微积分II综合复习题

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1.（8分）设$f(x)=\frac{\pi-x}{2}$, $x\in(0,2\pi)$，又设$f(0)=f(2\pi)=0$，将f延拓成$\mathbb{R}$ 上以2π 为周期的周期函数，仍记为$f(x)$。试将$f(x)$展开成Fourier级数$\frac{a_0}{2}+\sum_{n=1}^{+\infty}[a_n\cos nx+b_n\sin nx]$。进一步计算级数$\sum_{n=1}^{+\infty}\frac{\sin n}{n}$ 的和。

解: 由公式及周期函数的积分性质

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}dx = 0;\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx\\
&= \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\cos nxdx = \frac{1}{2\pi}\int_0^{2\pi}(\pi-x)d(\frac{1}{n}\sin nx)\\
&= \frac{1}{2n\pi}(\pi-x)\sin nx\Big|_0^{2\pi} + \frac{1}{2n\pi}\int_0^{2\pi}\sin nxdx = 0;\ n=1,2,\cdots
\end{aligned}
$$

1.（8分）设$f(x)=\frac{\pi-x}{2}$, $x\in(0,2\pi)$，又设$f(0)=f(2\pi)=0$，将f延拓成$\mathcal{R}$ 上以2π为周期的周期函数，仍记为$f(x)$。试将$f(x)$展开成Fourier级数$\frac{a_0}{2}+\sum_{n=1}^{+\infty}[a_n\cos nx+b_n\sin nx]$。进一步计算级数$\sum_{n=1}^{+\infty}\frac{\sin n}{n}$ 的和。

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nxdx\\
&= \frac{1}{\pi}\int_{0}^{2\pi}\frac{\pi-x}{2}\sin nxdx=\frac{1}{2\pi}\int_{0}^{2\pi}(\pi-x)d(-\frac{1}{n}\cos nx)\\
&= -\frac{1}{2n\pi}(\pi-x)\cos nx\Big|_0^{2\pi}-\frac{1}{2n\pi}\int_0^{2\pi}\cos nxdx=\frac{1}{n};\ n=1,2,\cdots
\end{aligned}$$

从而$f(x)\sim\sum_{n=1}^{+\infty}\frac{\sin nx}{n}$；由函数$f(x)$满足Dirichlet条件，

$$\sum_{n=1}^{+\infty}\frac{\sin n}{n}=\frac{f(1-0)+f(1+0)}{2}=f(1)=\frac{\pi-1}{2}.$$

2.（7分）(1)求幂级数$\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n(n-1)}$ 的和函数; (2) 求级数$\sum_{n=2}^{+\infty}\frac{(-1)^n}{n(n-1)}$的和;

解: (1). 幂级数$\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n(n-1)}$的收敛域为$[-1,1]$; 记

$$S(x)=\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n(n-1)}$$

$S(0)=0$. 当$x\in(-1,1)$时(为什么？)

$$S'(x)=\left(\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n(n-1)}\right)'=\sum_{n=2}^{+\infty}(-1)^n\frac{x^{n-1}}{n-1};$$

$$S'(0)=0,S''(x)=\left(\sum_{n=2}^{+\infty}(-1)^n\frac{x^{n-1}}{n-1}\right)'=\sum_{n=2}^{+\infty}(-1)^n x^{n-2}=\frac{1}{1+x};$$

从而$S'(x)=S'(x)-S'(0)=\int_0^x S''(x)dx=\int_0^x \frac{1}{1+x}dx=\ln(1+x)$;

$$\begin{aligned} S(x) &= S(x)-S(0)=\int_0^x S'(x)dx=\int_0^x \ln(1+x)dx \\ &= x\ln(1+x)-\int_0^x \frac{x}{1+x}dx=(x+1)\ln(1+x)-x; \end{aligned}$$

由于$S(x)$在区间$[-1,1]$上连续，由连续性

$$S(1)=\lim_{x\to 1^-} S(x)=\lim_{x\to 1^-}[(x+1)\ln(1+x)-x]=2\ln 2-1;$$

$$S(-1)=\lim_{x\to(-1)^+} S(x)=\lim_{x\to(-1)^+}[(x+1)\ln(1+x)-x]=1;$$

从而和函数为

$$\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n(n-1)}=S(x)=\begin{cases}(x+1)\ln(1+x)-x, x\in(-1,1]\\ 1, x=-1\end{cases}$$

(2). $\sum_{n=2}^{+\infty}\frac{(-1)^n}{n(n-1)}=S(1)=2\ln 2-1.$

3. (5分)设$\{a_n\}$是一个严格单调递减的正数数列，证明无穷级数$\sum_{n=1}^{+\infty}(-1)^n\frac{a_n+1}{n}$ 条件收敛。

证明: • $\frac{a_n+1}{n}\geq\frac{1}{n}$且级数$\sum_{n=1}^{+\infty}\frac{1}{n}$发散，由正项级数的比较判别法知：无穷级数$\sum_{n=1}^{+\infty}(-1)^n\frac{a_n+1}{n}$不是绝对收敛的；

• $\frac{a_n+1}{n}\geq 0$、$\{\frac{a_n+1}{n}\}$单调递减且$\lim_{n\to+\infty}\frac{a_n+1}{n}=0$, 由交错级数的莱布尼兹判别法，无穷级数$\sum_{n=1}^{+\infty}(-1)^n\frac{a_n+1}{n}$是收敛的；

从而，无穷级数$\sum_{n=1}^{+\infty}(-1)^n\frac{a_n+1}{n}$ 条件收敛。

4.(5分)证明：直角坐标系$Oxyz$中的向量场$\vec{u}(x,y,z)=\{yz,xz,xy\}$是一个无旋场。

证明:向量场$\vec{u}$的旋度

$$
\begin{aligned}
rot\vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \\
&= (x-x)\vec{i}-(y-y)\vec{j}+(z-z)\vec{k}=\vec{0};
\end{aligned}
$$

所以向量场$\vec{u}(x,y,z)=\{yz,xz,xy\}$是一个无旋场。

5. （7分）设S为上半单位球面$z=\sqrt{1-x^2-y^2}$，取内侧为正侧，计算第二型曲面积分$\iint_S dydz+dzdx+dxdy$.

解法1: 取$S_1: z=0\,(x^2+y^2\leq 1)$(上侧)，由曲面$S$与$S_1$所围立体为$\Omega$。则$S\cup S_1$ 是立体Ω边界曲面的内侧. 由高斯定理，

$$\iint_S+\iint_{S_1} dydz+dzdx+dxdy=\iint_{S\cup S_1} dydz+dzdx+dxdy$$

$$=\ -\iiint_\Omega\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right]dV=-\iiint_\Omega 0dV=0;$$

利用投影法，

$$\iint_{S_1} dydz=0,\ \iint_{S_1} dzdx=0,\ \iint_{S_1} dxdy=\iint_{x^2+y^2\leq 1} dxdy=\pi;$$

从而

$$\iint_S dydz+dzdx+dxdy=-\iint_{S_1} dydz+dzdx+dxdy=-\pi.$$

5. （7分）设S为上半单位球面$z=\sqrt{1-x^2-y^2}$，取内侧为正侧，计算第二型曲面积分$\iint_S dydz+dzdx+dxdy$.

解法2: $S:\ z=\sqrt{1-x^2-y^2}\,(x^2+y^2\le 1)$内侧的单位法向量为

$$\vec{n_0}=\{\cos\alpha,\cos\beta,\cos\gamma\}=-\frac{\{x,y,z\}}{\sqrt{x^2+y^2+z^2}}=-\{x,y,z\}$$

从而, 转化为第一型曲面积分, $D=\{x^2+y^2\le 1\}$,

$$\iint_S dydz+dzdx+dxdy=-\iint_S(x+y+z)dS;$$

$S:\ z=\sqrt{1-x^2-y^2}\,((x,y)\in D)$得$dS=\frac{1}{\sqrt{1-x^2-y^2}}dxdy$,

$$\iint_S dydz+dzdx+dxdy=-\iint_S(x+y+z)dS$$

$$=\ -\iint_D\frac{x+y+\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}dxdy=-\iint_D\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}dxdy$$

$$=\ -\iint_D dxdy=-\pi.$$

6.(10分) 设$R > 0$. (1). 设$D_1 = \{x^2 + y^2 \le R^2, x \ge 0, y \ge 0\}$,
计算二重积分$\iint_{D_1} e^{-x^2-y^2} dxdy$;

(2). 设$D_2 = \{(x, y) : 0 \le x \le R, 0 \le y \le R\}$,
计算$\lim_{R\to+\infty} \iint_{D_2} e^{-x^2-y^2} dxdy$, 由此求出$\int_0^{+\infty} e^{-x^2} dx$.

解: (1). $D_1 = \{(r, \theta) : 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le R\}$,

$$\iint_{D_1} e^{-x^2-y^2} dxdy = \iint_{D_1} e^{-r^2} \cdot rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^R re^{-r^2} dr = \frac{\pi(1 - e^{-R^2})}{4};$$

(2). 记$D_3=\{x^2+y^2\le 2R^2, x\ge 0, y\ge 0\}$, 由二重积分的性质

$$\iint_{D_1} e^{-x^2-y^2}dxdy \le \iint_{D_2} e^{-x^2-y^2}dxdy \le \iint_{D_3} e^{-x^2-y^2}dxdy;$$

$$\lim_{R\to+\infty}\iint_{D_1} e^{-x^2-y^2}dxdy=\lim_{R\to+\infty}\frac{\pi(1-e^{-R^2})}{4}=\frac{\pi}{4};$$

$$\lim_{R\to+\infty}\iint_{D_3} e^{-x^2-y^2}dxdy=\lim_{R\to+\infty}\frac{\pi(1-e^{-2R^2})}{4}=\frac{\pi}{4};$$

由极限的夹逼准则得$\lim_{R\to+\infty}\iint_{D_2} e^{-x^2-y^2}dxdy=\frac{\pi}{4}$;

$$\frac{\pi}{4}=\lim_{R\to+\infty}\iint_{D_2} e^{-x^2-y^2}dxdy=\lim_{R\to+\infty}\int_0^R dy\int_0^R e^{-x^2-y^2}dx$$

$$=\lim_{R\to+\infty}\left(\int_0^R e^{-x^2}dx\right)^2=\left(\int_0^{+\infty} e^{-x^2}dx\right)^2,$$

得$\int_0^{+\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

7.(10分)设$f(u,v)$在平面$\mathbb{R}^2$上有连续的一阶偏导数，
对$x\in\mathbb{R}$有$f(x,x^2)=2018$。又设对$x\in\mathbb{R}$，$f_1'(x,x^2)=x$。
(1). 当$t\neq 0$时求$f_2'(t,t^2)$; (2). 求$f_2'(0,0)$;

解: (1). $f(x,x^2)=2018$关于x求导数得：

$$f_1'(x,x^2)+2xf_2'(x,x^2)=0 \implies x+2xf_2'(x,x^2)=0$$

当$t\neq 0$时$f_2'(t,t^2)=-\frac{1}{2}$;

(2). $f(u,v)$有连续的一阶偏导数, 由连续的定义

$$f_2'(0,0)=\lim_{t\to 0}f_2'(t,t^2)=\lim_{t\to 0}-\frac{1}{2}=-\frac{1}{2}.$$

8. (7分)设$f(u,v)$在点(x_0,y_0)处可微，$\vec{\ell_1}$、$\vec{\ell_2}$、$\vec{\ell_3}$、$\vec{\ell_4}$ 为平面上四个互异的单位向量，且满足$\sum_{n=1}^{4}\vec{\ell_n}=(0,0)$.
求$\sum_{n=1}^{4}\frac{\partial f}{\partial \vec{\ell_n}}(x_0,y_0)$.

解: 由在点(x_0,y_0)处可微得

$$\frac{\partial f}{\partial \vec{\ell_n}}(x_0,y_0)=grad\, f(x_0,y_0)\cdot\vec{\ell_n}.$$

从而

$$\sum_{n=1}^{4}\frac{\partial f}{\partial \vec{\ell_n}}(x_0,y_0)=\sum_{n=1}^{4} grad\, f(x_0,y_0)\cdot\vec{\ell_n}=grad\, f(x_0,y_0)\cdot\sum_{n=1}^{4}\vec{\ell_n}=0.$$

9.(8分)设Γ为抛物线$2x=\pi y^2$上从点$O(0,0)$到点$B(\frac{\pi}{2},1)$的有向弧段，计算第二型曲线积分

$$\int_\Gamma (2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy.$$

解法1： $\Gamma:\ x=\frac{\pi}{2}y^2,\ y=y,\ y:\ 0\to 1;$

$$\begin{aligned}
I &= \int_\Gamma (2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy \\
&= \int_0^1 \left(\pi y^5-y^2\cos\left(\frac{\pi}{2}y^2\right)\right)d\left(\frac{\pi}{2}y^2\right) \\
&\quad +\int_0^1 \left(1-2y\sin\left(\frac{\pi}{2}y^2\right)+\frac{3}{4}\pi^2y^6\right)dy \\
&= \int_0^1 \left(1+\frac{7}{4}\pi^2y^6-\pi y^3\cos\left(\frac{\pi}{2}y^2\right)-2y\sin\left(\frac{\pi}{2}y^2\right)\right)dy
\end{aligned}$$

9.(8分)设Γ为抛物线$2x=\pi y^2$上从点$O(0,0)$到点$B(\frac{\pi}{2},1)$的有向弧段，计算第二型曲线积分

$$\int_\Gamma (2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy.$$

$$\begin{aligned}
I &= \int_0^1\left(1+\frac{7}{4}\pi^2y^6-\pi y^3\cos\left(\frac{\pi}{2}y^2\right)-2y\sin\left(\frac{\pi}{2}y^2\right)\right)dy \\
&= 1+\frac{1}{4}\pi^2-\int_0^1\left(\pi y^3\cos\left(\frac{\pi}{2}y^2\right)+2y\sin\left(\frac{\pi}{2}y^2\right)\right)dy(\text{取}\frac{\pi}{2}y^2=t) \\
&= 1+\frac{1}{4}\pi^2-\int_0^{\pi/2}\frac{2}{\pi}(t\cos t+\sin t)dt \\
&= 1+\frac{1}{4}\pi^2-\frac{2}{\pi}t\sin t\Big|_{t=0}^{t=\frac{\pi}{2}}=\frac{1}{4}\pi^2;
\end{aligned}$$

9.(8分)设Γ为抛物线$2x=\pi y^2$上从点$O(0,0)$到点$B(\frac{\pi}{2},1)$的有向弧段，计算第二型曲线积分

$$\int_\Gamma (2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy.$$

解法2: $P(x,y)=2xy^3-y^2\cos x$, $Q(x,y)=1-2y\sin x+3x^2y^2$. 在单连通区域$\mathbb{R}^2$ 内

$$\frac{\partial Q}{\partial x}=-2y\cos x+6xy^2=\frac{\partial P}{\partial x};$$

从而第二型曲线积分与路径无关；取原函数

$$u(x,y)=\int_0^x P(x,0)dx+\int_0^y Q(x,y)dy$$

$$=\int_0^x 0dx+\int_0^y(1-2y\sin x+3x^2y^2)dy=y-y^2\sin x+x^2y^3.$$

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$$\int_\Gamma (2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy=u\Big|_{(0,0)}^{(\frac{\pi}{2},1)}=\frac{\pi^2}{4}.$$

10.(8分) 设V为$\mathbb{R}^3$中锥面$z=\sqrt{x^2+y^2}$与平面$z=1$所围的有界闭区域（锥体），计算三重积分$\iiint_V(\sqrt{x^2+y^2}+z)dxdydz$.

解法1: (投影法) $V=\{(x,y)\in D, \sqrt{x^2+y^2}\le z\le 1\}$,

$$D=\{x^2+y^2\le 1\}=\{(r,\theta):\ 0\le\theta\le 2\pi, 0\le r\le 1\};$$

$$\begin{aligned}
& \iiint_V(\sqrt{x^2+y^2}+z)dxdydz \\
= & \iint_D dxdy\int_{\sqrt{x^2+y^2}}^{1}(\sqrt{x^2+y^2}+z)dz \\
= & \iint_D\left[\sqrt{x^2+y^2}-\frac{3}{2}(x^2+y^2)+\frac{1}{2}\right]dxdy \\
= & \int_0^{2\pi}d\theta\int_0^1\left[r-\frac{3}{2}r^2+\frac{1}{2}\right]\cdot rdr=\frac{5\pi}{12}.
\end{aligned}$$

10.(8分)设V为$\mathbb{R}^3$中锥面$z=\sqrt{x^2+y^2}$与平面$z=1$所围的有界闭区域（锥体）计算三重积分$\iiint_V(\sqrt{x^2+y^2}+z)dxdydz$.

解法2:(截面法)$V=\{0\le z\le 1,(x,y)\in D_z\}$,

$$D_z=\{x^2+y^2\le z^2\}=\{(r,\theta):\ 0\le\theta\le 2\pi, 0\le r\le z\};$$

$$\begin{aligned}
&\iiint_V(\sqrt{x^2+y^2}+z)dxdydz\\
=&\int_0^1 dz\iint_{D_z}(\sqrt{x^2+y^2}+z)dxdy\\
=&\int_0^1 dz\int_0^{2\pi}d\theta\int_0^z(r+z)\cdot rdr\\
=&\int_0^1\frac{5\pi}{3}z^3dz=\frac{5\pi}{12}.
\end{aligned}$$

11.(7分)设有一抛物面Σ：$z=\frac{1}{2}(x^2+y^2)\,(0\le z\le 1)$，已知面密度为$z+\frac{1}{2}$，求其质量$m$.

解: $m=\iint_\Sigma \left(z+\frac{1}{2}\right)dS$; $\Sigma:\ z=\frac{1}{2}(x^2+y^2)$, $(x,y)\in D$,

$$D=\{x^2+y^2\le 2\}=\{(r,\theta):\ 0\le\theta\le 2\pi,\ 0\le r\le\sqrt{2}\},$$

$$dS=\sqrt{1+x^2+y^2}dxdy,$$

$$\begin{aligned}
m &= \iint_\Sigma \left(z+\frac{1}{2}\right)dS \\
&= \iint_D \left(\frac{1}{2}(x^2+y^2)+\frac{1}{2}\right)\cdot\sqrt{1+x^2+y^2}dxdy \\
&= \frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(r^2+1)^{3/2}\cdot rdr=\frac{9\sqrt{3}-1}{5}\pi.
\end{aligned}$$

12.(5分)设D为平面上的一个有界闭区域，$u(x,y)$在D上连续，在D的内部每点处存在偏导数，$u|_{\partial D}=0$，且满足$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u$。证明：$\forall(x,y)\in D$，$u(x,y)=0$.

解: 利用格林公式

$$\begin{aligned}
&\iint_D 2u^2dxdy=\iint_D 2u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)dxdy\\
=&\iint_D\left(\frac{\partial u^2}{\partial x}-\frac{\partial(-u^2)}{\partial y}\right)dxdy\\
=&\oint_{\partial D}-u^2dx+u^2dy=0,
\end{aligned}$$

从而

$$\iint_D 2u^2dxdy=0\Longrightarrow u(x,y)=0,\forall(x,y)\in D.$$

13.(5分)设$u(x,y)$在平面上有连续的二阶偏导数，$F(s,t)$有连续的一阶偏导数，且$\forall(x,y)\in\mathbb{R}^2$ 有$F\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)=0$。对$\forall(s,t)\in\mathbb{R}^2$有$\left(\frac{\partial F}{\partial s}(s,t),\frac{\partial F}{\partial t}(s,t)\right)\neq(0,0)$。证明：

$$\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=0.$$

解: 对$F\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)=0$关于$x$、$y$分别求导，

$$\frac{\partial F}{\partial s}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial F}{\partial t}\cdot\frac{\partial^2 u}{\partial x\partial y}=0,\ \frac{\partial F}{\partial s}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial F}{\partial t}\cdot\frac{\partial^2 u}{\partial y^2}=0;$$

由$\left(\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right)\neq(0,0)$，

$$\begin{vmatrix}\frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\partial y}\\ \frac{\partial^2 u}{\partial x\partial y} & \frac{\partial^2 u}{\partial y^2}\end{vmatrix}=0,\iff\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=0.$$

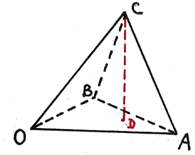
14.(8分)设$\vec{n}=\{\cos\alpha,\cos\beta,\cos\gamma\}$是一个给定的单位向量，C是平面$x\cos\alpha+y\cos\beta+z\cos\gamma=1$ 上的一条分段光滑的简单闭曲线，所围有界区域D的面积为A，设C 正向与单位向量$\vec{n}$符合右手法则。证明：

$$\oint_C (z\cos\beta-y\cos\gamma)dx+(x\cos\gamma-z\cos\alpha)dy+(y\cos\alpha-x\cos\beta)dz=2A.$$

证明：利用斯托克斯公式，

$$\begin{aligned}
&\oint_C (z\cos\beta-y\cos\gamma)dx+(x\cos\gamma-z\cos\alpha)dy+(y\cos\alpha-x\cos\beta)dz\\
=&\iint_D \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z\cos\beta-y\cos\gamma) & (x\cos\gamma-z\cos\alpha) & (y\cos\alpha-x\cos\beta) \end{vmatrix}\\
=&\iint_D 2\cos\alpha dydz+2\cos\beta dzdx+2\cos\gamma dxdy\\
=&\iint_D 2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)dS=\iint_D 2dS=2A.
\end{aligned}$$

1. 已知四面体OABC顶点$O(0,0,0)$，$A(1,2,3)$，$B(0,-1,2)$，$C(2,1,0)$，求四面体OABC的体积及顶点C在O、A、B三点所决定的平面上投影点D的坐标。



解：$\overrightarrow{OA}=\{1,2,3\}$,

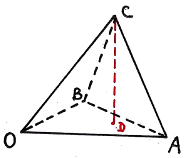
$\overrightarrow{OB}=\{0,-1,2\}$,

$\overrightarrow{OC}=\{2,1,0\}$,

$$V_{OABC}=\frac{1}{6}\left|(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}\right|=\frac{1}{6}\left\|\begin{vmatrix}1&2&3\\0&-1&2\\2&1&0\end{vmatrix}\right\|=2;$$

方法1. 记$\overrightarrow{OD} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$

$$= \{\lambda, 2\lambda - \mu, 3\lambda + 2\mu\},$$



则$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$$= \{-2 + \lambda, -1 + 2\lambda - \mu, 3\lambda + 2\mu\};$$

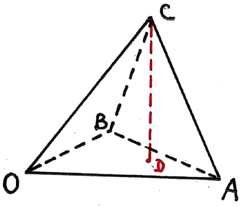
由$\overrightarrow{CD} \perp \overrightarrow{OA}$, $\overrightarrow{CD} \perp \overrightarrow{OB}$得

$$\begin{aligned} 0 &= \overrightarrow{CD} \cdot \overrightarrow{OA} = -4 + 14\lambda + 4\mu; \\ 0 &= \overrightarrow{CD} \cdot \overrightarrow{OB} = 1 + 4\lambda + 5\mu; \end{aligned}$$

解得

$$\lambda = \frac{4}{9}, \mu = -\frac{5}{9}, \overrightarrow{OD} = \{\frac{4}{9}, \frac{13}{9}, \frac{2}{9}\};$$

从而$D(\frac{4}{9}, \frac{13}{9}, \frac{2}{9})$.



方法2：

$$
\begin{aligned}
\vec{n} &= \overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = \{7, -2, -1\}; \\
\overrightarrow{DC} &= \left(\overrightarrow{OC}\right)_{\vec{n}} \frac{\vec{n}}{|\vec{n}|} = \left(\overrightarrow{OC} \cdot \frac{\vec{n}}{|\vec{n}|}\right) \frac{\vec{n}}{|\vec{n}|} \\
&= \frac{\overrightarrow{OC} \cdot \vec{n}}{|\vec{n}|^2} \vec{n} = \frac{2}{9} \vec{n} = \{\frac{14}{9}, \frac{-4}{9}, \frac{-2}{9}\};
\end{aligned}
$$

从而由$C(2,1,0)$得$D(\frac{4}{9}, \frac{13}{9}, \frac{2}{9})$.

2. 设圆C为球面$x^2+y^2+z^2=a^2$与平面$x+z=a$的交线，a为正实数。求圆C在xoy平面上的投影曲线，并求圆C的圆心及半径。

解法1：圆$C:\begin{cases} x^2+y^2+z^2=a^2, \\ x+z=a, \end{cases}$ 消去z 得投影柱面

$$x^2+y^2+(a-x)^2=a^2 \Longrightarrow (x-\frac{1}{2}a)^2+\frac{1}{2}y^2=\frac{1}{4}a^2;$$

圆C在xoy平面上的投影曲线为（椭圆曲线）

$$L:\begin{cases} (x-\frac{1}{2}a)^2+\frac{1}{2}y^2=\frac{1}{4}a^2, \\ z=0, \end{cases};$$

圆C的圆心投影点为$(\frac{1}{2}a,0)$, 得圆C的圆心为$(\frac{1}{2}a,0,\frac{1}{2}a)$; 注意到点$(0,0,a)$在圆$C$上，从而半径为

$$R=\sqrt{\left(\frac{a}{2}\right)^2+0^2+\left(\frac{a}{2}\right)^2}=\frac{a}{\sqrt{2}}.$$

2. 设圆C为球面$x^2+y^2+z^2=a^2$与平面$x+z=a$的交线，a为正实数。求圆C在xoy平面上的投影曲线，并求圆C的圆心及半径。

解法2：圆$C:\begin{cases} x^2+y^2+z^2=a^2, \\ x+z=a, \end{cases}$，解得$M_1(0,0,a)$在圆$C$上；

设$M_2(x,y,z)\in C$，$d=M_1M_2=\sqrt{x^2+y^2+(z-a)^2}$. 下面求$M_2(x,y,z)\in C$时$d$的极值，即函数

$d^2=x^2+y^2+(z-a)^2$在条件

$$x^2+y^2+z^2-a^2=0,\ x+z-a=0$$

下的极值；引进拉格朗日函数

$L(x,y,z,\lambda,\mu)=$

$x^2+y^2+(z-a)^2+\lambda\left(x^2+y^2+z^2-a^2\right)+\mu(x+z-a)$;

$$\begin{cases} \frac{\partial L}{\partial x}=2x+2\lambda x+\mu=0; & \frac{\partial L}{\partial y}=2y+2\lambda y=0; \\ \frac{\partial L}{\partial z}=2(z-a)+2\lambda z+\mu=0; & \frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-a^2=0; \\ \frac{\partial L}{\partial \mu}=x+z-a=0; & \end{cases}$$

解得拉格朗日函数的驻点为$(0,0,a,0,0)$及$(a,0,0,-2,2a)$;

2. 设圆C为球面$x^2+y^2+z^2=a^2$与平面$x+z=a$的交线，a为正实数。求圆C在xoy平面上的投影曲线，并求圆C的圆心及半径。

拉格朗日函数的驻点为$(0,0,a,0,0)$及$(a,0,0,-2,2a)$; 由实际意义知，$M_2(a,0,0)$时$d_{最大}=\sqrt{2}a$; 从而得圆C的圆心为$(\frac{1}{2}a,0,\frac{1}{2}a)$（$M_1$ 与M_2 的中点），半径为$R=\frac{1}{2}d=\frac{a}{\sqrt{2}}$.

3. 求曲面$S: z=x^2+\frac{1}{4}y^2+3$上平行于平面$\pi: 2x+y+z=0$ 的切平面方程。

解：设切点$M(x_0,y_0,z_0)$,

$$S: F(x,y,z)=z-x^2-\frac{1}{4}y^2-3=0;$$

曲面在M处法向量为$\vec{n_1}=\{-2x_0,-\frac{1}{2}y_0,1\}$；平面$\pi$的法向量为$\vec{n_2}=\{2,1,1\}$. 由切平面平行于平面$\pi$ 以及$M\in S$得

$$\frac{-2x_0}{2}=\frac{-\frac{1}{2}y_0}{1}=\frac{1}{1},\ z_0-x_0^2-\frac{1}{4}y_0^2-3=0;$$

解得$x_0=-1, y_0=-2, z_0=5$, $\vec{n_1}=\{2,1,1\}$, 切平面方程为

$$2(x+1)+(y+2)+(z-5)=0 \iff 2x+y+z=1.$$

4. 设$z=z(x,y)$是由$xyz+\sqrt{x^2+y^2+z^2}=1+\sqrt{3}$所确定的隐函数，求$z=z(x,y)$在$P(1,1,1)$处的全微分。

解法1：同时求微分

$$
\begin{aligned}
& d(xyz)+d\sqrt{x^2+y^2+z^2}=d(1+\sqrt{3}) \\
\Longrightarrow\quad & yzdx+xzdy+xydz+\frac{1}{2\sqrt{x^2+y^2+z^2}}d(x^2+y^2+z^2)=0 \\
\Longrightarrow\quad & yzdx+xzdy+xydz+\frac{2xdx+2ydy+2zdz}{2\sqrt{x^2+y^2+z^2}}=0
\end{aligned}
$$

取$(x,y,z)=(1,1,1)$得

$$
dx+dy+dz+\frac{1}{\sqrt{3}}(dx+dy+dz)=0\Longrightarrow dz|_{(1,1,1)}=-dx-dy.
$$

4. 设$z=z(x,y)$是由$xyz+\sqrt{x^2+y^2+z^2}=1+\sqrt{3}$所确定的隐函数，求$z=z(x,y)$在$P(1,1,1)$处的全微分。

解法2：关于x求导（注意$z=z(x,y)$)

$$\frac{\partial(xyz)}{\partial x}+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}=\frac{\partial(1+\sqrt{3})}{\partial x}$$

$$\Longrightarrow \quad yz+xy\frac{\partial z}{\partial x}+\frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial(x^2+y^2+z^2)}{\partial x}=0$$

$$\Longrightarrow \quad yz+xy\frac{\partial z}{\partial x}+\frac{2x+2z\frac{\partial z}{\partial x}}{2\sqrt{x^2+y^2+z^2}}=0$$

取$(x,y,z)=(1,1,1)$得

$$1+\frac{\partial z}{\partial x}+\frac{1}{\sqrt{3}}(1+\frac{\partial z}{\partial x})=0\Longrightarrow\frac{\partial z}{\partial x}|_{(1,1,1)}=-1.$$

类似，$\frac{\partial z}{\partial y}|_{(1,1,1)}=-1$；从而

$$dz|_{(1,1,1)}=\frac{\partial z}{\partial x}|_{(1,1,1)}dx+\frac{\partial z}{\partial y}|_{(1,1,1)}dy=-dx-dy.$$

5. 求函数$f(x,y)=x^3-4x^2+2xy-y^2$的极值点。

解：$\begin{cases} f'_x=3x^2-8x+2y=0, \\ f'_y=2x-2y=0, \end{cases}$，得驻点为$(0,0)$、$(2,2)$;

$$f''_{xx}=6x-8,\ f''_{xy}=2,\ f''_{yy}=-2;$$

- 驻点$(0,0)$:

$$A=f''_{xx}(0,0)=-8,\ B=f''_{xy}(0,0)=2,\ C=f''_{yy}(0,0)=-2;$$

由$AC-B^2=12>0$且$A<0$得: $(0,0)$是$f(x,y)$的极大值点；

- 驻点$(2,2)$:

$$A=f''_{xx}(2,2)=4,\ B=f''_{xy}(2,2)=2,\ C=f''_{yy}(2,2)=-2;$$

由$AC-B^2=-12<0$得: $(2,2)$不是函数$f(x,y)$的极值点；

6. 设周期为2π的函数$f(x)=\begin{cases}-1, -\pi<x<0\\ 1, 0\le x\le\pi\end{cases}$，求$f(x)$以$2\pi$为周期的傅里叶级数，并利用展开式求级数$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$的和。

解：$T=2\ell=2\pi\Longrightarrow \ell=\pi$;

$$
\begin{aligned}
a_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=0,\\
a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx=0, n=1,2,\cdots;\\
b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx\\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin nxdx=\frac{2\left(1-(-1)^n\right)}{\pi n}, n=1,2,\cdots;
\end{aligned}
$$

从而$f(x)$的傅里叶级数为

$$
f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]=\sum_{n=1}^{\infty}\frac{2\left(1-(-1)^n\right)}{\pi n}\sin nx;
$$

6. 设周期为2π的函数$f(x)=\begin{cases}-1, -\pi<x<0\\ 1, 0\le x\le\pi\end{cases}$，求$f(x)$以$\pi$为周期的傅里叶级数，并利用展开式求级数$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ 的和。

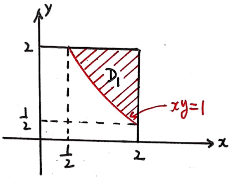
$$f(x)\sim\sum_{n=1}^{\infty}\frac{2\left(1-(-1)^n\right)}{\pi n}\sin nx;$$

取$x=\frac{\pi}{2}$, 由Dirichlet定理

$$\begin{aligned}
&\sum_{j=0}^{\infty}\frac{4(-1)^j}{\pi(2j+1)}=\sum_{j=0}^{\infty}\frac{4}{\pi(2j+1)}\sin\frac{(2j+1)\pi}{2}\\
=\ &\sum_{n=1}^{\infty}\frac{2\left(1-(-1)^n\right)}{\pi n}\sin\frac{n\pi}{2}=\frac{f(\frac{\pi}{2}+0)+f(\frac{\pi}{2}-0)}{2}=\frac{1+1}{2}=1;\\
&\Longrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}=\frac{\pi}{4};
\end{aligned}$$

7. 计算$\iint_D \max\{xy,1\}d\sigma$，其中

$D=\{(x,y):\ 0\le x\le 2, 0\le y\le 2\}$。



解： 取$D_1=D\cap\{xy\ge 1\}$,

$D_2=D\cap\{xy\le 1\}$;

$$\iint_D \max\{xy,1\}d\sigma=\iint_{D_1}\max\{xy,1\}d\sigma+\iint_{D_2}\max\{xy,1\}d\sigma$$

$$=\iint_{D_1}xyd\sigma+\iint_{D_2}d\sigma$$

$$=\iint_{D_1}xyd\sigma+\iint_{D}d\sigma-\iint_{D_1}d\sigma$$

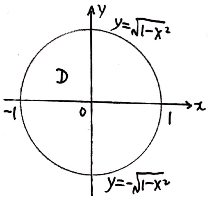
$$=\int_{1/2}^{2}dx\int_{1/x}^{2}(xy-1)dy+4=\frac{19}{4}+\ln 2;$$

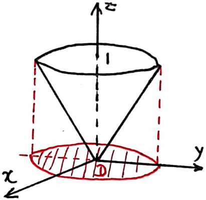
8. 计算$\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2+z^2}dz$.

解：$\Omega = \{(x,y) \in D,\ \sqrt{x^2+y^2} \le z \le 1\}$,

$D = \{(x,y):\ -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$;

$$\begin{aligned} I &= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2+z^2}dz \\ &= \iiint_{\Omega} \sqrt{x^2+y^2+z^2}dxdydz; \end{aligned}$$



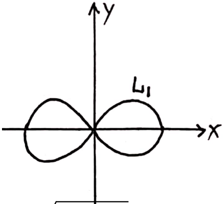


$$\Omega = \{0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{4}, 0 \le r \le \frac{1}{\cos\varphi}\};$$

$$\Omega = \{0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{4}, 0 \le r \le \frac{1}{\cos\varphi}\};$$

$$\begin{aligned}
I &= \iiint_\Omega \sqrt{x^2+y^2+z^2}dxdydz \\
&= \iiint_\Omega r \cdot r^2 \sin\varphi dr d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{1}{\cos\varphi}} r^3 \sin\varphi dr \\
&= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \frac{\sin\varphi}{4\cos^4\varphi} d\varphi = \frac{1}{6}(2\sqrt{2}-1)\pi;
\end{aligned}$$

9. 设L为双纽线$(x^2+y^2)^2=a^2(x^2-y^2)$，a为正实数，求曲线积分$\oint_L |x|ds$。



解：由对称性

$I=\oint_L |x|ds=4\oint_{L_1}|x|ds=4\int_{L_1} xds;$

引进极坐标(r,θ),

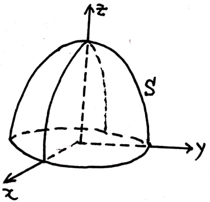
$$(x^2+y^2)^2=a^2(x^2-y^2)\Longleftrightarrow r=a\sqrt{\cos 2\theta};$$

$$L_1:\ x=a\sqrt{\cos 2\theta}\cos\theta,\ y=a\sqrt{\cos 2\theta}\sin\theta, 0\le\theta\le\frac{\pi}{4},$$

$$ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta=\frac{a}{\sqrt{\cos 2\theta}}d\theta;$$

$$I\ =\ 4\int_{L_1} xds=4\int_0^{\frac{\pi}{4}} a^2\cos\theta d\theta=2\sqrt{2}a^2.$$

10. 设S是半球面$z=\sqrt{R^2-x^2-y^2}$，$R>0$，计算曲面积分$I=\iint_S(x+y+z+1)^2dS$。



解：$I=\iint_S(x^2+y^2+z^2+1+2xy+2yz+2xz+2x+2y+2z)dS$,

由对称性: $\iint_S(2xy+2yz+2xz+2x+2y)dS=0$;

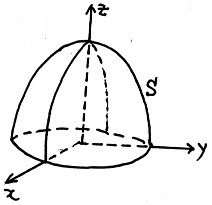
$$
\begin{aligned}
I &= \iint_S(x^2+y^2+z^2+1+2z)dS \\
&= \iint_S(x^2+y^2+z^2+1)dS+\iint_S 2zdS \\
&= \iint_S(R^2+1)dS+\iint_S 2zdS=(R^2+1)\cdot 2\pi R^2+\iint_S 2zdS
\end{aligned}
$$

$I = (R^2+1)\cdot 2\pi R^2 + \iint_S 2z dS;$

$S:\ z = \sqrt{R^2-x^2-y^2},$

$\quad (x,y)\in D = \{x^2+y^2\le R^2\},$

$dS = \frac{R}{\sqrt{R^2-x^2-y^2}}dxdy;$



$$\begin{aligned} I &= 2\pi(R^2+1)R^2 + \iint_D 2\sqrt{R^2-x^2-y^2}\cdot\frac{R}{\sqrt{R^2-x^2-y^2}}dxdy \\ &= 2\pi(R^2+1)R^2 + \iint_D 2Rdxdy = 2\pi R^2(R^2+R+1); \end{aligned}$$

11. 设在上半平面$D=\{(x,y):\ y>0\}$内，函数$f(x,y)$具有连续的一阶偏导数，且对任何$t>0$都有$f(tx,ty)=t^{-2}f(x,y)$。证明:
对D内的任意分段光滑的有向简单闭曲线L，都有

$$\oint_L yf(x,y)dx-xf(x,y)dy=0.$$

证明：在等式$f(tx,ty)=t^{-2}f(x,y)$二侧关于t求导

$$xf_1'(tx,ty)+yf_2'(tx,ty)=-2t^{-3}f(x,y),$$

取$t=1$得

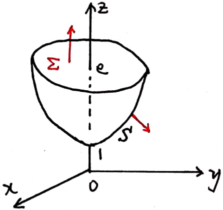
$$xf_1'(x,y)+yf_2'(x,y)=-2f(x,y);$$

在单连通区域D内$P(x,y)=yf(x,y)$，$Q(x,y)=-xf(x,y)$;

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2f(x,y)-xf_1'(x,y)-yf_2'(x,y)=0,$$

从而，$\oint_L yf(x,y)dx-xf(x,y)dy=0$。

12. S是曲线$\begin{cases} z = e^y \\ x = 0 \end{cases}$ $(0 \le y \le 1)$围绕z轴旋转生成的旋转曲面、下侧，求$I = \iint_S 4xzdydz - 2yzdzdx + (x^2 - z^2)dxdy$.



解法1：旋转曲面

$S : z = e^{\sqrt{x^2+y^2}}(1 \le z \le e)$、下侧；

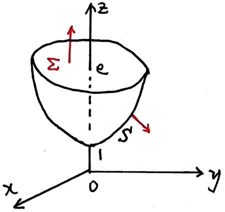
取$\Sigma : z = e(x^2 + y^2 \le 1)$、上侧；

由曲面S与Σ所围立体记为Ω.

由高斯公式及投影法

$$
\begin{aligned}
I &= \iint_{S\cup\Sigma} - \iint_{\Sigma} 4xzdydz - 2yzdzdx + (x^2 - z^2)dxdy \\
&= \iiint_{\Omega}(4z - 2z - 2z)dV - \left(0 + 0 + \iint_{x^2+y^2\le 1}(x^2 - e^2)dxdy\right) \\
&= e^2\pi - \int_0^{2\pi} d\theta \int_0^1 r^2\cos^2\theta \cdot rdr = e^2\pi - \frac{1}{4}\pi.
\end{aligned}
$$

12. S是曲线$\begin{cases} z = e^y \\ x = 0 \end{cases}$ $(0 \leq y \leq 1)$围绕z轴旋转生成的旋转曲面、下侧，求$I = \iint_S 4xzdydz - 2yzdzdx + (x^2 - z^2)dxdy$.



解法2：旋转曲面

$S: z = e^{\sqrt{x^2+y^2}}(1 \leq z \leq e)$、下侧；

下侧单位法向量

$$\vec{n}^0 = \frac{\left\{xe^{\sqrt{x^2+y^2}}, ye^{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2} right\}}{\sqrt{x^2+y^2}\sqrt{1+e^{2\sqrt{x^2+y^2}}}};$$

$$I = \iint_S \frac{e^{\sqrt{x^2+y^2}}\frac{(4x^2z-2y^2z)}{\sqrt{x^2+y^2}} - (x^2 - z^2)}{\sqrt{1 + e^{2\sqrt{x^2+y^2}}}} dS;$$

$$I=\iint_S \frac{e^{\sqrt{x^2+y^2}}\frac{(4x^2z-2y^2z)}{\sqrt{x^2+y^2}}-(x^2-z^2)}{\sqrt{1+e^{2\sqrt{x^2+y^2}}}}dS;$$

$$S:\ z=e^{\sqrt{x^2+y^2}},(x,y)\in D=\{x^2+y^2\le 1\}$$

$$dS=\sqrt{1+e^{2\sqrt{x^2+y^2}}}dxdy;$$

$$I=\iint_D\left[e^{2\sqrt{x^2+y^2}}\frac{(4x^2-2y^2)}{\sqrt{x^2+y^2}}-x^2+e^{2\sqrt{x^2+y^2}}\right]dxdy;$$

利用

$$\iint_D \frac{x^2 e^{2\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} dxdy = \iint_D \frac{y^2 e^{2\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} dxdy,$$

$$\iint_D x^2 dxdy = \iint_D y^2 dxdy,$$

$$\begin{aligned} I &= \iint_D \left[e^{2\sqrt{x^2+y^2}} \frac{(x^2+y^2)}{\sqrt{x^2+y^2}} - \frac{1}{2}(x^2+y^2) + e^{2\sqrt{x^2+y^2}} \right] dxdy \\ &= \iint_D \left[e^{2r}(r+1) - \frac{1}{2} r \right] \cdot rdrd\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 \left[e^{2r}(r^2+r) - \frac{1}{2} r^2 \right] dr = e^2\pi - \frac{1}{4}\pi. \end{aligned}$$

13. 在变力$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ 的作用下，质点由原点沿直线运动到椭圆面$x^2 + \frac{1}{3}y^2 + \frac{1}{6}z^2 = 1$上第一卦限上的点$P(a, b, c)$，问$a$、$b$、$c$取何值时力$\vec{F}$所做的功$W$最大，并求$W$的最大值。

解：直线OP：$x = at,\ y = bt,\ z = ct,\ t: 0 \to 1$;

$$\begin{aligned} W &= \int_{OP} \vec{F} \cdot d\vec{s} = \int_{OP} yzdx + zxdy + xydz \\ &= \int_0^1 3abct^2 dt = abc; \end{aligned}$$

我们要计算：当$a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2 = 1$且$a > 0$、$b > 0$ 及$c > 0$ 时求$W = abc$的最大值。引进拉格朗日函数

$$L(a, b, c, \lambda) = abc + \lambda(a^2 + \tfrac{1}{3}b^2 + \tfrac{1}{6}c^2 - 1);$$

$$\begin{cases} \frac{\partial L}{\partial a} = bc + 2\lambda a = 0, & \frac{\partial L}{\partial b} = ac + \frac{2}{3}\lambda b = 0, \\ \frac{\partial L}{\partial c} = ab + \frac{1}{3}\lambda c = 0, & \frac{\partial L}{\partial \lambda} = a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2 - 1 = 0 \end{cases}$$

得拉格朗日函数的驻点$(a, b, c, \lambda) = (\frac{1}{\sqrt{3}}, 1, \sqrt{2}, -\sqrt{\frac{3}{2}})$. 结合实际问题，当$(a, b, c) = (\frac{1}{\sqrt{3}}, 1, \sqrt{2})$时$W$有最大值$\frac{\sqrt{6}}{3}$.

14. 设P为椭球面S：$x^2+y^2+z^2-yz=1$上的动点，S在点P处的切平面与xoy平面垂直，求点P的轨迹C，并计算曲面积分$I=\iint_\Sigma \frac{(x+3)|y-2z|}{\sqrt{4+y^2+z^2-4yz}}dS$，Σ是椭球面S位于曲线C上方部分。

解：设点$P(x,y,z)$，则

$$x^2+y^2+z^2-yz=1---------(1)$$

椭球面S在$P(x,y,z)$处的法向量为$\vec{n}=\{2x,2y-z,2z-y\}$，由切平面与xoy平面垂直得

$$\vec{n}\cdot\vec{k}=0\Longrightarrow 2z-y=0-----------(2)$$

从而点P的轨迹为$C:\begin{cases} x^2+y^2+z^2-yz=1 \\ 2z-y=0 \end{cases}$. 点P的轨迹$C$在$xoy$平面上的投影曲线为$\begin{cases} x^2+\frac{3}{4}y^2=1 \\ z=0 \end{cases}$;

14. 设P为椭球面S：$x^2+y^2+z^2-yz=1$上的动点，S在点P处的切平面与xoy平面垂直，求点P的轨迹C，并计算曲面积分$I=\iint_{\Sigma}\frac{(x+3)|y-2z|}{\sqrt{4+y^2+z^2-4yz}}dS$，Σ是椭球面S位于曲线C上方部分。

点P的轨迹C在xoy平面上的投影曲线为$\begin{cases} x^2+\frac{3}{4}y^2=1 \\ z=0 \end{cases}$；

$$\Sigma: z=\frac{1}{2}y+\sqrt{1-x^2-\frac{3}{4}y^2}, (x,y)\in D=\{x^2+\frac{3}{4}y^2\le 1\}\text{且}2z-y\ge 0;$$

$$dS=\sqrt{1+(z'_x)^2+(z'_y)^2}dxdy=\frac{\sqrt{4+y^2+z^2-4yz}}{2z-y}dxdy;$$

$$\begin{aligned} I &= \iint_{\Sigma}\frac{(x+3)|y-2z|}{\sqrt{4+y^2+z^2-4yz}}dS=\iint_{\Sigma}\frac{(x+3)(2z-y)}{\sqrt{4+y^2+z^2-4yz}}dS \\ &= \iint_{D}(x+3)dxdy(\text{由对称性})=\iint_{D}3dxdy=2\sqrt{3}\pi. \end{aligned}$$

15. 幂级数$\sum_{n=0}^{\infty} a_n x^n$在$(-\infty,+\infty)$内收敛，和函数$y(x)$满足:

$$y'' - 2xy' - 4y = 0,\ y(0) = 0, y'(0) = 1.$$

(I) 证明$a_{n+2} = \frac{2}{n+1} a_n$，n=1, 2, $\cdots$

(II) 求$y(x)$的表达式。

解：(I). 在$(-\infty,+\infty)$内$y = \sum_{n=0}^{\infty} a_n x^n$(其收敛半径为$R = +\infty$),

$$\begin{cases} y'(x) = (\sum_{n=0}^{\infty} a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1}; \\ y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n; \end{cases}$$

$$\begin{cases} 0 = y'' - 2xy' - 4y = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2na_n - 4a_n\right] x^n \\ 0 = y(0) = a_0, \qquad 1 = y'(0) = a_1; \end{cases}$$

$$\begin{cases} 0 = y'' - 2xy' - 4y = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2na_n - 4a_n \right] x^n \\ 0 = y(0) = a_0, \qquad 1 = y'(0) = a_1; \end{cases}$$

比较系数得

$$\begin{cases} (n+2)(n+1)a_{n+2} = (2n+4)a_n \\ a_0 = 0, a_1 = 1 \end{cases} \Longrightarrow \begin{cases} a_{n+2} = \frac{2}{n+1} a_n \\ a_0 = 0, a_1 = 1 \end{cases}$$

(II) 由$a_{n+2}=\frac{2}{n+1}a_n$，$a_0=0$推得

$$a_0=a_2=\cdots=a_{2k}=0, k=1,2,\cdots;$$

由$a_{n+2}=\frac{2}{n+1}a_n$，$a_1=1$推得

$$a_3=\frac{2}{2}a_1,\ a_5=\frac{2}{4}a_3=\frac{2}{4}\frac{2}{2}a_1=\frac{1}{2!}a_1,$$
$$a_7=\frac{1}{3!}a_1,\cdots,a_{2k+1}=\frac{1}{k!}, k=0,1,2\cdots;$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}a_nx^n=x+\frac{1}{1!}x^3+\frac{1}{2!}x^5+\frac{1}{3!}x^7+\cdots \\ &= x\left[1+\frac{1}{1!}x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\cdots\right]=xe^{x^2}; \end{aligned}$$

1. 有二次曲面$S: x^2+xy+y^2-z^2=1$，求曲面S在曲面上点$(1,-1,0)$处的切平面方程。

解：$S: F(x,y,z)=x^2+xy+y^2-z^2-1=0$, 在点$(1,-1,0)$处的法向量

$$\vec{n}=\{F'_x, F'_y, F'_z,\}_{(1,-1,0)}=\{1,-1,0\};$$

从而, 切平面方程为

$$(x-1)+(-1)(y+1)+0(z-0)=0 \Longleftrightarrow x-y-2=0;$$

2. 求幂级数$\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n+1}$的收敛半径与和函数;

解: $\rho=\lim_{n\to+\infty}\left|\frac{\frac{(-1)^{n+1}}{n+2}}{\frac{(-1)^n}{n+1}}\right|=1$, 收敛半径为$R=\frac{1}{\rho}=1$, 收敛域$(-1,1]$; 对$x\in(-1,1]$, 记和函数为

$$S(x)=\sum_{n=2}^{+\infty}(-1)^n\frac{x^n}{n+1};$$

- 当$x\in(-1,1)$时

$$\begin{aligned}xS(x) &= \sum_{n=2}^{+\infty}(-1)^n\frac{x^{n+1}}{n+1}=\sum_{n=2}^{+\infty}(-1)^n\left(\int_0^x t^n dt\right)\\ &= \int_0^x\left(\sum_{n=2}^{+\infty}(-1)^n t^n\right)dt\text{(这里要求}|x|<1\text{, 问: 为什么?)}\\ &= \int_0^x\frac{t^2}{1+t}dt=\frac{1}{2}x^2-x+\ln(x+1).\end{aligned}$$

从而，当$x \in (-1,1)$时

$$S(x) = \begin{cases} \frac{\ln(x+1)}{x} + \frac{1}{2}x - 1, 0 < |x| < 1\text{时} \\ 0, x = 0\text{时} \end{cases}$$

- 当$x = 1$时，利用$S(x)$是收敛域$(-1,1]$上连续函数，

$$S(1) = \lim_{x\to 1^-} S(x) = \lim_{x\to 1^-}\left[\frac{\ln(x+1)}{x} + \frac{1}{2}x - 1\right] = -\frac{1}{2} + \ln 2;$$

综合：和函数

$$S(x) = \begin{cases} \frac{\ln(x+1)}{x} + \frac{1}{2}x - 1, x \in (-1,0)\cup(0,1]\text{时} \\ 0, x = 0\text{时} \end{cases}$$

3. 试叙述无穷级数$\sum_{n=1}^{+\infty} a_n$条件收敛的定义。试判别命题“若无穷级数$\sum_{n=1}^{+\infty} a_n$条件收敛，则无穷级数$\sum_{n=1}^{+\infty} (a_n)^2$收敛”是否正确，若正确请证明，若不正确请给出反例；

解：如果无穷级数$\sum_{n=1}^{+\infty} a_n$收敛、但无穷级数$\sum_{n=1}^{+\infty} |a_n|$发散，则称无穷级数$\sum_{n=1}^{+\infty} a_n$条件收敛；

命题不正确。

反例：无穷级数$\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt[3]{n}}$条件收敛, 但级

数$\sum_{n=1}^{+\infty} \left(\frac{(-1)^n}{\sqrt[3]{n}}\right)^2 = \sum_{n=1}^{+\infty} \frac{1}{n^{2/3}}$发散；

4. 设$f(x)=\begin{cases}1, x\in[0,\pi]\\ -2, x\in(\pi,2\pi)\end{cases}$，将函数$f$ 延拓成R 上以2π为周期的周期函数（仍记为f）．试将f展开成傅里叶级数

$$\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx),$$

并求傅里叶级数的和函数$S(x)\,(0\le x\le 2\pi)$.

解：$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=-1;$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=0,\ n=1,2,3,\cdots$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx=\frac{3[1-(-1)^n]}{n\pi},\ n=1,2,3,\cdots$$

傅里叶级数为

$$f(x)\sim-\frac{1}{2}+\sum_{n=1}^{+\infty}\frac{3[1-(-1)^n]}{n\pi}\sin nx;$$

由定理

$$S(x)=\frac{f(x-0)+f(x+0)}{2}=\begin{cases}-\frac{1}{2}, x=0,\pi,2\pi\\ 1, x\in(0,\pi)\\ -2, x\in(\pi,2\pi)\end{cases}$$

5. 设$D=\{1\le x^2+y^2\le 4\}$, S为曲面$z=\sqrt{x^2+y^2}$（其中$(x,y)\in D$）的上（内）侧，试计算第二类曲面积分$I=\iint_S dydz+dzdx+dxdy$.

解法1：曲面S $x^2+y^2-z^2=0$的单位法向量为$\pm\frac{\{x,y,-z\}}{\sqrt{x^2+y^2+z^2}}$；上（内）侧的单位法向量为

$$\vec{n^o}=\{\cos\alpha,\cos\beta,\cos\gamma\}=\frac{\{-x,-y,z\}}{\sqrt{x^2+y^2+z^2}}.$$

$$\begin{aligned}I &= \iint_S dydz+dzdx+dxdy=\iint_S(\cos\alpha+\cos\beta+\cos\gamma)dS\\ &= \iint_S\frac{-x-y+z}{\sqrt{x^2+y^2+z^2}}dS(S:\ z=\sqrt{x^2+y^2},(x,y)\in D,dS=\sqrt{2}dx\\ &= \iint_D\left(-\frac{x+y}{\sqrt{x^2+y^2}}+1\right)dxdy=\iint_D dxdy=3\pi.\end{aligned}$$

5. 设$D=\{1\le x^2+y^2\le 4\}$, S为曲面$z=\sqrt{x^2+y^2}$（其中$(x,y)\in D$）的上（内）侧，试计算第二类曲面积分$I=\iint_S dydz+dzdx+dxdy$.

解法2：记$S_1: z=2,(x,y)\in D_1=\{x^2+y^2\le 4\}$, 下侧；
$S_2: z=1,(x,y)\in D_2=\{x^2+y^2\le 1\}$, 上侧；
由曲面S、S_1及S_2所围区域为Ω, $S\cup S_1\cup S_2$为Ω表面内侧，利用高斯公式

$$\iint_S+\iint_{S_1}+\iint_{S_2} dydz+dzdx+dxdy=-\iiint_\Omega odxdydz=0,$$

$$\Longrightarrow I=-\iint_{S_1} dydz+dzdx+dxdy-\iint_{S_2} dydz+dzdx+dxdy.$$

由投影法

$$\iint_{S_1} dydz+dzdx+dxdy=-\iint_{D_1} dxdy=-4\pi;$$

$$\iint_{S_2} dydz+dzdx+dxdy=\iint_{D_2} dxdy=\pi;$$

$$\Longrightarrow I=4\pi-\pi=3\pi.$$

6. C是平面曲线$y=\int_0^x \sqrt{t^2}dt$, $x\in[1,\sqrt{\frac{\pi}{2}}]$, 求第一类曲线积分$\int_C xds$.

解：曲线C : $x=u, y=\int_0^u \sqrt{t^2}dt$, $u\in[1,\sqrt{\frac{\pi}{2}}]$,

$$ds=\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}du=\sqrt{1+\sin u^2}du,$$

$$\begin{aligned}\int_C xds &= \int_1^{\sqrt{\frac{\pi}{2}}} u\sqrt{1+\sin u^2}du\\ &= \frac{1}{2}\int_1^{\frac{\pi}{2}}\sqrt{1+\sin v}dv=\frac{1}{2}\int_1^{\frac{\pi}{2}}\left(\sin\frac{v}{2}+\cos\frac{v}{2}\right)dv\\ &= \left(\sin\frac{v}{2}-\cos\frac{v}{2}\right)\Big|_{v=1}^{v=\frac{\pi}{2}}=\cos\frac{1}{2}-\sin\frac{1}{2}.\end{aligned}$$

7.设函数$f(x,y)$在$\mathbb{R}^2$上有一阶连续偏导数，且对任何$t>0$、$(x,y)\in\mathbb{R}^2$有$f(tx,ty)=t^3f(x,y)$。试证明：对任何$(x,y)\in\mathbb{R}^2$有

$$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=3f(x,y).$$

解：已知对任何$t>0$、$(x,y)\in\mathbb{R}^2$有$f(tx,ty)=t^3f(x,y)$，关于t求导

$$xf_1'(tx,ty)+yf_2'(tx,ty)=3t^2f(x,y);$$

取$t=1$得

$$xf_1'(x,y)+yf_2'(x,y)=3f(x,y);$$

8. 设$f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，试求$f(x,y)$在$(0,0)$处沿方向$\vec{\ell}=\{\cos\alpha,\sin\alpha\}$(其中$\alpha\in[0,2\pi)$)的方向导数。

解：由定义

$$\left.\frac{\partial f}{\partial\vec{\ell}}\right|_{(0,0)}=\lim_{\rho\to0^+}\frac{f(\rho\cos\alpha,\rho\sin\alpha)-f(0,0)}{\rho}$$

$$=\lim_{\rho\to0^+}\frac{\rho^3\cos^2\alpha\sin\alpha}{\rho(\rho^4\cos^4\alpha+\rho^2\sin^2\alpha)}=\begin{cases}\frac{\cos^2\alpha}{\sin\alpha},\alpha\neq0,\alpha\neq\pi\\ 0,\ \alpha=0\text{或}\alpha=\pi\end{cases}$$

- 不能用公式

$$\left.\frac{\partial f}{\partial\vec{\ell}}\right|_{(0,0)}=\left.\frac{\partial f}{\partial x}\right|_{(0,0)}\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_{(0,0)}\sin\alpha$$

因为函数$f(x,y)$在$(0,0)$处不可微分。

9. 设γ为圆柱螺线$x=\cos t, y=\sin t, z=2t$位于$t\in[0,\pi]$的弧段，沿t增加的方向为γ的正向；γ_1为曲线γ位于$t\in[0,\pi/2]$的弧段且与γ同向；设L是γ在点$(0,1,\pi)$处的切线，L_1为切线L上以$(0,1,\pi)$为起点、长度为$\sqrt{5}$、与γ正向一致的有向直线段；试求第二类曲线积分$I=\int_{\gamma_1\cup L_1} yzdx+xzdy+xydz$.

解法1. $\gamma_1: x=\cos t, y=\sin t, z=2t$, 起点$t=0$、终点$t=\pi/2$;在点$(0,1,\pi)$处与$\gamma$同向的单位切向量$\vec{s}=\{-\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\}$, 有向直线段

$$L_1: x=-\frac{1}{\sqrt{5}}t, y=1, z=\pi+\frac{2}{\sqrt{5}}t,$$

起点$t=0$、终点$t=\sqrt{5}$;

$$\begin{aligned}
I &= \int_{\gamma_1} yzdx+xzdy+xydz+\int_{L_1} yzdx+xzdy+xydz \\
&= \int_0^{\pi/2}(-2t\sin^2 t+2t\cos^2 t+2\sin t\cos t)dt+\int_0^{\sqrt{5}}\left(-\frac{\pi}{\sqrt{5}}-\frac{4}{5}t\right. \\
&= \int_0^{\pi/2}(2t\cos 2t+\sin 2t)dt+-\pi-2=t\sin 2t\Big|_{t=0}^{t=\pi/2}=-(\pi+2).
\end{aligned}$$

解法2：

在R^3上$P(x,y,z)=yz$、$Q(x,y,z)=xz$、$R(x,y,z)=xy$满足

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \{0,0,0\},$$

从而第二类曲线积分$I=\int_{\gamma_1\cup L_1} yzdx+xzdy+xydz$与路径无关；记$\gamma_1$起点为$A(1,0,0)$，在点B$(0,1,\pi)$处与$\gamma$同向的单位切向量$\vec{s}=\{-\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\}$，则有向直线段$L_1$的终点$C(-1,1,2+\pi)$；利用积分与路径无关，可求

$$I=\int_{直线AC} yzdx+xzdy+xydz$$

或利用牛顿—莱布尼兹公式：到$(x_0,y_0,z_0)=(0,0,0)$求原函数

$$u(x,y,z)=\int_{x_0}^{x} P(x,y_0,z_0)dx+\int_{y_0}^{y} Q(x,y,z_0)dy+\int_{z_0}^{z} P(x,y,z)dz=xyz$$

$$I=\int_{A(1,0,0)}^{C(-1,1,2+\pi)} yzdx+xzdy+xydz=xyz\Big|_{A(1,0,0)}^{C(-1,1,2+\pi)}=-(2+\pi);$$

10. 求三重累次积分$I=\int_0^1 dx\int_0^1 dy\int_y^1 \frac{e^{-z^2}}{x^2+1}dz$.

解：记$D_{yz}=\{0\le y\le 1, y\le z\le 1\}$,

则$I=\int_0^1 dx\iint_{D_{yz}}\frac{e^{-z^2}}{x^2+1}dydz$; 由$D_{yz}=\{0\le z\le 1, 0\le y\le z\}$,

$$
\begin{aligned}
I &= \int_0^1 dx\iint_{D_{yz}}\frac{e^{-z^2}}{x^2+1}dydz \\
&= \int_0^1 dx\int_0^1 dz\int_0^z \frac{e^{-z^2}}{x^2+1}dy \\
&= \int_0^1 dx\int_0^1 \frac{ze^{-z^2}}{x^2+1}dz=\frac{1}{2}(1-\frac{1}{e})\int_0^1\frac{dx}{x^2+1} \\
&= \frac{\pi}{8}(1-\frac{1}{e});
\end{aligned}
$$

11. 求密度为常数c的抛物曲面Σ：$z=\frac{1}{2}(x^2+y^2)(0\le z\le 1)$的重心坐标；

解：由对称性，$\bar{x}=\bar{y}=0$;

$\Sigma: z=\dfrac{1}{2}(x^2+y^2)\ (x,y)\in D=\{x^2+y^2\le 2\}; dS=\sqrt{1+x^2+y^2}dxdy.$

$$\iint_\Sigma cdS = c\iint_D \sqrt{1+x^2+y^2}dxdy$$

$$= c\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\sqrt{1+r^2}\cdot rdr = c\pi(2\sqrt{3}-\frac{2}{3}).$$

$$\iint_\Sigma czdS = c\iint_D \frac{1}{2}(x^2+y^2)\sqrt{1+x^2+y^2}dxdy$$

$$= \frac{c}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r^2\sqrt{1+r^2}\cdot rdr = c\pi(\frac{4\sqrt{3}}{5}-\frac{2}{15}).$$

$$\bar{z}=\frac{\iint_\Sigma czdS}{\iint_\Sigma cdS}=\frac{55+9\sqrt{3}}{130},$$

从而重心坐标为$(0,0,\frac{55+9\sqrt{3}}{130})$.

12. 设$y=y_1(x)$、$y=y_2(x)$是区间$[0,1]$上的连续函数，$D=\{0\le x\le 1, y_1(x)\le y\le y_2(x)\}$, 函数在包含$D$的一个开区域内有一阶连续的偏导数，的边界曲线为逆时时针方向。试证明：

$$\oint_{\partial D} u(x,y)dx = -\iint_D \frac{\partial u}{\partial y}dxdy;$$

解：$C_1: x=t, y=y_1(t)$, 起点$t=0$, 终点$t=1$;
$C_2: x=1, y=t$, 起点$t=y_1(1)$, 终点$t=y_2(1)$;
$C_3: x=t, y=y_2(t)$, 起点$t=1$, 终点$t=0$;
$C_4: x=0, y=t$, 起点$t=y_2(0)$, 终点$t=y_1(0)$;

$$\begin{aligned}\oint_{\partial D} u(x,y)dx &= \int_{C_1} u(x,y)dx + \int_{C_2} u(x,y)dx \\ &\quad + \int_{C_3} u(x,y)dx + \int_{C_4} u(x,y)dx \\ &= \int_0^1 u(t,y_1(t))dt + \int_{y_1(1)}^{y_2(1)} 0dt + \int_1^0 u(t,y_2(t))dt + \int_{y_2(0)}^{y_1(0)} 0dt \\ &= \int_0^1 u(t,y_1(t))dt - \int_0^1 u(t,y_2(t))dt;\end{aligned}$$

$$\iint_D \frac{\partial u}{\partial y} dxdy = \int_0^1 dx \int_{y_1(x)}^{y_2(x)} \frac{\partial u}{\partial y} dy$$

$$= \int_0^1 \left(u(x, y_2(x)) - u(x, y_1(x))\right) dx = -\oint_{\partial D} u(x, y) dx.$$

13. $f(x,y)=(xy)^{\frac{2}{3}}$. (1)求$\frac{\partial f}{\partial x}(0,0)$、$\frac{\partial f}{\partial y}(0,0)$；(2). 求证：$f(x,y)$在$(0,0)$处可微；

解：(1) $\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0$;

$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0$;

(2). $\Delta f=f(\Delta x,\Delta y)-f(0,0)=(\Delta x\,\Delta y)^{\frac{2}{3}}$;

$$\frac{\Delta f-0\Delta x-0\Delta y}{\rho}=\frac{(\Delta x\,\Delta y)^{\frac{2}{3}}}{\rho},$$

其中$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$; 利用

$$0\le\frac{(\Delta x\,\Delta y)^{\frac{2}{3}}}{\rho}\le\frac{(\frac{1}{2}\rho^2)^{\frac{2}{3}}}{\rho}\le\rho^{\frac{1}{3}},$$

$\lim_{\rho\to 0^+}\rho^{\frac{1}{3}}=0$及夹逼准则得

$$\lim_{\rho\to 0^+}\frac{\Delta f-0\Delta x-0\Delta y}{\rho}=0\Longrightarrow\Delta f=0\delta x+0\Delta y+o(\rho);$$

由微分的定义，$df|_{(0,0)}=0dx+0dy=0$;

14. 设函数$f(x,y)$在单位圆盘$D=\{x^2+y^2\le 1\}$上有一阶连续的偏导函数，且在D上$|f(x,y)|\le 1$; 求证：存在$(x^*,y^*)\in\{x^2+y^2<1\}$满足

$$\left|\frac{\partial f}{\partial x}(x^*,y^*)\right|^2+\left|\frac{\partial f}{\partial y}(x^*,y^*)\right|^2\le 16.$$

证明：取$F(x,y)=f(x,y)+2(x^2+y^2)$, 则由$f(x,y)$在单位圆盘D上有一阶连续的偏导函数可得：函数$F(x,y)$在单位圆盘D上连续，从而$F(x,y)$在单位圆盘D上取到最小值m. 注意到$F(x,y)|_{x^2+y^2=1}\ge 1$、$F(0,0)\le 1$，有$m\le F(0,0)\le 1$.

- 如果$m=F(0,0)$, $(0,0)$是$F(x,y)$的极小值点，

$$0=\frac{\partial F}{\partial x}(0,0)=\frac{\partial f}{\partial x}(0,0),0=\frac{\partial F}{\partial y}(0,0)=\frac{\partial f}{\partial y}(0,0);$$

取$(x^*,y^*)=(0,0)$;

- 如果$m < F(0,0) \le 1$，由$F(x,y)|_{x^2+y^2=1} \ge 1$知：存在$(x^*,y^*) \in \{x^2+y^2<1\}$ 使得$m = F(x^*,y^*)$,

$$0 = \frac{\partial F}{\partial x}(x^*,y^*) = \frac{\partial f}{\partial x}(x^*,y^*) + 4x^*,$$

$$0 = \frac{\partial F}{\partial y}(x^*,y^*) = \frac{\partial f}{\partial y}(x^*,y^*) + 4y^*;$$

从而

$$\left|\frac{\partial f}{\partial x}(x^*,y^*)\right|^2 + \left|\frac{\partial f}{\partial y}(x^*,y^*)\right|^2 = 16((x^*)^2 + (y^*)^2) \le 16;$$