Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\operatorname{rank}(AB) = \operatorname{rank}(B)$ if A is invertible

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Suppose $g = U \Sigma V^T$. Since U and V are invertible, $\operatorname{rank}(g) = \operatorname{rank}(\Sigma)$

$= \#$ of non-zero singular values

Theorem: $\text{Range}(g) = \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ where $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix}$; $r = \text{rank}(g)$

$\text{Null}(g) = \text{span}\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ where $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$.

Proof: Exercise.

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Later!

How to compute SVD

Let $A \in M_{m\times n}$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \\ \hline & & 0 & \end{pmatrix} \in M_{m\times n}$$

Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\dots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}.$$

Now, $\operatorname{eig}(A^*A)$ are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

Since

$$\sigma_1 \vec{u}_1 = A\vec{v}_1,$$

we have

$$\vec{u}_1 = \frac{1}{\sqrt{17}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \\ \frac{4}{\sqrt{34}} & 0 & \mathbf{u}_3 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \end{pmatrix}$$

for some vector $\mathbf{u}_3$ orthonormal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. One possibility is

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \leq n$.

We need the following theorem.

Theorem: Let $B \in M_{n \times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

Note that $gg^T \in M_{m \times m}$ and $g^T g \in M_{n \times n}$ are symmetric.

$\therefore \exists$ n pairwise orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of $g^T g$.

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Note that $g g^T (g \vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i (g \vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Note that $g^T g$ is positive-definite and hence all of its eigenvalues must be +ve.

$\therefore$ $\lambda_i > 0$ for $i = 1, 2, \ldots, r$.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T (g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T (\lambda_i \vec{v}_i) = \lambda_i$.

$$\therefore \quad \|g\vec{v}_i\| = \sigma_i$$

Define $\vec{u}_i = \dfrac{g\vec{v}_i}{\sigma_i}$. Then: $\|\vec{u}_i\| = 1$.

Also, $\vec{u}_i \cdot \vec{u}_j = \dfrac{(g\vec{v}_i)^T (g\vec{v}_j)}{\sigma_i \sigma_j} = \dfrac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i \sigma_j} = \dfrac{\lambda_j \underbrace{\vec{v}_i^T \vec{v}_j}_{= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}}{\sigma_i \sigma_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases} \sigma_i & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

In matrix form,

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n \times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}$$

Extend $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Extend $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$.

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m\times m} \underbrace{g}_{m\times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n\times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}}_{m\times n} \neq \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^TU = UU^T = I$; $V^TV = VV^T = I$. $\therefore$ $g = U\Lambda^{1/2}V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Recap on the proof of existence:

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \le n$.

We need the following theorem.

Theorem: Let $B \in M_{n\times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

Note that $gg^T \in M_{m\times m}$ and $g^Tg \in M_{n\times n}$ are symmetric.

$\therefore \exists$ n pairwise $\boxed{\text{orthonormal eigenvectors } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \text{ of } g^Tg.}$

Find orthonormal basis of g^Tg

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i (g\vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Note that g^Tg is positive-definite and hence all of its eigenvalues must be +ve.

$\therefore$ $\lambda_i > 0$ for $i = 1, 2, \dots, r$.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i \vec{v}_i) = \lambda_i$.

Define $\vec{u}_i$

$\therefore$ $\|g\vec{v}_i\| = \sigma_i$

$\boxed{\text{Define } \vec{u}_i = \dfrac{g\vec{v}_i}{\sigma_i}}$. Then: $\|\vec{u}_i\| = 1$.

$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Also, $\vec{u}_i \cdot \vec{u}_j = \dfrac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i \sigma_j} = \dfrac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i \sigma_j} = \dfrac{\lambda_j \vec{v}_i^T \vec{v}_j}{\sigma_i \sigma_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases} \sigma_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

In matrix form,

$$\boxed{\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n \times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}}$$

Form preliminary matrix decomposition

Extend $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Extend $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$.

Extend basis

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n \times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}}_{m \times n} \stackrel{\text{def}}{=} \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^T U = U U^T = I$; $V^T V = V V^T = I$. $\therefore$ $g = U \Lambda^{1/2} V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Remark:

1. Note that $gg^T = U\Lambda^{1/2}\underbrace{V^TV}_{I}\Lambda^{1/2}U^T = U\Lambda U^T$

$\therefore$ U consists of eigenvectors of gg^T.

Note that $g^Tg = V\Lambda^{1/2}\underbrace{U^TU}_{I}\Lambda^{1/2}V^T = V\Lambda V^T$

$\therefore$ V consists of eigenvectors of g^Tg.

2. Note that $$g = U\underline{\underline{\Lambda}}^{1/2}V^T = \sum_{i=1}^{r}\sigma_i\, U \begin{array}{cc} \begin{array}{c} i\text{th} \\ \downarrow \end{array} & \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} & \leftarrow i\text{th} \end{array} V^T = \sum_{i=1}^{r}\sigma_i \vec{u}_i \vec{v}_i^T$$

$\vec{u}_i\vec{v}_i^T$ is called the eigen-image of g under SVD.

3. For $N \times N$ image, the required storage is:

$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r\text{ terms}} = (2N+1)r$$

Definition: For any k $(0 \le k \le r)$, we define

$$g_k = \sum_{j=1}^{k}\sigma_j \vec{u}_j \vec{v}_j^T \quad (\text{rank-}k \text{ approximation of } g)$$

Error of the approximation by SVD

Theorem: Let $f = \sum_{j=1}^{r} \sigma_j \vec{u}_j \vec{v}_j^T$ be the SVD of a $M \times N$ image f. For any $k < r$,

and $f_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^T$, we have: $\| f - f_k \|_F^2 = \sum_{i=k+1}^{r} \underset{\lambda_i}{\sigma_i^2}$

Proof: Let $f = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$.

Let $D \equiv f - f_k = \sum_{i=k+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \in M_{M \times N}$.

Then, the m-th row, n-th col entry of D is given by:

$$D_{mn} = \sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \in \mathbb{R} \quad \text{where} \quad \vec{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iM} \end{pmatrix} ; \quad \vec{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iN} \end{pmatrix}$$

$$\therefore D_{mn}^2 = \left(\sum_{i=k+1}^{r} \sigma_i u_{im} v_{in} \right)^2 = \sum_{i=k+1}^{r} \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn}.$$

Thus, $\|D\|_F^2 = \sum_m \sum_n D_{mn}^2$

$$= \sum_m \sum_n \left(\sum_{i=k+1}^{r} \sigma_i^2 \, u_{im}^2 \, v_{in}^2 + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j \, u_{im} v_{in} u_{jm} v_{jn} \right)$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 \underbrace{\sum_m u_{im}^2}_{1} \underbrace{\sum_n v_{in}^2}_{1} + 2 \sum_{i=k+1}^{r} \sum_{\substack{j=k+1 \\ j \neq i}}^{r} \sigma_i \sigma_j \underbrace{\sum_m u_{im} u_{jm}}_{0} \underbrace{\sum_n v_{in} v_{jn}}_{0}$$

$$= \sum_{i=k+1}^{r} \sigma_i^2 = \lambda_i$$

Remark:

- To approximate an image using SVD, arrange the eigenvalues λ_i in decreasing order and remove the last few terms in $\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$
- rank - k approximation is the optimal approximation using k-terms (in term of F-norm) (or with rank-k image)

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

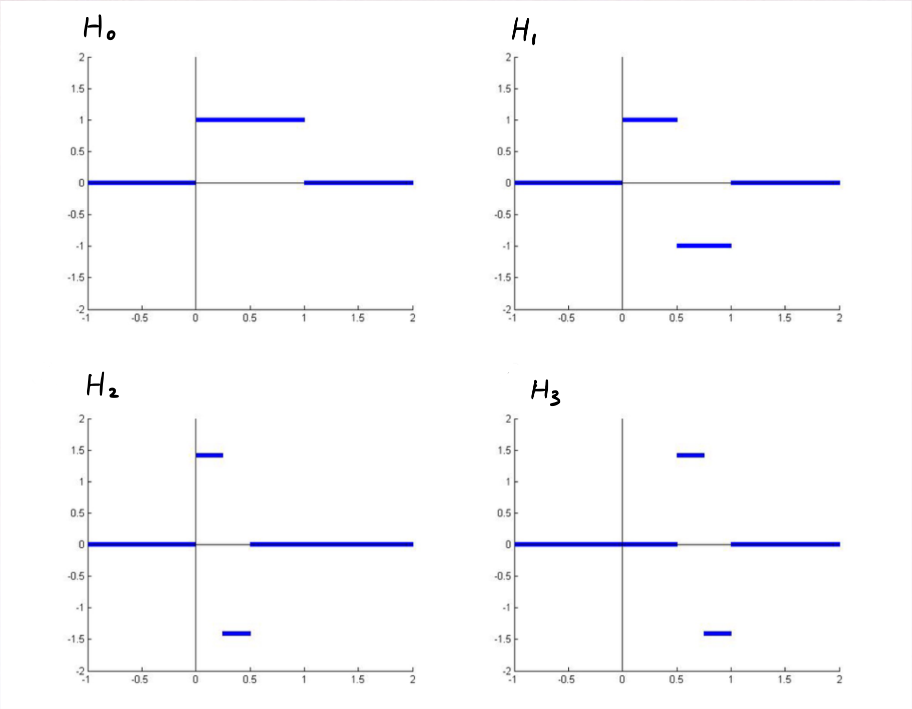
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that: $$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^M - 1}{\underbrace{e^{j2\pi\left(\frac{t}{M}\right)} - 1}_{\text{if } t\neq 0}} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

$\therefore$ (*) becomes: $$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \le k, l \le N-1$ and $U = (U_{kl})_{0 \le k,l \le N-1} \in M_{N \times N}$

U is clearly symmetric and also:

$$\hat{g} = U g U \quad \text{(DFT in matrix form)}$$

Note that: $\frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi (x_2 - x_1)\alpha}{N}} = \frac{1}{N^2} N \delta(x_2 - x_1)$

$$= \frac{1}{N} \delta(x_2 - x_1)$$

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\,T} \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^{N}$ is orthogonal but NOT orthonormal!

$\therefore \quad \mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^T$ ← Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

$$\hat{g} = \mathcal{U} g \mathcal{U}$$

$$\Rightarrow \quad \mathcal{U}^* \hat{g} \mathcal{U}^* = (\mathcal{U}^*\mathcal{U}) g (\mathcal{U}\mathcal{U}^*)$$

$$= \left(\frac{1}{N}\right) g \left(\frac{1}{N}\right)$$

$$\therefore \quad (N\mathcal{U})^* \hat{g} (N\mathcal{U})^* = g \quad //$$

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l}$$

$$= \left(e^{-j2\pi\left(\frac{kl}{4}\right)} \right)$$