2007)
- Clark and Oswald (1994) Wolfers (2003) Clark, Diener, Georgellis, and Lucas (2008) Aghion, Akcigit, Deaton, and Roulet (2015)

Optimal Labor Taxation

- Mirrlees (1971), Saez (2001), Kleven, Kreiner, and Saez (2009)

Taxation in Search-and-Matching Models

- Chéron, Hairault, and Langot (2008) Carbonnier et al. (2014), Breda, Haywood, and Haomin (2016)

Model

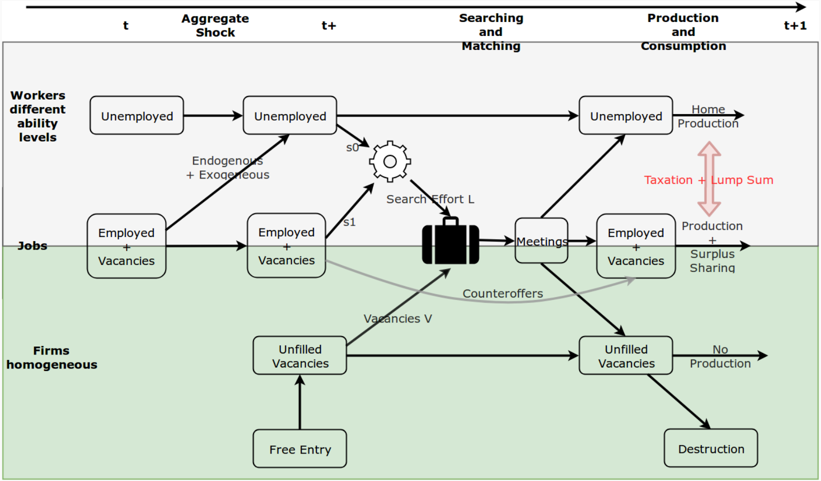

Figure 3: Timing of events

Model: Stochastic Equilibrium

Finding an equilibrium involves solving a system of "coupled" fixed point:

$$\begin{cases} \big(\underbrace{\triangle(x,w,z,h(.))}_{\text{Worker surplus}}, \underbrace{S(x,w,z,h(.))}_{\text{Joint surplus}}\big) = \Phi(\triangle(x,w,z,h(.)), S(x,w,z,h(.)), h(.)|h(.)) \\ \underbrace{h(.)}_{\text{Distribution of Employment}} = \Gamma(h(.)|\triangle(x,w,z,h(.)), S(x,w,z,h(.))) \end{cases}$$

- Similar to a mean field game with "common noise"
- Coupling through the probability of a meeting $\lambda(z,h)$

Model

Model: Steady-state

Fix $z = z^*$

$\approx$ Mean field game without common noise:

$$\begin{cases} \big(\underbrace{\triangle(x,w,z^*,h^*(.))}_{\text{Worker surplus}}, \underbrace{S(x,w,z^*,h^*(.))}_{\text{Joint surplus}}\big) \\ = \Phi^*(\triangle(x,w,z^*,h^*(.)), S(x,w,z^*,h^*(.)), h^*(.)|h^*(.)) \\ \qquad \underbrace{h^*(.)}_{\text{Distribution of Employment}} \qquad = \Gamma^*(h^*(.)|\triangle(x,w,z^*,h^*(.)), S(x,w,z^*,h^*(.))) \end{cases}$$

Discussion existence of a steady-state

Resolution method

We use the 3-step (Reiter, 2009) method:

1. Provide a finite representation of the model

Replace infinite dimensional (S, $\triangle$, h) objects by discrete value on grids: $F(\mathbf{X_t}, \mathbf{X_{t-1}}, \eta_\mathbf{t}, \varepsilon_\mathbf{t}) \to$ **Linear interpolation** for S, $\triangle$ and h: $\mathbf{X_t}$ contains values on grid $(S_{ij}, \triangle_{ij}, h_k)_t$ + aggregates at time t.

2. Solve for a steady-state of the discrete model

- Solve for S and $\triangle$ holding fixed h
- Solve for h holding fixed S and $\triangle$

3. Linearize F around its steady-state and use a rational expectation solver

$$F_1(\mathbf{X_t} - \mathbf{X_{ss}}) + F_2(\mathbf{X_{t-1}} - \mathbf{X_{ss}}) + F_3\eta_\mathbf{t} + F_4\varepsilon_\mathbf{t} = \mathbf{0}$$

Flat tax counter-factual

Flat tax counter-factual: Idea

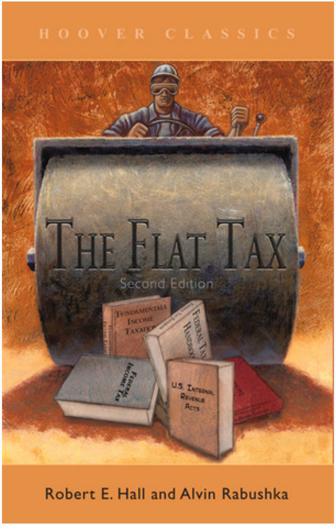

Figure 4: The Hall–Rabushka flat tax (1985)

Flat tax counter-factual: experiment

1. Estimate the model using Italian data
2. Find a flat tax such that the government revenue = constant
3. Simulate "step function" and "flat" tax economies and compare

Flat tax counter-factual: experiment

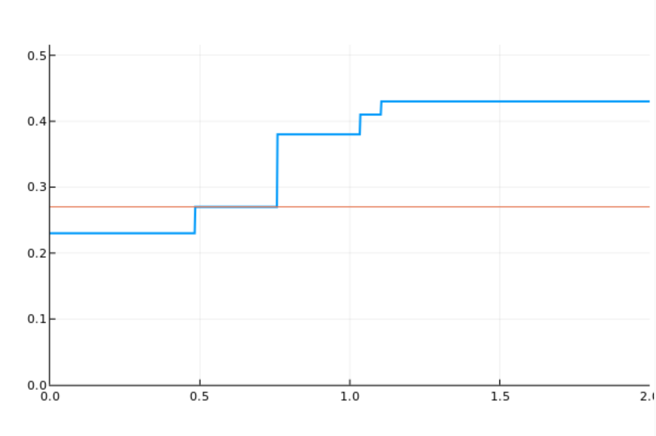

Figure 5: Marginal tax rate: step function versus flat tax

Flat tax counter-factual: Amplification

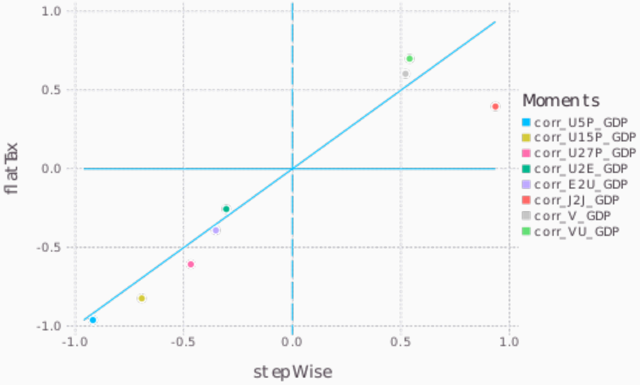

Figure 6: Amplification

Flat tax counter-factual: Levels

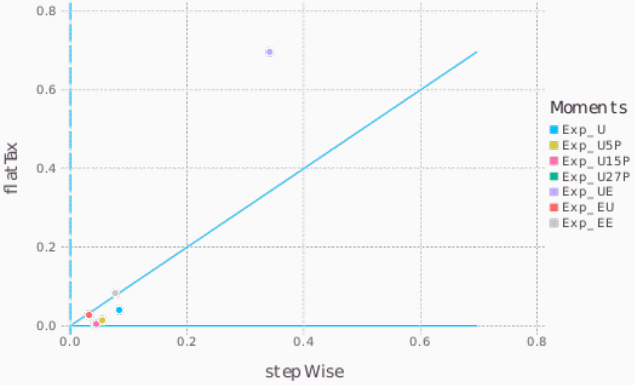

Figure 7: Levels

Flat tax counter-factual: Volatility

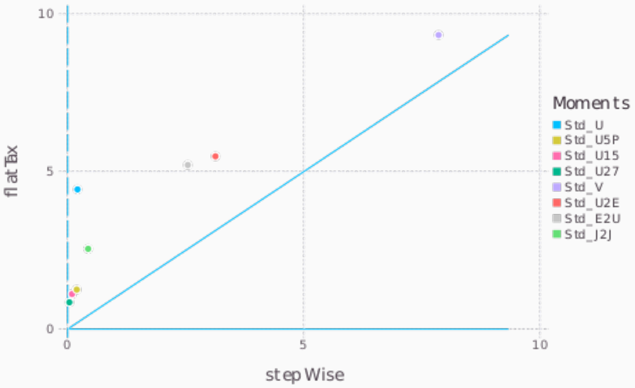

Figure 8: Volatility

Conclusion

Conclusion

Main Question

What are the impacts of payroll taxation on labor income risks along the business cycle?

Preliminary Answers

- payroll taxation as a tool to mitigate labor income shocks
- trade-off level - volatility

Work In Progress

- Estimation
- Counter-factuals
- Firm heterogeneity

Questions?

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Model

Workers, Firms and Production

- continuum of infinitely-lived workers differing in their individual productivity, indexed by x.
- distribution of ability x is exogenous and denoted by $\ell(x)$.
- firms are identical and can freely enter the market, incurring an exogenous cost $c(v)$ when posting v vacancies.
- when matched firms and workers produce a per period output $p(x, z_t)$.

Shocks:

- aggregate productivity z follows and AR(1) process

Back

Model

Frictional matching market

- Number of per period meetings, M_t, is determined by a matching function $M(L_t, V_t)$ increasing, concave and homogeneous of degree 1
- V_t is the aggregate number of vacancies and aggregate search effort L_t.

$$L_t = \int_0^1 u_{t+}(x)dx + s\int_0^1 h_{t+}(x)dx$$

- Unemployed and employed workers meet a job with probability $\lambda_t = M_t/L_t$ and $s\lambda_t$
- Free entry determines the number of vacancies

Back

Model

Vacancy creation

State variable $\Omega_t = (z_t, h_t(.))$

$$\lambda_t = M_t/L_t = M(1, V_t/L_t) = f(\Omega_t)$$

Search effort:

$$L_t = L(\Omega_t) = 1 - (1-s)(1-\delta)\int_0^1 \mathbb{1}\left\{S(x, w, \Omega_t) \geq 0\right\} h(x)dx$$

Free entry condition $c'(V_t) = q_t J_t$ with $c(.)$ a strictly increasing and convex cost function, J_t the expected value of a filled vacancy,q_t the probability for meeting a worker

$$V_t = V(\Omega_t) = (c')^{-1}\Big(M(\frac{1}{V_t L_t}, \frac{1}{L_t})\int_0^1 (\ell(x) - (1-\delta)h(x)) \max\{S(x, w, \Omega_t), 0\}\Big)$$

L.h.s strictly increasing in V_t, r.h.s strictly decreasing in V_t. $\rightarrow$ V_t uniquely determined at every period t.

Back

Model - Wage Setting

Wages are set as in (Robin, 2011):

- Unemployed workers receive their reservation wage $\phi^0(x, \Omega_t)$
- Bertrand competition when an employee meets another firm $\rightarrow$ employed workers receive the firm's reservation wage $\phi^1(x, \Omega_t)$

At time t, only 2 wages are offered:

$$\phi_t^1(x) \equiv \phi^1(x, \Omega_t) \text{ s.t. } \triangle(x, \phi^1(x), \Omega_t)) = S(x, \phi^1(x, \Omega_t), \Omega_t).$$

$$\phi_t^0(x) \equiv \phi^0(x, \Omega_t) \text{ s.t. } \triangle(x, \phi^0(x, \Omega_t), \Omega_t) = 0.$$

Back

Dynamic system - Match Surplus

State variable $\Omega_t = (z_t, h_t(.))$

$$
\begin{aligned}
S(x, w, \Omega_t) = & \underbrace{p(x, z_t) - \tau_w(w)w - b(x)}_{\text{flow value}} + \\
& \frac{1-\delta}{1+r}\mathbb{E}\Big[\underbrace{\mathbb{1}\left\{S(x, w, \Omega_{t+1}) < 0\right\} \max\{0, S(x, \phi^1_{t+1}(x), \Omega_{t+1})\}}_{\text{renegotiation or separation if surplus at } w<0} \\
& + \mathbb{1}\left\{S(x, w, \Omega_{t+1}) \geq 0\right\} \big[s\lambda(\Omega_{t+1}) \underbrace{\left(S(x, \phi^1_{t+1}(x), \Omega_{t+1})\right)}_{\text{continuation value if poaching}} \\
& + (1 - s\lambda(\Omega_{t+1})) \quad \underbrace{\left(A^S(x, w, \Omega_{t+1})\right)\big]}_{\text{continuation value if no poaching}} \quad \Big]
\end{aligned}
\tag{5}
$$

$$
A^S(x, w, \Omega_{t+1}) = \begin{cases} S(x, w, \Omega_{t+1}), & \text{if } 0 \leq \triangle(x, w, \Omega_{t+1}) \leq S_{t+1}(x, w, \Omega_{t+1}) \\ S(x, \phi^1_{t+1}(x), \Omega_{t+1}), & \text{if } \triangle(x, w, \Omega_{t+1}) > S(x, w, \Omega_{t+1}) \\ S(x, \phi^0_{t+1}(x), \Omega_{t+1}), & \text{if } \triangle(x, w, \Omega_{t+1}) < 0 \end{cases}
$$

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Dynamic system - Worker Surplus

$$\triangle(x, w, \Omega_t) = \underbrace{[1 - \tau_w(w)]w - b(x)}_{\text{flow value}} +$$

$$\frac{1-\delta}{1+r}\mathbb{E}\Big[\underbrace{\mathbb{1}\left\{S(x, w, \Omega_{t+1}) < 0\right\} \max\{0, S(x, \phi^1_{t+1}(x), \Omega_{t+1})\}}_{\text{renegotiation or separation if surplus at } w < 0}$$

$$+\, \mathbb{1}\left\{S(x, w, \Omega_{t+1}) \geq 0\right\}\Big[s\lambda(\Omega_{t+1})\underbrace{\big(S(x, \phi^1_{t+1}(x), \Omega_{t+1})\big)}_{\text{continuation value if poached}}$$

$$+\, (1 - s\lambda(\Omega_{t+1})) \quad \underbrace{\big(A^{\triangle}(x, w, \Omega_{t+1})\big)\Big]}_{\text{continuation value if not poached}} \quad \Big]$$

$$A^w(x, w, \Omega_t) = \begin{cases} \triangle(x, w, \Omega_t), & \text{if } 0 \leq \triangle(x, w, \Omega_t) \leq S(x, w, \Omega_t) \\ S(x, \phi^1_{t+1}(x), \Omega_t), & \text{if } \triangle(x, w, \Omega_t) > S(x, w, \Omega_t) \\ 0, & \text{if } \triangle(x, w, \Omega_t) < 0 \end{cases}$$

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Idea of a proof

Discussion existence of a steady-state

Fix $z = z^*$. Study the function ϕ defined by:

$$\phi : E \to C(K) \times C(K) \to E$$

$$\phi : m \to (S(x, w; z^*, m), \triangle(x, w; z^*, m)) \to \tilde{m} \tag{6}$$

- with E a convex closed and compact subset of the space of measures
- $C(K)$ the space of continuous functions $f : K \to \mathcal{R}$ with K the compact set in $\mathcal{R}^2$ defined by $K = [0, 1] \times [\underline{w}, \bar{w}]$, ($\underline{w}$ and $\bar{w}$ a lower bound and an upper bound on the wages)

Schauder fixed-point theorem: if the function ϕ is continuous, there exists a fixed point $\phi(h^*) = h^*$.

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Idea of a proof

Discussion existence of a steady-state

Have to show that the function ψ is continuous

$$\psi : E \rightarrow C(K) \times C(K)$$

$$\psi : m \rightarrow (S(x, w; z^*, m), \triangle(x, w; z^*, m)) \tag{7}$$

- ψ is the function that assigns to a measure m the solution of the fixed point problem $(S(x, w, z^*, m), \triangle(x, w, z^*, m)) = T(S(x, w, z^*, m), \triangle(x, w, z^*, m))$

T a contraction?

Continuity obvious?

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Idea of a proof

Discussion existence of a steady-state

Have to show that the function γ, assigning a measure $\tilde{m}$ to the functions S and $\triangle$, is continuous

$$\gamma : C(K) \times C(K) \to E$$

$$\gamma : (S(x, w; z^*, h), \triangle(x, w; z^*, h)) \to \tilde{h} \tag{8}$$

$$\forall x \in [0,1] : h^*(x) = \begin{cases} 0 & \text{if } S(x, w; z^*, h^*) \geq 0 \\ \frac{\lambda(z^*, h^*) l(x)}{1-(1-\delta)(1-\lambda(z^*, h^*))} & \text{otherwise} \end{cases} \tag{9}$$

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