PRINT Your Name: ______________________________, ______________________________
(last) (first)

SIGN Your Name: ______________________________

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Name of the person sitting to your left: ______________________________

Name of the person sitting to your right: ______________________________

- After the exam starts, please *write your student ID (or name) on every odd page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- The questions vary in difficulty, so if you get stuck on any question, it might help to leave it and try another one.
- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement, a short argument.) Note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-6, do give arguments, proofs or clear descriptions if requested. If there is a box do use it for your answer.
- You may consult only *one sheet of notes*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- There are **12** single sided pages including the cover sheet on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 6 questions (with 54 parts) on this exam worth a total of 152 points.**

$\boxed{\text{Do not turn this page until your instructor tells you to do so.}}$

SID:

1. TRUE or FALSE?: 2pts each

For each of the questions below, answer TRUE or FALSE. No need to justify answer.

Please fill in the appropriate bubble!

1. $(\neg(P \wedge \neg Q)) \equiv (P \implies Q)$

   ◯ True

   ◯ False

2. $\forall n \in \mathbb{N}, \exists n' \in \mathbb{N},\ n' > n.$

   ◯ True

   ◯ False

For the following, assume $Q(x,y)$ and $P(x)$ are predicates over the domain of x, y.

3. $(\forall x, \exists y, Q(x,y) \wedge P(x)) \implies \forall x, P(x)$

   ◯ True

   ◯ False

4. $(\forall n \in \mathbb{N}, \neg P(n+1) \implies \neg P(n)) \implies (\neg P(0) \vee \forall n \in \mathbb{N}, P(n)).$

   ◯ True

   ◯ False

For the next two parts, we consider two pairings S and T for a stable marriage instance and form a graph as follows; We take the vertices to be the people and the edges connect a man m, and woman w, if the pair (m,w) is in S or T (or both).

5. For any cycle in the graph for two pairings S and T, either all the men in the cycle prefer S or all the men in the cycle prefer T.

   ◯ True

   ◯ False

6. If S is the male-optimal pairing, and T is the female-optimal pairing, and the graph of S and T consists of a single cycle then there are exactly two stable pairings.

   ◯ True

   ◯ False

7. In a stable marriage instance with n men and n women, let M be every woman's last choice. For all $1 \leq i \leq n$, there is a set of preference lists where M can end up with his i^{th} choice after running the traditional stable marriage algorithm.

   ◯ True

   ◯ False

8. For every n, there is a stable marriage instance with a stable pairing where every man and woman is paired with their least favorite in the preference list.

○ True

○ False

9. No woman can have her optimal partner in the traditional marriage algorithm.

○ True

○ False

10. There are at most 2 stable pairings for any stable matching instance: the male optimal one and female optimal one.

○ True

○ False

11. There is no stable pairing for any stable roommates instance with $2n$ people.

○ True

○ False

12. Say I take a walk in any connected undirected graph, making sure I only traverse edges I haven't traversed before. Suppose I get stuck at a vertex v because there are no unused edges I can use to leave v. If I never see the same vertex twice in the walk, v must have degree 1.

○ True

○ False

13. For a directed graph, the sum of the outdegrees equals the sum of indegrees.

○ True

○ False

14. Recall, adding an edge to a tree creates a cycle. Is it true or false that adding an edge to a tree also ensures that there are two different simple paths between **any** pair of vertices in the tree?

○ True

○ False

15. Any graph where every vertex has degree at least 2 is connected.

○ True

○ False

16. A graph where every vertex has degree at least d requires at least d colors to vertex color it.

○ True

○ False

17. Any **connected** graph with average degree strictly less than 2 is a tree.

○ True

○ False

18. If $\gcd(x,y) = d$ and $\gcd(a,b) = d'$, then $\gcd(ax,by) = dd'$

○ True

○ False

19. If m is not prime, then for every x there is no multiplicative inverse for $x \pmod{m}$.

○ True

○ False

20. If $m > n$, then $\gcd(m,n) = \gcd(m-n,n)$.

○ True

○ False

21. For a prime p, any a that is relatively prime has $\{a, a^2, \ldots, a^{p-1}\}$ being all distinct modulo p.

○ True

○ False

2. An expression or number: 3 points each. Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

1. Let P= "We should be honest.", and Q = "We should be dedicated.", and R ="We should be overconfident." How should one write "We should be honest or dedicated, but not overconfident." as an expression involving P, Q and R.

2. How many faces are in an n-vertex planar graph with $n+10$ edges?

3. Consider a planar graph that has 30 faces. Add a vertex of degree 3 to the graph such that the resulting graph is planar as well. How many faces does the resulting graph have?

4. What is the minimum number of leaves in an n-vertex tree?

5. Consider a bipartite *planar* graph G that has v vertices, for $v > 2$. What is the maximum number of edges, expressed in terms of v, that G could have? (**Answer is an expression or number.**)

6. What is the maximum number of connected components that can result from removing the edges in a length-k cycle from a connected graph?

7. What time is 1000 hours after 1:00 PM? (You should include AM/PM.)

8. How many edges does one need to remove from a 4-dimensional hypercube to get a tree? (**Answer is a number.**)

9. What is the length of the shortest path (in number of edges) from the vertex 0101 to the vertex 1001 in a hypercube?

10. What is the maximum number of edges in a disconnected graph on n vertices?

11. What is the smallest number of edges one needs to add to a graph on k vertices of odd degree such that it has an Eulerian tour? (You can use parallel edges if needed. For example, a two-vertex Eulerian graph must consist of at least two edges between the same two vertices.)

12. What is $\sum_{i=1}^{70} i^2 \pmod 5$?

13. How **many solutions** are there to $5x = 5 \pmod{10}$? (**Answer is a number**)

14. *How many* solutions are there to $5x = 2 \pmod{10}$? (**Answer is a number.**)

For the following 3 parts, consider that $ax + bm = d$, where gcd$(x, m) = d$, and d may be larger than 1.

15. Consider the equation $xu = v \pmod m$. It has a solution for u if and only if v is a ________. (Your answer may involve the variables a, b, x, m and d.)

16. If u is a solution to the equation $xu = v \pmod m$ as above, write an expression for u in terms of possibly a, b, x, v, d, m and possibly including the *mod* function.

17. Again consider the equation $xu = v \pmod m$. How many solutions are there $\pmod m$ if there is at least one? (Answer is an expression in terms of possibly a, b, x, v, d.)

3. Make 3 Friends! (2/3/3/3/5)

Consider a process of building a n-vertex undirected graph with $n > 3$. Begin with a triangle, and then repeatedly add a vertex and 3 edges from that new vertex to 3 different previous vertices. Note that many different graphs can be generated by this process as the choice of which previous vertices to add edges to is unspecified. (Perhaps think of a party, where people arrive and choose 3 previous people to befriend. This would be the resulting undirected friend graph.)

1. **True/False** Any graph generated by this process is connected.

   ◯ True

   ◯ False

2. What is the maximum possible degree of a vertex in some n-vertex graph generated by this process?

3. What is the minimum number of colors needed to color any n vertex graph generated in this fashion? (The next two answers depend on this so if you skip this, also skip them, though reading them may help.)

4. Give an example of a graph that can be generated by this process that takes at least this many colors.

5. Give an algorithm to generate an coloring that achieves your answer for part 3 for any graph generated according the process above. (You will recieve no credit if your answer is wrong for part 3 of this question. Apologies.)

4. Some Proofs.(5/5/5/6)

1. Prove: If $y < x$ then $x \pmod{y} \leq x/2$.
   (Recall, $x \pmod{y} = x - \lfloor x/y \rfloor y$.)

2. Prove: For $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

3. Show that if (M,W) are partners in both the male-optimal and the female optimal stable pairings, then they must be partners in every stable pairing.

4. The complement of an undirected graph $G=(V,E)$ is $\overline{G}=(V,\overline{E})$, where the set of edges $\overline{E}$ is the set of edges NOT in G. (For example, the complement of a complete graph K_n is a graph with no edges and n vertices).

   Show that given any undirected graph G and its complement $\overline{G}$, at least one of them is connected.

   (Hint: Start with G being disconnected and show $\overline{G}$ must be connected. Also explain why this is sufficient.)

5. Coins. 2/4/4

Suppose we place n coins **in a circle** where n **is odd.** You are allowed to take two consecutive coins, which either both have heads facing up or both have tails facing up, and flip them over. You can do this operation as many times as desired. Prove that it is possible to reach a configuration where all the coins face the same way after a finite number of operations.

1. How does the number of maximally contiguous groups of heads h compare with the number of such groups t for tails? (Answer should be a statement involving h and t. Answer only will be graded.)

2. Show that at least one group of maximally contiguous heads or tails is even in length.

3. Give a proof that you can reach a configuration where all coins face the same way.

6. Globally Optimal Marriage, perhaps.(2/2/2/6)

In this problem, we address a different version of the marriage problem, one where given a set of preference lists for n-men and n-women one minimizes total dissatisfaction. The **dissatisfaction score** of a person is the position of their partner on that person's preference list. Then, the overall dissatisfaction score is the sum over everybody's individual score.

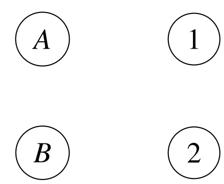
For example, for a 2-man, 2-woman instance where everyone gets their favorite person, position 1, then the score is 4. If each woman gets her favorite choice, position 1, and each man gets his least favorite choice, position 2, the total score is 6.

1. Define a directed graph for an instance and an pairing S of the men and women as follows. For each $\{m,w\} \in S$, add a directed edge from m to w. For each $\{m,w\} \notin S$ add a directed edge from w to m. How many directed edges are there in this graph in an n-man, n-woman instance?

2. Define a weight for a directed edge (m,w) to be the sum of the position of w in m's preference list and the position of m in w's preference list. Define the weight of the directed edge (w,m) to be the negation of the value for (m,w).

   Draw in the directed edges and label your edges with weights for the instance below for the pairing $\{(A,1),(B,2)\}$.

A:	1	2	1:	A	B
B:	2	1	2:	B	A



3. A pairing minimizes dissatisfaction if and only if all directed cycles in the graph have __________ cost.

4. Prove your answer for the previous part. (Only tackle this problem if you are confident of your answer to part 2.)