# First Principles: A Mathematics Handbook

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## Contents

*For my brother, Samuel,*
*whose strength of character, humour, and curiosity*
*continue to inspire my work and life.*

# Introduction

## Purpose of This Guide

This handbook aims to be a concise yet comprehensive reference for key mathematical first principles, covering topics essential for high school mathematics, university preparation, and STEM-related fields. It is designed for fast navigation, allowing readers to look up topics easily, find clear explanations, quickly learn new topics or revise specific ones.

## How to Use This Handbook

Each chapter focuses on a single topic, beginning with definitions and essential formulas. Mathematical notation follows a standardised convention throughout. Where topics are built upon, cross-references to other sections are included for further exploration or for revisiting relevant material.

## Notation and Conventions

- Definitions are shown in *italics*.
- Variables are italicised (e.g. $x, y, z$).
- Constants such as $e$ or $\pi$ retain their standard mathematical meaning.
- Essential formulas are highlighted in boxes (e.g. $\boxed{e^{i\pi} + 1 = 0}$).
- All angles are expressed in radians unless stated otherwise.

# 1 Exponents and Logarithms

## 1.1 Exponents

An exponent is written as $a^n$, where $a$ is the *base*, and $n$ is the *power* (also known as *index* or *exponent*).

Exponents follow a set of algebraic rules:

| **Rules of Exponents** |
|---|
| $a^m \cdot a^n = a^{m+n}$ |
| $a^m \div a^n = a^{m-n}$ |
| $(a^m)^n = a^{mn}$ |
| $a^0 = 1$ |
| $a^1 = a$ |
| $a^{-m} = \frac{1}{a^m}$ |
| $a^{\frac{1}{m}} = \sqrt[m]{a}$ |
| $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ |
| $(ab)^n = a^n \cdot b^n$ |
| $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ |

## 1.2 Logarithms

The logarithmic function $\log_a b = x$ is the solution to the exponential equation $a^x = b$. In other words, $\log_a b$ gives the power to which the base $a$ needs to be raised in order to produce $b$. Therefore,

$$\boxed{a^x = b \Longleftrightarrow \log_a b = x}$$

**Base restrictions:** Unless stated otherwise, $a > 0, a \neq 1$, and $b > 0$. (An exception would be for complex logarithms, for example.)

- The logarithm with base $e$ is written as $\ln x$ (natural logarithm).
- The logarithm with base 10 is typically written as $\log x$ (common logarithm).

| **Laws of logarithms** |
|---|
| $\log_a x + \log_a y = \log_a (xy)$ |
| $\log_a x - \log_a y = \log_a (\frac{x}{y})$ |
| $n \log_a x = \log_a (x^n)$ |

### 1.2.1 Logarithmic scales

When dealing with incredibly small or large numbers, it becomes difficult to distinguish them, so they can be scaled using logarithms.

## 1.3 Euler's number and the natural logarithm

Euler's number ($e \approx 2.718\ldots$) is an irrational constant, which arises in situations involving growth or decay, such as compound interest or radioactive decay.

A definition for $e$ can be found in finance, through continuous compounding:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

where $n$ is the number of compounding periods per unit time, and $e$ is the resulting constant as $n$ grows infinitely large (Euler's number). See **4. Finance** for more information.

The *natural logarithm*, written as $\ln x$, is the logarithm with base $e$:

$$\boxed{\ln x = \log_e x}$$

> **Note:** $e^x$ and $\ln x$ have unique properties in calculus. See **Chapter** [[**14. Differentiation**]], and **Chapter** [[**15. Integration**]] for more information.

# 2 Sequences and Series

A numerical *sequence* is an ordered list of numbers which can often be described by a formula or recurrence relation. Each number within the sequence is called a *term*. The general term within a sequence is called the $n^{th}$ term.

A numerical *series* is the sum of a range of terms in a sequence. A *finite series* describes a series which has $n$ terms ($u_1, u_2, u_3, \ldots, u_n$, where $u_n$ is the $n^{th}$ term of the sequence), and thus also has a finite sum. An *infinite series* describes a series which has infinite terms ($u_1, u_2, u_3, \ldots$), which can *converge* to a finite sum or *diverge* to infinity.

## 2.1 Sigma and Product Notation

*Sigma notation* can be useful when describing a series concisely. For example, the sum:

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_{100}$$

can be rewritten using sigma notation as:

$$\sum_{n=1}^{100} u_n$$

In general,

$$\boxed{\sum_{n=a}^{b} u_n}$$

represents the sum of all terms from $n = a$ to $n = b$.

Sigma notation also follows certain important algebraic properties:

**Addition Property:**

$$\boxed{\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n}$$

**Constant Multiplication Property:**

Given $c$ is a constant:

$$\boxed{\sum_{n=1}^{k} c \cdot u_n = c \cdot \sum_{n=1}^{k} u_n}$$

*Product notation* is similar to sigma notation. It also represents a repeated operation, but it denotes multiplication rather than summation. For example, the product:

$$u_1 \cdot u_2 \cdot u_3 \cdot \dots \cdot u_{100}$$

can be written concisely as:

$$\prod_{n=1}^{100} u_n$$

In general, the product of all terms from $n = a$ to $n = b$ can be written as:

$$\boxed{\prod_{n=a}^{b} u_n}$$

## 2.2 Arithmetic Sequences and Series

An *arithmetic sequence* (sometimes called an arithmetic *progression*) is a sequence where each term differs from the previous term by a constant value, called the *common difference*, typically denoted as $d$.

The general formula for the $n^{th}$ term of an arithmetic sequence is given by:

$$\boxed{u_n = u_1 + (n-1)d}$$

where: - $u_n$ is the $n^{th}$ term, - $d$ is the common difference, - $n$ is the position of the term (e.g., for the 5th term, $n = 5$).

The sum of a finite *arithmetic series* is given by:

$$\boxed{S_n = \frac{n}{2}(u_1 + u_n) \quad \text{OR} \quad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)}$$

where $S_n$ is the sum of the first $n$ terms.

## 2.3 Geometric Sequences and Series

A *geometric sequence* (sometimes called a geometric *progression*) is a sequence where each term is the product of the previous term and a constant value, called the *common ratio*, typically denoted as $r$.

The general formula for the $n^{th}$ term of a geometric sequence is given by:

$$\boxed{u_n = u_1 r^{n-1}}$$

The sum of a finite *geometric series* is given by:

$$\boxed{S_n = \frac{u_1(r^n - 1)}{r - 1} \quad \text{OR} \quad S_n = \frac{u_1(1 - r^n)}{1 - r}} \quad \text{where} \quad r \neq 1.$$

where $r$ is the common ratio.

If $|r| < 1$, the geometric series converges as $n \to \infty$, resulting in an infinite geometric series with sum:

$$\boxed{S_\infty = \frac{u_1}{1 - r}} \quad \text{where} \quad |r| < 1.$$

# 3 Functions

A *relation* between two variables $x$ and $y$, is any set of points which are on the $(x, y)$ plane.

There exist four different types of relations:

- One to one
- One to many
- Many to one
- Many to many

A *function* is defined as a mapping onto a single value. Therefore one to one, and many to one relations are considered to be functions.

## 3.1 Testing if a relation is a function

1. **Algebraic Method:** If we substitute any value of $x$ and it results in a singular $y$-value, then it can be considered a function.
2. **Graphical Method (Vertical Line Test):** If we are given a plot of a function on the $(x, y)$ plane, where the $x$-axis is parallel with the horizontal, and the $y$-axis is parallel with the vertical, then if we are able to draw a vertical line anywhere on the graph, and it only intersects the plot once, it is a function. If the vertical line intersects the plot more than once, then it cannot be considered a function.

## 3.2 Domain and Range

## 3.3 Linear functions

*Linear functions* are functions which can be written in the form:

$$\boxed{y = mx + c} \quad \text{OR} \quad \boxed{y = m(x - x_1) + y_1}$$

## 3.4 Quadratic functions

### 3.4.1 Line of symmetry

For functions in the form $f(x) = ax^2 + bx + c$, the axis of symmetry of is

$$\boxed{x = -\frac{b}{2a}}$$

### 3.4.2 Discriminant

$$\boxed{\Delta = b^2 - 4ac}$$

### 3.4.3 The Quadratic formula

The solutions to $ax^2 + bx + c = 0$ are:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \quad \text{where } a \neq 0$$

# 4 Finance

## 4.1 Important Concepts

- *Interest* is money paid at a regular rate in order to use lent money, or to delay the repayment of debt.
- *Inflation* is the increase in the price of goods and services with respect to time. Inflation means that a fixed amount of capital will be able to purchase fewer goods and services as prices increase.
- *Depreciation* is the decrease in the value of an asset with respect to time.
- The *principal* is the original loan amount that is borrowed.
- The *principal portion* of a payment is the portion that reduces the outstanding loan balance (principal).
- The *interest portion* is the payment covering the interest charged for that period.
- *Amortisation* is a process whereby a loan is repaid through a series of regular payments. In an amortised loan, interest for each period is calculated on the outstanding loan balance, therefore as regular payments reduce the principal, the interest portion of each payment decreases over time, whilst the principal portion increases.

## 4.2 Parameters and Conventions

In order to work with financial formulas, it is crucial to first understand the standard notation on which they depend.

| Calculator Label | Symbol in This Book | Meaning |
|---|---|---|
| N | $N$ | Total number of compounding periods |
| I% | $r$ | Nominal annual interest rate (%) |
| | $r_{\text{eff}}$ | Effective annual interest rate (%) |
| PV | $V_0$ | Present value of the loan/investment |
| PMT | $M$ | Payment per period |
| FV | $V_f$ | Future value of loan/investment |
| P/Y | $n_p$ | Number of payments per year |
| C/Y | $n_c$ | Number of compounding periods per year |

> **Note:** This topic is primarily solved using calculators or software tools which allow for the automation of iterative calculations. Regardless, it is important to recognise the meaning of each parameter and their relation to the relevant formulae.

## 4.3 Interest Rates

*Interest rates* are a measure of the cost of borrowing money, or the return on investment over a set period of time.

$$\boxed{\text{Interest Paid} = \text{Total Repayments - Amount Borrowed}}$$

### 4.3.1 Nominal and Effective Rates

The *nominal annual interest rate* ($r$) is the yearly rate of interest which does not consider compounding. Financial calculators often denote this as I%.

The *effective annual interest rate* ($r_{\text{eff}}$) considers the effect of compounding throughout the year. It reflects the percentage increase in value, or cost of debt, over one year.

The formula for the effective annual interest rate is given as follows:

$$\boxed{r_{\text{eff}} = \left(1 + \frac{r}{n_c}\right)^{n_c} - 1}$$

## 4.4 Compound Interest

*Compound interest* is when the interest is calculated using any accumulated interest from previous periods in addition to the original principal.

The general formula for compound interest is:

$$\boxed{V_f = V_0 \left(1 + \frac{r}{n_c}\right)^{n_c t}}$$

where $t$ is the time in years.

## 4.5 Continuous Compounding

When compounding occurs infinitely often (as $n_c \to \infty$), the formula becomes:

$$\boxed{V_f = V_0 e^{rt}}$$

# 5 Sets

A *set* is a collection of distinct objects, called *elements* (or *members*). They are typically denoted by a capital letter (such as $A, B, C$, etc.), and their elements can be written inside curly brackets. For example:

$$A = \{1, 2, 3, 4, e, 9.\overline{9}, 8086\}$$

Here, $1 \in A$ denotes that "1 is an element of $A$", and $5 \notin A$ means that "5 is not an element of $A$".

## 5.1 Types of Sets

| Name | Description | Example |
|---|---|---|
| **Finite set** | Contains a finite number of elements. | $B = \{1, 2, 3\}$ |
| **Infinite set** | Contains an infinite number of elements | $\mathbb{N} = \{1, 2, 3, \dots\}$ |
| **Singleton set** | Contains exactly one element. | $C = \{256\}$ |
| **Universal set** | Contains all objects under consideration in a certain context. Denoted as $U$. | |
| **Empty set** | (Sometimes called the *null set*) Contains no elements, and is denoted as $\emptyset$ or $\{\}$ | |

## 5.2 Subsets

A set $A$ is considered a *subset* of $B$ if every element in $A$ is also an element of $B$. In which case:

$$A \subseteq B$$

which means: "$A$ is a subset of or equal to $B$".

If $A$ is a subset of $B$, but $A \neq B$, then $A$ is considered a *proper subset* of $B$, shown as:

$$A \subset B$$

The empty set ($\emptyset$) is a subset of every set, including singletons, since it has no elements.

### 5.2.1 -> mutual exclusivity

$$A \cap B = \emptyset$$

# 6 Combinatorics

## 6.1 Basic Counting Principles

### 6.1.1 Multiplication Principle

The *Fundamental Counting Principle* (or the *Multiplication Principle*) states that if an event can occur in $m$ ways, and another *independent* event is able to occur in $n$ ways, then the total number of unique ways both events can occur is:

$$m \cdot n$$

For example, if there are $m$ Formula 1 drivers, and $n$ Formula 1 cars, each driver can be paired with any car. Therefore there are a total of $mn$ different driver–car combinations that can be made.

This principle can be generalised to any number of events. If there are $k$ independent events, where the $i^{\text{th}}$ event can occur in $n_i$ ways, then the total number of unique ways all of the events can occur is:

$$\boxed{\prod_{i=1}^{k} n_i}$$

### 6.1.2 Addition Principle

The *Addition Principle* states that given an event can occur in $m$ ways, and another *mutually exclusive* event (recall mutual exclusivity from **Chapter** [[**5. Sets**]]) can occur in $n$ ways, then the total number of ways either event can occur is:

$$m + n$$

Which can be generalised in the same way. If there are $k$ mutually exclusive events, where the the $i^{\text{th}}$ event can occur in $n_i$ ways, then the total number of possible outcomes is:

$$\boxed{\sum_{i=1}^{k} n_i}$$

## 6.2 Factorials

The *factorial* of a positive integer $n$ is written as $n!$, and defined as the product of all positive integers from 1 to $n$:

$$\boxed{n! = \prod_{i=1}^{n} i, \quad n \geq 1}$$

By convention, $0! = 1$.

## 6.3 Permutations

A *permutation* is a unique ordering of a set of objects. For example $ABC$, and $CBA$ are both permutations of three letters $A$, $B$, and $C$.

> **Note:** If you take $AAB$, swapping the position of the two $A$s is still considered the same permutation, since the result is still $AAB$.

The number of permutations of $n$ distinct objects taken in groups of $r$ at a time, is written as $nPr$, is given by:

$$\boxed{nPr = \frac{n!}{(n-r)!} \quad \text{(Assuming no identical objects)}}$$

From here we can see that if all $r$ objects are used ($r = n$):

$$nPr = \frac{n!}{(n-n)!} = \frac{n!}{(0)!} = \frac{n!}{1} = n!$$

So the special case where all objects are used ($r = n$):

$$\boxed{nPr = n! \quad \text{when } r = n}$$

### 6.3.1 Permutations with Identical Objects

The formula for the number of permutations given earlier assumes all objects are distinct. In situations where some objects are identical (e.g. the example given earlier with $AAB$), there are fewer distinct permutations, as some are repeated.

The general formula is:

$$\boxed{\frac{n!}{\prod_{i=1}^{r} n_i!}}$$

where $n$ is the total number of objects, $n_1, n_2, \ldots, n_r$ are the counts for each group of identical objects.

**Worked Example 6A**

Find the number of permutations of the letters in the word: MISSISSIPPI.

We use the general formula for permutations with identical objects:

$$\frac{n!}{\prod_{i=1}^{r} n_i!}$$

MISSISSIPPI has $n = 11$ letters, with 4 'I's, 4 'S's, 2 'P's, and 1 'M'.

So the number of permutations is:

$$\frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 4! \cdot 2}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4! \cdot 2}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{24 \cdot 2}$$

Simplifying the denominator: $24 \cdot 2 = 48 = 6 \cdot 8$, which can be cancelled from the numerator:

$= 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5$
$= 990 \cdot 7 \cdot 5$
$= 34\,650$ permutations

## 6.4 Combinations

A *combination* is a set of objects, meaning that order does not matter. For example $\{A, B, C\}$ and $\{C, B, A\}$ are the same combination of three letters, irrespective of the fact they are different permutations.

Since each group of $r$ objects can be arranged in $r!$ different orders, all representing the same combination, the number of combinations of $n$ distinct objects in groups of $r$, written as $nCr$ or $\binom{n}{r}$ ("n choose r"), is given by:

$$\boxed{nCr = \binom{n}{r} = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}}$$

We also define that $nC0 = 1$ (there is one way to choose nothing), and $nCn = 1$

(there is one way to choose all objects):

$$\boxed{\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1}$$

## 6.5 Partitions of Sets

A *partition* of a set is a way to group its elements into non-empty subsets, where each element belongs to exactly one subset.

For example, the set $\{A, B, C\}$ has the following partitions:

- $\{\{A, B, C\}\}$
- $\{\{A\}\{B, C\}\}$
- $\{\{B\}\{A, C\}\}$
- $\{\{C\}\{A, B\}\}$
- $\{\{A\}\{B\}\{C\}\}$

Therefore there are 5 different ways to partition the set $\{A, B, C\}$ in total.

## 6.6 Partitions of Objects (Stars and Bars)

The *stars and bars* method is a way to count the number of ways $n$ identical objects can be distributed into $r$ distinct groups. The method involves representing the $n$ objects as *"stars"* ( $\star$ ), and the partitions between the groups as *"bars"* ( | ).

So taking another example of 5 objects and 3 groups, may look like:

$$\star \star \mid \star \mid \star \star$$

This represents the grouping of $(2, 1, 2)$.

Since $r - 1$ bars are required to create $r$ groups, there are a total of $n + r - 1$ slots (positions). The number of ways to do this is:

$$\boxed{\binom{n + r - 1}{r - 1}}$$

## 6.7 Binomial Expansion

The *binomial theorem* gives a formula allowing us to expand powers of a binomial expression of the form $(a + b)^n$, without having to multiply out each term by hand. It states that (for any non-negative integer $n$):

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r}$$

For example, the binomial expansion for $n = 3$ is:

$$(a+b)^3 = \binom{3}{0} a^3 + \binom{3}{1} a^2 b + \binom{3}{2} a b^2 + \binom{3}{3} b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

### 6.7.1 Pascal's Triangle

The coefficients of $\binom{n}{r}$ which appear in the expansion can also be found through *Pascal's triangle*, where each number is the sum of the two numbers directly above it:

$$\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

## 6.8 Multinomial Theorem

The *multinomial theorem* is a generalisation of the binomial theorem to allow for the expansion of powers of a sum with more than two terms. This allows it to be applied to expressions such as $(a + b + c + \dots)^n$. It is formally defined as:

$$\boxed{\left(\sum_{i=1}^{m} x_i\right)^n = \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,k_2,\dots,k_m \geq 0}} \frac{n!}{\prod_{j=1}^{m} k_j!} \prod_{r=1}^{m} x_r^{k_r}}$$

Although this form is relatively cluttered, so typically, the *multinomial* coefficient is introduced:

$$\boxed{\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{\prod_{i=1}^{m} k_i!}}, \text{ where } k_1 + k_2 + \dots + k_m = n$$

Notice that for $m = 2$, this becomes the binomial coefficient:

$$\binom{n}{k_1, k_2} = \frac{n!}{k_1!k_2!},$$

Since $k_1 + k_2 = n$, we can write $k_2 = n - k_1$, giving:

$$\binom{n}{k_1, k_2} = \frac{n!}{k_1!(n - k_1)!},$$

Therefore, for $m = 2$, the multinomial coefficient reduces to the binomial coefficient:

$$\binom{n}{k_1, k_2} = \binom{n}{k_1}, \quad \text{(since } k_2 = n - k_1\text{)}$$

# 7 Sectors and Radians

# 8 Geometry

# 9 Trigonometry

# 10 Statistics

# 11 Probability

# 12 Random Variables and Probability Distributions

# 13 Hypothesis Testing

# 14 Differentiation

## 15 Integration

## 16 Differential Equations

## 17 Complex Numbers

# 18 Vectors

# 19 Matrices

## 19.1 Eigenvalues and Eigenvectors

## 19.2 Power Formula

$$\boxed{A^n = PD^nP^{-1}}$$

where $A$ is a diagonalisable square matrix, $n \in \mathbb{N}$, $P$ is the matrix of eigenvectors of $A$, and $D$ is the diagonal matrix of the corresponding eigenvalues

# 20 Graph Theory

## 20.1 Adjacency Matrices

$a_{ij} = 1$ if there exists an edge from vertex $i$ to vertex $j$, otherwise $a_{ij} = 0$, for an unweighted graph.

If the graph is weighted, $a_{ij} = w_{ij}$, where $w_{ij}$ represents the weight of the edge from $i$ to $j$, or 0 if there exists no edge.

## 20.2 Transition Matrices

$a_{ij}$ is the probability of moving from vertex $i$ to vertex $j$ in a given step. *(**Note:** Some curricula define $a_{ij}$ as the probability of moving from $j$ to $i$)*

# 21 Modelling

## 21.1 Logistic Functions

The general form of a logistic function is:

$$\boxed{\frac{L}{1 + Ce^{-kx}}}, \quad \text{where} \quad L, C, k \in \mathbb{R}^+,$$

and $L$ is the function's maximum value, $C$ is a constant resultant from the initial conditions, and $k$ is the growth rate.

# Appendix 1. Formula Reference

| **Exponents and Logarithms** | |
|---|---|
| *Rules of Exponents* | $a^m \cdot a^n = a^{m+n}$ |
| | $a^m \div a^n = a^{m-n}$ |
| | $(a^m)^n = a^{mn}$ |
| | $a^0 = 1,$ for all $a$ in $\mathbb{C}, a \neq 0$ |
| | $a^1 = a$ |
| | $a^{-m} = \frac{1}{a^m}$ |
| | $a^{\frac{1}{m}} = \sqrt[m]{a}$ |
| | $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ |
| | $(ab)^n = a^n \cdot b^n$ |
| | $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ |
| *Laws of Logarithms (real case: $a, x, y > 0$, and $a \neq 1$)* | $a^x = b \Longleftrightarrow \log_a b = x$ |
| | $\log_a x + \log_a y = \log_a (xy)$ |
| | $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ |
| | $n \log_a x = \log_a (x^n)$ |

| **Sequences and Series** | |
|---|---|
| *Sigma notation properties* | |
| Addition property | $\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n$ |
| Constant Multiplication Property | $\sum_{n=1}^{k} c \cdot u_n = c \cdot \sum_{n=1}^{k} u_n$ |
| *Arithmetic Sequences and Series* | |
| The $n^{\text{th}}$ term of an arithmetic sequence | $u_n = u_1 + (n-1)d$ |
| The sum of a finite arithmetic series | $S_n = \dfrac{n}{2}(u_1 + u_n)$ |
| | OR $\quad S_n = \dfrac{n}{2}\,(2u_1 + (n-1)d)$ |
| *Geometric Sequences and Series* | |
| The $n^{\text{th}}$ term of a geometric sequence | $u_n = u_1 r^{n-1}$ |
| The sum of a finite arithmetic series (where $r \neq 1$) | $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$ |
| | OR $\quad S_n = \dfrac{u_1(1 - r^n)}{1 - r}$ |
| The sum of an infinite geometric series | $S_\infty = \dfrac{u_1}{1 - r}, \quad$ where $\quad \lvert r \rvert < 1$ |

| **Functions** | |
|---|---|
| Equation of a straight line | $y = mx + c$ OR $y = m(x - x_1) + y_1$ |
| Line of symmetry | If $f(x) = ax^2 + bx + c$, the axis of symmetry of is $x = -\frac{b}{2a}$ |
| Discriminant | $\Delta = b^2 - 4ac$ |
| Quadratic formula | The solutions to $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$ |

| **Finance** | |
|---|---|
| Interest paid | Interest Paid = Total Repayments − Amount Borrowed |
| Effective annual interest rate | $r_{\text{eff}} = \left(1 + \frac{r}{n_c}\right)^{n_c} - 1$ |
| Compound interest (discrete) | $V_f = V_0 \left(1 + \frac{r}{n_c}\right)^{n_c t}$ |
| Compound interest (continuous) | $V_f = V_0 e^{rt}$ |

| **Sets** | |
|---|---|
| *De Morgan's Laws* | |
| Union | $(A \cup B)^c = A^c \cap B^c$ |
| Intersection | $(A \cap B)^c = A^c \cup B^c$ |

| **Combinatorics** |
|---|
| Fundamental Counting Principle |
| $n$ factorial |
| Binomail theorem |

| **Sectors and Radians** | |
|---|---|
| Radians and degrees equivalence | $\pi$ radians $= 180^\circ$ |
| Arc length | $l = r\theta$ |
| Sector area | $A = \frac{1}{2}r^2\theta$ |

| **Geometry** | |
|---|---|
| Distance between two points $a$ and $b$ in $n$-dimensional space | $\sqrt{\sum_{i=1}^{n}\left(b_i-a_i\right)^2}$ |
| Midpoint of a line segment with endpoints $a$ and $b$ in $n$-dimensional space | $\left(\frac{1}{2}\left(a_i+b_i\right)\right)_{1\le i\le n}$ |
| *Spheres of radius $r$* | |
| Surface area | $A=4\pi r^2$ |
| Volume | $V=\frac{4}{3}\pi r^3$ |
| *Pyramids of height $h$* | |
| Volume | $V=\frac{1}{3}Ah$ |
| *Parallelogram* | |
| Area (Given $\mathbf{v}$ and $\mathbf{w}$ are adjacent sides of a parallelogram) | $A=\lvert\mathbf{v}\times\mathbf{w}\rvert$ |
| *Cone* | |
| Surface area of the curved surface | $A=\pi rl$ |
| Volume (right cone) | $V=\frac{1}{3}\pi r^2 h$ |

| **Trigonometry** | |
|---|---|
| *Rules for triangles with sides $a, b, c,$ and angles $A, B, C,$ where $X$ is opposite to $x$* | |
| Sine Rule | $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ |
| Cosine Rule | $c^2=a^2+b^2-2ab\cos C$ |
| Triangle Area | $\dfrac{1}{2}ab\sin C$ |
| *Trigonometric Identities* | |
| Pythagorean Identity | $\cos^2\theta+\sin^2\theta=1$ |
| Tangent Identity | $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ |
| Double-Angle Identity | $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ |
| Reciprocal Identities | $\sin\theta=\dfrac{1}{\csc\theta}$ |
| | $\cos\theta=\dfrac{1}{\sec\theta}$ |
| | $\tan\theta=\dfrac{1}{\cot\theta}$ |

| **Statistics** | |
|---|---|
| Percentage error | $\varepsilon=\lvert\dfrac{v_a-v_e}{v_e}\rvert\cdot 100$ |
| *Descriptive statistics* | |
| Interquartile Range | $\mathrm{IQR}=Q_3-Q_1$ |

| **Statistics** | |
| --- | --- |
| Arithmetic Mean | $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$ |
| *Sampling* | |
| Unbiased estimator of the population variance | $\hat{\sigma}^2 = \dfrac{n}{n-1} s_n^2$ |

| **Probability** | |
| --- | --- |
| Probability of an event $E$ | $\mathbb{P}(E) = \dfrac{\lvert E \rvert}{\lvert U \rvert}$ |
| Combined events | $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ |
| Mutually exclusive events | $\mathbb{P}(E \cap F) = 0$ |
| Probability of $E$ given $F$ | $\mathbb{P}(E \mid F) = \dfrac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$ |

| **Random Variables and Probability Distributions** | |
| --- | --- |
| Expected value of a single discrete random variable $X$ | $\mathbb{E}[X] = \sum_i x_i \, \mathbb{P}(X = x_i)$ |
| *Linear Transformation of a single random variable $X$* | |
| Expected Value | $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ |
| Variance | $\mathrm{Var}[aX + b] = a^2 \, \mathrm{Var}[X]$ |
| *Linear combinations of random variables $X_1, X_2, \ldots, X_n$* | |
| Expected Value | $\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$ |
| *Linear combinations of independent random variables $X_1, X_2, \ldots, X_n$* | |
| Variance | $\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \, \mathrm{Var}[X_i]$ |
| *Uniform Distribution* | |
| $X \sim \mathcal{U}(a, b)$ | $a =$ lower bound<br>$b =$ upper bound |
| Mean ($\mu$) | $\mathbb{E}[X] = \dfrac{a+b}{2}$ |
| Variance ($\sigma^2$) | $\mathrm{Var}[X] = \dfrac{(b-a)^2}{12}$ |
| *Normal (Gaussian) Distribution* | |
| $X \sim \mathcal{N}(\mu, \sigma^2)$ | $\mu =$ mean |

| **Random Variables and Probability Distributions** | |
|---|---|
| | $\sigma^2$ = variance |
| Central Limit Theorem (CLT) | For large $n$, $\bar{X} \sim \mathcal{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ (approximately) |
| *Binomial Distribution* | |
| $X \sim \mathcal{B}(n, p)$ | $n$ = number of trials |
| | $p$ = probability of success |
| Mean ($\mu$) | $\mathbb{E}[X] = np$ |
| Variance ($\sigma^2$) | $\mathrm{Var}[X] = np(1-p)$ |
| *Poisson Distribution* | |
| $X \sim \mathcal{P}(\lambda)$ | $\lambda$ = mean and variance |

| **Hypothesis Testing** | |
|---|---|
| $z$ (mean, $\sigma$ known) | $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ |
| $t$ (mean, $\sigma$ unknown) | $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}, \quad df = n - 1$ |
| $z$ (proportion) | $z = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ |
| $\chi^2$ (GoF/independence) | $\chi^2 = \sum \dfrac{(O-E)^2}{E}$ |
| *Two sample* | |
| $t$ (two-sample, pooled variance) | $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \quad s_p^2 = \dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}$ |
| $z$ (two-sample proportions) | $z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}}, \quad \hat{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$ |

| **Differentiation** | |
|---|---|
| Derivative of $f(x)$ using first principles | $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ |
| Derivative of $x^n$ | $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ |
| *Standard Derivatives* | |

| **Differentiation** | |
|---|---|
| | $f(x) = \sin x \Rightarrow f'(x) = \cos x$<br>$f(x) = \cos x \Rightarrow f'(x) = -\sin x$<br>$f(x) = \tan x \Rightarrow f'(x) = \dfrac{1}{\cos^2 x} = \sec^2 x$<br>$f(x) = \sin x \Rightarrow f'(x) = \cos x$<br>$f(x) = \sin x \Rightarrow f'(x) = \cos x$ |
| *Standard Differentiation Rules* | |
| Chain rule | $\dfrac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$<br>$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ |
| Product rule | $\dfrac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x)$ |
| Quotient rule | $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$ |
| L'Hôpital's Rule | $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \lim_{x \to a} \dfrac{f'(x)}{g'(x)}$, for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms, with $g'(x) \neq 0$ |

| **Integration** | |
|---|---|
| Integral of $x^n$ | $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$ |
| Area of the region between the $x$-axis and $f(x)$, for the range $(a, b)$ | $A = \displaystyle\int_a^b y \, dx$ |
| Trapezoidal Rule | $\displaystyle\int_a^b y \, dx \approx \frac{h}{2}(y_0 + y_n + 2(y_1 + \cdots + y_{n-1}))$ |
| *Standard Integrals* | |
| | $\int \dfrac{1}{x} \, dx = \ln\lvert x\rvert + C$<br>$\int \sin x \, dx = -\cos x + C$<br>$\int \cos x \, dx = \sin x + C$<br>$\int e^x \, dx = e^x + C$<br>$\int \tan x \, dx = -\ln\lvert\cos x\rvert + C$<br>$\int \cot x \, dx = \ln\lvert\sin x\rvert + C$<br>$\displaystyle\int \sec^2 x \, dx = \tan x + C$<br>$\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$<br>$\displaystyle\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$ |

| **Integration** | |
|---|---|
| | $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$ |

| **Differential Equations** | |
|---|---|
| Euler's method (step size $h$) | $y_{n+1} = y_n + hf(x_n, y_n)$ |
| | $x_{n+1} = x_n + h$ |
| Coupled linear differential equations (diagonalisable case) | $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\vec{p}_1 + C_2 e^{\lambda_2 t}\vec{p}_2$ |

| **Complex Numbers** | |
|---|---|
| Rectangular form | $z = a + bi$ |
| Polar (modulus-argument) form | $z = r(\cos\theta + i\sin(\theta))$ |
| Exponential (Euler) form | $z = re^{i\theta}$ |
| Compliment conjugate | $\bar{z} = a - bi$ |
| De Moivre's theorem | $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ |

| **Vectors** | |
|---|---|
| Magnitude | $\lvert\mathbf{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$ |
| | $\lvert\mathbf{v}\rvert = \sqrt{\sum_{i=1}^{n} v_i^2}, \quad v_i = i^{\text{th}}$ component |
| Scalar product | $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$ |
| | $\mathbf{v} \cdot \mathbf{w} = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\cos\theta$, $\theta =$ angle between $\mathbf{v}, \mathbf{w}$ |
| Vector product | Given $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ |
| | $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$ |
| | $\lvert\mathbf{v} \times \mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\sin\theta$ |

| **Vectors** | |
|---|---|
| Vector equation of a line | $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}$ |
| Parametric form of a line | $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n$ |

| **Matrices** | |
|---|---|
| Matrix addition | $(A+B)_{ij} = a_{ij} + b_{ij}$ |
| Matrix multiplication | $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ <br> OR $\quad AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{1\le i\le m,\ 1\le j\le p}$ |
| Determinant of a 2×2 matrix | $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ |
| Determinant of a 3×3 matrix | $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$ <br> $a(ei - fh) - b(di - fg) + c(dh - eg)$ |
| Invertibility condition | $A$ is invertible if $\det A \neq 0$ |
| Power formula of a diagonalisible matrix $A$ | $A^n = PD^nP^{-1}$, $n \in \mathbb{N}$, $P =$ eigenvectors, $D =$ eigenvalues |
| *2D Transformation Matrices* | (Assuming $x$-axis is horizontal, and $y$-axis is vertical) |
| Stretch with scale factor $h$ horizontally, and $v$ vertically | $\begin{pmatrix} h & 0 \\ 0 & v \end{pmatrix}$, centred at the origin |
| Anticlockwise rotation of angle $\theta$ | $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, about the origin |
| Reflection in the line through the origin with angle $\theta$ from the $x$-axis | $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ |
| Shear with factor $h$ horizontally, and $v$ vertically | $\begin{pmatrix} 1 & h \\ v & 1 \end{pmatrix}$, about the origin |

| **Graph Theory** | |
|---|---|
| Adjacency matrix | $a_{ij} = 1$ if edge $i \to j$, otherwise 0. If weighted, $a_{ij} = w_{ij}$ |
| Transition matrix | $a_{ij} =$ probability of moving $i \to j$ (convention can vary, however) |

| **Graph Theory** | |
|---|---|
| State after $n$ transitions (row-stochastic convention) | $s_n = s_0 T^n$ |

| **Modelling** | |
|---|---|
| Logistic function | $\dfrac{L}{1 + Ce^{-kx}}$, where $L, C, k \in \mathbb{R}^+$ |
| *Volume of revolution in the range* $(a, b)$ | |
| about the $x$-axis | $V = \pi \displaystyle\int_a^b [f(x)]^2 \, dx$ |
| about the $y$-axis | $V = \pi \displaystyle\int_a^b [f(y)]^2 \, dy$ |

# Appendix 2. Worked Examples Template Page (TEMP)

| **Worked Example I** |
|---|
| Find $u_{10}$ for an arithmetic sequence where $u_1 = 3$ and $d = 5$.<br>We use the $n^{th}$ term formula:<br>$u_n = u_1 + (n-1)d$<br>$u_{10} = 3 + (10-1) \cdot 5 = 3 + 45 = 48$ |