级数— 傅里叶级数

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一、周期函数的傅里叶级数

- 十八世纪法国科学家傅里叶在研究热量传播时引进了傅里叶级数概念；
- 给定周期为$T=2\ell$的三角函数系

$$1, \cos\frac{\pi x}{\ell}, \sin\frac{\pi x}{\ell}, \cos\frac{2\pi x}{\ell}, \sin\frac{2\pi x}{\ell}, \cdots, \cos\frac{n\pi x}{\ell}, \sin\frac{n\pi x}{\ell}, \cdots$$

问： 一个以$T=2\ell$为周期的函数$f(x)$，是否一定可表示为三角级数

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]?$$

由函数$f(x)$如何确定系数a_0、a_1、$\cdots$；b_1、b_2、$\cdots$？

- 傅里叶级数理论以及它的推广形式（傅里叶变换）已经成为现代数学、物理、信息、数值计算等领域研究的重要工具；

给定周期为$T=2\ell$的三角函数系

$$1, \cos\frac{\pi x}{\ell}, \sin\frac{\pi x}{\ell}, \cos\frac{2\pi x}{\ell}, \sin\frac{2\pi x}{\ell}, \cdots, \cos\frac{n\pi x}{\ell}, \sin\frac{n\pi x}{\ell}, \cdots$$

以及$T=2\ell$为周期的函数$f(x)$，如何确定系数a_0、a_1、$\cdots$；b_1、b_2、$\cdots$ 使得

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]?$$

必要性

线性代数知识回顾： 设V是一个n维内积（欧氏）空间，$\varphi_1, \varphi_2, \cdots, \varphi_n$ 是一组正交基，$f \in V$且

$$f = a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n$$

则

$$\begin{aligned}
(f, \varphi_k) &= (a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n, \varphi_k) \\
&= a_1(\varphi_1, \varphi_k) + a_2(\varphi_2, \varphi_k) + \cdots + a_n(\varphi_n, \varphi_k) \\
&= a_k(\varphi_k, \varphi_k) \qquad \Longrightarrow a_k = \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)}
\end{aligned}$$

基本思想：　借用**代数观点** 记集合

$$V=\{f(x):\ f(x)\text{以}T=2\ell\text{为周期的连续函数}\},$$

按通常的函数相加、数乘运算$\Longrightarrow$ 集合V是一个线性空间；

- 定义

$$(f,g)=\int_{-\ell}^{\ell}f(x)g(x)dx,\ \forall f(x),g(x)\in V;$$

$\Longrightarrow$ $(\cdot,\cdot)$是线性空间V的一个内积运算；

- 周期为$T=2\ell$的三角函数系

$$1,\cos\frac{\pi x}{\ell},\sin\frac{\pi x}{\ell},\cos\frac{2\pi x}{\ell},\sin\frac{2\pi x}{\ell},\cdots,\cos\frac{n\pi x}{\ell},\sin\frac{n\pi x}{\ell},\cdots$$

是内积空间V的一组正交向量；

- $f=a_1\varphi_1+a_2\varphi_2+\cdots+a_n\varphi_n+\cdots,$

$$\Longrightarrow a_k=\frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)}=\frac{\int_{-\ell}^{\ell}f(x)\cdot\varphi_k(x)dx}{\int_{-\ell}^{\ell}\varphi_k(x)\cdot\varphi_k(x)dx};$$

三角函数系的正交性

周期为$T=2\ell$的函数集合定义内积

$$(f,g)=\int_{-\ell}^{\ell} f(x)g(x)dx,\ \forall f(x),g(x)\in V;$$

则周期为$T=2\ell$的三角函数系

$$1,\cos\frac{\pi x}{\ell},\sin\frac{\pi x}{\ell},\cos\frac{2\pi x}{\ell},\sin\frac{2\pi x}{\ell},\cdots,\cos\frac{n\pi x}{\ell},\sin\frac{n\pi x}{\ell},\cdots$$

正交，即满足

- $\int_{-\ell}^{\ell}\cos\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx=0,\ j=0,1,2,\cdots,k=1,2,\cdots$
- $\int_{-\ell}^{\ell}\cos\frac{j\pi x}{\ell}\cos\frac{k\pi x}{\ell}dx=\begin{cases}0,\ j\neq k;\\ 2\ell,\ k=j=0;\\ \ell,\ k=j=1,2,\cdots;\end{cases}$
- $\int_{-\ell}^{\ell}\sin\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx=\begin{cases}0,\ j\neq k;\\ \ell,\ k=j=1,2,\cdots;\end{cases}$

- $\int_{-\ell}^{\ell}\cos\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx=0,\ j=0,1,2,\cdots,k=1,2,\cdots$

证明： • $j=0,\ k=1,2,\cdots$时

$$\int_{-\ell}^{\ell}\sin\frac{k\pi x}{\ell}dx=-\frac{\ell}{k\pi}\cos\frac{k\pi x}{\ell}\Big|_{-\ell}^{\ell}=0;$$

- $j=k=1,2,\cdots$时

$$\int_{-\ell}^{\ell}\cos\frac{k\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx=-\frac{\ell}{4k\pi}\cos\frac{2k\pi x}{\ell}\Big|_{-\ell}^{\ell}=0;$$

- 当$k\neq j,\ j$、$k=1,2,\cdots$时

$$\int_{-\ell}^{\ell}\cos\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx=\int_{-\ell}^{\ell}\frac{1}{2}[\sin\frac{(k+j)\pi x}{\ell}-\sin\frac{(k-j)\pi x}{\ell})dx=0;$$

- $\int_{-\ell}^{\ell} \cos\frac{j\pi x}{\ell}\cos\frac{k\pi x}{\ell}dx = \begin{cases} 0,\ j \neq k; \\ 2\ell,\ k = j = 0; \\ \ell,\ k = j = 1, 2, \cdots; \end{cases}$

证明: •当$j = k = 0$时

$$\int_{-\ell}^{\ell} \cos\frac{j\pi x}{\ell}\cos\frac{k\pi x}{\ell}dx = \int_{-\ell}^{\ell} dx = 2\ell;$$

•当$j = k \neq 0$时

$$\int_{-\ell}^{\ell} \cos\frac{k\pi x}{\ell}\cos\frac{k\pi x}{\ell}dx = \int_{-\ell}^{\ell} \frac{1}{2}\left(1 + \cos\frac{2k\pi x}{\ell}\right) dx = \ell;$$

•当$j \neq k, j \neq 0, k \neq 0$时

$$\int_{-\ell}^{\ell} \cos\frac{k\pi x}{\ell}\cos\frac{k\pi x}{\ell}dx = \int_{-\ell}^{\ell} \frac{1}{2}\left(\cos\frac{(k+j)\pi x}{\ell} + \cos\frac{(k-j)\pi x}{\ell}\right) dx =$$

•当$j = 0, k \neq 0$时

$$\int_{-\ell}^{\ell} \cos\frac{k\pi x}{\ell}dx = 0;$$

- $\int_{-\ell}^{\ell} \sin\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx = \begin{cases} 0,\ j\neq k; \\ \ell,\ k=j=1,2,\cdots; \end{cases}$

证明: •当$j\neq k$时

$$\int_{-\ell}^{\ell} \sin\frac{j\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx = \int_{-\ell}^{\ell}\frac{1}{2}\left(\cos\frac{(k-j)\pi x}{\ell} - \cos\frac{(k+j)\pi x}{\ell}\right)dx = 0;$$

•当$j=k=1,2,\cdots$时

$$\int_{-\ell}^{\ell} \sin\frac{k\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx = \int_{-\ell}^{\ell}\frac{1}{2}\left(1-\cos\frac{2\pi x}{\ell}\right)dx = \ell;$$

傅里叶级数的定义

设$f(x)$是一个$T=2\ell$的周期函数，取

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx,\ n=0,1,2,\cdots;$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx,\ n=1,2,\cdots;$$

则三角级数

$$\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]$$

称为$T=2\ell$周期函数$f(x)$的傅里叶级数，记为

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]$$

问： $f(x)$的傅里叶级数的和函数是否为$f(x)$？即$S(x)=f(x)$？

狄利克雷定理

设$f(x)$是一个$T=2\ell$周期函数，

(1). 在$[-\ell,\ell]$内$f(x)$仅有有限个单调区间;

(2). 在$[-\ell,\ell]$内$f(x)$连续、或最多仅有有限个第一类间断点;

则$f(x)$ 的傅里叶级数

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]$$

收敛到f在x处的左、右极限的平均值, 即

$$\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]=S(x)=\frac{f(x-0)+f(x+0)}{2}.$$

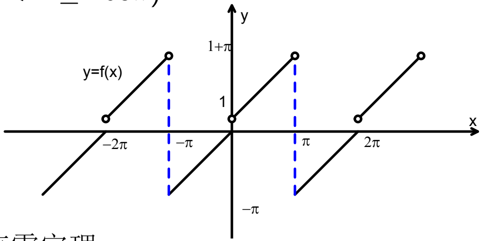
- 如果f在x处连续，$\Longrightarrow f(x-0)=f(x+0)=f(x)$; 从而

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]=S(x)=f(x).$$

- 如果f在x处第一间断点，

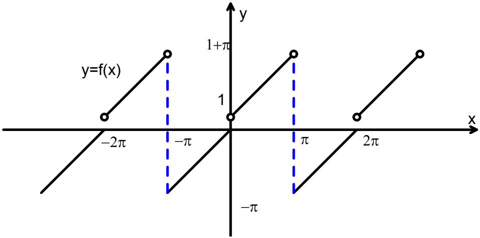
$$\begin{aligned} f(x) &\sim \frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right] \\ &= S(x)=\frac{f(x-0)+f(x+0)}{2}. \end{aligned}$$

例1(1). 已知级数$\frac{a_0}{2}+\sum_{n=1}^{+\infty}[a_n\cos nx+b_n\sin nx]$是$2\pi$周期函数$f(x)=\begin{cases} x, & -\pi\le x\le 0,\\ 1+x, & 0<x<\pi,\end{cases}$ 的傅里叶级数，傅里叶级数的和函数为$S(x)$，求$S(0)$? $S(\pi/2)$? $S(99\pi)$? $S(-\frac{97}{2}\pi)$? 以及$S(x)(101\pi<x\le 103\pi)$?



解：由狄利克雷定理，

$$
\begin{aligned}
S(0) &= \frac{f(0-0)+f(0+0)}{2}=\frac{1}{2},\ S(\pi/2)=f(\pi/2)=1+\pi/2\\
S(99\pi) &= S(50\times 2\pi-\pi)=S(-\pi)=\frac{f(-\pi-0)+f(-\pi+0)}{2}\\
&= \frac{f(\pi-0)+f(-\pi+0)}{2}=\frac{1}{2}
\end{aligned}
$$



$$S(-\frac{97}{2}\pi) = S(-24\times 2\pi - \pi/2) = S(-\pi/2) = -\pi/2;$$

$$S(x) = S(x - 102\pi + 102\pi)(\text{记}t = x - 102\pi, t \in [-\pi, \pi])$$

$$= S(t + 102\pi) = S(t) = \begin{cases} t, -\pi < t < 0; \\ 1 + t, 0 < t < \pi; \\ \frac{1}{2}, t = -\pi, t = 0, t = \pi; \end{cases}$$

$$= \begin{cases} x - 102\pi, 101\pi < x < 102\pi; \\ 1 + x - 102\pi, 102\pi < x < 103\pi; \\ \frac{1}{2}, x = 101\pi, x = 102\pi, x = 103\pi; \end{cases}$$

例1(2). $f(x)$是以2π为周期的函数,

在$(-\pi,\pi]$上$f(x)=\begin{cases} x, 0\le x\le\pi, \\ 0, -\pi<x<0, \end{cases}$,求$f(x)$ 的傅里叶级数,

以及级数在$x=\pi$ 或$x=\frac{3\pi}{2}$处的和。

解: $T=2\ell=2\pi$得$\ell=\pi$,

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx=\frac{1}{\pi}\int_0^{\pi}xdx=\frac{1}{2}\pi;$$

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_0^{\pi}x\cos nxdx$$

$$=\frac{1}{n\pi}x\sin nx\Big|_0^{\pi}-\int_0^{\pi}\frac{1}{n\pi}\sin nxdx=\frac{(-1)^n-1}{\pi n^2}, n=1,2,\cdots;$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_0^{\pi}x\sin nxdx$$

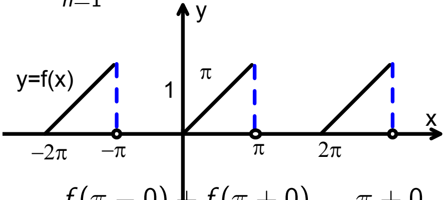
$$=-\frac{1}{n\pi}x\cos nx\Big|_0^{\pi}+\int_0^{\pi}\frac{1}{n\pi}\cos nxdx=\frac{(-1)^{n-1}}{n}, n=1,2,\cdots;$$

$T=2\ell=2\pi$得$\ell=\pi$,

$$a_0=\frac{1}{2}\pi,\ a_n=\frac{(-1)^n-1}{\pi n^2},\ b_n=\frac{(-1)^{n-1}}{n},$$

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right];$$

$$f(x)\sim\frac{1}{4}\pi+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{(-1)^{n-1}}{n}\sin nx\right]$$



$$S(\pi)=\frac{f(\pi-0)+f(\pi+0)}{2}=\frac{\pi+0}{2}=\frac{\pi}{2},$$

$$S(\frac{3\pi}{2})=S(-\frac{\pi}{2})=\frac{f(-\frac{\pi}{2}-0)+f(-\frac{\pi}{2}+0)}{2}=0.$$

例1(3). $f(x)$是以4为周期的函数，在$(0,4]$上的定义为$f(x)=\begin{cases}1, 0<x\leq 2,\\ 0, 2<x\leq 4,\end{cases}$，求$f(x)$的傅里叶级数，以及级数在$x=16$处的和。

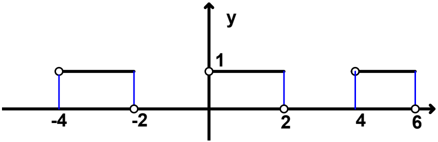
解： $T=2\ell=4$得$\ell=2$,

$$\begin{aligned}
a_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx=\frac{1}{2}\int_{0}^{2\ell} f(x)dx=\frac{1}{2}\int_{0}^{2} dx=1;\\
a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\ell}\int_{0}^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx\\
&= \frac{1}{2}\int_{0}^{4} f(x)\cos\frac{n\pi x}{2}dx=\frac{1}{2}\int_{0}^{2}\cos\frac{n\pi x}{2}dx=0, n=1,2,\cdots;\\
b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{2}\int_{0}^{4} f(x)\sin\frac{n\pi x}{2}dx\\
&= \frac{1}{2}\int_{0}^{2}\sin\frac{n\pi x}{2}dx=\frac{1}{n\pi}\left[1-(-1)^n\right], n=1,2,\cdots;
\end{aligned}$$

$\ell = 2,\ a_0 = 1,\ a_n = 0,\ b_n = \frac{1}{n\pi}\left[1-(-1)^n\right], n = 1, 2, \cdots;$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right]$$

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi x}{2}.$$



$$S(18) = S(4\times 4+2) = S(2) = \frac{f(2-0)+f(2+0)}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

二、有限区间上的傅里叶级数

- 大量实际问题中出现的函数不是周期函数，如何推广傅里叶级数到一般非周期函数？

基本思想： 给定区间$(-\ell,\ell]$上的函数$f(x)$（延拓）$\Longrightarrow$ 延拓为$T=2\ell$ 周期函数$F(x)$, 使得在$(-\ell,\ell]$上$F(x)=f(x)$ $\Longrightarrow$求出$F(x)$ 的傅里叶级数

$$F(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right]$$

把$F(x)$的傅里叶级数限制在区间$(-\ell,\ell]$上得$f(x)$的傅里叶级数，即在区间$(-\ell,\ell]$上

$$f(x)=F(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right].$$

如何求傅里叶级数的系数？
给定区间$(-\ell,\ell]$上的函数$f(x)$，$T=2\ell$周期函数$F(x)$满足$F(x)=f(x), x\in(-\ell,\ell]$;

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx, n=0,1,\cdots$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx, n=1,\cdots$$

在区间$(-\ell,\ell]$上

$$f(x)=F(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right].$$

定理： 给定区间$(-\ell,\ell]$上的函数$f(x)$, 取

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx, n = 0, 1, \cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx, n = 1, \cdots$$

则在有限区间$(-\ell,\ell]$上函数$f(x)$的傅里叶级数为

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\right].$$

例2(1). 求$f(x)=\begin{cases} x,\ -\pi\le x<0 \\ 0,\ 0\le x\le\pi \end{cases}$ 的区间$(-\pi,\pi]$ 上的傅里叶级数、及级数的和$S(\pi/2)$, $S(\pi)$, $S(11\pi)$, $S(x)(101\pi\le x\le 102\pi)$?

解：$\ell=\pi$,

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}xdx=-\frac{\pi}{2},$$

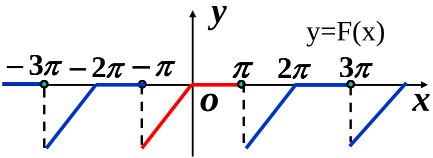
$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{0}x\cos nxdx=\frac{1-(-1)^n}{n^2\pi},$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{0}x\sin nxdx=\frac{(-1)^{n+1}}{n}.$$

$$f(x)\sim-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right];$$

$$f(x) \sim -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2\pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right] = S(x);$$

注意：$S(x)$是$f(x)$的2π周期延拓函数$F(x)$的级数之和；



$$S(\pi/2) = 0, \; S(\pi) = -\frac{\pi}{2},$$

$$S(11\pi) = S(2 \times 2\pi + \pi) = S(\pi) = -\frac{\pi}{2}.$$

$$S(x) = S(x - 102\pi) = \begin{cases} x - 102\pi, \; 101\pi < x \leq 102\pi \\ -\frac{\pi}{2}, \; x = 101\pi \end{cases}$$

例2(2). 求$f(x)=\pi^2-x^2$的区间$(-\pi,\pi]$上的傅里叶级数及它的和$S(\pi/2)$,$S(\pi)$?

解：$\ell=\pi$；

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)dx=\frac{4}{3}\pi^2,$$

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\cos nxdx=\frac{4(-1)^{n-1}}{n^2},$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\sin nxdx=0.$$

$$\Longrightarrow f(x)\sim\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4(-1)^{n-1}}{n^2}\cos nx$$

例2(2). 求$f(x)=\pi^2-x^2$的区间$(-\pi,\pi]$上的傅里叶级数及它的和$S(\pi/2)$,$S(\pi)$?

$f(x)=\pi^2-x^2$的2π周期延拓为$F(x)$, 则

$$S(\pi/2)=\frac{f(\pi/2-0)+f(\pi/2+0)}{2}=\frac{3}{4}\pi^2,$$

$$S(\pi)=\frac{F(\pi-0)+F(\pi+0)}{2}=\frac{f(\pi-0)+f(-\pi+0)}{2}=0,$$

$$S(-3\pi)=S(-4\pi+\pi)=S(\pi)=0.$$

例2(3). 求$f(x)=x$的区间$(2,4]$上的傅里叶级数及它的和$S(1/2)$, $S(4)$及$S(x)(17\le x\le 18)$?

解：$4-2=T=2\ell$得$\ell=1$; $f(x)=x(2<x\le 4)$的$T=2$周期延拓为$F(x)$,

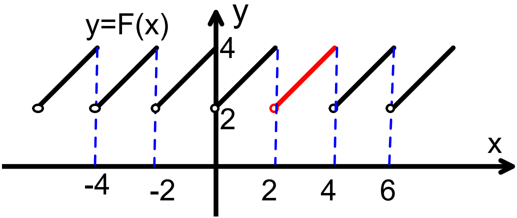
$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)dx=\int_2^4F(x)dx=\int_2^4xdx=6,$$

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\cos\frac{n\pi x}{\ell}dx=\int_2^4x\cos n\pi xdx=0,$$

$$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\sin\frac{n\pi x}{\ell}dx=\int_2^4x\sin n\pi xdx=-\frac{2}{n\pi}.$$

当$2<x\le 4$时，$f(x)=F(x)\sim 3+\sum_{n=1}^{\infty}\frac{-2}{n\pi}\sin n\pi x$;

当$2 < x \leq 4$时，$f(x) = F(x) \sim 3 + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin n\pi x = S(x)$;



$$S(1/2) = S(2 + \frac{1}{2}) = \frac{F(5/2-0) + F(5/2+0)}{2} = f(5/2) = \frac{5}{2},$$

$$S(4) = \frac{F(4-0) + F(4+0)}{2} = \frac{f(4-0) + f(2+0)}{2} = 3,$$

$$S(-2) = S(-2 + 3 \times 2) = S(4) = 3.$$

$$S(x) = S(x-14) = \begin{cases} x - 14, \ 17 \leq x < 18 \\ 3, \ x = 18 \end{cases}$$

三、正弦级数与余弦级数

正弦级数与余弦级数

给定区间$[-\ell, 0)$上的函数$f(x)$,能否仅用正弦函数系

$$\sin\frac{\pi x}{\ell}, \sin\frac{2\pi x}{\ell}, \cdots, \sin\frac{n\pi x}{\ell}, \cdots$$

来表示？ 或仅用余弦函数系

$$1, \cos\frac{\pi x}{\ell}, \cos\frac{2\pi x}{\ell}, \cdots, \cos\frac{n\pi x}{\ell}, \cdots$$

来表示？

$$\text{正弦级数}: f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell},$$

$$\text{余弦级数}: f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}.$$

$$正弦级数: f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell},$$

正弦级数： 给定$[-\ell, 0]$上的函数$f(x) \Longrightarrow$(奇延拓)$\tilde{f}(x) = \begin{cases} f(x), -\ell \le x \le 0, \\ -f(-x), 0 < x < \ell, \end{cases} \Rightarrow$ (周期$T = 2\ell$ 延拓)$F(x)$

$\Longrightarrow F(x)$的傅里叶级数

$$F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right],$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} F(x) \cos \frac{n\pi x}{\ell} dx = 0, n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} F(x) \sin \frac{n\pi x}{\ell} dx = \frac{2}{\ell} \int_{-\ell}^{0} f(x) \sin \frac{n\pi x}{\ell} dx, n = 1, 2, \cdots$$

$\Longrightarrow$限制在区间$[-\ell, 0]$上$f(x) = F(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$.

定理： 给定$[-\ell, 0]$上的函数$f(x)$, 取

$$b_n = \frac{2}{\ell}\int_{-\ell}^{0} f(x)\sin\frac{n\pi x}{\ell}dx, n = 1, 2, \cdots$$

在区间$[-\ell, 0]$上函数$f(x)$的正弦级数为

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}.$$

定理： 给定$[0, \ell]$上的函数$f(x)$, 取

$$b_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx, n = 1, 2, \cdots$$

在区间$[0, \ell]$上函数$f(x)$的正弦级数为$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$.

$$余弦级数: f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell},$$

余弦级数： 给定$[-\ell, 0]$上的函数$f(x) \Longrightarrow$(偶延拓)

$\tilde{f}(x) = \begin{cases} f(x), -\ell \leq x \leq 0, \\ f(-x), 0 < x < \ell, \end{cases} \Longrightarrow$ (周期$T = 2\ell$ 延拓)$F(x) \Longrightarrow$

$F(x)$的傅里叶级数

$$F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right],$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x) \cos\frac{n\pi x}{\ell} dx = \frac{2}{\ell}\int_{-\ell}^{0} f(x) \cos\frac{n\pi x}{\ell} dx, n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x) \sin\frac{n\pi x}{\ell} dx = 0, n = 1, 2, \cdots$$

$\Longrightarrow$限制在区间$[-\ell, 0]$上

$$f(x) = F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell};$$

定理： 给定$[-\ell,0]$上的函数$f(x)$，取

$$a_n=\frac{2}{\ell}\int_{-\ell}^{0}f(x)\cos\frac{n\pi x}{\ell}dx,\ n=0,1,2,\cdots$$

在区间$[-\ell,0]$上函数$f(x)$的余弦级数为

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}.$$

定理： 给定$[0,\ell]$上的函数$f(x)$，取

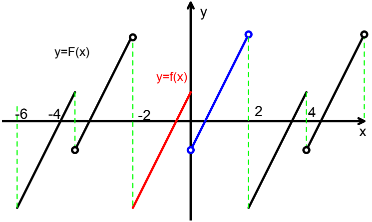
$$a_n=\frac{2}{\ell}\int_{0}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx,\ n=0,1,2,\cdots$$

在区间$[0,\ell]$上函数$f(x)$的余弦级数为

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}.$$

例3(1). 在区间$[-2,0]$上$f(x)=1+2x$, 而$f(x)$的正弦级数$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$的和函数为$S(x)$, 求$S(-1),S(2)$, $S(x)(16\le x\le 18)$?

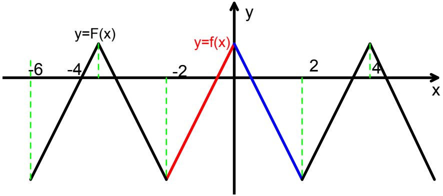
解：$f(x)$的奇延拓得$\tilde{f}$, 再以$T=4$为周期延拓得$F(x)$,



$$
\begin{aligned}
S(-1) &= \frac{F(-1-0)+F(-1+0)}{2} = -1;\\
S(1) &= -S(-1)=1;\ \ S(2)=-S(-2)=0;\\
S(x) &= S(x-16) = -S(16-x) = \begin{cases} -f(16-x), -2<16-x<0 \\ 0,\ 16-x=-2\text{或}0 \end{cases}\\
&= \begin{cases} 2x-33,\ 16<x<18 \\ 0,\ x=16\text{或}18 \end{cases}
\end{aligned}
$$

例3(2). 在区间$[-2,0]$上$f(x)=1+2x$, 而$f(x)$的余弦级数$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}$的和函数为$S(x)$, 求$S(-1)$,$S(2)$, $S(x)(16\le x\le 18)$?

解：$f(x)$的偶延拓得$\tilde{f}$, 再以$T=4$为周期延拓得$F(x)$,



$$
\begin{aligned}
S(-1) &= \frac{F(-1-0)+F(-1+0)}{2}=-1;\\
S(1) &= S(-1)=-1;\ S(2)=S(-2)=-3;\\
S(x) &= S(x-16)=S(16-x)=f(16-x)=33-2x;
\end{aligned}
$$

例3(3). $f(x)=\cos 2x(0\le x\le 3)$, 试求$f(x)$的正弦与余弦级数?

解: $\ell=3$,

•正弦级数:

$$b_n=\frac{2}{3}\int_0^3\cos 2x\sin\frac{n\pi x}{3}dx=\frac{1}{3}\int_0^3[\sin(\frac{n\pi}{3}+2)x-\sin(\frac{n\pi}{3}-2)x]dx$$

$$\Longrightarrow\quad b_n=\frac{12((-1)^n\cos 6-1)}{n^2\pi^2-36},$$

$$f(x)\sim\sum_{n=1}^{\infty}\frac{12((-1)^n\cos 6-1)}{n^2\pi^2-36}\sin\frac{n\pi x}{3}.$$

例3(3). $f(x)=\cos 2x(0\le x\le 3)$, 试求$f(x)$的正弦与余弦级数?

•余弦级数: $\ell=3$,

$$
\begin{aligned}
a_0 &= \frac{2}{3}\int_0^3 \cos 2x dx = \frac{1}{3}\sin 6; \\
a_n &= \frac{2}{3}\int_0^3 \cos 2x \cos\frac{n\pi x}{3} dx \\
&= \frac{1}{3}\int_0^3 \left[\cos(\frac{n\pi}{3}+2)x+\cos(\frac{n\pi}{3}-2)x\right] dx \\
&= \frac{2n\pi(-1)^{n-1}\sin 6}{n\pi^2-36}, n=1,2,\cdots
\end{aligned}
$$

$$
f(x)\sim \frac{1}{6}\sin 6+\sum_{n=1}^{\infty}\frac{2n\pi(-1)^{n-1}\sin 6}{n^2\pi^2-36} cos\frac{n\pi x}{3}.
$$

例3(4). 试将函数$f(x) = x(0 < x \leq 2)$展开成正弦与余弦级数？并求级数$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$与$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$之和。

解：•正弦级数: $\ell = 2$,

$$b_n = \frac{2}{2}\int_0^2 x \sin \frac{n\pi x}{2} dx = -\frac{4}{n\pi}\cos n\pi = (-1)^{n+1}\frac{4}{n\pi}, \ n = 1, 2, \cdots;$$

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi} \sin \frac{n\pi x}{2}.$$

记$A(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi} \sin \frac{n\pi x}{2}$, 则

$$A(1) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi} \sin \frac{n\pi}{2}$$

$$= \sum_{k=0}^{\infty} (-1)^{(2k+1)+1}\frac{4}{(2k+1)\pi} \sin(k\pi + \frac{\pi}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k 4}{(2k+1)\pi};$$

$$\Longrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}A(1) = \frac{\pi}{4};$$

例3(4). 试将函数$f(x)=x(0<x\le 2)$展开成正弦与余弦级数？并求级数$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$与$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$之和。

解：•余弦级数: $\ell=2,\quad a_0=\frac{2}{2}\int_0^2 xdx=2;$

$$a_n=\frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx=\frac{4}{n^2\pi^2}(\cos n\pi-1)=\frac{4[(-1)^n-1]}{n^2\pi^2},n=1,2,\cdots$$

$$f(x)\sim 1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}cos\frac{n\pi x}{2}.$$

记$B(x)=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}cos\frac{n\pi x}{2}$, 则

$$B(0)=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}=1+\sum_{k=0}^{\infty}\frac{4\cdot(-2)}{(2k+1)^2\pi^2};$$

$$\Longrightarrow\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{1-B(0)}{8}\pi^2=\frac{1}{8}\pi^2;$$