Introduction to RBC models

TA Session 2

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Objective of the second TA session

At the end of this session you should be able to :

Part I

- Solve a very special case of the RBC model

Part II

- Understand the Euler's theorem

Part III

- Solve the RBC model with labor and without capital

Homework

Solve analytically the problem in a very special case

$$
\begin{aligned}
U(C_t) &= \log(C_t) \\
Z_t F(K_t, 1) &= Z_t K_t^{\alpha} \\
\delta &= 1
\end{aligned}
$$

Firms

In every period, firms maximize their profits with respect to capital :

$$\max_{K_t} \ \pi_t = Y_t - \tilde{R}_t K_t - W_t$$

Subject to

$$Y_t = Z_t F(K_t, 1) = Z_t K_t^{\alpha}$$

Firms

Then, substituting the constraint in the objective function :

$$\max_{K_t} \ \pi_t = Z_t K_t^{\alpha} - \tilde{R}_t K_t - W_t$$

Derive the first order condition :

$$\begin{aligned}
\frac{\partial \pi_t}{\partial K_t} = 0 \iff & Z_t \alpha K_t^{\alpha-1} - \tilde{R}_t = 0 \\
\iff & Z_t \alpha K_t^{\alpha-1} = \tilde{R}_t \\
\iff & \alpha \frac{Y_t}{K_t} = \tilde{R}_t
\end{aligned}$$

Households

The representative household chooses $\{C_t, S_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize :

$$\mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t U(C_t) \right)$$

subject to

$$\begin{aligned} C_t + S_t &\leqslant \tilde{R}_t K_t + W_t + \pi_t \\ K_{t+1} &= (1-\delta) K_t + S_t \\ K_{t+1} \geqslant 0 \ ; \ K_t &> 0 \textit{ given} \end{aligned}$$

Households

Lagrangian

As usual, combine the two constraints, use $\delta = 1$ and $\pi_t = 0$ and set up the lagrangian

$$\mathcal{L} = \mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \log(C_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t (\tilde{R}_t K_t + W_t - C_t - K_{t+1}) \right)$$

And derive the first order conditions :

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \iff \frac{1}{C_t} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \iff \beta \, \mathbb{E}_t \left[\lambda_{t+1} \tilde{R}_{t+1} \right] = \lambda_t$$

Households

The Euler Equation

As usual, combine the two first order conditions to obtain the Euler equation :

$$\frac{1}{C_t} = \beta\,\mathbb{E}_t\left[\frac{1}{C_{t+1}}\tilde{R}_{t+1}\right]$$

Moreover, we know that :

$$\mathbb{E}_t\,\tilde{R}_{t+1} = \mathbb{E}_t\,Z_{t+1}\alpha K_{t+1}^{\alpha-1}$$

So, we have the following system :

$$\frac{1}{C_t} = \beta\,\mathbb{E}_t\left[\frac{1}{C_{t+1}}\tilde{R}_{t+1}\right] \tag{1}$$

$$K_{t+1} = \tilde{R}_t K_t + W_t - Ct = Z_t K_t^{\alpha} - Ct \tag{2}$$

Households

How do we solve this ?

Conjecture : $C_t = mY_t$

Then, using (1) :

$$\begin{aligned} \frac{1}{mY_t} &= \mathbb{E}_t(\frac{\beta}{mY_{t+1}}\alpha\frac{Y_{t+1}}{K_{t+1}}) \\ \iff \frac{1}{Y_t} &= \mathbb{E}_t(\frac{\beta\alpha}{K_{t+1}}) \\ \iff \frac{1}{Y_t} &= \frac{\beta\alpha}{K_{t+1}} \\ \iff K_{t+1} &= Y_t\beta\alpha \end{aligned}$$

Households

How do we solve this ?

Using (2) :

$$\begin{aligned} C_t &= Y_t - K_{t+1} \\ \iff C_t &= Y_t - Y_t\beta\alpha \\ \iff C_t &= (1-\beta\alpha)Y_t \\ \iff C_t &= mY_t \end{aligned}$$

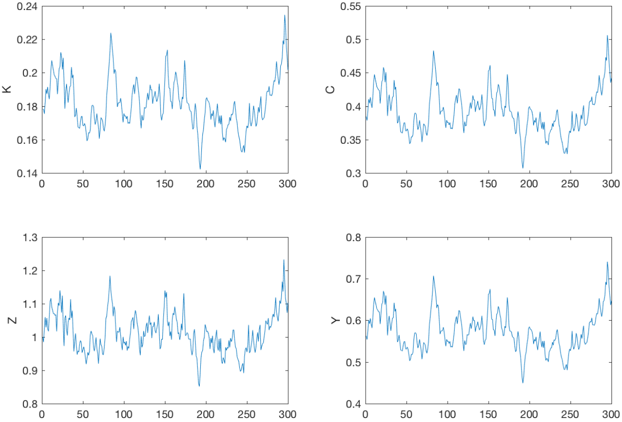
With m $= (1-\beta\alpha)$ and the conjecture is verified

Finally :

$$\begin{aligned} C_t &= (1-\alpha\beta)Z_tK_t^{\alpha} \\ K_{t+1} &= \beta\alpha Z_tK_t^{\alpha} \end{aligned}$$

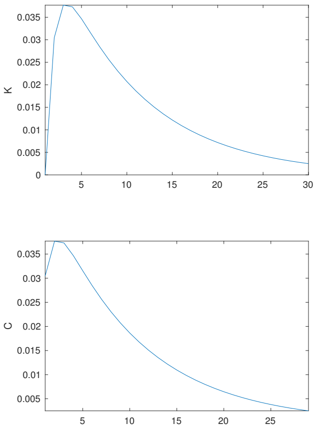
We can do a lot of things with a solved model :

For instance, simulate the model :



We can do a lot of things with a solved model :

Or show its reaction to shocks :



Part II

The Euler's theorem

Let's prove that $\pi_t = 0$ under specific conditions, met in the model presented above.

We used a production function that features **constant return to scale** and then one can apply the Euler's theorem.

With $f(x, y)$ an homogeneous function of degree 1 so that $\lambda f(x, y) = f(\lambda x, \lambda y)$:

$$f(x, y) = x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y}$$

The Euler's theorem

You know that $Z_t F(K_t, L_t) = Z_t K_t^{\alpha} L_t^{1-\alpha}$ has **constant return to scale** ie is homogeneous of degree 1. Moreover, we know that when firms maximize their profits and that they are price takers each factors is remunerated according to its marginal product $Z_t \dfrac{\partial F(K_t, L_t)}{\partial K_t} = \tilde{R}_t$ and $Z_t \dfrac{\partial F(K_t, L_t)}{\partial L_t} = W_t$. Then,

$$Z_t F(K_t, L_t) = K_t \tilde{R}_t + W_t L_t$$

Then :

$$\pi_t = Z_t F(K_t, L_t) - K_t \tilde{R}_t - W_t L_t = 0$$

Then, the total remuneration accruing to the factors exhausts output and profit is equal to zero.

Part III

The RBC model with labor

The representative household chooses $\{C_t, S_t, N_t\}_{t=0}^{\infty}$ to maximize :

$$\mathbb{E}_0\left(\sum_{t=0}^{\infty}\beta^t U(C_t, L_t)\right)$$

subject to

$$C_t = W_t N_t \quad (\textit{Flow budget constraint})$$
$$L_t + N_t = 1 \quad (\textit{Time endowment})$$

Take :

$$U(C_t, L_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} - \varphi\frac{(1-L_t)^{1+\psi}}{1+\psi} & \textit{if } \gamma \neq 1 \\ \log(C_t) - \varphi\frac{(1-L_t)^{1+\psi}}{1+\psi} & \textit{if } \gamma = 1 \end{cases}$$

The RBC model with labor

If $\gamma \neq 1$

Then, the representative household chooses $\{C_t, S_t, N_t\}_{t=0}^{\infty}$ to maximize :

$$\mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \frac{(1-L_t)^{1+\psi}}{1+\psi} \right) \right)$$

subject to

$$C_t = W_t N_t$$
$$L_t + N_t = 1$$

The RBC model with labor

Substitute the constraints in the objective function :

The representative household chooses $\{N_t\}_{t=0}^{\infty}$ to maximize :

$$\mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \left(\frac{(W_t N_t)^{1-\gamma}}{1-\gamma} - \varphi \frac{N_t^{1+\psi}}{1+\psi} \right) \right)$$

One can see the trade-off between leisure and consumption appeared

The RBC model with labor

Take the derivative of the objective with respect to N_t :

$$
\begin{aligned}
W_t^{1-\gamma} N_t^{-\gamma} &= \varphi N_t^{\psi} \\
\iff W_t^{1-\gamma} &= \varphi N_t^{\psi+\gamma} \\
\iff N_t &= \left(\frac{1}{\varphi}\right)^{\frac{1}{\psi+\gamma}} W_t^{\frac{1-\gamma}{\psi+\gamma}}
\end{aligned}
$$

The response of N with respect to W_t depends on the value of the parameters.

The RBC model with labor

If $\gamma = 1$

The representative household chooses $\{N_t\}_{t=0}^{\infty}$ to maximize :

$$\mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \log(W_t N_t) - \varphi \frac{N_t^{1+\psi}}{1+\psi}) \right)$$

Then,

$$\frac{W_t}{W_t N_t} = \varphi N_t^{\psi}$$
$$\iff N_t = \left(\frac{1}{\varphi} \right)^{\frac{1}{1+\psi}}$$

N does not depend on the wage.