MMAT 5390: Mathematical Imaging

Lecture 4:
Image decomposition by Haar/Walsh Transform

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Recap: Haar transform

For details, please refer to Lecture note Chapter 2

Definition of Haar functions:

The Haar functions are defined recursively as follows:

$$H_0(t) \equiv 1 \quad \text{for } 0 \le t < 1$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \end{cases}$$

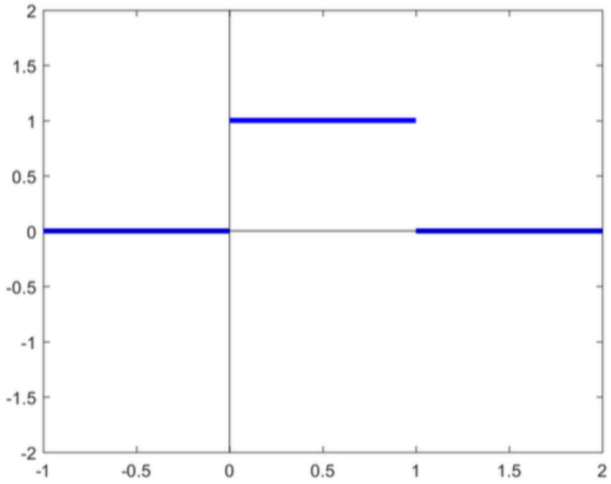
$$H_{2^p+n}(t) \equiv \begin{cases} \sqrt{2^p} & \text{if } \dfrac{n}{2^p} \le t < \dfrac{n+0.5}{2^p} \\ -\sqrt{2^p} & \text{if } \dfrac{n+0.5}{2^p} \le t < \dfrac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \cdots ; n = 0, 1, 2, \cdots , 2^p - 1$

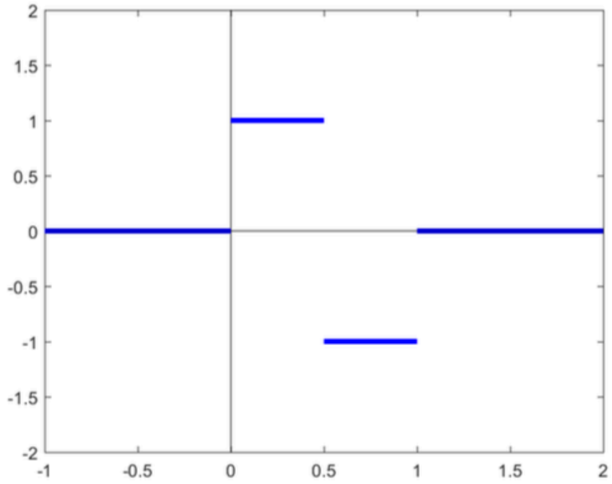
Recap: Haar transform

For details, please refer to Lecture note Chapter 2

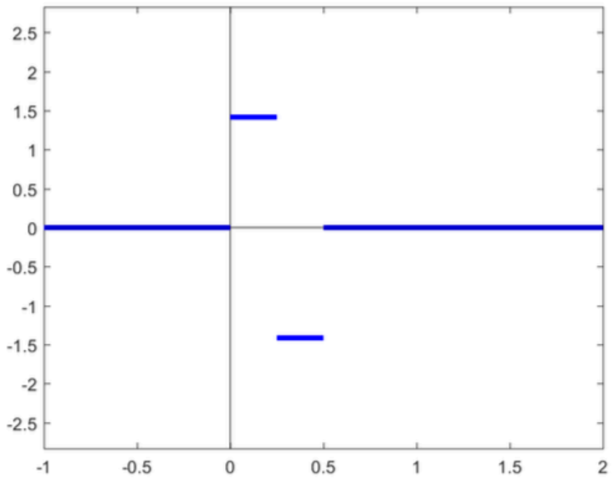
Definition of Haar functions:



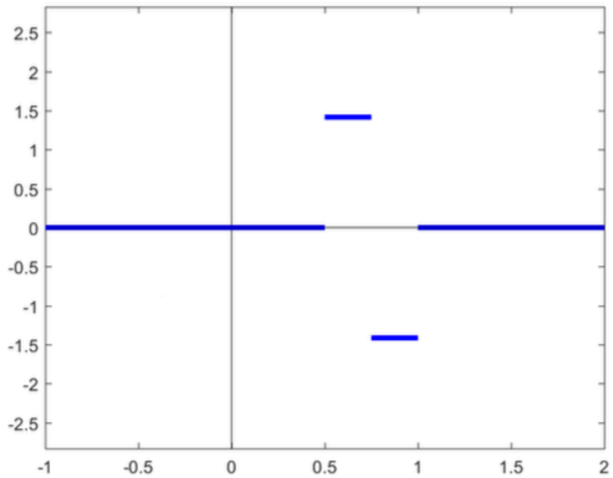
H_0



H_1



H_2



H_3

Recap: Haar transform

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Definition of Haar transforms: (N = power of 2)

Definition 3.3: (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is performed as follows. Divide t variable by the size of matrix $= N$. That's:

Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \cdots, N-1$

We obtain the Haar Transform matrix:

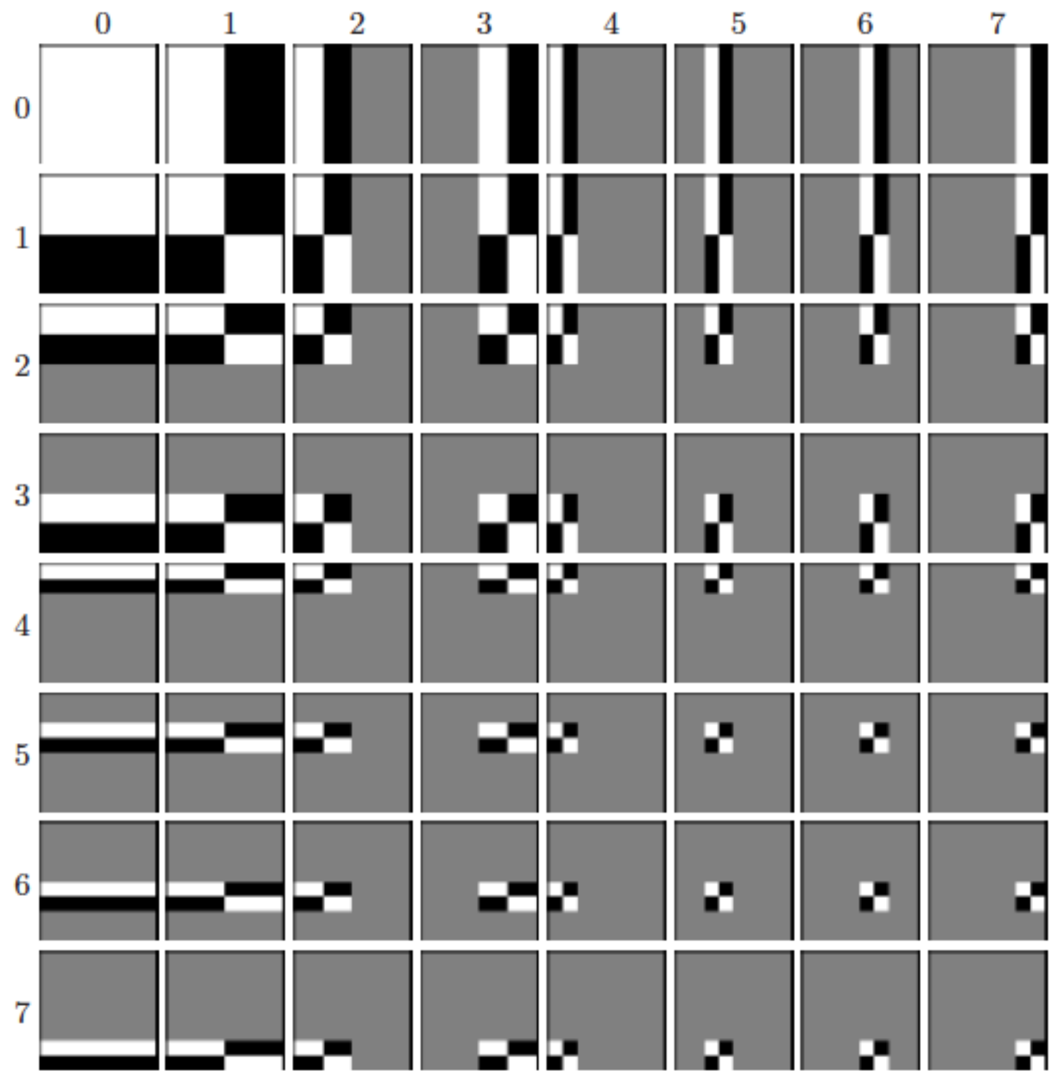
$$\tilde{H} \equiv \frac{1}{\sqrt{N}} H \qquad \text{where } H \equiv (H(k,i))_{0 \leq k, 1 \leq N-1}$$

(Then $\tilde{H}^T \tilde{H} = I$)

The Haar Transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{H} f \tilde{H}^T$$

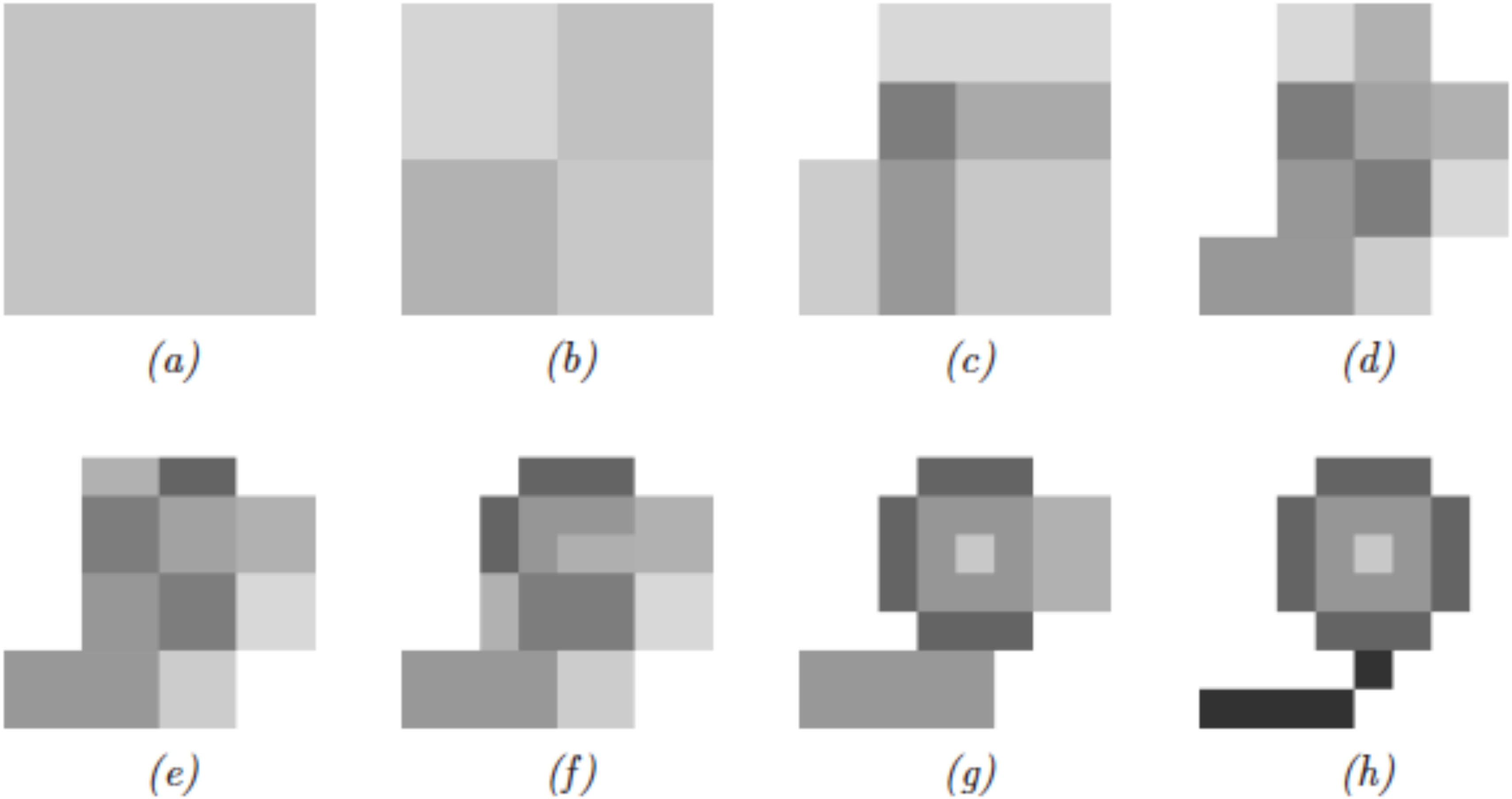
Haar transform elementary images



Haar transform basis image. White = positive; Black = negative; Grey = 0
The i-th row j-th column elementary image is given by:

The outer product of $\tilde{H}(i,:)$ and $\tilde{H}(j,:)$

Reconstruction using Haar decomposition



(a) = using 1x1 elementary images (first 1 row and first 1 column elementary images;

(b) = using 2x2 elementary images (first 2 rows and first 2 column elementary images…

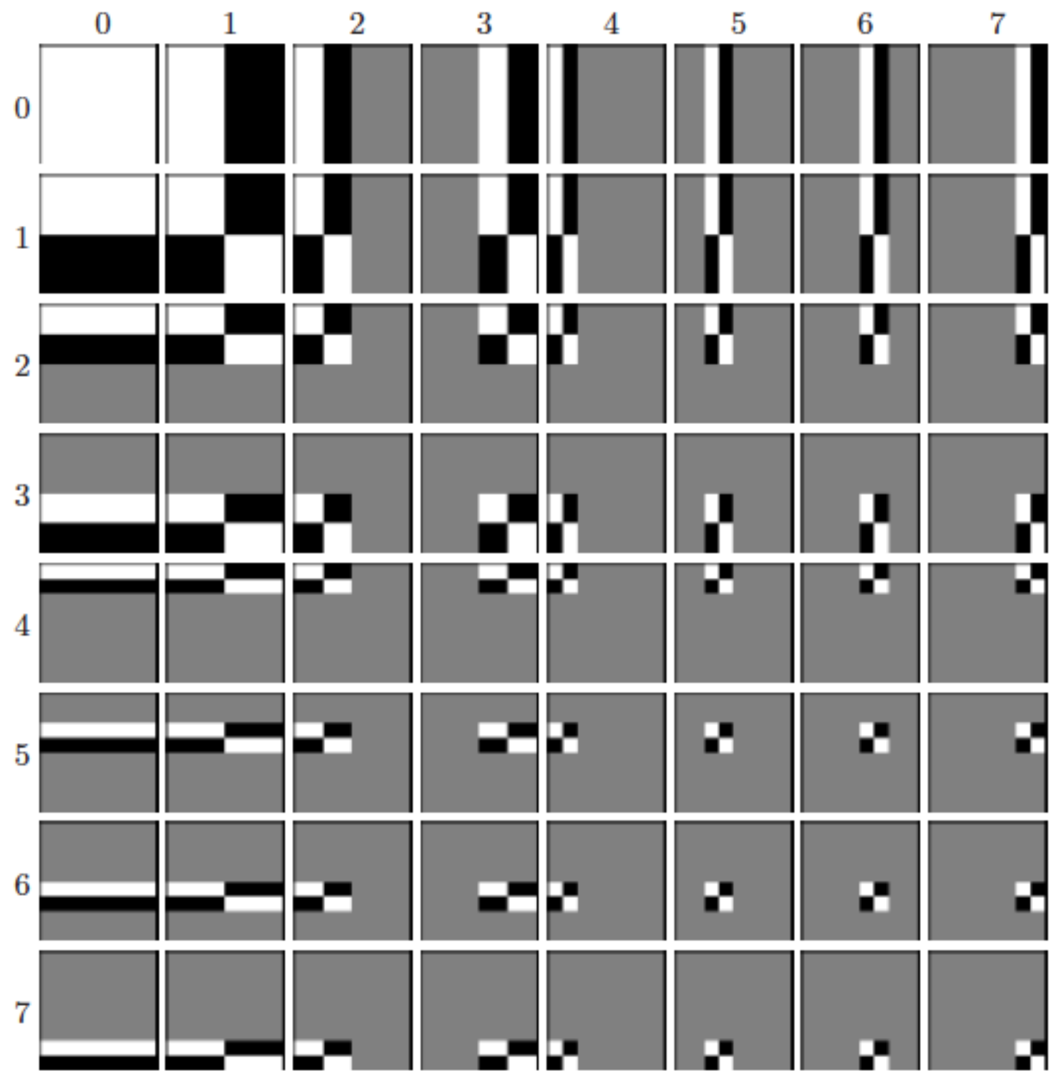
And so on…

Error under Haar decomposition

$$\sum_{all\ pixels} (reconstructed\ pixel - original\ pixel)^2$$

$Square\ error\ for\ image\ 'a:$	366394
$Square\ error\ for\ image\ 'b:$	356192
$Square\ error\ for\ image\ 'c:$	291740
$Square\ error\ for\ image\ 'd:$	222550
$Square\ error\ for\ image\ 'e:$	192518
$Square\ error\ for\ image\ 'f:$	174625
$Square\ error\ for\ image\ 'g:$	141100
$Square\ error\ for\ image\ 'h:$	0

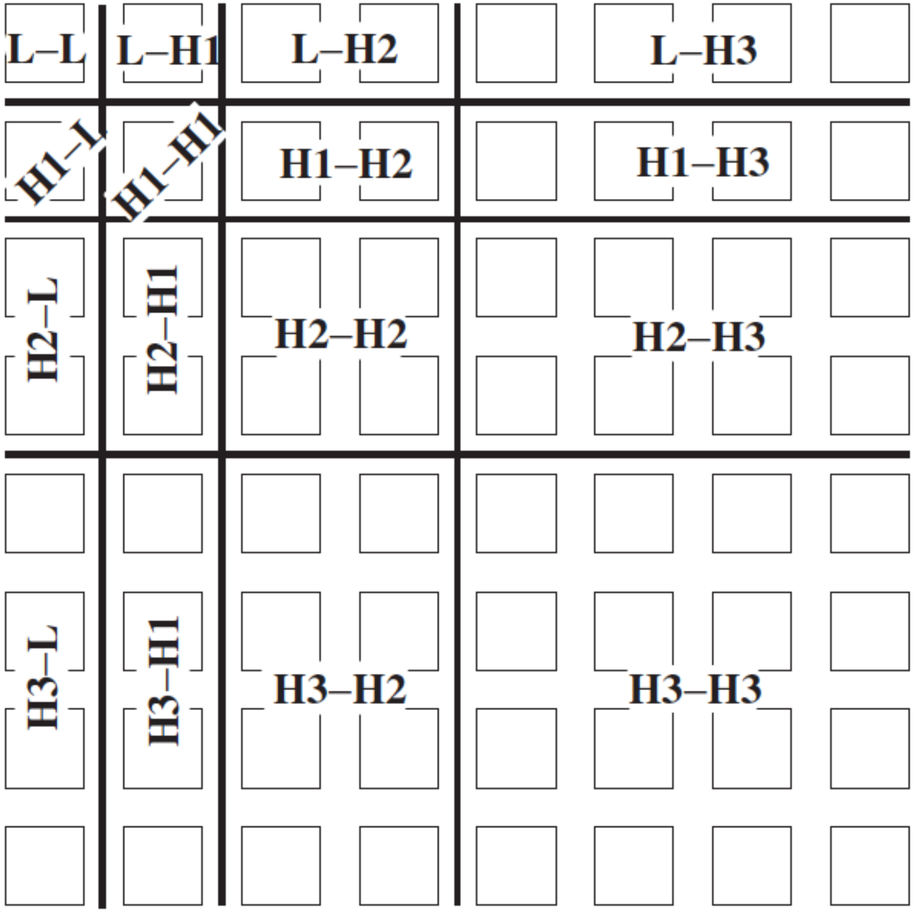
More about Haar transform



Haar transform basis image. White = positive; Black = negative; Grey = 0

More about Haar Transform

What are the coefficients associated to different elementary images representing?



The thick lines divide them into sets of elementary images of the same resolution. Letters L and H are used to indicate low and high resolution, respectively. The numbers next to letter H indicates which level of high resolution. The pairs of letters used indicate which resolution we have along the vertical and horizontal axis. For example, pair L-H2 indicates that the corresponding panels have low resolution along the vertical axis, but high second order resolution along the horizontal axis.

More examples

Discrete Haar transform

Original | Compressed (16:1)

More examples

Discrete Haar transform

Original

Compressed (16:1)

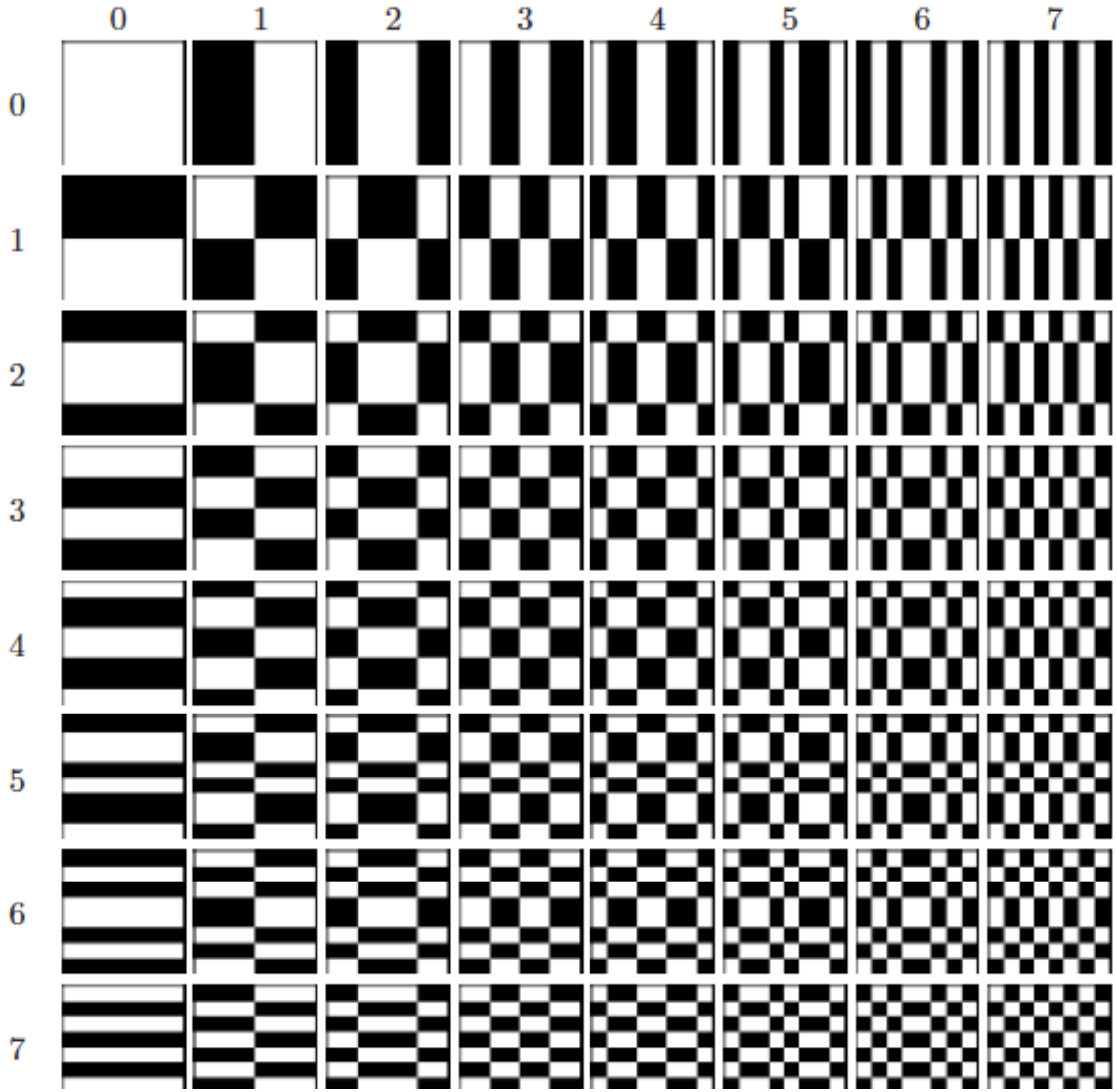
More examples

Discrete Haar transform

Original

Compressed (16:1)

Elementary images of Walsh transform

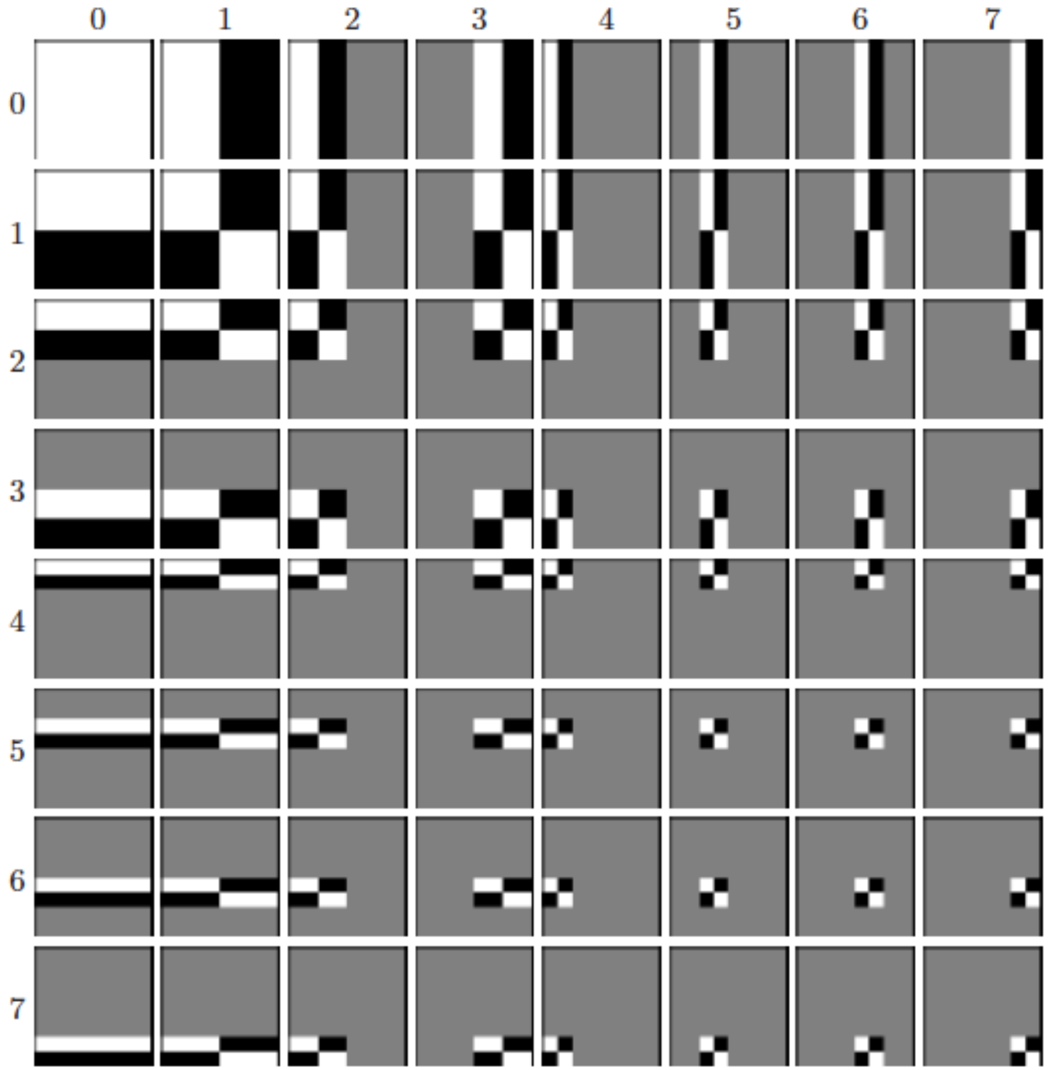


Here, the i-th row j-th column image represents the elementary image I_{ij}^{W}.

where I_{ij}^{W} is the elementary image given by taking the outer product of the i-th row and j-th row of $\tilde{W}$.

Walsh transform elementary images. White = positive; Black = negative; Grey = 0

Compared with Haar transform

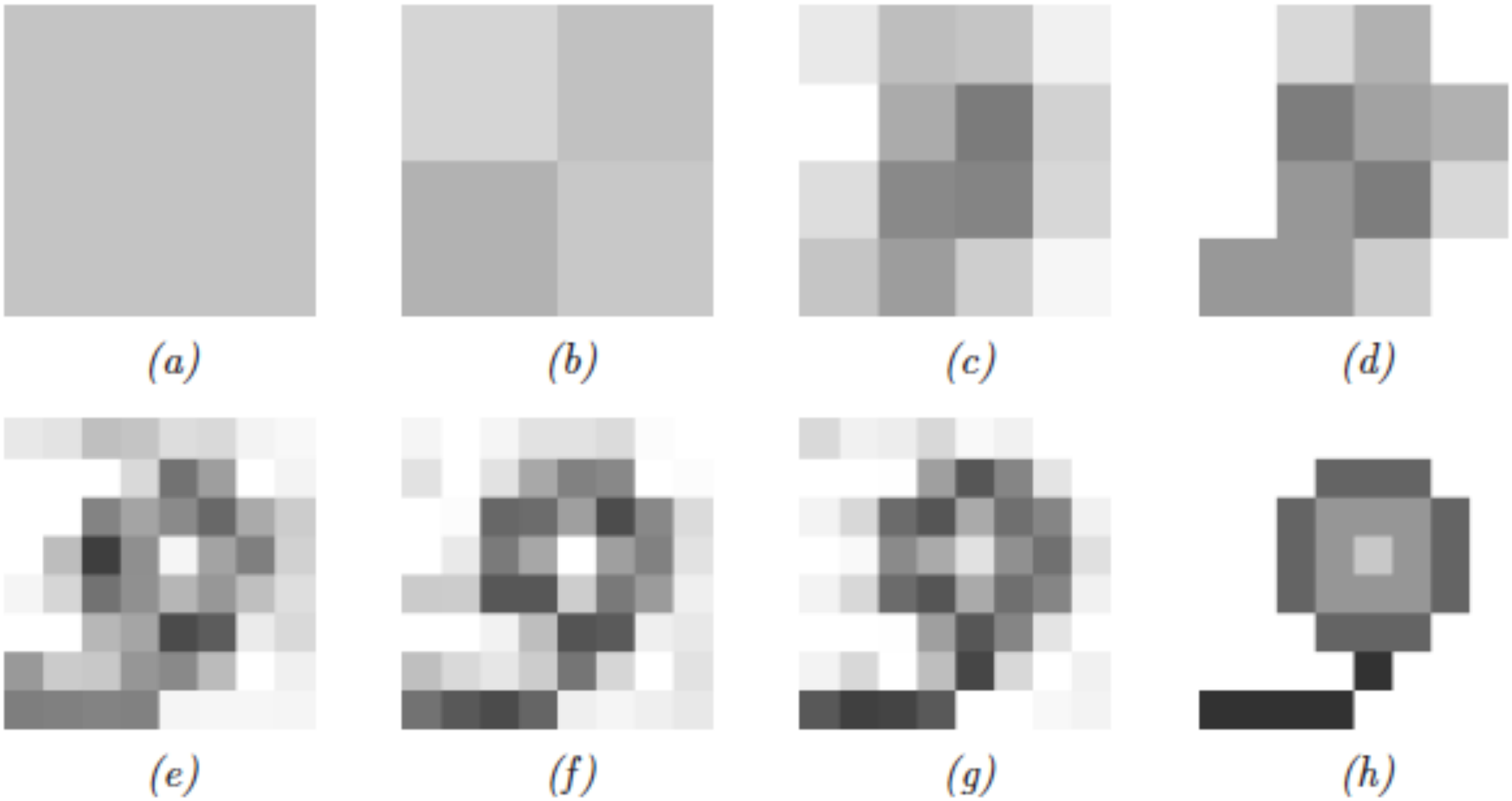


Here, the i-th row j-th column image represents the elementary image I_{ij}^H.

where I_{ij}^H is the elementary image given by taking the outer product of the i-th row and j-th row of $\tilde{H}$.

Haar transform basis image. White = positive; Black = negative; Grey = 0

Reconstruction using Walsh decomposition



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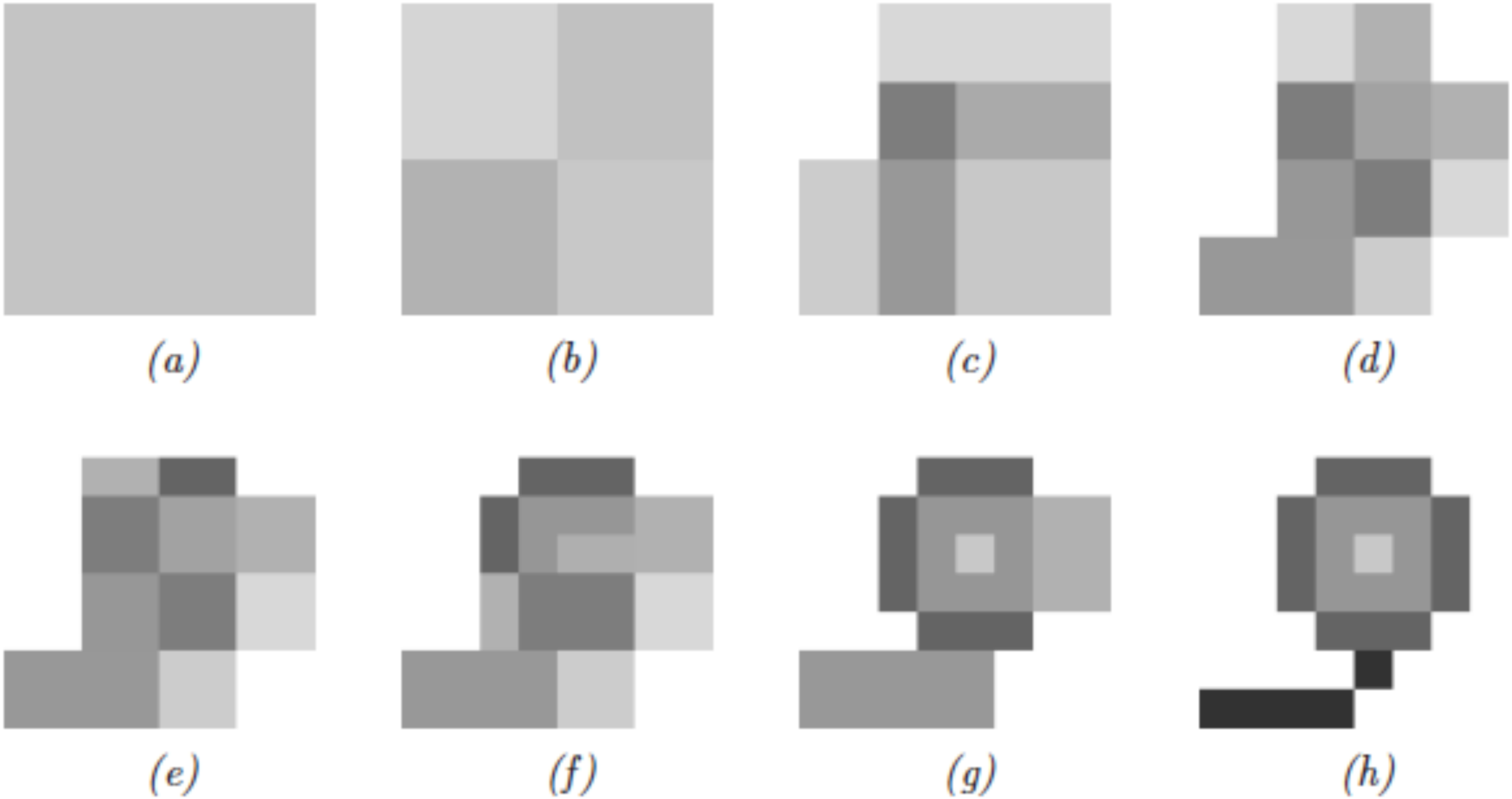
And so on…

Compared with Haar decomposition



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And so on…

Error under Walsh decomposition

$$\sum_{all\ pixels} (reconstructed\ pixel - original\ pixel)^2$$

$$Square\ error\ for\ image\ a: \quad 366394$$

$$Square\ error\ for\ image\ b: \quad 356190$$

$$Square\ error\ for\ image\ c: \quad 262206$$

$$Square\ error\ for\ image\ d: \quad 222550$$

$$Square\ error\ for\ image\ e: \quad 148029$$

$$Square\ error\ for\ image\ f: \quad 92078$$

$$Square\ error\ for\ image\ g: \quad 55905$$

$$Square\ error\ for\ image\ h: \quad 0$$

A quick revision: Image decomposition

What is image decomposition:

Given an image f, we write f as a linear combination of elementary images:

coefficients

$$f = \sum_{i=i}^{M} \sum_{j=1}^{N} g_{ij} I_{ij}.$$

Elementary images

(By truncating the terms with small coefficients, we can compress the image while preserving the important details)

Main technique for image decomposition:

Suppose $\boxed{f = UgV^T}$. Write:

$$U \equiv \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix} \text{ and } V \equiv \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \\ | & | & & | \end{pmatrix}$$

Then:

$$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{u}_i \vec{v}_j^T$$

which is the linear combination of elementary images $\vec{u}_i \vec{v}_j^T$.

A quick revision: **Image decomposition**

Main goal:

$$\boxed{f = UgV^T}.$$

HOW TO CHOOSE U and V?
WHAT IS THE REQUIREMENT of g?

So far we have learnt:

1. **SVD**

Diagonal

$$f = U\Sigma V^T$$

unitary

2. **Haar transform**

Sparse (hopefully)

$$f = \tilde{H}^T g \tilde{H}$$

Haar transform matrix

3. **Walsh transform**

Sparse (hopefully)

$$f = \tilde{W}^T g \tilde{W}$$

Walsh transform matrix

Recap: Definition of DFT

1D and 2D Discrete Fourier Transform:

Definition 5.1: The $1D$ discrete Fourier transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, \cdots, N-1$, is defined as:

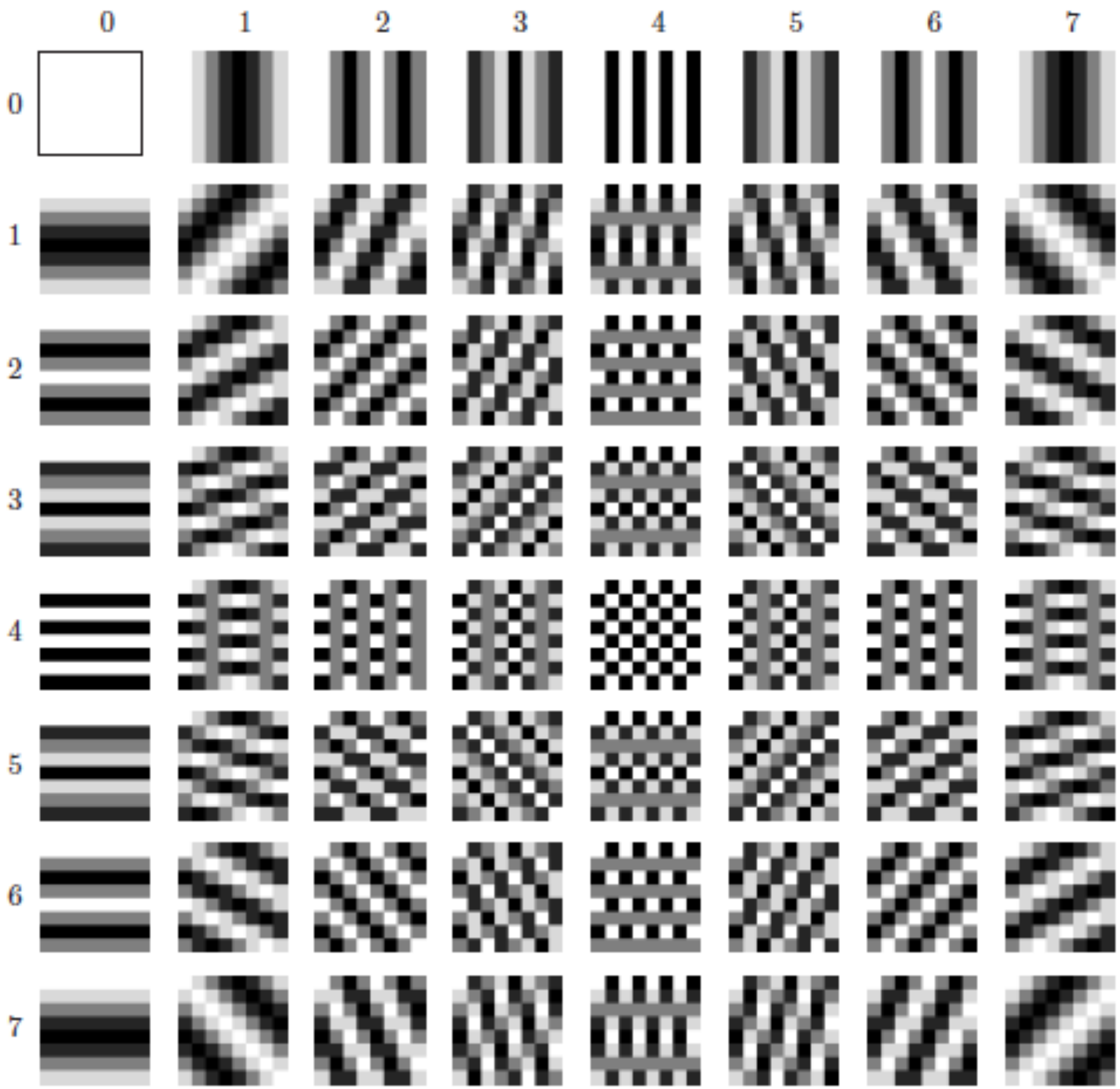
$$F(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-j\frac{2\pi mk}{N}} \qquad \text{where} \qquad \left(\begin{array}{rl} j &= \sqrt{-1} \\ e^{j\theta} &= \cos\theta + i\sin\theta \end{array} \right)$$

The $2D$ discrete Fourier transform for an $M \times N$ image $g(k, l)$, defined at $k = 0, 1, \cdots, M-1$ and $l = 0, 1, \cdots, N-1$, is defined as:

$$\hat{g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k, l) e^{-j2\pi(\frac{km}{M} + \frac{ln}{N})} \qquad (*)$$

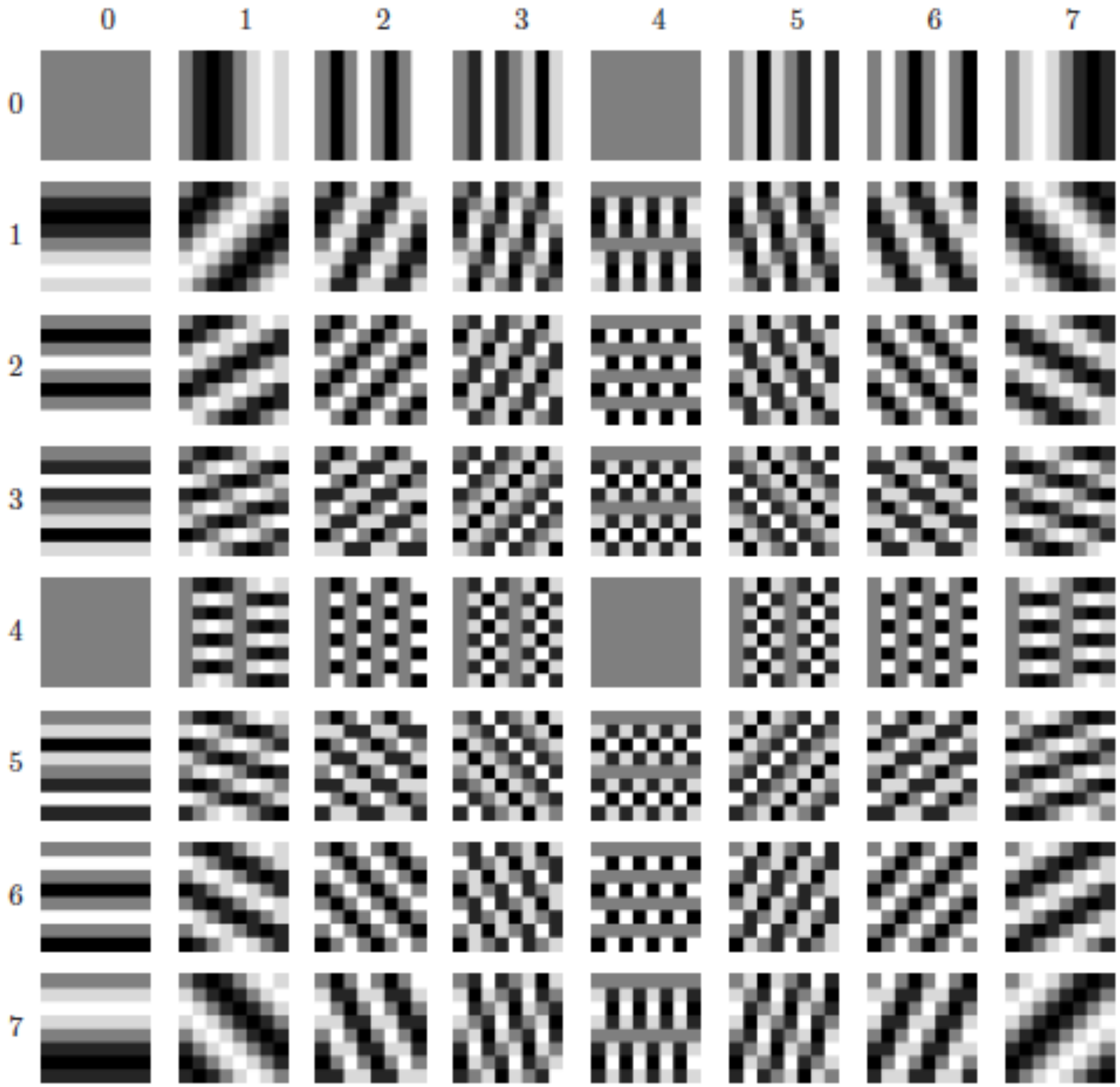
For details, please refer to Lecture Note Chapter 2

Elementary images of DFT decomposition



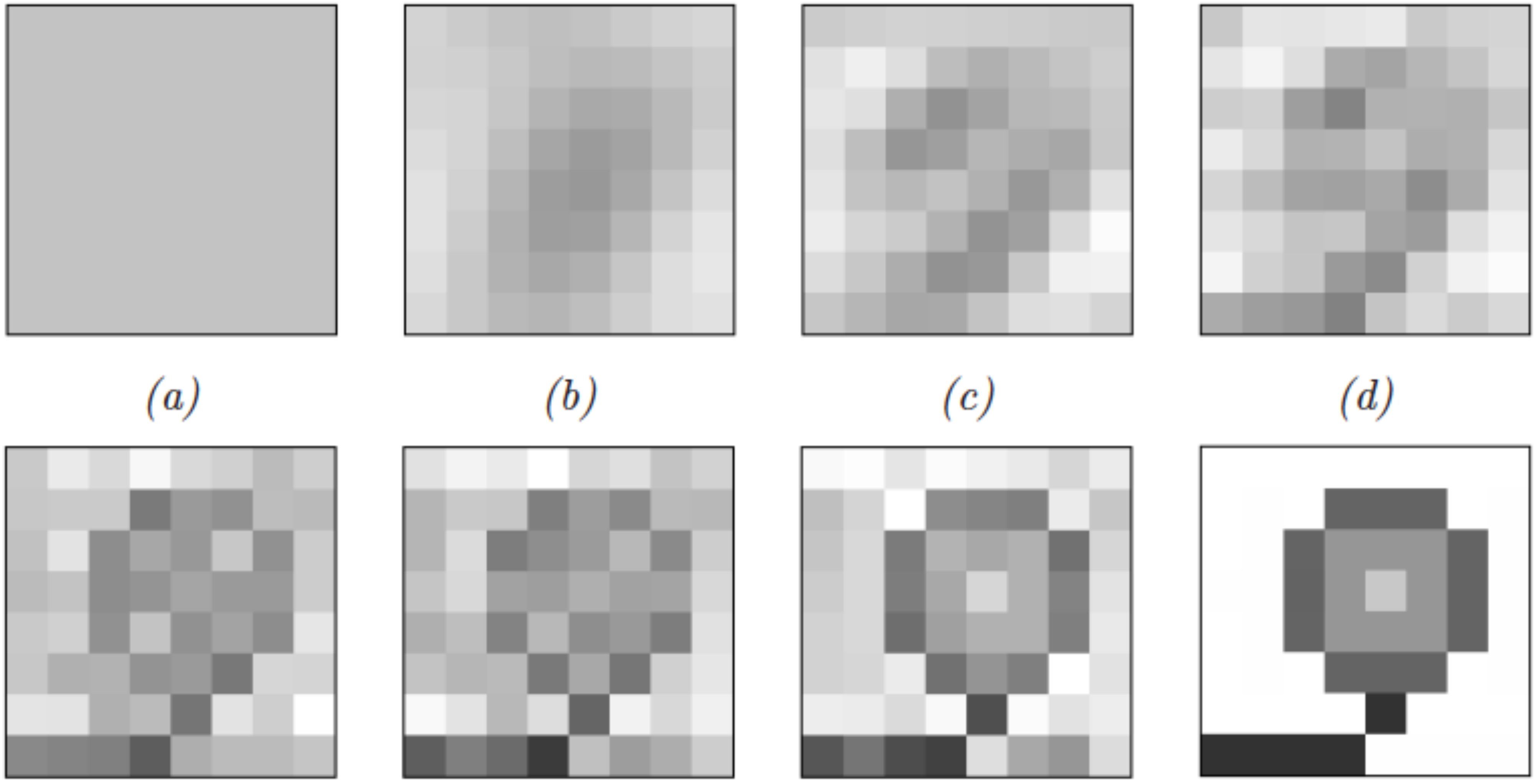
Real part of the Fourier transform basis images

Elementary images of DFT decomposition



Imaginary part of the Fourier transform basis images

Reconstruction w/ DFT decomposition



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Comparison of errors

The flower example:

	0	1	2	3	4	5	6
SVD	230033	118412	46673	11882			
Haar	366394	356192	291740	222550	192518	174625	141100
Walsh	366394	356190	262206	222550	148029	92078	55905
DFT	366394	285895	234539	189508	141481	119612	71908

Real example

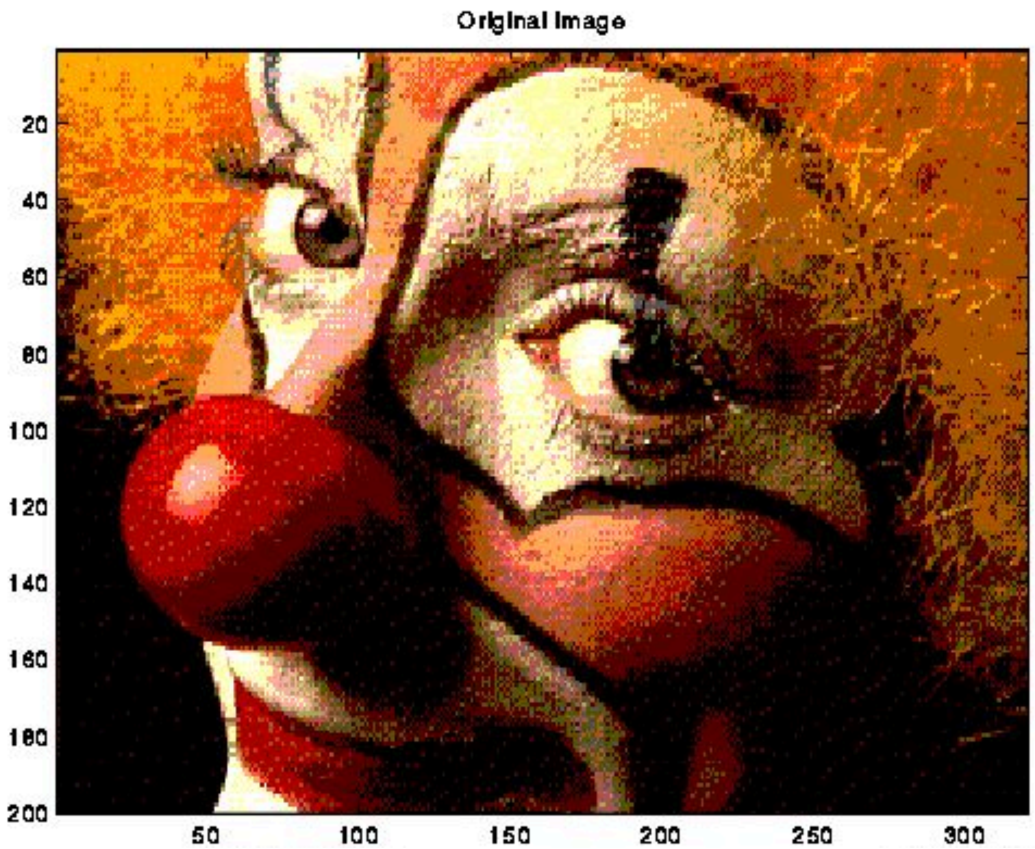
Original:

Compressed: 13.6:1

Truncating the small coefficients

Real example

Original:



Compressed: