STAT 4710J

Homework 2

1.

(a) $\hat{y} = \bar{y} + r\sigma_y \frac{x-\bar{x}}{\sigma_x}$

$e_i = y_i - \hat{y}_i$

$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} y_i - \hat{y}_i = \sum_{i=1}^{n} y_i - \bar{y} - \sigma_y \frac{x_i - \bar{x}}{\sigma_x}.$

Since $\sum_{i=1}^{n} y_i - \bar{y} = \sum_{i=1}^{n} x_i - \bar{x} = 0.$

$\sum_{i=1}^{n} e_i = 0.$

(b) $e_i = y_i - \hat{y}_i$

$\frac{1}{n}\sum_{i=1}^{n} e_i = \frac{1}{n}\sum y_i - \frac{1}{n}\sum \hat{y}_i$

$= \bar{y} - \bar{\hat{y}} = 0 \quad \Rightarrow \bar{y} = \bar{\hat{y}}$

(c) $\hat{y} = \bar{y} + r\sigma_y \frac{x-\bar{x}}{\sigma_x}$

Let $x = \bar{x} \quad \Rightarrow \hat{y} = \bar{y}$

Hence $(\bar{x}, \bar{y})$ is on the simple linear regression line

2.

(a) $e = Y - \hat{Y} = Y - X\hat{\theta}$

Because $e \perp \text{span}(X)$ $\mathbb{1} \in \text{span}(X)$.,

$e \cdot \mathbb{1} = 0 \Rightarrow \sum_{i=1}^{n} e_i = 0$

(b) Still, $e \perp \text{span}(X)$ $\vec{x} \in \text{span}(X)$

e and $\vec{x}$ are orthogonal

(c) Because $\hat{Y} \in \text{span}(X)$ $e \perp \text{span}(X)$

3、 $R(\gamma) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \gamma x_i)^2$

$$R'(\gamma) = \frac{1}{n} \sum_{i=1}^{n} 2(y_i - \gamma x_i) \cdot (-x_i) = 0$$

$$= \sum_{i=1}^{n} (\gamma x_i^2 - x_i y_i) = 0$$

$$\Rightarrow \quad \gamma \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\Rightarrow \quad \gamma = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

4. $\hat{\gamma} = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\overline{xy}}{\overline{x^2}}$

(a) False

Because $\mathbb{1} \notin \text{span}(x)$

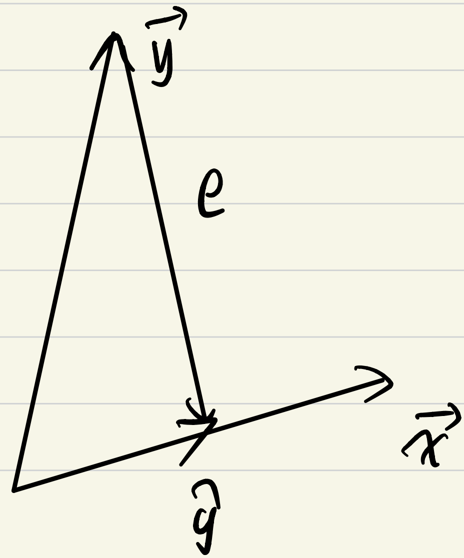
(b) True, We want e to be small.

$(\vec{y} - \hat{y}) \perp \vec{x} \Rightarrow e \perp \vec{x}$

(c) True, $\hat{y} \parallel \vec{x} \Rightarrow e \perp \hat{y}$

(d) False, $\hat{y} = \hat{\gamma} \cdot x$

$\hat{\gamma} \cdot \bar{x} = \frac{\overline{xy}}{\overline{x^2}} \cdot \bar{x} \neq \bar{y}$



5.

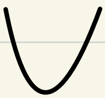
(a) quadratic γ

(b) $g_i'(\gamma) = \frac{2}{n}(y_i - \gamma x_i)\cdot(-x_i)$

$g_i''(\gamma) = \frac{2}{n}(-x_i)^2 = \frac{2x_i^2}{n} > 0$. non-negative

Thus g_i is a convex function

(c) By showing $g_i''(x) > 0$, we know that



$g_i'(x)$ is monogomous increasing,

thus $\begin{cases} g_i'(x) < 0, & x < x_0 \\ g_i'(x) > 0 & x > x_0 \end{cases}$, where $g'(x_0) = 0$

when $x < x_0$ $g_i'(x) < 0$, $g_i(x)$ decreases

when $x > x_0$ $g_i'(x) > 0$ $g_i(x)$ increases.

Thus $\frac{dg(x)}{dx} = 0$ minimizes $g(x)$

(d)

i. Suppose $f(x) = g(x) + h(x)$

$g(c x_1 + (1-c)x_2) \le c\, g(x_1) + (1-c)\, g(x_2)$

$h(c x_1 + (1-c)x_2) \le c\, h(x_1) + (1-c)\, h(x_2)$

Add, $f(c x_1 + (1-c)x_2) \le c\, f(x_1) + (1-c)\, f(x_2)$

Thus, f is a convex function

ii. Prove by induction,

Base case: $n = 2$, $g(x) + h(x)$ is convex

Inductive case: Suppose sum of n convex functions is convex function, we need to prove sum of n+1 convex function is convex.

$$f_1 + f_2 + \cdots f_n + f_{n+1}$$
$$= \underbrace{(f_1 + \cdots + f_n)}_{\text{convex}} + f_{n+1}$$

Thus, sum of n+1 convex function is convex, So the statement is True.

(e) First of all, MSE is a convex function by (a),(b). And by (c), we know that the critical point would minimize MSE function.