MMAT 5390: Mathematical Imaging

Lecture 7:
Even Discrete Cosine Transform &
Image enhancement in the frequency domain

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Even Discrete Cosine Transform

1D and 2D Even Discrete Cosine Transform:

1D: The 1D EDCT (for a signal f of size M) is given by

$$\hat{f}_{ec}(m) = \frac{1}{2M} \sum_{k=-M}^{M-1} f(k) \cos \frac{\pi m(2k+1)}{2M}$$

where $0 \leq m \leq 2M-1$.

2D: Let f be an $M \times N$ image. Reflect it about its left and top broader to get a $2M \times 2N$ image. (indices are shifted by $\frac{1}{2}$)

The EDCT of f is given by

$$\hat{f}_{ec}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

with $0 \leq m \leq 2M-1, 0 \leq n \leq 2N-1$

For details, please refer to Lecture Note Chapter 2

Inverse Even Discrete Cosine Transform

Inverse Even Discrete Cosine Transform:

2D:

$$f(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{ec}(m,n)\cos\frac{\pi m(2k+1)}{2M}\cos\frac{\pi n(2l+1)}{2N}$$

$$\text{for } C(0)=1, C(m)=C(n)=2 \text{ for } m,n\neq 0$$

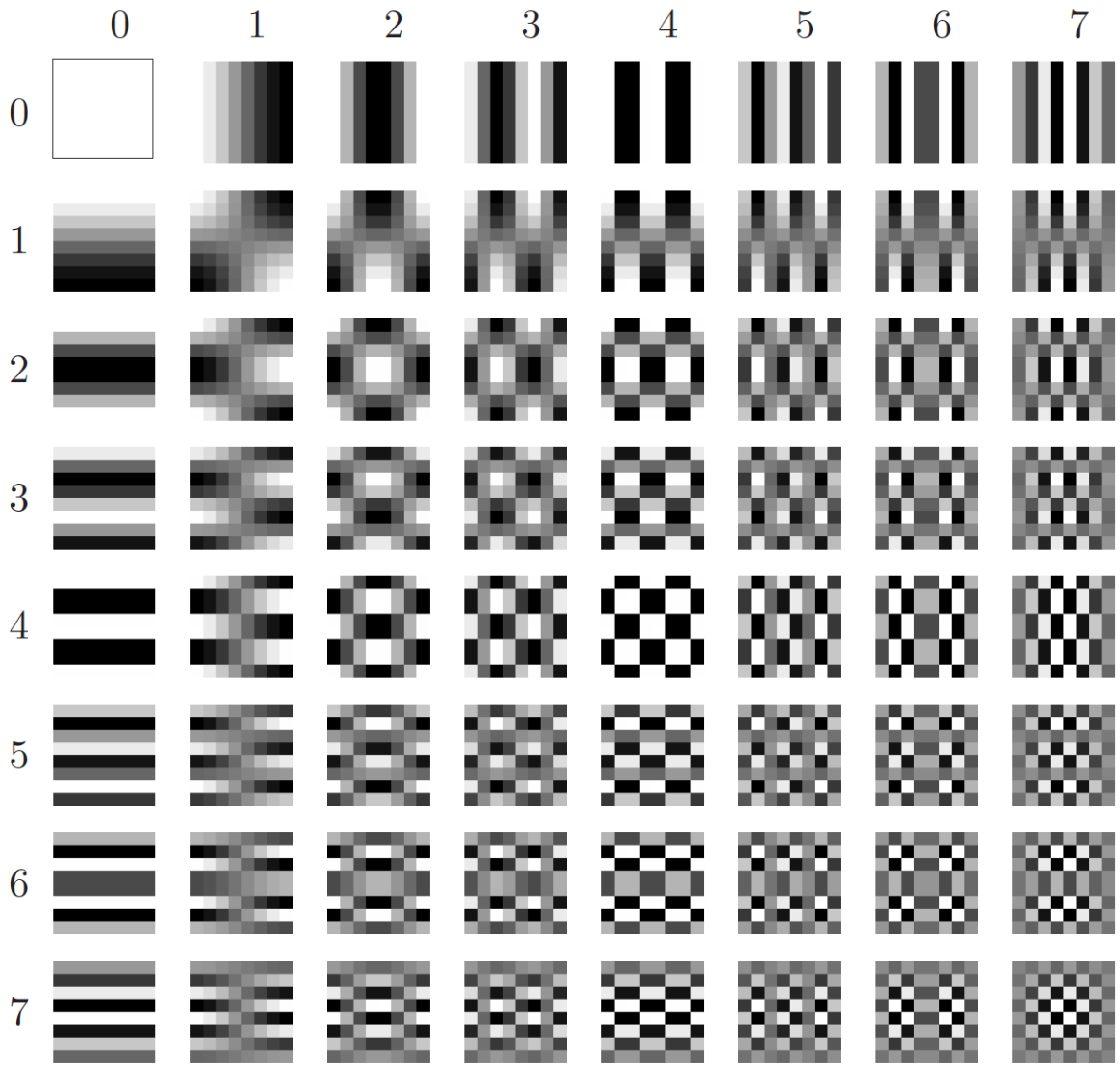
Image Decomposition

$$f = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}_{ec}(m,n)\vec{T}_m\vec{T'}_n^T$$

$$\text{where: } \vec{T}_m^T = \begin{pmatrix} T_m(0) \\ T_m(1) \\ \vdots \\ T_m(M-1) \end{pmatrix}, \vec{T'}_n^T = \begin{pmatrix} T'_n(0) \\ T'_n(1) \\ \vdots \\ T'_n(N-1) \end{pmatrix} \text{ with } T_m(k) = C(m)\cos\frac{\pi m(2k+1)}{2M}$$

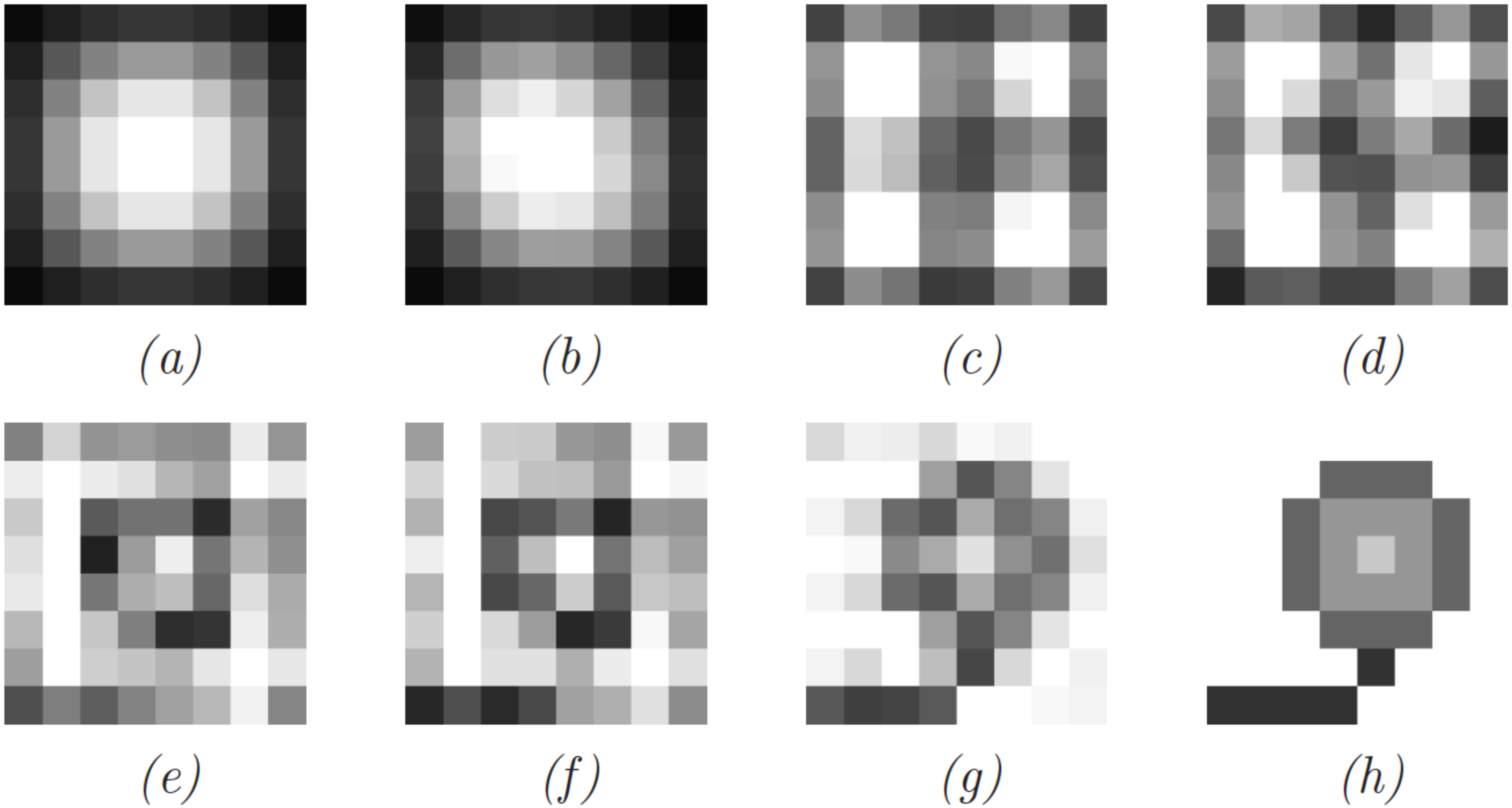
$$\text{and } T'_n(k) = C(n)\cos\frac{\pi n(2k+1)}{2N}.$$

Elementary images of EDCT decomposition



The basis images in terms of which any 8×8 image is expanded by EDCT.

Reconstruction w/ EDCT decomposition



(a) = using 1x1 elementary images (first 1 row and first 1 column elementary images;

(b) = using 2x2 elementary images (first 2 rows and first 2 column elementary images…and so on…

Other similar transforms

Odd Discrete Cosine Transform:

$$\hat{f}_{oc}(m,n) \equiv \frac{1}{(2M-1)(2N-1)} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} C(k)C(l) f(k,l) \cos \frac{2\pi mk}{2M-1} \cos \frac{2\pi nl}{2N-1}$$

Even Discrete Sine Transform:

$$\hat{f}_{es}(m,n) \equiv -\frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \sin \frac{\pi m(2k+1)}{2M} \sin \frac{\pi n(2l+1)}{2N}$$

Odd Discrete Sine Transform:

$$\hat{f}_{os}(m,n) \equiv -\frac{4}{(2M+1)(2N+1)} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \sin \frac{2\pi m(k+1)}{2M+1} \sin \frac{2\pi n(l+1)}{2N+1}$$

All of them have explicit formula for their inverses.

(For details, please refer to Lecture Note Chapter 2)

Other similar transforms

Inverse Odd Discrete Cosine Transform:

$$f(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{oc}(m,n)\cos\frac{2\pi mk}{2M-1}\cos\frac{2\pi nl}{2N-1}$$

where $C(0) = 1, C(m) = C(n) = 2$ if $m, n \neq 0$

Inverse Even Discrete Sine Transform:

$$f(k,l) = -\sum_{m=1}^{M}\sum_{n=1}^{N} S(m)S(n)\,\hat{f}_{es}(m,n)\sin\frac{\pi m(2k+1)}{2M}\sin\frac{\pi n(2l+1)}{2N}$$

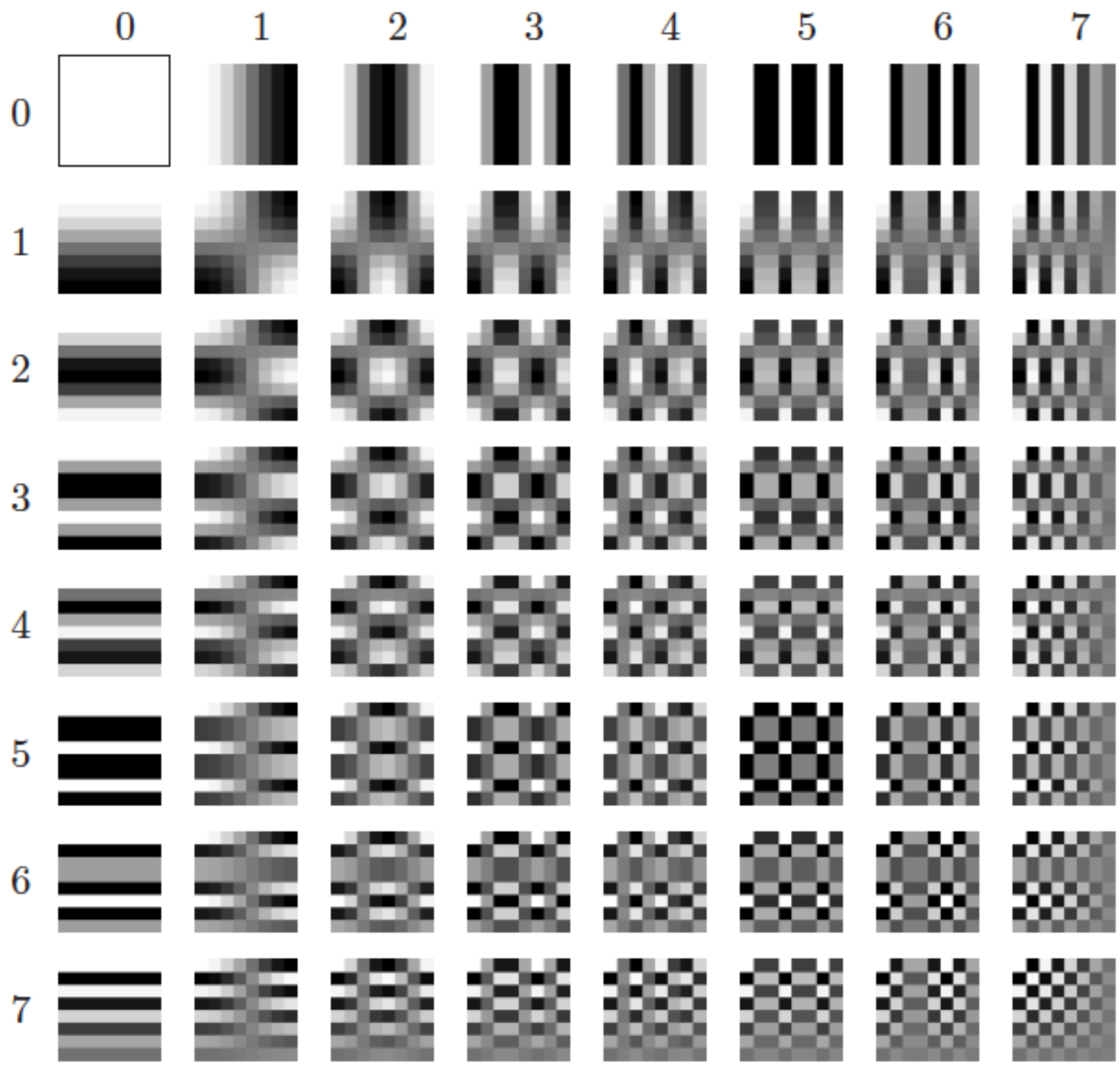
where $S(M) = S(N) = 1$, $S(m) = S(n) = 2$ for $m \neq M, n \neq N$.

Inverse Odd Discrete Sine Transform:

$$f(k,l) = -16\sum_{m=1}^{M}\sum_{n=1}^{N} \hat{f}_{os}(m,n)\sin\frac{2\pi m(k+1)}{2M+1}\sin\frac{2\pi n(l+1)}{2N+1}$$

(For details, please refer to Lecture Note Chapter 2)

Elementary images of ODCT decomposition



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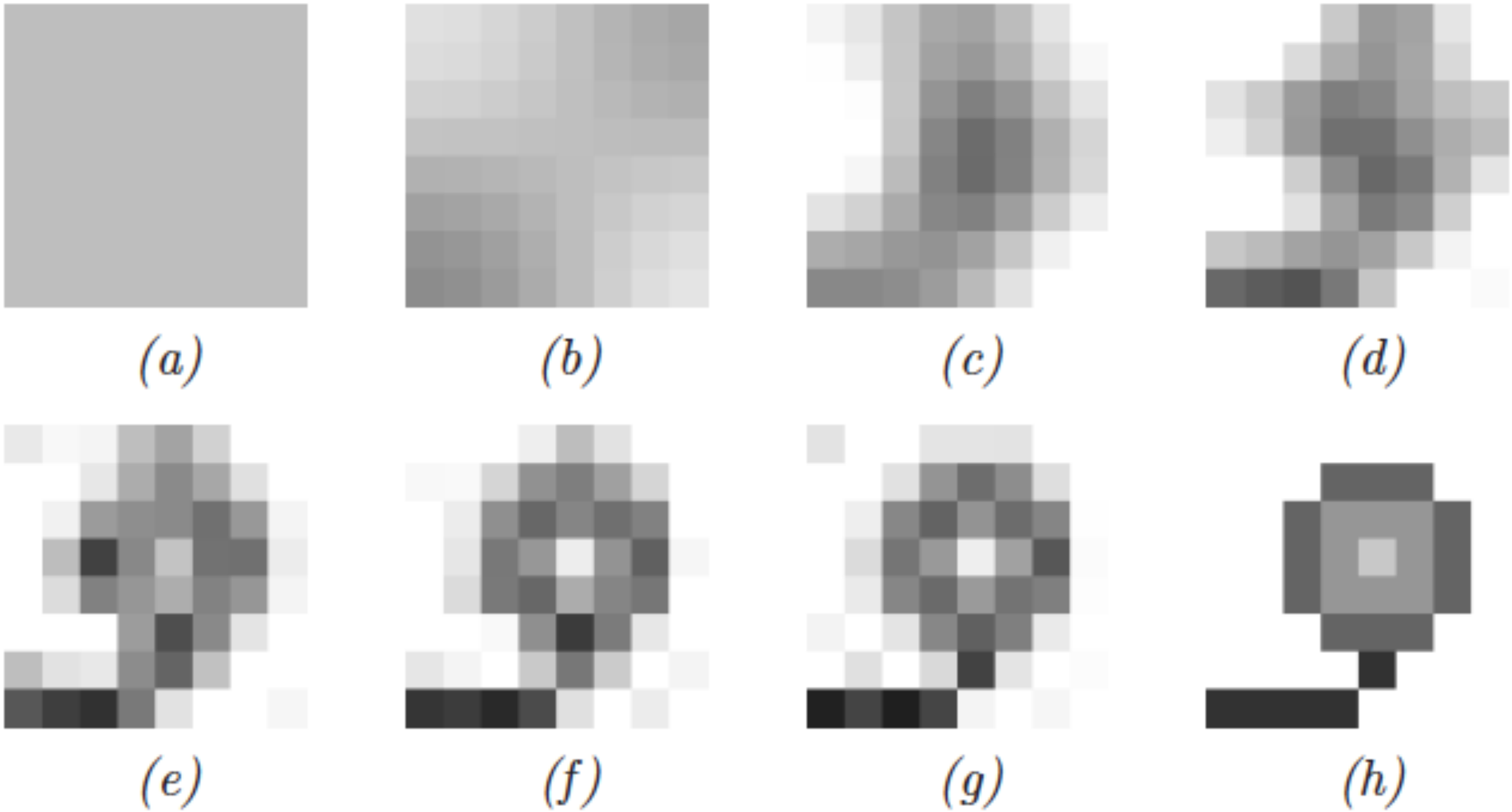
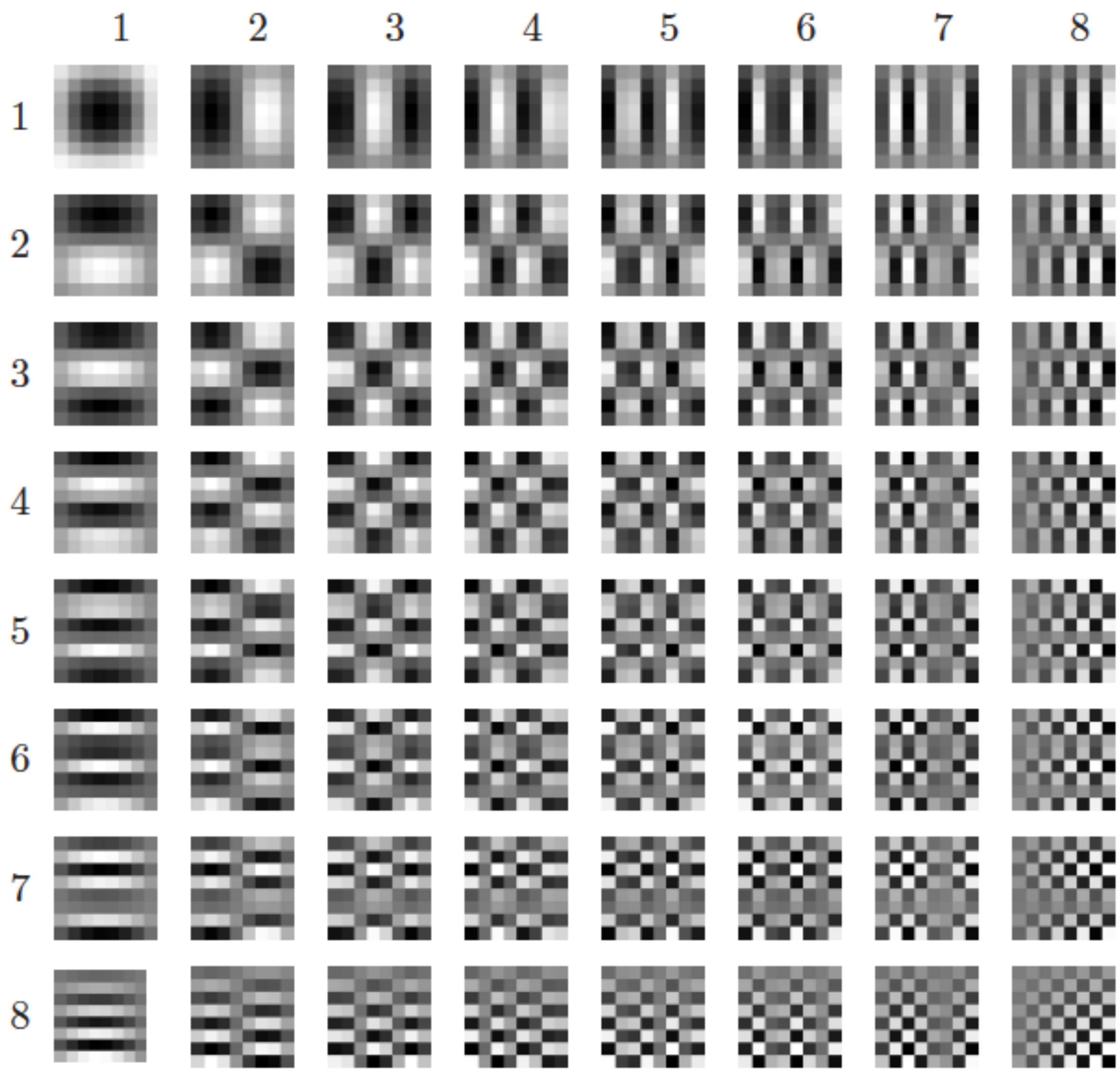
Reconstruction w/ ODCT decomposition



(a) = using 1x1 elementary images (first 1 row and first 1 column elementary images;

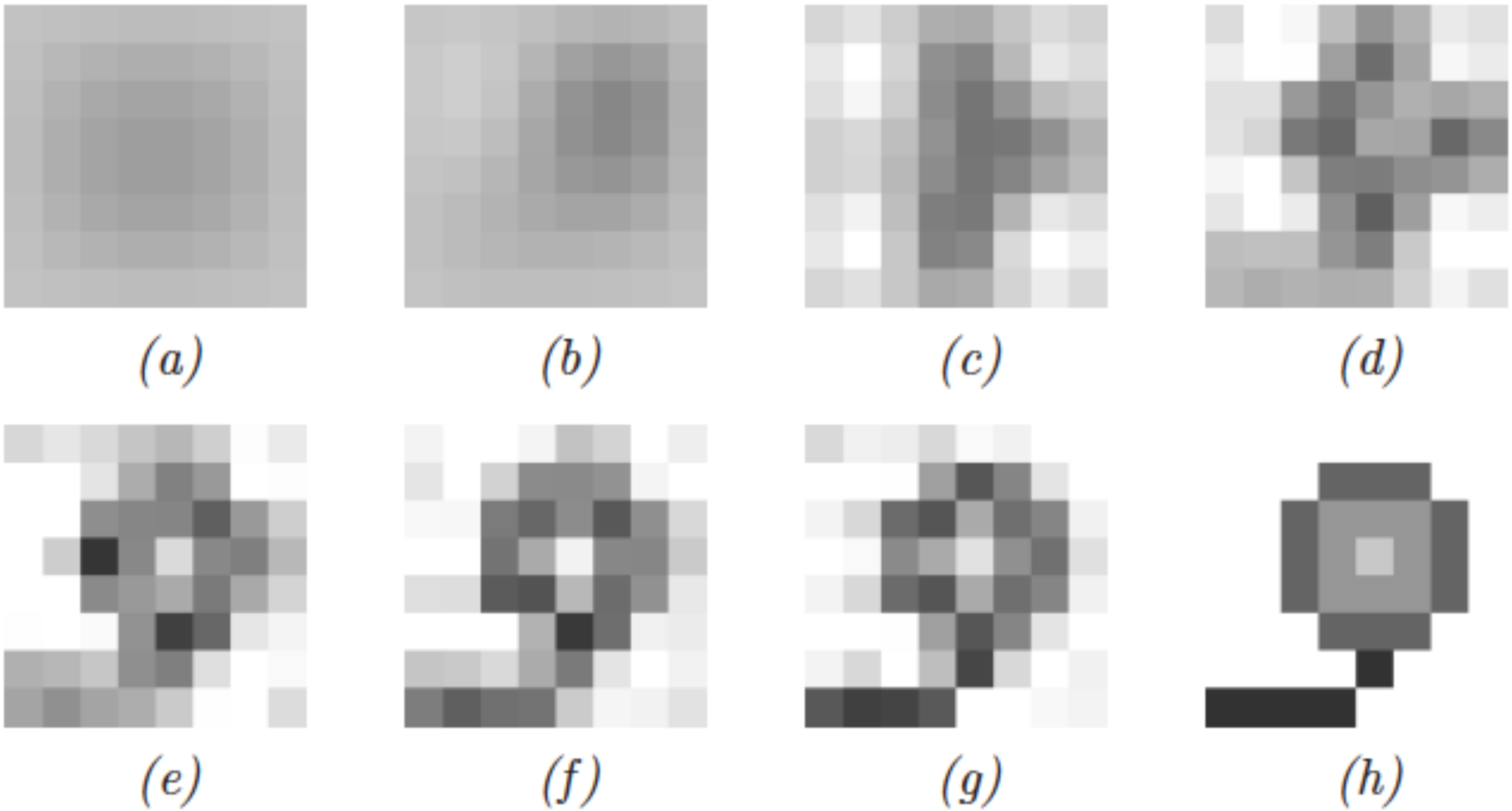
(b) = using 2x2 elementary images (first 2 rows and first 2 column elementary images…and so on…

Elementary images of EDST decomposition



The basis images in terms of which any 8×8 image is expanded by EDCT.

Reconstruction w/ EDST decomposition



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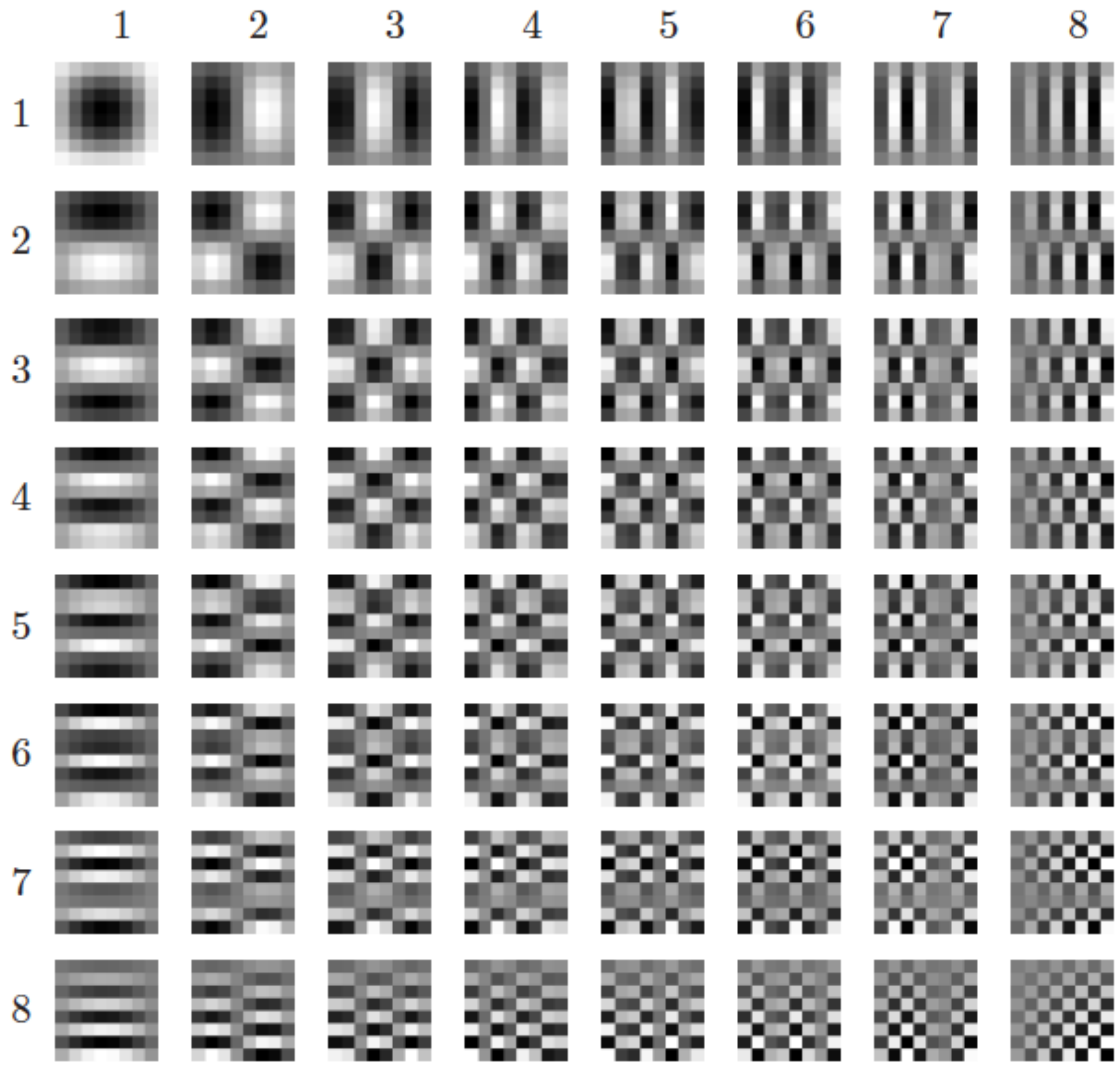
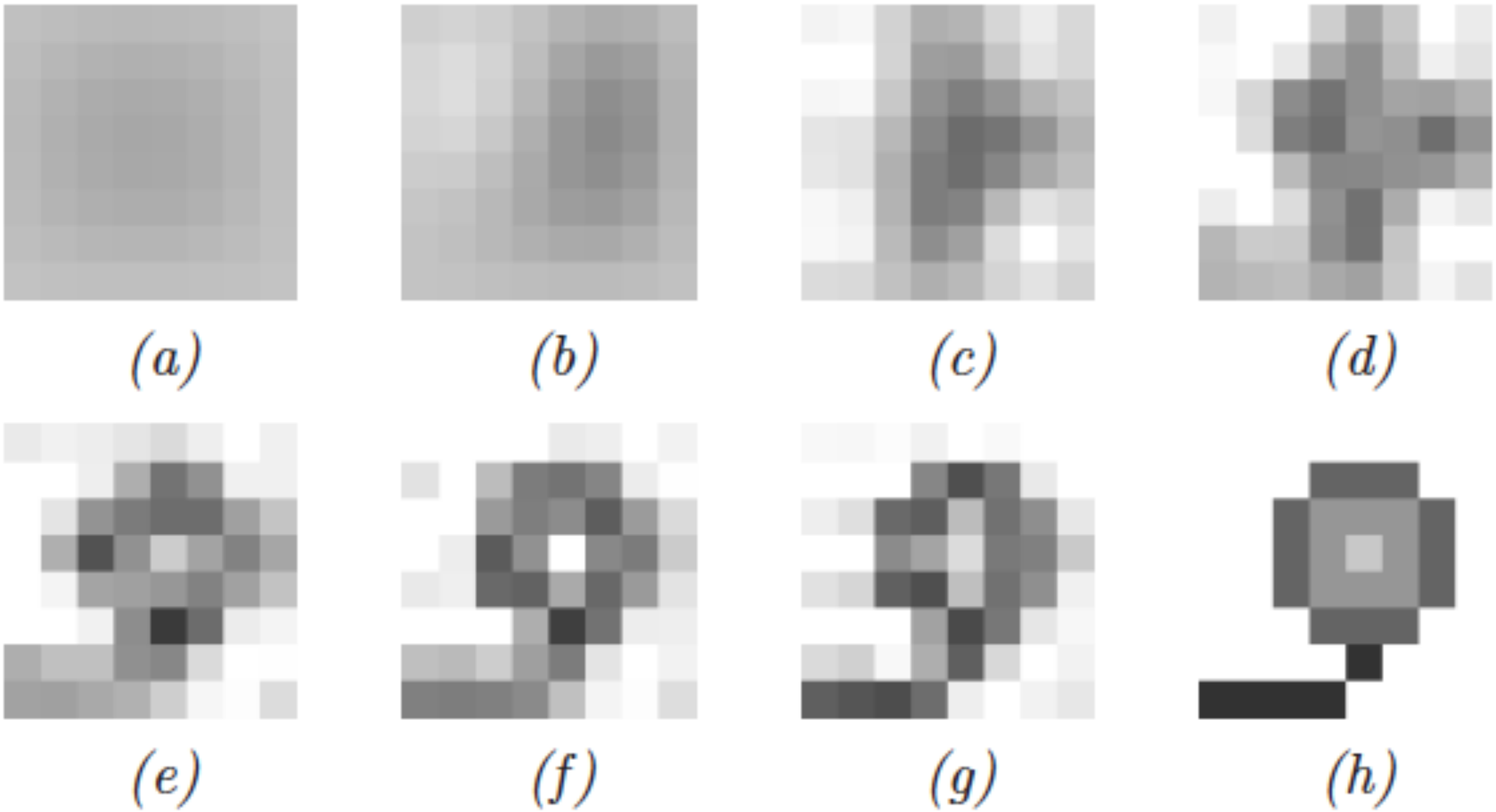
Elementary images of ODST decomposition



The basis images in terms of which any 8×8 image is expanded by EDCT.

Reconstruction w/ ODST decomposition



(a) = using 1x1 elementary images (first 1 row and first 1 column elementary images;

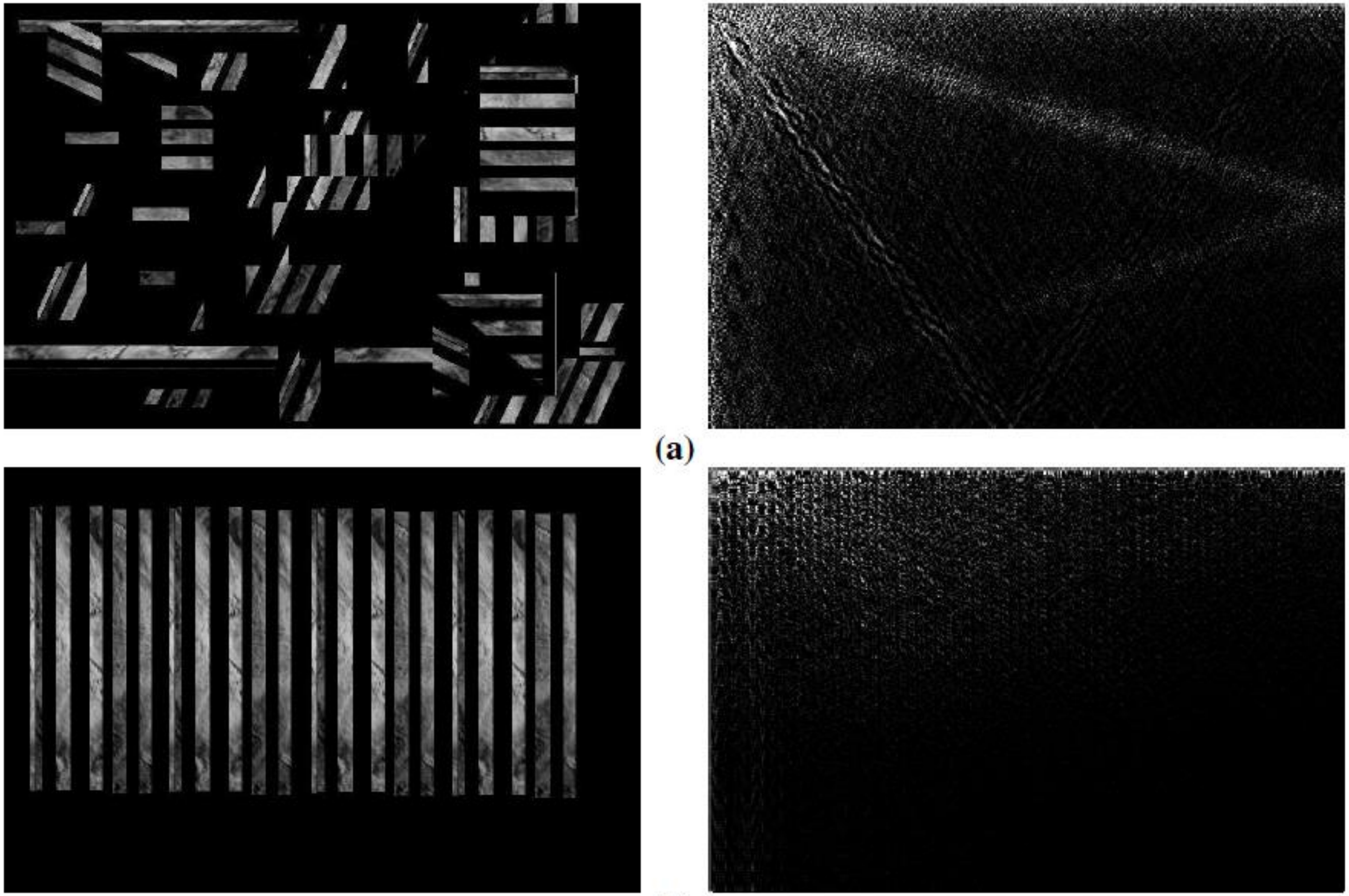
(b) = using 2x2 elementary images (first 2 rows and first 2 column elementary images…and so on…

Comparison of errors

The flower example:

	0	1	2	3	4	5	6
SVD	230033	118412	46673	11882			
Haar	366394	356192	291740	222550	192518	174625	141100
Walsh	366394	356190	262206	222550	148029	92078	55905
DFT	366394	285895	234539	189508	141481	119612	71908
EDCT	366394	338683	216608	173305	104094	49179	35662
ODCT	368946	342507	221297	175046	96924	55351	39293
EDST	341243	328602	259157	206923	153927	101778	55905
ODST	350896	326264	254763	205803	159056	109829	67374

More example on DCT decomposition



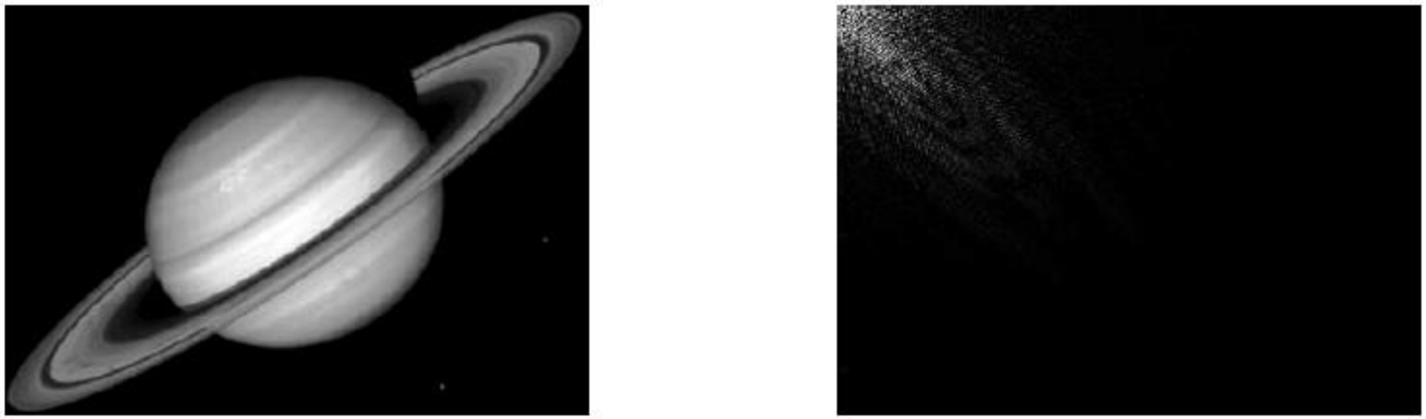
(a)

(b)

(a) Uncorrelated image and its DCT; (b) Correlated image and its DCT.

More example on DCT decomposition

Original image DCT

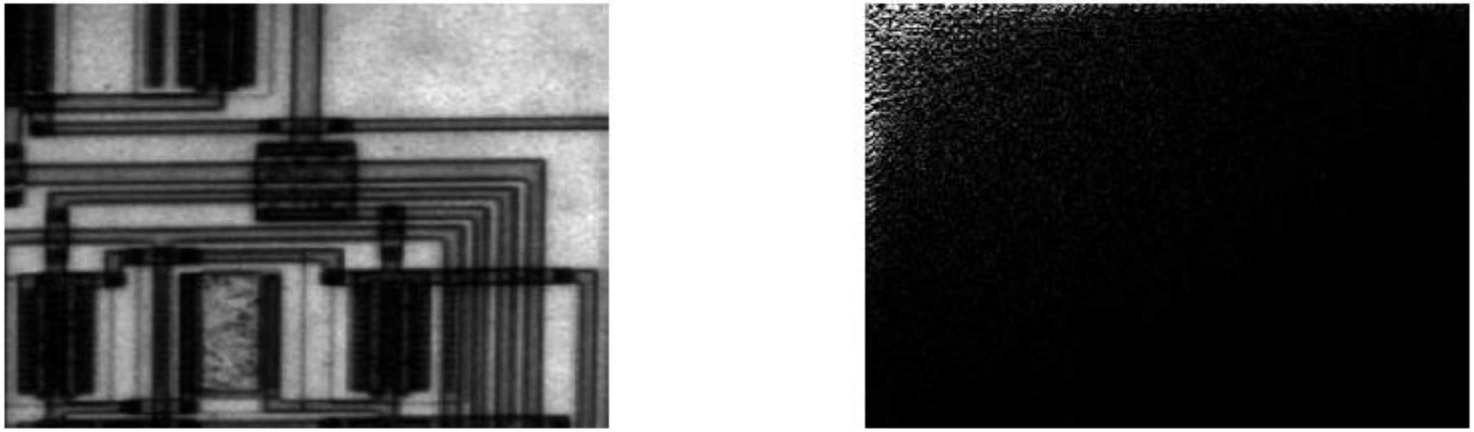


(a)

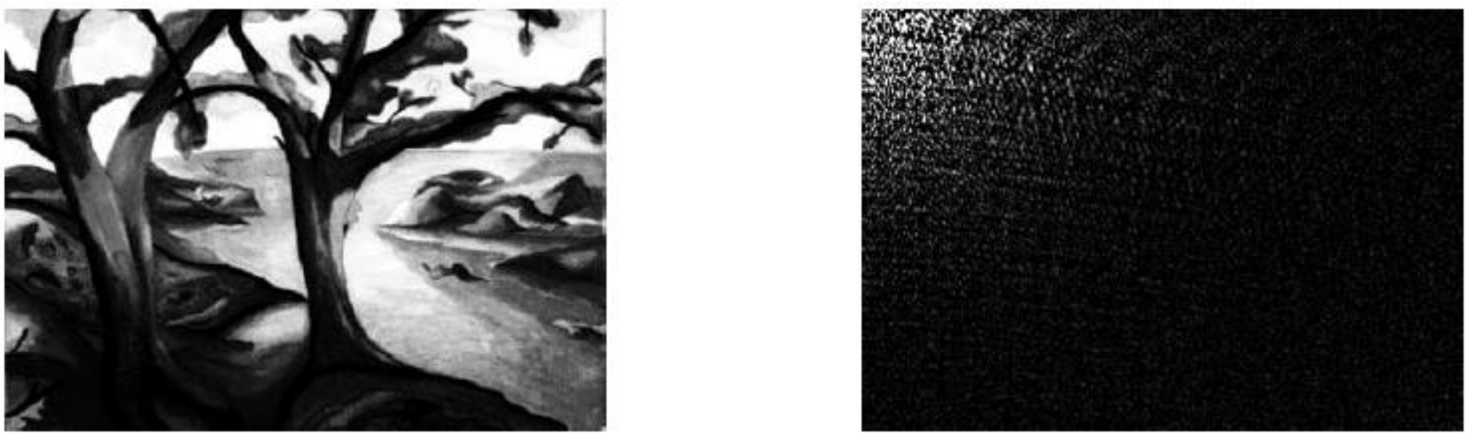
(b)

More example on DCT decomposition

Original image | DCT



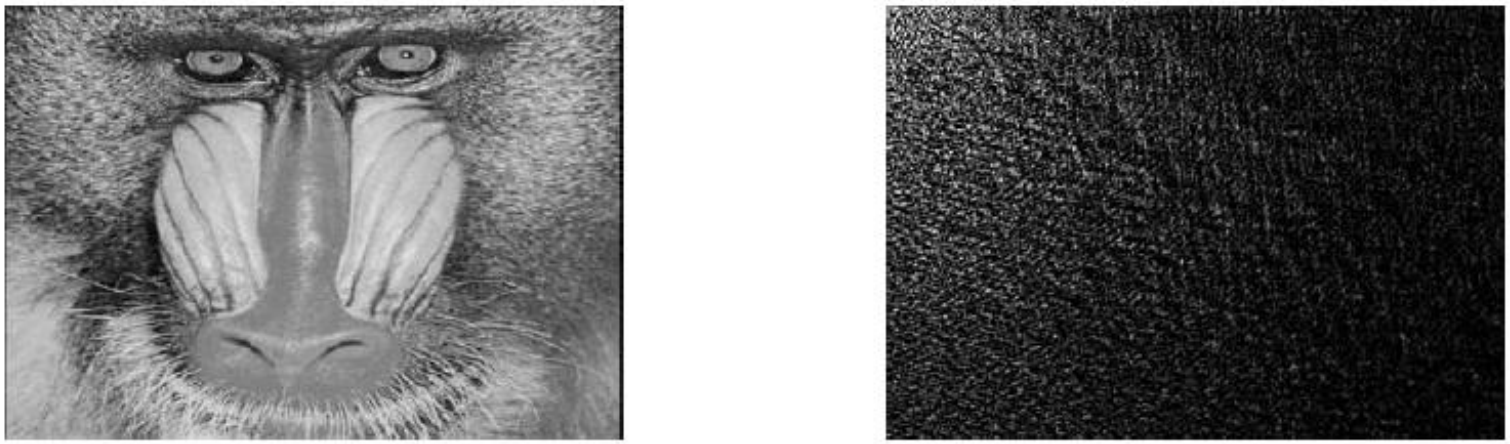
(c)



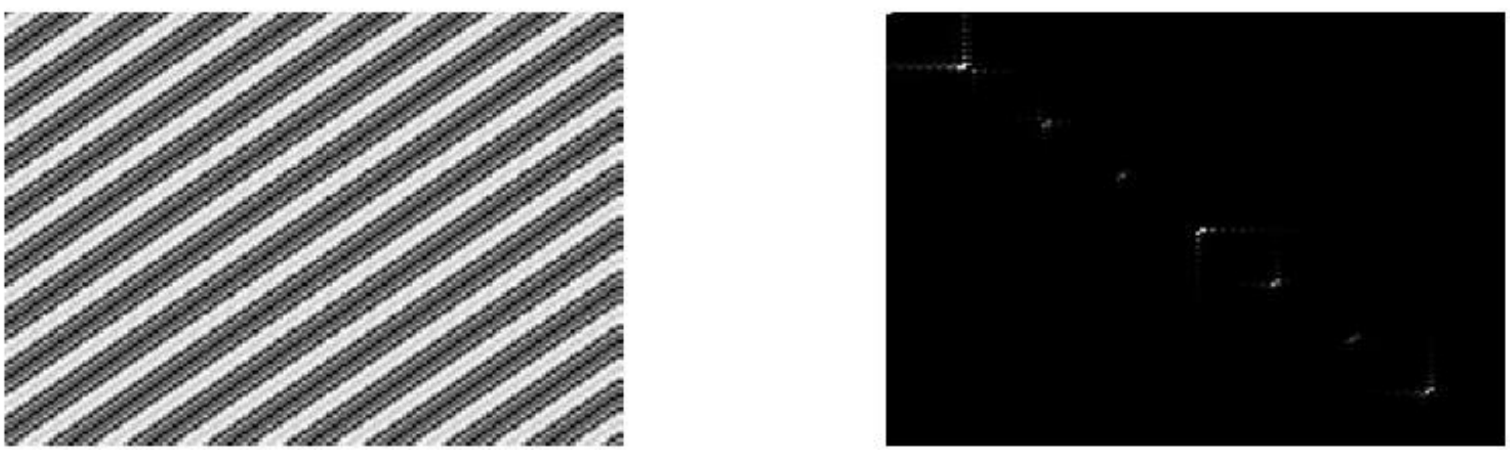
(d)

More example on DCT decomposition

Original image DCT



(e)



(f)

More example on DCT decomposition

(a)

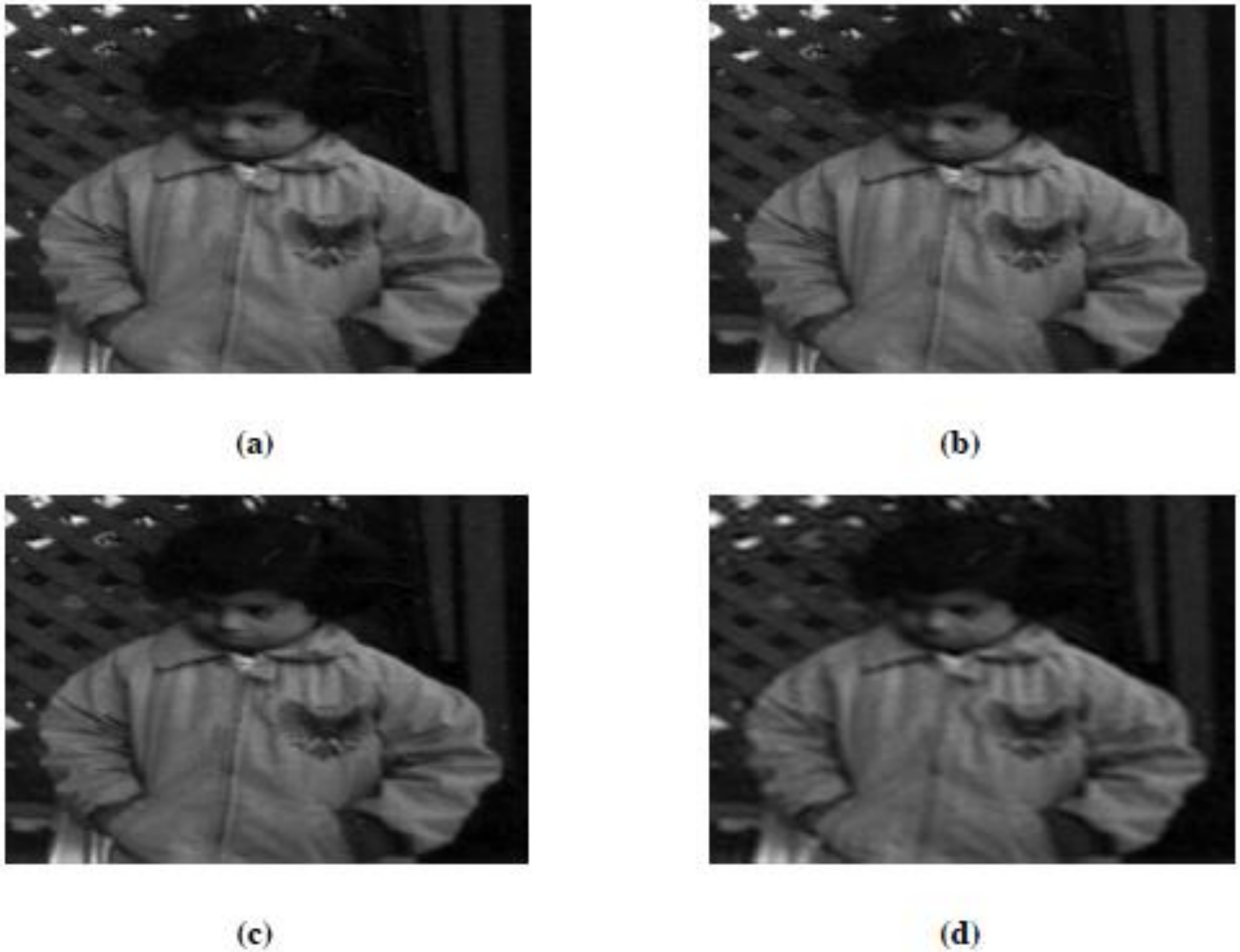
(b)

(c)

(d)

Inverse DCT of Saturn; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

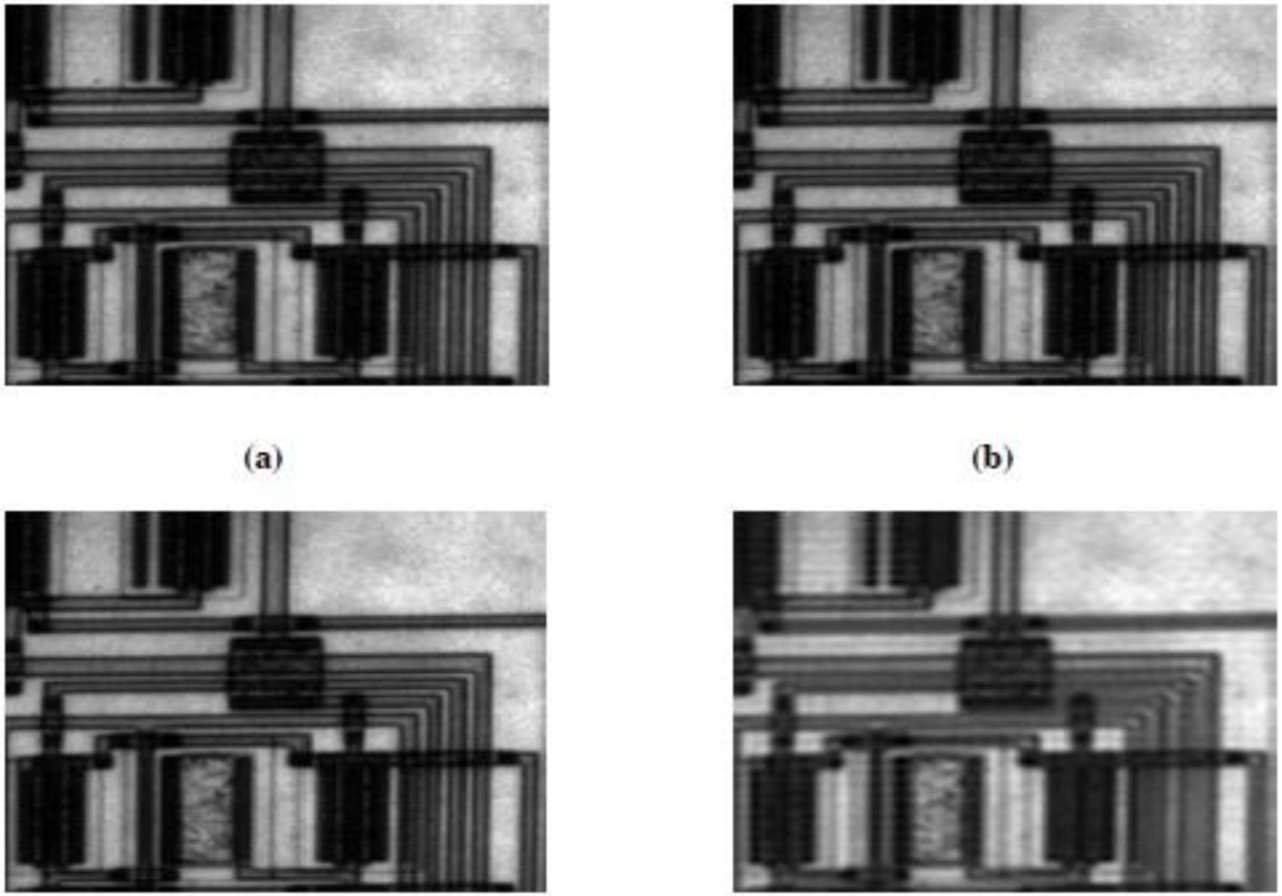
More example on DCT decomposition



(c) (d)

Inverse DCT of Child; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

More example on DCT decomposition



(a) (b) (c) (d)

Inverse DCT of Circuit; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

More example on DCT decomposition

(a)

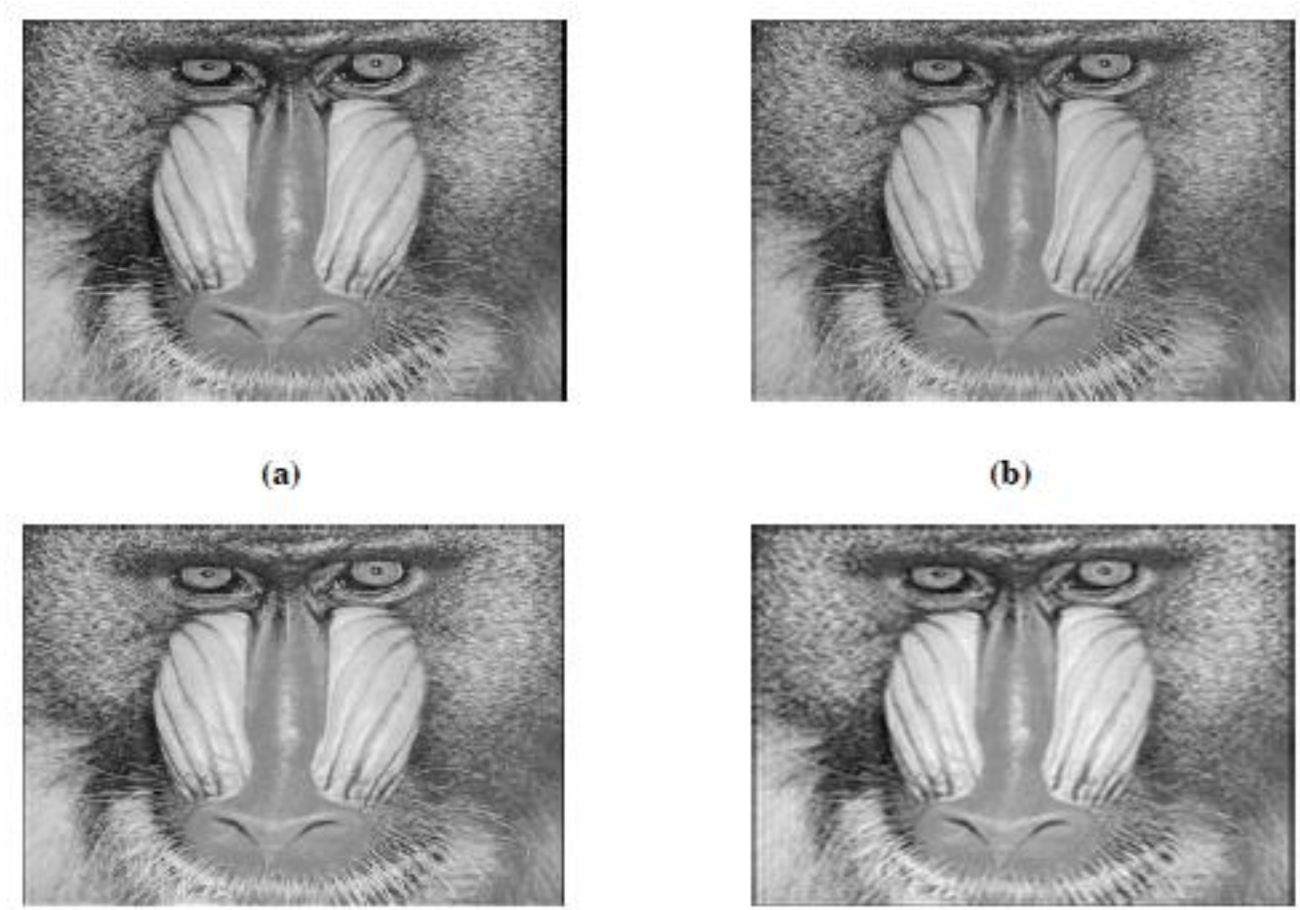
(b)

(c)

(d)

Inverse DCT of Trees; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

More example on DCT decomposition

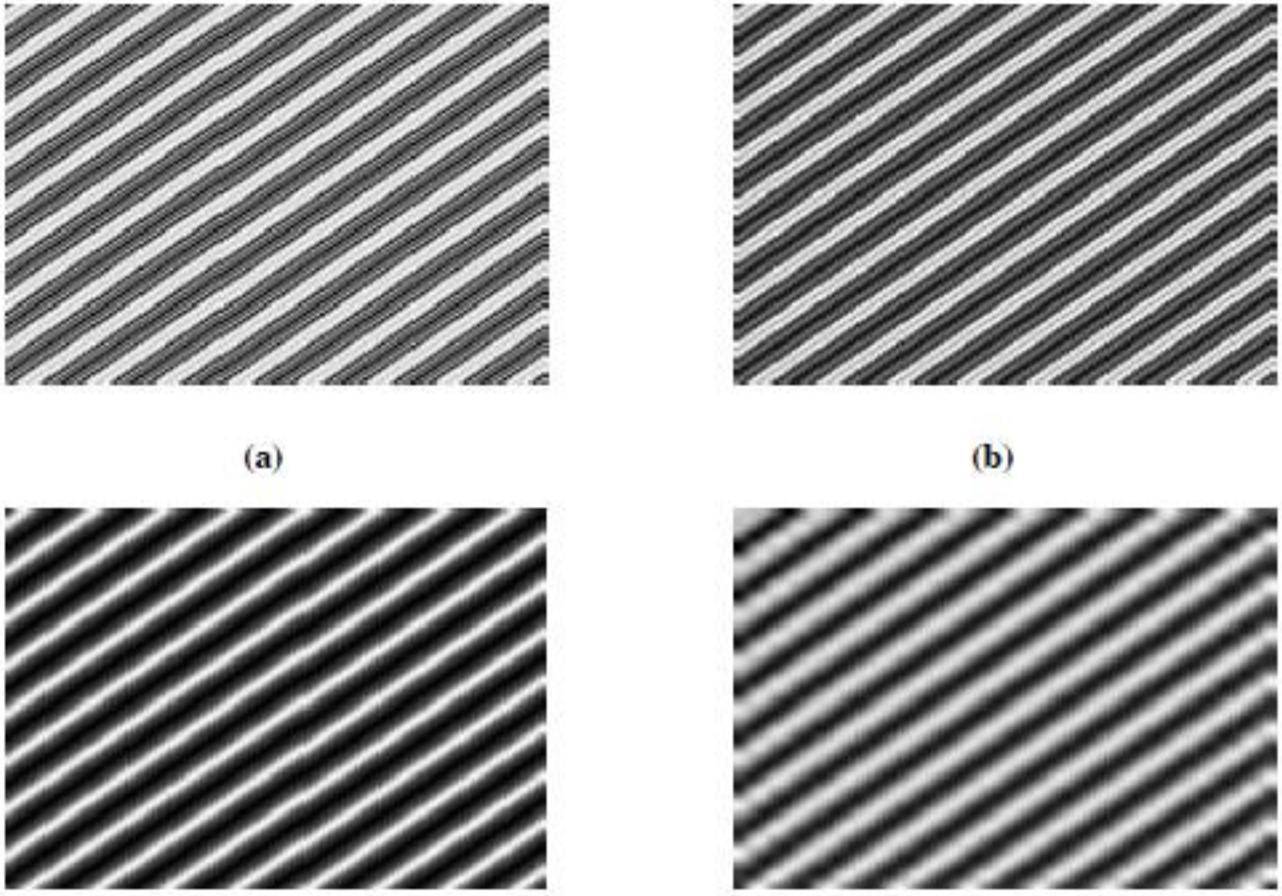


(a) (b)

(c) (d)

. Inverse DCT of Baboon; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

More example on DCT decomposition



(a) (b) (c) (d)

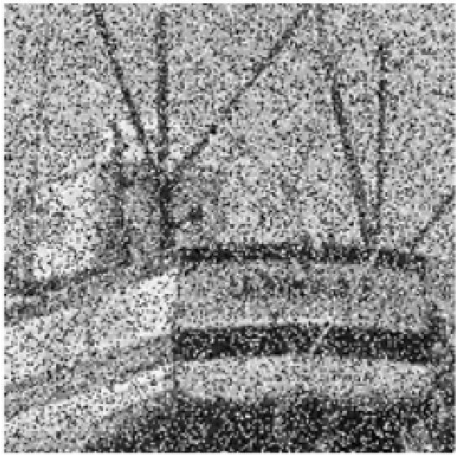
Inverse DCT of sine wave; (a) DCT(100%); (b) DCT(75%); (c) DCT(50%); (d) DCT(25%).

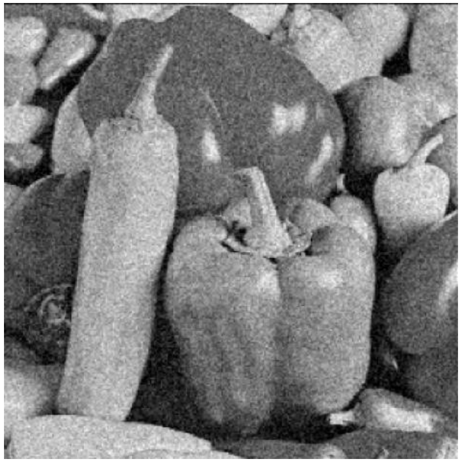
Image Enhancement

- **What is image enhancement?**
  - Image enhancement is the process by which we improve an image so that it looks subjectively better.
- **How?**
  - An image is enhanced when we:
    - remove additive noise and interference;
    - remove multiplicative interference;
    - increase its contrast;
    - decrease its blurring.
  - Some standard methods:
    - smoothing or low pass filtering;
    - sharpening or high pass filtering;
    - histogram manipulation and
    - algorithms that remove noise while avoid blurring the image.

Image Enhancement

- **We will consider two image enhancement problems:**
  - Image denoising





  - Image deblurring

Image Enhancement

- **We will consider two image enhancement problems:**
  - Image denoising

  - Image deblurring

Image Enhancement

- **Linear filtering:**
  - Modifying a pixel value (in the spatial domain) by a linear combination of neighborhood values.
- **Operations in spatial domain v.s. operations in frequency domains:**
  - Linear filtering (matrix multiplication in spatial domain) = discrete convolution
  - In the frequency domain, it is equivalent to multiplying the Fourier transform of the image with a certain function that “kills” or modifies certain frequency components

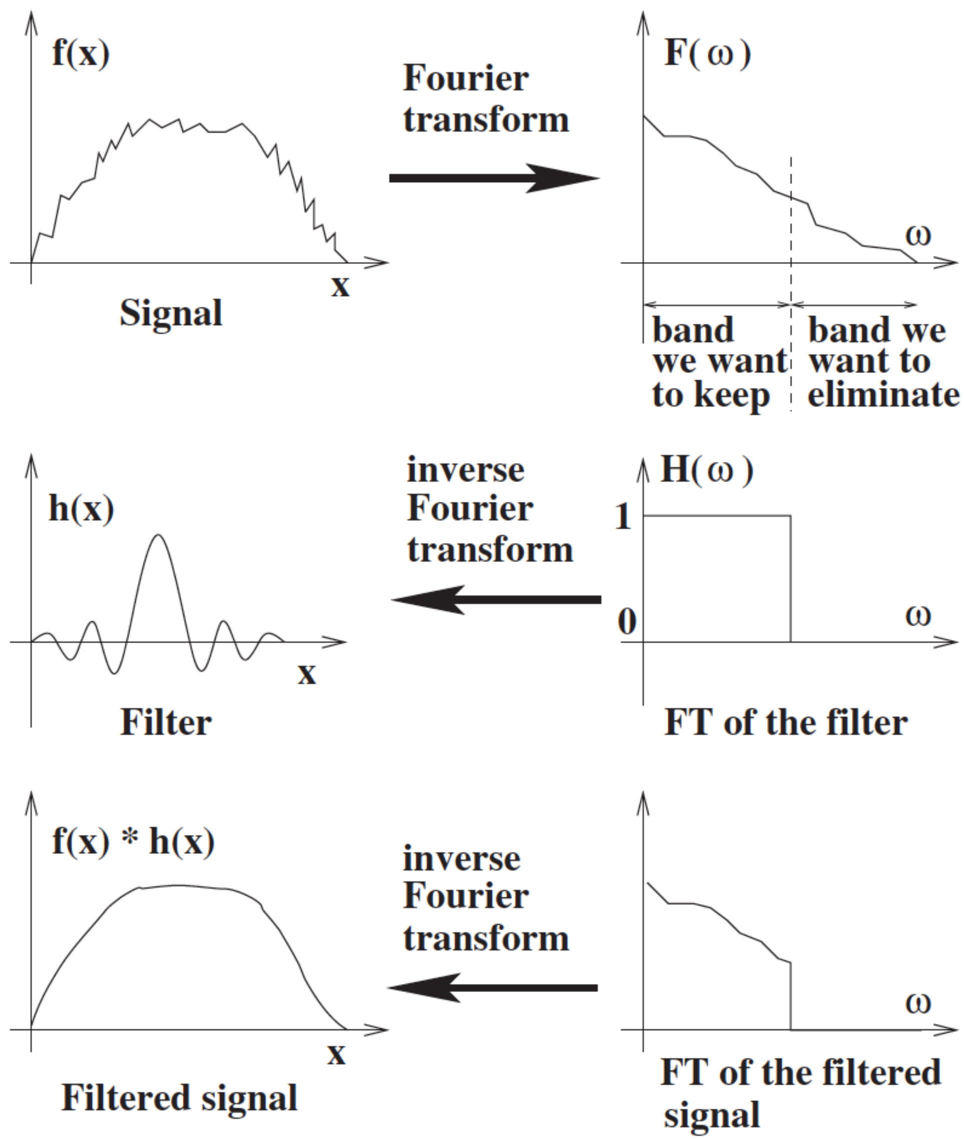
Spatial transform v.s. frequency transform

- **Discrete convolution:**

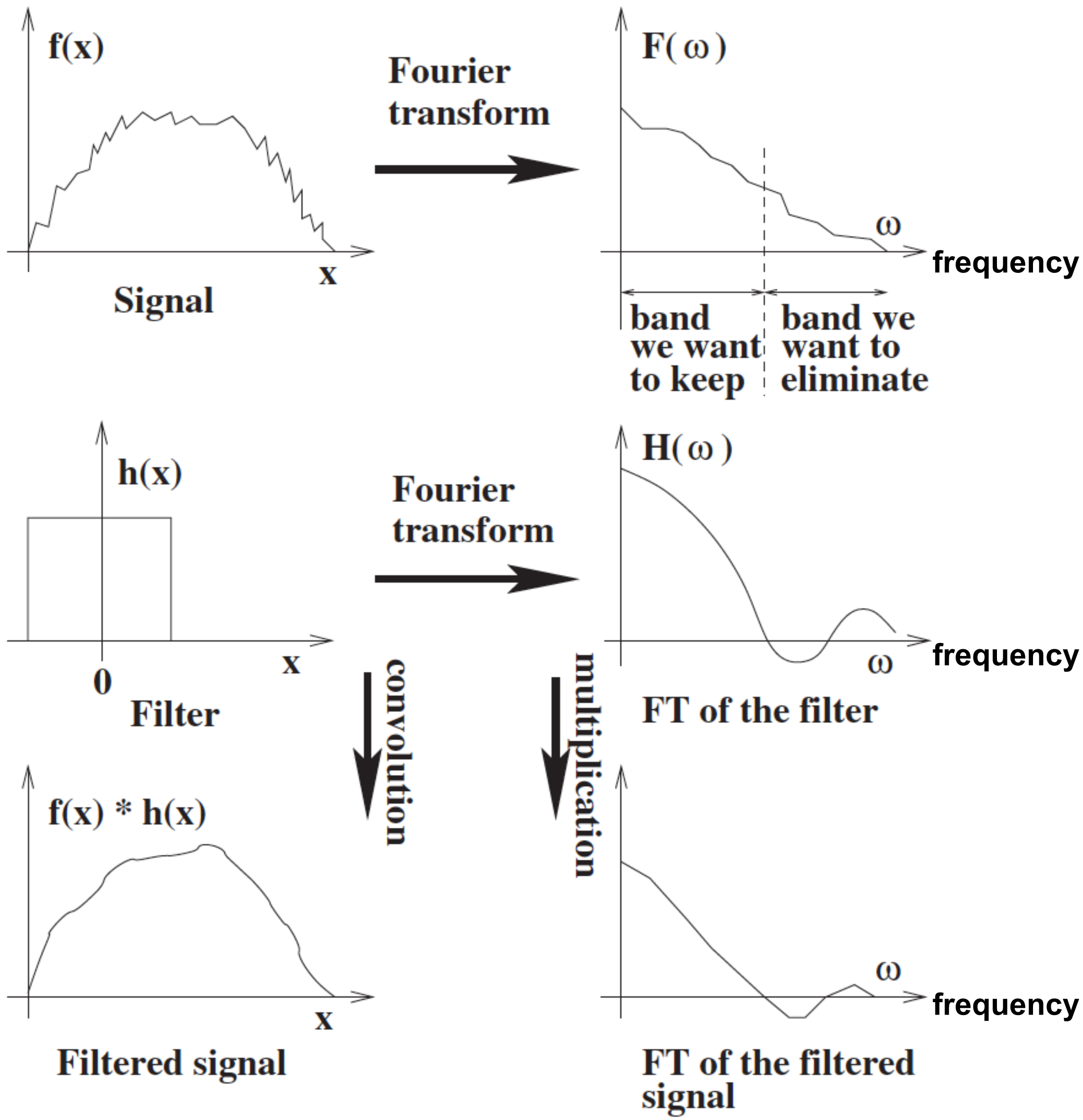
$$v(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} g(n-n', m-m')w(n', m')$$

- **DFT of Discrete convolution: Product of fourier transform**
- **DFT(convolution of f and w) = C*DFT(f)*DFT(w)**
- Multiplying the Fourier transform of the image with a certain function that "kills" or modifies certain frequency components

Spatial transform v.s. frequency transform



Spatial transform v.s. frequency transform

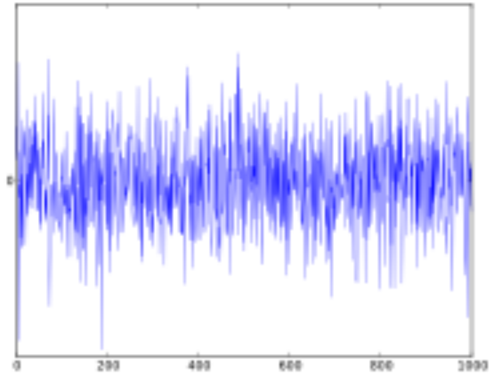


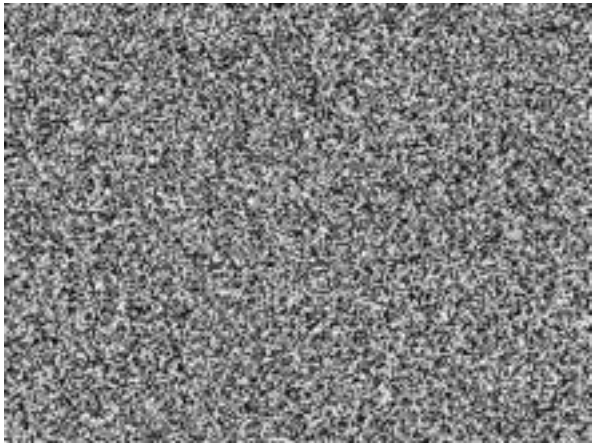
Gaussian noise

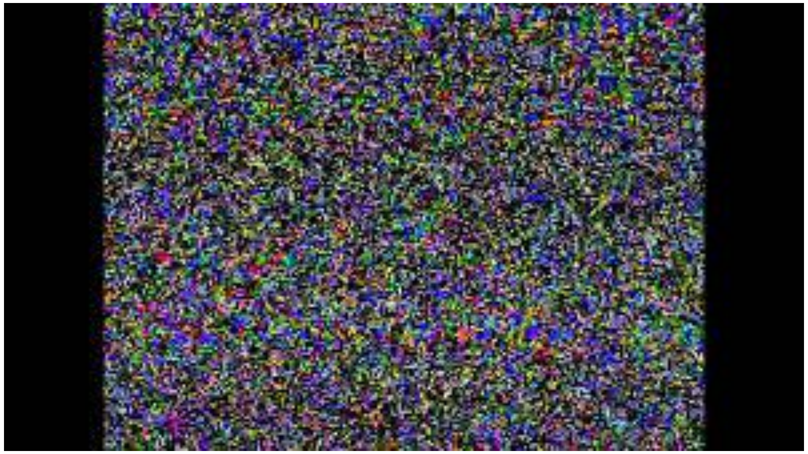
- **Example of Gaussian noises:**

White noise

- **Example of white noises:**

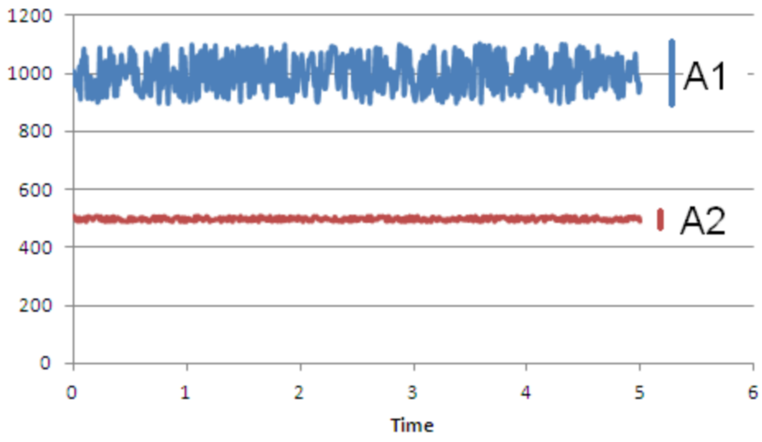


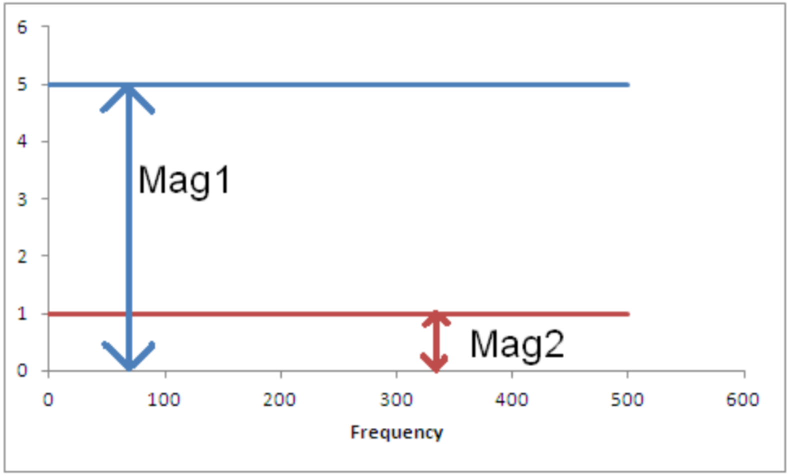


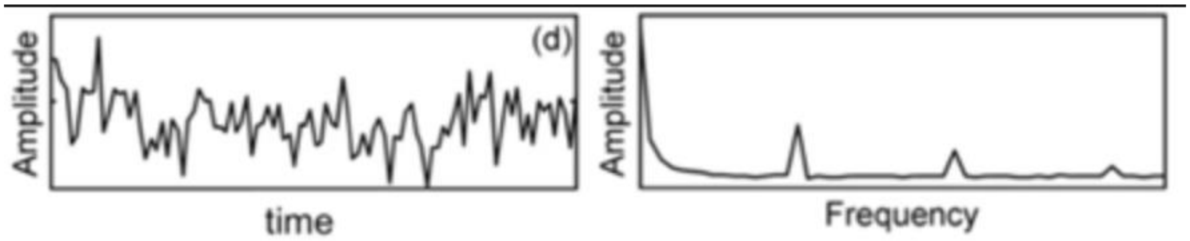


White noise

- Example of white noises:

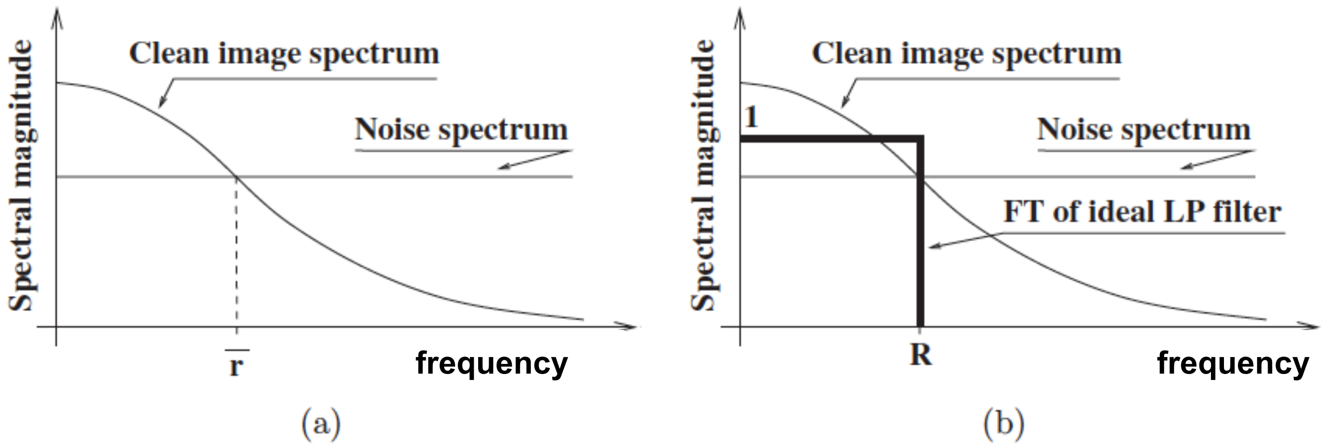






Noises as high frequency component

Why noises are often considered as high frequency component?



(a) Clean image spectrum and Noise spectrum (Noise dominates the high-frequency component);
(b) Filtering of high-frequency component

Spatial transform v.s. frequency transform

- **Discrete convolution:**

$$v(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} g(n-n', m-m') w(n', m')$$

- **DFT of Discrete convolution: Product of fourier transform**
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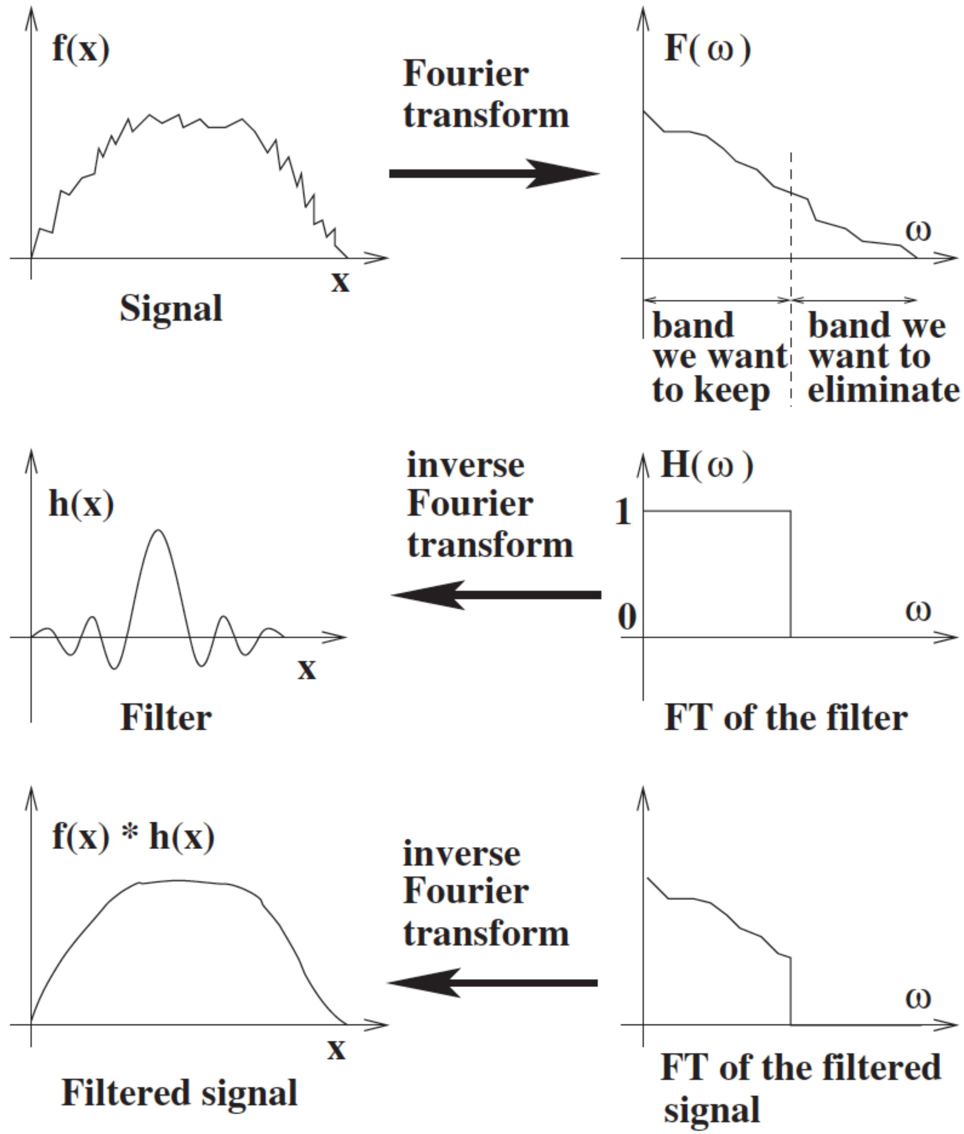
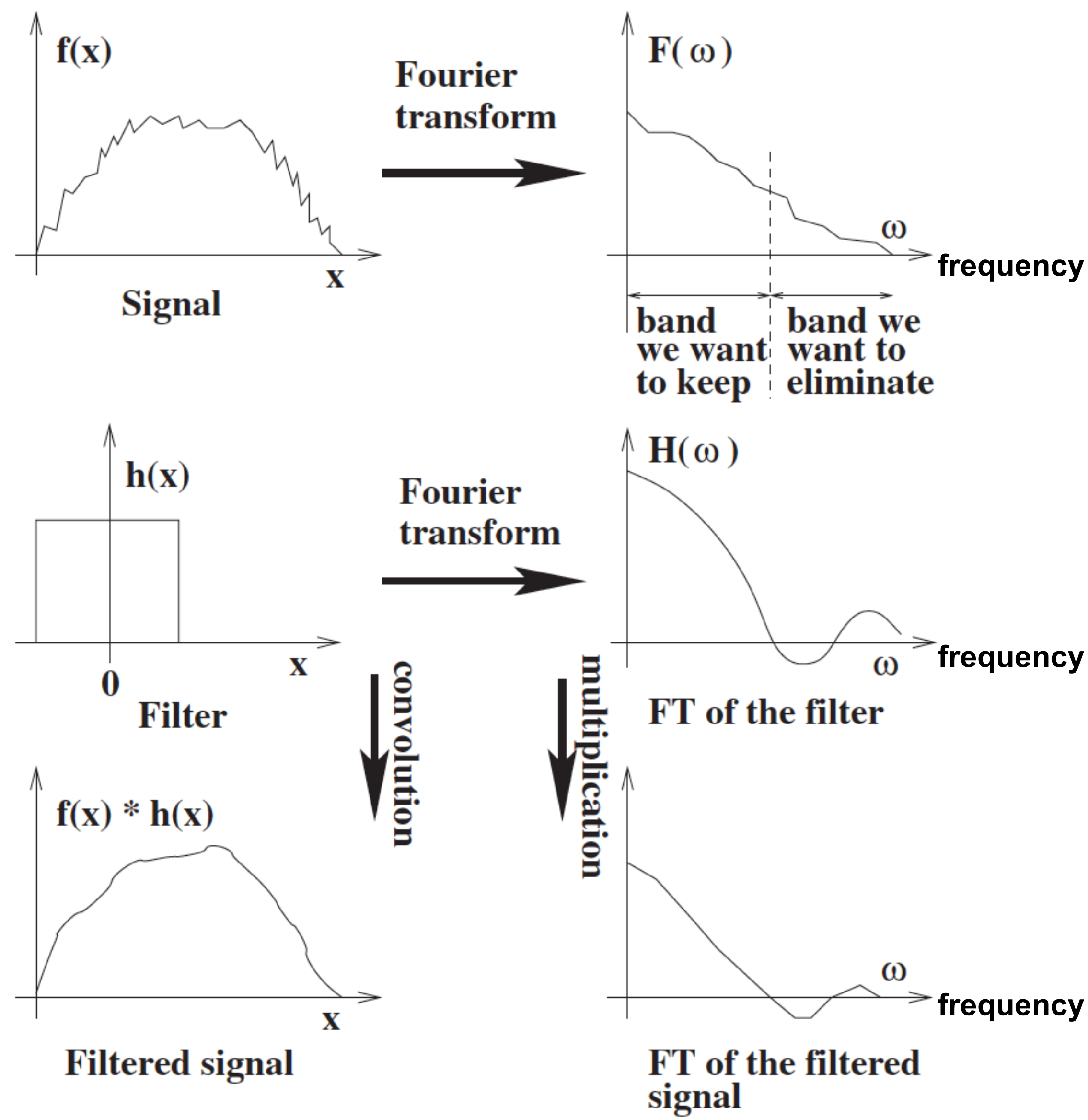
Spatial transform v.s. frequency transform



Spatial transform v.s. frequency transform



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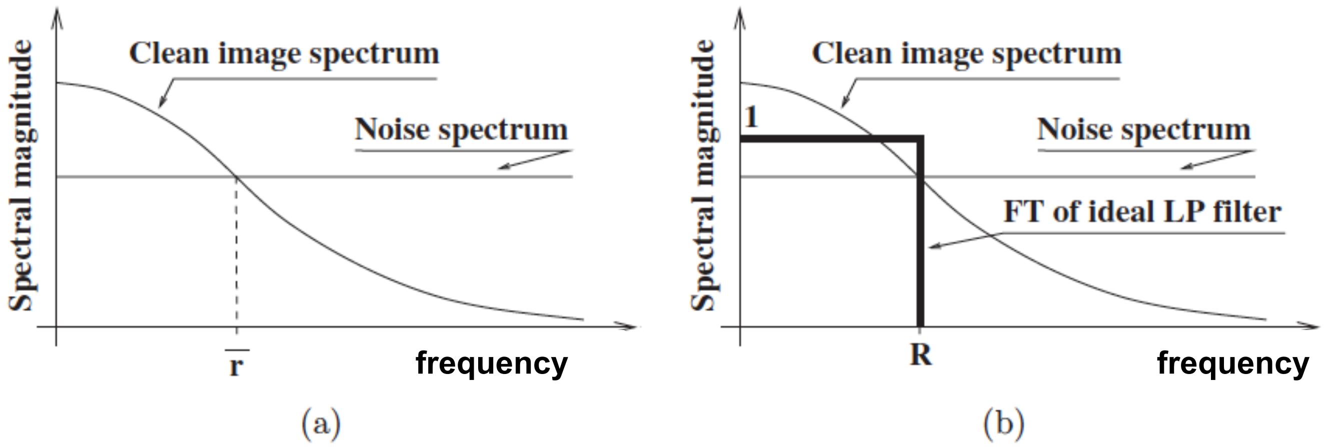
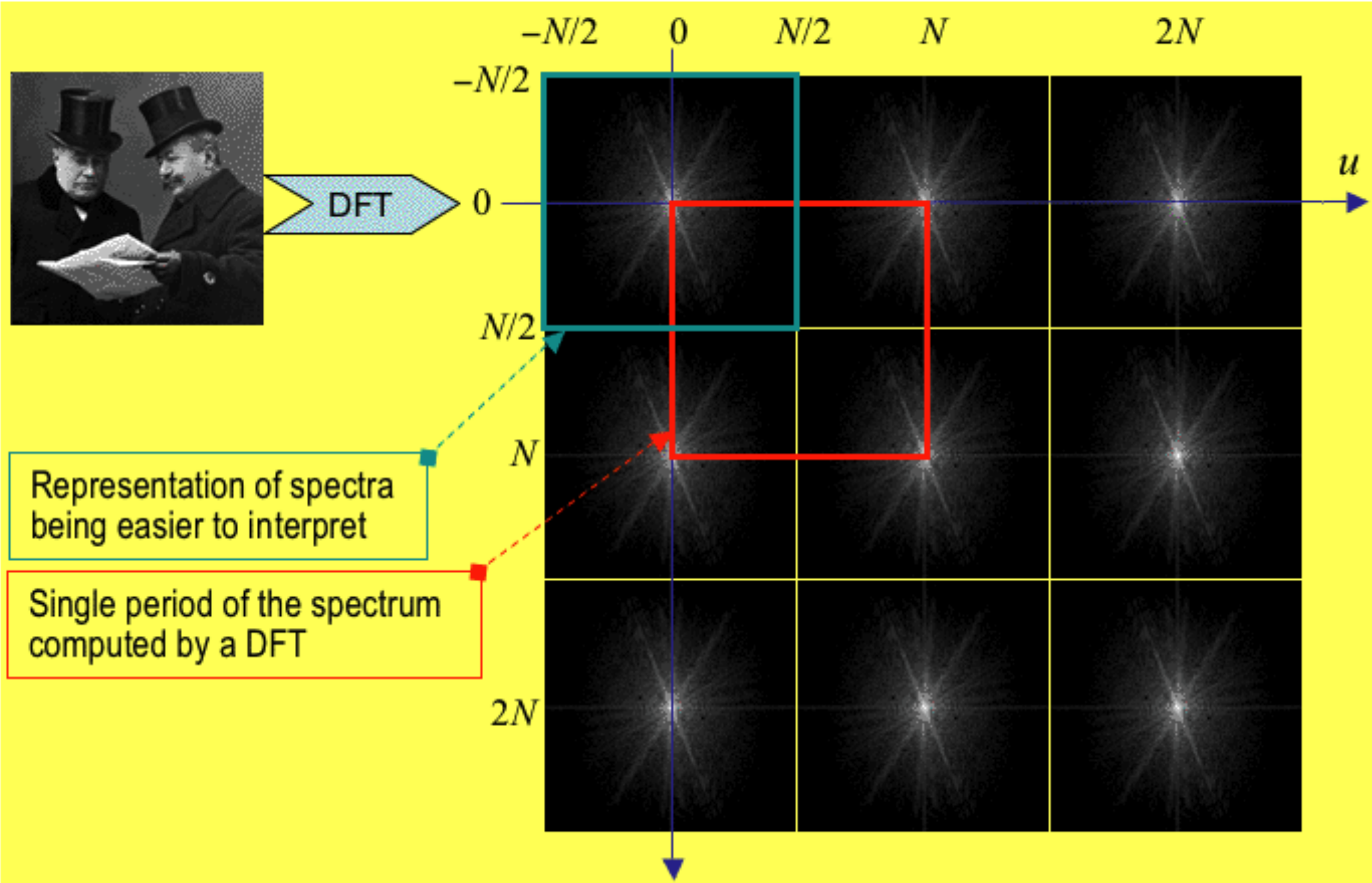
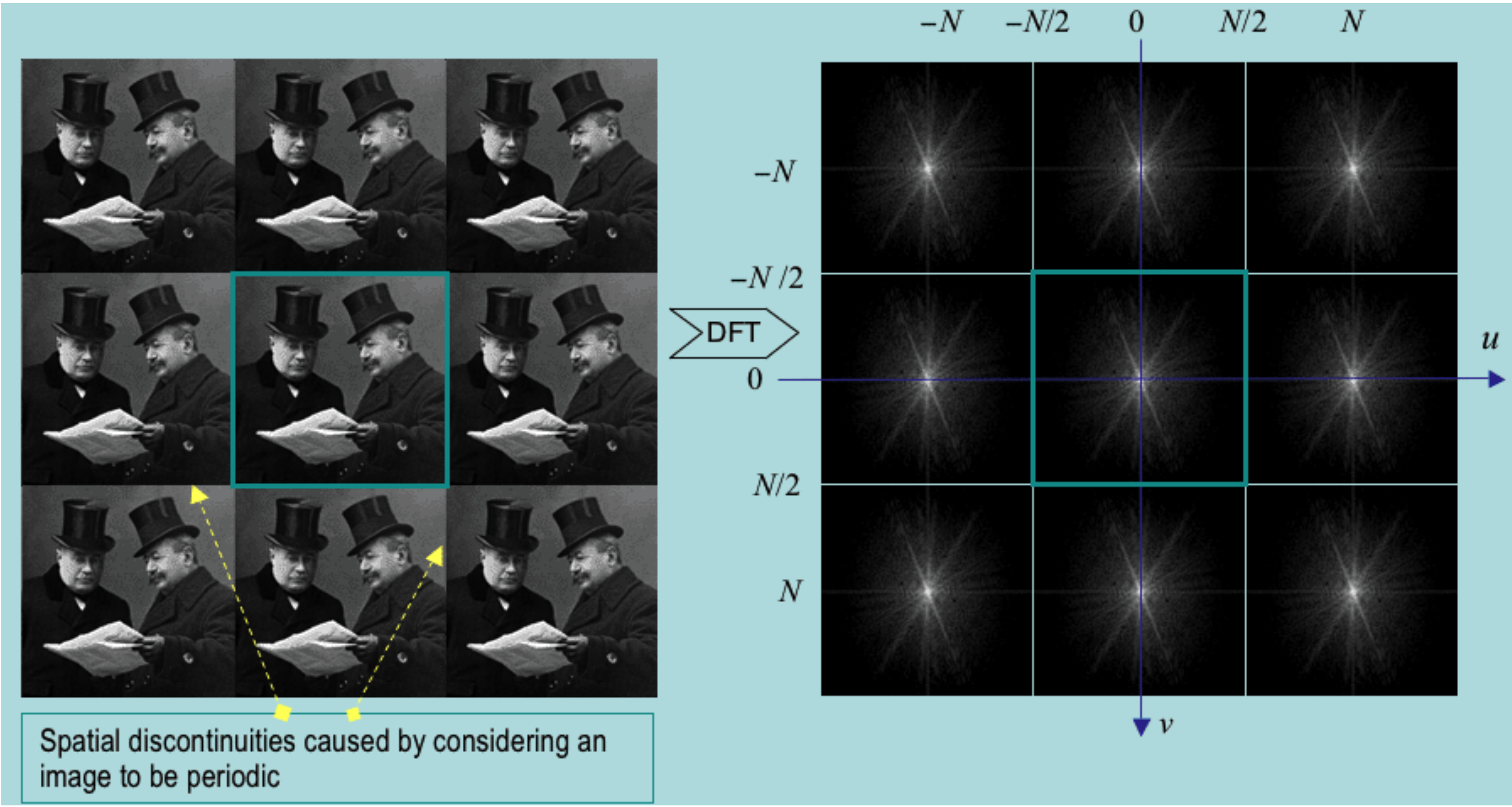


(a) Clean image spectrum and Noise spectrum (Noise dominates the high-frequency component);
(b) Filtering of high-frequency component

Frequency spectrum of an image

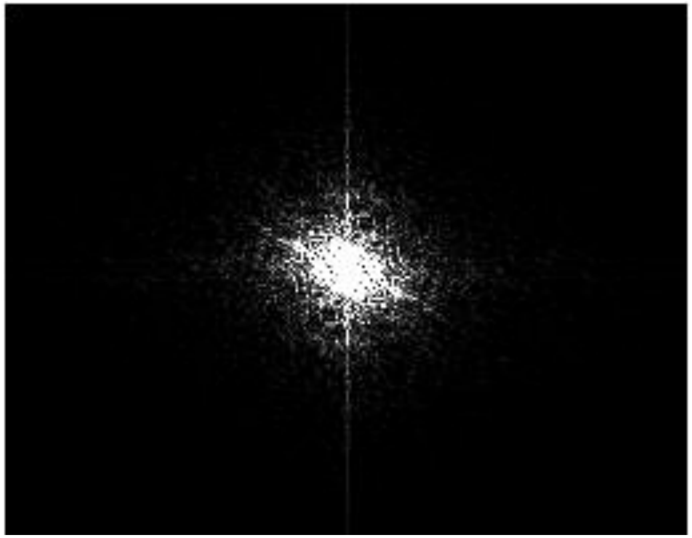


Frequency spectrum of an image



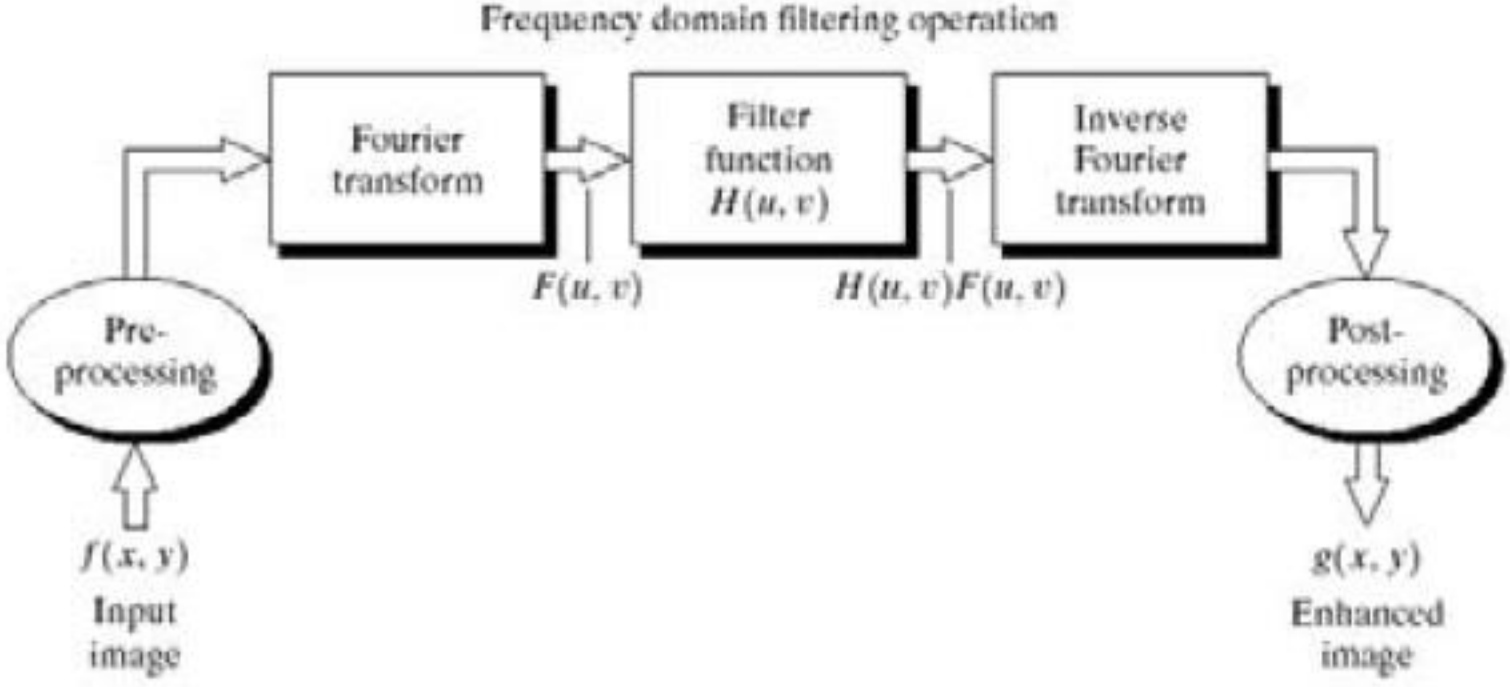
Frequency spectrum of an image

Original image

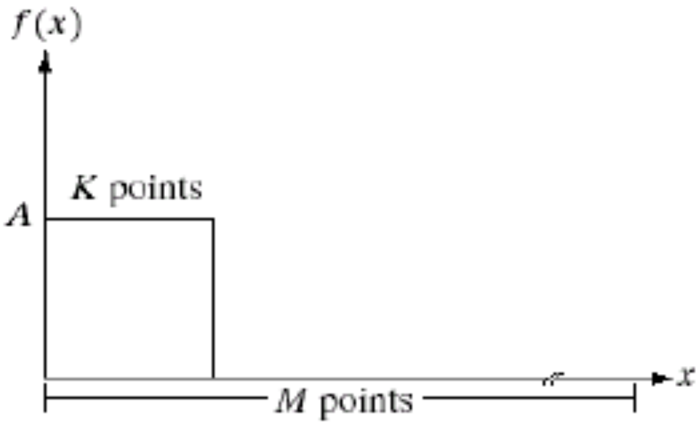


Spectrum

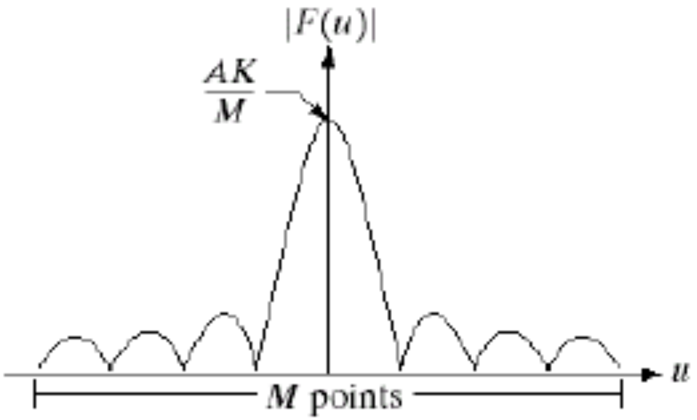
Key steps for image enhancement in the frequency domain



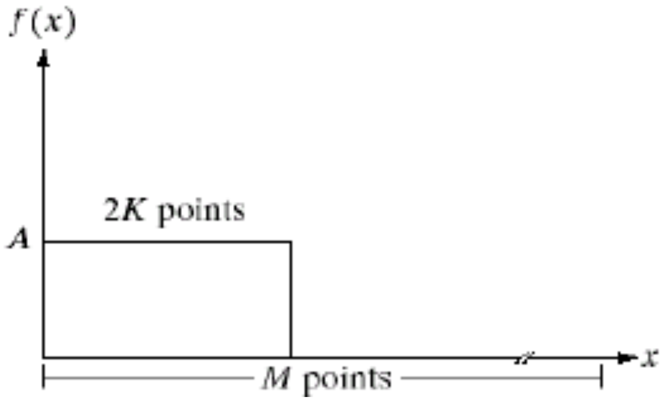
Relationship between spatial and frequency domain

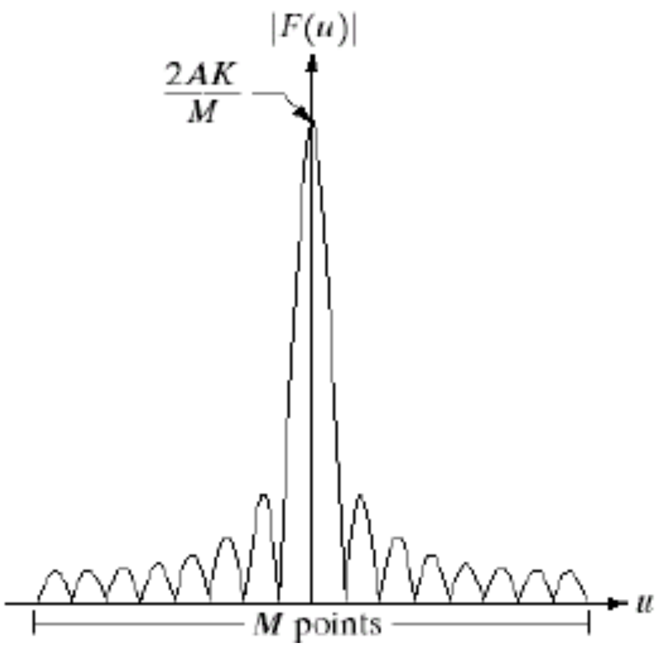


Flat filter

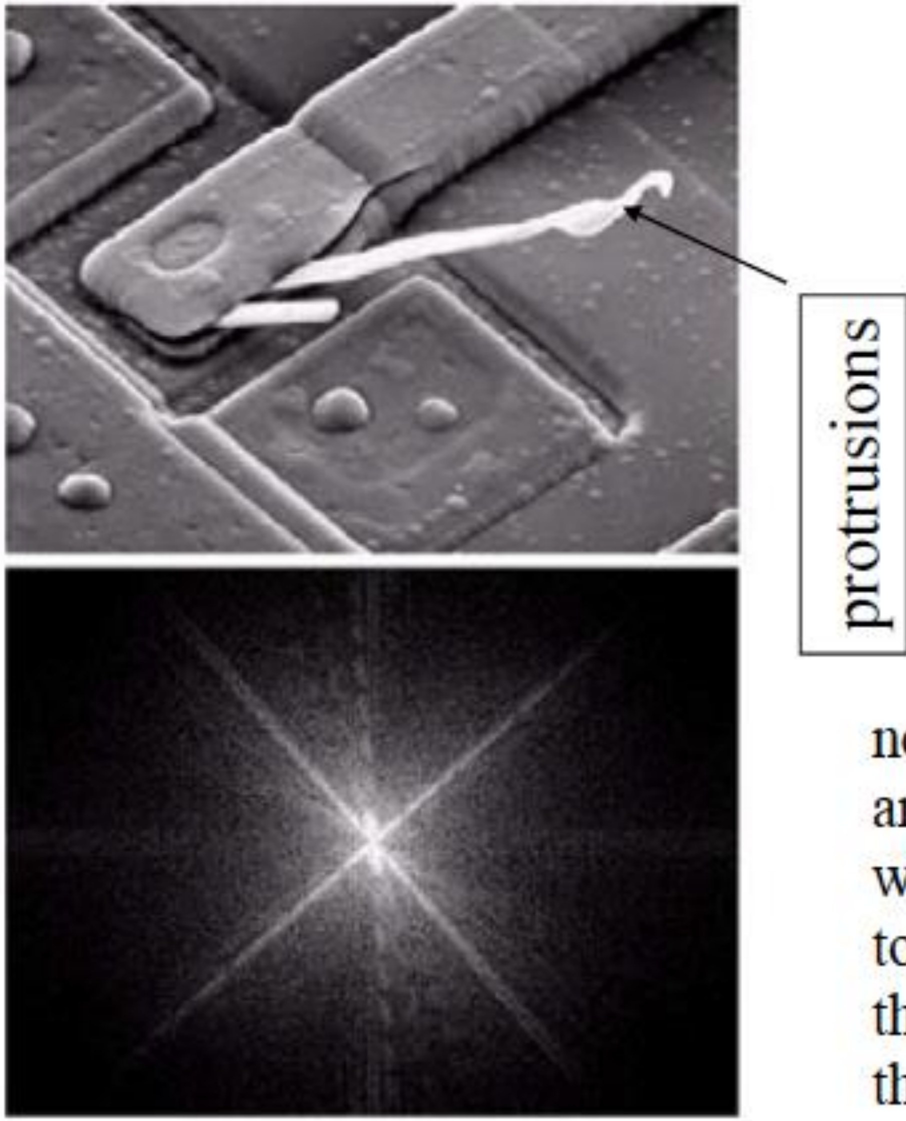


Low pass filtering





Spatial and frequency domain



SEM: scanning electron Microscope

(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

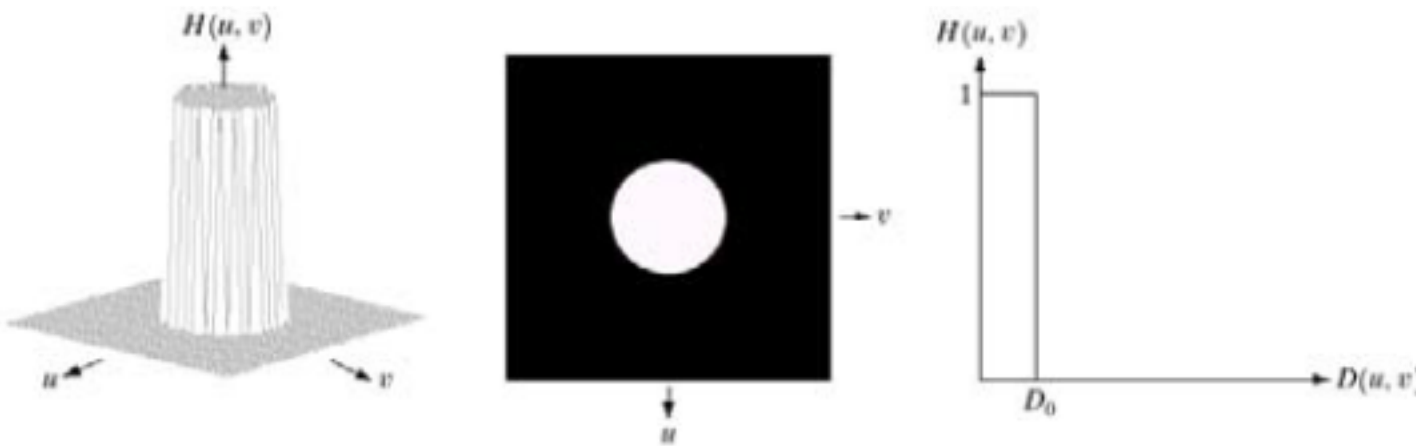
notice the ±45° components and the vertical component which is slightly off-axis to the left! It corresponds to the protrusion caused by thermal failure above.

Ideal Low Pass Filter

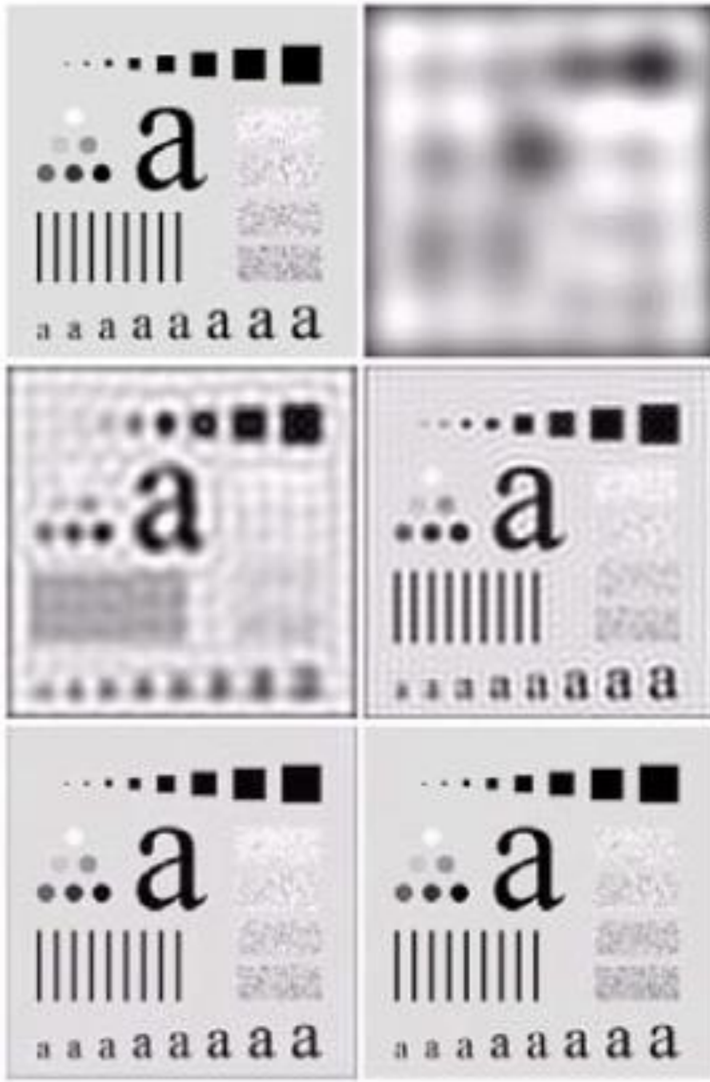
Ideal low-pass filter

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) \succ D_0 \end{cases}$$

D_0 is the cutoff frequency and $D(u,v)$ is the distance between (u,v) and the frequency origin.



Ideal Low Pass Filter

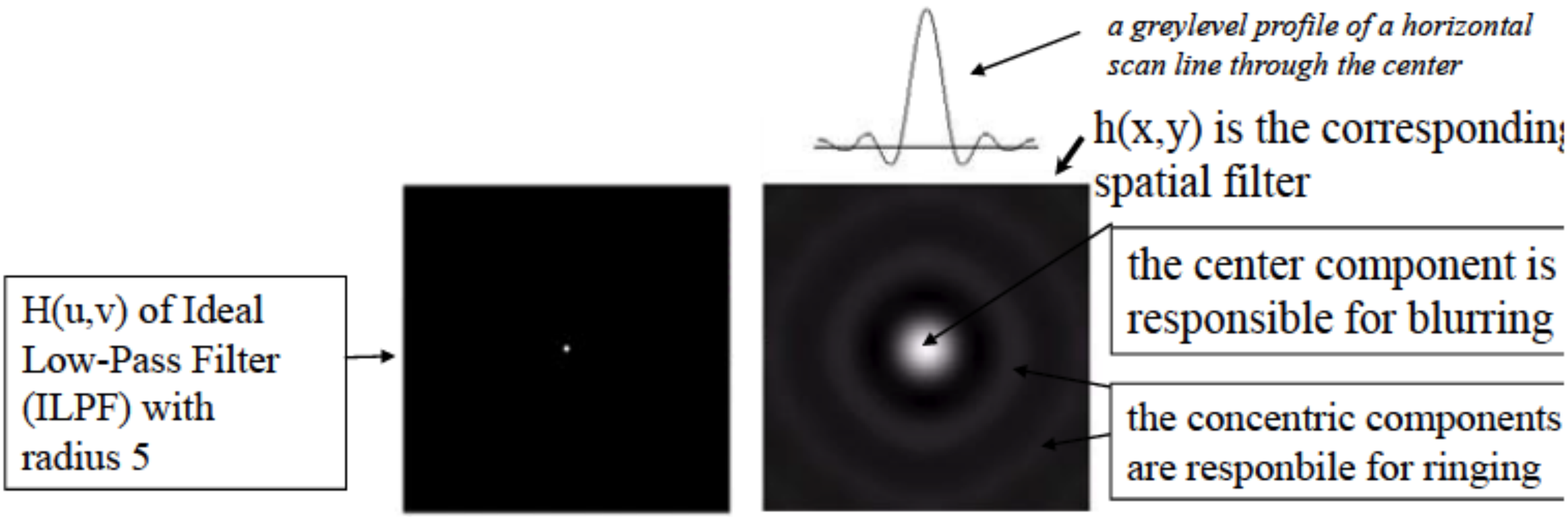


useless, even though only 8% of image power is lost!

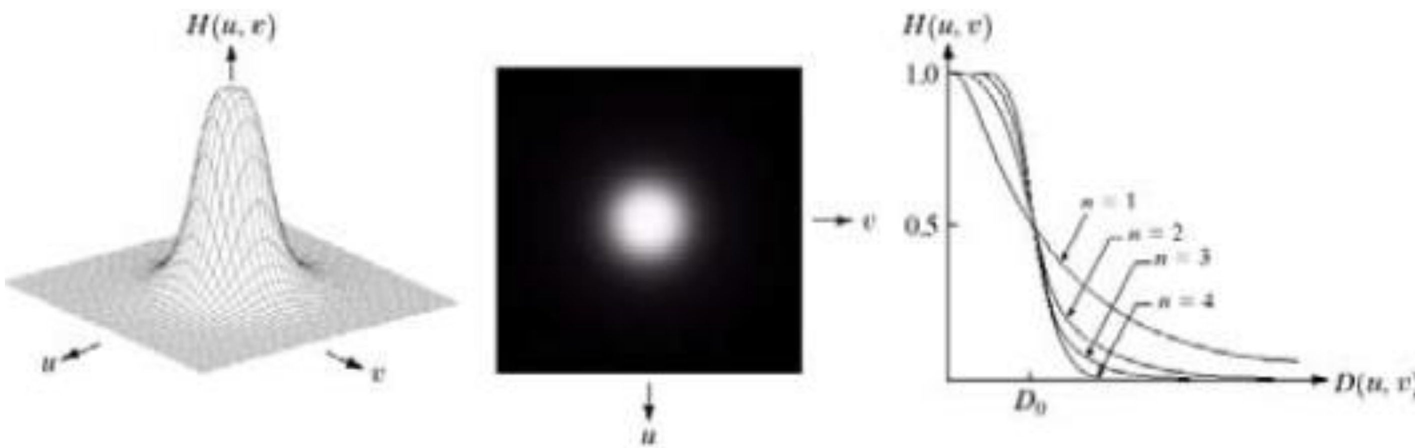
notice both blurring and ringing!

Ideal Low Pass Filter with larger and larger radii D0

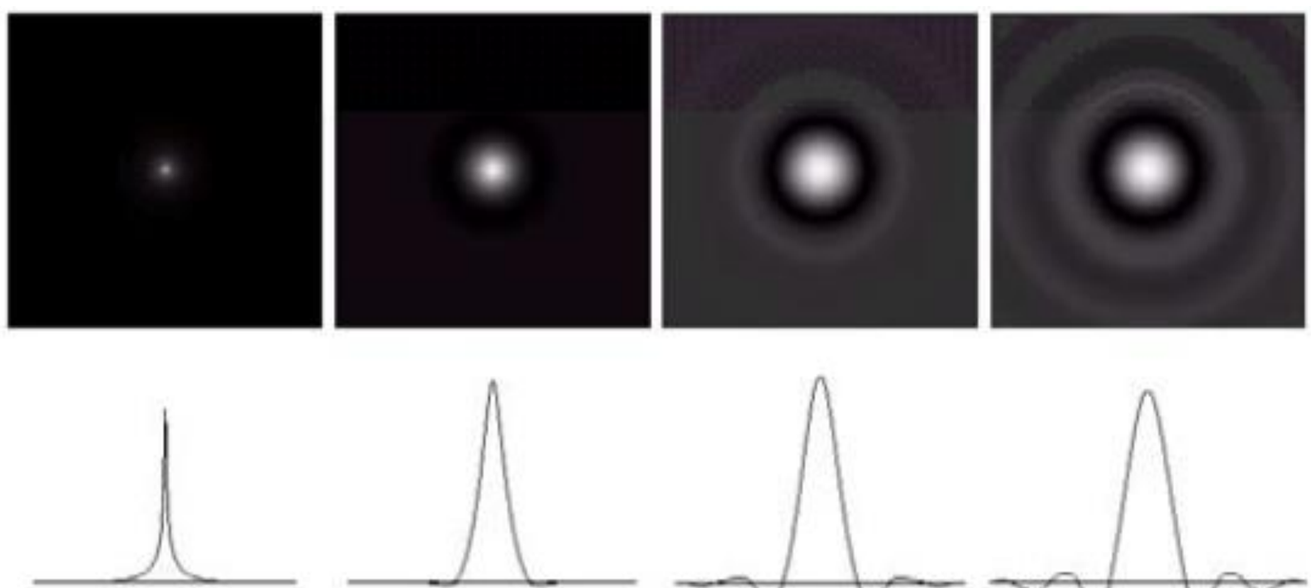
Explanation of ringing effect



Butterworth Low Pass Filter

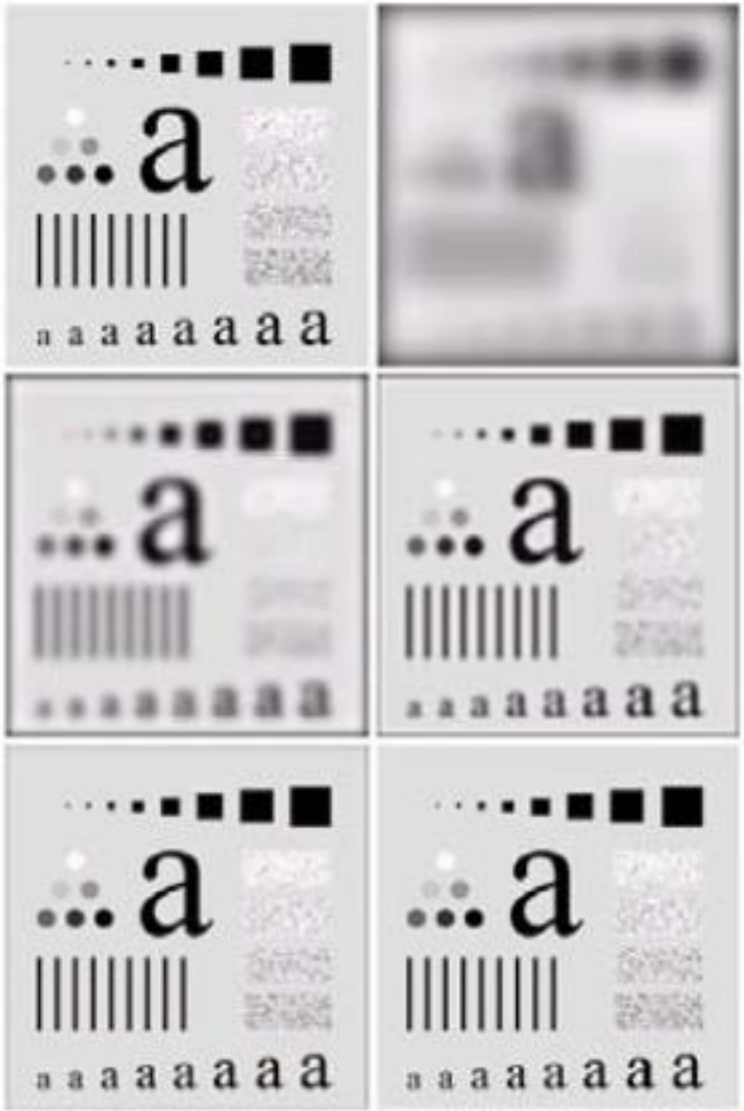


$$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{n}}$$



n=1 **n=2** **n=5** **n=20**

Butterworth Low Pass Filter

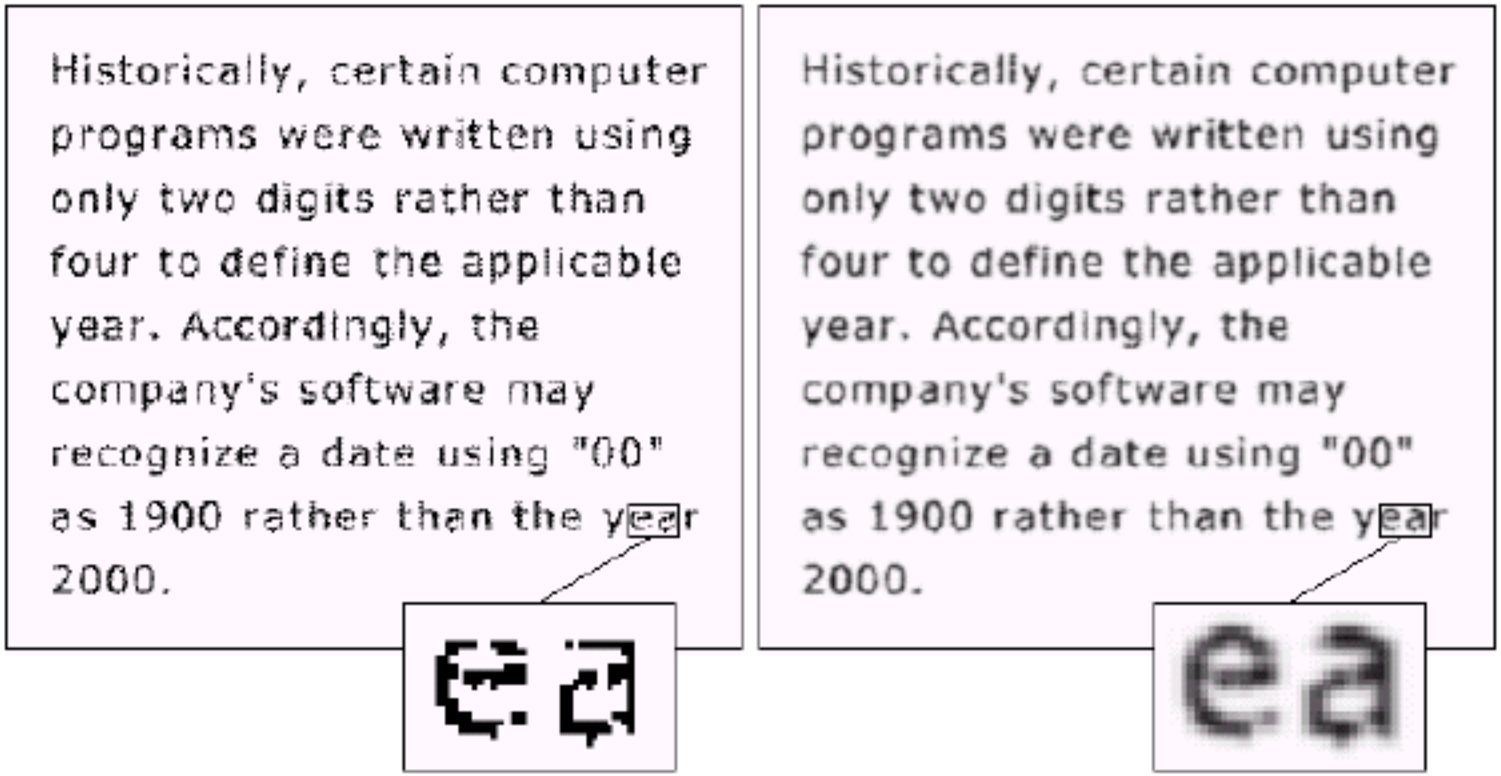


Butterworth Low Pass Filter with larger and larger radii D0

Gau)

Gaussian Low Pass Filter

Applications: fax transmission, duplicated documents and old records.



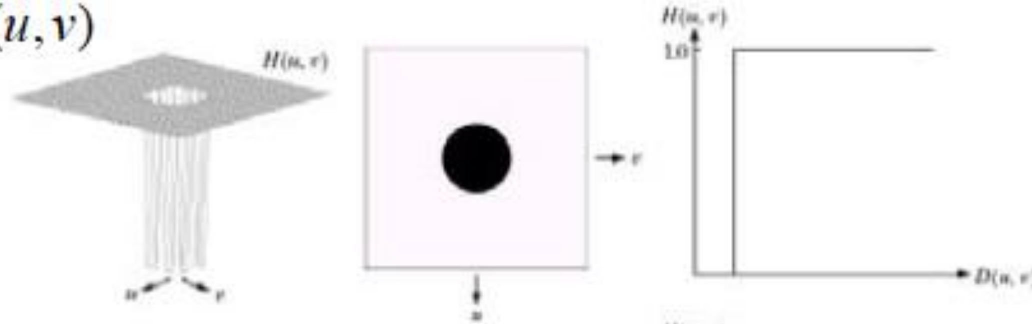
GLPF with D_0=80 is used.

Application: Low Pass Filter

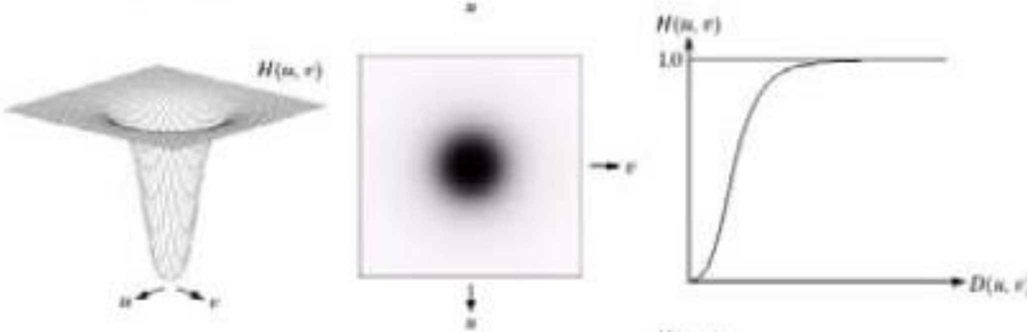
A LPF is also used in printing, e.g. to smooth fine skin lines in faces.

High Pass Filter

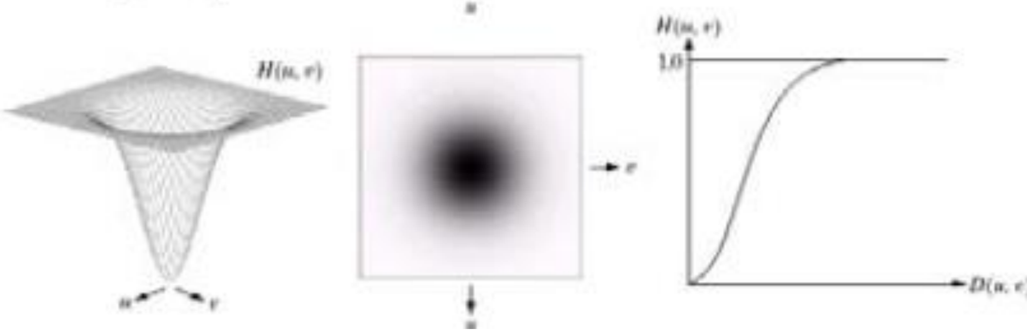
$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$



ideal high-pass filter

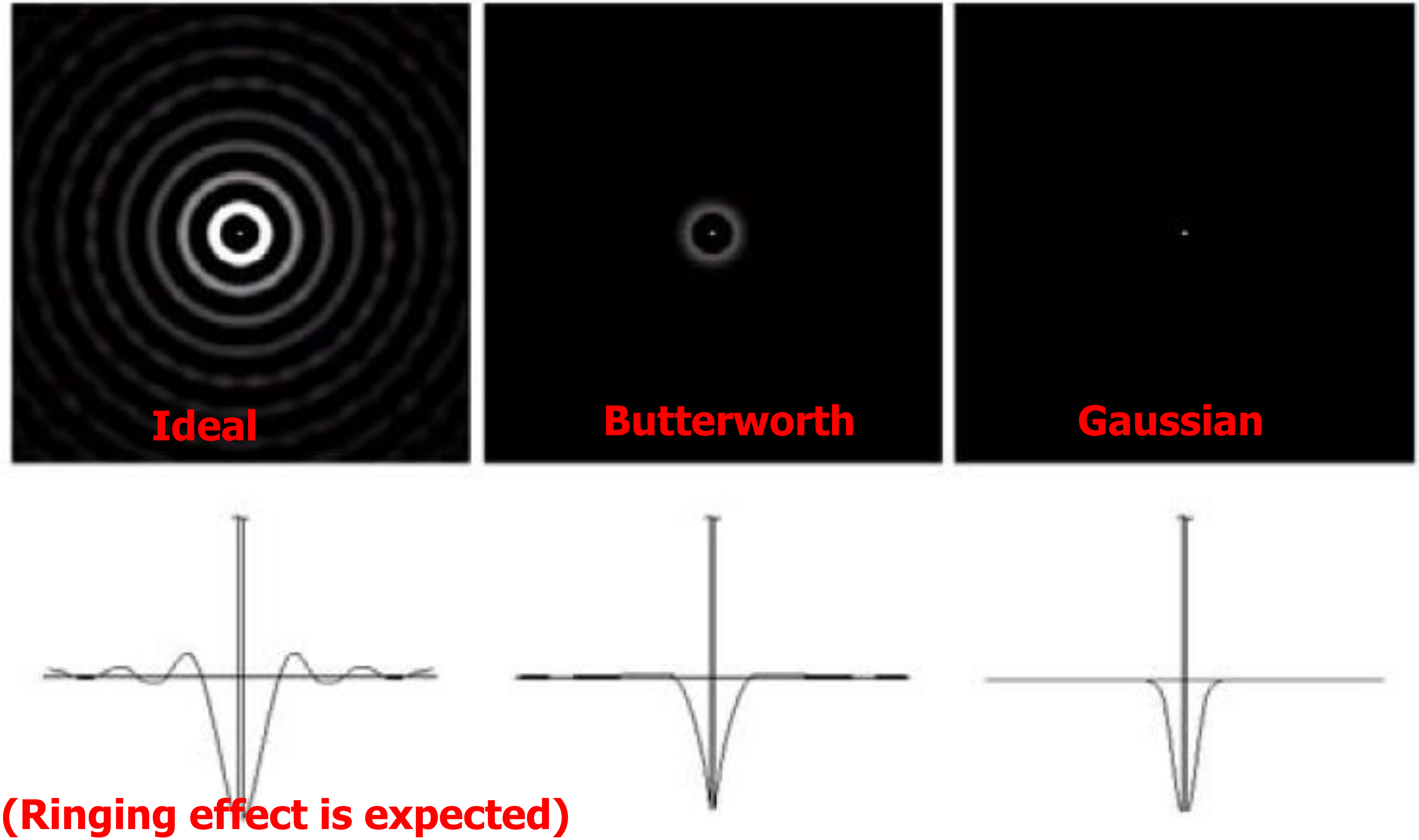


Butterworth high-pass

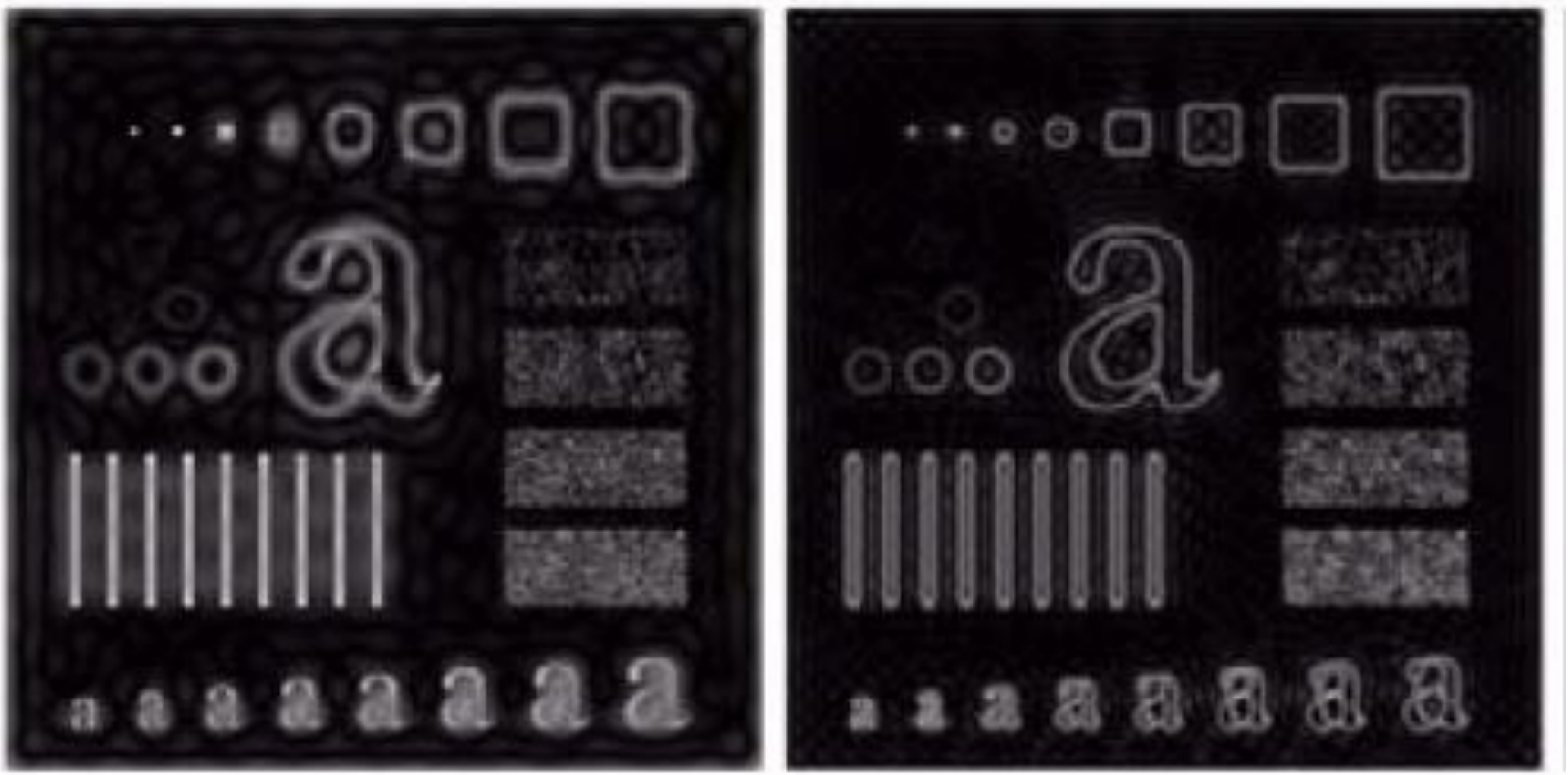


Gaussian high-pass

Spatial representation of High Pass Filter



Ideal High Pass Filter



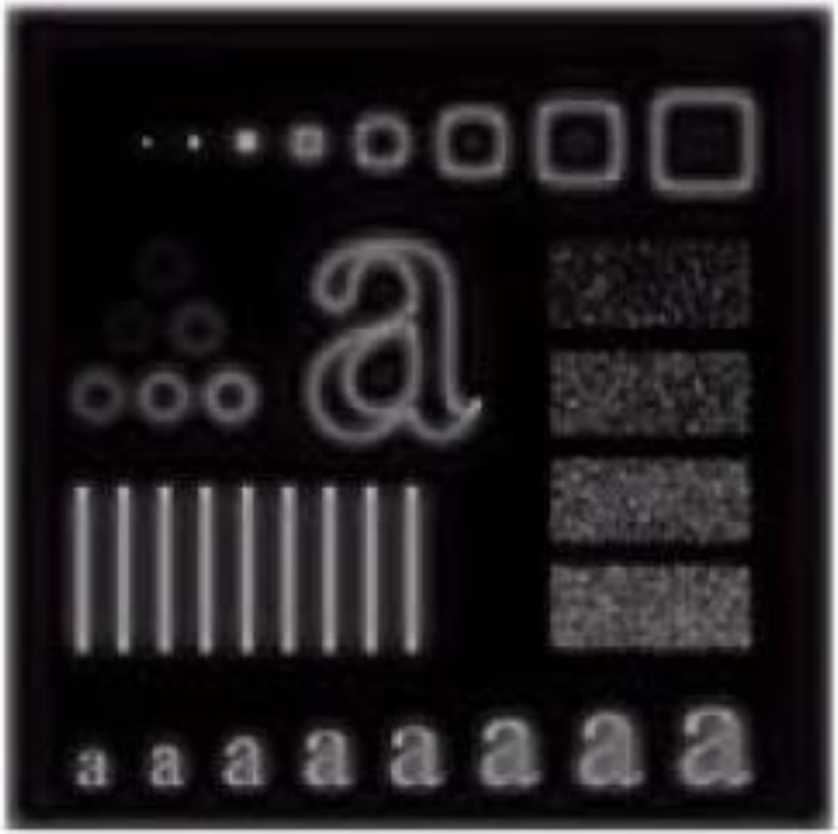
D0 = 15

D0 = 30

(Ringing effect is observed)

Ideal High Pass Filter

Butterworth High Pass Filter

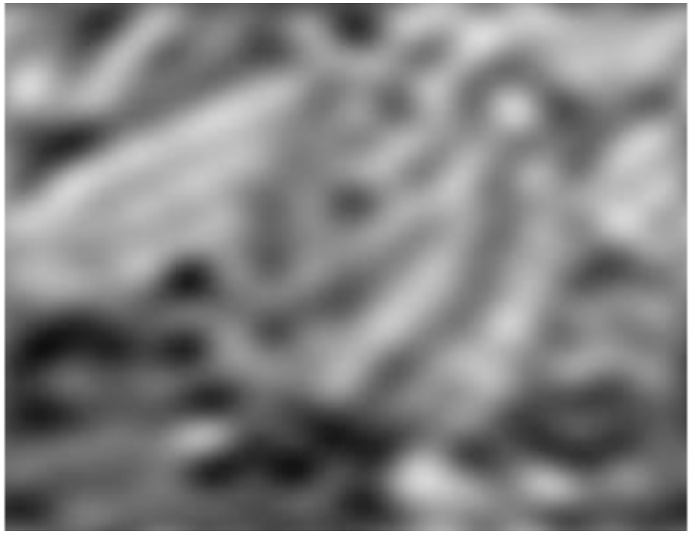


D0 = 15

D0 = 30

Comparison: High Pass Filter

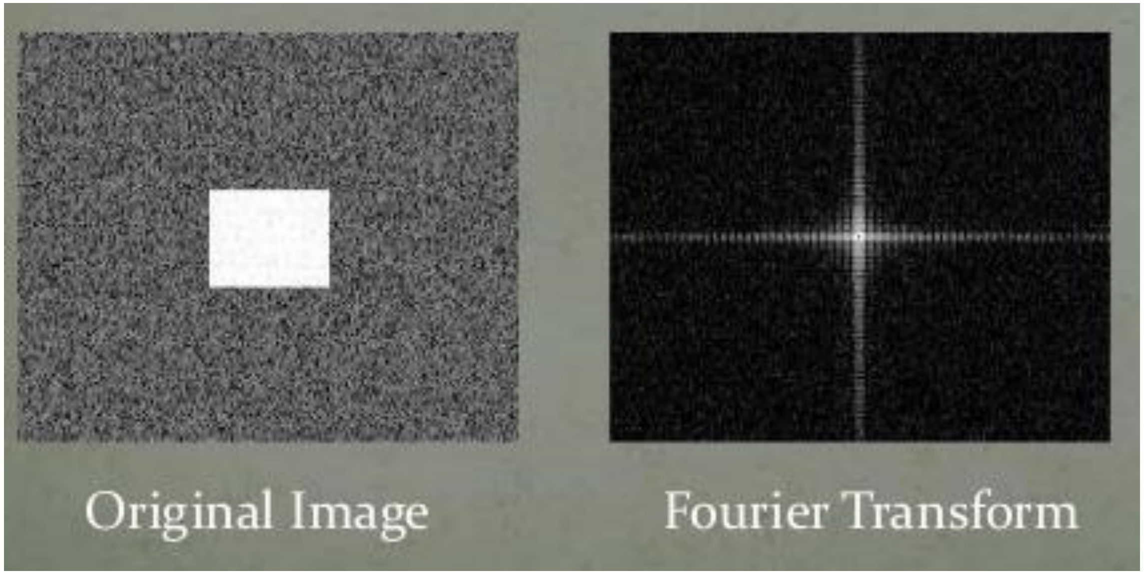
High-pass filtering



Low-pass filtering

Image denoising examples

Ideal filtering



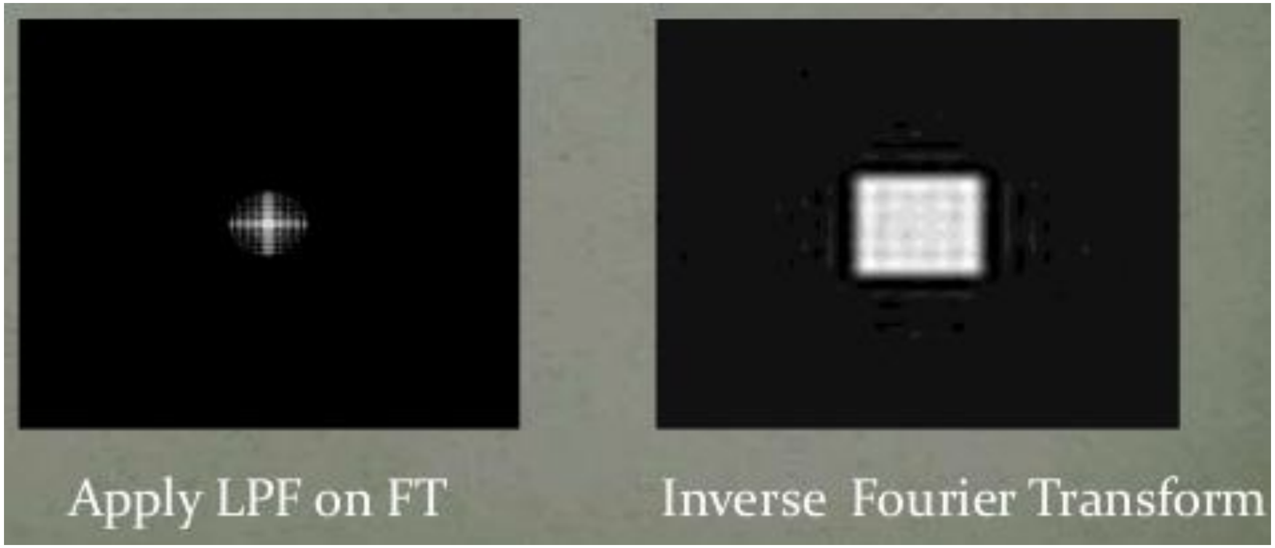
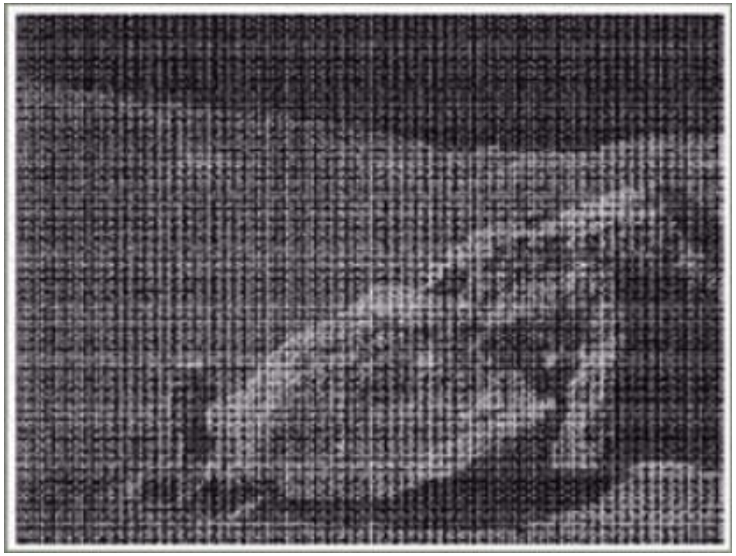
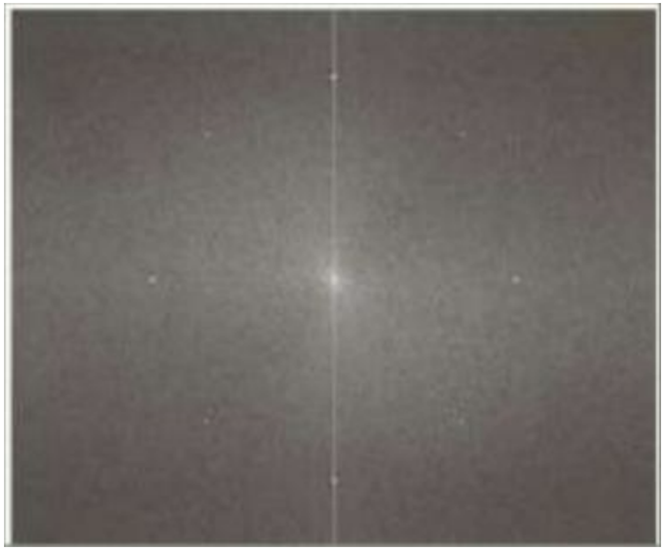


Image denoising examples

Ideal filtering



Noisy image



Frequency domain

Denoised

Image denoising examples

Butterworth filtering

Noisy

Image denoising examples

Butterworth filtering

Denoised

Image denoising examples

Gaussian filtering

noisy

denoised

$(\sigma=1)$

denoised

$(\sigma=1.5)$

Image denoising examples