Math 5390: Mathematical Imaging

Lecture 3:
Image decomposition by Haar/Walsh Transform

Prof. Ronald Lok Ming Lui
Department of Mathematics,
The Chinese University of Hong Kong

Recap: Haar transform

For details, please refer to Lecture note Chapter 2

Definition of Haar functions:

The Haar functions are defined recursively as follows:

$$H_0(t) \equiv 1 \quad \text{for } 0 \le t < 1$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \end{cases}$$

$$H_{2^p+n}(t) \equiv \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \cdots; n = 0, 1, 2, \cdots, 2^p - 1$

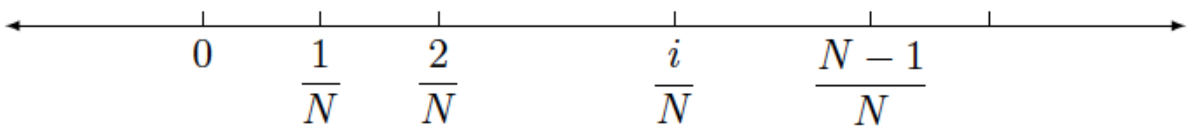
Recap: Haar transform

For details, please refer to Lecture Note Chapter 2

Definition of Haar transforms: (N = power of 2)

Definition 3.3: (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is performed as follows. Divide t variable by the size of matrix $= N$. That's:



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \cdots, N-1$

We obtain the Haar Transform matrix:

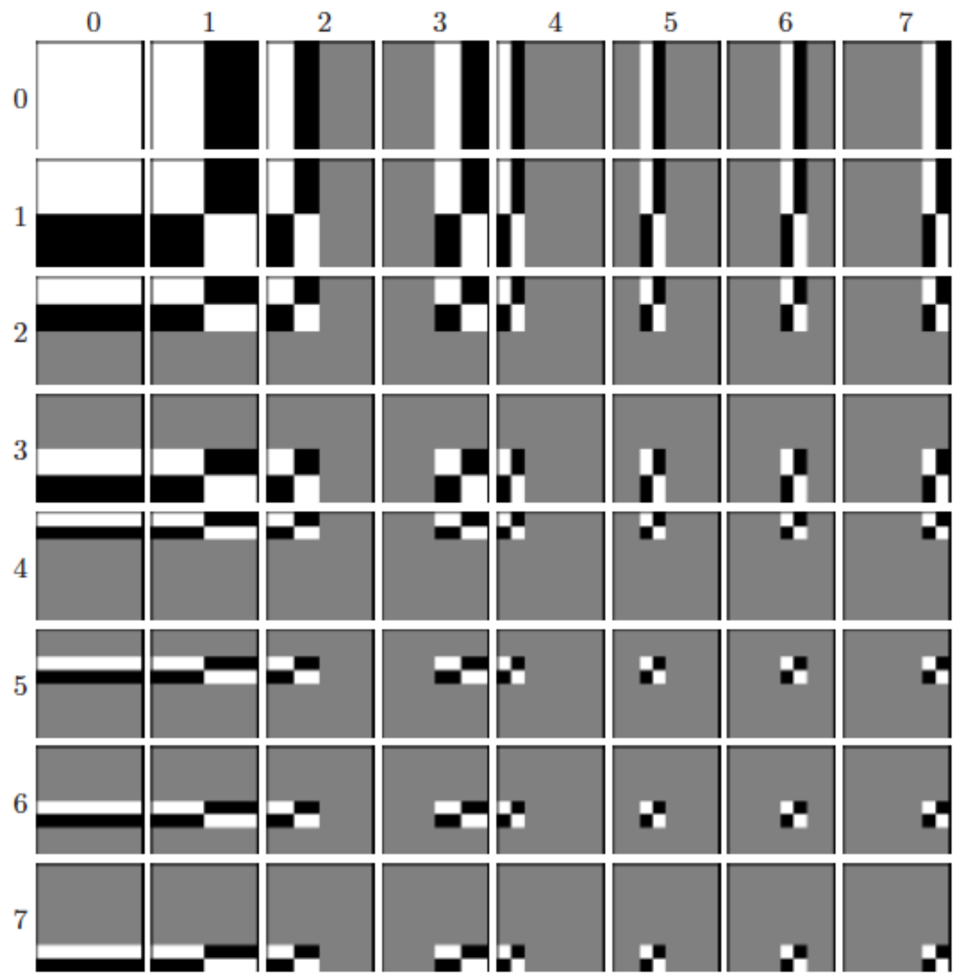
$$\tilde{H} \equiv \frac{1}{\sqrt{N}} H \qquad \text{where } H \equiv (H(k,i))_{0 \le k, 1 \le N-1}$$

(Then $\tilde{H}^T \tilde{H} = I$)

The Haar Transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{H} f \tilde{H}^T$$

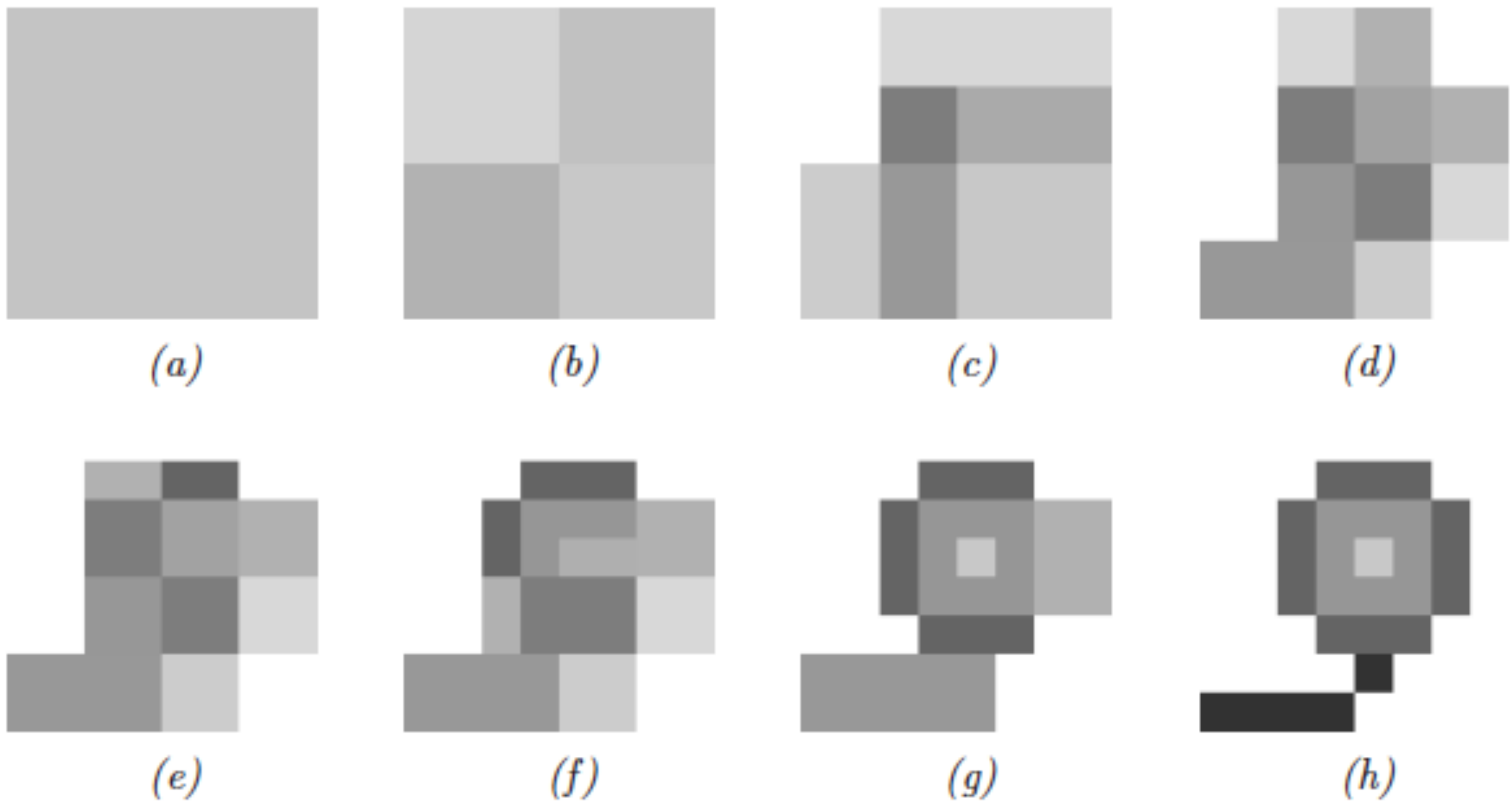
Haar transform elementary images



Haar transform basis image. White = positive; Black = negative; Grey = 0
The i-th row j-th column elementary image is given by:

The outer product of $\tilde{H}(i,:)$ and $\tilde{H}(j,:)$

Reconstruction using Haar decomposition



(a) = using 1x1 elementary images (first 1 row and first 1 column elementary images;

(b) = using 2x2 elementary images (first 2 rows and first 2 column elementary images…

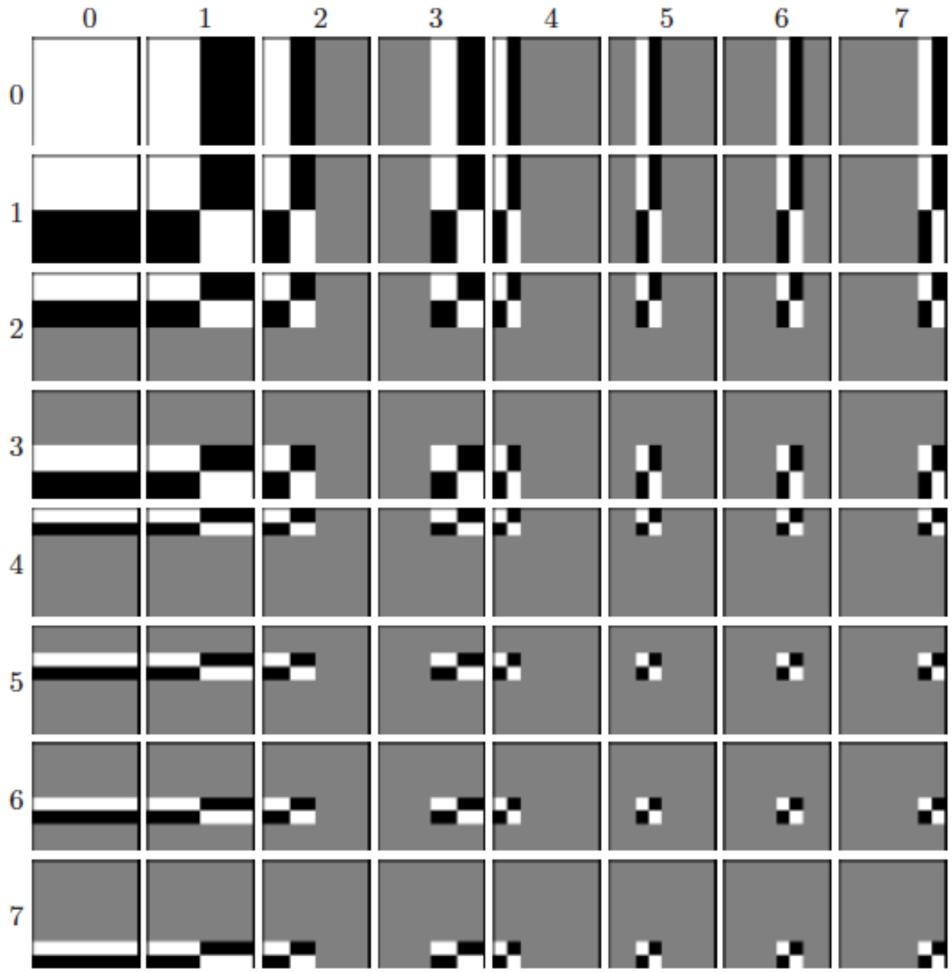
And so on…

Error under Haar decomposition

$$\sum_{all\ pixels} (reconstructed\ pixel - original\ pixel)^2$$

Square error for image 'a:	366394
Square error for image 'b:	356192
Square error for image 'c:	291740
Square error for image 'd:	222550
Square error for image 'e:	192518
Square error for image 'f:	174625
Square error for image 'g:	141100
Square error for image 'h:	0

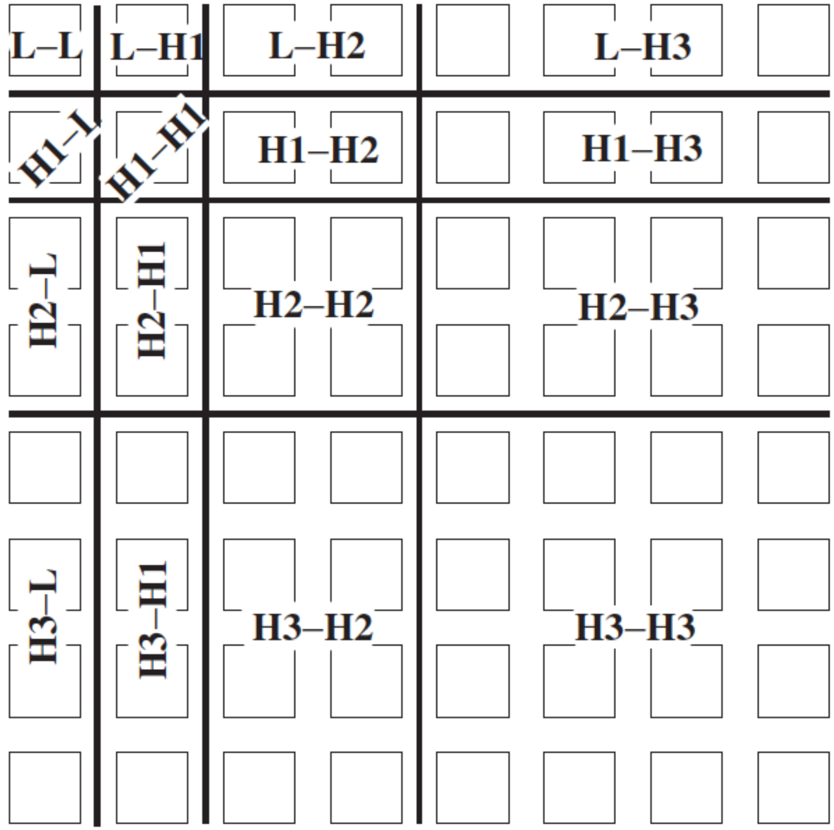
More about Haar transform



Haar transform basis image. White = positive; Black = negative; Grey = 0

More about Haar Transform

What are the coefficients associated to different elementary images representing?



The thick lines divide them into sets of elementary images of the same resolution. Letters L and H are used to indicate low and high resolution, respectively. The numbers next to letter H indicates which level of high resolution. The pairs of letters used indicate which resolution we have along the vertical and horizontal axis. For example, pair L-H2 indicates that the corresponding panels have low resolution along the vertical axis, but high second order resolution along the horizontal axis.

More examples

Discrete Haar transform

Original

Compressed

More examples

Discrete Haar transform

Original

Compressed (16:1)

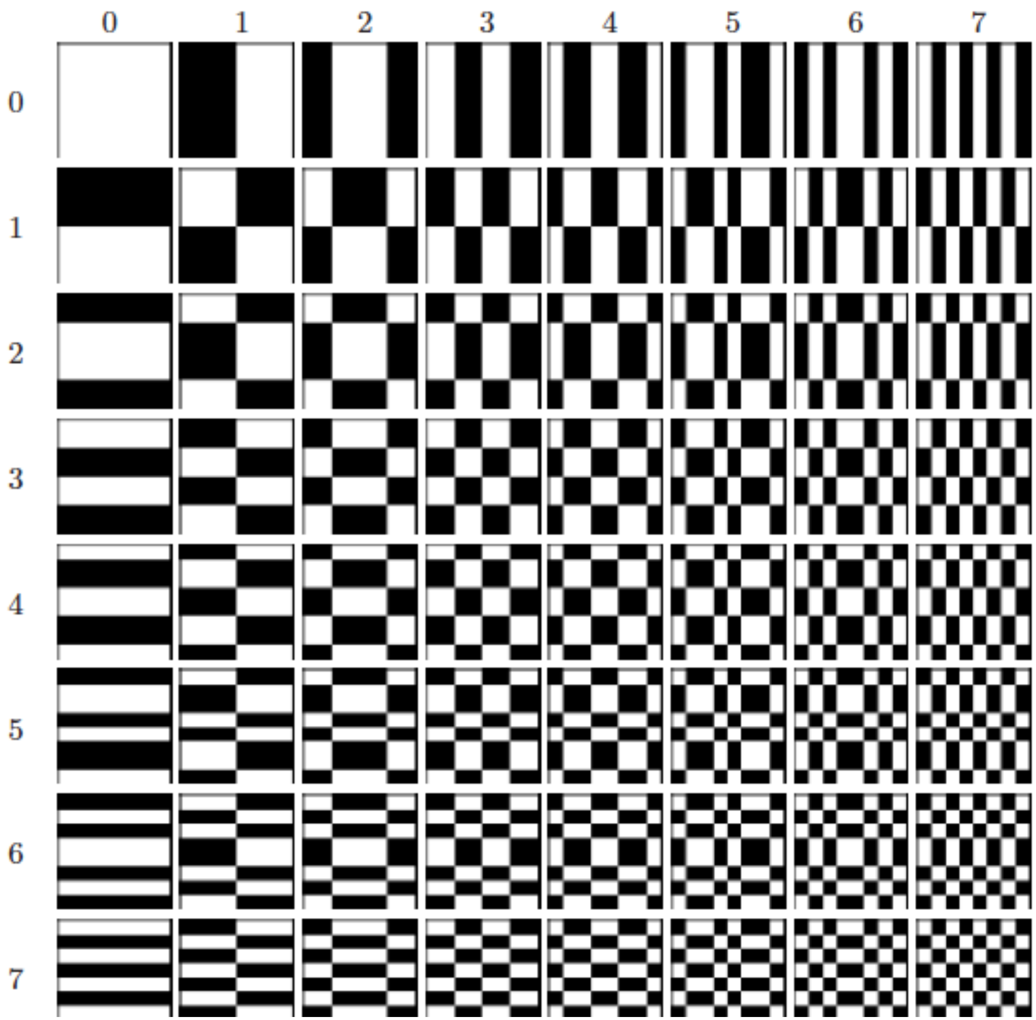
More examples

Discrete Haar transform

Original

Compressed (16:1)

Elementary images of Walsh transform

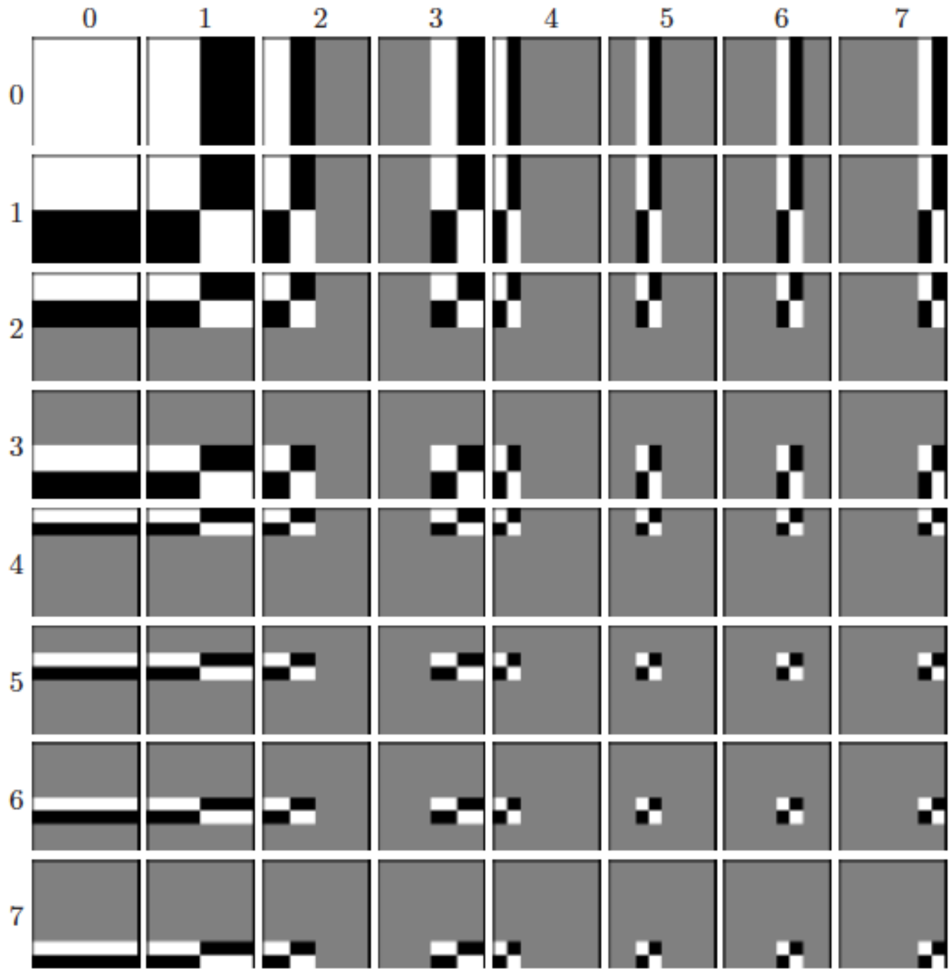


Here, the i-th row j-th column image represents the elementary image I_{ij}^W.

where I_{ij}^W is the elementary image given by taking the outer product of the i-th row and j-th row of $\tilde{W}$.

Walsh transform elementary images. White = positive; Black = negative; Grey = 0

Compared with Haar transform

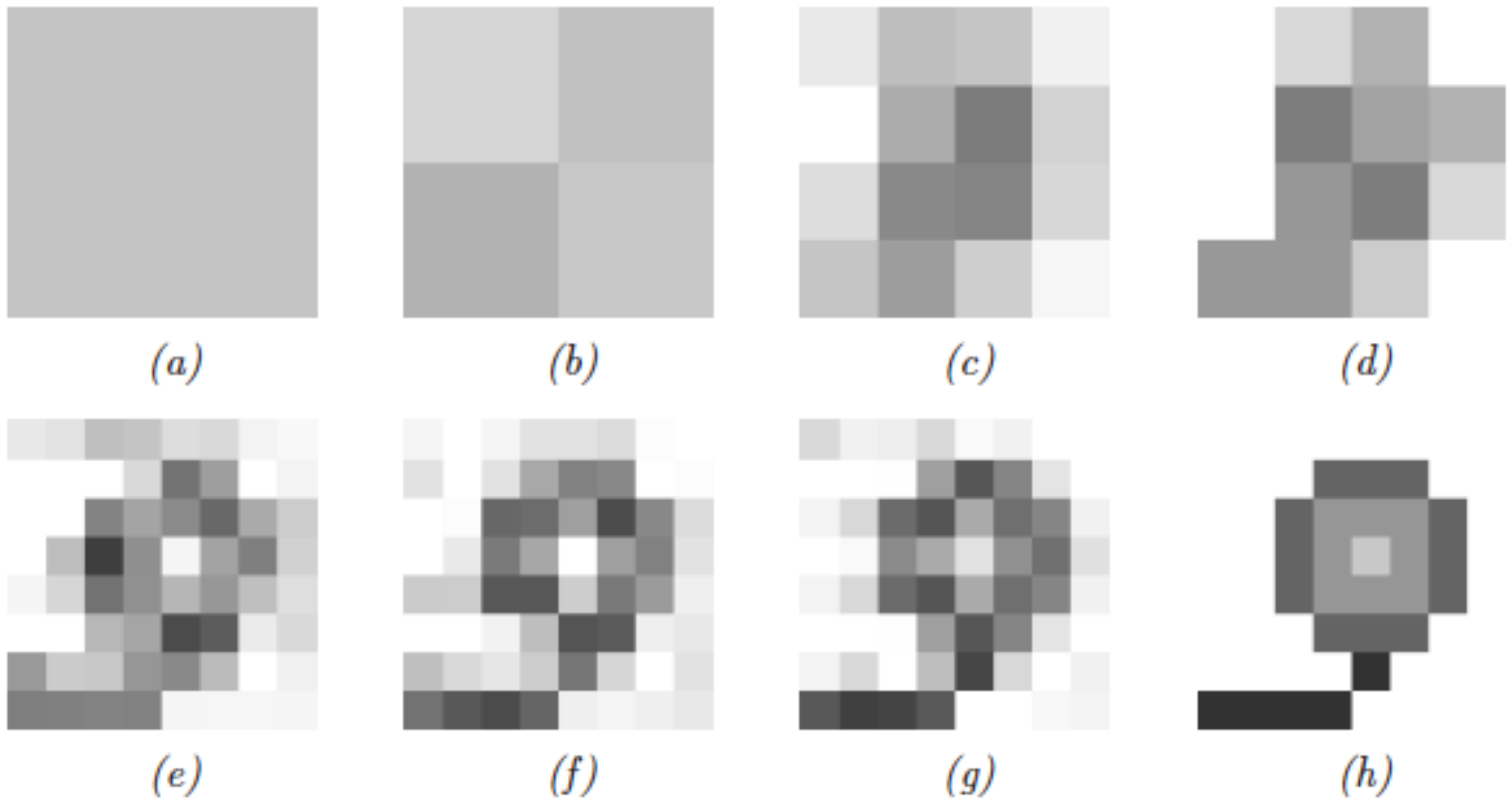


Here, the i-th row j-th column image represents the elementary image I_{ij}^{H}.

where I_{ij}^{H} is the elementary image given by taking the outer product of the i-th row and j-th row of $\tilde{H}$.

Haar transform basis image. White = positive; Black = negative; Grey = 0

Reconstruction using Walsh decomposition



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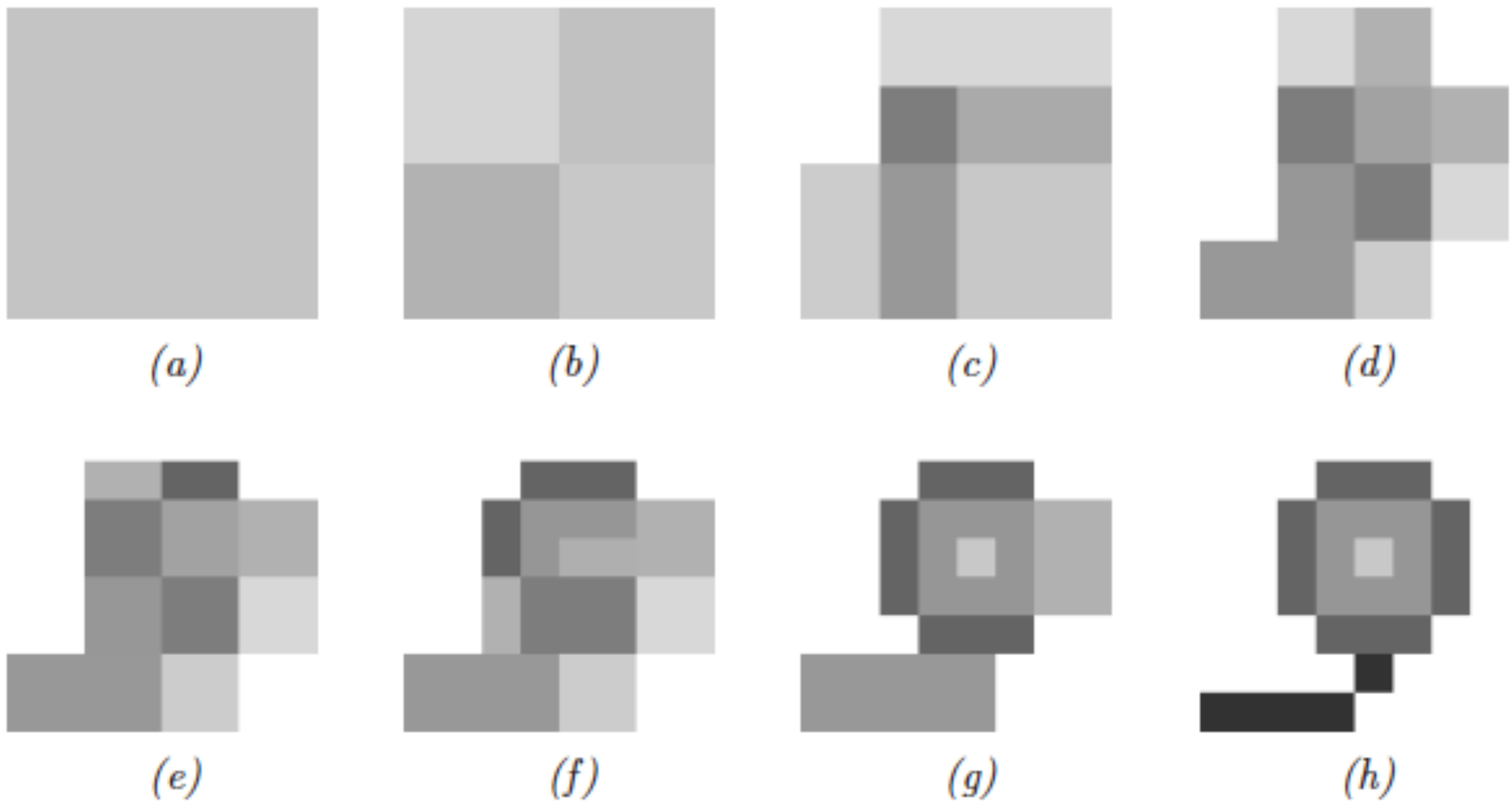
And so on…

Compared with Haar decomposition



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Error under Walsh decomposition

$$\sum_{all\ pixels} (reconstructed\ pixel - original\ pixel)^2$$

Square error for image a:	366394
Square error for image b:	356190
Square error for image c:	262206
Square error for image d:	222550
Square error for image e:	148029
Square error for image f:	92078
Square error for image g:	55905
Square error for image h:	0