Alan | Advocate Cap. New York, United States

DUAL CURVE STRIPPING

The credit and liquidity crisis of 2007 has triggered a number of inconsistencies in the interest rate market, questioning some of the standard methods and assumptions used to price and hedge interest rate derivatives. It has been shown that using a single risk-free curve (constructed from market instruments referencing underlying rates of varying tenors) to forecast and discount cash flows is not theoretically correct. Standard market practice has evolved to a multi-curve approach, using different curves to forecast and discount cash flows. The risk-free discount curve is calculated by the Overnight-Indexed Swap (OIS) curve. In this paper, I researched on dual curve bootstrapping method namely OIS-LIBOR bootstrapping on swap pricing after financial crisis.

What is OIS discounting

An overnight indexed swap (OIS) is an interest rate swap where the periodic floating payment is generally based on a return calculated from a daily compound interest investment. The reference for a daily compounded rate is an overnight rate (or overnight index rate) and the exact averaging formula depends on the type of such rate.

The index rate is typically the rate for overnight unsecured lending between banks, for example the Federal funds rate for US dollars, Eonia for Euros or Sonia for sterling. The fixed rate of OIS is typically an interest rate considered less risky than the corresponding interbank rate (LIBOR) because there is limited counterparty risk.

Before 2007, single curve works very well in derivative pricing since the spread of OIS and LIBOR is relatively small. However, after 2007, industry realized the difference between OIS and LIBOR grows dramatically which affects almost all related financial instruments pricing. And this difference (spread) comes from the credit risk in banks and becomes an important measure of the health of financial industry.

Why is OIS discounting

Constructing a single yield curve to price and hedge interest rate sensitive instruments was standard market practice before the financial crisis in 2007. Interbank credit and liquidity issues were assumed negligible and not considered for pricing. However, after 2007, people saw a huge difference in spread which comes from the credit and liquidity risk, see graph of spread below.

As we can see from the graph, after 2007, there is a long trend in the Spread until 2012. This can be explained by the reacting from market of the credit crisis. And another long trend shows up around 2015 because of the EURO crisis and OIL crash. So, indeed the spread between LIBOR and OIS plays a more and more important role in modern financial industry especially in derivatives pricing.

Since the financial crisis of 2007, the Classical Pricing Framework outlined before has been replaced by the modern pricing approach (Multiple-Curve Framework). This entails the construction of multiple zero-coupon curves, one for each tenor. For example, in Europe the following curves are constructed:

- 1-month nominal swap curve, referencing the 1-month EURIBOR rate
- 3-month nominal swap curve, referencing the 3-month EURIBOR rate
- 12-month nominal swap curve, referencing the 12-month EURIBOR rate
- N-year nominal swap curve, referencing the N-year EURIBOR rate

Each curve is constructed using a set of instruments homogeneous in underlying rate tenor, which provides a consistent representation of credit and liquidity risk. In the multiple curve framework, there are forecasting or tenor curves, shown in the examples above and there is a discounting curve. The forecasting curves are yield curves used to compute forward rates, and the discounting curve is used to discount future cash flows. The forecasting curve will be selected according to the underlying rate tenor of the selected interest rate derivative. The choice of discounting curve is however less clear.

Derivatives are priced in a risk-neutral framework, using risk-free rates for discounting. Before the 2007 credit crunch EURIBOR and EURIBOR swap rates were used as proxies for risk-free rates. The validity of this approach was questioned during the credit crisis, as banks became reluctant to lend to each other because of credit concerns. EURIBOR rates carry credit and liquidity risk premium, which differ across tenors. The banking sector is no longer considered risk-free. The value of a derivative must be considered along with credit risk and collateral agreements. When derivative dealers trade in the non-centrally cleared, i.e. over-the-counter (OTC) market, credit has become a major concern, and this additional risk needs to be taken into account in pricing. The interest rate paid on cash collateral also affects pricing. The standard approach to factor in credit risk is to first calculate the default-free value of the derivative, and then make credit and other valuation adjustments.

How is OIS discounting

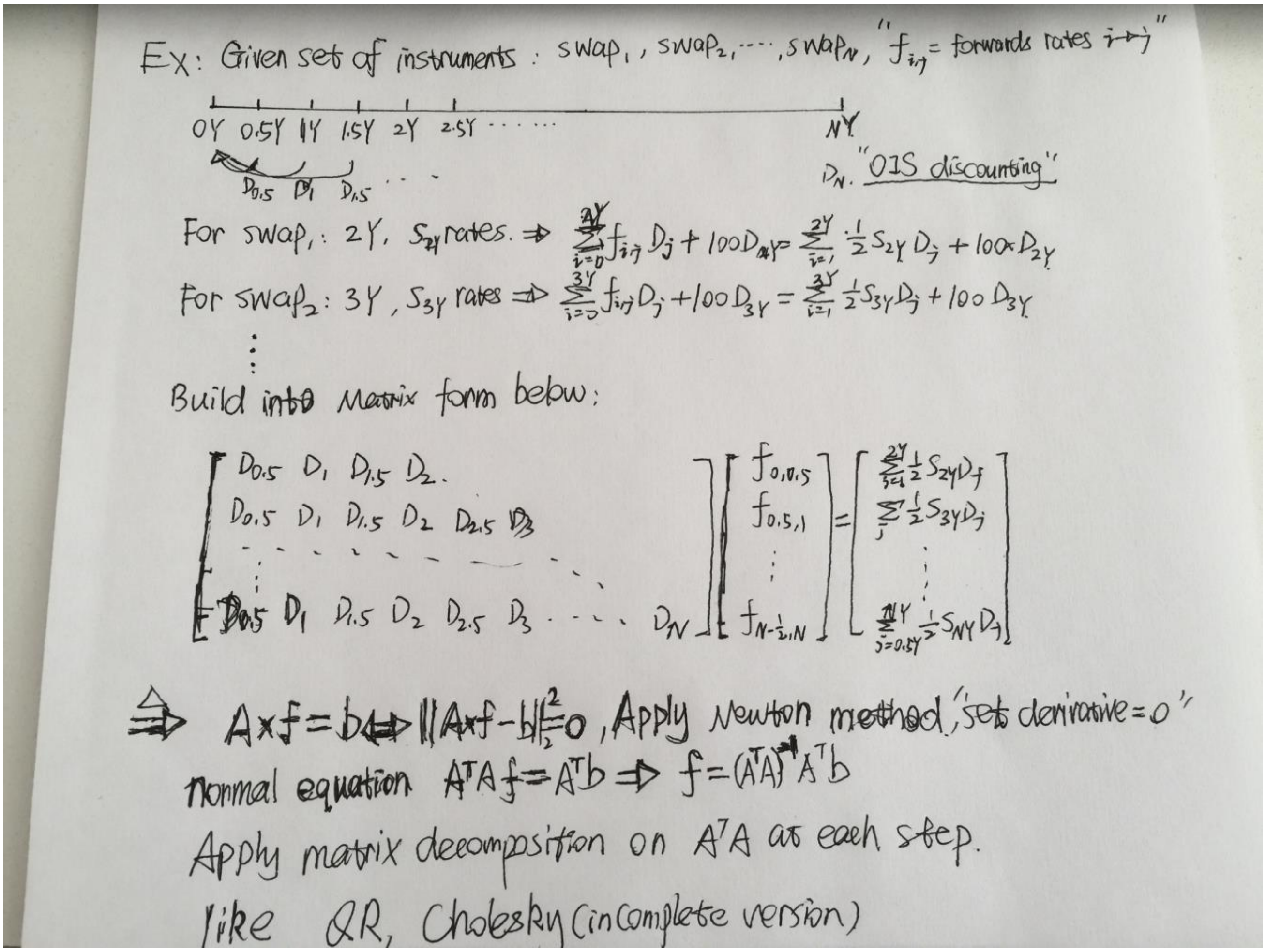
To illustrate how to apply OIS discounting in dual curve bootstrapping we are referring to the standard multi-curve framework.

Multi-Curve Framework:

The post-crisis, multiple curve framework is summarized below, following Ametrano and Bianchetti (2013).

1. Select the relevant OIS instruments to build the OIS swap curve (the discounting curve).
2. Select the separate sets of vanilla interest rate inputs, which will be used to construct the tenor curves. Each set of instruments should be homogeneous in their underlying rate tenor. There is typically a 1-month, 3-month, 6-month and 12-month curve, depending on the market.
3. Build multiple separate forecasting curves and the discounting curve using the inputs selected in (1 and 2) along with the discounting curve and a set of bootstrapping rules.
4. The forecasting curves are then used to calculate the forward rates and corresponding cash flows.
5. The discount factors are calculated from the discount curve calculated in 1.
6. Calculate the default-free price of the derivative using the relevant pricing formula and curves and make the required value adjustments.

Mathematically, we have the following diagram:



Ex: Given set of instruments: $swap_1, swap_2, \cdots, swap_N$, "$f_{i\to j}$ = forwards rates $i \to j$"

0Y 0.5Y 1Y 1.5Y 2Y 2.5Y ······ NY

$D_{0.5}$ D_1 $D_{1.5}$ · · · D_N. "OIS discounting"

For $swap_1$: 2Y, S_{2Y} rates. $\Rightarrow \sum_{i=0}^{2Y} f_{i\to j} D_j + 100 D_{2Y} = \sum_{j=1}^{2Y} \frac{1}{2} S_{2Y} D_j + 100 \times D_{2Y}$

For $swap_2$: 3Y, S_{3Y} rates $\Rightarrow \sum_{i=0}^{3Y} f_{i\to j} D_j + 100 D_{3Y} = \sum_{j=1}^{3Y} \frac{1}{2} S_{3Y} D_j + 100 D_{3Y}$

⋮

Build into Matrix form below:

$$\begin{bmatrix} D_{0.5} & D_1 & D_{1.5} & D_2 & & & & \\ D_{0.5} & D_1 & D_{1.5} & D_2 & D_{2.5} & D_3 & & \\ \cdots & & & & & & \ddots & \\ D_{0.5} & D_1 & D_{1.5} & D_2 & D_{2.5} & D_3 & \cdots & D_N \end{bmatrix} \begin{bmatrix} f_{0,0.5} \\ f_{0.5,1} \\ \vdots \\ f_{N-\frac{1}{2},N} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{2Y} \frac{1}{2} S_{2Y} D_j \\ \sum_j \frac{1}{2} S_{3Y} D_j \\ \vdots \\ \sum_{j=0.5Y}^{NY} \frac{1}{2} S_{NY} D_j \end{bmatrix}$$

$\Rightarrow A \times f = b \Leftrightarrow \|A \times f - b\|_2^2 = 0$, Apply Newton method, "set derivative = 0"

normal equation $A^T A f = A^T b \Rightarrow f = (A^T A)^{-1} A^T b$

Apply matrix decomposition on $A^T A$ at each step.

like QR, Cholesky (incomplete version)

However, to solve this requires much computational effort which takes much time. In this implementation, I will refer to Bloomberg Approximation formula as below to calculate adjusted forward rates:

$$\alpha = \mathrm{forward}^{\mathrm{OIS}} - \mathrm{forward}^{\mathrm{LIBOR}}$$

$$\alpha \approx - {\Delta_{rate}^{T} * d_{swap}^{T} * T^2} \Big/ {2 * 10000}$$

Results and Conclusion

I implemented both optimization method and approximation method, results comparison below:

Experimental data based on LIBOR curve and OIS curve start on 2018-08-20. We solve this linear system in Python with least square programming. Following the below steps:

- Bootstrap OIS curve with the procedure in the reference.
- Query a set of market traded instruments (Swaps) labeled as $\{s_1, s_2, s_3, \ldots, s_n\}$.
- Calculate NPV of this set of instruments discounting OIS.
- Build discount factors matrix with OIS discounting.
- Use Python Least Square Package solve for forward rates.

Optimization result

Given market OIS quote below:

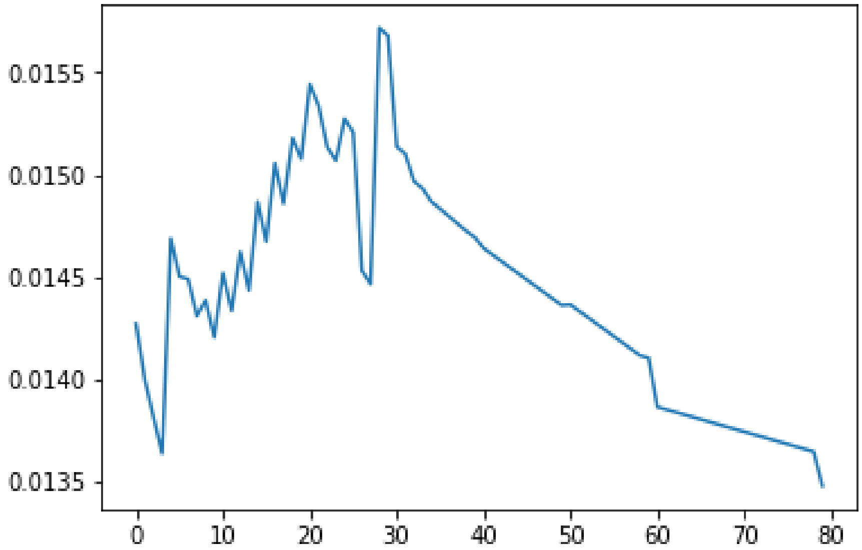
Maturity	Rate
2020-08-20	2.7871
2021-08-20	2.83
2022-08-20	2.8421
2023-08-20	2.8454
2024-08-20	2.8517
2025-08-20	2.8589
2026-08-20	2.8698
2027-08-20	2.882
2028-08-20	2.8948
2029-08-20	2.9096
2030-08-20	2.9178
2031-08-20	2.9267
2032-08-20	2.925
2033-08-20	2.9377
2034-08-20	2.9425
2035-08-20	2.945
2038-08-20	2.9465

2043-08-20	2.9382
2048-08-20	2.9243
2058-08-20	2.8843
2059-08-20	2.8843

Given market SWAP quote below:

Maturity	Rate
2020-08-20	2.7871
2021-08-20	2.83
2022-08-20	2.8421
2023-08-20	2.8454
2024-08-20	2.8517
2025-08-20	2.8589
2026-08-20	2.8698
2027-08-20	2.882
2028-08-20	2.8948
2029-08-20	2.9096
2030-08-20	2.9178
2031-08-20	2.9267
2032-08-20	2.925
2033-08-20	2.9377
2034-08-20	2.9425
2035-08-20	2.945
2038-08-20	2.9465
2043-08-20	2.9382
2048-08-20	2.9243
2058-08-20	2.8843
2059-08-20	2.8843

We get the adjusted forward rate below:

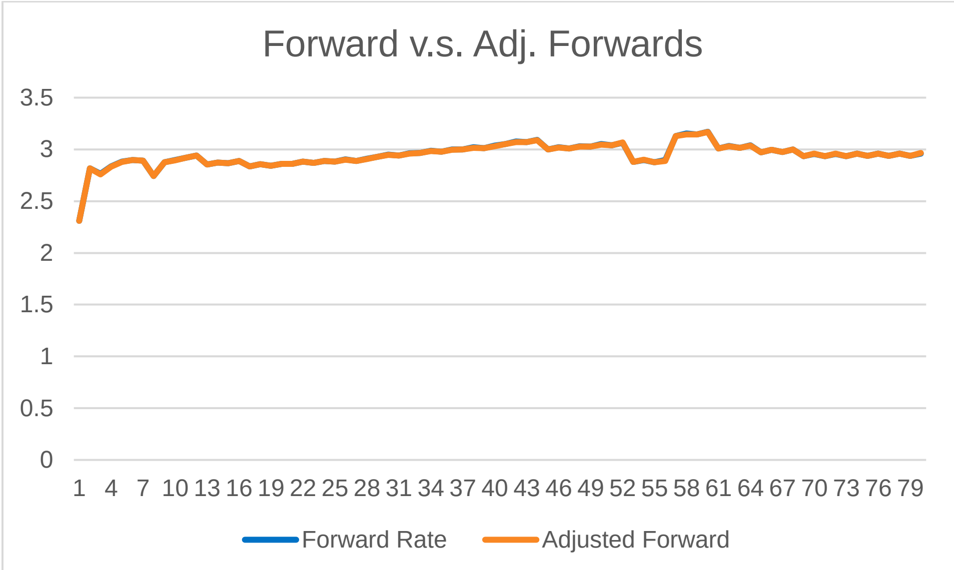


As we can see the forward rate generated by optimization method is quite rugged which makes it not in line with reality. But the totally error is eliminated between fix leg and float leg. Since we have more variables than constraints.

However, there is an obvious issue with this implementation. We cannot calculate sensitivity risk metrics like PV01 and DV01 in this method. Since if we recalculate the forward rate after shocking we still hold the zero NPV between fix and float legs. Which means their effects will cancel each other and level us zero sensitivity.

Approximation result

In this section I implemented approximation method with forward rate adjustments “Alpha”s.



The forward adjustments based on the "Alpha" method is very small which still leaves the gap between fix leg and float leg.

Currency	Start Date	Maturity	Notional(k)	NPV(k)	PV01(k)
USD	8/20/2018	10	100,000	104,491	90
USD	8/20/2018	10	100,000	-102,863	-4
Summary		20	200,000	1,628	86

Error with 100MM notional principal we get 1,628,000 difference in NPV. Relative error computed as 1.55% which is huge. We still have to modify the "Alpha" calculation in order to close this gap.

Conclusion:

While the optimization method gives zero gap in pricing but we cannot apply it into risk calculation. On the other hand, we calculate risk metrics with approximated method it is hard to determine how to calculate the adjusted "Alpha" to close this gap. But this method gives us close enough risk metrics as we saw in Bloomberg terminal the detailed comparison listed below, with USD currency:

Currency	Maturity	Coupon (%)	Fixed OIS PV01	Floating OIS PV01	Net PV01	BBG PV01 OIS	Diff to BBG PV01 (%)
USD	2	2.7871	19,547	192	19,355	19,376	-0.11%
USD	3	2.8300	28,964	216	28,749	28,705	0.15%
USD	4	2.8421	38,174	270	37,904	37,873	0.08%
USD	5	2.8454	47,198	367	46,830	46,716	0.24%
USD	6	2.8517	56,047	511	55,536	55,317	0.40%
USD	7	2.8589	64,679	657	64,022	63,784	0.37%
USD	8	2.8698	73,118	827	72,292	71,965	0.45%
USD	9	2.8820	81,359	1,012	80,347	79,973	0.47%
USD	10	2.8948	89,443	1,253	88,190	87,819	0.42%
USD	11	2.9096	97,359	1,523	95,835	95,398	0.46%
USD	12	2.9178	105,069	1,793	103,276	102,780	0.48%
USD	13	2.9267	112,644	2,108	110,536	109,992	0.49%
USD	14	2.9250	119,991	2,408	117,583	117,019	0.48%
USD	15	2.9377	127,282	2,811	124,472	123,921	0.44%
USD	16	2.9425	134,401	3,220	131,181	130,575	0.46%
USD	17	2.9450	141,357	3,633	137,724	137,064	0.48%
USD	20	2.9465	161,328	4,944	156,384	155,613	0.50%
USD	25	2.9382	192,201	7,595	184,606	183,690	0.50%
USD	30	2.9243	220,039	10,360	209,680	208,605	0.52%
USD	40	2.8843	267,853	15,899	251,954	250,549	0.56%

As we can see from the table, the largest difference is about 50 BPS which is always acceptable in risk analysis model. Also, the floating leg has some duration left which comes from the basis gap between OIS and LIBOR rates as well as the artificial alpha we are using.