First Principles: A Mathematics Handbook

Alan D. Smith

Contents

For my brother, Samuel,
whose strength of character, humour, and curiosity
continue to inspire my work and life.

1 Introduction

1.1 Purpose of This Guide

This handbook aims to be a concise yet comprehensive reference for key mathematical first principles, covering topics essential for high school mathematics, university preparation, and STEM-related fields. It is designed for fast navigation, allowing readers to look up topics easily, find clear explanations, quickly learn new topics or revise specific ones.

1.2 How to Use This Handbook

Each chapter focuses on a single topic, beginning with definitions and essential formulas. Mathematical notation follows a standardised convention throughout. Where topics are built upon, cross-references to other sections are included for further exploration or for revisiting relevant material.

1.3 Notation and Conventions

- Definitions are shown in *italics*.
- Variables are italicised (e.g. x, y, z).
- Constants such as e or π retain their standard mathematical meaning.
- Essential formulas are highlighted in boxes (e.g. $\boxed{e^{i\pi} + 1 = 0}$)

2 Exponents and Logarithms

2.1 Exponents

An exponent is written as a^n, where a is the *base*, and n is the *power* (also known as *index* or *exponent*).

Exponents follow a set of algebraic rules:

Rules of Exponents
$a^m \cdot a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{mn}$
$a^0 = 1$
$a^1 = a$
$a^{-m} = \frac{1}{a^m}$
$a^{\frac{1}{m}} = \sqrt[m]{a}$
$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$
$(ab)^n = a^n \cdot b^n$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

2.2 Logarithms

The logarithmic function $\log_a b = x$ is the solution to the exponential equation $a^x = b$. In other words, $\log_a b$ gives the power to which the base a needs to be raised in order to produce b. Therefore,

$$\boxed{a^x = b \iff \log_a b = x}$$

Base restrictions: Unless stated otherwise, $a > 0, a \neq 1$, and $b > 0$. (An exception would be for complex logarithms, for example.)

- The logarithm with base e is written as $\ln x$ (natural logarithm).
- The logarithm with base 10 is typically written as $\log x$ (common logarithm).

Laws of logarithms
$\log_a x + \log_a y = \log_a (xy)$
$\log_a x - \log_a y = \log_a (\frac{x}{y})$
$n \log_a x = \log_a (x^n)$

2.2.1 Logarithmic scales

When dealing with incredibly small or large numbers, it becomes difficult to distinguish them, so they can be scaled using logarithms.

2.3 Euler's number and the natural logarithm

Euler's number ($e \approx 2.718\ldots$) is an irrational constant, which arises in situations involving growth or decay, such as compound interest or radioactive decay.

A definition for e can be found in finance, through continuous compounding:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

where n is the number of compounding periods per unit time, and e is the resulting constant as n grows infinitely large (Euler's number).

The *natural logarithm*, written as $\ln x$, is the logarithm with base e:

$$\boxed{\ln x = \log_e x}$$

> **Note:** e^x and $\ln x$ have unique properties in calculus. See **Chapter 12. Differentiation**, and **Chapter 13. Integration** for more information.

3 Sequences and Series

A numerical *sequence* is an ordered list of numbers which can often be described by a formula or recurrence relation. Each number within the sequence is called a *term*. The general term within a sequence is called the n^{th} term.

A numerical *series* is the sum of a range of terms in a sequence. A *finite series* describes a series which has n terms ($u_1, u_2, u_3, \dots, u_n$, where u_n is the n^{th} term of the sequence), and thus also has a finite sum. An *infinite series* describes a series which has infinite terms ($u_1, u_2, u_3, \dots$), which can *converge* to a finite sum or *diverge* to infinity.

Sigma notation can be useful when describing a series concisely. For example, the sum:

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_{100}$$

can be rewritten using sigma notation as:

$$\sum_{n=1}^{100} u_n$$

In general,

$$\boxed{\sum_{n=a}^{b} u_n}$$

represents the sum of all terms from $n = a$ to $n = b$.

Sigma notation also follows certain important algebraic properties:

Addition Property:

$$\boxed{\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n}$$

Constant Multiplication Property:

Given c is a constant:

$$\boxed{\sum_{n=1}^{k} c \cdot u_n = c \cdot \sum_{n=1}^{k} u_n}$$

3.1 Arithmetic Sequences and Series

An *arithmetic sequence* (sometimes called an arithmetic *progression*) is a sequence where each term differs from the previous term by a constant value, called the *common difference*, typically denoted as d.

The general formula for the n^{th} term of an arithmetic sequence is given by:

$$\boxed{u_n = u_1 + (n-1)d}$$

where: - u_n is the n^{th} term, - d is the common difference, - n is the position of the term (e.g., for the 5th term, $n = 5$).

The sum of a finite *arithmetic series* is given by:

$$\boxed{S_n = \frac{n}{2}(u_1 + u_n) \quad \text{OR} \quad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)}$$

where S_n is the sum of the first n terms.

3.2 Geometric Sequences and Series

A *geometric sequence* (sometimes called a geometric *progression*) is a sequence where each term is the product of the previous term and a constant value, called the *common ratio*, typically denoted as r.

The general formula for the n^{th} term of a geometric sequence is given by:

$$\boxed{u_n = u_1 r^{n-1}}$$

The sum of a finite *geometric series* is given by:

$$\boxed{S_n = \frac{u_1(r^n - 1)}{r-1} \quad \text{OR} \quad S_n = \frac{u_1(1-r^n)}{1-r}} \quad \text{where} \quad r \neq 1.$$

where r is the common ratio.

If $|r| < 1$, the geometric series converges as $n \to \infty$, resulting in an infinite geometric series with sum:

$$\boxed{S_\infty = \frac{u_1}{1-r}}$$

4 Finance

Inflation is the increase in the price of goods and services with respect to time. Inflation means that a fixed amount of capital will be able to purchase fewer goods and services as prices increase.

Depreciation is the decrease in the value of an asset with respect to time.

The *principal* is the original loan amount that is borrowed.

The *principal portion* of a payment is the portion that reduces the outstanding loan balance (principal).

The *interest portion* is the payment covering the interest charged for that period.

Amortisation is a process whereby a loan is repaid through a series of regular payments. In an amortised loan, interest for each period is calculated on the outstanding loan balance, therefore as regular payments reduce the principal, the interest portion of each payment decreases over time, whilst the principal portion increases.

$$\boxed{\text{Interest Paid} = \text{Total Repayments - Amount Borrowed}}$$

> **Note:** This topic is primarily solved using calculators or software tools which allow for the automation of iterative calculations. Regardless, it is important to recognise the meaning of each parameter and their relation to the relevant formulae.

Variable	Meaning
N	Total number of compounding periods
I%	Interest rate per year (%)
PV	Present value of the loan/investment
PMT	Payment per period
FV	Future value of loan/investment
P/Y	Number of payments per year
C/Y	Number of compounding periods per year

5 Sets

A *set* is a collection of distinct objects, called *elements* (or *members*). They are typically denoted by a capital letter (such as A, B, C, etc.), and their elements can be written inside curly brackets. For example:

$$A = \{1, 2, 3, 4, e, 9.\overline{9}, 8086\}$$

Here, $1 \in A$ denotes that "1 is an element of A", and $5 \notin A$ means that "5 is not an element of A".

5.1 Types of Sets

Name	Description	Example
Finite set	Contains a finite number of elements.	$B = \{1, 2, 3\}$
Infinite set	Contains an infinite number of elements	$\mathbb{N} = \{1, 2, 3, \dots\}$
Singleton set	Contains exactly one element.	$C = \{256\}$
Universal set	Contains all objects under consideration in a certain context. Denoted as U.	
Empty set	(Sometimes called the *null set*) Contains no elements, and is denoted as $\emptyset$ or $\{\}$	

5.2 Subsets

A set A is considered a *subset* of B if every element in A is also an element of B. In which case:

$$A \subseteq B$$

which means: "A is a subset of or equal to B".

If A is a subset of B, but $A \neq B$, then A is considered a *proper subset* of B, shown as:

$$A \subset B$$

The empty set ($\emptyset$) is a subset of every set, including singletons, since it has no elements.

6 Sectors and Radians

7 Trigonometry

8 Descriptive Statistics

9 Probability

10 Random Variables and Probability Distributions

11 Hypothesis Testing

12 Functions

A *relation* between two variables x and y, is any set of points which are on the (x, y) plane.

There exist four different types of relations: - One to one - One to many - Many to one - Many to Many

A *function* is defined as a mapping onto a single value. Therefore one to one, and many to one relations are considered to be functions.

12.1 Testing if a relation is a function

1. **Algebraic Method:** If we substitute any value of x and it results in a singular y-value, then it can be considered a function.
2. **Graphical Method (Vertical Line Test):** If we are given a plot of a function on the (x, y) plane, where the x-axis is parallel with the horizontal, and the y-axis is parallel with the vertical, then if we are able to draw a vertical line anywhere on the graph, and it only intersects the plot once, it is a function. If the vertical line intersects the plot more than once, then it cannot be considered a function.

12.2 Domain and Range

13 Differentiation

14 Integration

15 Differential Equations

16 Complex Numbers

17 Vectors

18 Matrices

19 Graph Theory

20 Modelling

21 Worked Examples Template Page (TEMP)

Worked Example I
Find u_{10} for an arithmetic sequence where $u_1 = 3$ and $d = 5$. We use the n^{th} term formula:$u_n = u_1 + (n-1)d$ $u_{10} = 3 + (10-1) \cdot 5 = 3 + 45 = 48$