TA Sessions : Business Cycle

Additional material : precision on the derivation of the demand function

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I had a lot of questions about "why does the integral drops out when one takes the derivative of the lagrangian with respect to : $C_s(i)$ ".

Ie, how can you go from the equations (1) to (2) ?

$$\mathcal{L}(C_s(i)) = \int_0^1 P_s(i)C_s(i)di - \lambda_s \left[\left(\int_0^1 (C_s(i))^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \right] \tag{1}$$

$$\implies \frac{\partial \mathcal{L}(C_s(i))}{\partial C_s(i)} = 0 \iff P_s(i) - \lambda_s \left[C_s(i)^{\frac{-1}{\theta}} \left(\int_0^1 (C_s(i))^{\frac{\theta-1}{\theta}} di \right)^{\frac{1}{\theta-1}} \right] = 0 \tag{2}$$

I show here how you can understand and demonstrate this result :

1 Recall : what does it mean to differenciate a function ?

1.1 The usual stuff

Solving an optimization problem means looking for the **stationary point** of the program that you consider.

You know that to solve this problem you have to look for the solution such that the derivative of the function (with respect to your choice variable) that you optimize is null. Basically, it means that you look for a solution such that when you "perturb" a little bit the function that you optimize using this choice variable, this function will not evolve at the first order. Ok, but why ?

Let's recall the definition of a derivative :

Define a function f such that :

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

The derivative of f in x_0 with respect to h is :

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3}$$

So, when a function is derivable :

$$f(x_0 + h) = f(x_0) + h \times f'(x_0) + o(h) \tag{4}$$

Recall that $o(h)$ is something that divided by h (if h is a scalar) or the norm of h (if h is a vector) goes to zero when h goes to zero.

Remark : this form should remind you of something (hopefully you should see, that it is the first order approximation of the function f in x_0). So, it is equivalent to look for a solution such as the derivative of f is null in x_0 or that its first order approximation does not evolve at the neighboorhood of x_0

Now define $f : (x, y) \in \mathbb{R}^2 \mapsto f(x, y) \in \mathbb{R}$

With $X = (x, y)$ and $H = (h_1, h_2)$

Taking the first order approximation of the function (ie approximating the function with a linear form) :

$$f(X + H) = f(X) + L_X(H) + o(H)$$

With, $L_X(h_1, h_2) = ah_1 + bh_2$ a linear function of H that depends on X.

Moreover, you know that $L_X(h_1, h_2) = df(H) = \partial_x f(X)h_1 + \partial_y f(X)h_2$

1.2 What does it mean to derive a functional with respect to a function

Remark : a functional is a function of function

In the optimization program that we are interested in, our variable is a function, we take the derivative of the lagrangian with respect to $C(i)$.

$$\begin{array}{rccc} C: & \in \mathbb{C} & \longrightarrow & \mathbb{R} \\ & i & \longmapsto & C(i) \end{array}$$

With $\mathbb{C}$ the space of continuous function.

An exemple

Let's considere $\mathcal{F}$ a function such that :

$$\begin{array}{rccc} \mathcal{F}: & \in \mathbb{C} & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \mathcal{F}(f) = \int_0^1 f(i)^2 di \end{array}$$

Then,

$$\begin{aligned} F(f + h) &= \int_0^1 (f + h)^2 di \\ &= \int_0^1 f^2 di + 2\int_0^1 (f \times h) di + \int_0^1 h^2 di \\ &= F(f) + \mathcal{L}_f(h) + o(h) \end{aligned}$$

Remark : h is now a function

With $\mathcal{L}_f(h)$ a linear function of h (that depends on f), and we saw that $\mathcal{L}_f(h) = d\mathcal{F}(f)(h)$

And with, $d\mathcal{F}(f) : h \mapsto 2\int_0^1 (f \times h) di$

Finally, you can see that :

$$\begin{aligned} d\mathcal{F}(f) = 0 &\iff \forall h \in \mathcal{C} \quad 2\int_0^1 (f \times h) di = 0 \\ \text{And this is true} &\iff f = 0 \end{aligned} \tag{5}$$

In order to see that this statement is true, take $h = f$ (you can do that because it's is true $\forall h$), then

$$\int_0^1 (f \times h) di = 0 \implies \int_0^1 f^2 di = 0$$

Then, f is necessary null (if the integral of a positive function is null the function is necessarily equal to zero).

2 An application to the households' program

First, remark that : $d\mathcal{L}(f)(h) = \left[\frac{\partial \mathcal{L}(f+\varepsilon h)}{\partial \varepsilon}\right]_{|\varepsilon=0}$

Indeed, :

$$\begin{aligned} \mathcal{L}(f + \varepsilon h) &= \mathcal{L}(f) + d\mathcal{L}(f)(\varepsilon h) + o(\varepsilon h) \\ &= \mathcal{L}(f) + \varepsilon d\mathcal{L}(f)(h) + o(\varepsilon) \end{aligned}$$

So,

$$\begin{aligned} \left[\frac{\partial \mathcal{L}(f+\varepsilon h)}{\partial \varepsilon}\right]_{|\varepsilon=0} &= d\mathcal{L}(f)(h) + o(1) \\ &= d\mathcal{L}(f)(h) \end{aligned}$$

Consider the lagrangian of our problem and study the effect of a small perturbation (in other words, a small variation) of this function in the direction h.

$$\mathcal{L}(C_s(i) + \varepsilon h(i)) = \int_0^1 \left(P_s(i)(C_s(i) + \varepsilon h(i))\right) di - \lambda_s \left[\left(\int_0^1 (C_s(i) + \varepsilon h(i))^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \right]$$

$$\frac{\partial \mathcal{L}(C_s(i) + \varepsilon h(i))}{\partial \varepsilon} = \int_0^1 P_s(i) h(i) di - \lambda_s \left[h(i) \int_0^1 (C_s(i) + \varepsilon h(i))^{\frac{-1}{\theta}} di \times \left(\int (C_s(i) + \varepsilon h(i))^{\frac{\theta-1}{\theta}} di \right)^{\frac{1}{\theta-1}} \right]$$

Then,

$$\left[\frac{\partial \mathcal{L}(C_s(i)+\varepsilon h(i))}{\partial \varepsilon}\right]_{|\varepsilon=0} = \int_0^1 P_s(i)h(i)di - \lambda_s\left[h(i)\int_0^1 (C_s(i))^{\frac{-1}{\theta}}di \times \left(\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}}di\right)^{\frac{1}{\theta-1}}\right]$$

And,

$$\begin{aligned}\frac{\partial \mathcal{L}(C_s)(h)}{\partial C_s(i)} &= \int_0^1 P_s(i)h(i)di - \lambda_s\left[h(i)\int_0^1 (C_s(i))^{\frac{-1}{\theta}} \times \left(\int C_s(i)^{\frac{\theta-1}{\theta}}di\right)^{\frac{1}{\theta-1}}\right] = 0 \\ \implies & \int_0^1 \left[\left(P_s(i) - \lambda_s\left[\int_0^1 (C_s(i))^{\frac{-1}{\theta}}\left(\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}}\right]\right) \times h(i)\right] di = 0\end{aligned}$$

And then,

$$\begin{aligned}&\int_0^1 \left[\left(P_s(i) - \lambda_s\left[\int_0^1 (C_s(i))^{\frac{-1}{\theta}}\left(\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}}\right]\right) \times h(i)\right] di = 0 \\ &\iff P_s(i) - \lambda_s\int_0^1 (C_s(i))^{\frac{-1}{\theta}} \times \left(\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}} = 0\end{aligned}$$

In the same way that in (5)

3 A general rule :

The maps of the following form $h \mapsto \int f \times h$ can be identified with the function f wich defines it in the way that two linears maps are equal if and only if the functions defining them are equal.

In other words :

$$\forall h, \int_I fh = \int_I gh \iff f = g$$

You can now use this rule without worring :

For every functional that takes the following form (all the functions CES for instance) :

$$\mathcal{F}(f) = \left[\int_I \varphi(f(x))dx\right]$$

You know that :

$$\begin{aligned}\mathcal{F}(f+\varepsilon h) &= \left[\int_I \varphi((f(x)+\varepsilon h(x))dx\right] \\ \frac{\partial \mathcal{F}(f+\varepsilon h)}{\partial \varepsilon}_{|\varepsilon=0} &= \left[\int_I \varphi'((f(x)h(x))dx\right]\end{aligned}$$

By definition, $\frac{\partial \mathcal{F}(f+\varepsilon h)}{\partial \varepsilon}_{|\varepsilon=0} = d\mathcal{F}(f)(h)$
Then,

$$d\mathcal{F}(f)(h) = \left[\int_I \varphi'((f(x)h(x))dx\right]$$

And,

$$d\mathcal{F}(f) = 0 \iff \forall x \;\; \varphi'((f(x)) = 0$$

To come back to our example, you can now directly see that :

$$\begin{aligned}\frac{\partial \int_0^1 P_s(i)C_s(i)di}{\partial C_s(i)} = 0 &\iff \frac{\partial (P_s(i)C_s(i))}{\partial C_s(i)} = 0 \\ &\iff P_s(i) = 0\end{aligned}$$