5 Proof or Disprove

(a) Prove. Direct Proof

Let $n \in \mathbb{N}$ be an odd number, so let $n = 2k + 1, k \in \mathbb{N}$.

So $n^2 + 2n = (2k+1)^2 + 2*(2k+1) = 4k^2 + 4k + 1 + 4k + 2 = 4k^2 + 8k + 3 = 2*(2k^2 + 4k + 1) + 1$
Since $k \in \mathbb{N}$, so $(2k^2 + 4k + 1) \in \mathbb{N}$, so $n^2 + 2n$ is odd.

Thus, the proposition is true.

Q.E.D.

(b) Prove. Proof by Cases

Let $x, y \in \mathbb{R}$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (1) $x >= y$; or (2) $x < y$.

Case (1): Since $x >= y$, so $|x - y| = x - y$ and $\min(x, y) = y$.

So $(x + y - |x - y|)/2 = (x + y - x + y)/2 = (2y)/2 = y = \min(x, y)$

Case (2): Since $x < y$, so $|x - y| = -x + y$, and $\min(x, y) = x$

So $(x + y - |x - y|)/2 = (x + y + x - y)/2 = (2x)/2 = x = \min(x, y)$

Thus, $\min(x, y) = (x + y - |x - y|)/2$

Q.E.D.

(c) Prove. Proof by Contradiction

We proceed by contradiction. Assume that the proposition is false, which means that for some $a, b \in \mathbb{R}, (a + b <= 10)$, and that $((a <= 7)$ or $(b <= 3))$ is false. Let our assertion R state that $(a + b <= 10)$.

Since $((a <= 7)$ or $(b <= 3))$ is false, so $(a > 7)$ and $(b > 3)$. So $a + b > 7 + 3 > 10$.

This implies $\neg R$. We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired.

Thus, the proposition is true.

Q.E.D.

(d) Prove. Proof by Contradiction

We proceed by contradiction. Assume that the proposition is false, which means that for some $r \in \mathbb{R}$, r is irrational and $r + 1$ is rational. Let our assertion R state that r is irrational. Since $r + 1$ is rational, by definition, let $r + 1 = \frac{p}{q}$ such that $p, q \in \mathbb{Z}$. So $r = r + 1 - 1 = \frac{p}{q} - 1 = \frac{p-q}{q}$. Since $p - q, q \in \mathbb{Z}$, so by definition, r is rational.

This implies $\neg R$. We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired.

Thus, the proposition is true.

Q.E.D.

(e) Disprove.

Consider $n = 6 \in \mathbb{Z}^+$.

So $10n^2 = 10 * 6^2 = 360$, and $n! = 6! = 720$.

Since $360 < 720$, so $10n^2 > n!$ is false for $n = 6 \in \mathbb{Z}^+$.

Thus, the proposition is false.

Q.E.D.