Math 5390: Mathematical Imaging

Lecture 2:
Mathematical Review on Linear Algebra

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Linear Algebra: Revisit

Determinant

$$
\begin{array}{ccccc}
 & & 0 & -105 & 0 \\
-7 & 6 & -5 & -7 & 6 \\
0 & -1 & -5 & 0 & -1 \\
0 & -3 & 8 & 0 & -3 \\
 & & 56 & 0 & 0
\end{array}
$$

$$
\therefore \begin{vmatrix} -7 & 6 & -5 \\ 0 & -1 & -5 \\ 0 & -3 & 8 \end{vmatrix} = 56 + 0 + 0 - (0 + -105 + 0) = 161
$$

Linear Algebra: Revisit

Determinant

Definition 1.1. Let A be a $n \times n$ matrix. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\boxed{\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j}),}$$

where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from removing row i and column j of A (called the **minor** of the entry of A in row i, column j). The scalar $\det(A)$ is called the **determinant** of A and is also denoted by $|A|$. The scalar

$$(-1)^{i+j} \det(\tilde{A}_{ij})$$

is called the **cofactor** of the entry of A in row i, column j.

Reduce the computation of determinant of a big matrix to a smaller matrix

$$\boxed{\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij}).}$$

Linear Algebra: Revisit

Properties of Determinant

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} + k\mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + k \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

(Linearity)

Linear Algebra: Revisit

Properties of Determinant

Theorem 1.4. *If A is a square matrix and B is a matrix obtained from A by interchanging any two rows of A, then* $\det(B) = -\det(A)$.

Theorem 1.5. *Let A be a square matrix, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then* $\det(B) = \det(A)$.

Corollary. *If a $n \times n$ matrix has rank less than n, then* $\det(A) = 0$.

Theorem 1.6. *For any two $n \times n$ matrices A and B,* $\det(AB) = \det(A)\det(B)$.

Corollary. *A square matrix is invertible if and only if* $\det(A) \neq 0$. *Furthermore, if A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Theorem 1.7. *For any square matrix A,* $\det(A^T) = \det(A)$.

Linear Algebra: Revisit

Eigenvalues and Eigenvectors

Definition 1.8. Let A be a $n \times n$ matrix. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$. The scalar λ is called the **eigenvalue** of A corresponding to the eigenvector $\mathbf{v}$.

Definition 1.9. Let A be a $n \times n$ matrix. The polynomial $f(t) = det(A - tI_n)$ is called the **characteristic polynomial** of A.