6 Uniform Means

(a) $\frac{1}{n+1}$

Given that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed uniform random variables on the interval $[0, 1]$ (where $n \in \mathbb{Z}^+$), and that $Y = \min\{X_1, X_2, \ldots, X_n\}$, so we have $0 \leq Y \leq 1$, and thus, we can use the tail sum formula to obtain:

$$\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y > y)\, dy = \int_0^1 \mathbb{P}(Y > y)\, dy$$

Now, since $X_1, X_2, \ldots, X_n$ are uniform i.i.ds, and $Y = \min\{X_1, X_2, \ldots, X_n\}$, so for any $y \in [0, 1]$, we can calculate an expression for $\mathbb{P}(Y > y)$ as: $\mathbb{P}(Y > y) = \mathbb{P}(\min\{X_1, X_2, \ldots, X_n\} > y) = \mathbb{P}(X_1 > y, X_2 > y, \ldots, X_n > y) = \mathbb{P}(X_1 > y) \cdot \mathbb{P}(X_2 > y) \cdot \cdots \cdot \mathbb{P}(X_n > y)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i > y) = \frac{1-y}{1-0} = 1 - y$. Thus,

$$\mathbb{P}(Y > y) = (1 - y) \cdot \cdots \cdot (1 - y) = (1 - y)^n$$

Therefore, we can now calculate the expectation of Y, which is:

$$\mathbb{E}(Y) = \int_0^1 \mathbb{P}(Y > y)\, dy = \int_0^1 (1 - y)^n\, dy = -\frac{(1 - y)^{n+1}}{n + 1}\bigg|_0^1 = -0 + \frac{1}{n + 1} = \frac{1}{n + 1}$$

(b) $\frac{n}{n+1}$

Given that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed uniform random variables on the interval $[0, 1]$ (where $n \in \mathbb{Z}^+$), and that $Z = \max\{X_1, X_2, \ldots, X_n\}$, so we have $0 \leq Z \leq 1$, and thus, we can use the formula for Cumulative Distribution Function to obtain:

$$\mathbb{E}(Z) = \int_{-\infty}^1 z \cdot f(z)\, dz = \int_0^1 z \cdot f(z)\, dz$$

Now, since $X_1, X_2, \ldots, X_n$ are uniform i.i.ds, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, so for any $z \in [0, 1]$, we can calculate an expression for $\mathbb{P}(Z \leq z)$ as: $\mathbb{P}(Z \leq z) = \mathbb{P}(\max\{X_1, X_2, \ldots, X_n\} \leq z) = \mathbb{P}(X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z) = \mathbb{P}(X_1 \leq z) \cdot \mathbb{P}(X_2 \leq z) \cdot \cdots \cdot \mathbb{P}(X_n \leq z)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i \leq z) = \frac{z-0}{1-0} = z$. Thus,

$$F(z) = \mathbb{P}(Z \leq z) = z \cdot \cdots \cdot z = z^n$$

Therefore, $f(z) = \frac{dF(z)}{dz} = nz^{n-1}$ we can now calculate the expectation of Z, which is:

$$\mathbb{E}(Z) = \int_0^1 z \cdot nz^{n-1}\, dz = \int_0^1 nz^n\, dz = \left(\frac{n}{n + 1} \cdot z^{n+1}\bigg|_0^1\right) = \frac{n}{n + 1}$$