多元函数的微分学

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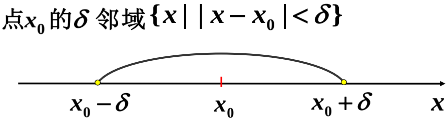
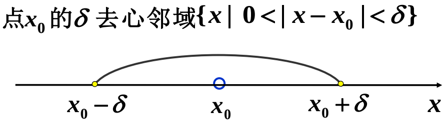
一. 多维空间、邻域

1. 一维空间$\mathbb{R} = \{x: -\infty < x < \infty\}$;

- 邻域$U(x_0, \delta) = \{x: |x - x_0| < \delta\}$;



- 去心邻域$\{x: 0 < |x - x_0| < \delta\}$;



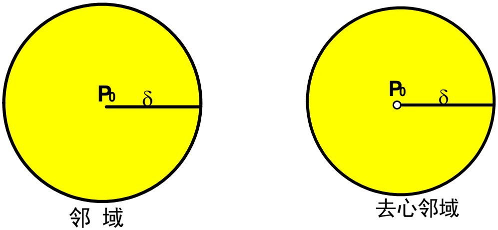
2. 二维空间$\mathbb{R}^2 = \{(x, y): -\infty < x < \infty, -\infty < y < \infty\}$;

记$P(x, y)$、$P_0(x_0, y_0)$为2维空间点,

$$|PP_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

为二点之间距离;

- 邻域$U(P_0, \delta) = \{P(x, y): |PP_0| < \delta\}$;



- 去心邻域$\{P(x, y): 0 < |PP_0| < \delta\}$;

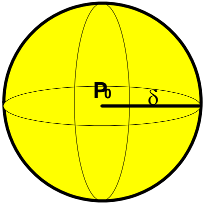
3. n维空间$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_j \in \mathbb{R}, j = 1, 2, \cdots, n.\}$;

记$P(x_1, x_2, \cdots, x_n)$、$P_0(x_1^0, x_2^0, \cdots, x_n^0)$为$n$维空间点;

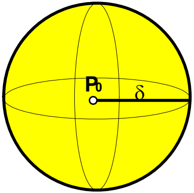
$$|PP_0| = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \cdots + (x_n - x_n^0)^2}$$

为二点之间距离;

- 邻域$U(P_0, \delta) = \{P(x, y) : |PP_0| < \delta\}$;



邻 域



去心邻域

- 去心邻域$\{P(x, y) : 0 < |PP_0| < \delta\}$;

定义： 给定平面点集E:

• $P_0(x_0,y_0)$称为点集E的内点：如果存在P_0的某邻域$U(P_0,\delta)\subseteq E$; 点集E的所有内点组成点集E的内点集E^o.

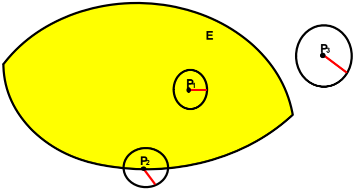
• $P_0(x_0,y_0)$称为点集E的外点：如果存在P_0的某邻域$U(P_0,\delta)$使得$E\cap U(P_0,\delta)=\varnothing$;

• $P_0(x_0,y_0)$称为点集E的边界点：如果存在P_0不是点集E的内点、也不是点集E的外点；点集E的所有边界点组成点集E的边界∂E。

- P_1集合E的内点；

  P_2集合E的边界点；

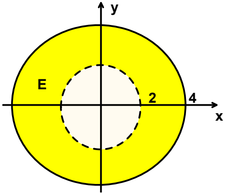
  P_3集合E的外点;



例1(1). 给定点集

$E = \{(x,y):\ 1 < x^2 + y^2 \le 4\}$,

求点集E的内点、外点及边界点？



解： 点集E的内点集$E^0 = \{(x,y):\ 1 < x^2 + y^2 < 4\}$, 点集$E$的外点集为

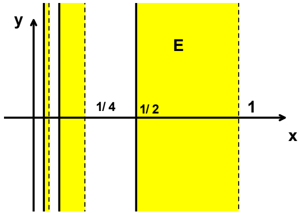
$$\{(x,y):\ x^2 + y^2 > 4\text{或}x^2 + y^2 < 1\},$$

点集E的边界$\partial E = \{(x,y):\ x^2 + y^2 = 4\text{或}x^2 + y^2 = 1\}$.

例1(2). 给定点集

$$E=\{(x,y):\ y\in\mathbb{R},\frac{1}{2\cdot 4^n}\le x<\frac{1}{4^n},n=0,1,2,\cdots\}$$

求点集E的内点及边界点？



解： 点集E的内点集

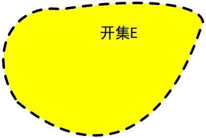
$$E^0=\{(x,y):\ y\in\mathbb{R},\frac{1}{2\cdot 4^n}<x<\frac{1}{4^n},n=0,1,2,\cdots\};$$

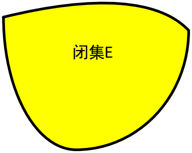
点集E的边界

$$\partial E=\{(x,y):\ y\in\mathbb{R},x=0\text{或}x=\frac{1}{2\cdot 4^n}\text{或}x=\frac{1}{4^n},n=0,1,2,\cdots\};$$

定义： 给定平面点集E:

- 点集E称为开集：如果内点集$E^o = E$, 即E中每点都是点集E的内点（或点集E的边界点全不属于E）；

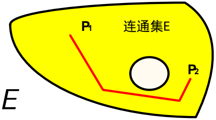




- 点集E称为闭集：如果点集E的边界点全属于E，即$\partial E \subset E$;

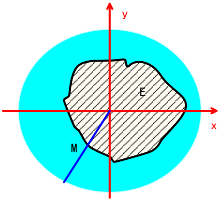
定义： 给定平面点集E:

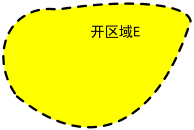
- 点集E称为连通的：如果对E中任何二点P_1、P_2, 总存在完全属于点集E的折线L连接这二点P_1、P_2;



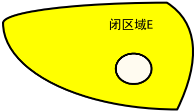
定义： 给定平面点集E:

- 点集E称为有界的：如果存在常数M，使得E中每点到原点的距离不超过M;





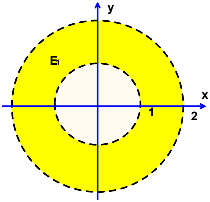
- 点集E称为开区域：如果点集E为开集且连通的；
- 点集E称为闭区域：如果点集E为闭集且连通的；

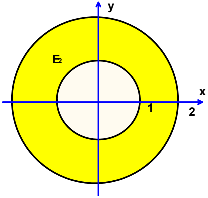


例1(3). (1). 点集$E_1=\{(x,y):1<x^2+y^2<4\}$的内点集

$$E_1^o=\{(x,y):1<x^2+y^2<4\}=E_1$$

从而点集E_1为开集；由于E_1是连通的，从而E_1为有界的开区域；





(2). 点集$E_2=\{(x,y):1\le x^2+y^2\le 4\}$的边界

$$\partial E_2=\{(x,y):x^2+y^2=4\text{或}x^2+y^2=1\}\subset E_2$$

从而点集E_2为闭集；由于E_2是连通的，从而E_1为有界的闭区域；

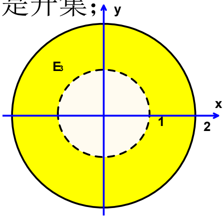
例1(3). (3). 点集$E_3=\{(x,y):\ 1<x^2+y^2\le 4\}$的边界

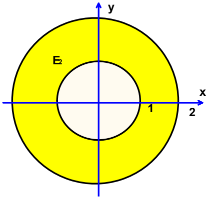
$$\partial E_3=\{(x,y):\ x^2+y^2=4\text{或}x^2+y^2=1\}\not\subset E_3$$

点集E_3为不是闭集；点集E_3的内点集

$$E_3^o=\{(x,y):\ 1<x^2+y^2<4\}\neq E_3$$

点集E_3为不是开集；





(4). 点集$E_4=\{(x,y):\ x^2+y^2<1\text{或}(x-2)^2+y^2<1\}$的内点集

$$E_4^o=\{(x,y):\ x^2+y^2<1\text{或}(x-2)^2+y^2<1\}=E_4$$

点集E_4为是开集；由于E_4不连通的，从而E_4为开集、但不是开区域；

定义： 给定n维空间中集合D及法则f，如果对于任何$P(x_1, x_2, \cdots, x_n) \in D$, 由法则$f$有且仅有一个实数$y$与$P(x_1, x_2, \cdots, x_n)$对应，则称法则$f$是定义在$D$上的$n$元函数，

定义： 给定n维空间中集合D及法则f，如果对于任何$P(x_1, x_2, \cdots, x_n) \in D$，由法则$f$有且仅有一个实数$y$与$P(x_1, x_2, \cdots, x_n)$对应，则称法则$f$是定义在$D$上的$n$元函数，记为$y = f(x_1, x_2, \cdots, x_n)$.

注： $(x_1, x_2, \cdots, x_n)$称为自变量，y为因变量，D为定义域。

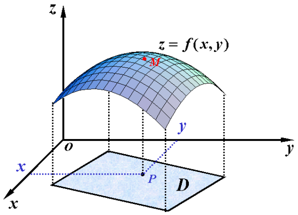
如：$z = x^2 + 3y^2$、$z = \sin(xy^3)$、$z = \begin{cases} 1, x \geq y\text{时}, \\ \sin x, x < y\text{时}, \end{cases}$ 都是二元函数；

$w = x^2 z + 3y^2$、$w = (x + y - z)\sin(xy^3)$都是三元函数；

注： 下面我们以二元函数$z = f(x, y)$为例来讨论，其结论对一般多元函数基本上仍适用。

注： (几何意义) 二元函数$z=f(x,y)$

$\Leftrightarrow$空间中的曲面$\Sigma: z-f(x,y)=0$;



注： 多元初等函数$\Leftrightarrow$由基本初等函数经过有限次四则运算、复合运算所得函数;

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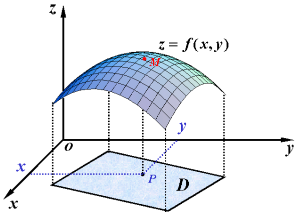


注： 多元初等函数$\Leftrightarrow$由基本初等函数经过有限次四则运算、复合运算所得函数；如函数
$z=\sin(xy)+e^{x+5y}$、$y=\ln(x^3y^2+x\sin y)$;

定义： 假设函数$z=f(x,y)$在$P_0(x_0,y_0)$的某去心邻域内有定义。如果点$P(x,y)$趋于定点$P_0(x_0,y_0)$时相应地函数值$f(x,y)$越来越接近（或等于）常数A, 则称$(x,y)\to(x_0,y_0)$ 时函数$f(x,y)$有极限A, 记为

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A, \text{或} \lim_{x\to x_0,y\to y_0} f(x,y)=A.$$

- $0<|PP_0|=\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$
  $\Leftrightarrow P(x,y)$趋于定点$P_0(x_0,y_0)$;
- $|f(x,y)-A|<\epsilon \Leftrightarrow f(x,y)$越来越接近（或等于）$A$;

注： （$\epsilon-\delta$定义）对任何$\epsilon>0$, 存在$\delta>0$, 当$P(x,y)$满足

$$0<|PP_0|=\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

时必有$|f(x,y)-A|<\epsilon$，则$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A$.

例2(1). $f(x,y) = xy\sin\left(\frac{1}{x^2+y^4}\right)$, 求证:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

解. 对任何$\epsilon > 0$, 取$\delta = \sqrt{\epsilon/2}$, 当$P(x,y)$满足

$$0 < |PP_0| = \sqrt{x^2+y^2} < \delta$$

时, 有

$$|f(x,y) - 0| = \left| xy\sin\left(\frac{1}{x^2+y^4}\right)\right| \leq \frac{1}{2}(x^2+y^2) < \frac{1}{2}\delta^2 = \epsilon.$$

由定义得$\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

例2(2). $f(x,y)=x^2+xy$, 求证: $\lim_{(x,y)\to(1,0)} f(x,y)=1$.

解. 对任何$\epsilon>0$, 取$\delta=\min\{\frac{1}{2},\frac{\epsilon}{6}\}$, 当$P(x,y)$满足

$$0<|PP_0|=\sqrt{(x-1)^2+y^2}<\delta$$

时, 有$|x-1|<\delta$, $|y|<\delta \Rightarrow$

$$|f(x,y)-1|=\left|x^2+xy-1\right|\le 3|x-1|+2|y|<5\delta\le\epsilon.$$

由定义得$\lim_{(x,y)\to(1,0)} f(x,y)=1$.

注： 多元函数极限性质、运算法则等完全类似于一元函数，如极限的四则运算、复合运算、夹逼准则等（除单调有界准则，为什么？）。

例2(3). $\lim_{(x,y)\to(0,2)} \frac{\sin xy}{(x^2+y)x}$

$= \lim_{(x,y)\to(0,2)} \frac{\sin xy}{xy} \cdot \lim_{(x,y)\to(0,2)} \frac{y}{x^2+y} = 1 \times \frac{2}{2} = 1;$

例2(4). $\lim_{(x,y)\to(\infty,2)} \left(1 + \frac{2}{x}\right)^{\frac{x^2}{x+y}}$

$$= \lim_{(x,y)\to(\infty,2)} \left[\left(1 + \frac{2}{x}\right)^{\frac{x}{2}}\right]^{\frac{2x}{x+y}}$$

$$= e^{\lim_{(x,y)\to(\infty,2)} \frac{2x}{x+y}} = e^{\lim_{(x,y)\to(\infty,2)} \frac{2}{1+y/x}} = e^2;$$

例2(5). $\lim_{(x,y)\to(0,2)} \frac{x\sin(xy^2)}{\sqrt[n]{1+x^2y}-1} = \lim_{(x,y)\to(0,2)} \frac{x\cdot(xy^2)}{\frac{x^2y}{n}}$

$= \lim_{(x,y)\to(\infty,2)} ny = 2n;$

例2(6). 求$\lim_{(x,y)\to(+\infty,+\infty)}(x^2+3y^2)e^{-(x+2y)}$?

解.当$x\geq 10$、$y\geq 10$时有$0\leq \frac{x^2+3y^2}{(x+2y)^2}\leq \frac{x^2+3y^2}{x^2+4y^2}\leq 1$.
记$t=x+2y$, 则

$$\lim_{(x,y)\to(+\infty,+\infty)}(x+2y)^2e^{-(x+2y)}=\lim_{t\to+\infty}\frac{t^2}{e^t}=0;$$

当$(x,y)\to(+\infty,+\infty)$时$(x+2y)^2e^{-(x+2y)}$是无穷小量、$\frac{x^2+3y^2}{(x+2y)^2}$是有界量；从而$(x^2+3y^2)e^{-(x+2y)}$是无穷小量，即

$$\lim_{(x,y)\to(+\infty,+\infty)}(x^2+3y^2)e^{-(x+2y)}=0.$$

例2(7). 求$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$?

解. 由$0 \le \left|\frac{x^2y}{x^2+y^2}\right| \le \frac{|x|\cdot\frac{1}{2}(x^2+y^2)}{x^2+y^2} = \frac{1}{2}|x|$; 及

$$\lim_{(x,y)\to(0,0)} 0 = 0, \quad \lim_{(x,y)\to(0,0)} \frac{1}{2}|x| = 0,$$

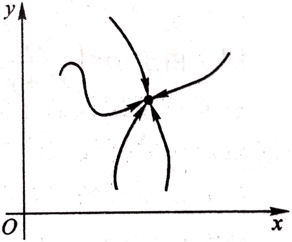
利用夹逼准则

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^2y}{x^2+y^2}\right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

注： 多元函数极限的讨论一般比较困难。

定义： 假设函数$z=f(x,y)$在$P_0(x_0,y_0)$的某去心邻域内有定义。如果点$P(x,y)$趋于定点$P_0(x_0,y_0)$时相应地函数值$f(x,y)$越来越接近（或等于）某常数A, 则称$(x,y)\to(x_0,y_0)$时函数$f(x,y)$有极限A,记为$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A$;

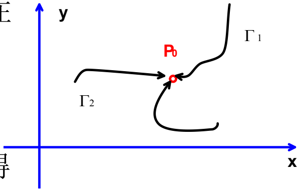
注： 多元函数极限的定义中，要求点$P(x,y)$沿任何方向、任何路径趋于$P_0(x_0,y_0)$时$f(x,y)$越来越接近（或等于）同一个常数A，即



$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A \quad\Leftrightarrow\quad \lim_{(x,y)\in\Gamma,(x,y)\to(x_0,y_0)} f(x,y)=A,$$

Γ是通过$P_0(x_0,y_0)$的任何一条曲线.

定理：（极限不存在的准则）如果存在通过$P_0(x_0,y_0)$的曲线Γ_1使得$\lim_{(x,y)\in\Gamma_1,(x,y)\to(x_0,y_0)} f(x,y)$不存在，或存在通过$P_0(x_0,y_0)$的曲线$\Gamma_1$、$\Gamma_2$使得

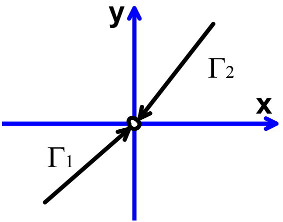


$$\lim_{(x,y)\in\Gamma_1,(x,y)\to(x_0,y_0)} f(x,y)=A,$$

$$\lim_{(x,y)\in\Gamma_2,(x,y)\to(x_0,y_0)} f(x,y)=B,\ A\neq B.$$

则$P(x,y)\to P_0(x_0,y_0)$时$f(x,y)$无极限。

例2(4). 求证：$(x,y)\to(0,0)$时 $f(x,y)=\frac{xy}{x^2+y^2}$无极限。



证明: 取$\Gamma_1: y=x$、$\Gamma_2: y=2x$它们都通过原点$(0,0)$，而

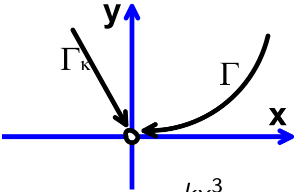
$$\lim_{(x,y)\in\Gamma_2,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=2x} f(x,y)=\lim_{x\to0}\frac{2x^2}{x^2+4x^2}=\frac{2}{5},$$

$$\lim_{(x,y)\in\Gamma_1,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=x} f(x,y)=\lim_{x\to0}\frac{x^2}{x^2+x^2}=\frac{1}{2}.$$

由定理，$(x,y)\to(0,0)$时$f(x,y)=\frac{xy}{x^2+y^2}$无极限。

例2(5). 求证：$(x,y)\to(0,0)$时

$f(x,y)=\frac{x^2y}{x^4+y^2}$无极限。



证明:取$\Gamma_k: y=kx$(k为实数),它们都通过原点$(0,0)$;

$$\lim_{(x,y)\in\Gamma_k,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=kx} f(x,y)=\lim_{x\to0}\frac{kx^3}{x^4+k^2x^2}=0,$$

(问：沿所有方向趋于$(0,0)$时函数$f(x,y)$都趋于$A=0$, 能否说明$\lim_{(x,y)\to(0,0)} f(x,y)=0$? 为什么？)

取$\Gamma_2: y=x^2$, 它通过原点$(0,0)$,

$$\lim_{(x,y)\in\Gamma_2,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=x^2} f(x,y)=\lim_{x\to0}\frac{x^4}{x^4+x^4}=\frac{1}{2}.$$

由定理，$(x,y)\to(0,0)$时$f(x,y)=\frac{x^2y}{x^4+y^2}$无极限。

例2(6). $f(x,y)=\frac{x^2y}{|x|^\beta+y^2}$，证明：$\beta\ge 4$时极限$\lim_{(x,y)\to(0,0)} f(x,y)$ 不存在？$0<\beta<4$时极限$\lim_{(x,y)\to(0,0)} f(x,y)$存在？并求极限。

解: 当$\beta>4$时，存在通过原点$O(0,0)$的曲线$\Gamma: y=|x|^{\beta/2}$, 满足

$$\lim_{(x,y)\in\Gamma,(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{1}{2}|x|^{2-\frac{\beta}{2}}=\infty,$$

由归结原理知：$\lim_{(x,y)\to(0,0)} f(x,y)$ 不存在；
当$\beta=4$时，存在通过原点$O(0,0)$的曲线$\Gamma_1: y=x^2$及直线$\Gamma_2: y=0$, 满足

$$\lim_{(x,y)\in\Gamma_1,(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{1}{2}=\frac{1}{2},$$

$$\lim_{(x,y)\in\Gamma_2,(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{0}{x^4}=0,$$

由归结原理知：$\lim_{(x,y)\to(0,0)} f(x,y)$ 不存在；

例2(6). $f(x,y)=\frac{x^2y}{|x|^\beta+y^2}$，证明: $\beta\geq 4$时极限$\lim_{(x,y)\to(0,0)} f(x,y)$ 不存在？ $0<\beta<4$时极限$\lim_{(x,y)\to(0,0)} f(x,y)$存在？并求极限。

解: 当$0<\beta<4$时，有$0\leq |f(x,y)|\leq \frac{1}{2}|x|^{2-\frac{\beta}{2}}$且

$$\lim_{(x,y)\to(0,0)} 0=0,\quad \lim_{(x,y)\to(0,0)} \frac{1}{2}|x|^{2-\frac{\beta}{2}}=0;$$

由夹逼准则，$\lim_{(x,y)\to(0,0)} f(x,y)=0$.

极限$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$，或$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$称为二元函数$f(x,y)$当$(x,y)\to(x_0,y_0)$时的**累次极限**.

如：$\lim_{y\to 1}\lim_{x\to 0}\frac{x}{\sin(xy+x^3y^2)} = \lim_{y\to 1}\lim_{x\to 0}\frac{x}{xy+x^3y^2}$

$$= \lim_{y\to 1}\frac{1}{y} = 1.$$

注： 当$(x,y)\to(x_0,y_0)$时（二重）极限与累次极限是否存在没有必然的联系；如

$(x,y)\to(0,0)$时$f(x,y)=\frac{x^2y}{x^4+y^2}$无（二重）极限，但是

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2y}{x^4+y^2} = \lim_{y\to 0}\frac{0}{y^2} = 0;$$

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{0}{x^4} = 0;$$

累次极限

如：由夹逼准则, $\lim_{(x,y)\to(0,0)} x\sin\frac{1}{y} = 0$, 但是

$$\lim_{x\to 0}\lim_{y\to 0} x\sin\frac{1}{y}\text{不存在};$$

$$\lim_{y\to 0}\lim_{x\to 0} x\sin\frac{1}{y} = \lim_{y\to 0} 0 = 0;$$

定理： 如果

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y),\quad \lim_{x\to x_0}\lim_{y\to y_0} f(x,y),\quad \lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

都存在，则它们必相等，即

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{y\to y_0}\lim_{x\to x_0} f(x,y).$$

自变量增量:

$$\Delta x = x - x_0,\ \Delta y = y - y_0;$$

因变量(函数值)增量:

$$\Delta z = f(x,y) - f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0);$$

问： $\Delta x \approx 0,\ \Delta y \approx 0$时有$\Delta z \approx 0$?

$$\Leftrightarrow\ \Delta x \to 0,\ \Delta y \to 0\text{时有}\Delta z \to 0?$$

$\Leftrightarrow \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \Delta z = 0$? 或等价地 $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$?

定义： 函数$f(x,y)$在定点$P_0(x_0,y_0)$的某个邻域内有定义，且$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$，则称函数$f(x,y)$在点$P_0(x_0,y_0)$处连续。

例3(1). 求证： 函数$f(x,y)=\begin{cases} \frac{x^3y}{x^4+y^4}, (x,y)\neq(0,0)\text{时}, \\ 0,\ (x,y)=(0,0)\text{时} \end{cases}$ 在点$(0,0)$间断(不连续).

定义： 函数$f(x,y)$在定点$P_0(x_0,y_0)$的某个邻域内有定义，且$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$，则称函数$f(x,y)$在点$P_0(x_0,y_0)$处连续。

例3(1). 求证：函数$f(x,y)=\begin{cases}\frac{x^3y}{x^4+y^4}, (x,y)\neq(0,0)\text{时}, \\ 0, (x,y)=(0,0)\text{时}\end{cases}$ 在点$(0,0)$间断(不连续).

解: 取$\Gamma_1: y=0$、$\Gamma_2: y=x$它们都通过原点$(0,0)$，而

$$\lim_{(x,y)\in\Gamma_1,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=0} f(x,y)=\lim_{x\to0}\frac{0}{x^4}=0,$$

$$\lim_{(x,y)\in\Gamma_2,(x,y)\to(0,0)} f(x,y)=\lim_{x\to0,y=x} f(x,y)=\lim_{x\to0}\frac{x^4}{x^4+x^4}=\frac{1}{2}.$$

由定理，$(x,y)\to(0,0)$时$f(x,y)=\frac{xy}{x^2+y^2}$无极限。从而$f(x,y)$在点$(0,0)$间断。

例3(2). 函数$f(x,y)=\begin{cases}\frac{x^3y}{x^2+y^2}, & (x,y)\neq(0,0)\text{时},\\ a, & (x,y)=(0,0)\text{时}\end{cases}$ 在点$(0,0)$处连续，求常数a?

解: 由$0\le |f(x,y)|=\left|\frac{x^3y}{x^2+y^2}\right|\le \frac{1}{2}x^2$、夹逼准则及$\lim_{(x,y)\to(0,0)}\frac{1}{2}x^2=0$得

$$\lim_{(x,y)\to(0,0)} f(x,y)=0.$$

由连续的定义得$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)$，从而$a=0$.

注: $f(x,y)$在点(x_0,y_0)间断, 等价于$f(x,y)$满足下列条件之一：

(1). $f(x,y)$在(x_0,y_0)处无定义；

或(2). $(x,y)\to(x_0,y_0)$时无极限；

或(3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)\neq f(x_0,y_0)$;

注: 连续函数通过有限次四则运算、复合运算后所得函数连续;

注: 基本初等函数在定义域内处处连续

$\Rightarrow$初等函数在定义域内处处连续;

$\frac{x^2y}{x^2+y^3}$、$xy^2\sin x+y$、$\arcsin\frac{y}{x}$等都是初等函数，在各自的定义域内处处连续。$\Rightarrow$

$$\lim_{(x,y)\to(2,1)}\frac{x^2y}{x^2+y^3}=\frac{4}{5},$$

$$\lim_{(x,y)\to(2,1)}\arcsin\frac{y}{x}=\arcsin\frac{1}{2}=\frac{\pi}{6}.$$

定理：（最值定理）有界、闭区域D上的连续函数$f(x,y)$必有最大值M、最小值m，即存在点$P_1(x_1,y_1)$、$P_2(x_2,y_2)\in D$满足

$$m=f(x_1,y_1)\le f(x,y)\le f(x_2,y_2)=M,\ (x,y)\in D.$$

注: $E=\{(x,y):\ x\in(-\infty,+\infty),y\ge 0\}$是闭区域但不是有界的，函数$f(x,y)=x+y$在D内连续，而$f(x,y)=x+y$取不到最大值、最小值。

定理：（介值定理）有界、闭区域D上的连续函数$f(x,y)$有最大值M、最小值m，则对于任何常数$c\in[m,M]$，至少存在点$P(\xi,\eta)\in D$使得$f(\xi,\eta)=c$. (等价地，有界、闭区域D上连续函数$f(x,y)$的值域为闭区间$[m,M]$).

- 偏导数研究多元函数值随某一个自变量变化的情况;
- 全微分研究多元函数值随所有自变量变化的情况。

定义： 函数$f(x,y)$在点$P_0(x_0,y_0)$的某邻域内有定义。

- 如果固定$y=y_0$时一元函数$f(x,y_0)$ 关于x在x_0可导，则称$f(x,y)$在点$P_0(x_0,y_0)$关于x偏可导，关于x的偏导数为

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=f'_x(x_0,y_0)=\left.\frac{df(x,y_0)}{dx}\right|_{x=x_0}=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}.$$

- 函数$f(x,y)$在点$P_0(x_0,y_0)$的某邻域内有定义。如果固定$x=x_0$时一元函数$f(x_0,y)$ 关于y在y_0可导，则称$f(x,y)$在点$P_0(x_0,y_0)$关于y偏可导，关于y的偏导数为

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}=f'_y(x_0,y_0)=\left.\frac{df(x_0,y)}{dy}\right|_{y=y_0}=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}.$$

•简单地说，

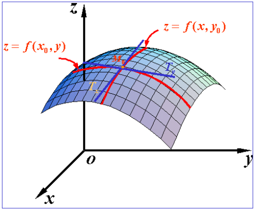
关于x偏导数$\Leftrightarrow$ $f(x,y)$作为x的一元函数求导数(y看作常数);

关于y偏导数$\Leftrightarrow$ $f(x,y)$作为y的一元函数求导数(x看作常数);

•偏导数的几何意义:

设$M_0(x_0,y_0,f(x_0,y_0))$为曲面$\Sigma: z=f(x,y)$上的一点，曲面Σ被平面$y=y_0$截得曲线

$$\Gamma: \begin{cases} z=f(x,y) \\ y=y_0 \end{cases},$$



曲线Γ在点M_0处切线的斜率等于

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f'_x(x_0,y_0);$$

曲面Σ被平面$x=x_0$截得曲线$L: \begin{cases} z=f(x,y) \\ x=x_0 \end{cases}$，曲线$L$在点$M_0$处切线的斜率等于$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f'_y(x_0,y_0);$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f'_x(x_0,y_0) = \left.\frac{df(x,y_0)}{dx}\right|_{x=x_0} = \lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f'_y(x_0,y_0) = \left.\frac{df(x_0,y)}{dy}\right|_{y=y_0} = \lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}.$$

关于x偏导数$\Leftrightarrow$ $f(x,y)$作为x的一元函数求导数(y看作常数);

关于y偏导数$\Leftrightarrow$ $f(x,y)$作为y的一元函数求导数(x看作常数);

- 一元函数求导方法：(1). 从定义出发；(2). 利用求导法则；

(回顾)基本初等函数的导数:

$$(C)' = 0; \quad (x^{\mu})' = \mu x^{\mu-1};$$

$$(a^x)' = a^x \ln a; \quad (e^x)' = e^x;$$

$$(\log_a x)' = \frac{1}{x \ln a}, \quad (\ln x)' = \frac{1}{x};$$

$$(\sin x)' = \cos x; \quad (\cos x)' = -\sin x;$$

$$(\tan x)' = \frac{1}{\cos^2 x}; \quad (\cot x)' = -\frac{1}{\sin^2 x};$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}; \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$$

$$(\arctan x)' = \frac{1}{1+x^2}; \quad (arc\cot x)' = -\frac{1}{1+x^2};$$

例1(1) $f(x,y)=x^3+2xy+(x-2)\arctan\frac{1}{x^2+y^2}$，求$f'_y(2,1)$?

解： $f'_y(2,1)=\left.\frac{df(2,y)}{dy}\right|_{y=1}=\left.\frac{d(8+4y)}{dy}\right|_{y=1}=4.$

例1(2) $f(x,y)=x^3+2xy+\sin(xy^2)$, 求$f'_x(\pi,1)$、$f'_y(\pi,1)$?

解： 由定义

$$\begin{aligned} f'_x(\pi,1) &= \left.\frac{df(x,1)}{dx}\right|_{x=\pi} = \left.\frac{d(x^3+2x+\sin x)}{dx}\right|_{x=\pi} \\ &= \left(3x^2+2+\cos x\right)_{x=\pi} = 1+3\pi^2. \end{aligned}$$

$$\begin{aligned} f'_y(\pi,1) &= \left.\frac{df(\pi,y)}{dy}\right|_{y=1} = \left.\frac{d(\pi^3+2\pi y+\sin(\pi y^2))}{dy}\right|_{y=1} \\ &= \left(2\pi+2\pi y\cos(\pi y^2)\right)_{y=1} = 0. \end{aligned}$$

例1(3) $z = x^2y + 2xy^2 - 3y^2$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$?

解： $\frac{\partial z}{\partial x} = 2xy + 2y^2$, $\quad \frac{\partial z}{\partial y} = x^2 + 4xy - 6y$.

例1(4) $z = \arcsin(xy^2)$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$?

解： $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-x^2y^4}}\left(xy^2\right)'_x = \frac{1}{\sqrt{1-x^2y^4}} \cdot y^2 = \frac{y^2}{\sqrt{1-x^2y^4}}$,

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-x^2y^4}}\left(xy^2\right)'_y = \frac{1}{\sqrt{1-x^2y^4}} \cdot 2xy = \frac{2xy}{\sqrt{1-x^2y^4}}.$$

例1(5) $z=(1+xy)^{y^2}$，求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$？

解： $\frac{\partial z}{\partial x}=y^2(1+xy)^{y^2-1}(1+xy)'_x=y^3(1+xy)^{y^2-1}$,

$$\begin{aligned}\frac{\partial z}{\partial y} &= \left[e^{y^2\ln(1+xy)}\right]'_y=e^{y^2\ln(1+xy)}\left[y^2\ln(1+xy)\right]'_y\\ &= (1+xy)^{y^2}\left[2y\ln(1+xy)+y^2\cdot\frac{1}{(1+xy)}(1+xy)'_y\right]\\ &= (1+xy)^{y^2}\left[2y\ln(1+xy)+\frac{xy^2}{(1+xy)}\right].\end{aligned}$$

例1(6) $z=\int_{x+y}^{xy^2} e^{t^2}dt$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$?

- $\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)dt\right] = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)$;

解： 由定义

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^{x^2y^4}\left[xy^2\right]'_x - e^{(x+y)^2}\left[x+y\right]'_x \\ &= y^2e^{x^2y^4} - e^{(x+y)^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= e^{x^2y^4}\left[xy^2\right]'_y - e^{(x+y)^2}\left[x+y\right]'_y \\ &= 2xye^{x^2y^4} - e^{(x+y)^2}.\end{aligned}$$

例1(6) $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+4y^2},\ (x,y)\neq(0,0)\text{时}\\ 0,\ (x,y)=(0,0)\text{时}\end{cases}$，

求 $\frac{\partial f}{\partial x}\Big|_{(0,0)}$、 $\frac{\partial f}{\partial y}\Big|_{(0,0)}$?

解： 由定义

$$\frac{\partial f}{\partial x}\Big|_{(0,0)}=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{x^2\sin\frac{1}{x^2}-0}{x}=0;$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)}=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{y^2\sin\frac{1}{4y^2}-0}{y}=0;$$

问：为什么不用求导法则？

例1(7) $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, (x,y)\neq(0,0)\text{时}\\ 0, (x,y)=(0,0)\text{时}\end{cases}$，求

证：$f(x,y)$在$(0,0)$处有偏导数但极限不存在（从而不连续）。

证明： 直线$\Gamma_1: y=0$、$\Gamma_2: y=x$都通过点$(0,0)$,

$$\lim_{(x,y)\to(0,0),(x,y)\in\Gamma_1} f(x,y)=\lim_{x\to0,y=0} f(x,y)=\lim_{x\to0}\frac{0}{x^2}=0,$$

$$\lim_{(x,y)\to(0,0),(x,y)\in\Gamma_2} f(x,y)=\lim_{x\to0,y=x} f(x,y)=\lim_{x\to0}\frac{x^2}{x^2+x^2}=\frac{1}{2},$$

从而，$(x,y)\to(0,0)$时$f(x,y)$无极限，得$f(x,y)$在$(0,0)$处不连续。

$$f'_x(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{0-0}{x-0}=0,$$

$$f'_y(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{0-0}{y-0}=0.$$

例1(8) $f(x,y)=|x|+|y|$，求证：$f(x,y)$在$(0,0)$处没有偏导数但连续。

证明： $f(x,y)=|x|+|y|=\sqrt{x^2}+\sqrt{y^2}$是一个初等函数且在$(0,0)$处有定义，从而$f(x,y)$在$(0,0)$处连续。
由定义，

$$f'_x(0,0)=\left.\frac{df(x,0)}{dx}\right|_{x=0}=\left.\frac{d|x|}{dx}\right|_{x=0}\ \text{不存在;}$$

$$f'_y(0,0)=\left.\frac{df(0,y)}{dy}\right|_{y=0}=\left.\frac{d|y|}{dy}\right|_{y=0}\ \text{不存在;}$$

注： ∘ 一元函数：可导⇒连续；

∘多元函数：可（偏）导与连续没有必然的联系，即

可（偏）导不一定连续，连续不一定可（偏）导；

高阶偏导数

设函数$f(x,y)$在区域D内处处可（偏）导，函数

$$(x,y)\in D \hookrightarrow f'_x(x,y),\ (x,y)\in D \hookrightarrow f'_y(x,y)$$

分别称为函数$f(x,y)$关于x的（一阶）偏导函数$f'_x(x,y)$、关于y的（一阶）偏导函数$f'_y(x,y)$;

●如果（一阶）**偏导函数**$f'_x(x,y)$在(x_0,y_0)处关于x还可偏导，则$\left.\frac{\partial f'_x}{\partial x}\right|_{(x_0,y_0)}$称为函数$f(x,y)$在$(x_0,y_0)$处关于$x$的2阶偏导数，记为

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = f''_{xx}(x_0,y_0) = \left.\frac{\partial f'_x}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{df'_x(x,y_0)}{dx}\right|_{x=x_0};$$

• 如果（一阶）**偏导函数**$f'_x(x,y)$在(x_0,y_0)处关于y还可偏导，则$\left.\frac{\partial f'_x}{\partial y}\right|_{(x_0,y_0)}$称为函数$f(x,y)$在$(x_0,y_0)$处关于$x$、$y$的2阶(混合)偏导数，记为

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(x_0,y_0)} = f''_{xy}(x_0,y_0) = \left.\frac{\partial f'_x}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{df'_x(x_0,y)}{dy}\right|_{y=y_0};$$

• 一阶**偏导函数**$f'_y(x,y)$在(x_0,y_0)处关于x还可偏导，则$\left.\frac{\partial f'_y}{\partial x}\right|_{(x_0,y_0)}$称为函数$f(x,y)$在$(x_0,y_0)$处关于$y$、$x$的2阶偏导数，记为

$$\left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(x_0,y_0)} = f''_{yx}(x_0,y_0) = \left.\frac{\partial f'_y}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{df'_y(x,y_0)}{dx}\right|_{x=x_0};$$

- 一阶**偏导函数**$f'_y(x,y)$在(x_0,y_0)处关于y还可偏导，

则 $\left.\frac{\partial f'_y}{\partial y}\right|_{(x_0,y_0)}$称为函数$f(x,y)$在$(x_0,y_0)$处关于$y$的2阶(混合)偏导数，记为

$$\left.\frac{\partial^2 f}{\partial^2 y}\right|_{(x_0,y_0)} = f''_{yy}(x_0,y_0) = \left.\frac{\partial f'_y}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{df'_y(x_0,y)}{dy}\right|_{y=y_0};$$

- 如果函数$f(x,y)$在区域D内处处可2阶（偏）导，函数

$$(x,y)\in D \hookrightarrow f''_{xx}(x,y),\ (x,y)\in D \hookrightarrow f''_{xy}(x,y),\cdots$$

分别称为函数$f(x,y)$关于x的（2阶）偏导函数$f''_{xx}(x,y)$、关于x、y 的（2阶）混合偏导函数$f''_{xy}(x,y),\cdots;$

定义: 2阶偏导函数在(x_0,y_0)处的偏导数称为函数$f(x,y)$在(x_0,y_0)处的3阶偏导数.

如:

$$\left.\frac{\partial^3 f}{\partial x\partial y\partial x}\right|_{(x_0,y_0)} = f_{xyx}^{(3)}(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}\right)\bigg|_{(x_0,y_0)};$$

$$\left.\frac{\partial^3 f}{\partial x^2\partial y}\right|_{(x_0,y_0)} = f_{xxy}^{(3)}(x_0,y_0) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)\bigg|_{(x_0,y_0)};$$

$$\left.\frac{\partial^3 f}{\partial y\partial x^2}\right|_{(x_0,y_0)} = f_{yxx}^{(3)}(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\partial x}\right)\bigg|_{(x_0,y_0)};$$

例2(1). $f(x,y)=x^3y^4+2y\sin x$, 求一阶偏导函数$f'_x(x,y)$、$f'_y(x,y)$? 2阶偏导数$f''_{xx}(\pi,0)$、$f''_{xy}(\pi,0)$?

解: 一阶偏导函数

$$f'_x(x,y)=\left(x^3y^4+2y\sin x\right)'_x=3x^2y^4+2y\cos x;$$

$$f'_y(x,y)=\left(x^3y^4+2y\sin x\right)'_y=4x^3y^3+2\sin x;$$

2阶偏导数

$$\begin{aligned} f''_{xx}(\pi,0) &= \left.\frac{\partial f'_x}{\partial x}\right|_{(\pi,0)} = \left(3x^2y^4+2y\cos x\right)'_x\Big|_{(\pi,0)} \\ &= \left(6xy^4-2y\sin x\right)\Big|_{(\pi,0)}=0; \end{aligned}$$

$$\begin{aligned} f''_{xy}(\pi,0) &= \left.\frac{\partial f'_x}{\partial y}\right|_{(\pi,0)} = \left(3x^2y^4+2y\cos x\right)'_y\Big|_{(\pi,0)} \\ &= \left(12x^2y^3+2\cos x\right)\Big|_{(\pi,0)}=-2; \end{aligned}$$

例2(2). $z = x^y$, 求偏导(函)数$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial x \partial y}$、$\frac{\partial^3 z}{\partial x \partial y \partial x}$?

解： 偏导(函)数

$$\frac{\partial z}{\partial x} = yx^{y-1};$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial x}\left[yx^{y-1}\right] = y(y-1)x^{y-2};$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}\left[yx^{y-1}\right] = x^{y-1} + yx^{y-1}\ln x;$$

$$\begin{aligned} \frac{\partial^3 z}{\partial x \partial y \partial x} &= \frac{\partial}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \frac{\partial}{\partial x}\left[x^{y-1} + yx^{y-1}\ln x\right] \\ &= (y-1)x^{y-2} + y(y-1)x^{y-2} + yx^{y-2} \\ &= (y^2 + y - 1)x^{y-2}; \end{aligned}$$

例2(3). 求证$u=\ln\sqrt{x^2+y^2}$满足Laplace方程$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

证明：

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2}\right)'_x \\ &= \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\left(x^2+y^2\right)'_x \\ &= \frac{2x}{2(x^2+y^2)}=\frac{x}{x^2+y^2};\end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]=\left(\frac{x}{x^2+y^2}\right)'_x=\frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2};$$

由对称性

$$\frac{\partial^2 u}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2};$$

从而$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

例2(4). $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2},\ (x,y)\neq(0,0)\text{时}\\ 0,\ (x,y)\neq(0,0)\text{时}\end{cases}$，求一阶偏导函数$f'_x(x,y)$、$f'_y(x,y)$? 2阶偏导数$f''_{xx}(0,0)$、$f''_{xy}(0,0)$?

解： $f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0$;

当$(x_0,y_0)\neq(0,0)$时

$$f'_x(x_0,y_0)=\frac{\partial}{\partial x}\left[\frac{xy(x^2-y^2)}{x^2+y^2}\right]\Bigg|_{(x_0,y_0)}=\frac{x^4y-y^5+2x^2y^3}{(x^2+y^2)^2}\Bigg|_{(x_0,y_0)}.$$

从而一阶偏导函数$f'_x(x,y)$为

$$f'_x(x,y)=\begin{cases}\frac{x^4y-y^5+2x^2y^3}{(x^2+y^2)^2},\ (x,y)\neq(0,0)\text{时}\\ 0,\ (x,y)\neq(0,0)\text{时}\end{cases}$$

例2(4). $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2},\ (x,y)\neq(0,0)\text{时}\\ 0,\ (x,y)\neq(0,0)\text{时}\end{cases}$，求一阶偏导函数$f'_x(x,y)$、$f'_y(x,y)$?2阶偏导数$f''_{xx}(0,0)$、$f''_{xy}(0,0)$?

解： $f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0;$
当$(x_0,y_0)\neq(0,0)$时

$$f'_y(x_0,y_0)=\frac{\partial}{\partial y}\left[\frac{xy(x^2-y^2)}{x^2+y^2}\right]\Bigg|_{(x_0,y_0)}=\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}\Bigg|_{(x_0,y_0)}.$$

从而一阶偏导函数$f'_y(x,y)$为

$$f'_y(x,y)=\begin{cases}\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2},\ (x,y)\neq(0,0)\text{时}\\ 0,\ (x,y)\neq(0,0)\text{时}\end{cases}$$

例2(4). $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}, (x,y)\neq(0,0)\text{时}\\ 0, (x,y)\neq(0,0)\text{时}\end{cases}$，求一阶偏导函数$f'_x(x,y)$、$f'_y(x,y)$?2阶偏导数$f''_{xx}(0,0)$、$f''_{xy}(0,0)$?

- 一阶偏导函数$f'_x(x,y)$为

$$f'_x(x,y)=\begin{cases}\frac{x^4y-y^5+2x^2y^3}{(x^2+y^2)^2}, (x,y)\neq(0,0)\text{时}\\ 0, (x,y)\neq(0,0)\text{时}\end{cases}$$

由定义，

$$\begin{aligned}f''_{xx}(0,0) &= \left[f'_x\right]'_x(0,0)=\lim_{x\to 0}\frac{f'_x(x,0)-f'_x(0,0)}{x-0}\\ &= \lim_{x\to 0}\frac{0-0}{x-0}=0;\end{aligned}$$

$$\begin{aligned}f''_{xy}(0,0) &= \left[f'_x\right]'_y(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y-0}\\ &= \lim_{y\to 0}\frac{-y-0}{y-0}=-1;\end{aligned}$$

例2(4). $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}, (x,y)\neq(0,0)\text{时}\\ 0, (x,y)\neq(0,0)\text{时}\end{cases}$，求一阶偏导函数$f'_x(x,y)$、$f'_y(x,y)$?2阶偏导数$f''_{xx}(0,0)$、$f''_{xy}(0,0)$?

- 一阶偏导函数$f'_y(x,y)$为

$$f'_y(x,y)=\begin{cases}\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}, (x,y)\neq(0,0)\text{时}\\ 0, (x,y)\neq(0,0)\text{时}\end{cases}$$

由定义，

$$\begin{aligned} f''_{yx}(0,0) &= \left[f'_y\right]'_x(0,0)=\lim_{x\to0}\frac{f'_y(x,0)-f'_y(0,0)}{x-0}\\ &= \lim_{x\to0}\frac{x-0}{x-0}=1;\end{aligned}$$

$$\begin{aligned} f''_{yy}(0,0) &= \left[f'_y\right]'_y(0,0)=\lim_{y\to0}\frac{f'_y(0,y)-f'_y(0,0)}{y-0}\\ &= \lim_{y\to0}\frac{0-0}{y-0}=0;\end{aligned}$$

定理： 若2阶混合偏导函数$f''_{xy}(x,y)$及$f''_{yx}(x,y)$在(x_0,y_0)处连续，则$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$.

如：$f(x,y)=xy+\frac{y}{x}$的偏导函数

$$f'_x=y-\frac{y}{x^2},\ f'_y=x+\frac{1}{x},$$

$$f''_{xy}=1-\frac{1}{x^2},\ f''_{yx}=1-\frac{1}{x^2}$$

注： 上述定理中“2阶混合偏导函数在(x_0,y_0)处连续”条件必需的，否则结论可能不再成立。如上例2(4)。

注： 类似结论对高阶混合导数也成立。如：

若3阶混合偏导函数$f^{(3)}_{xxy}(x,y)$及$f^{(3)}_{xyx}(x,y)$ 在(x_0,y_0)处连续，则$f^{(3)}_{xxy}(x_0,y_0)=f^{(3)}_{xyx}(x_0,y_0)$.

定义： 函数$z=f(x,y)$在$P_0(x_0,y_0)$的某邻域内有定义，记（自变量增量）

$$\Delta x=x-x_0,\ \Delta y=y-y_0,$$

相应地函数值增量（全增量）

$$\Delta z=f(x,y)-f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0).$$

如果存在常数A、B使得，当$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}\to 0$时，

$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=A\Delta x+B\Delta y+o(\rho).$$

则称函数$z=f(x,y)$在$P_0(x_0,y_0)$处**可微**，主要部分$A\Delta x+B\Delta y$称为函数在$P_0(x_0,y_0)$处的**全微分**，记为

$$dz=A\Delta x+B\Delta y.$$

注： 当$(\Delta x)^2+(\Delta y)^2\to 0$

$$\Delta z=dz+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)\Leftrightarrow\frac{\Delta z-(A\Delta x+B\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\to 0.$$

例3(1). 讨论$f(x,y)=x^2+3xy$在$(1,2)$处是否可微？并求全微分df及在$(1,2)$处的偏导数？

解： $\Delta x=x-1$、$\Delta y=y-2$,

$\Delta f=f(1+\Delta x,2+\Delta y)-f(1,2)=8\Delta x+3\Delta y+(\Delta x)^2+3\Delta x\Delta y.$

记$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，$\Rightarrow\ \max\{|\Delta x|,|\Delta y|\}\le\rho\le|\Delta x|+|\Delta y|$, 利用

$$0\le\frac{(\Delta x)^2+3\Delta x\Delta y}{\rho}\le\frac{\rho^2+3\rho^2}{\rho}=4\rho$$

及夹逼准则得

$$\lim_{\rho\to 0}\frac{(\Delta x)^2+3\Delta x\Delta y}{\rho}=0\Leftrightarrow(\Delta x)^2+3\Delta x\Delta y=o(\rho).$$

$$\Rightarrow\ \Delta f=8\Delta x+3\Delta y+o(\rho).$$

由定义，$f(x,y)$在$(1,2)$处是可微且$df=8\Delta x+3\Delta y$.

例3(1).讨论$f(x,y)=x^2+3xy$在$(1,2)$处是否可微?并求的全微分df及在$(1,2)$处的偏导数？

例3(1).讨论$f(x,y)=x^2+3xy$在$(1,2)$处是否可微?并求的全微分df及在$(1,2)$处的偏导数？

解： $f(x,y)$在$(1,2)$处是可微且$df=8\Delta x+15\Delta y$。注意

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)}=(2x+3y)|_{(1,2)}=8,\ \left.\frac{\partial f}{\partial y}\right|_{(1,2)}=(3x+3y^2)|_{(1,2)}=15$$

从而

$$df=8\Delta x+15\Delta y=\left.\frac{\partial f}{\partial x}\right|_{(1,2)}\Delta x+\left.\frac{\partial f}{\partial x}\right|_{(1,2)}\Delta y.$$

问： 上面结果对任意函数$f(x,y)$都成立？（答案：是的。具体见下面）

• 讨论特殊函数$f(x,y)=x$及$g(x,y)=y$在(x_0,y_0)处的微分。

$$\Delta f=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=\Delta x=\Delta x+0\cdot\Delta y+o(\rho)$$

$$\Rightarrow dx=df=\Delta x+0\Delta y=\Delta x.$$

类似计算得

$$dy=dg=0\Delta x+\Delta y=\Delta y.$$

▶利用上述结论，全微分也可记为

$$dz=A\Delta x+B\Delta y\ \Leftrightarrow\ dz=Adx+Bdy.$$

如：$f(x,y)=x^2+3xy$在$(1,2)$处可微、且

$$df=8\Delta x+3\Delta y\text{ 或 }df=8dx+3dy;$$

定理： 如果函数$f(x,y)$在(x_0,y_0)处可微分，则$f(x,y)$在(x_0,y_0)处可偏导，且

$$df\big|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} dx + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} dy.$$

证明： 如果函数$z=f(x,y)$在(x_0,y_0)处可微分，由全微分定义，存在常数A、B使得$df|_{(x_0,y_0)} = A\Delta x + B\Delta y$，即当$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}\to 0$时

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = A\Delta x + B\Delta y + o(\rho).$$

$$\begin{aligned}\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} &= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{A\Delta x + o(|\Delta x|)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(A+\frac{o(|\Delta x|)}{|\Delta x|}\cdot\frac{|\Delta x|}{\Delta x}\right) = A;\end{aligned}$$

类似，

$$\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} = \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{B\Delta y+o(|\Delta y|)}{\Delta y}$$
$$= \lim_{\Delta y\to 0}\left(B+\frac{o(|\Delta y|)}{|\Delta y|}\cdot\frac{|\Delta y|}{\Delta y}\right) = B;$$

从而$f(x,y)$在(x_0,y_0) 处可偏导，且

$$df|_{(x_0,y_0)} = Adx+Bdy = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}dx + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}dy.$$

注：对多元函数：可微分必可偏导；**但可导并不一定可微分；**

例3(2). 求证：$f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, (x,y)\neq(0,0)\text{时},\\ 0, (x,y)=(0,0)\text{时},\end{cases}$ 在$(0,0)$处可偏导、但是不可微。

证明： $f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0;$

$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0;$

从而$f(x,y)$在$(0,0)$处可偏导。下面证明：$f(x,y)$在$(0,0)$处不可微。（反证法）

若$f(x,y)$在$(0,0)$处可微，由定理得：当$\rho\to 0$时

$$\Delta f=f(\Delta x,\Delta y)-f(0,0)=\left.\frac{\partial f}{\partial x}\right|_{(0,0)}dx+\left.\frac{\partial f}{\partial y}\right|_{(0,0)}dy+o(\rho)=o(\rho);$$

$$\Leftrightarrow\frac{\Delta x\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=o(\rho)\Leftrightarrow\frac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}=\frac{o(\rho)}{\rho}\to 0.$$

$$\Leftrightarrow \frac{\Delta x \Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = o(\rho) \Leftrightarrow \frac{\Delta x \Delta y}{(\Delta x)^2+(\Delta y)^2} = \frac{o(\rho)}{\rho} \to 0.$$

而$\rho = \sqrt{(\Delta x)^2+(\Delta y)^2} \to 0 \Leftrightarrow (\Delta x, \Delta y) \to (0,0)$

$$\Leftrightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x \Delta y}{(\Delta x)^2+(\Delta y)^2} = 0. \cdots\cdots (*)$$

$$\lim_{\substack{(\Delta x, \Delta y)\to(0,0)\\ \Delta y = 0}} \frac{\Delta x \Delta y}{(\Delta x)^2+(\Delta y)^2} = \lim_{\Delta x\to 0} \frac{0}{(\Delta x)^2} = 0;$$

$$\lim_{\substack{(\Delta x, \Delta y)\to(0,0)\\ \Delta y = \Delta x}} \frac{\Delta x \Delta y}{(\Delta x)^2+(\Delta y)^2} = \lim_{\Delta x\to 0} \frac{(\Delta x)^2}{(\Delta x)^2+(\Delta x)^2} = \frac{1}{2};$$

知：$(\Delta x, \Delta y) \to (0,0)$时$\frac{\Delta x \Delta y}{(\Delta x)^2+(\Delta y)^2}$无极限，这与$(*)$矛盾！
即$f(x,y)$在$(0,0)$处不可微。

- 函数$f(x,y)$在(x_0,y_0)处可微分$\Longrightarrow$ $f(x,y)$ 在(x_0,y_0)处可偏导，且

$$df\big|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} dx + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} dy.$$

- 但是，$f(x,y)$ 在(x_0,y_0)处可偏导，**不能保证**函数$f(x,y)$在(x_0,y_0)处可微分；

定理： 如果函数$f(x,y)$在(x_0,y_0)处的可微分，则$f(x,y)$在(x_0,y_0) 处连续。

证明： 如果函数$z=f(x,y)$在(x_0,y_0)处的可微分，则存在常数A、B 使得$df|_{(x_0,y_0)}=A\Delta x+B\Delta y$, 即当$\rho\to 0$时

$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=A\Delta x+B\Delta y+o(\rho).$$

而$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}\to 0$时 $\Leftrightarrow$ $(\Delta x,\Delta y)\to(0,0)$时

$$f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)+A\Delta x+B\Delta y+o(\rho)\to f(x_0,y_0).$$

即

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0).$$

由定义，$f(x,y)$在(x_0,y_0) 处连续。

定理： 如果函数$f(x,y)$在(x_0,y_0)处的可微分，则$f(x,y)$在(x_0,y_0) 处连续。

注： 函数在(x_0,y_0)可微分$\Rightarrow$函数连续；**但是**函数连续不一定可微分；

如： $f(x,y)=|x|+|y|$在$(0,0)$处连续、但是在在$(0,0)$处不可导、不可微（问：为什么？）

定理： 如果函数$f(x,y)$的**偏导函数**$f'_x(x,y)$、$f'_y(x,y)$在(x_0,y_0)处连续，则$f(x,y)$ 在(x_0,y_0) 处可微分。

证明： 记$\Delta x = x - x_0$、$\Delta y = y - y_0$，

$$
\begin{aligned}
\Delta f &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\
&= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0)] \\
&\quad + [f(x_0+\Delta x, y_0) - f(x_0,y_0)] \\
&= f'_y(x_0+\Delta x, y_0+\theta_1\Delta y)\cdot\Delta y + f'_x(x_0+\theta_2\Delta x, y_0)\cdot\Delta x \\
&= f'_y(x_0,y_0)\cdot\Delta y + f'_x(x_0,y_0)\cdot\Delta x + r(\Delta x,\Delta y).
\end{aligned}
$$

其中θ_1、$\theta_2\in(0,1)$，

$$
\begin{aligned}
r(\Delta x,\Delta y) &= \left[f'_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f'_y(x_0,y_0)\right]\Delta y \\
&\quad + \left[f'_x(x_0+\theta_2\Delta x, y_0) - f'_x(x_0,y_0)\right]\Delta x;
\end{aligned}
$$

$$\Delta f = f_x'(x_0, y_0) \cdot \Delta x + f_y'(x_0, y_0) \cdot \Delta y + r(\Delta x, \Delta y),$$

$$\begin{aligned} r(\Delta x, \Delta y) &= \left[f_y'(x_0 + \Delta x, y_0 + \theta_1 \Delta y) - f_y'(x_0, y_0)\right] \Delta y \\ &\quad + \left[f_x'(x_0 + \theta_2 \Delta x, y_0) - f_x'(x_0, y_0)\right] \Delta x; \end{aligned}$$

记$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$，则$|\Delta x| \le \rho$、$|\Delta x| \le \rho$，

$$\begin{aligned} 0 \le \frac{|r(\Delta x, \Delta y)|}{\rho} &\le \left|f_y'(x_0 + \Delta x, y_0 + \theta_1 \Delta y) - f_y'(x_0, y_0)\right| \\ &\quad + \left|f_x'(x_0 + \theta_2 \Delta x, y_0) - f_x'(x_0, y_0)\right|; \end{aligned}$$

偏导函数$f_x'(x, y)$、$f_y'(x, y)$在(x_0, y_0)处连续$\Rightarrow$

$$\lim_{\rho \to 0} \left|f_y'(x_0 + \Delta x, y_0 + \theta_1 \Delta y) - f_y'(x_0, y_0)\right| = 0,$$

$$\lim_{\rho \to 0} \left|f_x'(x_0 + \theta_2 \Delta x, y_0) - f_x'(x_0, y_0)\right| = 0,$$

结合夹逼定理，$\lim_{\rho \to 0} \frac{r(\Delta x, \Delta y)}{\rho} = 0 \Leftrightarrow r(\Delta x, \Delta y) = o(\rho)$，

$$\Delta f = f_x'(x_0, y_0) \cdot \Delta x + f_y'(x_0, y_0) \cdot \Delta y + o(\rho),$$

结合夹逼定理，$\lim_{\rho\to 0}\frac{r(\Delta x,\Delta y)}{\rho}=0 \Leftrightarrow r(\Delta x,\Delta y)=o(\rho)$,

$$\Delta f = f'_x(x_0,y_0)\cdot\Delta x + f'_y(x_0,y_0)\cdot\Delta y + o(\rho),$$

$\Rightarrow f(x,y)$ 在(x_0,y_0) 处可微分,

$$df|_{(x_0,y_0)} = f'_x(x_0,y_0)dx + f'_y(x_0,y_0)dy.$$

例3(3). 求

证：$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, (x,y)\neq(0,0)\text{时},\\ 0,\ (x,y)=(0,0)\text{时},\end{cases}$

在$(0,0)$处可微、但偏导函数$f'_x(x,y)$、$f'_y(x,y)$在$(0,0)$处不连续。

证明： 记$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，由$0\le\frac{|f(\Delta x,\Delta y)|}{\rho}\le\rho\left|\sin\frac{1}{\rho^2}\right|\le\rho$

得: 当$\rho\to0$时$f(\Delta x,\Delta y)=o(\rho)$,

$$\Delta f=f(0+\Delta x,0+\Delta y)-f(0,0)=0\Delta x+0\Delta y+o(\rho),$$

得$f(x,y)$在$(0,0)$处可微且$df|_{(0,0)}=0dx+0dy=0.$

- 显然，$f'_x(0,0)=0$ (问：为什么？)；
- 当$(x,y)\neq(0,0)$时

$$f'_x(x,y)=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2},$$

$$\Rightarrow f'_x(x,y)=\begin{cases}2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}, (x,y)\neq(0,0)\text{时},\\ 0,\ (x,y)=(0,0)\text{时},\end{cases}$$

$$f'_x(x,y)=\begin{cases} 2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2},\ (x,y)\neq(0,0)\text{时}, \\ 0,\ (x,y)=(0,0)\text{时}, \end{cases}$$

下面证明：偏导函数$f'_x(x,y)$在$(0,0)$处不连续（无极限）。

•取$\Gamma: x=y^2$, 它通过原点$(0,0)$.

当$(x,y)\in\Gamma$且$(x,y)\to(0,0)$时

$$f'_x(x,y)=\begin{cases} 2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}, (x,y)\neq(0,0)\text{时}, \\ 0,\ (x,y)=(0,0)\text{时}, \end{cases}$$

下面证明：偏导函数$f'_x(x,y)$在$(0,0)$处不连续（无极限）。

•取$\Gamma: x=y^2$, 它通过原点$(0,0)$.
当$(x,y)\in\Gamma$且$(x,y)\to(0,0)$时$\cos\frac{1}{x^2+y^2}=\cos\frac{1}{y^4+y^2}$无极限，而

$$2x\sin\frac{1}{x^2+y^2}=2y^2\sin\frac{1}{y^4+y^2}\to 0,\ \frac{2x}{x^2+y^2}=\frac{2y^2}{y^4+y^2}\to 2;$$

利用极限性质得：当$(x,y)\in\Gamma$且$(x,y)\to(0,0)$时$f'_x(x,y)$无极限
$\Rightarrow$ 当$(x,y)\to(0,0)$时$f'_x(x,y)$无极限 $\Rightarrow$ 偏导函数$f'_x(x,y)$在$(0,0)$处不连续.

•对偏导函数$f'_y(x,y)$可类似讨论：

$$f'_y(x,y)=\begin{cases} 2y\sin\frac{1}{x^2+y^2}-\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}, (x,y)\neq(0,0)\text{时}, \\ 0,\ (x,y)=(0,0)\text{时}, \end{cases}$$

定理： 如果函数$f(x,y)$的**偏导函数**$f'_x(x,y)$、$f'_y(x,y)$在(x_0,y_0)处连续，则$f(x,y)$ 在(x_0,y_0) 处可微分。

注： **偏导函数**$f'_x(x,y)$、$f'_y(x,y)$在(x_0,y_0)处连续 $\Rightarrow$ $f(x,y)$ 在(x_0,y_0) 处可微分；反之不然。

- 一元函数：

$$\begin{aligned} \text{函数}f(x)\text{在}x_0\text{可微} \quad &\Leftrightarrow \text{函数}f(x)\text{在}x_0\text{可导} \\ &\Rightarrow \text{函数}f(x)\text{在}x_0\text{连续} \end{aligned}$$

- 多元函数：

$$\begin{aligned} &\text{偏导函数}f'_x(x,y)\text{、}f'_y(x,y)\text{在}(x_0,y_0)\text{连续} \\ &\Rightarrow \text{函数}f(x,y)\text{在}(x_0,y_0)\text{可微} \\ &\Rightarrow \begin{cases} \Rightarrow \text{函数}f(x,y)\text{在}(x_0,y_0)\text{可偏导} \\ \Rightarrow \text{函数}f(x,y)\text{在}(x_0,y_0)\text{连续} \end{cases} \end{aligned}$$

例3(4). 求证：函数$f(x,y)=x^3y^2\sin(x+y)$处处可微，且求其微分$df|_{(\pi,\pi)}$？

证明： 利用

$$f'_x(x,y)=3x^2y^2\sin(x+y)+x^3y^2\cos(x+y),$$

$$f'_y(x,y)=2x^3y\sin(x+y)+x^3y^2\cos(x+y).$$

由偏导函数$f'_x(x,y)$、$f'_y(x,y)$处处连续（初等函数且处处有定义），函数$f(x,y)$处处可微，且

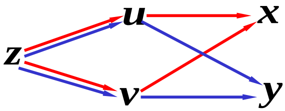
$$df|_{(x_0,y_0)}=f'_x(x_0,y_0)dx+f'_y(x_0,y_0)dy;$$

特别在点(π,π)处可微，且

$$df|_{(\pi,\pi)}=f'_x(\pi,\pi)dx+f'_y(\pi,\pi)dy=\pi^5dx+\pi^5dy;$$

复合函数求导法

定理： 设函数$z=f(u,v)$在(u_0,v_0)处可微分，$u=\varphi(x,y)$及$v=\psi(x,y)$在(x_0,y_0)处可偏导，且



$$u_0=\varphi(x_0,y_0),\ v_0=\psi(x_0,y_0)$$

则复合函数 $z=f\left(\varphi(x,y),\psi(x,y)\right)$

在(x_0,y_0)处可偏导，

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)};$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}=\left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)};$$

注： u、v为中间变量，x、y为最终变量；

证明：记$\Delta x = x - x_0$，$\rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}$,

$\Delta u = u - u_0 = \varphi(x, y_0) - \varphi(x_0, y_0)$, $\Delta v = v - v_0 = \psi(x, y_0) - \psi(x_0, y_0)$

由$z = f(u, v)$在(u_0, v_0)处**可微分**得

$$\begin{aligned}\Delta z &= f(u_0 + \Delta u, v_0 + \Delta v) - f(u_0, v_0) \\ &= \left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)} \Delta u + \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)} \Delta v + o(\rho);\end{aligned}$$

由u、v在(x_0, y_0)处可偏导

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{x \to x_0} \frac{\varphi(x, y_0) - \varphi(x_0, y_0)}{x - x_0} = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)};$$

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \lim_{x \to x_0} \frac{\psi(x, y_0) - \psi(x_0, y_0)}{x - x_0} = \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)};$$

$$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)},\ \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)};$$

因此有

$$\lim_{\Delta x\to 0}\frac{\rho}{|\Delta x|}=\lim_{x\to x_0}\sqrt{\left(\frac{\Delta u}{\Delta x}\right)^2+\left(\frac{\Delta v}{\Delta x}\right)^2}=C\Rightarrow\lim_{\Delta x\to 0}\rho=0;$$

其中$C=\sqrt{\left(\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)}\right)^2+\left(\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)}\right)^2}$;

我们已经证明：$\lim_{\Delta x\to 0}\frac{\rho}{|\Delta x|}=C$，$\lim_{\Delta x\to 0}\rho=0$.

$$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)},\quad \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)};$$

$$\Delta z=\left.\frac{\partial z}{\partial u}\right|_{(u_0,u_0)}\Delta u+\left.\frac{\partial z}{\partial v}\right|_{(u_0,u_0)}\Delta v+o(\rho);$$

$$\begin{aligned}
&\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}=\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}\\
=&\left.\frac{\partial z}{\partial u}\right|_{(u_0,u_0)}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,u_0)}\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\\
&+\lim_{\Delta x\to 0}\frac{o(\rho)}{\rho}\cdot\frac{\rho}{|\Delta x|}\cdot\frac{|\Delta x|}{\Delta x}\\
=&\left.\frac{\partial z}{\partial u}\right|_{(u_0,u_0)}\cdot\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,u_0)}\cdot\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)};
\end{aligned}$$

- “函数$z=f(u,v)$在(u_0,v_0)处**可微分**” 这一条件是不能少。

如: $z=f(u,v)=u^{1/3}v^{1/3}$在$(u_0,v_0)=(0,0)$处可偏导、但是不可微分,

$$\left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)}=0,\ \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)}=0;$$

取$u=(x+y)^2$、$v=(x+y)$ 在$(x_0,y_0)=(0,0)$处可偏导. 复合函数为$z=(x+y)$, 它在$(x_0,y_0)=(0,0)$处

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0)}=1,\quad \left.\frac{\partial z}{\partial y}\right|_{(0,0)}=1;$$

而利用公式

$$\left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}=0\cdot 0+0\cdot 1=0$$

$$\left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}+\left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)}\cdot\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}=0\cdot 0+0\cdot 1=0.$$

例4(1) $z=(x+y)^{xy}$, 求$\frac{\partial z}{\partial x}$?

解法1.(直接法）$z=e^{xy\ln(x+y)}$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^{xy\ln(x+y)}(xy\ln(x+y))'_x\\ &= (x+y)^{xy}\left(y\ln(x+y)+\frac{xy}{x+y}\right);\end{aligned}$$

解法2.(复合函数求导法）取$u=x+y$、$v=xy$、$z=u^v$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\\ &= vu^{v-1}+yu^v\ln u = xy(x+y)^{xy-1}+y(x+y)^{xy}\ln(x+y)\\ &= (x+y)^{xy}\left(y\ln(x+y)+\frac{xy}{x+y}\right);\end{aligned}$$

例4(2)已知$f(u,v)$有一阶连续的偏导函数，$z=f(x^2,xy)$，求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$？

解: $u=x^2$, $v=xy$, $z=f(u,v)$.

$$f_1'(u,v)=\frac{\partial f(u,v)}{\partial u},\ f_2'(u,v)=\frac{\partial f(u,v)}{\partial v},$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=f_1'\cdot 2x+f_2'\cdot y=2xf_1'+yf_2';$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=f_1'\cdot 0+f_2'\cdot x=xf_2';$$

例4(3)已知$f(u,v)$有二阶连续的偏导函数，$z=f(x^2-y^2,xy)$，求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$、$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial x\partial y}$？

解: 记$u=x^2-y^2$, $v=xy$, $z=f(u,v)$.

$f_1'(u,v)=\frac{\partial f(u,v)}{\partial u}$, $f_2'(u,v)=\frac{\partial f(u,v)}{\partial v}$, $f_{11}''(u,v)=\frac{\partial^2 f(u,v)}{\partial u^2}$,

$f_{12}''(u,v)=\frac{\partial^2 f(u,v)}{\partial u\partial v}$, $f_{21}''(u,v)=\frac{\partial^2 f(u,v)}{\partial v\partial u}$, $f_{22}''(u,v)=\frac{\partial^2 f(u,v)}{\partial v^2}$.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\
&= f_1'(u,v)\cdot 2x+f_2'(u,v)\cdot y\\
&= 2xf_1'(x^2-y^2,xy)+yf_2'(x^2-y^2,xy);\\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\\
&= f_1'(u,v)\cdot(-2y)+f_2'(u,v)\cdot x\\
&= -2yf_1'(x^2-y^2,xy)+xf_2'(x^2-y^2,xy);
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 2xf_1'(u,v) + yf_2'(u,v);\\
\frac{\partial^2 z}{\partial x^2} &= 2f_1'(u,v) + 2x\frac{\partial}{\partial x}\left[f_1'(u,v)\right] + y\frac{\partial}{\partial x}\left[f_2'(u,v)\right]\\
&= 2f_1' + 2x\left[\frac{\partial f_1'}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1'}{\partial v}\frac{\partial v}{\partial x}\right] + y\left[\frac{\partial f_2'}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2'}{\partial v}\frac{\partial v}{\partial x}\right]\\
&= 2f_1' + 2x\left[f_{11}''\cdot 2x + f_{12}''\cdot y\right] + y\left[f_{21}''\cdot 2x + f_{22}'\cdot y\right]\\
&= 2f_1' + 4x^2 f_{11}'' + 4xyf_{12}'' + 2xyf_{22}'';\\
\frac{\partial^2 z}{\partial x\partial y} &= 2x\frac{\partial}{\partial y}\left[f_1'(u,v)\right] + f_2'(u,v) + +y\frac{\partial}{\partial y}\left[f_2'(u,v)\right]\\
&= 2x\left[\frac{\partial f_1'}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_1'}{\partial v}\frac{\partial v}{\partial y}\right] + f_2' + y\left[\frac{\partial f_2'}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_2'}{\partial v}\frac{\partial v}{\partial y}\right]\\
&= 2x\left[f_{11}''\cdot(-2y) + f_{12}''\cdot x\right] + f_2' + +y\left[f_{21}''\cdot(-2y) + f_{22}'\cdot x\right]\\
&= f_2' - 4xyf_{11}'' + (2x^2 - 2y^2)f_{12}'' + xyf_{22}'';
\end{aligned}
$$

例4(4) 已知$f(u,v,w)$有一阶连续的偏导函数，$z=f(x^2,\sin y,xy)$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$？

解:

$$\frac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot 0+f_3'\cdot y=2xf_1'+yf_3';$$

$$\frac{\partial z}{\partial y}=f_1'\cdot 0+f_2'\cdot\cos y+f_3'\cdot x=\cos yf_2'+xf_3';$$

例4(5) 已知$f(u,v)$有二阶连续的偏导函数，$z=f(xy,x+y^2)$，求$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial x\partial y}$？

解:

$$\frac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot 1=yf_1'(xy,x+y^2)+f_2'(xy,x+y^2);$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=y\left[f_{11}''\cdot y+f_{12}''\cdot 1\right]+\left[f_{21}''\cdot y+f_{22}''\cdot 1\right]\\&=y^2f_{11}''+2yf_{12}''+f_{22}''\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=f_1'+y\left[f_{11}''\cdot x+f_{12}''\cdot 2y\right]+\left[f_{21}''\cdot x+f_{22}''\cdot 2y\right]\\&=f_1'+xyf_{11}''+(x+2y^2)f_{12}''+2yf_{22}''\end{aligned}$$

例4(6) 已知$f(x,y)$在$(1,1)$处可微，$f(1,1)=1$，$\frac{\partial f}{\partial x}\Big|_{(1,1)}=2$，$\frac{\partial f}{\partial y}\Big|_{(1,1)}=3$. 记$\varphi(x)=f[x,f(x,x)]$，求$\frac{d\varphi(x)}{dx}\Big|_{x=1}$?

解:

$$
\begin{aligned}
\frac{d\varphi(x)}{dx}\Big|_{x=1} &= \left[f_1'[x,f(x,x)]+f_2'[x,f(x,x)]\cdot\frac{\partial f(x,x)}{\partial x}\right]_{x=1} \\
&= f_1'[1,f(1,1)]+f_2'[1,f(1,1)]\cdot\big(f_1'(x,x)+f_2'(x,x)\big)_{x=1} \\
&= f_1'[1,1]+f_2'(1,1)\cdot\big(f_1'(1,1)+f_2'(1,1)\big)=2+3(2+3)=12.
\end{aligned}
$$

例4(7) 取$u=\ln\sqrt{x^2+y^2}$, $v=\arctan\frac{y}{x}$. 函数$z=z(x,y)$满足

$$(x+y)\frac{\partial z}{\partial x}-(x-y)\frac{\partial z}{\partial y}=0.$$

试求函数$z=z(u,v)$所满足的方程。

解： $z=z(u,v)=z(x,y)$, $\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$,

$$\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},\ \frac{\partial v}{\partial x}=-\frac{y}{x^2+y^2},\ \frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}.$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{x}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{y}{x^2+y^2}\frac{\partial z}{\partial v};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{y}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{x}{x^2+y^2}\frac{\partial z}{\partial v};$$

$$\Rightarrow 0=(x+y)\frac{\partial z}{\partial x}-(x-y)\frac{\partial z}{\partial y}=2\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v};$$

$$\Rightarrow 2\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=0.$$

例4(8) $z=f(x,y)$有二阶连续偏导数，且满足

$$4\frac{\partial^2 z}{\partial x^2}+12\frac{\partial^2 z}{\partial x\partial y}+5\frac{\partial^2 z}{\partial y^2}=0.$$

试求常数a、b,使得在变换$u=x+ay$、$v=x+by$下函数$z=z(u,v)$满足的方程$\frac{\partial^2 z}{\partial u\partial v}=0$。

解： $z=z(u,v)=z(x,y)$;

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=a\frac{\partial z}{\partial u}+b\frac{\partial z}{\partial v};$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]=\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right]$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2};$$

$u=x+ay$、$v=x+by$,

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]=\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right]$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}$$

$$=a\frac{\partial^2 z}{\partial u^2}+(a+b)\frac{\partial^2 z}{\partial u\partial v}+b\frac{\partial^2 z}{\partial v^2};$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]=\frac{\partial}{\partial y}\left[a\frac{\partial z}{\partial u}+b\frac{\partial z}{\partial v}\right]$$

$$=a\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+a\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}+b\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y}+b\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}$$

$$=a^2\frac{\partial^2 z}{\partial u^2}+2ab\frac{\partial^2 z}{\partial u\partial v}+b^2\frac{\partial^2 z}{\partial v^2};$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2};$$

$$\frac{\partial^2 z}{\partial x \partial y} = a\frac{\partial^2 z}{\partial u^2} + (a+b)\frac{\partial^2 z}{\partial u \partial v} + b\frac{\partial^2 z}{\partial v^2};$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial u^2} + 2ab\frac{\partial^2 z}{\partial u \partial v} + b^2\frac{\partial^2 z}{\partial v^2};$$

$$\begin{aligned}\Rightarrow \qquad & 0 = 4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 5\frac{\partial^2 z}{\partial y^2} \\ = \quad & (4+12a+5a^2)\frac{\partial^2 z}{\partial u^2} + (8+12a+12b+10ab)\frac{\partial^2 z}{\partial u \partial v} \\ & +(4+12b+5b^2)\frac{\partial^2 z}{\partial v^2}.\end{aligned}$$

取$4+12a+5a^2=0$, $4+12b+5b^2=0$
及$8+12a+12b+10ab\neq 0$, 解得$a=-2$、$b=-\frac{2}{5}$.

例4(8) $z=f(x,y)$有二阶连续偏导数，且满足

$$4\frac{\partial^2 z}{\partial x^2}+12\frac{\partial^2 z}{\partial x\partial y}+5\frac{\partial^2 z}{\partial y^2}=0.$$

试求常数a、b,使得在变换$u=x+ay$、$v=x+by$下函数$z=z(u,v)$满足的方程$\frac{\partial^2 z}{\partial u\partial v}=0$。

注： $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u\partial v}=0$

$$\Rightarrow\frac{\partial z}{\partial v}=A(v),$$

$$z=\int A(v)dv+G(u)=F(v)+G(u)$$

$$\Rightarrow z=F(x-\frac{2}{5}y)+G(x-2y),$$

其中F、G为任意可微的一元函数。

全微分运算

由全微分的定义：

- 全微分的四则运算：

$$d(f \pm g) = df \pm dg;\ d(fg) = gdf + fdg,\ d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}.$$

- 一阶微分形式不变性：

$z = f(u, v)$、$u = \varphi(x, y)$及$v = \psi(x, y)$为可微函数，
则$z = f(\varphi(x, y), \psi(x, y))$可微，且

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy;$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy;$$

证明： $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$, $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$;

$$\begin{aligned}
dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\
&= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.
\end{aligned}$$

例5(1) $z = (x+y)^{xy}$, 试求dz、$\frac{\partial z}{\partial x}$及$\frac{\partial z}{\partial y}$?

解： $u = x + y$、$v = xy$、$z = u^v$,

$$
\begin{aligned}
dz &= du^v = v\, u^{v-1} du + u^v \ln u\, dv \\
&= xy \cdot (x+y)^{xy-1} d(x+y) + (x+y)^{xy} \ln(x+y)\, d(xy) \\
&= xy \cdot (x+y)^{xy-1} (dx + dy) + (x+y)^{xy} \ln(x+y)\, (ydx + ydx) \\
&= \left(xy \cdot (x+y)^{xy-1} + y(x+y)^{xy} \ln(x+y)\right) dx \\
&\quad + \left(xy \cdot (x+y)^{xy-1} + x(x+y)^{xy} \ln(x+y)\right) dy
\end{aligned}
$$

$$
\Rightarrow \frac{\partial z}{\partial x} = xy \cdot (x+y)^{xy-1} + y(x+y)^{xy} \ln(x+y),
$$

$$
\frac{\partial z}{\partial y} = xy \cdot (x+y)^{xy-1} + x(x+y)^{xy} \ln(x+y).
$$

例5(2) 函数f是可微的，$z=f(x^2+y^2,xy)$，试求dz、$\frac{\partial z}{\partial x}$及$\frac{\partial z}{\partial y}$?

解:

$$
\begin{aligned}
dz &= f_1' \cdot d(x^2+y^2) + f_2' \cdot d(xy) \\
&= f_1' \cdot (2xdx + 2ydy) + f_2' \cdot (ydx + xdy) \\
&= \left(2xf_1' + yf_2'\right) dx + \left(2yf_1' + xf_2'\right) dy;
\end{aligned}
$$

$$
\Rightarrow \frac{\partial z}{\partial x} = 2xf_1' + yf_2',
$$

$$
\frac{\partial z}{\partial y} = 2yf_1' + xf_2'.
$$

例5(3) 函数f是可微的，$z=f(x+y,x,xy)$，试求dz、$\frac{\partial z}{\partial x}$及$\frac{\partial z}{\partial y}$?

解：

$$\begin{aligned} dz &= f_1' \cdot d(x+y) + f_2' \cdot d(x) + f_3' \cdot d(xy) \\ &= f_1' \cdot (dx+dy) + f_2' \cdot dx + f_3' \cdot (ydx + xdy) \\ &= \left(f_1' + f_2' + yf_3'\right) dx + \left(f_1' + xf_3'\right) dy; \end{aligned}$$

$$\Rightarrow \frac{\partial z}{\partial x} = f_1' + f_2' + yf_3',$$

$$\frac{\partial z}{\partial y} = f_1' + xf_3'.$$

隐函数存在定理： 假设二元函数$F(x,y)$在$P_0(x_0,y_0)$的邻域内有一阶连续的偏导函数，

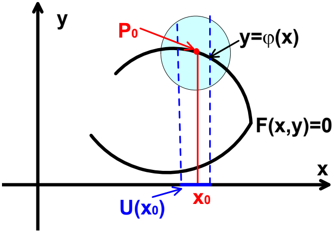
$$F(x_0,y_0)=0 \text{ 且 } F'_y(x_0,y_0)\neq 0.$$

则存在x_0的邻域$U(x_0)$、及$U(x_0)$内唯一的隐函数$y=\varphi(x)$满足

$$F(x,\varphi(x))=0,\ \varphi(x_0)=y_0;$$

隐函数$y=\varphi(x)$在$U(x_0)$内有连续的导数，且

$$\frac{dy}{dx}=\frac{d\varphi(x)}{dx}=-\frac{F'_x(x,\varphi(x))}{F'_y(x,\varphi(x))}.$$



证明省略。

例6(1). 函数$F(x,y)=x^2+y^2-1$在$(x_0,y_0)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ 的邻域内有一阶连续的偏导数$F'_x(x,y)=2x$, $F'_y(x,y)=2y$.

$$F(x_0,y_0)=F(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=0,\ F'_y(x_0,y_0)=\sqrt{2}\neq 0.$$

由隐函数存在定理，在邻域$(x_0-\delta,x_0+\delta)$内
由$F(x,y)=x^2+y^2-1=0$确定唯一解（隐函数）$y=\varphi(x)=\sqrt{1-x^2}$，它满足$\varphi(x_0)=\frac{1}{\sqrt{2}}=y_0$及

$$\frac{d\varphi(x)}{dx}=-\frac{x}{\sqrt{1-x^2}}=-\frac{F'_x(x,\varphi(x))}{F'_y(x,\varphi(x))}.$$

隐函数存在定理： 假设三元函数$F(x,y,z)$在(x_0,y_0,z_0)的邻域内有一阶连续的偏导函数，

$$F(x_0,y_0,z_0)=0 \text{ 且 } F_z'(x_0,y_0,z_0)\neq 0.$$

则存在(x_0,y_0)的邻域$U((x_0,y_0))$、及$U((x_0,y_0))$内唯一的隐函数$z=\varphi(x,y)$满足

隐函数存在定理： 假设三元函数$F(x,y,z)$在(x_0,y_0,z_0)的邻域内有一阶连续的偏导函数，

$$F(x_0,y_0,z_0)=0 \text{ 且 } F_z'(x_0,y_0,z_0)\neq 0.$$

则存在(x_0,y_0)的邻域$U((x_0,y_0))$、及$U((x_0,y_0))$内唯一的隐函数$z=\varphi(x,y)$满足

$$F(x,y,\varphi(x,y))=0,\ \varphi(x_0,y_0)=z_0$$

且隐函数$z=\varphi(x,y)$在$U((x_0,y_0))$内有连续的偏导数，

$$\frac{\partial\varphi(x,y)}{\partial x}=-\frac{F_x'(x,y,\varphi(x,y))}{F_z'(x,y,\varphi(x,y))},\ \frac{\partial\varphi(x,y)}{\partial y}=-\frac{F_y'(x,y,\varphi(x,y))}{F_z'(x,y,\varphi(x,y))}.$$

证明省略。

例6(2). 由$z^3-3xyz=9$确定隐函数$z=z(x,y)$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial x}$?

解法1: $F(x,y,z)=z^3-3xyz-9=0$,由隐函数存在定理

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}=-\frac{-3yz}{3z^2-3xy}=\frac{yz}{z^2-xy},$$

$$\frac{\partial z}{\partial y}=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}=-\frac{-3xz}{3z^2-3xy}=\frac{xz}{z^2-xy}.$$

解法2:(隐函数求导法则）隐函数$z=z(x,y)$满足

$$(z(x,y))^3-3xyz(x,y)-9=0$$

(x、y相互独立，z与(x,y)有关），同时关于x、y求偏导得

$$3z^2\frac{\partial z}{\partial x}-3yz-3xy\frac{\partial z}{\partial x}=0\Rightarrow\frac{\partial z}{\partial x}=\frac{yz}{z^2-xy}.$$

$$3z^2\frac{\partial z}{\partial y}-3xz-3xy\frac{\partial z}{\partial y}=0\Rightarrow\frac{\partial z}{\partial y}=\frac{xz}{z^2-xy}.$$

例6(3). F有连续的偏导函数，由$F(x+yz,x+z)=0$确定隐函数$z=z(x,y)$, 求$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial x}$?

解法1 $G(x,y,z)=F(x+yz,x+z)$,由隐函数存在定理

$$\frac{\partial z}{\partial x}=-\frac{G'_x(x,y,z)}{G'_z(x,y,z)}=-\frac{F'_1+F'_2}{yF'_1+F'_2};$$

$$\frac{\partial z}{\partial y}=-\frac{G'_y(x,y,z)}{G'_z(x,y,z)}=-\frac{zF'_1}{yF'_1+F'_2}.$$

解法2(隐函数求导法则）隐函数$z=z(x,y)$满足

$$F(x+yz,x+z)=0$$

(x、y相互独立，z与(x,y)有关），同时关于x、y求偏导得

$$F'_1\cdot\left(1+y\frac{\partial z}{\partial x}\right)+F'_2\cdot\left(1+\frac{\partial z}{\partial x}\right)=0\Rightarrow\frac{\partial z}{\partial x}=-\frac{F'_1+F'_2}{yF'_1+F'_2};$$

$$F'_1\cdot\left(z+y\frac{\partial z}{\partial y}\right)+F'_2\cdot\frac{\partial z}{\partial y}=0\Rightarrow\frac{\partial z}{\partial y}=-\frac{zF'_1}{yF'_1+F'_2}.$$

例6(4). 设$f(x,y,z)$、$\varphi(x,y,z)$有连续的偏导函数，由

$$u=f(x,y,z),\varphi(x^2,e^y,z)=0,y=\sin x$$

确定隐函数$z=z(x)$,求$\frac{dz}{dx}$及$\frac{du}{dx}$?

解： 由$\varphi(x^2,e^y,z)=0,y=\sin x$确定隐函数$z=z(x)$，同时关于$x$求导得

$$2x\varphi_1'+e^{\sin x}\cdot\cos x\cdot\varphi_2'+\varphi_3'\cdot\frac{dz}{dx}=0,$$

$$\Rightarrow\frac{dz}{dx}=-\frac{2x\varphi_1'+e^{\sin x}\cdot\cos x\cdot\varphi_2'}{\varphi_3'};$$

$$\begin{aligned}\frac{du}{dx}&=f_1'+f_2'\cdot\frac{dy}{dx}+f_3'\frac{dz}{dx}\\&=f_1'+\cos x\cdot f_2'-f_3'\cdot\frac{2x\varphi_1'+e^{\sin x}\cdot\cos x\cdot\varphi_2'}{\varphi_3'}.\end{aligned}$$

例6(5).由$e^z-xyz=0$确定隐函数$z=z(x,y)$,求$\frac{\partial^2 z}{\partial x^2}$及$\frac{\partial^2 z}{\partial x\partial y}$?

解：（x、y相互独立，z与(x,y)有关）关于x求偏导得

$$e^z\frac{\partial z}{\partial x}-yz-xy\frac{\partial z}{\partial x}=0\Rightarrow\frac{\partial z}{\partial x}=\frac{yz}{e^z-xy};$$

关于y求偏导得

$$e^z\frac{\partial z}{\partial y}-xz-xy\frac{\partial z}{\partial y}=0\Rightarrow\frac{\partial z}{\partial y}=\frac{xz}{e^z-xy};$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{yz}{e^z-xy}\right)\\&=\frac{y\frac{\partial z}{\partial x}(e^z-xy)-yz\left(e^z\frac{\partial z}{\partial x}-y\right)}{(e^z-xy)^2}\\&=\frac{(y^2z+yz^2)(e^z-xy)-y^2z^2e^z}{(e^z-xy)^3};\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{yz}{e^z - xy}\right) \\
&= \frac{(z + y\frac{\partial z}{\partial y})(e^z - xy) - yz\left(e^z\frac{\partial z}{\partial y} - x\right)}{(e^z - xy)^2} \\
&= \frac{(xyz + ze^z)(e^z - xy) - xyz^2 e^z}{(e^z - xy)^3};
\end{aligned}
$$

例6(6). 设φ有二阶导数且$\varphi' \neq 0$, 由$x^2 + y^2 - z = \varphi(x + y + z)$确定隐函数$z = z(x, y)$, $u = \frac{1}{x-y}\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$. 求$dz$及$\frac{\partial u}{\partial x}$?

解: 同时求微分得

$$d(x^2+y^2-z) = d\varphi(x+y+z) \Rightarrow 2xdx+2ydy-dz = \varphi'\cdot(dx+dy+dz)$$

$$\Rightarrow dz = \frac{1}{1+\varphi'}\left[(2x-\varphi')dx + (2y-\varphi')dy\right];$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{2x-\varphi'}{1+\varphi'}, \ \frac{\partial z}{\partial y} = \frac{2y-\varphi'}{1+\varphi'};$$

$$\Rightarrow u = \frac{1}{x-y}\left(\frac{2x-\varphi'}{1+\varphi'} - \frac{2y-\varphi'}{1+\varphi'}\right) = \frac{2}{1+\varphi'(x+y+z)};$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{2}{(1+\varphi')^2}\cdot\frac{\partial(1+\varphi'(x+y+z))}{\partial x} = -\frac{2}{(1+\varphi')^2}\cdot\varphi''\cdot\left(1+\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{2(2x+1)\varphi''}{(1+\varphi')^3}.$$

例6(7). 由$u^2+v^2-x^2-y^2=1$及$-u+v-xy=0$ 确定二个隐函数$u=u(x,y)$、$v=v(x,y)$. 求$\frac{\partial u}{\partial x}$?$\frac{\partial v}{\partial x}$?

解法1. x、y相互独立，u、v与(x,y)有关。关于x求偏导得

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}-2x=0,\ -\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}-y=0$$

解得

$$\frac{\partial u}{\partial x}=\frac{x-yv}{u+v},\ \frac{\partial v}{\partial x}=\frac{x+yu}{u+v}.$$

解法2. 同时求微分

$$2udu+2vdv-2xdx-2ydy=0,\ -du+dv-ydx-xdy=0;$$

解得

$$du=\frac{x-yv}{u+v}dx+\frac{y-xv}{u+v}dy,\ dv=\frac{x+yu}{u+v}dx+\frac{y+xu}{u+v}dy;$$

$$\Rightarrow\frac{\partial u}{\partial x}=\frac{x-yv}{u+v},\ \frac{\partial v}{\partial x}=\frac{x+yu}{u+v}.$$

例6(8). 取x作为函数，而y和z作为自变量，试变换下列方程

$$(x-z)\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$$

解： 记$z=f(x,y)$, $x=\varphi(y,z)$, 从而$z=f(\varphi(y,z),y)$. 关于z求偏导得

$$1=f_1'\cdot\varphi_2'\Rightarrow\frac{\partial z}{\partial x}=f_1'=\frac{1}{\varphi_2'}=\frac{1}{\varphi_z'};$$

关于y求偏导得

$$0=f_1'\cdot\varphi_1'+f_2'\Rightarrow\frac{\partial z}{\partial y}=f_2'=-\varphi_1'f_1'=-\frac{\varphi_y'}{\varphi_z'};$$

$$\Rightarrow 0=(x-z)\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{\varphi_z'}\left((x-z)-y\varphi_y'\right)$$

$$\Rightarrow\varphi_y'=\frac{x-z}{y}\Rightarrow\frac{\partial x}{\partial y}=\frac{x-z}{y}.$$

例6(9). 取x作为函数，而$u=y-z$和$v=y+z$作为自变量，试变换下列方程$(y-z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=0$.

解： 记$z=\varphi(x,y)$, $x=f(u,v)=f(y-z,y+z)$, 从而

$$x=f(y-\varphi(x,y),y+\varphi(x,y)).$$

关于x求偏导得

$$1=f'_u\cdot(-\varphi'_x)+f'_v\cdot(\varphi'_x)=(f'_v-f'_u)\cdot\varphi'_x;$$

关于y求偏导得

$$0=f'_u\cdot(1-\varphi'_y)+f'_v\cdot(1+\varphi'_y)=(f'_v-f'_u)\cdot\varphi'_x;$$

由上面二式解得

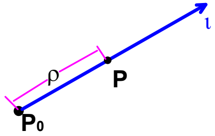
$$\frac{\partial z}{\partial x}=\varphi'_x=\frac{1}{f'_v-f'_u},\ \frac{\partial z}{\partial y}=\varphi'_y=\frac{f'_v+f'_u}{f'_v-f'_u};$$

代入$(y-z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=0$得

$$f'_u+f'_v=\frac{y-z}{y+z}=\frac{u}{v}\Rightarrow\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}=\frac{u}{v}.$$

方向导数与梯度

定义： 设函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$的某邻域内有定义，$\vec{\ell}$是从P_0出发的射线，$P(x,y,z)$为射线$\vec{\ell}$上的一点，记$\rho=|P_0P|$为点P_0与点P之间的距离。



若极限

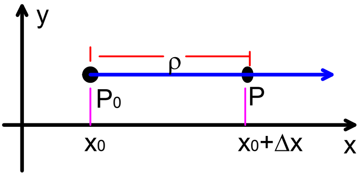
$$\lim_{\rho\to0^+}\frac{u(P)-u(P_0)}{\rho}=\lim_{\rho\to0^+}\frac{\Delta_{\vec{\ell}}u}{\rho}$$

存在，则称这个极限为函数$u(x,y,z)$

在$P_0(x_0,y_0,z_0)$处沿方向$\vec{\ell}$的方向导数，记为$\left.\frac{\partial u}{\partial\vec{\ell}}\right|_{P_0}$，

$$\left.\frac{\partial u}{\partial\vec{\ell}}\right|_{P_0}=\lim_{\rho\to0^+}\frac{u(P)-u(P_0)}{\rho}.$$

例7(1). 分别求函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$处沿x轴正方向$\vec{i}$与x轴反方向$-\vec{i}$的方向导数?

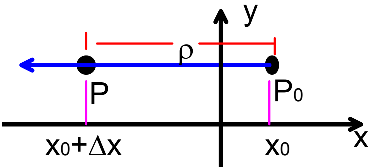


解： 设$P(x_0+\Delta x, y_0, z_0)$为经过$P_0(x_0,y_0,z_0)$且沿$x$轴正方向$\vec{i}$的射线上的任一点，$\Delta x > 0$且$\rho = \Delta x$，

$$\left.\frac{\partial u}{\partial \vec{i}}\right|_{P_0} = \lim_{\rho \to 0^+} \frac{u(x_0+\Delta x, y_0, z_0) - u(x_0, y_0, z_0)}{\rho}$$

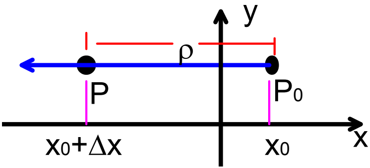
$$= \lim_{\Delta x \to 0^+} \frac{u(x_0+\Delta x, y_0, z_0) - u(x_0, y_0, z_0)}{\Delta x} = \left.\frac{\partial u}{\partial x}\right|_{P_0}.$$

例7(1). 分别求函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$处沿x轴正方向$\vec{i}$与x轴反方向$-\vec{i}$的方向导数?



设$P(x_0+\Delta x,y_0,z_0)$为经过$P_0(x_0,y_0,z_0)$且沿x轴反方向$-\vec{i}$的射线上的任一点,

例7(1). 分别求函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$处沿x轴正方向$\vec{i}$与x轴反方向$-\vec{i}$的方向导数?



设$P(x_0+\Delta x,y_0,z_0)$为经过$P_0(x_0,y_0,z_0)$且沿x轴反方向$-\vec{i}$的射线上的任一点，$\Delta x<0$且$\rho=-\Delta x$，

$$\left.\frac{\partial u}{\partial(-\vec{i})}\right|_{P_0}=\lim_{\rho\to0^+}\frac{u(x_0+\Delta x,y_0,z_0)-u(x_0,y_0,z_0)}{\rho}$$

$$=\lim_{\Delta x\to0^-}\frac{u(x_0+\Delta x,y_0,z_0)-u(x_0,y_0,z_0)}{-\Delta x}=-\left.\frac{\partial u}{\partial x}\right|_{P_0}.$$

例7(2). 求函数$u(x,y,z)=|x|+|y|+|z|$在$P_0(0,0,0,)$处沿$\vec{\ell}=\{a,b,c\}$ $(a^2+b^2+c^2\neq 0)$的方向导数?

解： 设$P(x,y,z)$为经过$P_0(0,0,0)$且沿$\vec{\ell}$轴方向的射线上的一点，则$\rho=|P_0P|$,

$$P(x,y,z)=P\left(\frac{a\rho}{\sqrt{a^2+b^2+c^2}},\frac{b\rho}{\sqrt{a^2+b^2+c^2}},\frac{c\rho}{\sqrt{a^2+b^2+c^2}}\right).$$

$$\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0}=\lim_{\rho\to 0^+}\frac{u(P)-u(P_0)}{\rho}=\lim_{\rho\to 0^+}\frac{|a\rho|+|b\rho|+|c\rho|}{\rho\sqrt{a^2+b^2+c^2}}$$
$$=\lim_{\rho\to 0^+}\frac{|a|+|b|+|c|}{\sqrt{a^2+b^2+c^2}}=\frac{|a|+|b|+|c|}{\sqrt{a^2+b^2+c^2}};$$

- $u(x,y,z)=|x|+|y|+|z|$在$P_0(0,0,0)$处没有偏导数，但是沿任何方向$\vec{\ell}$的方向导数$\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0}$都存在。

•函数$u(x,y,z)=|x|+|y|+|z|$在$P_0(0,0,0,)$处没有偏导数，但$P_0(0,0,0,)$处沿任何方向的方向导数都存在;

例7(3). 求函数$u(x,y,z)=\begin{cases}1, & xyz=0\text{时}\\ 0, & xyz\neq 0\text{时}\end{cases}$ 在$P_0(0,0,0)$ 处有偏导数，但是沿$\vec{\ell}=\{1,1,1\}$ 的方向导数不存在。

解： 由定义

$$u'_x(0,0,0)=\lim_{x\to 0}\frac{u(x,0,0)-u(0,0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0;$$

类似，$u'_y(0,0,0)=0,\ u'_z(0,0,0)=0;$

$$\left.\frac{\partial u}{\partial \ell}\right|_{(0,0,0)}=\lim_{\rho\to 0^+}\frac{u\left(\frac{\rho}{\sqrt{3}},\frac{\rho}{\sqrt{3}},\frac{\rho}{\sqrt{3}}\right)-u(0,0,0)}{\rho}=\lim_{\rho\to 0^+}\frac{0-1}{\rho}=\infty;$$

沿$\vec{\ell}=\{1,1,1\}$ 的方向导数 $\left.\frac{\partial u}{\partial \ell}\right|_{(0,0,0)}$不存在.

定理： 设函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$处可微分，方向$\vec{\ell}$的方向角为α、β、γ. 则

$$\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0}=\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\cos\gamma.$$

- 函数$u(x,y,z)$在$P_0(x_0,y_0,z_0)$处可微分时，才能用上述公式；事实上，例7(3)说明：函数在P_0处有偏导数并不能保证沿任何方向都有方向导数；

证明: 与$\vec{\ell}$同方向的单位矢量$\vec{\ell^o} = \{\cos\alpha, \cos\beta, \cos\gamma\}$,
设$P(x,y,z)$为经过$P_0(x_0,y_0,z_0)$且沿$\vec{\ell}$方向的射线上的一点,
则$\rho = |P_0P|$,

$$P(x,y,z) = P(x_0+\rho\cos\alpha, y_0+\rho\cos\beta, z_0+\rho\cos\gamma).$$

$$\begin{aligned}
&\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0} \\
&= \lim_{\rho\to 0^+}\frac{u(x_0+\rho\cos\alpha, y_0+\rho\cos\beta, z_0+\rho\cos\gamma) - u(x_0,y_0,z_0)}{\rho} \\
&= \lim_{\rho\to 0^+}\frac{\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\rho\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\rho\cos\beta + \left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\rho\cos\gamma + o(\rho)}{\rho} \\
&= \lim_{\rho\to 0^+}\left[\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta + \left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\cos\gamma + \frac{o(\rho)}{\rho}\right] \\
&= \left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta + \left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\cos\gamma;
\end{aligned}$$

例7(3). 求函数$u(x,y,z)=\ln(x+\sqrt{y^2+z^2})$在$A(1,0,1)$处沿$A$到$B(3,-2,2)$方向的方向导数?

解: $\vec{AB}=\{2,-2,1\}$,$\{\cos\alpha,\cos\beta,\cos\gamma\}=\frac{\vec{AB}}{|\vec{AB}|}=\{\frac{2}{3},-\frac{2}{3},\frac{1}{3}\}$.

$$\left.\frac{\partial u}{\partial x}\right|_A=\left.\frac{1}{x+\sqrt{y^2+z^2}}\right|_A=\frac{1}{2};$$

$$\left.\frac{\partial u}{\partial y}\right|_A=\left.\frac{y}{\sqrt{y^2+z^2}(x+\sqrt{y^2+z^2})}\right|_A=0;$$

$$\left.\frac{\partial u}{\partial z}\right|_A=\left.\frac{z}{\sqrt{y^2+z^2}(x+\sqrt{y^2+z^2})}\right|_A=\frac{1}{2};$$

$$\begin{aligned}\left.\frac{\partial u}{\partial \vec{AB}}\right|_A &= \left.\frac{\partial u}{\partial x}\right|_A\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_A\cdot\cos\beta+\left.\frac{\partial u}{\partial z}\right|_A\cdot\cos\gamma\\ &= \frac{1}{2}\times\frac{2}{3}+0\times\left(-\frac{2}{3}\right)+\frac{1}{2}\times\frac{1}{3}=\frac{1}{2};\end{aligned}$$

- 可微分三元函数$u=f(x,y,z)$, 方向$\vec{\ell}$的方向余弦为$\vec{\ell^o}=\frac{\vec{\ell}}{|\vec{\ell}|}=\{\cos\alpha,\cos\beta,\cos\gamma\}$, 则在$P_0(x_0,y_0,z_0)$处

$$\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0}=\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\cos\gamma.$$

- 可微分二元函数$u=f(x,y)$,方向$\vec{\ell}$的方向余弦为$\vec{\ell^o}=\frac{\vec{\ell}}{|\vec{\ell}|}=\{\cos\alpha,\cos\beta\}$, 则在$P_0(x_0,y_0)$处

$$\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{P_0}=\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta.$$

例7(4). 由$x^2+2y^2+3z^2+xy-z-9=0$确定隐函数$z=z(x,y)$, 求函数$z(x,y)$在$P_0(1,-2)$处沿方向$\vec{\ell}=\vec{i}+3\vec{j}$的方向导数?

解: 当$(x_0,y_0)=(1,-2)$时，解得$z_0=z(x_0,y_0)=1$; 在

$$x^2+2y^2+3z^2+xy-z-9=0$$

分别关于x、y求偏导（x与y独立变量，z与(x,y)有关）

$$2x+6z\frac{\partial z}{\partial x}+y-\frac{\partial z}{\partial x}=0, \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{P_0}=0;$$

$$4y+6z\frac{\partial z}{\partial y}+x-\frac{\partial z}{\partial y}=0, \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{P_0}=\frac{7}{5};$$

$$\vec{\ell}=\vec{i}+3\vec{j} \Rightarrow \{\cos\alpha,\cos\beta\}=\frac{\vec{\ell}}{|\vec{\ell}|}=\{\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\};$$

$$\Rightarrow \left.\frac{\partial z}{\partial \vec{\ell}}\right|_{P_0}=\left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta=\frac{21}{5\sqrt{10}}.$$

三元函数$u=f(x,y,z)$在$P_0(x_0,y_0,z_0)$处可微分，则沿方向$\vec{\ell^o}=\{\cos\alpha,\cos\beta,\cos\gamma\}$的方向导数为

$$\begin{aligned}\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0} &= \left.\frac{\partial u}{\partial x}\right|_{P_0}\cdot\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{P_0}\cdot\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{P_0}\cdot\cos\gamma \\ &= \left\{\left.\frac{\partial u}{\partial x}\right|_{P_0},\left.\frac{\partial u}{\partial y}\right|_{P_0},\left.\frac{\partial u}{\partial z}\right|_{P_0}\right\}\cdot\{\cos\alpha,\cos\beta,\cos\gamma\}=grad\ u|_{P_0}\cdot\vec{\ell^o}.\end{aligned}$$

记

$$grad\ u|_{P_0}=\left\{\left.\frac{\partial u}{\partial x}\right|_{P_0},\left.\frac{\partial u}{\partial y}\right|_{P_0},\left.\frac{\partial u}{\partial z}\right|_{P_0}\right\},$$

称为$u=f(x,y,z)$在$P_0(x_0,y_0,z_0)$处的**梯度**.

$$\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0}=grad\ u|_{P_0}\cdot\vec{\ell^o}:$$

梯度及含义

$$\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0} = grad\ u|_{P_0} \cdot \vec{\ell^o}:$$

▶方向导数 $\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0}$ 为梯度$grad\ u|_{p_0}$在$\vec{\ell^o}$方向的投影.

由 $\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0} = |grad\ u|_{P_0}| \cdot \cos < grad\ u|_{P_0}, \vec{\ell^o} >$得:

- 方向$\vec{\ell^o}$与$grad\ u|_{P_0}$同向时，方向导数 $\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0}$ 最大为$|grad\ u|_{P_0}|$;
- 方向$\vec{\ell^o}$与$grad\ u|_{P_0}$反向时，方向导数 $\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0}$ 最小

为$-|grad\ u|_{P_0}|$;

- 方向$\vec{\ell^o}$与$grad\ u|_{P_0}$垂直时，方向导数 $\left.\frac{\partial u}{\partial \vec{\ell^o}}\right|_{P_0} = 0$;

例7(5). 求函数$u=\ln(x^2+y^2+z^2)$在$P_0(1,2,-2)$处的梯度？问：函数u在$P_0(1,2,-2)$处沿什么方向的方向导数最大、最小？

解: 由定义

$$\begin{aligned} grad\ u|_{P_0} &= \left\{ \left.\frac{\partial u}{\partial x}\right|_{P_0}, \left.\frac{\partial u}{\partial y}\right|_{P_0}, \left.\frac{\partial u}{\partial z}\right|_{P_0} \right\} \\ &= \left.\frac{\{2x,2y,2z\}}{x^2+y^2+z^2}\right|_{P_0} = \left\{\frac{2}{9},\frac{4}{9},-\frac{4}{9}\right\}, \end{aligned}$$

•函数u在$P_0(1,2,-2)$处沿方向$\vec{\ell}=grad\ u|_{P_0}=\{\frac{2}{9},\frac{4}{9},-\frac{4}{9}\}$时方向导数为最大，等于$|grad\ u_{P_0}|=\frac{2}{3}$;

•函数u在$P_0(1,2,-2)$处沿方向$\vec{\ell}=-grad\ u|_{P_0}=\{-\frac{2}{9},-\frac{4}{9},\frac{4}{9}\}$时方向导数为最小，等于$-|grad\ u_{P_0}|=-\frac{2}{3}$;

一.泰勒定理（公式）

▶一元函数：$f(x)$在区间(a,b)内有$n+1$阶导数，$x_0\in(a,b)$，则

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\cdots \\ &\quad +\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x); \end{aligned}$$

其中$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, $\xi=x_0+\theta(x-x_0)$, $\theta\in(0,1)$;

▶多元函数？

泰勒定理

(以二元函数$f(x,y)$为例）记号：

$$\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial y}\right)^\ell f(x_0,y_0)=\left.\frac{\partial^{m+\ell}f(x,y)}{\partial x^m\partial y^\ell}\right|_{(x_0,y_0)};$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)=\sum_{\ell=0}^{m}C_m^\ell h^\ell k^{m-\ell}\left(\frac{\partial}{\partial x}\right)^\ell\left(\frac{\partial}{\partial y}\right)^{m-\ell}f(x_0,y_0);$$

例8(1). 取$f(x,y)=x^3y^5$, 求$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(2,1)$.

解. 由规定

$$
\begin{aligned}
& \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(2,1) \\
= & \left(h\frac{\partial}{\partial x}\right)^3 f(2,1)+3\left(h\frac{\partial}{\partial x}\right)^2\left(k\frac{\partial}{\partial y}\right) f(2,1) \\
& +3\left(h\frac{\partial}{\partial x}\right)\left(k\frac{\partial}{\partial y}\right)^2 f(2,1)+\left(k\frac{\partial}{\partial y}\right)^3 f(2,1) \\
= & h^3\left.\frac{\partial^3 f}{\partial x^3}\right|_{(2,1)}+3h^2k\left.\frac{\partial^3 f}{\partial x^2\partial y}\right|_{(2,1)} \\
& +3hk^2\left.\frac{\partial^3 f}{\partial x\partial y^2}\right|_{(2,1)}+k^3\left.\frac{\partial^3 f}{\partial y^3}\right|_{(2,1)} \\
= & 6h^3+180h^2k+720hk^2+240k^3;
\end{aligned}
$$

泰勒定理(带拉格朗日余项)

泰勒定理： 函数$f(x,y)$在D内有$n+1$阶偏导数，$(x_0,y_0)\in D$，记$h=x-x_0$, $k=y-y_0$. 则对$(x,y)\in D$，

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(\xi,\eta);$$

其中$(\xi,\eta)=(x_0+\theta h,y_0+\theta k)\in D$, $\theta\in(0,1)$;

证明： 取$g(t)=f(x_0+th,y_0+tk)$, 则$g(0)=f(x_0,y_0)$,

$g'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$,

$g^{(2)}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$, $\cdots\cdots$

$g^{(n)}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$,

$g^{(n+1)}(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(\xi,\eta)$. 利用

$g(t)=g(0)+g'(0)t+\frac{g^{(2)}(0)}{2!}t^2+\cdots+\frac{g^{(n)}(0)}{n!}t^n+R_n(t)$;

其中$R_n(t)=\frac{g^{(n+1)}(\theta t)}{(n+1)!}t^{n+1}$, $\theta\in(0,1)$; 由$f(x,y)=g(1)$得

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$
$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$
$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(\xi,\eta);$$

其中$(\xi,\eta)=(x_0+\theta h,y_0+\theta k)\in D$, $\theta\in(0,1)$;

泰勒定理： 函数$f(x,y)$在(x_0,y_0)有n阶偏导数，记$h=x-x_0$, $k=y-y_0$. 则

$$
\begin{aligned}
f(x,y) &= f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0) \\
&+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots \\
&+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+o(\rho^n);
\end{aligned}
$$

其中$\rho=\sqrt{h^2+k^2}\to 0$.

例8(2). 求函数$f(x,y)=\sin(2x+y)$在$(0,0)$处带皮亚诺余项的3阶泰勒定理？

解： $f(0,0)=0,\ f_x'(0,0)=2,\ f_y'(0,0)=1,\ f_{xx}^{(2)}(0,0)=0,$
$f_{xx}^{(2)}(0,0)=0,\ f_{xy}^{(2)}(0,0)=0,\ f_{yy}^{(2)}(0,0)=0,\ f_{xxx}^{(3)}(0,0)=-8,$
$f_{xxy}^{(3)}(0,0)=-4,\ f_{xyy}^{(3)}(0,0)=-2,\ f_{yyy}^{(3)}(0,0)=-1;$

$$\begin{aligned}f(x,y)&=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)\\&+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\frac{1}{3!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^3 f(0,0)+o(\rho^3);\end{aligned}$$

其中$\rho=\sqrt{x^2+y^2}\to 0.$

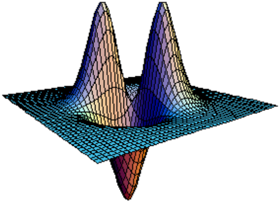
$$\begin{aligned}\sin(2x+y)&=0+(2x+y)+\frac{1}{2!}\left(x^2\cdot 0+2xy\cdot 0+y^2\cdot 0\right)+\\&+\frac{1}{3!}\left(-8x^3-12x^2y-6xy^2-y^3\right)+o(\rho^3);\end{aligned}$$

定义： 假设函数$f(x,y)$在$P_0(x_0,y_0)$的某邻域$U(P_0)$内有定义，若对$(x,y)\in U(P_0)$ 有

$$f(x,y)\geq f(x_0,y_0)\ (\text{或}f(x,y)\leq f(x_0,y_0)),$$

则称$f(x_0,y_0)$为函数$f(x,y)$的极小值(或极大值)，(x_0,y_0)称为函数的极小值点(或极大值点)。

• 函数$f(x,y)=\frac{xy}{e^{x^2+y^2}}$的极值



定理： (x_0,y_0)为函数$f(x,y)$的极值点、且$f(x,y)$在(x_0,y_0)有偏导数，则(x_0,y_0)必是驻点，即$f'_x(x_0,y_0)=f'_y(x_0,y_0)=0$.

证明： 不妨假设(x_0,y_0)为函数$f(x,y)$的极大值点，
取$g(x)=f(x,y_0)$, 则$x=x_0$为一元函数$g(x)$极大值点, 从而$0=g'(x_0)=f'_x(x_0,y_0)$. 类似可证：$f'_y(x_0,y_0)=0$.

►定理仅仅是一必要条件，而不是充分条件；如$f(x,y)=x^3y$满足$f'_x(0,0)=f'_y(0,0)=0$, 但是$(0,0)$不是$f(x,y)=x^3y$的极大值点、也不是极小值点。

定理:（充分条件）设(x_0, y_0)为函数$f(x, y)$的驻点，在(x_0, y_0)处有二阶偏导数，记

$$A = f''_{xx}(x_0, y_0),\ B = f''_{xy}(x_0, y_0),\ C = f''_{yy}(x_0, y_0).$$

则 (1). $AC - B^2 > 0$时, (x_0, y_0)为函数$f(x, y)$的极值点；
当$A > 0$时(x_0, y_0)为极小值点，当$A < 0$时(x_0, y_0)为极大值点；

(2). $AC - B^2 < 0$时, (x_0, y_0)不是函数$f(x, y)$的极值点；

(3). $AC - B^2 = 0$时, 不能用本定理；

▶上述定理仅仅对二元函数适用，对三元及以上函数必须作适当改变。

▶考虑函数$f_1(x, y) = x^4y^4$、$f_2(x, y) = -x^4y^4$及$f_3(x, y) = x^3y^3$,
在$(0, 0)$处同时满足$AC - B^2 = 0$,而点$(0, 0)$是f_1的极小值点、f_2的极大值点、不是f_3的极值点;

例8(3).求$f(x,y)=x^3+y^3-3xy$的极值?

解: 由$f'_x=3x^2-3y=0$, $f'_y=3y^2-3x=0$ 得驻点$(0,0)$及$(1,1)$;

$$f''_{xx}=6x,\ f''_{xy}=-3,\ f''_{yy}=6y.$$

•驻点$(0,0)$:

$$A=f''_{xx}(0,0)=0,\ B=f''_{xy}(0,0)=-3,\ C=f''_{yy}(0,0)=0.$$

满足$AC-B^2=-9<0$, 驻点$(0,0)$不是极值点;

•驻点$(1,1)$:

$$A=f''_{xx}(1,1)=6,\ B=f''_{xy}(1,1)=-3,\ C=f''_{yy}(1,1)=6.$$

满足$AC-B^2=27>0$且$A=6>0$, 驻点$(1,1)$是极小值点, 极小值$f(1,1)=-1$.

例8(4). 求$f(x,y)=x^3-y^3+3x^2+3y^2-9x$的极值？

解： 由$f_x'=3x^2+6x-9=0$, $f_y'=-3y^2+6y=0$ 得驻点$(-3,0)$、$(-3,2)$、$(1,0)$及$(1,2)$;

$$f_{xx}''=6x+6,\ f_{xy}''=0,\ f_{yy}''=-6y+6.$$

驻点	A	B	C	$AC-B^2$	f
(–3, 0)	–12	0	6	–	无极值
(1, 0)	12	0	6	+	极小值 –5
(–3, 2)	–12	0	–6	+	极大值 31
(1, 2)	12	0	–6	–	无极值

例8(5). 由$x^2-6xy+10y^2-2yz-z^2+18=0$确定隐函数$z=z(x,y)$, 求$z=z(x,y)$的极值？

解： 由$x^2-6xy+10y^2-2yz-z^2+18=0$关于$x$、$y$求偏导得

$$2x-6y-2y\frac{\partial z}{\partial x}-2z\frac{\partial z}{\partial x}=0,\Rightarrow \frac{\partial z}{\partial x}=\frac{x-3y}{y+z}$$

$$-6x+20y-2z-2y\frac{\partial z}{\partial y}-2z\frac{\partial z}{\partial y}=0,\Rightarrow \frac{\partial z}{\partial y}=\frac{10y-3x-z}{y+z};$$

由$\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$得驻点$(9,3)$(这时$z=3$)及$(-9,-3)$(这时$z=-3$);

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x-3y}{y+z}\right)=\frac{(y+z)-(x-3y)\frac{\partial z}{\partial x}}{(y+z)^2},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{x-3y}{y+z}\right) = \frac{-3(y+z)-(x-3y)(1+\frac{\partial z}{\partial y})}{(y+z)^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(10-\frac{\partial z}{\partial y})(y+z)-(10y-3x-z)(1+\frac{\partial z}{\partial y})}{(y+z)^2},$$

- 驻点$(9,3)$(这时$z=3$):

$$A = z''_{xx}(9,3) = \frac{1}{6},\ B = z''_{xy}(9,3) = -\frac{1}{2},\ C = z''_{yy}(9,3) = \frac{5}{3}.$$

满足$AC-B^2=\frac{1}{36}>0$且$A=\frac{1}{6}>0$, 驻点$(9,3)$是极小值点，极小值$z(9,3)=3$

- 驻点$(-9,-3)$(这时$z=-3$):

$$A = z''_{xx}(-9,-3) = -\frac{1}{6},\ B = z''_{xy}(-9,-3) = \frac{1}{2},\ C = z''_{yy}(-9,-3) = -\frac{5}{3}.$$

满足$AC-B^2=\frac{1}{36}>0$且$A=-\frac{1}{6}<0$, 驻点$(-9,-3)$是极大值点，极大值$z(-9,-3)=-3$.

•求最值的一般方法：

◦求函数在D内的所有驻点处的函数值；

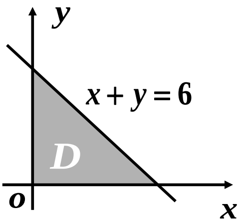
◦求函数在D的边界上的最大值和最小值

◦比较驻点处的函数值与边界上的最大值和最小值，其中最大者即为最大值，最小者即为最小值.

例8(6). 求二元函数

$$z = f(x, y) = x^2 y(4 - x - y)$$

在直线$x + y = 6$，x轴和y轴所围成的闭区域D上的最大值与最小值.



解： •先求函数在D内的驻点处的函数值. 解方程组

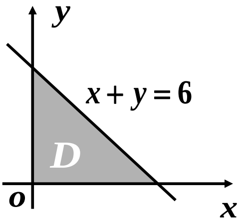
$$\begin{cases} f_x'(x, y) = 2xy(4 - x - y) - x^2 y = 0 \\ f_y'(x, y) = x^2(4 - x - y) - x^2 y = 0 \end{cases}$$

得区域D内唯一驻点$(2, 1)$, $f(2, 1) = 4$;

例8(6). 求二元函数

$$z = f(x,y) = x^2y(4-x-y)$$

在直线$x+y=6$，x轴和y轴所围成的闭区域D上的最大值与最小值.



•再求$f(x,y)$在D边界上的最值.

- 在边界$x=0$和$y=0$上$f(x,y)=0$;
- 在边界$x+y=6$上，即$y=6-x(0\le x\le 6)$时

$$z = f(x,6-x) = 2(x^3-6x^2)\,(0\le x\le 6);$$

由$\frac{dz}{dx}=6x(x-4)=0$得驻点$x=0$、$x=4$; 比较$f(0,6)=0$、$f(4,2)=-64$、$f(6,0)=0$得：函数$f(x,y)$在边界$x+y=6$上最大值$f(0,6)=f(6,0)=0$、最小值$f(4,2)=-64$;

•比较得：$f(x,y)$在区域D上的最大值$f(2,1)=4$、最小值$f(4,2)=-64$。

例8(7). 用铁皮做一个体积为V的长方体形无盖水箱，问：长、宽、高为何值时表面积最小？

直接解法： 设长、宽、高分别为$x>0$、$y>0$、$z>0$, 当$xyz=V$时,求表面积$S=2xz+2yz+xy$的最小值(条件极值）？ 由条件$xyz=V$得$z=\frac{V}{xy}$代入得

$$S=\frac{2V(x+y)}{xy}+xy,\ x>0,\ y>0$$

（无条件极值）。由

$$S'_x=y-\frac{2V}{x^2}=0,\ S'_y=x-\frac{2V}{y^2}=0$$

得（唯一）驻点$(x,y)=(\sqrt[3]{2V},\sqrt[3]{2V})$(这时$z=\frac{\sqrt[3]{2V}}{2}$). 根据实际问题，最小值一定存在，知当$x=y=\sqrt[3]{2V}$, $z=\frac{\sqrt[3]{2V}}{2}$时S取最小值$3\,(2V)^{2/3}$.

例8(7)可描述为：　在条件$\varphi(x, y, z) = 0$下, 如何求目标函数$u = f(x, y, z)$的最值？

▶在很多实际问题中，从限制条件中很难详细解出$z = \psi(x, y)$的表达式，直接解法有一定的困难；

问题： 在条件$\varphi(x,y,z)=0$下求目标函数$u=f(x,y,z)$的最值？

假设目标函数$u=f(x,y,z)$在(x_0,y_0,z_0)处取到极值，
由$\varphi(x,y,z)=0$得隐函数为$z=\psi(x,y)$. 则$\varphi(x_0,y_0,z_0)=0$
且$u=f(x,y,\psi(x,y))$在(x_0,y_0)取到极值. 从而

$$0=\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}=f'_x(x_0,y_0,z_0)+f'_z(x_0,y_0,z_0)\cdot\psi'_x(x_0,y_0)$$

$$0=\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}=f'_y(x_0,y_0,z_0)+f'_z(x_0,y_0,z_0)\cdot\psi'_y(x_0,y_0)$$

利用

$$\psi'_x(x_0,y_0)=-\frac{\varphi'_x(x_0,y_0,z_0)}{\varphi'_z(x_0,y_0,z_0)},\ \psi'_y(x_0,y_0)=-\frac{\varphi'_y(x_0,y_0,z_0)}{\varphi'_z(x_0,y_0,z_0)}.$$

$$0=f_x'(x_0,y_0,z_0)+f_z'(x_0,y_0,z_0)\cdot\left(-\frac{\varphi_x'(x_0,y_0,z_0)}{\varphi_z'(x_0,y_0,z_0)}\right),$$

$$0=f_y'(x_0,y_0,z_0)+f_z'(x_0,y_0,z_0)\cdot\left(-\frac{\varphi_y'(x_0,y_0,z_0)}{\varphi_z'(x_0,y_0,z_0)}\right).$$

记$\lambda_0=-\frac{f_z'(x_0,y_0,z_0)}{\varphi_z'(x_0,y_0,z_0)}$, 拉格朗日函数

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda\varphi(x,y,z).$$

则

$$\begin{cases} f_x'(x_0,y_0,z_0)+\lambda_0\varphi_x'(x_0,y_0,z_0)=0,\\ f_y'(x_0,y_0,z_0)+\lambda_0\varphi_y'(x_0,y_0,z_0)=0,\\ f_z'(x_0,y_0,z_0)+\lambda_0\varphi_z'(x_0,y_0,z_0)=0,\\ \varphi(x_0,y_0,z_0)=0, \end{cases} \Leftrightarrow \begin{cases} L_x'(x_0,y_0,z_0,\lambda_0)=0,\\ L_y'(x_0,y_0,z_0,\lambda_0)=0,\\ L_z'(x_0,y_0,z_0,\lambda_0)=0,\\ L_\lambda'(x_0,y_0,z_0,\lambda_0)=0, \end{cases}$$

从而有

定理: (必要条件)若在条件$\varphi(x,y,z)=0$下求目标函数$u=f(x,y,z)$有极值点(x_0,y_0,z_0)，则存在常数λ_0使得(x_0,y_0,z_0,λ_0)是拉格朗日函数

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda\varphi(x,y,z)$$

的驻点。

•问：若(x_0,y_0,z_0,λ_0)是拉格朗日函数

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda\varphi(x,y,z)$$

的驻点，如何判定(x_0,y_0,z_0)是条件$\varphi(x,y,z)=0$下目标函数$u=f(x,y,z)$的极值点（条件极值）？这是一个较困难的问题，一般需要根据具体的实际情况来确定。

问题： 在条件$\varphi(x,y,z)=0$下，求目标函数$u=f(x,y,z)$的最值？

求条件条件极值一般步骤：

(1). 引进拉格朗日函数

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda\varphi(x,y,z)$$

(2). 求拉格朗日函数的驻点(x_0,y_0,z_0,λ_0);

(3). 结合实际问题确定(x_0,y_0,z_0)为所求极值点；

▶要从理论上严格判别(x_0,y_0,z_0)为所求极值点，是一个非常困难的问题；

▶注意条件形式：$\varphi(x,y,z)=0$。

例8(4).用铁皮做一个体积为V的长方体形无盖水箱，问：长、宽、高为何值时表面积最小？

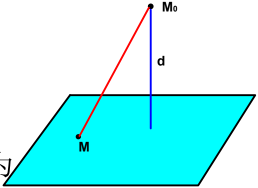
解：设长、宽、高分别为$x>0$、$y>0$、$z>0$, 则问题转化为：$xyz-V=0$时, 求表面积$S=2xz+2yz+xy$的最小值(条件极值)？引进拉格朗日函数

$$L(x,y,z,\lambda)=2xz+2yz+xy+\lambda(xyz-V);$$

$$\Rightarrow \begin{cases} L'_x(x,y,z,\lambda)=2z+y+\lambda yz=0, \\ L'_y(x,y,z,\lambda)=2z+x+\lambda xz=0, \\ L'_z(x,y,z,\lambda)=2x+2y+\lambda xy=0, \\ L'_\lambda(x,y,z,\lambda)=xyz-V=0, \end{cases}$$

解得拉格朗日函数的驻点$(\sqrt[3]{2V},\sqrt[3]{2V},\frac{\sqrt[3]{2V}}{2},\lambda_0)$; 由实际问题可以知条件极值一定存在最小值，从而$(\sqrt[3]{2V},\sqrt[3]{2V},\frac{\sqrt[3]{2V}}{2})$ 为所求极小值点，$S_{极小}=3\,(2V)^{2/3}$.

例8(8). 求平面外一点$M_0(x_0, y_0, z_0)$到平面$Ax + By + Cz + D = 0$ 的距离d?



解： 点M_0到平面上点$M(x, y, z)$的距离为

$$|M_0M| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

d^2为目标函数

$$f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

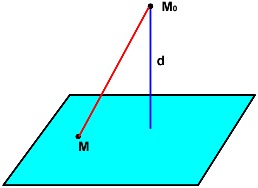
在条件$Ax + By + Cz + D = 0$下的最小值。引进拉格朗日函数

$$L(x, y, z, \lambda) = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2+\lambda(Ax+By+Cz+D);$$

$$\begin{cases} L'_x(x, y, z, \lambda) = 2(x - x_0) + \lambda A = 0, \\ L'_y(x, y, z, \lambda) = 2(y - y_0) + \lambda B = 0, \\ L'_z(x, y, z, \lambda) = 2(z - z_0) + \lambda C = 0, \\ L'_\lambda(x, y, z, \lambda) = Ax + By + Cz + D = 0, \end{cases}$$

解得拉格朗日函数的驻点

例8(8). 求平面外一点$M_0(x_0, y_0, z_0)$到平面$Ax + By + Cz + D = 0$ 的距离d?



拉格朗日函数的驻点

$$(x_0 - \frac{1}{2}A\lambda_0, y_0 - \frac{1}{2}B\lambda_0, z_0 - \frac{1}{2}C\lambda_0, \lambda_0),$$

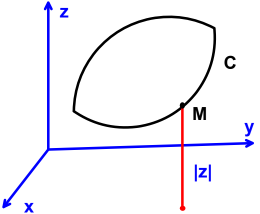
其中$\lambda_0 = \frac{2(Ax_0+By_0+Cz_0)}{A^2+B^2+C^2}$. 由实际问题可以知条件极值一定存在最小值，从而$(x_0 - \frac{1}{2}A\lambda_0, y_0 - \frac{1}{2}B\lambda_0, z_0 - \frac{1}{2}C\lambda_0)$为所求极小值点，

$$d = \frac{1}{2}|\lambda_0|\sqrt{A^2 + B^2 + C^2} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

例8(9). 求曲线

$$C:\begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z=5 \end{cases}$$

上到xoy平面最近与最远的点？



解： 转化为：函数$u=z^2$

在条件$x^2+y^2-2z^2=0$ 与$x+y+3z-5=0$下的条件极值。引进拉格朗日函数

$$L(x,y,z,\lambda,\mu)=z^2+\lambda(x^2+y^2-2z^2)+\mu(x+y+3z-5);$$

$$\begin{cases} L'_x(x,y,z,\lambda,\mu)=2\lambda x+\mu=0, \cdots\cdots(1) \\ L'_y(x,y,z,\lambda,\mu)=2\lambda y+\mu=0, \cdots\cdots(2) \\ L'_z(x,y,z,\lambda,\mu)=2z-4\lambda z+3\mu=0, \cdots\cdots(3) \\ L'_\lambda(x,y,z,\lambda,\mu)=x^2+y^2-2z^2=0, \cdots\cdots(4) \\ L'_\mu(x,y,z,\lambda,\mu)=x+y+3z-5=0, \cdots\cdots(5) \end{cases}$$

$(1)-(2)\Rightarrow \lambda(x-y)=0 \Rightarrow \lambda=0$或$x=y$;

$$\begin{cases} L'_x(x,y,z,\lambda,\mu) = 2\lambda x + \mu = 0, \cdots\cdots (1) \\ L'_y(x,y,z,\lambda,\mu) = 2\lambda y + \mu = 0, \cdots\cdots (2) \\ L'_z(x,y,z,\lambda,\mu) = 2z - 4\lambda z + 3\mu = 0, \cdots\cdots (3) \\ L'_\lambda(x,y,z,\lambda,\mu) = x^2 + y^2 - 2z^2 = 0, \cdots\cdots (4) \\ L'_\mu(x,y,z,\lambda,\mu) = x + y + 3z - 5 = 0, \cdots\cdots (5) \end{cases}$$

$(1)-(2) \Rightarrow \lambda(x-y) = 0 \Rightarrow \lambda = 0$或$x = y$；

- 若$\lambda = 0 \Rightarrow \mu = 0, (3)-(4) \Rightarrow x = y = z = 0$矛盾！
- 若$x = y$, $(4)-(5) \Rightarrow$

$$(x,y,z) = (1,1,1)\text{或}(x,y,z) = (-5.-5,-5).$$

从而拉格朗日函数的驻点为$(1,1,1,\frac{1}{5},-\frac{2}{5})$及$(-5.-5,-5,\frac{1}{5},2)$;
由实际问题可以知条件极值一定存在最小值及最大值，从而最近点为$(1,1,1)$、最远点为$(-5.-5,-5)$。

例8(10). 在区域$D=\{x^2+y^2-xy\le 75\}$上定义函数$h(x,y)=75-x^2-y^2+xy$.

(I). $M(x,y)\in D$, $g(x,y)$是函数$h(x,y)$在点M处最大的方向导数，求函数$g(x,y)$?

(II). 求函数$g(x,y)$在区域D边界$x^2+y^2-xy=75$上的最大、最小值点?

解： (I). 函数$h(x,y)$在点M处最大的方向导数

$$g(x,y)=|grad\ h|=|\{-2x+y,-2y+x\}|=\sqrt{(y-2x)^2+(x-2y)^2};$$

(II). $g(x,y)$在区域D边界$x^2+y^2-xy=75$上的最值点等同于

$$g^2(x,y)=5x^2+5y^2-8xy$$

在条件$x^2+y^2-xy=75$下的最值点; 引进拉格朗日函数

$$L(x,y,\lambda)=5x^2+5y^2-8xy+\lambda(x^2+y^2-xy-75);$$

引进拉格朗日函数

$$L(x,y,\lambda)=5x^2+5y^2-8xy+\lambda(x^2+y^2-xy-75);$$

$$\begin{cases} L'_x(x,y,\lambda)=10x-8y+\lambda(2x-y)=0,\\ L'_y(x,y,\lambda)=10y-8x+\lambda(2y-x)=0,\\ L'_\lambda(x,y,\lambda)=x^2+y^2-xy-75=0,\end{cases}$$

解得拉格朗日函数的驻点为

$$(5\sqrt{3},5\sqrt{3},\lambda_1),\ (-5\sqrt{3},-5\sqrt{3},\lambda_2),\ (5,-5,\lambda_3),\ (-5,5,\lambda_4),$$

它们为所有可能的极值点；直接计算得：

$g(5\sqrt{3},5\sqrt{3})=g(-5\sqrt{3},-5\sqrt{3})=5\sqrt{6}$,

$g(5,-5)=g(-5,5)=15\sqrt{2}$;

知$(5\sqrt{3},5\sqrt{3})$、$(-5\sqrt{3},-5\sqrt{3})$是最小值点，$(5,-5)$、$(-5,5)$是最大值点。

一. 空间曲线的切线与法平面

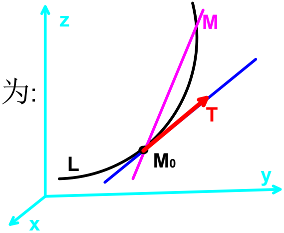
给定空间曲线（参数形式）

$$L: x=\varphi(t), y=\psi(t), z=h(t), \alpha<t<\beta;$$

如何求曲线L上点$M_0(x_0, y_0, z_0)$(相应于参数t_0)处的切线方程?

在曲线L上M_0附近任取一点M(相应于参数t); $M(\varphi(t), \psi(t), h(t))$,

$M_0(\varphi(t_0), \psi(t_0), h(t_0))$,则割线$M_0M$方程为:



$$\frac{x-x_0}{\varphi(t)-\varphi(t_0)}=\frac{y-y_0}{\psi(t)-\psi(t_0)}=\frac{z-z_0}{h(t)-h(t_0)};$$

$$\frac{x-x_0}{\frac{\varphi(t)-\varphi(t_0)}{t-t_0}}=\frac{y-y_0}{\frac{\psi(t)-\psi(t_0)}{t-t_0}}=\frac{z-z_0}{\frac{h(t)-h(t_0)}{t-t_0}};$$

当M沿曲线L趋于M_0时(等价于$t\to t_0$)，割线M_0M趋于$M_0(x_0, y_0, z_0)$处的切线。

割线M_0M为

$$\frac{x-x_0}{\frac{\varphi(t)-\varphi(t_0)}{t-t_0}}=\frac{y-y_0}{\frac{\psi(t)-\psi(t_0)}{t-t_0}}=\frac{z-z_0}{\frac{h(t)-h(t_0)}{t-t_0}};$$

令$t\to t_0$得$M_0(x_0,y_0,z_0)$处的切线方程为

$$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{h'(t_0)};$$

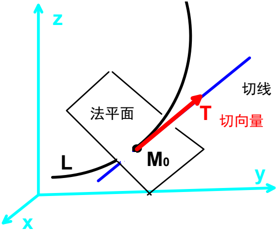
• 空间曲线

$L: x=\varphi(t), y=\psi(t), z=h(t)$

在 $M_0(x_0,y_0,z_0)$(相应于参数 t_0)处的

切线方程为

$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{h'(t_0)}$;



• $\vec{T}=\{\varphi'(t_0),\psi'(t_0),h'(t_0)\}$称为曲线L上点 M_0 处的**切向量**;

• 过点 M_0 且沿 M_0 处的切向量 $\vec{T}$ 方向的直线称为曲线L在点 M_0 处的切线,

$$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{h'(t_0)};$$

• 过点 M_0 且垂直于切向量 $\vec{T}$ 的平面为曲线L在点 M_0 处的**法平面**;

$$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+h'(t_0)(z-z_0)=0;$$

例9(1).求曲线$x=t,\ y=-t^2,\ z=t^3$上与平面$x+2y+z=4$平行的切线方程?

解： 设切线方程的切点为$M_0(t_0,-t_0^2,t_0^3)$, 则切向量为$\vec{T}=\{1,-2t_0,3t_0^2\}$;

切线//平面 $\Leftrightarrow \{1,-2t_0,3t_0^2\}\cdot\{1,2,1\}=0 \Leftrightarrow t_0=1$或$t_0=\dfrac{1}{3}$;

- $t_0=1$时，切向量$\vec{T}=\{1,-2,3\}$, 切点$M_0(1,-1,1)$, 切线方程为

$$\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-1}{3};$$

- $t_0=\frac{1}{3}$时，切向量$\vec{T}=\{1,-\frac{2}{3},\frac{1}{3}\}$, 切点$M_0(\frac{1}{3},-\frac{1}{9},\frac{1}{27})$, 切线方程为

$$\frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{9}}{-\frac{2}{3}}=\frac{z-\frac{1}{27}}{\frac{1}{3}};$$

例9(2). 求曲线 $\begin{cases} x^2+y^2+z^2=4 \\ x-2y+\sqrt{2}z=2 \end{cases}$ 在 $M_0(1,1,\sqrt{2})$ 处的切线方程与法平面方程？

解： 由 $\begin{cases} x^2+y^2+z^2=4 \\ x-2y+\sqrt{2}z=2 \end{cases}$ 确定隐函数 $\begin{cases} y=y(x) \\ z=z(x) \end{cases}$. 曲线有参数表示

$$x=x,\ y=y(x),\ z=z(x)$$

在 M_0 处的切向量 $\vec{T}=\{1,y'(x),z'(x)\}_{x=1}$; 由隐函数求导得

$$2x+2yy'+2zz'=0,\ 1-2y'+\sqrt{2}z'=0,$$

当 $(x,y,z)=(1,1,\sqrt{2})$ 时解得 $y'|_{x=1}=0,\ z'|_{x=1}=-\frac{\sqrt{2}}{2}$. 在 M_0 处的切向量 $\vec{T}=\{1,0,-\frac{\sqrt{2}}{2}\}$，切线方程为

$$\frac{x-1}{1}=\frac{y-1}{0}=\frac{z-\sqrt{2}}{-\frac{\sqrt{2}}{2}} \Leftrightarrow \frac{x-1}{2}=\frac{y-1}{0}=\frac{z-\sqrt{2}}{-\sqrt{2}};$$

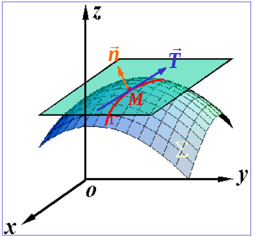
法平面方程为

$$(x-1)+0\cdot(y-1)-\frac{\sqrt{2}}{2}(z-\sqrt{2})=0 \Leftrightarrow 2x-\sqrt{2}z=0.$$

给定曲面Σ：$F(x,y,z)=0$及曲面Σ上点$M(x_0,y_0,z_0)$. 假设

$\Gamma:\ x=x(t),\ y=y(t),\ z=z(t)$

为曲面Σ上通过点$M(x_0,y_0,z_0)$（相应参数为t_0）的任意一条曲线,



由$F(x(t),y(t),z(t))=0$在$t=t_0$时关于t求导得

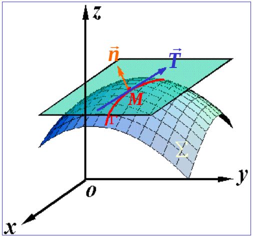
$$F'_x(x_0,y_0,z_0)\cdot\left.\frac{dx}{dt}\right|_{t=t_0}+F'_y(x_0,y_0,z_0)\cdot\left.\frac{dy}{dt}\right|_{t=t_0}+F'_z(x_0,y_0,z_0)\cdot\left.\frac{dz}{dt}\right|_{t=t_0}=0$$

$$\Rightarrow\{F'_x,F'_y,F'_z\}_M\cdot\left\{\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\}_{t=t_0}=0;$$

$$\left\{F'_x, F'_y, F'_z\right\}_M \cdot \left\{\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\}_{t=t_0} = 0;$$

注意到$\vec{n} = \left\{F'_x, F'_y, F'_z\right\}_M$与曲线Γ无关；

$\vec{T} = \left\{\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\}_{t=t_0}$是曲线Γ 在$M$ 处的切向量；



- 等式$\vec{n} \cdot \vec{T} = 0$说明：

曲面Σ上通过点M的所有曲线在M_0处的切线位于同一个平面上（过M且以$\vec{n}$为法向的平面）。

给定曲面Σ：$F(x, y, z) = 0$及曲面Σ上点$M(x_0, y_0, z_0)$,

- $\vec{n} = \left\{F'_x, F'_y, F'_z\right\}_M$称为曲面Σ在点$M$处的**法向量**；
- 过点M且与法向量$\vec{n}$垂直的平面称为曲面Σ在点M处的**切平面**：

$$F'_x(x_0, y_0, z_0)\cdot(x-x_0)+F'_y(x_0, y_0, z_0)\cdot(y-y_0)+F'_z(x_0, y_0, z_0)\cdot(z-z_0) = 0;$$

- 过点M且与法向量$\vec{n}$平行的直线称为曲面Σ在点M处的**法线**:

$$x-x_0 \qquad y-y_0 \qquad z-z_0$$

例9(3). 求曲面$x^2+2y^2=21-3z^2$平行于平面$x+4y+6z=0$的切平面方程?

解: 记切点为$M_0(x_0,y_0,z_0)$, 曲面在M_0处的法向量为$\vec{n}=\{2x_0,4y_0,6z_0\}$, 则

$$x_0^2+2y_0^2=21-3z_0^2,\ \text{且}\ \{2x_0,4y_0,6z_0\}//\{1,4,6\},$$

解得切点为

$$(x_0,y_0,z_0)=(1,2,2)\ \text{或}\ (x_0,y_0,z_0)=(-1,-2,-2);$$

- 过切点$(1,2,2)$的切平面方程

$$(x-1)+4(y-2)+6(z-2)=0\ \Leftrightarrow\ x+4y+6z-21=0;$$

- 过切点$(-1,-2,-2)$的切平面方程

$$(x+1)+4(y+2)+6(z+2)=0\ \Leftrightarrow\ x+4y+6z+21=0;$$

例9(4).平面π与曲面$z=x^2+y^2$相切于$M_0(1,-2,5)$，直线$L:\begin{cases} x+y+b=0 \\ x+ay-z-3=0 \end{cases}$ 在平面π上。求a、b?

解： 曲面$z=x^2+y^2$在$M_0(1,-2,5)$处的法向量$\vec{n}=\{2x,2y,-1\}_{M_0}=\{2,-4,-1\}$，平面$\pi$的方程为

$$2(x-1)-4(y+2)-(z-5)=0 \Leftrightarrow 2x-4y-z-5=0;$$

在直线L上取不同二点

$$A(-b,0,-3-b),\ B(-b-1,1,a-b-4),$$

点A与B都在平面π上，

$$\Rightarrow -2b+3+b-5=0,-b-a-7=0$$

$$\Rightarrow a=-5,\ b=-2.$$