Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N \times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

Imaging problems

① Find a suitable transformation $T \ni$ $g := T(\underset{\uparrow \text{ noisy image}}{f})$ becomes good!

② Given a noisy/distorted image g and transformation T, find original clean image f.

$$\underset{\text{known}}{g} = \underset{\text{known}}{T}(\underset{\text{unknown}}{f}) + \underset{\text{unknown}}{n}$$

(Inverse problem)

Definition: (Linear image transformation)

$\mathcal{O}: \mathcal{I} \to \mathcal{I}$ is linear $\iff$ $\mathcal{O}(af+g) = a\,\mathcal{O}(f) + \mathcal{O}(g)$ for $\forall f, g \in \mathcal{I}$; $\forall a \in \mathbb{R}$

Take $f \in \mathcal{I}$. Let

$$f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(i,j) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(i,j) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{N} f(x,y) \begin{array}{c c} \begin{array}{ccccc} & & \downarrow y & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow x \\ \\ \\ \end{array} \end{array}$$

Let $g = \mathcal{O}(f)$. Assume $\mathcal{O}$ is linear, then:

$$g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y) \left[\mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \right) \right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta) \quad \text{where}$$

$$h(x,\alpha,y,\beta) = \left[\mathcal{O}(P_{xy}) \right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{c c} \begin{array}{ccccc} & & y^{th}\downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow x^{th} \\ \\ \\ \end{array} \end{array}$$

Remark: $h(x, \alpha, y, \beta)$ = how much input value at (x, y) influence the output value at (α, β).

Pixel (x,y) affecting pixel (α, β) by a weight $h(x, \alpha, y, \beta)$.

Definition: (Point spread function)

$h(\cdot, \alpha, \cdot, \beta)$ is called the PSF at (α, β).

Fix α, β. Let x, y as variables!

Definition: (Shift-invariant)

Shift invariant $\Longleftrightarrow$ $h(x, \alpha, y, \beta) = h(\alpha - x, \beta - y)$ for $\forall\, 1 \leq x, y, \alpha, \beta \leq N$

Definition: (Convolution) Let $f, g \in \mathcal{I}$.

Convolution of f and g $\Leftrightarrow$ $f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, g(\alpha - x, \beta - y)$

(Assume f and g are periodically extended: $\begin{cases} f(x+iN, y+jN) = f(x,y) \\ g(x+iN, y+jN) = g(x,y) \end{cases} \forall\, i,j \in \mathbb{Z}$)

Theorem: PSF is shift-invariant $\Rightarrow$ the operator $\mathcal{O}$ is a convolution with the input image.

Proof: Let $g := \mathcal{O}(f)$. $g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y) \underbrace{h(x, \alpha, y, \beta)}_{h(\alpha - x,\, \beta - y)}$

$= f * h(\alpha, \beta)$

Remark:

- $f * h = h * f$ (exercise)
- Convolution is important for understanding image blur.

Definition: (Separable)

Separable $\Leftrightarrow h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ for $\forall \quad 1 \le x, y, \alpha, \beta \le N$.

Theorem: Suppose PSF is separable. Then:

The operator $\mathcal{O}$ consists of two matrix multiplication.

Proof:

$$g(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta)$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h_c(x,\alpha)\, h_r(y,\beta)$$

$$= \underbrace{\sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\left(\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta) \right)}_{\text{matrix multiplication}}}_{\text{matrix multiplication}}$$

Recall: $\boxed{(AB)_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}}$

Representation of $\mathcal{O}$ by a matrix H :

We can write :

$$
\begin{aligned}
g(\alpha,\beta) = {} & f(1,1)\,h(1,\alpha,1,\beta) + f(2,1)\,h(2,\alpha,1,\beta) + \dots + f(N,1)\,h(N,\alpha,1,\beta) \\
& + f(1,2)\,h(1,\alpha,2,\beta) + \dots + f(N,2)\,h(N,\alpha,2,\beta) \\
& \dots \\
& + f(1,N)\,h(1,\alpha,N,\beta) + \dots + f(N,N)\,h(N,\alpha,N,\beta)
\end{aligned}
$$

Each (α,β) is associated to a linear equations.

Arrange:

$$
\vec{f} = \begin{pmatrix} f(1,1) \\ \vdots \\ f(N,1) \\ f(1,2) \\ \vdots \\ f(N,2) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} ; \quad \vec{g} = \begin{pmatrix} g(1,1) \\ g(2,1) \\ \vdots \\ g(N,1) \\ g(1,2) \\ \vdots \\ g(N,2) \\ \vdots \\ g(1,N) \\ g(2,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

In matrix form, let

$$
\vec{g} = \begin{pmatrix} g(1,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

Then : $\vec{g} = H \vec{f}$

$N^2 \times N^2$ matrix

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$3 \times 3 \text{ image} = \begin{array}{c} \\ f_{13} \\ f_{23} \\ f_{33} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} f_{31} & f_{32} & f_{33} \end{array} \\ \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \\ \begin{array}{ccc} f_{11} & f_{12} & f_{13} \end{array} \end{array} \begin{array}{c} \\ f_{11} \\ f_{21} \\ f_{31} \\ \\ \end{array}$$

$$g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \; ; \; g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}$$

etc ...

By careful examination, we see that

$$\begin{bmatrix}
0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\
1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\
1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\
0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\
0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\
1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\
0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0
\end{bmatrix}$$

What is $h(2,3,2,1)$?
What is $h(1,2,3,3)$?

$h(2,3,2,1) = 0$

$h(1,2,3,3) = 1/4$.

Recall:

$$H = \begin{pmatrix}
\begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}} \end{pmatrix} & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=1 \end{pmatrix}} \end{pmatrix} \\
\begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}} \end{pmatrix} & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=2 \end{pmatrix}} \end{pmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=N \end{pmatrix}} \end{pmatrix} & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=N \end{pmatrix}} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha\downarrow & \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=N \end{pmatrix}} \end{pmatrix}
\end{pmatrix}$$

By careful examination, we see that:

$$
\begin{array}{cc}
 & y \longrightarrow \\
\beta \big\downarrow & H = \begin{pmatrix}
\begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=1 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=2 \end{pmatrix} \end{pmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=N \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=N \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=N \end{pmatrix} \end{pmatrix}
\end{pmatrix}
\end{array}
$$

Meaning of

col of small block — row of small block — col of block matrix — row of block matrix

$$h(x, \alpha, y, \beta)$$

$$
\begin{pmatrix} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j \end{pmatrix} \end{pmatrix} := \begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix} \in M_{N\times N}
$$

Definition: H is called the <u>transformation matrix</u> of $\mathcal{O}$.

Example 1.2 Consider an image transformation on a 2×2 image. Suppose the matrix representation of the image transformation is given by:

$$H = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 \\ 3 & 0 & 4 & 0 \\ 6 & 3 & 8 & 4 \end{pmatrix}.$$

Prove that the image transformation is separable. Find g_1 and g_2 such that:

$$h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta).$$

Solution: H for a 2×2 image:

$$\begin{pmatrix} \overset{x \to}{\alpha \downarrow \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}} & \overset{x \to}{\alpha \downarrow \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}} \\ \overset{x \to}{\alpha \downarrow \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}} & \overset{x \to}{\alpha \downarrow \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}} \end{pmatrix} \in M_{4 \times 4}$$

H is separable $\Leftrightarrow$ $h(x, \alpha, y, \beta) = g_1(x, \alpha)\, g_2(y, \beta)$.

Easy to check:

if H is separable: $H = \begin{pmatrix} g_2(1,1) G_1 & g_2(2,1) G_1 \\ g_2(1,2) G_1 & g_2(2,2) G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_1(2,2) \end{pmatrix}$

$$
\begin{pmatrix}
\begin{pmatrix} h(1,1,1,1) & h(2,1,1,1) \\ h(1,2,1,1) & h(2,2,1,1) \end{pmatrix} & \begin{pmatrix} h(1,1,2,1) & h(2,1,2,1) \\ h(1,2,2,1) & h(2,2,2,1) \end{pmatrix} \\
\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}
\end{pmatrix}
$$

$$
\begin{pmatrix} g_1(1,1)\,g_2(1,1) & g_1(2,1)\,g_2(1,1) \\ g_1(1,2)\,g_2(1,1) & g_1(2,2)\,g_2(1,1) \end{pmatrix} = g_2(1,1) \underbrace{\begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_2(1,1) \end{pmatrix}}_{G_1}
$$

$$
g_2(2,1)\, G_1
$$

In our case, $H = \begin{pmatrix} 2G_1 & 1G_1 \\ 3G_1 & 4G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\therefore$

$$\begin{array}{ll} g_1(1,1) = 1 & g_2(1,1) = 2 \\ g_1(2,1) = 0 & g_2(2,1) = 1 \\ g_1(1,2) = 2 & g_2(1,2) = 3 \\ g_1(2,2) = 1 \;// & g_2(2,2) = 4 \;// \end{array}$$