I worked alone without getting any help, except asking questions on Piazza and reading the Notes of this course.

1 Short Answer: Graphs

(a) 3

Since there is no loop in a tree by definition, so a degree 3 node n would be connecting to 3 other nodes, n_1, n_2, n_3, such that no two nodes among n_1, n_2, n_3 would be connected after n is removed (or else there would be a loop). Thus, n_1, n_2, n_3 are in three different parts, and since a tree is connected by definition, so the 3 parts n_1, n_2, n_3 are in, respectively, must be the only 3 connected components.

(b) 7

For a n-vertex tree, G, by definition, G is connected and has $n-1$ edges. So, after Bob's and Alice's movements, there would be $n-1+10-5=n+4$ edges in G. On the other hand, since the resulting graph has three connected components, C_1, C_2, C_3 and assume each component has n_1, n_2, n_3 vertices, respectively. WLOG, consider component C_1, which has n_1 vertices. Since we want to remove all cycles in the resulting graph (with minimum removal), by the fourth equivalent definition of a tree, the optimal option is to turn C_1 into a tree, which would make it have n_1-1 edges. Similarly, the final C_2 and C_3 should have n_2-1 and n_3-1 edges, respectively.

Then, since we are considering graph G, so the number of vertices remains the same, which means that $n_1+n_2+n_3=n$. Thus, the total number edges after removing all cycles would be $n_1-1+n_2-1+n_3-1=n_1+n_2+n_3-3=n-3$ edges. Thus, this implies that we need to remove $(n+4)-(n-3)=7$ edges.

(c) False

Consider $n=3$ as a counterexample. For K_3, the complete graph on 3 vertices forms a triangle, so K_3 has 3 edges. On the other hand, using Lemma 5.1, the number of edges in a 3-dimensional hypercube is $3*2^{3-1}=3*2^2=12$. Since when $n=3, 3<12$, so the proposition is false.

(d) $\frac{n-1}{2}$

For a complete graph with n vertices, by definition, every pair of vertices is connected, so we have that the degree of every vertex is $n-1$, and since there are n vertices, with each edge connecting 2 vertices (or contributing 2 degrees), so there are $\frac{n(n-1)}{2}$ edges in the graph. On the other hand, by definition of a Hamiltonian cycle, since each vertex appears exactly once and that there is a final edge connecting the first and last same vertex, so there would be $n-1+1=n$ edges in each Hamiltonian cycle. Then, desiring the least number of Hamiltonian cycles, we expect them to all be edge-disjoint. Therefore, x, the number of Hamiltonian cycles needed to cover the complete graph is at least $\frac{\frac{n(n-1)}{2}}{n}=\frac{n(n-1)}{2}$.

(e) A set of **two** Hamiltonian cycles, H_1 and H_2.

$$H_1=\Big\{(0,1),(1,2),(2,3),(3,4),(4,0)\Big\}$$

$$H_2=\Big\{(0,2),(2,4),(4,1),(1,3),(3,0)\Big\}$$

2 Eulerian Tour and Eulerian Walk

(a) No, there isn't.

Consider vertex 3, which has a degree of 3, so it's not even degree, which means that the graph G is not even degree. Using Theorem 5.1, an undirected graph G has an Eulerian tour if and only if G is even degree and connected, so there isn't an Eulerian tour in the graph.

(b) Yes, there is.

Consider a walk W: {3, 4}, {4, 2}, {2, 3}, {3, 1}, {1, 4}, {4, 8}, {8, 1}, {1, 2}, {2, 6}, {6, 1}, {1, 7}, {7, 8}, {8, 6}, {6, 7}.

Since W is a walk that uses each edge exactly once, so by definition, W is an Eulerian walk in the graph.

(c) The condition is that there is exactly zero or exactly two vertices with odd degree, and G is connected.

Proof. In other words, there is an Eulerian walk in an undirected graph G if and only if there is exactly zero or exactly two vertices with odd degree, and G is connected. To prove this, we must establish two directions: if, and only if.

Only if. We give a proof by cases for this situation, i.e., given the assumption that G is connected, and that there is exactly zero or exactly two vertices with odd degree, so there would be two cases: either (1) G has exactly zero odd vertices, or (2) G has exactly two odd vertices.

Case 1: Let G have exactly zero odd vertices, which means that G is even degree, then using Theorem 5.1, G has an Eulerian tour. Since by definition, an Eulerian tour is an Eulerian walk, so G has an Eulerian walk.

Case 2: Let u, v be the two odd vertices. Just like our proof for Theorem 5.1, we can use a recursive algorithm for finding an Eulerian walk.

Start with u_0, we claim a walk W that doesn't use an edge twice, would always get stuck at v_0. Just like our proof for the claim in Theorem 5.1, after we leave u_0 initially, every vertex in G except for v_0 has even degree, which means that whenever we enter a vertex $v^* \neq v_0$, there is at least one edge we haven't used that's incident to v^*, which means that the walk wouldn't be stuck at v^*. Thus, the walk would always get stuck at v_0. This walk would not always be an Eulerian walk. However, after the initial walk, the "unused" degrees of all vertices in G is even, so similar to our proof in Theorem 5.1, a walk that starts at an arbitrary vertex v and doesn't use an edge twice would be a cycle that gets stuck at v. Thus, again, similar to our proof in Theorem 5.1, we could always find a set of **edge disjoint** tours, $T_1, T_2, ..., T_k, k \geq 1$, along with our initial walk W from u_0 to v_0 that covers all the edges in G.

Then, since G is connected by assumption, so again, similar to our proof in Theorem 5.1, we could splice together W and $T_1, T_2, ..., T_k$ so that it forms an Eulerian walk.

If. We give a direct proof for the forward direction, i.e., if an undirected graph G has an Eulerian walk, then G is connected and has exactly 0 or exactly 2 odd degree vertices.

Assume that G has an Eulerian walk W. By definition of Eulerian tours, so every vertex must have an edge adjacent to it, which implies that G is connected.

Then, let W traverse the vertices in this way: $v_0, v_1, ..., v_w$. Excluding the first and last vertices, for any vertex $v_i, 0 < v < w$, the edges $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ can be paired up. So every time a vertex is reached in the middle of W, there would always be two edges adjacent to it, which implies that all the vertices (except v_0, v_w) are even vertices. Now, we divide the situation into two cases, and exactly one of which must be true: (1) $v_0 = v_w$; or (2) $v_0 \neq v_w$.

Case (1): If $v_0 = v_w$, then by definition, W is a tour, which means that W is an Eulerian tour. Using Theorem 5.1, we have that G is even, which implies that there is 0 odd vertices in G.

Case (2): If $v_0 \neq v_w$, then consider v_0. Suppose it has appeared $k, k \in \mathbb{N}$ times in the walk besides being the initial vertex, then using our proof above, so it has a degree of $2k + 1$, which is an odd number, so v_0 is an odd vertex. Similarly, v_w is also an odd vertex, and they're the only two odd vertices in G, which implies that G has exactly two odd vertices.

Thus, we have proved that if G has an Eulerian walk, then G is connected, and G has exactly zero or exactly two odd vertices.

Therefore, the condition is that there is exactly zero or exactly two vertices with odd degree, and G is connected.

Q.E.D.

3 Bipartite Graph

Proof. To prove this, we must establish two directions: if, and only if.

Only if. We give a proof by contradiction for the forward direction, i.e., if a graph G is bipartite, then it has no tours of odd length. Assume, for a contradiction, that for a bipartite graph G_1, there is a tour T of odd length.

By definition of partite, consider the 2 disjoint sets of vertices of G, say L and R, such that no 2 vertices in the same set have an edge between them. WLOG, let our tour T begin in a vertex $v_0 \in R$, and let our i^{th} edge be $\{v_{i-1}, v_i\}$. We claim that for any odd-numbered edge, let it be the $(2n+1)^{th}$ edge $\{v_{2n}, v_{2n+1}\}$, $n \in \mathbb{N}$, then vertex $v_{2n+1} \in L$.

We proceed by induction on n.

Base case (n = 0): Consider edge $\{v_0, v_1\}$. Since there must not be an edge between 2 vertices in the same set (as stated above by definition), so $v_1 \in L$.

Thus, the base case is correct.

Inductive Hypothesis: Assume that the claim, for the $(2n+1)^{th}$ edge $\{v_{2n}, v_{2n+1}\}$ and $n \in \mathbb{N}$, $v_{2n+1} \in L$, is true for $n = k \in \mathbb{N}$.

Inductive Step: We prove the claim for $n = k+1 \geq 1$. Since $v_{2k+1} \in L$, and by definition of bipartite, so edge v_{2k+1}, v_{2k+2} connects two vertices in different sets, which implies that $v_{2k+2} \in R$. Thus, similarly, since v_{2k+2}, v_{2k+3} is an edge, so $v_{2k+3} \in L$, which gives us that $v_{2(k+1)+1} \in L$.

Thus, by the principle of mathematical induction, the claim holds.

Thus, the last vertex of the tour T would always be in the set R, while the first vertex of the tour T, by assumption, is in the set L, which implies that T is not a tour.

Therefore, there isn't a tour T of odd length in a bipartite graph, so we have that if G is bipartite, then G has no tours of odd length.

If. We give a Proof by Cases by providing a bipartition on the set of vertices of G where G is a graph with no tours of odd length, and we'll show that G is bipartite.

We proceed by cases. Let us divide our proof into three cases, exactly one of which must be true: (1) G has no tours; or (2) G has tours, but no tours of odd length.

Case (1): With the assumption that G has no tours, consider any path in G, so it doesn't have repeating vertices (or else it would result in a sub-path that is a tour). Similarly, G have no cycles. Thus, G only have walks. Let $W_1, W_2, ..., W_k$ be the edge disjoint walks of G. Let's start with W_1. Let the sequence of vertices that W_1 traverse be $v_{1,1}, v_{1,2}, ..., v_{1,w1}$. Let L, R be the two sets we will be putting our vertices into, and we define them such that no 2 vertices in the same set have an edge between them. First, let $v_{1,1} \in L$, and then alternate R, L to put in the vertices W_1 traverses. In general, $v_{1,2i-1} \in L, v_{1,2j} \in R$ where $i, j \in \mathbb{Z}^+$. Since all the edges in W_1 are of the form $\{v_{1,k}, v_{1,k+1}\}, k \in \mathbb{Z}^+$, so this assignment of vertices would not violate our assumption regarding L and R.

Then, consider W_2. Again, Let the sequence of vertices that W_2 traverse be $v_{2,1}, v_{2,2}, ..., v_{2,w2}$. There are two situations: either no vertex in $W-2$ have already been assigned into L or R, or there are vertices in $W-2$ that have already been assigned into L or R.

If none of the vertices in $W-2$ have already been assigned into L or R, then we repeat the process for vertices in W_2 just as we did for W_1.

Else if some vertices in $W - 2$ have already been assigned into L or R. We claim that there is at most 1 such vertex. We proceed by contradiction. Assume that there are two vertices u, v that have already been assigned into L or R. Then, there is a sub-walk of W_2 from u to v, and similarly, there is a sub-walk of W_1 from u to v, which contradicts our assumption that there should be no tours in G. Thus, there is at most 1 vertex that have been assigned already. WLOG, assume this vertex $v_{2,i} \in L$. So, similar to how we assigned vertices of W_1, we would assign all vertices that are adjacent to $v_{2,i}$ into R, then assign all vertices that are adjacent to them into L again, and continue this process. This would give us a bipartition that does not violate the assumption.

Now, consider W_3. Let the sequence of vertices that W_3 traverse be $v_{3,1}, v_{3,2}, ..., v_{3,w2}$. For situations like having zero or one vertices that have already been assigned, we will utilize a similar strategy as discussed above. Yet, there will only be one exception: suppose both $v_{3,i} \in W_i$ and $v_{3,j} \in W_j$ have already been assigned. We have proved that they could not have been assigned in the same walk W_k, or else it would lead to a tour, and thus contradicting our assumption. Start with $v_{3,i}$, and WLOG, let $v_{3,i} \in L$. Label its neighbors into R, and repeat this process. When we reach v_{3_j}, if our new assignment is the same as its previous assignment, then the algorithm would lead to a successful bipartition; on the other hand, if our new assignment is different as its previous assignment, then we would flip the assignment of the entire walk W_j. This would lead to a successful bipartition as v_{3_j} would now have the same assignment as its previous assignment, and the labeled W_j would still be successfully bipartitioned. Continue this process for all walks $W_4, ..., W_k$, and we will have successfully bipartitioned all the vertices of G.

Thus, we could always find a bipartition if G has no tours, which implies that if G has no tours, then G is bipartite.

Case (2): With the assumption that G has tours and no tours of odd length, we have that all the tours $T_1, T_2, ..., T_t$ are of even length, which means that for any arbitrary tour $T_x, 1 \leq x \leq t$, and let the vertices T_x traverse be $v_{x,1}, v_{x,2}, ..., v_{x,wx}, v_{x,1}$ (since a tour is closed). Since T_x is of even length, so there are an even number of edges in T_x, denoted as $\{v_{x,1}, v_{x,2}\}, \{v_{x,2}, v_{x,3} ..., \{v_{x,wx-1}, v_{x,wx}\}, \{v_{x,wx}, v_{x,1}\}$. Thus, this implies that wx is even, which means that we could assign the vertices of T_x into disjoint sets L and R just like how we assigned the vertices of walks in Case (1). Thus, similar to our proof in Case (1), we have that if G only has no tours of odd length, then G is still bipartite.

Therefore, we have that if G has no tours of odd length, then G is bipartite.

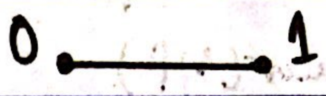
Therefore, we conclude that a graph is bipartite if and only if it has no tours of odd length.

Q.E.D.

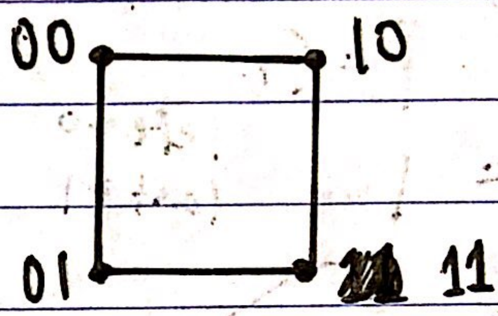
4. Hypercubes.

(a).

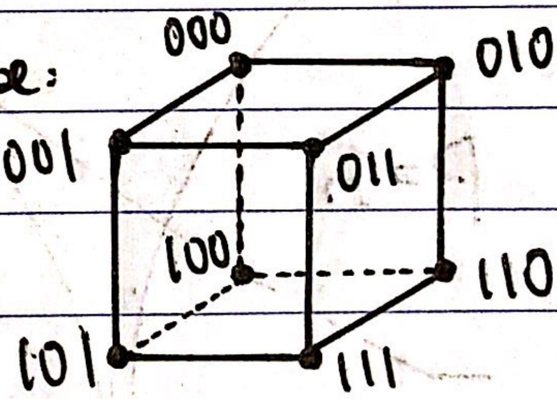
1 - dimensional hypercube:



2 - dimensional hypercube:



3 - dimensional hypercube:



4 Hypercubes

(b) Direct Proof

Proof. We give a direct proof by providing a bipartition on the set of vertices of G where G is an n-dimensional hypercube with $n \geq 1$. We claim that this assignment of vertices would create a valid bipartition: let L, R be two disjoint sets of vertices of G. Let $s_0 = 0^n$ (the n-bit string that is entirely comprised of 0) be in L. Then, for any $\{0,1\}^n$ string s_1 that has an even number of different bit position compared to s_0, we would have $s_1 \in L$; for any $\{0,1\}^n$ string s_2 that has an odd number of different bit position compared to s_0, we would have $s_2 \in R$.

Since by definition of hypercubes, two vertices x and y are connected by edge $\{x, y\}$ if and only if x and y differ in exactly one bit position, and since in our assignment of vertices, WLOG, consider the set L. For any two vertices $u, v \in L$, we have that u and v would always differ in an even number of bit positions, which implies that u and v would not differ in exactly one bit position, and thus, they wouldn't be connected by an edge. Similarly, for any two vertices in R, they wouldn't be connected by an edge. Thus, this assignment of vertices would create vertex disjoint sets L and R such that no 2 vertices in the same set have an edge between them.

Therefore, for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Q.E.D.

5 Triangulated Planar Graph

(a) 12

Let the triangulated planar graph G have V vertices, F faces, and E edges. Using Euler's formula, so we have $V + F = E + 2$.

Then, since every face has 3 sides, and since each edge is always shared by two faces, so $2E = 3F$, which gives us that $E = \frac{3}{2}F$. Thus, in this situation, we have that Eulers's formula is equivalent to $V + F = \frac{3}{2}F + 2$, which means that $V = \frac{1}{2}F + 2$. Moreover, consider any arbitrary vertex v, and let its charge be $c(v)$. By our definition of "charge," $c(v) = 6 -$ degree(v), which implies that $c = 6 -$ the number of edges connected to v. Since every edge connects two vertices, so the sum of the degrees of all vertices is twice the number of edges, or $2E$. Thus, the sum of the charges on all the vertices, $\sum C = \sum\limits_{v\in V} c(v) = \sum\limits_{v\in V}(6 - \text{degree}(v)) = \sum\limits_{v\in V}(6) - \sum\limits_{v\in V} \text{degree}(v) = 6V - 2E = 6(\frac{1}{2}F + 2) - 3F = 3F + 12 - 3F = 12$.

Therefore, the sum of the charges on all the vertices is 12.

(b) Charge of degree five vertex: **1**; charge of degree six vertex: **0**.

Since we define charge on any vertex v as the value of $6 -$ degree(v), so the charge of a degree 5 vertex is $6 - 5 = 1$, and the charge of a degree 6 vertex is $6 - 6 = 0$.

(c) Proof by contradiction.

Proof. We proceed by contradiction. Assume that the proposition is false, which means that there exists a triangulated planar graph G such that G does not have any vertex of degree 1, 2, 3, 4, and that G does not have two degree 5 vertices which are adjacent, and that G does not have a degree 5 vertex and a degree 6 vertex which are adjacent, under the assumption that after discharging all degree 5 vertices, there is a degree 5 vertex with positive remaining charge. Let R be the assertion that there is a degree 5 vertex with positive remaining charge.

Consider any degree 5 vertex $v \in G$. By definition of the degree of a vertex, so v has 5 "neighbors." Since there is no vertex of 1, 2, 3 or 4, and that v could not be adjacent to any degree 5 or degree 6 vertices, so for any neighbor v_n of v, we have that v_n has a degree $d \geq 7$. So, v_n has a charge $c \leq 6 - 7 = -1$. In other words, since v_n was chosen arbitrarily, so all 5 neighbors of v has a negative charge. Thus, after discharging v, we have that v shifted all of its charge to its neighbors, which implies that v would have a charge of 0, so v does not have positive remaining charge. Since we chose v arbitrarily as well, so there would be no degree 5 vertex with positive remaining charge, which implies $\neg R$.

We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired. Therefore, a situation like this is impossible, which implies that the statement regarding triangulated planar graph is true under the assumption that, after discharging all degree 5 vertices, there is a degree 5 vertex with positive remaining charge.

Q.E.D.

(d) **Yes**, it does. The possible degrees of that vertex are: **1, 2, 3, 4, 7**.

Since we have proved earlier in part (a) that the sum of the charges on all the vertices is 12, so there must exist vertices with positive charge after discharging.

For the second part of the question, WLOG, consider a vertex v_1 with degree 1, so v_1 has a charge of $6 - 1 = 5 > 0$. Since v_1 was never discharged, so it remains with a charge of 5, so vertices with degree 1 would have positive charge after discharging. Similarly, since $6 - 2 = 4, 6 - 3 = 3, 6 - 4 = 2$, so degree 2 vertices, degree 3 vertices, and degree 4 vertices would remain with a charge of 4, 3, 2, respectively after discharging, so they would all have positive charges.

Degree 7 vertices could possibly have a positive charge after discharging the degree 5 vertices because consider $v_7 \in G$, a degree 7 vertex, so the initial charge of v_7 is $6 - 7 = -1 < 0$. Consider this situation: let all of v_7's "neighbors," or all the vertices it's incident to (let them be $v_{7,1}, v_{7,2}, v_{7,3}, v_{7,4}, v_{7,5}, v_{7,6}, v_{7,7}$) are degree 5 vertices. WLOG, consider $v_{7,1}$, which has a degree of 5, so a charge of $6 - 5 = 1$. Since v_7 has a negative charge and is incident to $v_{7,1}$ (in other words, it is the neighbor of $v_{7,1}$), so $v_{7,1}$ would shift 1/5 of its charge, which is $\frac{1}{5}$ to v_7. Similarly, $v_{7,2}, v_{7,3}, v_{7,4}, v_{7,5}, v_{7,6}, v_{7,7}$ would also each give a charge of $\frac{1}{5}$ to v_7, so v_7 would end up with a charge of $-1 + \frac{1}{5} * 7 = -1 + \frac{7}{5} = \frac{2}{5} > 0$. Thus, it is possible for degree 7 vertices to have positive charges after discharging the degree 5 vertices.

Now we'll show that vertices with any other degree is not possible.

First of all, by definition of the triangulated planar graphs, no vertex could begin with a degree $d \leq 0$.

Consider degree 5 vertices, our assumption indicates that "no degree 5 vertices have positive charge after discharging the degree 5 vertices."

Consider any degree 6 vertices, v_6, so v_6 begin with a charge of $6 - 6 = 0$. By definition, a degree 5 vertice would shift its charge to a neighbor if and only if that neighbor has a negative charge. Yet, v_6 doesn't have a negative charge, so it wouldn't receive any charge from the discharging process. Thus, degree 6 vertices would remain with a charge of 0, so they don't have positive charges after discharging.

Consider any vertex v^* with a degree $d^* \geq 8$, so it has a a charge of $c^* \leq 6 - d^*$. On the other hand, the maximum amount of charge d^* could receive during the discharging of degree 5 vertices is the number of its neighbors, d^*, times $\frac{1}{5}$, which is the maximum charge it could receive from any of its neighbors. Thus, the maximum amount of charge d^* could receive is $\frac{1}{5}d^*$. Thus, the maximum charge v^* could have in the end is $6 - d^* + \frac{1}{5}d^* = 6 - \frac{4}{5}d^* \leq 6 - \frac{4}{5} * 8 = -\frac{2}{5} < 0$. Thus, this implies that any vertex with a degree greater than or equal to 8 cannot have positive charge after discharging.

Therefore, we have shown that both (1) there definitely exist vertices with positive charge after discharging and (2) the only possible degrees are 1, 2, 3, 4 and 7.

(e) It might have **6 or 7** neighbors of degree 5.

Consider any arbitrary degree 7 vertex v_7, by our definition of charge, so it has a charge of $6-7 = -1$. Let us denote the number of its neighbors of degree 5 as n. Thus, as I proved in part (d) above, if $n = 7$, which means that all 7 neighbors of v_7 is degree 5, then v_7 would end up with a charge of $\frac{2}{5} > 0$, a positive charge. Moreover, if $n = 6$, so exactly 6 of its neighbors is degree 5, then similarly, v_7 would receive a charge of $\frac{1}{5}$ from exactly 6 neighbors, so it would end up with a charge of $(-1) + \frac{1}{5} * 6 = \frac{1}{5} > 0$, a positive charge.

Now we'll show that if v_7 has any other degree, or in other words, if $n \neq 6$ and $n \neq 7$, then it would not have a positive charge after discharging all degree 5 vertices.

First, by definition of degree, it is not possible for v_7 to have 8 or more neighbors of degree 5, so $n \leq 8$. Similarly, by definiton, $n \geq 0$, so we have that $0 \leq n \leq 5$. Thus, similar to our logic above, the amount of charge v_7 could receive from its neighbors, $c_{receive} \leq \frac{1}{5} * 5 = 1$. Thus, the eventual charge of $v_7, c \leq (-1) + 1 = 0$, so v_7 would not have a positive charge. Thus, this gives us that v_7 would not have a positive charge if $n \neq 6$ and $n \neq 7$.

(f) Yes, they are.

Again, let the graph be G and consider our v_7 from part (e), which is a degree 7 vertex that has positive charge after discharging the degree 5 vertices. Let its neighbors be $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ in clockwise order. Since we have proved in part (e) that it has either 6 or 7 neighbors of degree 5, there is at most one neighbor not of degree 5. If there is one such neighbor (not of degree 5), WLOG, let it be u_7. Thus, under either situation, u_1, u_2 are two of v_7's neighbors of degree 5 such that there is no other edges between $\{v_7, u_1\}$ and $\{v_7, u_2\}$.

Then, we claim that there must be a walk in G from u_1 to u_2 that doesn't involve v_7 (in other words, there must be a cycle including both u_1 and u_2). We proceed by contradiction. Let R be the assertion that G is a triangulated planar graph.

Since there is no cycle including both u_1 and u_2, so there must be a unbounded face F^* bordered by $\{v_7, u_1\}$ and $\{v_7, u_2\}$. Consider the face bordered by edges $\{v_7, u_1\}$ and $\{v_7, u_2\}$ that is not closed, so the number of edges bordering it must be at least $2 + 2 = 4 > 3$, contradicting the definition of a triangulated planar graph, thus implying $\neg R$.

We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired. Therefore, a situation like this is impossible, which implies that there must be a cycle including both u_1 and u_2.

Now, assume that the cycle we found including both u_1 and u_2 has greater than or equal to 4 edges, then the face bounded by this cycle would have at least 4 edges bordering it; again, this violates our definition of a triangulated planar graph, so the cycle we found must have exactly 3 edges, and since we already have that $\{v_7, u_1\}$ and $\{v_7, u_2\}$ are two edges of this cycle, so the last edge must be $\{u_1, u_2\}$, which gives us the conclusion that two of these degree 5 vertices are adjacent.

Q.E.D.

(g) Proof by Cases.

Proof. We proceed by cases. Let us divide our proof into three cases, exactly one of which must be true: using the definition of "charge" and the process of discharging all degree 5 vertices, so (1) there is a degree 5 vertex with positive remaining charge; or (2) no degree 5 vertices remain with positive charge, and that there is a degree 7 vertex with positive charge; or (3) no degree 5 vertices remain with positive charge, and that there is no degree 7 vertex with positive charge. For clearer notation, let us denote the proposition we wish to prove as P, which says that every triangulated planar graph contains either (a) a vertex of degree 1, 2, 3, 4, (b) two degree 5 vertices which are adjacent, or (c) a degree 5 and a degree 6 vertices which are adjacent.

Case (1): With the assumption that there is a degree 5 vertex with positive remaining charge after discharging all degree 5 vertices, as we proved in part (c), so we have that P is true.

Case (2): With the assumption that no degree 5 vertices have positive charge after discharging the degree 5 vertices, and that there exists a degree 7 vertex with positive charge, as we proved in part (f), we have that two of the degree 5 neighbors of the degree 7 vertex are adjacent. Thus, with the the 2^{nd} condition of P's conclusion satisfied, we have that P is true.

Case (3): With the assumption that no degree 5 vertices have positive charge after discharging the degree 5 vertices, and that there is no degree 7 vertex with positive charge. Since we proved in part (d) that if no degree 5 vertices have positive charge after discharging all the degree 5 vertices, then there must still exist vertices with positive charge, and the only possible degrees of these vertices are 1, 2, 3, 4 and 7. Since we have that there is no degree 7 vertex with positive charge after discharging, so there must be vertices with degree 1, 2, 3 or 4, thus satisfying the 1^{st} condition of P's conclusion, so we have that P is true.

Thus, the proposition P, every triangulated planar graph contains either (a) a vertex of degree 1, 2, 3, 4, (b) two degree 5 vertices which are adjacent, or (c) a degree 5 and a degree 6 vertices which are adjacent, is true, and the proof is complete.

Q.E.D.