Lecture 8: Recall:

Example of Low-pass filters for image denoising

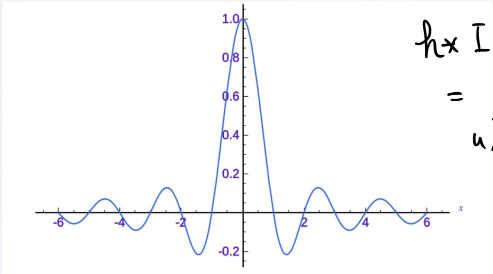
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, $\mathcal{F}^{-1}(H(u,v))$ looks like:



$h * I(x,y)$

$$= \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of I has an effect on $h * I(x,y)$!!

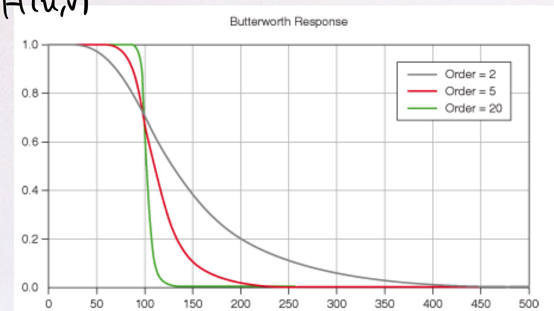
Good: Simple

Bad: Produce ringing effect!

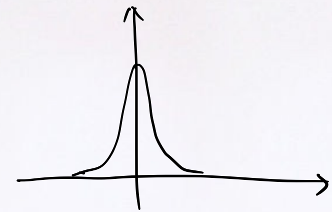
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer):

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^n}$$

$H(u,v)$ in 1-dim



$\mathcal{F}^{-1}(H(u,v))$ in 1-dim

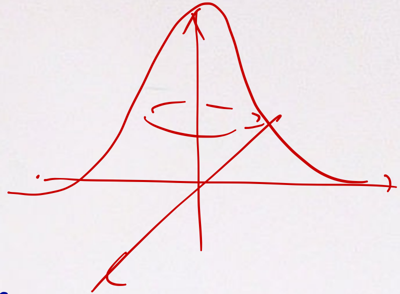


Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$$H(u,v) = \exp\left(-\frac{\overset{u^2+v^2}{D(u,v)}}{2\sigma^2}\right)$$

σ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n}$$

$(H(u,v) = 0 \text{ if } D(u,v) = 0)$

Choose the right n

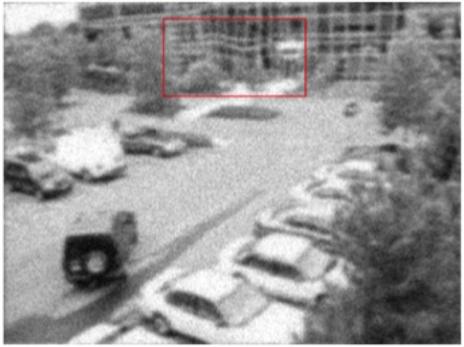
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

Remark:

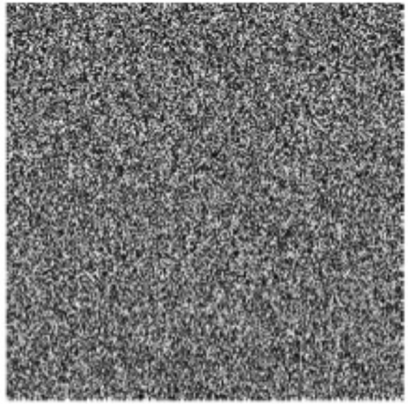
Examples of degradation function $H(u,v)$

Example:

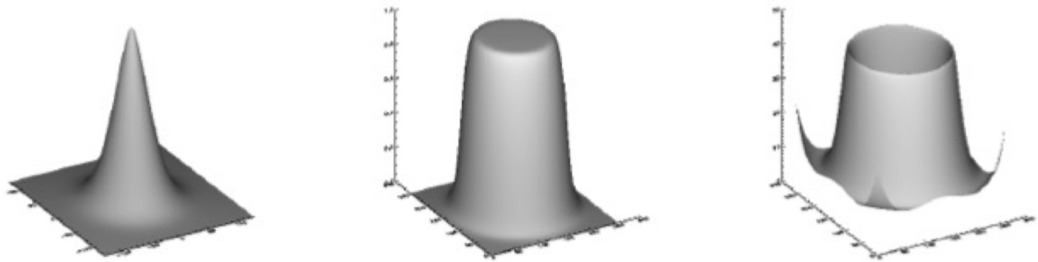
Image deblurring in the frequency domain: (Assume H is known)

Original

Blurred image



Direct inverse filtering



$H(u,v)$ $B(u,v)$: D $= 90, n = 8$ Inverse B/H

Original Image $G(u,v)$ Blurred using D $= 90, n = 8$ Restored with a best D and n.

What does Wiener filtering do mathematically?

We'll show: Wiener filter minimizes the mean square error:

$$\varepsilon^2\Big(\underset{\text{original}}{\underset{\uparrow}{f}},\ \underset{\text{Restored}}{\underset{\uparrow}{\hat{f}}}\Big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2\, dx\, dy$$

(We assume the continuous case to avoid complicated indices)

Assume that f and n are spatially uncorrelated:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, n(x+r, y+s)\, dx\, dy \quad \text{for all } r, s.$$

(Sketch of proof)

We need to use: Parseval Theorem:

$$\varepsilon^2(f,\hat{f}) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2 \, dx\, dy = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v) - \hat{F}(u,v)|^2 \, du\, dv \quad \text{for some constant } C$$

where $F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j(xu+yv)}$, $\hat{F}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(x,y)\, e^{-j(xu+yv)}\, dx\, dy$

Since f and n are spatially uncorrelated, we can show that:

$$\varepsilon^2(f, \hat{f}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(1-WH)F|^2 + |WN|^2 \, du \, dv$$

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-WH)F \, \overline{W} \, \overline{N} \, du \, dv = 0 \right)$$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\,G(u,v)$$

$$\left(H = DFT(h);\quad G(u,v) = DFT(g);\quad P(u,v) = DFT(p) \text{ where } \right.$$

$$\left. p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \right)$$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{H(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Using the fact that both D and L are block-circulant, we can check that:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,

$$\Lambda_D(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\ddot{H} = DFT(h)$.

and

$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} \,,$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

Combining these information and substitute into the "governing" equation:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g},$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

We can check that:

①

$$\boxed{\Lambda_D^* \Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & & \\ & N^4|H(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|H(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}}$$

②

$$\boxed{\Lambda_L^* \Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & & \\ & N^4|P(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|P(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}}$$

③ $\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)}$ where $F = DFT(f), G = DFT(g)$.

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2+\gamma|P(u,v)|^2]NF(u,v)=N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2\frac{|H(u,v)|^2+\gamma|P(u,v)|^2}{\overline{H(u,v)}}F(u,v)=G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f})=(L\vec{f})^T(L\vec{f})$$

Subject to the constraint that:

$$\|\underbrace{\vec{g}-H\vec{f}}_{\vec{n}}\|^2=\varepsilon$$

(allow fixed amount of noise)