Math 5390: Mathematical Imaging

Lecture 2:
More about Image Transformation

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Image transformation

Convolution

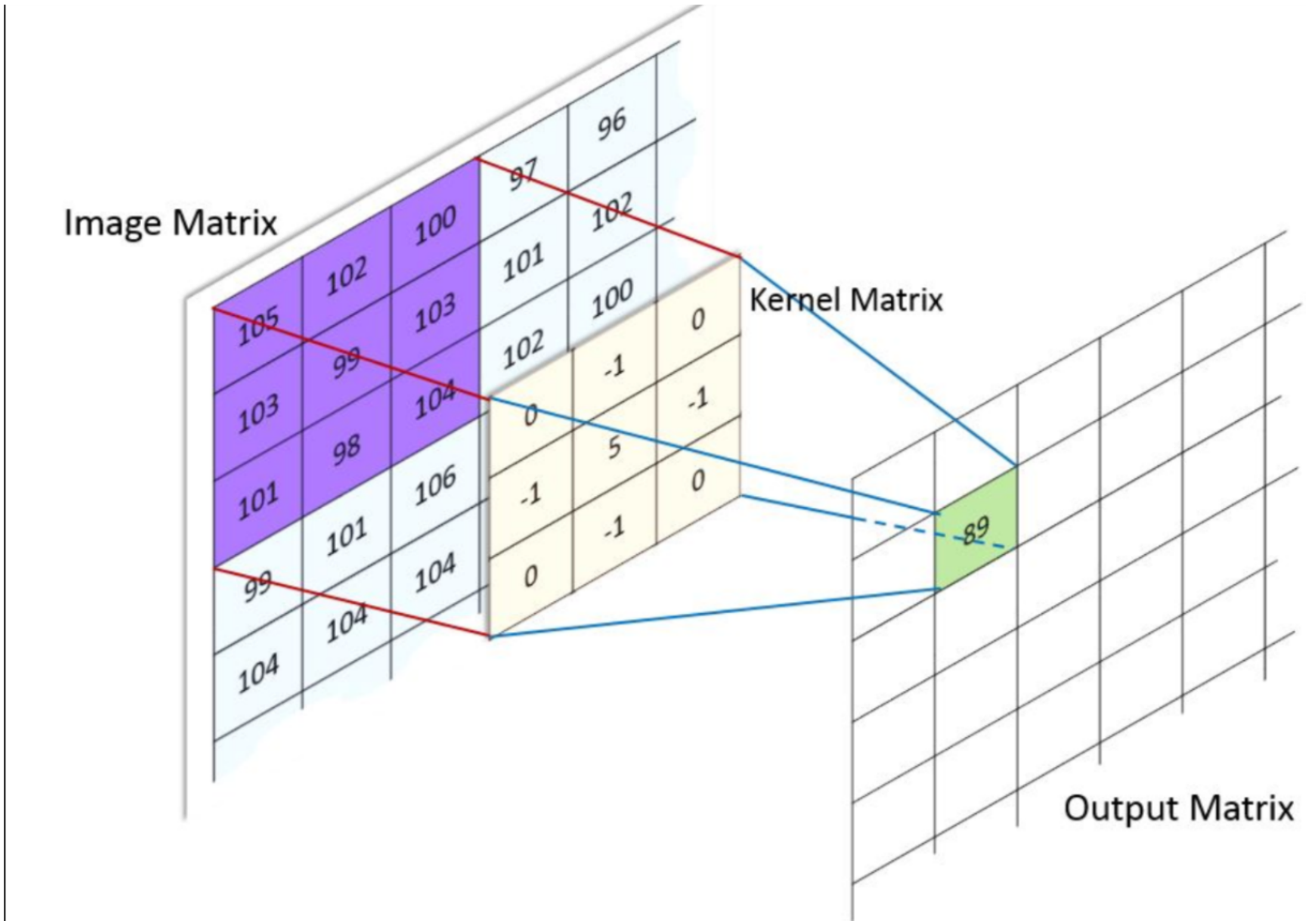


Image transformation

Convolution

Figure 1: The original grayscale image

Image transformation

Convolution

105	102	100	97	96
103	99	103	101	102
101	98	104	102	100
99	101	106	104	99
104	104	104	100	98

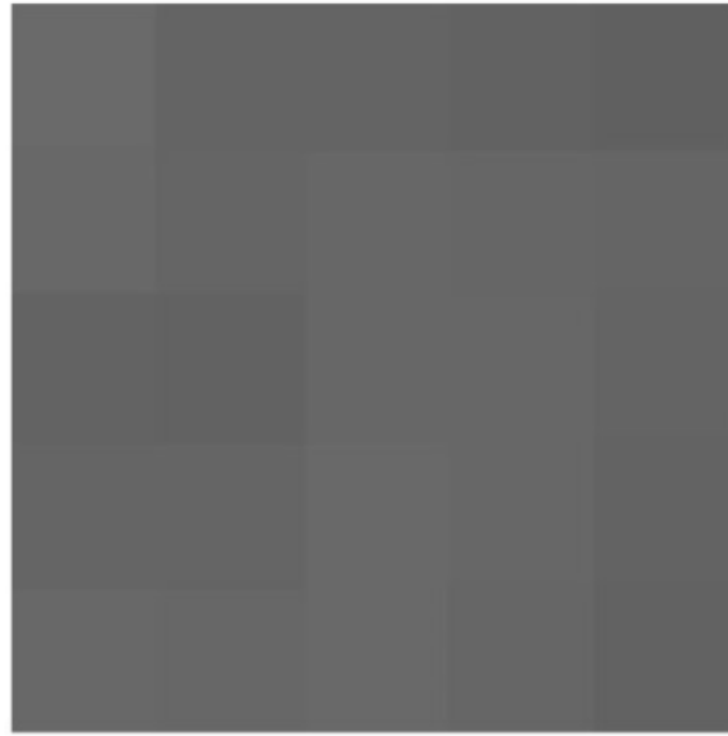


Figure 2: The first 5 columns and rows of the image in Figure 1

Image transformation

Convolution

$$Kernel = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

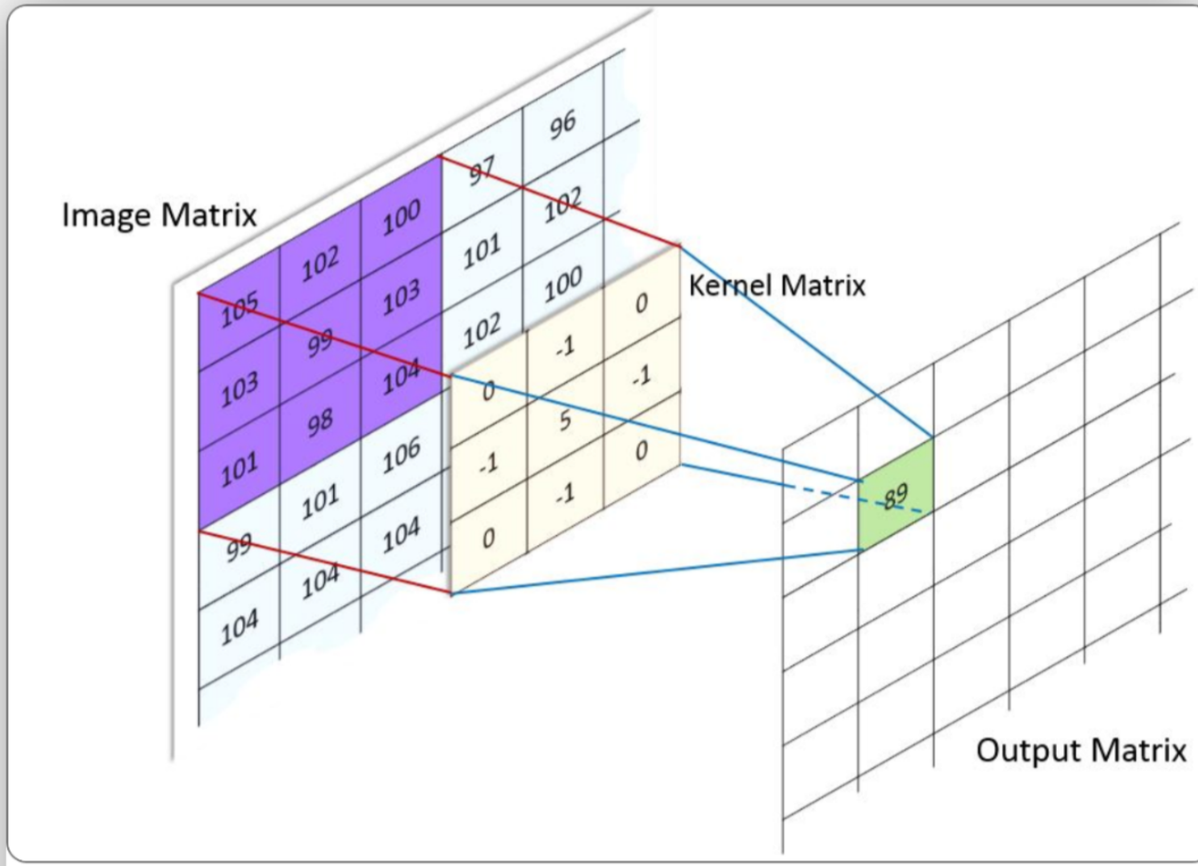


Figure 3: To calculate the value of convolution output at pixel (2,2), center the kernel at the same pixel position on the image matrix

Image transformation

Convolution

Figure 1: The original grayscale image

Image transformation

Convolution

Figure 6: Sharpened image

Image transformation

Convolution

$$Kernel = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Image transformation

Convolution

SVD

For details, please refer to Supplementary note 2!

- **SVD**

Singular Value Decomposition (SVD)

Let g be an image (can be general $m \times n$ image).

Assume $g^T g$ is of rank r. Then g can be written as

$$g = U\Lambda^{1/2}V^T$$

where $U \in M_{m\times m}$ and $V \in M_{n\times n}$ are orthogonal matrices ($UU^T = U^TU = I$ and $VV^T = V^TV = I$) and $\Lambda^{1/2}$ is a diagonal $n \times n$ matrix.

- **An image can be decomposed as:**

$$g = U\Lambda^{1/2}V^T = \sum_{i=1}^{r} \lambda_i^{1/2} \vec{u}_i \vec{v}_i^T .$$

Eigen-image

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

Show the different stages of the SVD of the following image:

$$
g = \begin{pmatrix}
255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\
255 & 255 & 255 & 100 & 100 & 100 & 255 & 255 \\
255 & 255 & 100 & 150 & 150 & 150 & 100 & 255 \\
255 & 255 & 100 & 150 & 200 & 150 & 100 & 255 \\
255 & 255 & 100 & 150 & 150 & 150 & 100 & 255 \\
255 & 255 & 255 & 100 & 100 & 100 & 255 & 255 \\
255 & 255 & 255 & 255 & 50 & 255 & 255 & 255 \\
50 & 50 & 50 & 50 & 255 & 255 & 255 & 255
\end{pmatrix}
$$

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

The image looks like:

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

Consider the eigenvalues of:

$$gg^T = \begin{pmatrix} 520200 & 401625 & 360825 & 373575 & 360825 & 401625 & 467925 & 311100 \\ 401625 & 355125 & 291075 & 296075 & 291075 & 355125 & 381125 & 224300 \\ 360825 & 291075 & 282575 & 290075 & 282575 & 291075 & 330075 & 205025 \\ 373575 & 296075 & 290075 & 300075 & 290075 & 296075 & 332575 & 217775 \\ 360825 & 291075 & 282575 & 290075 & 282575 & 291075 & 330075 & 205025 \\ 401625 & 355125 & 291075 & 296075 & 291075 & 355125 & 381125 & 224300 \\ 467925 & 381125 & 330075 & 332575 & 330075 & 381125 & 457675 & 258825 \\ 311100 & 224300 & 205025 & 217775 & 205025 & 224300 & 258825 & 270100 \end{pmatrix}$$

Eigenvalues are:

$$\begin{array}{llll} 2593416.500 & 111621.508 & 71738.313 & 34790.875 \\ 11882.712 & 0.009 & 0.001 & 0.000 \end{array}$$

We take first 5 eigenvalues!!

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

The corresponding first five eigenvectors are:

$$\begin{pmatrix} 0.441 & -0.167 & -0.080 & -0.388 & 0.764 \\ 0.359 & 0.252 & -0.328 & 0.446 & 0.040 \\ 0.321 & 0.086 & 0.440 & 0.034 & -0.201 \\ 0.329 & 0.003 & 0.503 & 0.093 & 0.107 \\ 0.321 & 0.086 & 0.440 & 0.035 & -0.202 \\ 0.359 & 0.252 & -0.328 & 0.446 & 0.040 \\ 0.407 & 0.173 & -0.341 & -0.630 & -0.504 \\ 0.261 & -0.895 & -0.150 & 0.209 & -0.256 \end{pmatrix}$$

$$\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4 \quad \mathbf{u}_5$$

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

The corresponding first five eigenvectors are:

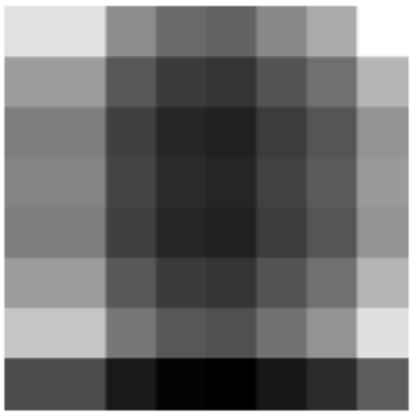
$\mathbf{v}_i$ can be computed by $g^T \mathbf{u}_i$

$$\begin{pmatrix} 0.410 & 0.389 & 0.264 & 0.106 & -0.012 \\ 0.410 & 0.389 & 0.264 & 0.106 & -0.012 \\ 0.316 & 0.308 & -0.537 & -0.029 & 0.408 \\ 0.277 & 0.100 & 0.101 & -0.727 & 0.158 \\ 0.269 & -0.555 & 0.341 & 0.220 & 0.675 \\ 0.311 & -0.449 & -0.014 & -0.497 & -0.323 \\ 0.349 & -0.241 & -0.651 & 0.200 & -0.074 \\ 0.443 & -0.160 & 0.149 & 0.336 & -0.493 \end{pmatrix}$$

Example of SVD decompsition of an image

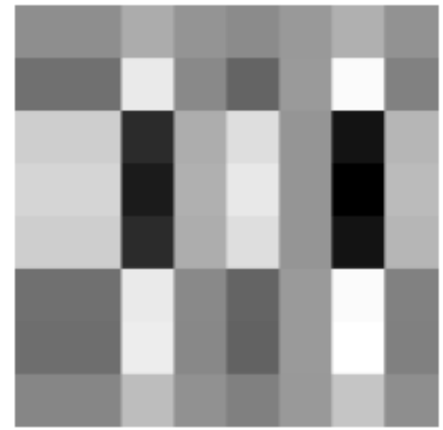
Example 2.1: SVD decomposition of an image

Compute the five eigenimages:



i=1

i=2



i=3

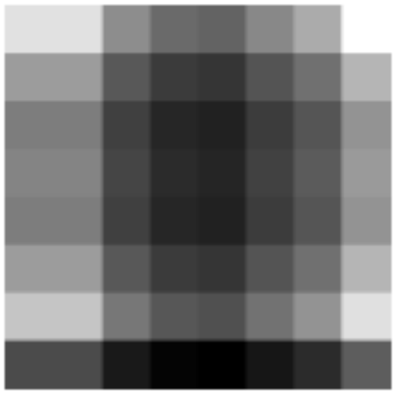
i=4

i=5

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

Compute the five eigen decomposition



a) k=1

b) k=2

c) k=3

d) k=4

e) k=5

Original

Example of SVD decompsition of an image

Example 2.1: SVD decomposition of an image

Error in the reconstruction:

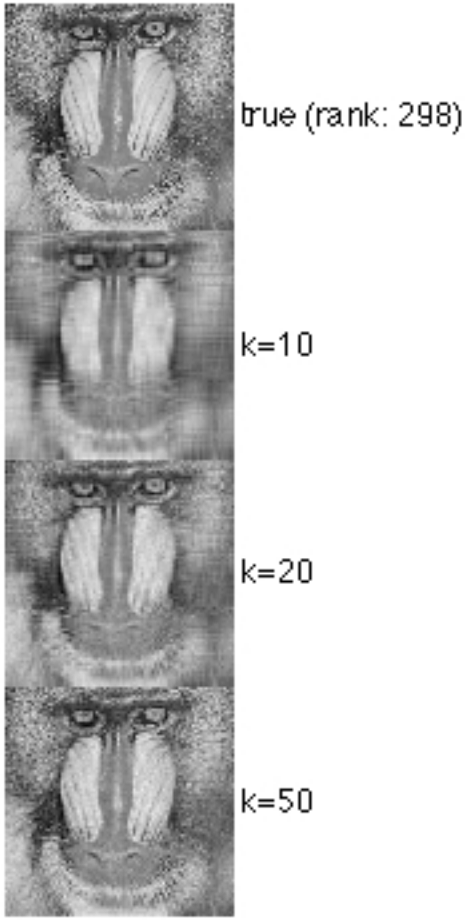
$$\sum_{all\ pixels} (reconstructed\ pixel - original\ pixel)^2$$

$$\begin{array}{lll}
\textit{Square error for image a:} & 230033.32 & (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 230033.41) \\
\textit{Square error for image b:} & 118412.02 & (\lambda_3 + \lambda_4 + \lambda_5 = 118411.90) \\
\textit{Square error for image c:} & 46673.53 & (\lambda_4 + \lambda_5 = 46673.59) \\
\textit{Square error for image d:} & 11882.65 & (\lambda_5 = 11882.71) \\
\textit{Square error for image e:} & &
\end{array}$$

Small error, 0.01!

Example of SVD decompsition of an image

SVD decomposition of an image



Work well for images with patterns

Work well for simple images

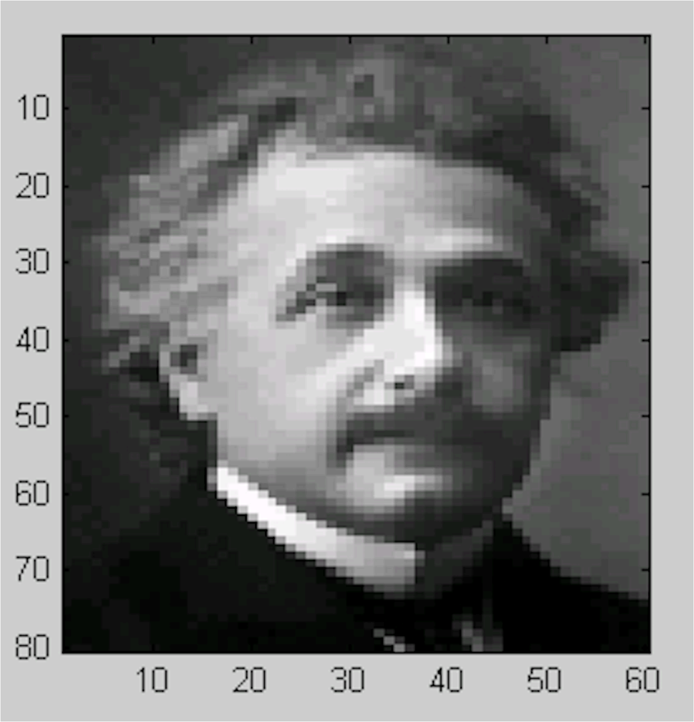
Example of SVD decompsition of an image

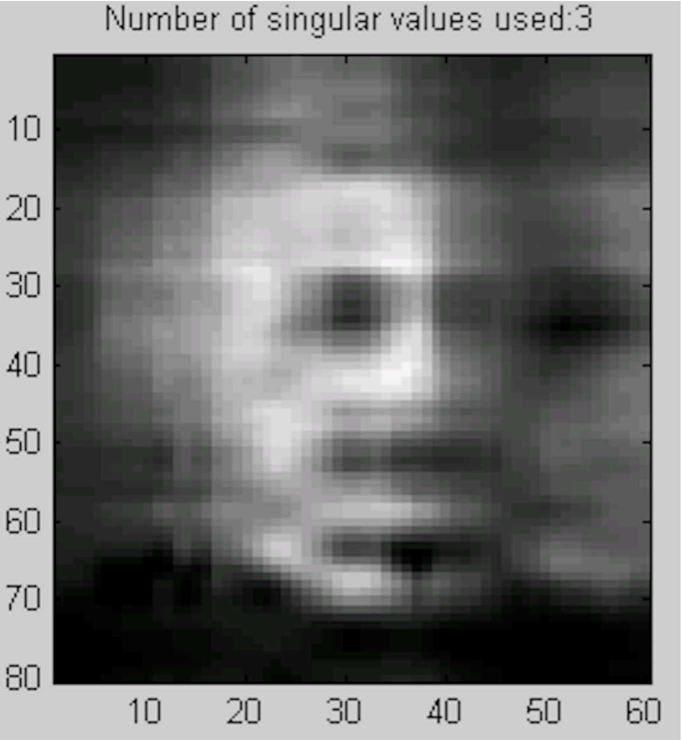
SVD decomposition of an image

Low rank approximation can capture key (big) object

Example of SVD decompsition of an image

SVD decomposition of an image





Low rank approximation can capture key (big) object