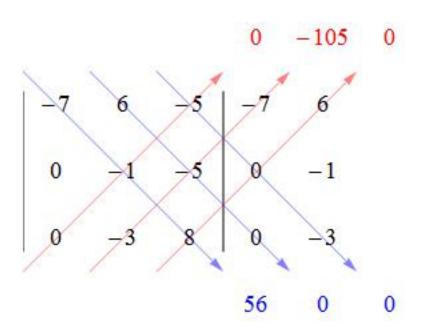


Math 5390: Mathematical Imaging

Lecture 2: Mathematical Review on Linear Algebra

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Determinant



$$\begin{vmatrix} -7 & 6 & -5 \\ 0 & -1 & -5 \\ 0 & -3 & 8 \end{vmatrix} = 56 + 0 + 0 - (0 + -105 + 0) = 161$$

Determinant

Definition 1.1. Let A be a $n \times n$ matrix. If n = 1, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j}),$$

where \tilde{A}_{ij} is the $(n-1) \times (n-1)$ matrix obtained from removing row i and column j of A (called the **minor** of the entry of A in row i, column j). The scalar $\det(A)$ is called the **determinant** of A and is also denoted by |A|. The scalar

$$(-1)^{i+j}\det(\tilde{A}_{ij})$$

is called the **cofactor** of the entry of A in row i, column j.

Reduce the computation of determinant of a big matrix to a smaller matrix

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij}).$$

Properties of Determinant

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} + k\mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{u} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + k \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{r-1} \\ \mathbf{v} \\ \mathbf{a}_{r+1} \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

(Linearity)

Properties of Determinant

Theorem 1.4. If A is a square matrix and B is a matrix obtained from A by interchanging any two rows of A, then det(B) = -det(A).

Theorem 1.5. Let A be a square matrix, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then det(B) = det(A).

Corollary. If a $n \times n$ matrix has rank less than n, then det(A) = 0.

Theorem 1.6. For any two $n \times n$ matrices A and B, det(AB) = det(A)det(B).

Corollary. A square matrix is invertible if and only if $det(A) \neq 0$. Furthermore, if A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$.

Theorem 1.7. For any square matrix A, $det(A^T) = det(A)$.

Eigenvalues and Eigenvectors

Definition 1.8. Let A be a $n \times n$ matrix. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a scalar λ such that $A\mathbf{v} = \lambda \mathbf{v}$. The scalar λ is called the **eigenvalue** of A corresponding to the eigenvector \mathbf{v} .

Definition 1.9. Let A be a $n \times n$ matrix. The polynomial $f(t) = det(A - tI_n)$ is called the **characteristic polynomial** of A.