Lecture 5

Recall:

Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j \sin\theta$)

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \leq k \leq M-1$, $0 \leq l \leq N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j 2\pi \left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j 2\pi \left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\,\hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\,\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that: $$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \underbrace{\frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^M - 1}{e^{j2\pi\left(\frac{t}{M}\right)} - 1}}_{\text{if } t\neq 0} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

$\therefore$ $(*)$ becomes: $$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \leq k, l \leq N-1$ and $U = (U_{kl})_{0 \leq k, l \leq N-1} \in M_{N \times N}$

U is clearly symmetric and also:

$$\hat{g} = U g U \quad \text{(DFT in matrix form)}$$

Note that: $\frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi(x_2 - x_1)\alpha}{N}} = \frac{1}{N^2} N \delta(x_2 - x_1)$

$= \frac{1}{N} \delta(x_2 - x_1)$

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\,T} \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^{N}$ is orthogonal but NOT orthonormal!

$$\therefore \quad \mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$$

$$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$$

$$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^{\,T}$$

Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

$$\hat{g} = \mathcal{U} g \mathcal{U}$$

$$\Rightarrow \quad \mathcal{U}^* \hat{g} \mathcal{U}^* = (\mathcal{U}^*\mathcal{U}) g (\mathcal{U}\mathcal{U}^*)$$

$$= \left(\frac{1}{N}\right) g \left(\frac{1}{N}\right)$$

$$\therefore \quad (N\mathcal{U})^* \hat{g} (N\mathcal{U})^* = g \;//$$

Example Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \left(U_{kl} \right)_{k,l} = \left(e^{-j2\pi\left(\frac{kl}{4}\right)} \right)$$

Remark:

Note that $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I$. $\therefore$ $\mathcal{U}$ is not unitary.

If we normalize $\mathcal{U}$ to $\tilde{\mathcal{U}} = \sqrt{N}\,\mathcal{U}$. Then $\tilde{\mathcal{U}}$ is unitary!

Some other definition of DFT:

(1D) $\hat{f}(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{k-1} f(k) e^{-j\left(\frac{2\pi m k}{N}\right)}$

(2D) $\hat{f}(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l)\, e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$

In this case, let $\hat{\mathcal{U}} = (\hat{\mathcal{U}}_{kl})_{0 \le k,l \le N-1}$; $\hat{\mathcal{U}}_{kl} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi kl}{N}}$. Then:

$\hat{f} = \hat{\mathcal{U}} f \hat{\mathcal{U}}$

Then, $\hat{\mathcal{U}} = \sqrt{N}\,\mathcal{U} = \tilde{\mathcal{U}}$.

$\therefore$ Normalizing the definition of DFT $\Rightarrow$ unitary $\tilde{\mathcal{U}}$ can be applied!

BUT: Inverse DFT must be adjusted!!

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$

$\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$

Then, the DFT of $g * w$ $= MN\, \mathrm{DFT}(g)\, \mathrm{DFT}(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf. after DFT!!

Proof:

Change of variables:

$n \to n'' = n - n'$

$m \to m'' = m - m'$

DFT of $g * w$ at (p, q)

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m')\, w(n', m')\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \underbrace{\frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n', m')\, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)}}_{\hat{w}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n, m)$ and $g(n, m-M) = g(n, m)$

$$\therefore\ T = \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j2\pi \frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi \frac{pn''}{N}} \overset{n'''=N+n''}{=\!=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N,\ m'')}_{g(n''',\, m'')}\, e^{-j2\pi\left(\frac{pn''}{N}\right)} \cancelto{1}{e^{j2\pi p}}$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)} = MN\,\hat{g}(p,q)$$

$$\therefore\ \widehat{g * w}\,(p,q) = MN\,\hat{g}(p,q)\,\hat{w}(p,q)$$

Remark: Conversely, if $x(n,m) = g(n,m)\,w(n,m)$

Then, $\hat{x}(k,l) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \hat{g}(p,q)\,\hat{w}(k-p,\, l-q)$ (Convolution of g and w)

2. Average value of image

Average value of $g = \bar{g} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) = \underbrace{\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi(0)}}_{\hat{g}(0,0)}$

3. DFT of a rotated image

Consider a $N \times N$ image g.

Then: $\hat{g}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$

Write k and l in polar coordinates:

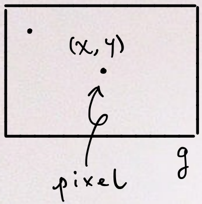
$k \equiv r\cos\theta \;;\; l = r\sin\theta$

Similarly, write $m \equiv \omega\cos\phi \;;\; n = \omega\sin\phi$.

Note that: $km + ln = r\omega(\cos\theta\cos\phi + \sin\theta\sin\phi) = r\omega\cos(\theta-\phi)$.

Denote $\mathcal{P}(g) = \{(r,\theta) : (r\cos\theta, r\sin\theta) \text{ is a pixel of } g\}$

(Polar coordinate set of g)



If $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, then $(r, \theta) \in \mathcal{P}(g)$.

Then: $$\underbrace{\hat{g}(m, n) = \hat{g}(\omega, \phi)}_{\text{Identify } \hat{g}(m,n) \text{ with } \hat{g}(\omega,\phi)} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \underbrace{g(r, \theta)}_{\text{Identify } g(k,l) \text{ with } g(r,\theta)} e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)}$$

Consider a rotated image $\tilde{g}(r, \theta) = g(r, \theta + \theta_0)$ where θ is defined between $-\theta_0$ to $\pi/2 - \theta_0$.

$\therefore$ image g is rotated clockwisely by θ_0.

DFT of $\tilde{g}$ is:

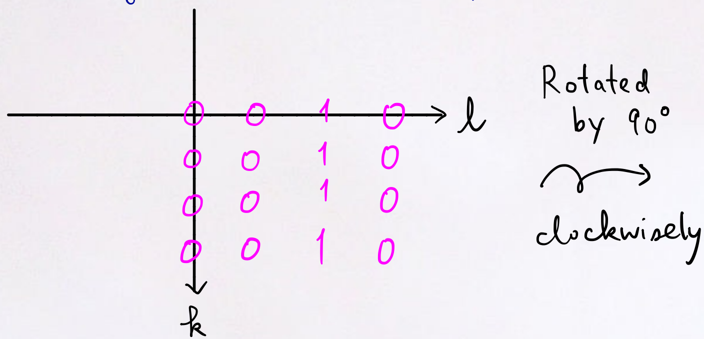
$$\hat{\tilde{g}}(\omega, \phi) = \frac{1}{N^2} \sum_{(r,\theta)\in\mathcal{P}(\tilde{g})} \underset{\substack{\parallel \\ g(r,\, \underbrace{\theta+\theta_0}_{\tilde{\theta}})}}{\tilde{g}(r, \theta)}\, e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)} = \frac{1}{N^2} \sum_{(r,\tilde{\theta})\in\mathcal{P}(g)} g(r, \tilde{\theta})\, e^{-j2\pi\left(\frac{r\omega\cos(\tilde{\theta}-\theta_0-\phi)}{N}\right)}$$

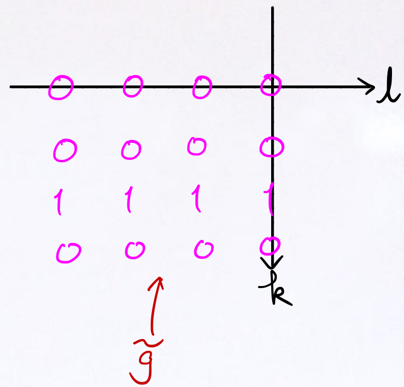
$\therefore \hat{\tilde{g}}(\omega, \phi) = \hat{g}(\omega, \phi + \theta_0)$. ($\phi$ is also defined between $-\theta_0$ to $\pi/2 - \theta_0$)

DFT of an image rotated by θ_0 = DFT of the original image rotated by θ_0.

Example: Let $g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then: $\hat{g} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note that g in the coordinate system:





Note that indices of $\tilde{g}$ are taken as: $\begin{cases} -3 \le l \le 0 \\ 0 \le k \le 3 \end{cases}$.

Now, DFT of $\tilde{g}$ = $\hat{\tilde{g}}$ (given by: $\displaystyle\sum_{k=0}^{3}\sum_{l=-3}^{0}\tilde{g}(k,l)\, e^{-j2\pi\left(\frac{km+ln}{4}\right)}$)

$$= \begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \downarrow k \\ \begin{array}{cccc} -3 & -2 & -1 & 0 \end{array} \xrightarrow{\ l\ } & \end{array}$$

$\therefore$ $0 \le k \le 3$

$-3 \le l \le 0$

4. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$-k_0 \le k' \le N-1-k_0$ and $-l_0 \le l' \le N-1-l_0$

Let $\tilde{g}$ be shifted image of g defined as:

$\tilde{g}(k,l) = g(k-k_0, l-l_0)$ where $0 \le k \le N-1$

$0 \le l \le N-1$

Then: $$\hat{\tilde{g}}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k-k_0, l-l_0)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\frac{1}{N^2}\sum_{k'=-k_0}^{N-1-k_0}\sum_{l'=-l_0}^{N-1-l_0} g(k', l')\, e^{-j2\pi\left(\frac{k'm+l'n}{N}\right)}}_{\hat{g}(m,n)}\, e^{-j2\pi\left(\frac{k_0 m+l_0 n}{N}\right)}$$

$$\therefore \quad \hat{g}(m,n) = \hat{g}(m,n)\, e^{-j2\pi\left(\frac{k_0 m + l_0 n}{N}\right)}$$

<u>Remark:</u> $\hat{g}(m-m_0,\, n-n_0) = \mathrm{DFT}\left(g \times e^{j2\pi\left(\frac{m_0 k + n_0 l}{N}\right)}\right)$ with carefully chosen indices!