Lecture 2:

Last time:

Shift-invariant $\longleftrightarrow$ Convolution

$$\left(f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, g(\alpha - x, \beta - y)\right)$$

Periodic extension is required

Separable $\longleftrightarrow$ 2 matrix multiplication

More about convolution

"Geometric" intrepretation of discrete convolution

Let $$f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, g(\alpha - x, \beta - y)$$

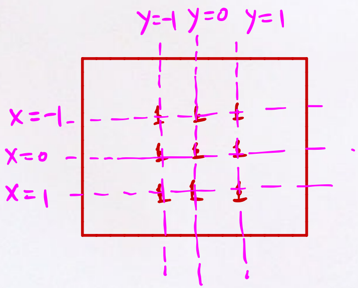
Consider a simple case where only several entries of g are non-zero.

Namely,

$g(0,0) = g(N,N) = 1$; $g(1,0) = g(1,N) = 1$; $g(-1,0) = g(N-1,N) = 1$

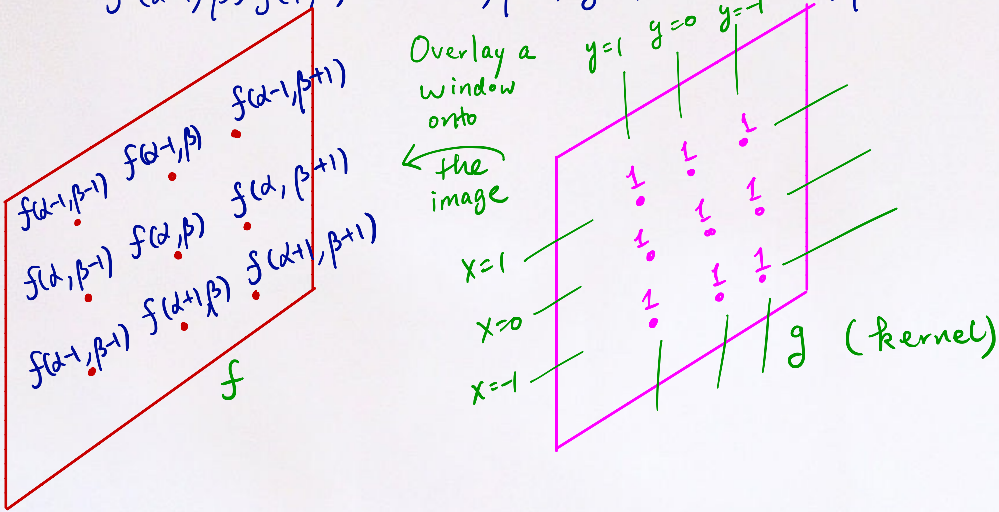
$g(0,1) = g(N,1) = 1$; $g(1,1) = g(1,1) = 1$; $g(-1,1) = g(N-1,1) = 1$

$g(0,-1) = g(N,N-1) = 1$; $g(1,-1) = g(1,N-1) = 1$; $g(-1,-1) = g(N-1,N-1) = 1$



Expand the summation:

$$
\begin{aligned}
f*g(\alpha,\beta) = {} & f(\alpha,\beta)\,g(0,0) + f(\alpha,\beta+1)\,g(0,-1) + f(\alpha,\beta-1)\,g(0,1) + \\
& f(\alpha+1,\beta)\,g(-1,0) + f(\alpha+1,\beta+1)\,g(-1,-1) + f(\alpha+1,\beta-1)\,g(-1,1) + \\
& f(\alpha-1,\beta)\,g(1,0) + f(\alpha-1,\beta+1)\,g(1,-1) + f(\alpha-1,\beta-1)\,g(1,1).
\end{aligned}
$$



Properties of shift-invariant/separable image transformation

Definition: (Circulant matrix)

A circulant matrix $V := \mathrm{circ}(\vec{v})$ associated to a vector $\vec{v} = (v_0, v_1, \ldots, v_{n-1})^T \in \mathbb{C}^n$ is a $n \times n$ matrix whose columns are given by iterations of shift operator T acting on $\vec{v}$. Here, $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_0 \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix}.$$

$\therefore$ k^{th} column is given by $T^{k-1}(\vec{v})$ $(k = 1, 2, \ldots, n)$

$$\therefore \quad V = \begin{pmatrix} v_0 & v_{n-1} & v_{n-2} & \cdots & v_1 \\ v_1 & v_0 & v_{n-1} & \cdots & v_2 \\ \vdots & \vdots & v_0 & & \vdots \\ & & \vdots & & v_{n-1} \\ v_{n-1} & v_{n-2} & v_{n-3} & \cdots & v_0 \end{pmatrix}$$

Definition:(Block circulant)

V is block-circulant $\Longleftrightarrow$ $V = \begin{pmatrix} H_0 & H_{n-1} & \cdots & H_1 \\ H_1 & H_0 & & H_2 \\ \vdots & \vdots & \cdots & \vdots \\ & & & H_{n-1} \\ H_{n-1} & H_{n-2} & \cdots & H_0 \end{pmatrix} \ni$

each H_i is a circulant matrix.

Theorem: If H = transf. matrix of shift-invariant operator,

then $H = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ A_{21} & \cdots & A_{2N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$ where each A_{ij} is a circulant matrix.

Proof: Consider

$$A_{ij} = \begin{array}{cc} & x \rightarrow \\ \alpha \downarrow & \left(\begin{pmatrix} y = j \\ \beta = i \end{pmatrix} \right) \end{array}$$

$$\therefore \; A_{ij} = \begin{pmatrix} h(1,1,j,i) & h(2,1,j,i) & \cdots & h(N,1,j,i) \\ h(1,2,j,i) & h(2,2,j,i) & \cdots & h(N,2,j,i) \\ \vdots & \vdots & & \vdots \\ h(1,N,j,i) & h(2,N,j,i) & \cdots & h(N,N,j,i) \end{pmatrix}$$

Shift-invariant $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g(\alpha - x, \beta - y)$ for some g.

$$\therefore \; A_{ij} = \begin{pmatrix} g(0, i-j) & g(\overset{N-1}{\cancel{-1}}, i-j) & \cdots & g(\overset{1}{\cancel{1-N}}, i-j) \\ h(1, i-j) & g(0, i-j) & \cdots & g(\overset{2}{\cancel{2-N}}, i-j) \\ \vdots & \vdots & & \vdots \\ h(N-1, i-j) & g(N-2, i-j) & \cdots & g(0, i-j) \end{pmatrix} \leftarrow \text{Circulant}$$

(Assume periodic property)

Properties of separable image transformation

Recall: Separable $h \iff h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$.

Let $g = Hf$. $\implies g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)}_{\text{Matrix multiplication}}$

transformation matrix

Consider $h_r = (h_r(y,\beta))_{1 \le y,\beta \le N} \in M_{N\times N}$

$h_c = (h_c(x,\alpha))_{1 \le x,\alpha \le N} \in M_{N\times N}$ Let $S = f\, h_r$.

$f = (f(x,y))_{1 \le x,y \le N} \in M_{N\times N}$

Easy to see: $g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha)\, S(x,\beta) = \sum_{x=1}^{N} h_c^T(\alpha,x)\, S(x,\beta)$

$\therefore\ g = h_c^T S = h_c^T f\, h_r$ (Matrix form)

Stacking operator

Definition: (Stacking operator)

Define: $\vec{V}_n = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{row } n \\ \\ \\ \\ \end{array}$ and $N_n = \begin{pmatrix} 0 \\ I_n \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow (n-1)N \times N \text{ zero matrix} \\ \leftarrow N \times N \text{ identity matrix} \\ \leftarrow (N-n) \times N \text{ zero matrix} \end{array}$

Let $f \in \mathcal{I}$. The <u>stacking operator</u> $\mathcal{S}$ on f is defined as:

$$\mathcal{S} f := \vec{f} := \sum_{n=1}^{N} N_n f \vec{V}_n$$

Remark: 1. $\mathcal{S} f \in M_{N^2 \times 1}$

2. The 1^{st} col of f forms the first N entries of $\mathcal{S} f$

The 2^{nd} col of f forms the next N entries of $\mathcal{S} f$ etc ...

Theorem: $\mathcal{S}$ is linear and $f = \sum_{n=1}^{N} N_n^T \vec{f}\, \vec{V}_n^T$

(exercise)

What is the meaning of $N_n f V_n$?

Consider: $f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and consider $N_2 f V_2$.

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad N_2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

Then: $f V_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$

$$N_2 f V_2 = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

etc...

Similarity between images

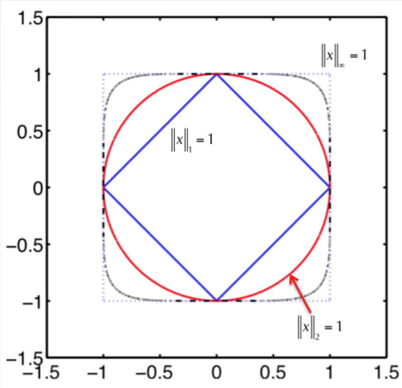
Need to define matrix norm $\|\cdot\|$ such that: for $\forall\, f, g \in \mathcal{I}$, we can define similarity between f and g as $\|f-g\|$.

Definition: A vector / matrix norm is a function $\|\cdot\| : \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) $\to \mathbb{R}$ so that for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) and $\alpha \in \mathbb{R}$, we have:

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.
2. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
3. $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$

Example:

$$\left.\begin{aligned} &\bullet\ \|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i| \qquad \vec{x} = (x_1, x_2, \dots, x_m)^T \\ &\bullet\ \|\vec{x}\|_2 = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \\ &\bullet\ \|\vec{x}\|_\infty = \max_{i=1,2,\dots,m} |x_i| \end{aligned}\right\} \text{Vector norm}$$



Definition: (induced matrix norm) Let $A \in \mathbb{R}^{m\times m}$. We define the <u>induced matrix norm</u> induced by a vector norm $\| \|_v$ to be the smallest $C \in \mathbb{R}$ such that

$$\| A x\|_v \le C \|x\|_v \text{ for } \forall \vec{x} \in \mathbb{R}^m.$$

Equivalently,

$$\|A\| = \sup_{\vec{x}\in\mathbb{R}^m, \vec{x}\neq 0} \frac{\|A\vec{x}\|_v}{\|\vec{x}\|_v} = \sup_{\substack{\vec{x}\in\mathbb{R}^m \\ \|\vec{x}\|_v = 1}} \|A\vec{x}\|_v .$$

Notation: We denote the matrix norm induced by the vector norm $\|\cdot\|_p$ by the same symbol $\|\cdot\|_p$

Another commonly used matrix norm

Definition: (Frobenius norm)

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$$

Let $\vec{a}_j$ = j-th col of A. We have: $\|A\|_F = \sqrt{\sum_{j=1}^{n}\|a_j\|_2^2} = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$

where $tr(\cdot)$ = trace of the matrix.

Theorem: The matrix 2-norm and Frobenius norm (F-norm) are invariant under multiplication by unitary matrices).

That is, for any $A \in \mathbb{R}^{m\times n}$, and unitary matrix $U \in \mathbb{R}^{m\times m}$, we have:

$$\|UA\|_2 = \|A\|_2 \quad \text{and} \quad \|UA\|_F = \|A\|_F.$$

Proof:

$$\|UA\vec{x}\|_2^2 = (UA\vec{x})^T(UA\vec{x}) = \vec{x}^T A^T U^T U A \vec{x}^T = \vec{x}^T A^T A \vec{x} = \|A\vec{x}\|_2^2$$

$$\therefore \|UA\|_2 = \sup_{\|\vec{x}\|_2 = 1} \|UA\vec{x}\|_2 = \sup_{\|\vec{x}\|=1} \|A\vec{x}\|_2 = \|A\|_2$$

Also, $\|UA\|_F = \sqrt{tr((UA)^T(UA))} = \sqrt{tr(A^T U^T U A)} = \sqrt{tr(A^T A)} = \|A\|_F$

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Image decomposition

Suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ (Separable).

Then: $g = h_c^T f h_r \Rightarrow f = (h_c^T)^{-1} g (h_r)^{-1}$

Write: $(h_c^T)^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $h_r^{-1} = \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_N^T & - \end{pmatrix}$

Then: $f = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij}\, \vec{u}_i \vec{v}_j^T}$ $M_{N\times N}$

Check that: $(h_c^T)^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} h_r^{-1} = \vec{u}_i \vec{v}_j^T$

(i,j)-entry

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m\times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\text{rank}(AB) = \text{rank}(B)$ if A is invertible

$\text{rank}(AB) = \text{rank}(A)$ if B is invertible.

Suppose $g = U \Sigma V^T$. Since U and V are invertible, $\text{rank}(g) = \text{rank}(\Sigma)$

$= \#$ of non-zero singular values