Lecture 9:

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{H(u,v) + \underbrace{\varepsilon \, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{= H(u,v)F(u,v) + N(u,v)} T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)
is small for (u,v) far away from $(0,0)$

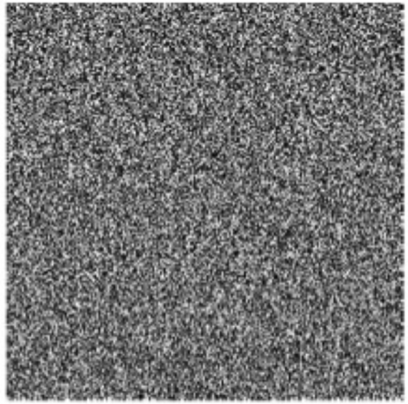
$\therefore \dfrac{N(u,v)}{H(u,v) + \varepsilon \, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$ — Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1 + \left(\dfrac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$.

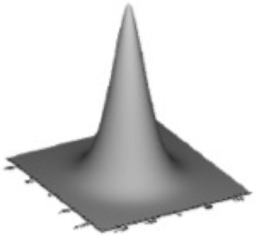
Then define: $\hat{F}(u,v) = T(u,v)\,G(u,v) \approx F(u,v)\,B(u,v) + \dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

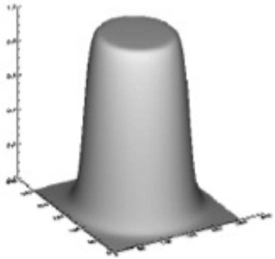
$\dfrac{N(u,v)\,B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

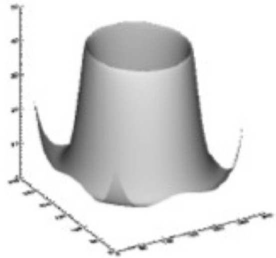
Bad: Has to choose D and n carefully.



$H(u, v)$



$B(u, v)$: $\mathrm{D} = 90, n = 8$



Inverse B/H

Original Image $G(u, v)$

Blurred using $\mathrm{D} = 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y) =$ inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We'll show: Wiener filter minimizes the mean square error:

$$\underset{\substack{\uparrow \\ \text{original}}}{\varepsilon^2(f}, \underset{\substack{\uparrow \\ \text{Restored}}}{\hat{f}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y) - \hat{f}(x,y)|^2 \, dx \, dy$$

(We assume the continuous case to avoid complicated indices)

$$\text{Let } \underset{\text{Observed}}{g} = \overset{\text{degradation}}{h} * \underset{\text{original}}{f} + \underset{\text{noise}}{n}$$

Assume that f and n are spatially uncorrelated:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, n(x+r, y+s) \, dx \, dy \quad \text{for all } r, s.$$

Define: $\hat{f}(x,y) = w(x,y) * g(x,y)$ for some $w(x,y)$

(FT of $\hat{f}$ is like: $W(u,v)\, G(u,v)$)

Goal: Find $W(u,v)$ such that $\varepsilon^2(f, \hat{f})$ is minimized.

(Recall: $\hat{f}$ is obtained as follows:

Step 1: Let $\hat{F}(u,v) = \underbrace{W(u,v)}_{\text{Filter}} G(u,v)$

Step 2: Compute iFT of $\hat{F}$ to get $\hat{f}$

$\therefore \hat{f} = w * g$ for some w.)

(Sketch of proof)

We need to use: Parseval Theorem:

$$\mathcal{E}^2(f,\hat{f}) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)-\hat{f}(x,y)|^2\,dx\,dy = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)-\hat{F}(u,v)|^2\,du\,dv \quad \text{for some constant } C$$

where $F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-j(xu+yv)}$, $\quad \hat{F}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(x,y)\,e^{-j(xu+yv)}\,dx\,dy$

Let $G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,e^{-j(xu+yv)}\,dx\,dy$ and $N(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(x,y)\,e^{-j(xu+yv)}\,dx\,dy$

Then: $\hat{F}(u,v) = W(u,v)\,G(u,v)$ (as $\hat{f}(x,y) = iFT(W(u,v)\,G(u,v))$)

So, $\hat{F}(u,v) = W(u,v)\,G(u,v) = W(u,v)\big(H(u,v)\,F(u,v) + N(u,v)\big)$

In other words, $F - \hat{F} = (1-WH)F - WN$

and $\mathcal{E}^2(f,\hat{f}) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(1-WH)F - WN|^2\,du\,dv$

Since f and n are spatially uncorrelated, we can show that:

$$\mathcal{E}^2(f, \hat{f}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(1-WH)F|^2 + |WN|^2 \, du\, dv$$

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-WH)F\,\overline{W}\,\overline{N} \, du\, dv = 0 \right)$$

Since we look for $w(x,y)$ (hence $W(u,v)$) such that $\mathcal{E}^2$ is minimized, we can regard $\mathcal{E}^2$ is dependent on W.

To minimize $\mathcal{E}^2(W)$, we consider:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}^2(W + tV) = 0 \text{ for all } V.$$

We get: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(1-\overline{W}\,\overline{H})H|F|^2V - (1-WH)\overline{H}|F|^2\overline{V} + \overline{W}|N|^2V + W|N|^2\overline{V} = 0$ for all V.

Put $V = -(1-WH)\overline{H}|F|^2 + W|N|^2$. Then: we have: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|-(1-WH)\overline{H}|F|^2 + W|N|^2\right|^2 du\, dv = 0.$

$$\therefore \quad -(1-WH)\bar{H}|F|^2 + W|N|^2 = 0$$

$$\Downarrow$$

$$W = \frac{\bar{H}}{|H|^2 + |N|^2/|F|^2}.$$

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\text{degradation}}{h} * f + \underset{\text{noise}}{n}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$ $\qquad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

D: transformation matrix of $h * f$

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ (or f) such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$$ subject to the constraint:

$$\|\vec{g} - H\vec{f}\|^2 = \varepsilon$$

- $\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$ ← Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ ← Deblur

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

$\xrightarrow{\text{Put } h=1}$ $\nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$

More generally, $\nabla^2 f = p * f$ ← discrete convolution

where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

<u>Remark:</u> $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ means we allow some fixed level of noise.

$\underset{\|\vec{n}\|^2}{\parallel}$

Assume $S(p * f) = L \vec{f}$

transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T (L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$\left(D^T D + \gamma L^T L\right)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2} G(u,v)$$

$$\left(H = DFT(h);\quad G(u,v) = DFT(g);\quad P(u,v) = DFT(p) \text{ where } p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \right)$$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Sketch of proof:

Recall: our problem is to minimize:

$$\vec{f}^T L^T L \vec{f} \text{ subject to } \|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

$$\| \vec{g} - D\vec{f} \|^2 = (\vec{g} - D\vec{f})^T (g - D\vec{f})$$

From calculus, the minimizer must satisfy:

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}} \left(\vec{f}^T L^T L \vec{f} + \lambda (\vec{g} - D\vec{f})^T (\vec{g} - D\vec{f}) \right) = 0 \quad \text{for}$$

where $\vec{f} = (f_1, f_2, \ldots, f_i, \ldots, f_{N^2})^T$ and λ is the Lagrange's multiplier.

Here, $\frac{\partial K}{\partial \vec{f}} = \left(\frac{\partial K}{\partial f_1}, \frac{\partial K}{\partial f_2}, \ldots, \frac{\partial K}{\partial f_{N^2}} \right)^T$

Easy to check:

- $\dfrac{\partial (\vec{f}^T \vec{a})}{\partial \vec{f}} = \vec{a}$
- $\dfrac{\partial (\vec{b}^T \vec{f})}{\partial \vec{f}} = \vec{b}$
- $\dfrac{\partial (\vec{f}^T A \vec{f})}{\partial \vec{f}} = (A + A^T)\vec{f}$

$$\vec{f}^{\,T}\vec{a} = (f_1, f_2, \dots, f_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 f_1 + a_2 f_2 + \dots + a_n f_n$$

$$\frac{\partial \vec{f}^{\,T}\vec{a}}{\partial f_j} = a_j$$

$$\therefore \frac{\partial \vec{f}^{\,T}\vec{a}}{\partial \vec{f}} \overset{\text{def}}{=} \left(\frac{\partial \vec{f}^{\,T}\vec{a}}{\partial f_1}, \dots, \frac{\partial \vec{f}^{\,T}\vec{a}}{\partial f_n} \right) = (a_1, a_2, \dots, a_n)$$

etc. . . .

$\therefore\ D = 0 \Rightarrow (2L^TL)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^TD\vec{f}) = 0$

$\Rightarrow (D^TD + \gamma L^TL)\vec{f} = D^T\vec{g}$ where $\gamma = \frac{1}{\lambda}$ and λ is the Lagrange's multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f}) = \varepsilon.$$

Now, we'll consider the frequency domain.

Note that D and L are transformation matrix of convolution.

$\therefore$ D and L are block-circulant.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Using the fact that both D and L are block-circulant, we can check that:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,
$$\Lambda_D(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\ddot{H} = DFT(h)$.

and
$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$$\Lambda_D = N^2 \begin{pmatrix} H(0,0) & & & & & & & \\ & H(1,0) & & & & & & \\ & & \ddots & & & & & \\ & & & H(N-1,0) & & & & \\ & & & & H(0,1) & & & \\ & & & & & H(1,1) & & \\ & & & & & & \ddots & \\ & & & & & & & H(N-1,1) & \\ & & & & & & & & \ddots \end{pmatrix}$$

1st col of DFT(h)

2nd col of DFT(h)

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} ,$$

We get :

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = H^T \vec{g},$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

We can check that:

①

$$\boxed{\Lambda_D^* \Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & & \\ & N^4|H(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|H(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}}$$

②

$$\boxed{\Lambda_L^* \Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & & \\ & N^4|P(1,0)|^2 & & & & \\ & & \ddots & & & \\ & & & N^4|P(N-1,0)|^2 & & \\ & & & & \ddots & \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}}$$

③ $\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)}$ where $F = DFT(f), G = DFT(g)$.

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\in \mathbb{R}^{N^2}}{\vec{g}} = \underset{\in M_{N^2 \times N^2}}{D} \vec{f} \in \mathbb{R}^{N^2}$)

Let $H = DFT(h) \in M_{N \times N}$

Diagonalization of D:

$$D = W_{N^2} \underbrace{\begin{bmatrix} \begin{pmatrix} H(0,0) & & & & & & & \\ & H(1,0) & & & & & & \\ & & \ddots & & & & & \\ & & & H(N-1,0) & & & & \\ & & & & H(0,1) & & & \\ & & & & & \ddots & & \\ & & & & & & H(N-1,1) & \\ & & & & & & & \ddots \\ & & & & & & & & H(0,N-1) \\ & & & & & & & & & \ddots \\ & & & & & & & & & & H(N-1,N-1) \end{pmatrix} \end{bmatrix}}_{\text{Stack } H \text{ to form the diagonal matrix.}} W^{-1}$$

Fact 2:

$$W^{-1}\vec{f} = \begin{pmatrix} F(0,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} = \mathcal{S}(F) \quad \text{where } F = DFT(f).$$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2\frac{|H(u,v)|^2 + \gamma|P(u,v)|^2}{\overline{H(u,v)}}F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

Subject to the constraint that:

$$\|\underbrace{\vec{g} - H\vec{f}}_{\vec{n}}\|^2 = \varepsilon$$

(allow fixed amount of noise)

Diagonalization of block-circulant matrix H

Let H be the block-circulant matrix as defined above. Define a matrix with elements:

$$W_N(k,n) := \frac{1}{\sqrt{N}} exp\left(\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Consider the **Kronecker product** $\otimes$ of W_N with itself:

$$W := W_N \otimes W_N$$

Recall that: Kronecker product is defined as:

The **Kronecker product** of two matrices are given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

$A = (a_{ij})_{0 \leq i,j \leq N-1}$

$B = (b_{ij})_{0 \leq i,j \leq N-1}$

W^{-1} can be easily computed!

Easy to check: $W^{-1} = W_N^{-1} \otimes W_N^{-1}$ where:

$$W_N^{-1}(k,n) := \frac{1}{\sqrt{N}} exp\left(-\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Let

$$\Lambda(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where H = DFT of the point spread function h, $\left\lfloor \frac{k}{N} \right\rfloor$ = largest integer smaller than or equal to $\frac{k}{N}$ and $\mathrm{mod}_N(k) = k(\mathrm{mod}\ N)$ (e.g. $10(\mathrm{mod}\ 3) = 1$)

Then, we can show that $H = W\Lambda W^{-1}$ and $H^{-1} = W\Lambda^{-1}W^{-1}$.

Also, $H^T = W\Lambda^* W^{-1}$. (Λ^* is the complex conjugate of Λ)

By direct calculation, it is easy to check that $W^{-1}\vec{g} = N\zeta(G)$ where $G = DFT(g)$.

Example: Assume that:

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$
W^{-1}\vec{g} = \frac{1}{3}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}
\begin{pmatrix}
g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22}
\end{pmatrix}
$$

$$
= \frac{1}{3}
\begin{pmatrix}
g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\
g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\begin{matrix}
= 3^2 G(0,0) \\
= 3^2 G(1,0) \\
\vdots \\
\\
\\
\end{matrix}
$$

$G = DFT(g)$

$\therefore \quad W^{-1}\vec{g} = 3\, \mathcal{S}(G)$