Lecture 4:

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

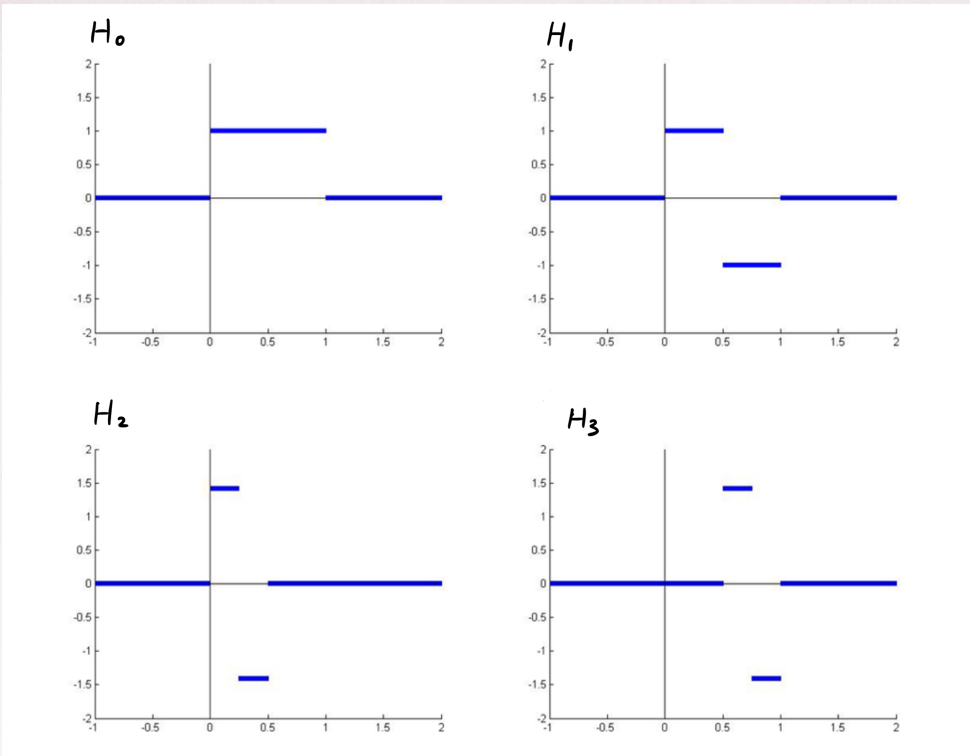
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots$; $n = 0, 1, 2, \dots, 2^p - 1$

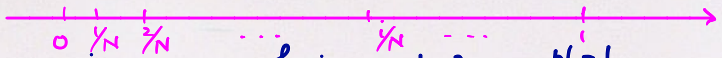
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

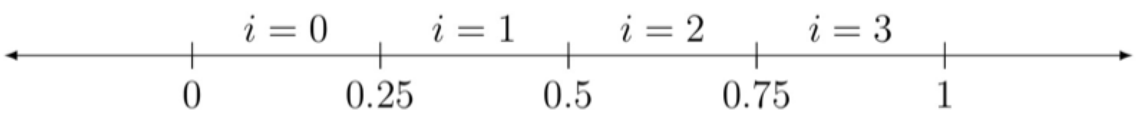
We obtain the Haar Transform matrix: $\widetilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \big(H(k,i)\big)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

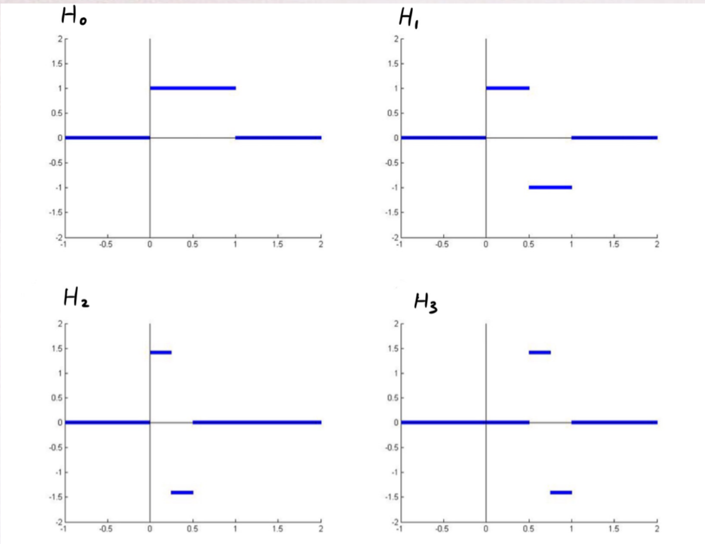
$$g = \widetilde{H} f \widetilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

More zeros

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T f \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

g ← transformed image

Let $\tilde{H} = \begin{pmatrix} - \vec{h}_1^T - \\ - \vec{h}_2^T - \\ \vdots \\ - \vec{h}_N^T - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{= I_{ij}^H}$

I_{ij}^T = elementary images under Haar Transform.

Definition: (Walsh function) The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q}\left\{W_j(2t) + (-1)^{j+q} W_j(2t-1)\right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \dots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

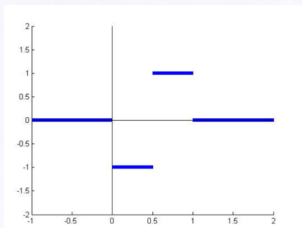
Example: Compute $W_1(t)$.

Put $j=0$, $q=1$. Then:

$$W_1(t) = (-1)^{\lfloor 0 \rfloor + 1}\left\{W_0(2t) + (-1)^1 W_0(2t-1)\right\} = (-1)\left\{W_0(2t) + (-1)^1 W_0(2t-1)\right\}$$

For $0 \le t < \frac{1}{2}$, $W_0(2t) = 1$, $W_0(2t-1) = 0$ $\Rightarrow$ $W_1(t) = -1$.

For $\frac{1}{2} \le t < 1$, $W_0(2t) = 0$, $W_0(2t-1) = 1$ $\Rightarrow$ $W_1(t) = 1$.



Definition: (Discrete Walsh transform)

The Walsh Transform of a $N \times N$ image is defined as follows.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

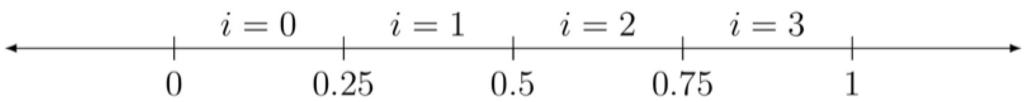
The Walsh transform matrix is: $\tilde{W} \equiv \frac{1}{\sqrt{N}} W$ where $W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

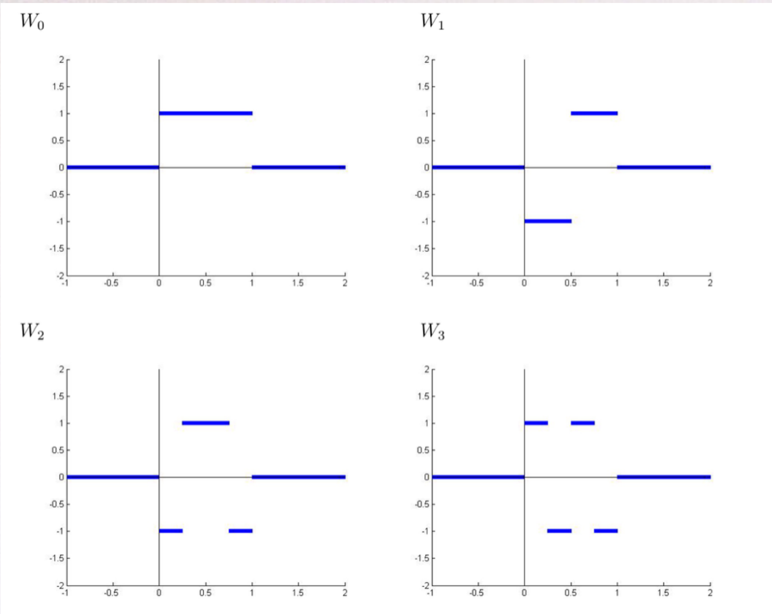
$$g = \tilde{W} f \tilde{W}^T$$

Example Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:



We can check that:



So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}} W = \frac{1}{2} W$$

$(\tilde{W}^T \tilde{W} = I)$

Example 2.7: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{W} f \tilde{W}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

More zeros in the coefficient matrix!

Remark: 1. Walsh transform is to transform an image to a "transformed image" with much more zeros.

Elementary images under Walsh transform:

Under Walsh Transform, $f = \tilde{W}^T g \tilde{W}$.

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{w}_i \vec{w}_j^T}_{= I_{ij}^W}$ where $\tilde{W} = \begin{pmatrix} - \vec{w}_1^T - \\ - \vec{w}_2^T - \\ \vdots \\ - \vec{w}_N^T - \end{pmatrix}$

I_{ij}^W = elementary images under Walsh transform.

Walsh functions and sine function

Definition: (Rademacher function)

A Rademacher function of order n $(n \neq 0)$ is defined as:

$$R_n(t) \equiv \text{sign}[\sin(2^n \pi t)] \quad \text{for} \quad 0 \le t \le 1.$$

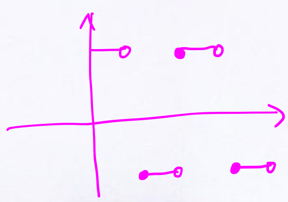
where $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$ and $\text{sign}(x) = 0$ if $x = 0$.

For $n = 0$, $R_0(t) \equiv 1$ for $0 \le x \le 1$.

Let $N = b_{m+1} 2^m + b_m 2^{m-1} + \dots + b_1 2^0$. Then, the R-Walsh function $\tilde{W}_N$ is given by:

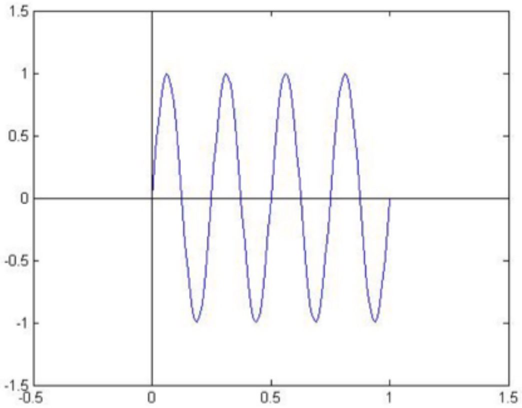
$$\tilde{W}_N = \prod_{i=1,\, b_i \neq 0}^{m+1} R_i(t)$$

(where the values at the jumps are defined such that the function is continuous from the right)

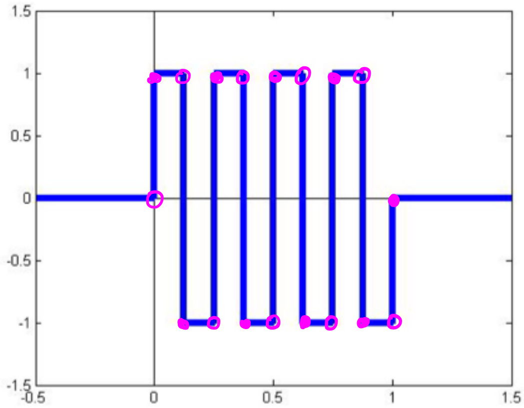


Example : Compute R-Walsh function $\tilde{W}_4$ using Rademacher function.

Consider $\sin(8\pi t)$:



Therefore, $R_3(t) =$



As $4 = \underbrace{1}_{b_3} \cdot 2^2 + \underbrace{0}_{b_2} \cdot 2^1 + \underbrace{0}_{b_1} \cdot 2^0$, we have

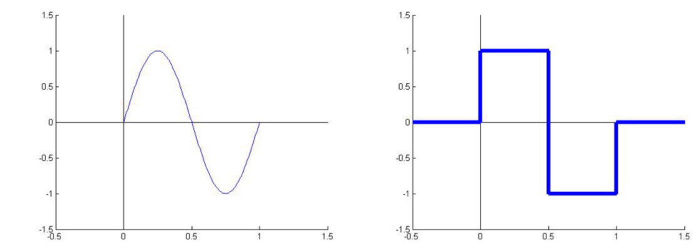
$$\tilde{W}_4 = \prod_{i=1, b_i \neq 0}^{3} R_i(t) = R_3(t)$$

$(W_{2j+q}(t) \equiv (-1)^{\lfloor j/2 \rfloor + q}\{W_j(2t) + (-1)^{j+q} W_j(2t-1)\})$

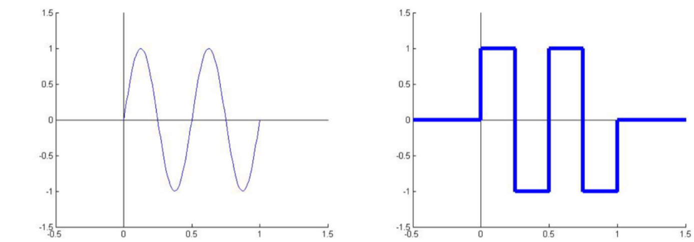
For $W_3(t)$: As $3 = \underbrace{1}_{b_2} \cdot 2^1 + \underbrace{1}_{b_1} \cdot 2^0$, we have

$$\tilde{W}_3(t) = \prod_{i=1, b_i \neq 0}^{2} R_i(t) = R_1(t)R_2(t)$$

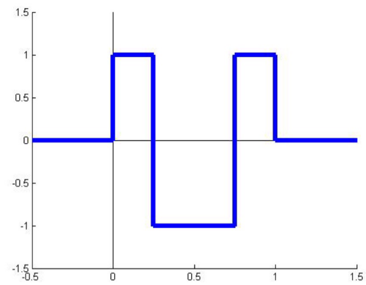
$R_1(t)$:



$R_2(t)$:



Therefore, $\tilde{W}_3(t)$:



Relationship between Walsh functions and R-Walsh functions

$$W_0(t) = \tilde{W}_0(t), W_1(t) = -\tilde{W}_1(t), W_2(t) = -\tilde{W}_3(t), W_3(t) = \tilde{W}_2(t),$$
$$W_4(t) = \tilde{W}_6(t), W_5(t) = -\tilde{W}_7(t), W_6(t) = -\tilde{W}_5(t), W_7(t) = \tilde{W}_4(t)$$

But how to get these formula?

How to determine Walsh from R -Walsh?

Write $i = b_{m+1} 2^m + b_m 2^{m-1} + \dots + b_1 2^0$.

Determine j whose binary representation is given by:

$$c_{m+1} \, c_m \dots c_1 \quad \text{where}$$

$$c_{m+1} = b_{m+1} \pmod 2, \quad c_k = (b_{k+1} + b_k) \pmod 2$$

Then: $W_i(t) = \pm \tilde{W}_j(t)$

The $\pm$ sign is determined from $W_i(0)$!

Example 2.9 Consider $W_7(t)$.

Check that $W_7(t) > 0$.
Now, $7 = 2^2 + 2^1 + 2^0$ so binary representation of 7 is 111.
Therefore, $j = 100$ (binary) $= 4$
Thus, $W_7(t) = \tilde{W}_4(t)$.

Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi mk}{N}} \quad \left(\text{where } j = \sqrt{-1},\ e^{j\theta} = \cos\theta + j\sin\theta\right)$$

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{MN}$!)

(no -ve sign)

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\,\hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that: $$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^{M}-1}{\underbrace{e^{j2\pi\left(\frac{t}{M}\right)}-1}_{\text{if } t\neq 0}} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

∴ (*) becomes: $$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \le k, l \le N-1$ and $\mathcal{U} = (U_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$

$\mathcal{U}$ is clearly symmetric and also:

$$\hat{g} = \mathcal{U} g\, \mathcal{U} \quad \text{(DFT in matrix form)}$$

Note that: $\displaystyle \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi (x_2 - x_1)\alpha}{N}} = \frac{1}{N^2} N\, \delta(x_2 - x_1)$

$$= \frac{1}{N} \delta(x_2 - x_1)$$

Let $\mathcal{U} = \left(\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_N\right)$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\,T}\, \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^{N}$ is orthogonal but NOT orthonormal!

$$\therefore \quad \mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$$

$$\therefore \quad g = (N\mathcal{U})^* \, \hat{g} \, (N\mathcal{U})^*$$

$$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \, \vec{w}_k \vec{w}_l^{\,T}$$

Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

$$\hat{g} = \mathcal{U} g \mathcal{U}$$

$$\Rightarrow \quad \mathcal{U}^* \hat{g} \mathcal{U}^* = (\mathcal{U}^*\mathcal{U}) \, g \, (\mathcal{U}\mathcal{U}^*)$$

$$= \left(\frac{1}{N}\right) g \left(\frac{1}{N}\right)$$

$$\therefore \quad (N\mathcal{U})^* \, \hat{g} \, (N\mathcal{U})^* = g \quad //$$

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l}$$

$$= \left(e^{-j2\pi\left(\frac{kl}{4}\right)} \right)$$