

Example of Low-pass filters for image denoising

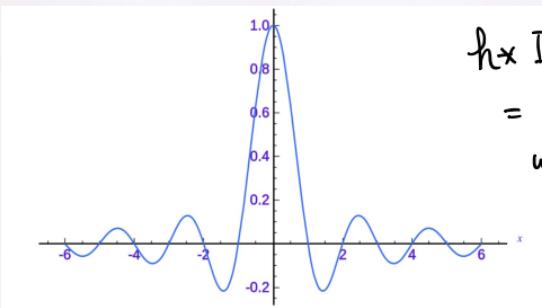
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-\frac{N}{2} \leq u \leq \frac{N}{2}-1$, $-\frac{N}{2} \leq v \leq \frac{N}{2}-1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \leq D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, $\mathcal{F}^{-1}(H(u,v))$ looks like:



$$h * I(x,y)$$

$$= \sum_{u,v} h(x-u, y-v) I(u,v)$$

every pixel values of
I has an effect on
 $h * I(x,y) !!$

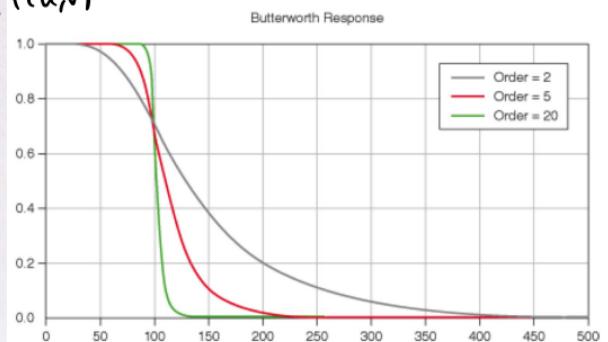
Good: Simple

Bad : Produce ringing effect!

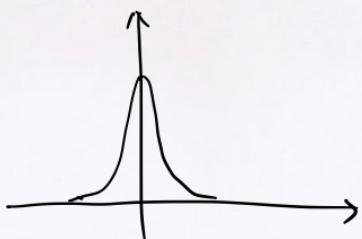
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer) :

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_o\right)^n}$$

$H(u,v)$ in 1-dim



$\tilde{f}^{-1}(H(u,v))$ in 1-dim



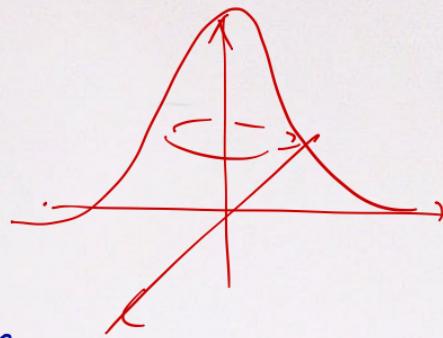
Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$$H(u, v) = \exp\left(-\frac{D(u, v)}{2\sigma^2}\right)$$

$$u^2 + v^2$$

σ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n} \quad (H(u,v) = 0 \text{ if } D(u,v) = 0)$$

Choose the right n

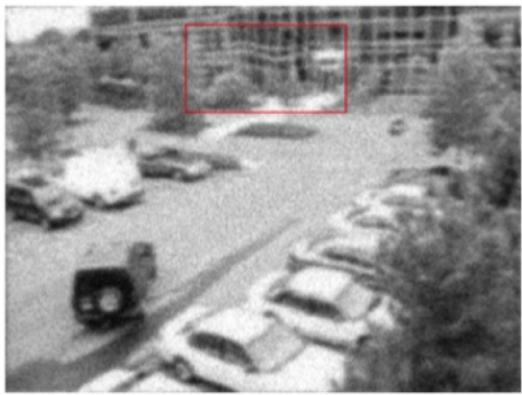
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

Image deblurring



Atmospheric turbulence



Motion Blur



Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = H(f) + n$

where H is the degradation function/operator and n is the additive noise.

Assumption on H :

1. H is position invariant:

Let $g(x, y) = H(f)(x, y)$ and let $\tilde{f}(x, y) := f(x - \alpha, y - \beta)$.

Then: $H(\tilde{f})(x, y) = g(x - \alpha, y - \beta)$

2. Linear: $H(f_1 + f_2) = H(f_1) + H(f_2)$

$H(\alpha f) = \alpha H(f)$ where α is a scalar multiplication.

3. Linearity can be extended to integral:

$$H\left(\iint \alpha(u, v) f(x-u, y-v) du dv\right) = \iint \alpha(u, v) H(f)(x-u, y-v) du dv$$

With the above assumption, consider an impulse signal:

$$\delta(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0) \\ 0 & \text{if } (x, y) \neq (0, 0) \end{cases}$$

Then: $f(x, y) = f * \delta(x, y) = \sum_{\alpha=-M_2}^{M_2-1} \sum_{\beta=-N_2}^{N_2-1} f(\alpha, \beta) \delta(x-\alpha, y-\beta)$

$$\therefore g(x, y) = H(f)(x, y)$$

$$= \sum_{\alpha=-M_2}^{M_2-1} \sum_{\beta=-N_2}^{N_2-1} f(\alpha, \beta) H(\delta)(x-\alpha, y-\beta) \quad (\text{by linearity and position-invariant})$$

$$= \sum_{\alpha=-M_2}^{M_2-1} \sum_{\beta=-N_2}^{N_2-1} f(\alpha, \beta) h(x-\alpha, y-\beta) \quad \text{where } h(x, y) = H(\delta)(x, y)$$

$$= f * h(x, y)$$

\therefore With the above assumption,

Degradation/Blur = Convolution

Remark:

1. h is called the point spread function

2. $\therefore g(x,y) = h * f(x,y) + n(x,y)$

In the frequency domain,

$$G(u,v) = c H(u,v) F(u,v) + N(u,v)$$

\nwarrow constant

\therefore Deblurring can be done by:

$$\text{Compute: } F(u,v) \approx \frac{G(u,v)}{cH(u,v)}$$

from observed image
 \downarrow
from known degradation

Obtain: $f(x,y) = \text{DFT}^{-1}(F(u,v))$

(Does NOT work very well due to noise!)

Examples of degradation function $H(u,v)$

1. Atmospheric turbulence blur:

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k = degree of turbulence

$k = 0.0025$ (severe)

$k = 0.001$ (mild)

$k = 0.00025$ (low turbulence)

2. Out of focus blur:

In the frequency domain, define $H(u,v)$ as the DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq D_o^2 \\ 0 & \text{otherwise} \end{cases}$$

In some situations, a simple model:

$$H(u,v) = \begin{cases} 1 & \text{if } u^2 + v^2 \leq D_o^2 \\ 0 & \text{otherwise} \end{cases} \quad (\text{Usually not accurate})$$

3. Uniform Linear Motion Blur

Assume $f(x, y)$ undergoes planar motion during acquisition.
(original)

Let $(x_0(t), y_0(t))$ be the motion components in the x- and y-directions
time

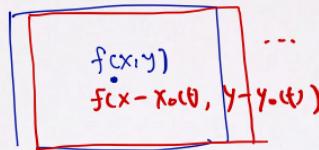
Let T be the total exposure time.

The observed image is given by:

$$g(x, y) = \int_0^T f(x - x_0(t), y - y_0(t)) dt$$

Now, let $G(u, v) = \text{DFT}(g)(u, v)$, then:

$$\begin{aligned} G(u, v) &= \frac{1}{N^2} \sum_x \sum_y g(x, y) e^{-j \frac{2\pi}{N} (ux + vy)} \\ &= \frac{1}{N^2} \sum_x \sum_y \int_0^T f(x - x_0(t), y - y_0(t)) dt e^{-j \frac{2\pi}{N} (ux + vy)} \\ &= \left(\sum_x \sum_y f(x - x_0(t), y - y_0(t)) e^{-j \frac{2\pi}{N} (ux + vy)} \right) dt \end{aligned}$$



Recall that $DFT(f(x-x_0, y-y_0)) = F(u,v) e^{-j\frac{2\pi}{N}(ux_0(t) + vy_0(t))}$

We have:
$$\begin{aligned} G(u,v) &= \int_0^T [F(u,v) e^{-j\frac{2\pi}{N}(ux_0(t) + vy_0(t))}] dt \\ &= F(u,v) \int_0^T e^{-j\frac{2\pi}{N}(ux_0(t) + vy_0(t))} dt \\ &= F(u,v) H(u,v) \end{aligned}$$

∴ Degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(ux_0(t) + vy_0(t))} dt$$

(Speeding problem !!)

Example: Suppose the camera is moving left horizontally with a constant speed c .

That is, the image at time t is given by:

$$I^*(x, y) = I(x, y - ct)$$

Then: the degradation function is given by:

$$H(u, v) = \int_0^T e^{-j\frac{2\pi}{N}(v(ct))} dt$$

Remark: Once the degradation function is known, the original image can be restored by: $IDFT\left(\frac{G(u, v)}{H(u, v)}\right)$ (given that there's no noise)

What if there is noise??

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \frac{1}{H(u,v) + \varepsilon \operatorname{sgn}(H(u,v))}$ ($\operatorname{sgn}(z) = 1$ if $\operatorname{Re}(z) \geq 0$ and $\operatorname{sgn}(z) = -1$ otherwise)
Avoid singularity

Compute $\hat{F}(u,v) = G(u,v) T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boost up noise

$$\hat{F}(u,v) = G(u,v) T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon \operatorname{sgn}(H(u,v))}$$
$$\frac{H(u,v)F(u,v) + N(u,v)}{H(u,v)}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (keep low frequencies)
is small for (u,v) far away from $(0,0)$

$\therefore \frac{N(u,v)}{H(u,v) + \varepsilon \operatorname{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$

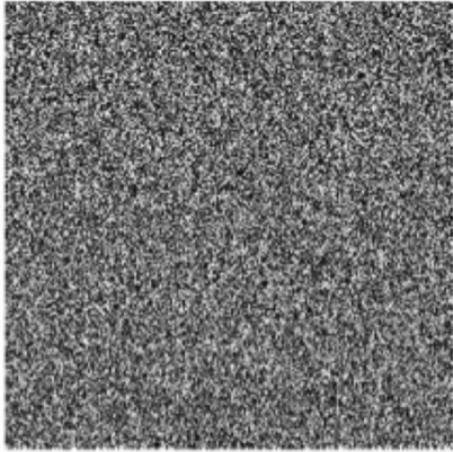
Large gain in
high frequencies
↓
Boost up noises!!



Original



Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u, v) = \frac{1}{1 + \left(\frac{u^2 + v^2}{D^2}\right)^n}$ and $T(u, v) = \frac{B(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))}$.

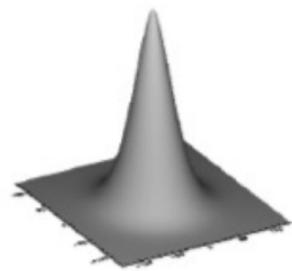
Then define: $\hat{F}(u, v) = T(u, v) G(u, v) \approx F(u, v) B(u, v) + \frac{N(u, v) B(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))}$

$$\frac{N(u, v) B(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))} \approx \frac{N(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))} \quad \text{for } (u, v) \approx (0, 0)$$

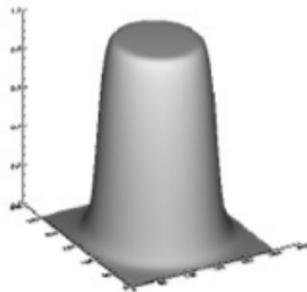
$\frac{N(u, v) B(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))}$ is small (as $B(u, v)$ is small) for (u, v) far away from $(0, 0)$.

$\frac{B(u, v)}{H(u, v) + \varepsilon \operatorname{sgn}(H(u, v))}$ suppresses the high-frequency gain.

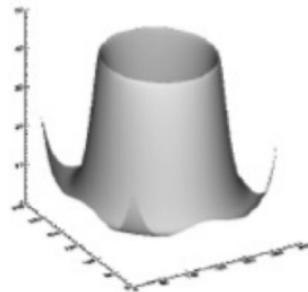
Bad: Has to choose D and n carefully.



$H(u, v)$



$B(u, v)$: $D = 90, n = 8$



Inverse B/H



Original Image $G(u, v)$



Blurred using $D = 90, n = 8$



Restored with a best D and n .