MMAT 5390: Mathematical Imaging

Lecture 1:
Introduction to mathematical image processing

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Some Useful Information

- **Lecturer:** Prof. Ronald Lui
- **Email:** lmlui@math.cuhk.edu.hk
- **Tel:** 3943-7975
- **Office:** Lady Shaw Building (LSB) 207
- **Lecture time:** Wed: 6:30pm-9pm
- **Textbook:** Based on ppt & lecture notes
- **Course website:** http://www.math.cuhk.edu.hk/course/1819/MMat5390
- **Other references:**
  - Image processing: the fundamentals by Maria Petrou and Costas Petrou [Free access of online version on CUHK library]
  - Fundamentals of Digital Image Processing: A Practical Approach with Examples in Matlab by Chris Solomon and Toby Breckon [Free access of online version on CUHK library]
  - Digital Image Processing (3rd ed.) by Rafael C. Gonzalez and Richard E. Woods [Available in CUHK bookstore]

Some Useful Information

- **Assessment scheme:**
  - **Homework assignment** **15%**
  - **Midterm (7:30pm-9pm, October 24, in class)** **35%**
  - **Final (6:30pm – 9pm, Dec 5 or 12?)** **50%**

- Relax + enjoy + develop interest in imaging!

What is our goal in Math 3360?

- Mathematical + Image Processing

IMAGE PROCESSING TASKS:
Denoising, Segmentation, Registration, Compression,…

$\Updownarrow$

MATHEMATICS:
Linear algebra, Calculus, transformation,…

What is our goal in Math 3360?

- **Topic to be covered:**
  - **Introduction to digital images and imaging problems:**
    - Digital images, point spread functions, image transformation, separable, shift-invariant etc.
  - **Image compression by image decomposition:**
    - SVD, Haar transform, Walsh tranform, DFT, DCT, EDCT, ODCT, FFT
  - **Image denoising:**
    - What's noise?
    - Denoising in the spatial domain: Linear/non-linear filtering, anisotropic diffusion, energy minimization method (famous ROF model);
    - Denoising in the frequency domain: High-pass/low-pass filtering
  - **Image deblurring:**
    - Inverse filtering, Weiner filter, Constrained least-square filtering
  - **Image segmentation:**
    - Active contour; Famous Chan-Vese segmentation model

Some tastes about IMAGING

- **Image denoising:**
  - Image can be corrupted by "noises" during transmission or error during capturing the image intensity
  - Reconstruct a "clean" (usually visually) image from the noisy one

Image destroyed by noises

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Image destroyed by noises

Restored image

Some tastes about IMAGING

- **Image denoising:**
  - **Where is the MATHEMATICS?**
  - **Minimization model:**

$$\min_u \int_\Omega \left(|\nabla u| + \frac{\lambda}{2}(u-f)^2\right).$$

  - **Solving PDE:**

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(f-u).$$

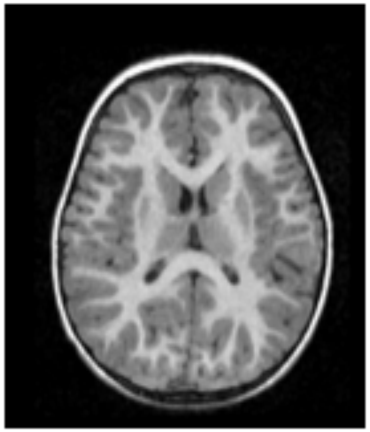
Don't worry about the mathematics! You will learn it (simple version) and find it easy later!

Image destroyed by noises

Restored image

Some tastes about IMAGING

- **Image segmentation:**
  - Image may contain too much information.
  - Need: extract useful information from an image.
  - Image segmentation aims to automatically extract important part or regions of an image.



MRI

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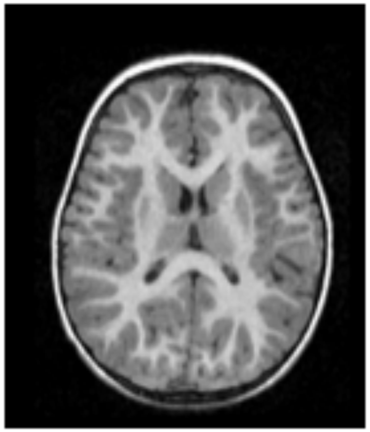

MRI

Segmented region 1

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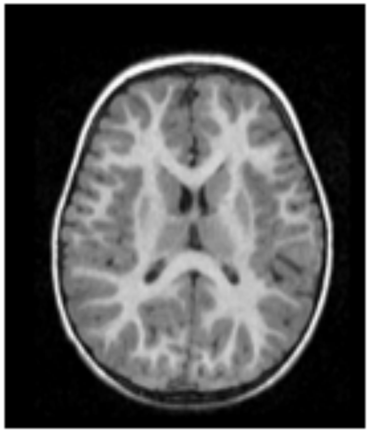


MRI

Segmented region 1

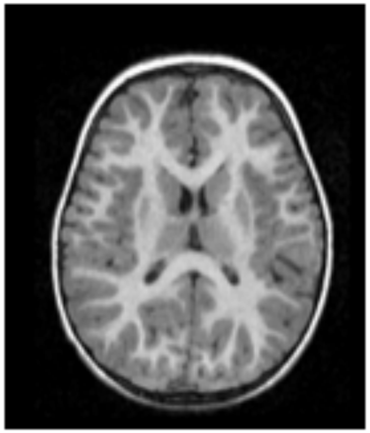
Segmented region 2

Some tastes about IMAGING

- **Image segmentation:**
  - Where is the Mathematics?
  - Minimization model:

$$\arg\min_{c_1,c_2,\varphi} \; \mu \int_\Omega \delta\big(\varphi(x)\big)|\nabla\varphi(x)|\,dx + \nu\int_\Omega H\big(\varphi(x)\big)\,dx$$
$$+\lambda_1\int_\Omega |f(x)-c_1|^2 H\big(\varphi(x)\big)\,dx + \lambda_2\int_\Omega |f(x)-c_2|^2\big(1-H\big(\varphi(x)\big)\big)\,dx$$

Don't worry about the mathematics! You *may* learn it (simple version)!



MRI

Segmented region 1

Segmented region 2

Some tastes about IMAGING

- **Image compression:**
  - Image compression aims to use less storage to represent an image.
  - Do you know familiar JPEG compression is actually based on mathematical theories? You will learn how it works in Math 3360.

Original image

Zoom in of the original image

Some tastes about IMAGING

- ## Image compression:
  - Image compression aims to use less storage to represent an image.
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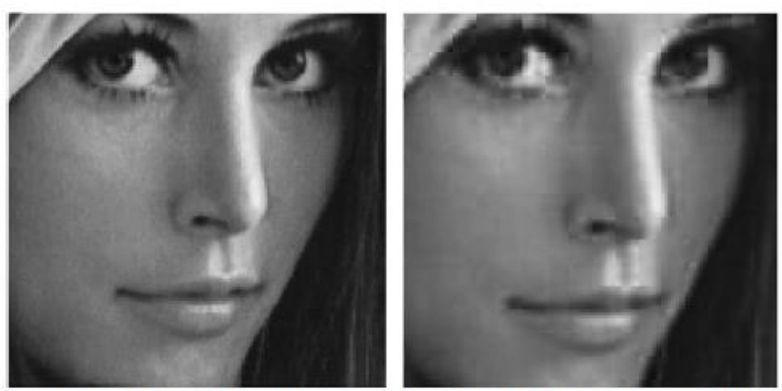
Original image *Compressed image*



Zoom in of the original image *Zoom in of the compressed image*

1% of the original storage

What is a digital image?

- **Mathematical definition:**
  - A 2D (grayscale) digital image is a 2D function defined on a 2D domain (usually rectangular domain):

$$f : \Omega \to \mathbb{R}$$

  - $f(x, y)$ is called the brightness/intensity/grey level;
  - (x,y) is the spatial coordinates of the image.
  - Thus, a 2D digital image looks like this:

$$f(x,y) = \begin{bmatrix} f(1,1) & f(1,2) & \dots & f(1,N) \\ f(2,1) & f(2,2) & \dots & f(2,N) \\ \vdots & \vdots & & \vdots \\ f(N,1) & f(N,2) & \dots & f(N,N) \end{bmatrix}$$

  - Each element in the matrix is called pixel (picture element);
  - Usually, $0 \leq f(x, y) \leq G - 1$ and $(N = 2^n, G = 2^m)$

IMAGE PROCESSING IS RELATED TO LINEAR ALGEBRA!!

What is a digital image?

- **Mathematical definition of color image:**
  - A 2D (color) digital image is a 2D function defined on a 2D domain (usually rectangular domain):

$$f := (f_R, f_G, f_B) : \Omega \to \mathbb{R}^3$$

  - f_R, f_G, f_B are the intensity/brightness/grey level corresponding to R, G and B respectively ;
  - Combination of R, G, B forms the full spectrum of color!

WE WILL FOCUS ON: Grayscale image!

How is a digital image formed?

- **Sensor:**
  - Each sensor captured the amount of photon of certain wavelength;
  - Typical color images consist of three color bands (RGB).
  - Reflected light of an object/phontons are captured by three different sets of sensors, each set made to have a different sensitivity function.

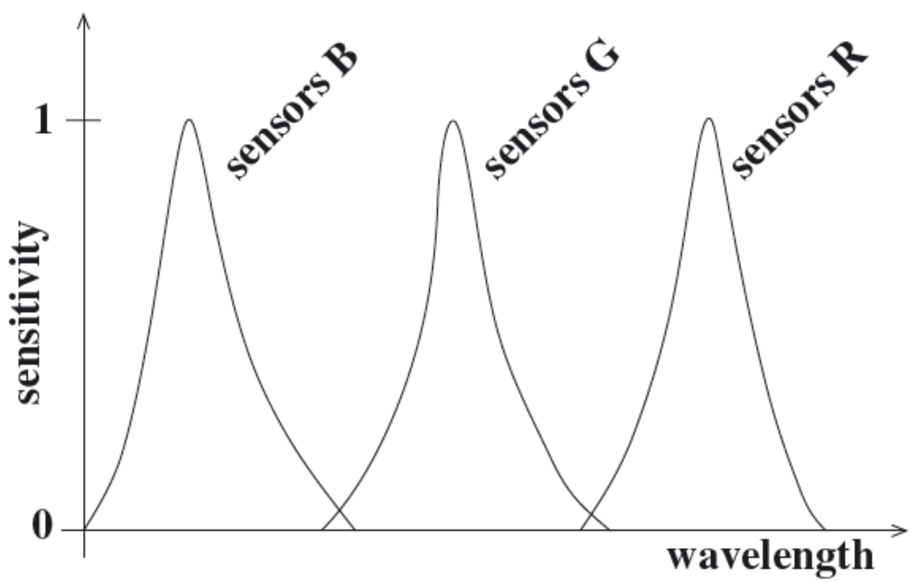


Figure 1: The spectrum of the light which reaches a sensor is multiplied with the sensitivity function of the sensor and recorded by the sensor. This recorded value is the brightness of the image in the location of the sensor and in the band of the sensor. This figure shows the sensitivity curves of three different sensor types.

How is a digital image formed?

- **What is the intensity values recorded by a sensor:**
  - Each spectrum of light is associated with an energy value E;
  - Each sensor has a sensitivity function S associated to different spectrum of light;
  - The intensity = sum of E*S

How is a digital image formed?

- **Example 1:**
  - A digital camera has a triple array of 3x3 sensors:

R G B	R G B	R G B
R G B	R G B	R G B
R G B	R G B	R G B

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

  - The wavelengths of the photons that reach the pixel locations of each triple sensor:

Location $(1,1)$: $\lambda_0, \lambda_9, \lambda_9, \lambda_8, \lambda_7, \lambda_8, \lambda_1, \lambda_0, \lambda_1, \lambda_1$
Location $(1,2)$: $\lambda_1, \lambda_3, \lambda_3, \lambda_4, \lambda_4, \lambda_5, \lambda_2, \lambda_6, \lambda_4, \lambda_5$
Location $(1,3)$: $\lambda_6, \lambda_7, \lambda_7, \lambda_0, \lambda_5, \lambda_6, \lambda_6, \lambda_1, \lambda_5, \lambda_9$
Location $(2,1)$: $\lambda_0, \lambda_1, \lambda_0, \lambda_2, \lambda_1, \lambda_1, \lambda_4, \lambda_3, \lambda_3, \lambda_1$
Location $(2,2)$: $\lambda_3, \lambda_3, \lambda_4, \lambda_3, \lambda_4, \lambda_4, \lambda_5, \lambda_2, \lambda_9, \lambda_4$
Location $(2,3)$: $\lambda_7, \lambda_7, \lambda_6, \lambda_7, \lambda_6, \lambda_1, \lambda_5, \lambda_9, \lambda_8, \lambda_7$
Location $(3,1)$: $\lambda_6, \lambda_6, \lambda_1, \lambda_8, \lambda_7, \lambda_8, \lambda_9, \lambda_9, \lambda_8, \lambda_7$
Location $(3,2)$: $\lambda_0, \lambda_4, \lambda_3, \lambda_4, \lambda_1, \lambda_5, \lambda_4, \lambda_0, \lambda_2, \lambda_1$
Location $(3,3)$: $\lambda_3, \lambda_4, \lambda_1, \lambda_0, \lambda_0, \lambda_4, \lambda_2, \lambda_5, \lambda_2, \lambda_4$

  - Sensitivity of the sensor:

Wavelength	Sensors B	Sensors G	Sensors R	Energy
λ_0	0.2	0.0	0.0	1.00
λ_1	0.4	0.2	0.1	0.95
λ_2	0.8	0.3	0.2	0.90
λ_3	1.0	0.4	0.2	0.88
λ_4	0.7	0.6	0.3	0.85
λ_5	0.2	1.0	0.5	0.81
λ_6	0.1	0.8	0.6	0.78
λ_7	0.0	0.6	0.8	0.70
λ_8	0.0	0.3	1.0	0.60
λ_9	0.0	0.0	0.6	0.50

How is a digital image formed?

- **Example 1.1: (Continued)**

Location $(1,1)$: $\lambda_0, \lambda_9, \lambda_9, \lambda_8, \lambda_7, \lambda_8, \lambda_1, \lambda_0, \lambda_1, \lambda_1$
Location $(1,2)$: $\lambda_1, \lambda_3, \lambda_3, \lambda_4, \lambda_4, \lambda_5, \lambda_2, \lambda_6, \lambda_4, \lambda_5$
Location $(1,3)$: $\lambda_6, \lambda_7, \lambda_7, \lambda_0, \lambda_5, \lambda_6, \lambda_6, \lambda_1, \lambda_5, \lambda_9$
Location $(2,1)$: $\lambda_0, \lambda_1, \lambda_0, \lambda_2, \lambda_1, \lambda_1, \lambda_4, \lambda_3, \lambda_3, \lambda_1$
Location $(2,2)$: $\lambda_3, \lambda_3, \lambda_4, \lambda_3, \lambda_4, \lambda_4, \lambda_5, \lambda_2, \lambda_9, \lambda_4$
Location $(2,3)$: $\lambda_7, \lambda_7, \lambda_6, \lambda_7, \lambda_6, \lambda_1, \lambda_5, \lambda_9, \lambda_8, \lambda_7$
Location $(3,1)$: $\lambda_6, \lambda_6, \lambda_1, \lambda_8, \lambda_7, \lambda_8, \lambda_9, \lambda_9, \lambda_8, \lambda_7$
Location $(3,2)$: $\lambda_0, \lambda_4, \lambda_3, \lambda_4, \lambda_1, \lambda_5, \lambda_4, \lambda_0, \lambda_2, \lambda_1$
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Wavelength	Sensors B	Sensors G	Sensors R	Energy
λ_0	0.2	0.0	0.0	1.00
λ_1	0.4	0.2	0.1	0.95
λ_2	0.8	0.3	0.2	0.90
λ_3	1.0	0.4	0.2	0.88
λ_4	0.7	0.6	0.3	0.85
λ_5	0.2	1.0	0.5	0.81
λ_6	0.1	0.8	0.6	0.78
λ_7	0.0	0.6	0.8	0.70
λ_8	0.0	0.3	1.0	0.60
λ_9	0.0	0.0	0.6	0.50

- Intensity:

$$
\begin{aligned}
E_R &= \begin{pmatrix} 2.645 & 2.670 & 3.729 \\ 1.167 & 4.053 & 4.576 \\ 4.551 & 1.716 & 1.801 \end{pmatrix} \\
E_G &= \begin{pmatrix} 1.350 & 4.938 & 4.522 \\ 2.244 & 4.176 & 4.108 \\ 2.818 & 2.532 & 2.612 \end{pmatrix} \\
E_B &= \begin{pmatrix} 1.540 & 5.047 & 1.138 \\ 4.995 & 5.902 & 0.698 \\ 0.536 & 4.707 & 5.047 \end{pmatrix}
\end{aligned}
$$

What is "Image resolution"?

- **Image resolution:**
  - Recall: A digital image looks like:

$$f(x,y) = \begin{bmatrix} f(1,1) & f(1,2) & \dots & f(1,N) \\ f(2,1) & f(2,2) & \dots & f(2,N) \\ \vdots & \vdots & & \vdots \\ f(N,1) & f(N,2) & \dots & f(N,N) \end{bmatrix}$$

where:

$$0 \leq f(x,y) \leq G-1 \quad (N = 2^n,\ G = 2^m)$$

  - (N,G) is called the image resolution.
  - Sometimes, (n,m) is referred to as image resolution as well.

What is "Image resolution"?

- **Effect on different image resolution:**
  - False contouring: reducing M

$m = 8$

$m = 7$

$m = 6$

$m = 5$

$m = 4$

$m = 3$

What is "Image resolution"?

- **Little Effect by m on a complicated image:**

256 grey levels ($m = 8$)

128 grey levels ($m = 7$)

64 grey levels ($m = 6$)

32 grey levels ($m = 5$)

What is "Image resolution"?

- **Effect on different image resolution:**
  - Checkerboard effect: reducing N

256 × 256 pixels

128 × 128 pixels

64 × 64 pixels

32 × 32 pixels

What is "Image contrast"?

- **Good image contrast means:**
  - grey values present in the image range from black to white;
  - making use of the full range of intensity/brightness to which the human vision system is sensitive.

How do we read a digital image in Matlab?

- **Keep in mind: imread & imwrite!**
- **Please attend TA session when you will learn MATLAB command to do mathematical imaging!**

Main tasks in image processing

- **Major tasks in imaging includes:**
  - **To improve the quality of an image in a subjective way, usually by increasing its contrast. This is called image enhancement.**
  - **To use as few bits as possible to represent the image, with minimum deterioration in its quality. This is called image compression.**
  - **To improve an image in an objective way, for example by reducing its blurring. This is called image restoration.**
  - **To extract explicit characteristics of the image which can be used to identify the contents of the image. This is called feature extraction.**

Image enhancement

Original

Enhanced

Image enhancement

Original

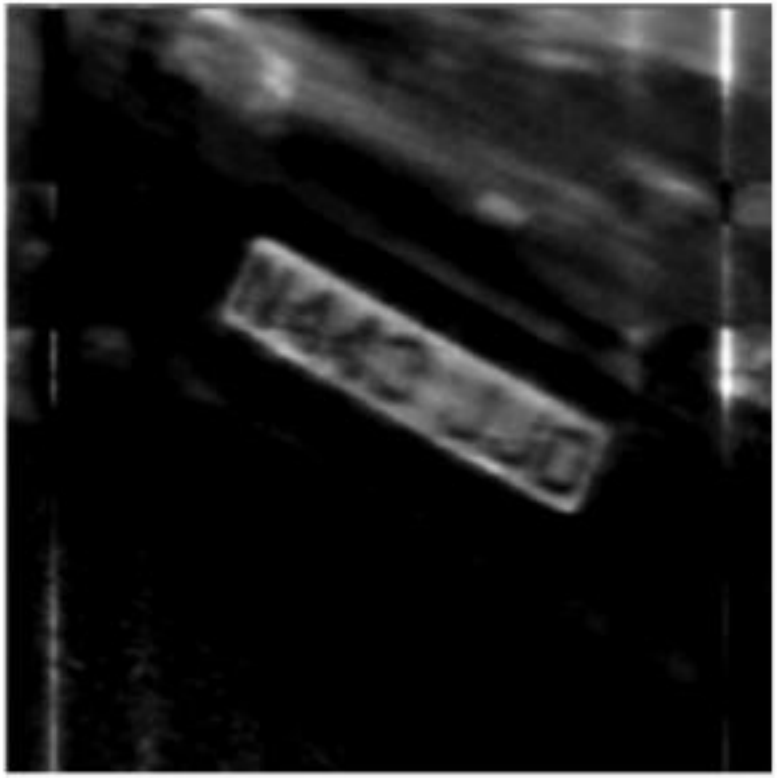
Deblurred

Image enhancement

Blurry image

Original

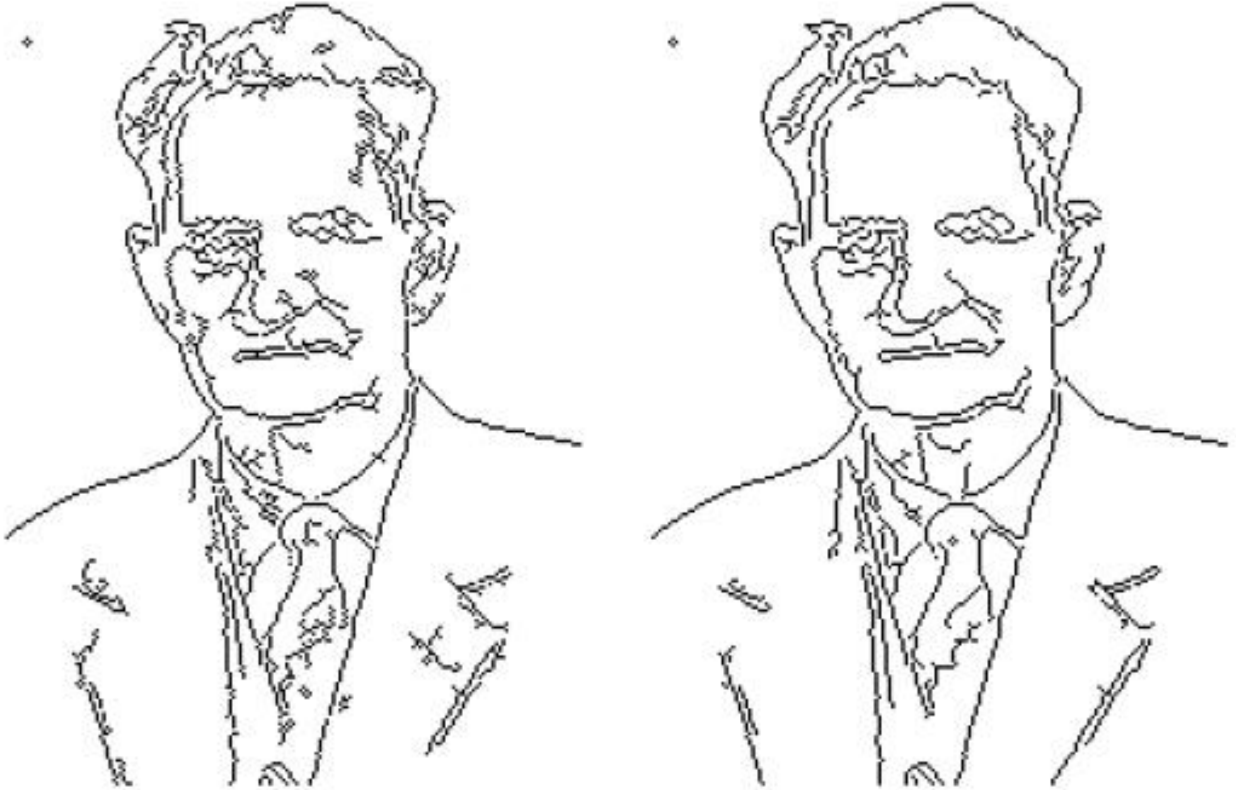
Deblurred image



Deblurred

Feature selection

Original image



Edge detection

Main tasks in image processing

- **Mathematical technique for imaging (in 1 sentence):**
  - **TRANSFORM an input image to a better image**
  - **Mathematically,**

$$\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$$

  - $\mathcal{O}$ **can be LINEAR:** $\mathcal{O}[af + bg] = a\mathcal{O}[f] + b\mathcal{O}[g]$
  - $\mathcal{O}$ **can be NON-LINEAR**

Now: some mathematics!

- Please refer to Chapter 1 in the lecture note.
- You will learn:
  - How linear operator to transform an image (to a better image) is defined?
  - Shift invariant v.s. convolution
  - Separable operator