First Principles: A Mathematics Handbook

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Contents

For my brother, Samuel,
whose strength of character, humour, and curiosity
continue to inspire my work and life.

Introduction

0.1 Purpose of This Guide

This handbook aims to be a concise yet comprehensive reference for key mathematical first principles, covering topics essential for high school mathematics, university preparation, and STEM-related fields. It is designed for fast navigation, allowing readers to look up topics easily, find clear explanations, quickly learn new topics or revise specific ones.

0.2 How to Use This Handbook

Each chapter focuses on a single topic, beginning with definitions and essential formulas. Mathematical notation follows a standardised convention throughout. Where topics are built upon, cross-references to other sections are included for further exploration or for revisiting relevant material.

0.3 Notation and Conventions

- Definitions are shown in *italics*.
- Variables are italicised (e.g. x, y, z).
- Constants such as e or π retain their standard mathematical meaning.
- Essential formulas are highlighted in boxes (e.g. $\boxed{e^{i\pi} + 1 = 0}$).
- All angles are expressed in radians unless stated otherwise.

1 Exponents and Logarithms

1.1 Exponents

An exponent is written as a^n, where a is the *base*, and n is the *power* (also known as *index* or *exponent*).

Exponents follow a set of algebraic rules:

Rules of Exponents
$a^m \cdot a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{mn}$
$a^0 = 1$
$a^1 = a$
$a^{-m} = \frac{1}{a^m}$
$a^{\frac{1}{m}} = \sqrt[m]{a}$
$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$
$(ab)^n = a^n \cdot b^n$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

1.2 Logarithms

The logarithmic function $\log_a b = x$ is the solution to the exponential equation $a^x = b$. In other words, $\log_a b$ gives the power to which the base a needs to be raised in order to produce b. Therefore,

$$\boxed{a^x = b \iff \log_a b = x}$$

Base restrictions: Unless stated otherwise, $a > 0, a \neq 1$, and $b > 0$. (An exception would be for complex logarithms, for example.)

- The logarithm with base e is written as $\ln x$ (natural logarithm).
- The logarithm with base 10 is typically written as $\log x$ (common logarithm).

Laws of logarithms
$\log_a x + \log_a y = \log_a (xy)$
$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
$n \log_a x = \log_a (x^n)$

1.2.1 Logarithmic scales

When dealing with incredibly small or large numbers, it becomes difficult to distinguish them, so they can be scaled using logarithms.

1.3 Euler's number and the natural logarithm

Euler's number ($e \approx 2.718\ldots$) is an irrational constant, which arises in situations involving growth or decay, such as compound interest or radioactive decay.

A definition for e can be found in finance, through continuous compounding:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

where n is the number of compounding periods per unit time, and e is the resulting constant as n grows infinitely large (Euler's number). See **4. Finance** for more information.

The *natural logarithm*, written as $\ln x$, is the logarithm with base e:

$$\boxed{\ln x = \log_e x}$$

> **Note:** e^x and $\ln x$ have unique properties in calculus. See **Chapter** [[**14. Differentiation**]], and **Chapter** [[**15. Integration**]] for more information.

2 Sequences and Series

A numerical *sequence* is an ordered list of numbers which can often be described by a formula or recurrence relation. Each number within the sequence is called a *term*. The general term within a sequence is called the n^{th} term.

A numerical *series* is the sum of a range of terms in a sequence. A *finite series* describes a series which has n terms ($u_1, u_2, u_3, \dots, u_n$, where u_n is the n^{th} term of the sequence), and thus also has a finite sum. An *infinite series* describes a series which has infinite terms ($u_1, u_2, u_3, \dots$), which can *converge* to a finite sum or *diverge* to infinity.

Sigma notation can be useful when describing a series concisely. For example, the sum:

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_{100}$$

can be rewritten using sigma notation as:

$$\sum_{n=1}^{100} u_n$$

In general,

$$\boxed{\sum_{n=a}^{b} u_n}$$

represents the sum of all terms from $n = a$ to $n = b$.

Sigma notation also follows certain important algebraic properties:

Addition Property:

$$\boxed{\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n}$$

Constant Multiplication Property:

Given c is a constant:

$$\boxed{\sum_{n=1}^{k} c \cdot u_n = c \cdot \sum_{n=1}^{k} u_n}$$

2.1 Arithmetic Sequences and Series

An *arithmetic sequence* (sometimes called an arithmetic *progression*) is a sequence where each term differs from the previous term by a constant value, called the *common difference*, typically denoted as d.

The general formula for the n^{th} term of an arithmetic sequence is given by:

$$\boxed{u_n = u_1 + (n-1)d}$$

where: - u_n is the n^{th} term, - d is the common difference, - n is the position of the term (e.g., for the 5th term, $n = 5$).

The sum of a finite *arithmetic series* is given by:

$$\boxed{S_n = \frac{n}{2}(u_1 + u_n) \quad \text{OR} \quad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)}$$

where S_n is the sum of the first n terms.

2.2 Geometric Sequences and Series

A *geometric sequence* (sometimes called a geometric *progression*) is a sequence where each term is the product of the previous term and a constant value, called the *common ratio*, typically denoted as r.

The general formula for the n^{th} term of a geometric sequence is given by:

$$\boxed{u_n = u_1 r^{n-1}}$$

The sum of a finite *geometric series* is given by:

$$\boxed{S_n = \frac{u_1(r^n - 1)}{r - 1} \quad \text{OR} \quad S_n = \frac{u_1(1 - r^n)}{1 - r}} \quad \text{where} \quad r \neq 1.$$

where r is the common ratio.

If $|r| < 1$, the geometric series converges as $n \to \infty$, resulting in an infinite geometric series with sum:

$$\boxed{S_\infty = \frac{u_1}{1 - r}} \quad \text{where} \quad |r| < 1.$$

3 Finance

3.1 Important Concepts

- *Interest* is money paid at a regular rate in order to use lent money, or to delay the repayment of debt.
- *Inflation* is the increase in the price of goods and services with respect to time. Inflation means that a fixed amount of capital will be able to purchase fewer goods and services as prices increase.
- *Depreciation* is the decrease in the value of an asset with respect to time.
- The *principal* is the original loan amount that is borrowed.
- The *principal portion* of a payment is the portion that reduces the outstanding loan balance (principal).
- The *interest portion* is the payment covering the interest charged for that period.
- *Amortisation* is a process whereby a loan is repaid through a series of regular payments. In an amortised loan, interest for each period is calculated on the outstanding loan balance, therefore as regular payments reduce the principal, the interest portion of each payment decreases over time, whilst the principal portion increases.

3.2 Parameters and Conventions

In order to work with financial formulas, it is crucial to first understand the standard notation on which they depend.

Calculator Label	Symbol in This Book	Meaning
N	N	Total number of compounding periods
I%	r	Nominal annual interest rate (%)
	r_{eff}	Effective annual interest rate (%)
PV	V_0	Present value of the loan/investment
PMT	M	Payment per period
FV	V_f	Future value of loan/investment
P/Y	n_p	Number of payments per year
C/Y	n_c	Number of compounding periods per year

> **Note:** This topic is primarily solved using calculators or software tools which allow for the automation of iterative calculations. Regardless, it is important to recognise the meaning of each parameter and their relation to the relevant formulae.

3.3 Interest Rates

Interest rates are a measure of the cost of borrowing money, or the return on investment over a set period of time.

$$\boxed{\text{Interest Paid} = \text{Total Repayments - Amount Borrowed}}$$

3.3.1 Nominal and Effective Rates

The *nominal annual interest rate* (r) is the yearly rate of interest which does not consider compounding. Financial calculators often denote this as I%.

The *effective annual interest rate* (r_{eff}) considers the effect of compounding throughout the year. It reflects the percentage increase in value, or cost of debt, over one year.

The formula for the effective annual interest rate is given as follows:

$$\boxed{r_{\text{eff}} = \left(1 + \frac{r}{n_c}\right)^{n_c} - 1}$$

3.4 Compound Interest

Compound interest is when the interest is calculated using any accumulated interest from previous periods in addition to the original principal.

The general formula for compound interest is:

$$\boxed{V_f = V_0 \left(1 + \frac{r}{n_c}\right)^{n_c t}}$$

where t is the time in years.

3.5 Continuous Compounding

When compounding occurs infinitely often (as $n_c \to \infty$), the formula becomes:

$$\boxed{V_f = V_0 e^{rt}}$$

4 Sets

A *set* is a collection of distinct objects, called *elements* (or *members*). They are typically denoted by a capital letter (such as A, B, C, etc.), and their elements can be written inside curly brackets. For example:

$$A = \{1, 2, 3, 4, e, 9.\overline{9}, 8086\}$$

Here, $1 \in A$ denotes that "1 is an element of A", and $5 \notin A$ means that "5 is not an element of A".

4.1 Types of Sets

Name	Description	Example
Finite set	Contains a finite number of elements.	$B = \{1, 2, 3\}$
Infinite set	Contains an infinite number of elements	$\mathbb{N} = \{1, 2, 3, \dots\}$
Singleton set	Contains exactly one element.	$C = \{256\}$
Universal set	Contains all objects under consideration in a certain context. Denoted as U.	
Empty set	(Sometimes called the *null set*) Contains no elements, and is denoted as $\emptyset$ or $\{\}$	

4.2 Subsets

A set A is considered a *subset* of B if every element in A is also an element of B. In which case:

$$A \subseteq B$$

which means: "A is a subset of or equal to B".

If A is a subset of B, but $A \neq B$, then A is considered a *proper subset* of B, shown as:

$$A \subset B$$

The empty set ($\emptyset$) is a subset of every set, including singletons, since it has no elements.

5 Sectors and Radians

6 Geometry

7 Trigonometry

8 Statistics

9 Probability

10 Random Variables and Probability Distributions

11 Hypothesis Testing

12 Functions

A *relation* between two variables x and y, is any set of points which are on the (x, y) plane.

There exist four different types of relations:

- One to one
- One to many
- Many to one
- Many to many

A *function* is defined as a mapping onto a single value. Therefore one to one, and many to one relations are considered to be functions.

12.1 Testing if a relation is a function

1. **Algebraic Method:** If we substitute any value of x and it results in a singular y-value, then it can be considered a function.
2. **Graphical Method (Vertical Line Test):** If we are given a plot of a function on the (x, y) plane, where the x-axis is parallel with the horizontal, and the y-axis is parallel with the vertical, then if we are able to draw a vertical line anywhere on the graph, and it only intersects the plot once, it is a function. If the vertical line intersects the plot more than once, then it cannot be considered a function.

12.2 Domain and Range

13 Differentiation

14 Integration

15 Differential Equations

16 Complex Numbers

17 Vectors

18 Matrices

18.1 WIP

18.2 Power Formula

Power formula	$A^n = PD^nP^{-1}$, where A is a diagonalisible square matrix, $n \in \mathbb{N}$, P is the matrix of eigenvectors of A, and D is the diagonal matrix of the corresponding eigenvalues

19 Graph Theory

Graph Theory	
Adjacency matrices	$a_{ij} = 1$ if there exists an edge from vertex i to vertex j, otherwise $a_{ij} = 0$, for an unweighted graph.
	If the graph is weighted, $a_{ij} = w_{ij}$, where w_{ij} represents the weight of the edge from i to j, or 0 if there exists no edge.
Transition matrices	a_{ij} is the probability of moving from vertex i to vertex j in a given step. *(**Note:** Some curricula define a_{ij} as the probability of moving from j to i)*

20 Modelling

20.1 Logistic Functions

The general form of a logistic function is:

$$\boxed{\frac{L}{1+Ce^{-kx}}}, \quad \text{where} \quad L, C, k \in \mathbb{R}^{+},$$

and L is the function's maximum value, C is a constant resultant from the initial conditions, and k is the growth rate.

Appendix 1. Formula Reference

Exponents and Logarithms	
Rules of Exponents	$a^m \cdot a^n = a^{m+n}$
	$a^m \div a^n = a^{m-n}$
	$(a^m)^n = a^{mn}$
	$a^0 = 1,$ for all a in $\mathbb{C}, a \neq 0$
	$a^1 = a$
	$a^{-m} = \frac{1}{a^m}$
	$a^{\frac{1}{m}} = \sqrt[m]{a}$
	$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$
	$(ab)^n = a^n \cdot b^n$
	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
Laws of Logarithms (real case: $a, x, y > 0$, and $a \neq 1$)	$a^x = b \Longleftrightarrow \log_a b = x$
	$\log_a x + \log_a y = \log_a (xy)$
	$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
	$n \log_a x = \log_a (x^n)$

Sequences and Series	
Sigma notation properties	
Addition property	$\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n$
Constant Multiplication Property	$\sum_{n=1}^{k} c \cdot u_n = c \cdot \sum_{n=1}^{k} u_n$
Arithmetic Sequences and Series	
The n^{th} term of an arithmetic sequence	$u_n = u_1 + (n-1)d$
The sum of a finite arithmetic series	$S_n = \dfrac{n}{2}(u_1 + u_n)$
	OR $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$
Geometric Sequences and Series	
The n^{th} term of a geometric sequence	$u_n = u_1 r^{n-1}$
The sum of a finite arithmetic series (where $r \neq 1$)	$S_n = \dfrac{u_1(r^n - 1)}{r - 1}$
	OR $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$
The sum of an infinite geometric series	$S_\infty = \dfrac{u_1}{1 - r}$, where $\lvert r \rvert < 1$

Finance	
Interest paid	Interest Paid = Total Repayments − Amount Borrowed
Effective annual interest rate	$r_{\text{eff}} = \left(1 + \frac{r}{n_c}\right)^{n_c} - 1$
Compound interest (discrete)	$V_f = V_0 \left(1 + \frac{r}{n_c}\right)^{n_c t}$
Compound interest (continuous)	$V_f = V_0 e^{rt}$

Sets	
De Morgan's Laws	
Union	$(A \cup B)^c = A^c \cap B^c$
Intersection	$(A \cap B)^c = A^c \cup B^c$

Sectors and Radians	
Radians and degrees equivalence	π radians $= 180^\circ$
Arc length	$l = r\theta$
Sector area	$A = \frac{1}{2} r^2 \theta$

Geometry	
Distance between two points a and b in n-dimensional space	$\sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$
Midpoint of a line segment with endpoints a and b in n-dimensional space	$\left(\frac{1}{2}(a_i + b_i)\right)_{1 \le i \le n}$
Spheres of radius r	
Surface area	$A = 4\pi r^2$
Volume	$V = \frac{4}{3}\pi r^3$
Pyramids of height h	
Volume	$V = \frac{1}{3}Ah$, where A is the area of the base, and h is perpendicular to the base plane
Parallelogram	
Area	$A = \lvert\mathbf{v} \times \mathbf{w}\rvert$, where $\mathbf{v}$ and $\mathbf{w}$ are two adjacent sides of a parallelogram

Trigonometry	
Rules for triangles with sides a, b, c, and angles A, B, C, where X is opposite to x	
Sine Rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine Rule	$c^2 = a^2 + b^2 - 2ab\cos C$
Triangle Area	$\frac{1}{2}ab\sin C$
Trigonometric Identities	
Pythagorean Identity	$\cos^2\theta + \sin^2\theta = 1$
Tangent Identity	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
Double-Angle Identity	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
Reciprocal Identities	$\sin\theta = \frac{1}{\csc\theta}$
	$\cos\theta = \frac{1}{\sec\theta}$
	$\tan\theta = \frac{1}{\cot\theta}$

Statistics	
Percentage error	$\varepsilon = \lvert\frac{v_a - v_e}{v_e}\rvert \cdot 100$
Descriptive statistics	
Interquartile Range	$\mathrm{IQR} = Q_3 - Q_1$
Arithmetic Mean	$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$
Sampling	
Unbiased estimator of the population variance	$\hat{\sigma}^2 = \frac{n}{n-1}s_n^2$

Probability	
Probability of an event E	$\mathbb{P}(E) = \frac{\lvert E\rvert}{\lvert U\rvert}$
Combined events	$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
Mutually exclusive events	$\mathbb{P}(E \cap F) = 0$
Probability of E given F	$\mathbb{P}(E\vert F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$

Random Variables and Probability Distributions	
Expected value of a single discrete random variable X	$\mathbb{E}[X] = \sum_i x_i \, \mathbb{P}(X = x_i)$
Linear Transformation of a single random variable X	
Expected Value	$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
Variance	$\mathrm{Var}[aX + b] = a^2 \, \mathrm{Var}[X]$
Linear combinations of random variables $X_1, X_2, \dots, X_n$	
Expected Value	$\mathbb{E}\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n a_i \mathbb{E}[X_i]$
Linear combinations of independent random variables $X_1, X_2, \dots, X_n$	
Variance	$\mathrm{Var}\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n a_i^2 \, \mathrm{Var}[X_i]$
Uniform Distribution	
$X \sim \mathcal{U}(a, b)$	$a =$ lower bound $b =$ upper bound
Mean (μ)	$\mathbb{E}[X] = \dfrac{a + b}{2}$
Variance (σ^2)	$\mathrm{Var}[X] = \dfrac{(b - a)^2}{12}$
Normal (Gaussian) Distribution	
$X \sim \mathcal{N}(\mu, \sigma^2)$	$\mu =$ mean $\sigma^2 =$ variance
Central Limit Theorem (CLT)	For large n, $\bar{X} \sim \mathcal{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ (approximately)
Binomial Distribution	
$X \sim \mathcal{B}(n, p)$	$n =$ number of trials $p =$ probability of success
Mean (μ)	$\mathbb{E}[X] = np$
Variance (σ^2)	$\mathrm{Var}[X] = np(1 - p)$
Poisson Distribution	
$X \sim \mathcal{P}(\lambda)$	$\lambda =$ mean and variance

Hypothesis Testing	
z (mean, σ known)	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
t (mean, σ unknown)	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}, \quad df = n - 1$

Hypothesis Testing	
z (proportion)	$z = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
χ^2 (GoF/independence)	$\chi^2 = \sum \dfrac{(O-E)^2}{E}$
Two sample	
t (two-sample, pooled variance)	$t = \dfrac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \quad s_p^2 = \dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$
z (two-sample proportions)	$z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n_1} + \frac{1}{n_2})}}, \quad \hat{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$

Functions	
Equation of a straight line	$y = mx + c$ OR $y = m(x - x_1) + y_1$
Line of symmetry	If $f(x) = ax^2 + bx + c$, the axis of symmetry of is $x = -\frac{b}{2a}$
Discriminant	$\Delta = b^2 - 4ac$
Quadratic formula	The solutions to $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$

Differentiation	
Derivative of $f(x)$ using first principles	$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
Derivative of x^n	$f(x) = x^n \;\Rightarrow\; f'(x) = nx^{n-1}$
Standard Derivatives	
	$f(x) = \sin x \Rightarrow f'(x) = \cos x$ $f(x) = \cos x \Rightarrow f'(x) = -\sin x$ $f(x) = \tan x \Rightarrow f'(x) = \dfrac{1}{\cos^2 x} = \sec^2 x$ $f(x) = \sin x \Rightarrow f'(x) = \cos x$ $f(x) = \sin x \Rightarrow f'(x) = \cos x$
Standard Differentiation Rules	

Differentiation	
Chain rule	$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
Product rule	$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x)$
Quotient rule	$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$
L'Hôpital's Rule	$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms, with $g'(x) \neq 0$

Integration	
Integral of x^n	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
Area of the region between the x-axis and $f(x)$, for the range (a, b)	$A = \int_a^b y \, dx$
Trapezoidal Rule	$\int_a^b y \, dx \approx \frac{h}{2}(y_0 + y_n + 2(y_1 + \cdots + y_{n-1}))$
Standard Integrals	
	$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$ $\int \sin x \, dx = -\cos x + C$ $\int \cos x \, dx = \sin x + C$ $\int e^x \, dx = e^x + C$ $\int \tan x \, dx = -\ln\lvert\cos x\rvert + C$ $\int \cot x \, dx = \ln\lvert\sin x\rvert + C$ $\int \sec^2 x \, dx = \tan x + C$ $\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$ $\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln\lvert\frac{x-a}{x+a}\rvert + C$ $\int \frac{1}{x^2 + 1} \, dx = \arctan(x) + C$

Differential Equations	
Coupled linear differential equations (diagonalisable case)	$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\vec{p}_1 + C_2 e^{\lambda_2 t}\vec{p}_2$

Complex Numbers	
Rectangular form	$z = a + bi$
Polar (modulus-argument) form	$z = r(\cos\theta + i\sin(\theta))$
Exponential (Euler) form	$z = re^{i\theta}$
Compliment conjugate	$\bar{z} = a - bi$
De Moivre's theorem	$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Vectors	
Magnitude	$\lvert\mathbf{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$
	$\lvert\mathbf{v}\rvert = \sqrt{\sum_{i=1}^n v_i^2}, \quad v_i = i^{\text{th}}$ component
Scalar product	$\mathbf{v}\cdot\mathbf{w} = \sum_{i=1}^n v_i w_i$
	$\mathbf{v}\cdot\mathbf{w} = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\cos\theta$, $\theta =$ angle between $\mathbf{v}, \mathbf{w}$
Vector product	Given $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\mathbf{v}\times\mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$
	$\lvert\mathbf{v}\times\mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\sin\theta$
Vector equation of a line	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \ \ \lambda\in\mathbb{R}$
Parametric form of a line	$x = x_0 + \lambda l, \ \ y = y_0 + \lambda m, \ \ z = z_0 + \lambda n$

Matrices	
Matrix addition	$(A+B)_{ij} = a_{ij} + b_{ij}$
Matrix multiplication	$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ OR $\quad AB = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]_{1 \le i \le m,\ 1 \le j \le p}$
Determinant of a 2×2 matrix	$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
Determinant of a 3×3 matrix	$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$ $a(ei - fh) - b(di - fg) + c(dh - eg)$
Invertibility condition	A is invertible if $\det A \neq 0$
Power formula of a diagonalisible matrix A	$A^n = PD^nP^{-1}$, $n \in \mathbb{N}$, $P =$ eigenvectors, $D =$ eigenvalues
2D Transformation Matrices	(Assuming x-axis is horizontal, and y-axis is vertical)
Stretch with scale factor h horizontally, and v vertically	$\begin{pmatrix} h & 0 \\ 0 & v \end{pmatrix}$, centred at the origin
Anticlockwise rotation of angle θ	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, about the origin
Reflection in the line through the origin with angle θ from the x-axis	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Shear with factor h horizontally, and v vertically	$\begin{pmatrix} 1 & h \\ v & 1 \end{pmatrix}$, about the origin

Graph Theory	
Adjacency matrix	$a_{ij} = 1$ if edge $i \to j$, otherwise 0. If weighted, $a_{ij} = w_{ij}$
Transition matrix	$a_{ij} =$ probability of moving $i \to j$ (convention can vary, however)
State after n transitions (row-stochastic convention)	$s_n = s_0 T^n$

Modelling	
Logistic function	$\dfrac{L}{1 + Ce^{-kx}}$, where $L, C, k \in \mathbb{R}^+$
Volume of revolution in the range (a, b)	

Modelling	
about the x-axis	$V = \pi \int_a^b [f(x)]^2 \, dx$
about the y-axis	$V = \pi \int_a^b [f(y)]^2 \, dy$

Appendix 2. Worked Examples Template Page (TEMP)

Worked Example I

Find u_{10} for an arithmetic sequence where $u_1 = 3$ and $d = 5$. We use the n^{th} term formula:$u_n = u_1 + (n-1)d$ $u_{10} = 3 + (10-1) \cdot 5 = 3 + 45 = 48$