6 Preserving Set Operations

(a)

We would show two parts: (1) $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$; and (2) $(f^-1(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$.

Part (1): For any element $e \in f^{-1}(A \cup B)$, by definition of inverse images, so $f(e) \in A \cup B$, so $f(e) \in A$ or $f(e) \in B$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (i) $f(e) \in A$; or (ii) $f(e) \notin A$.

Case (i): Since $f(e) \in A$, so by definition, $e \in f^{-1}(A)$, so $e \in (f^{-1}(A) \cup f^{-1}(B))$

So $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$

Case (ii): Since $f(e) \notin A$, and since $f(e) \in A$ or $f(e) \in B$, so $f(e) \in B$.

So by definition, $e \in f^{-1}(B)$, so $e \in (f^{-1}(A) \cup f^{-1}(B))$

So $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$

Thus, we have that $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \cup f^{-1}(B))$, so $e \in f^{-1}(A)$ or $e \in f^{-1}(B)$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (i) $e \in f^{-1}(A)$; or (ii) $e \notin f^{-1}(A)$.

Case (i): Since $e \in f^{-1}(A)$, so by definition, $f(e) \in A$, so $f(e) \in A \cup B$.

So by definition, $e \in f^{-1}(A \cup B)$.

Case (ii): Since $e \notin f^{-1}(A)$, and since $e \in f^{-1}(A)$ or $e \in f^{-1}(B)$, so $e \in f^{-1}(B)$.

So by definition, $f(e) \in B$, so $f(e) \in A \cup B$.

So by definition, $e \in f^{-1}(A \cup B)$.

Thus, we have that $(f^{-1}(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$.

Since $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$ and $(f^{-1}(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$, so we have that $f^{-1}(A \cup B) = (f^{-1}(A) \cup f^{-1}(B))$.

Q.E.D.

(b)

We would show two parts: (1) $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$; and (2) $(f^-1(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$.

Part (1): For any element $e \in f^{-1}(A \cap B)$, by definition of inverse images, so $f(e) \in A \cap B$, so $f(e) \in A$ and $f(e) \in B$. Since $f(e) \in A$, by definition, so $e \in f^{-1}(A)$. Similarly, $e \in f^{-1}(B)$. So we have $e \in (f^{-1}(A) \cap f^{-1}(B))$, which implies that $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \cap f^{-1}(B))$, so $e \in f^{-1}(A)$ and $e \in f^{-1}(B)$. Since $e \in f^{-1}(A)$, so by definition of inverse images, $f(e) \in A$. Similarly, $f(e) \in B$. So $f(e) \in A \cap B$. So by definition, $e \in f^{-1}(A \cap B)$, which implies that $(f^-1(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$.

Since $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$ and $(f^{-1}(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$, so we have that $f^{-1}(A \cap B) = (f^{-1}(A) \cap f^{-1}(B))$.

Q.E.D.

(c)

We would show two parts: (1) $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$; and (2) $(f^-1(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$.

Part (1): For any element $e \in f^{-1}(A \setminus B)$, by definition of inverse images, so $f(e) \in A \setminus B$, so $f(e) \in A$ and $f(e) \notin B$. Since $f(e) \in A$, by definition, so $e \in f^{-1}(A)$. Similarly, $e \notin f^{-1}(B)$. So we have $e \in (f^{-1}(A) \setminus f^{-1}(B))$, which implies that $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \setminus f^{-1}(B))$, so $e \in f^{-1}(A)$ and $e \notin f^{-1}(B)$. Since $e \in f^{-1}(A)$, so by definition of inverse images, $f(e) \in A$. Similarly, $f(e) \notin B$. So $f(e) \in A \setminus B$. So by definition, $e \in f^{-1}(A \setminus B)$, which implies that $(f^-1(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$.

Since $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$ and $(f^{-1}(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$, so we have that $f^{-1}(A \setminus B) = (f^{-1}(A) \setminus f^{-1}(B))$.

Q.E.D.

(d)

We would show two parts: (1) $f(A \cup B) \subseteq (f(A) \cup f(B))$; and (2) $(f(A) \cup f(B)) \subseteq f(A \cup B)$.

Part (1): Consider any element $e \in f(A \cup B)$. By definition of images, so there exists some $x \in A \cup B$ such that $e = f(x)$. WLOG, let $x \in A$. So by definition, $e \in f(A)$, so $e \in f(A) \cup f(B)$, which implies that $f(A \cup B) \subseteq (f(A) \cup f(B))$.

Part (2): Consider any element $e \in f(A) \cup f(B)$. WLOG, let $e \in f(A)$. By definition, so $\exists x \in A$ such that $e = f(x)$. Since $x \in A$, so $x \in A \cup B$, so by definition, $e \in f(A \cup B)$, which implies that $(f(A) \cup f(B)) \subseteq f(A \cup B)$.

Thus, we have that $f(A \cup B) = (f(A) \cup f(B))$.

Q.E.D.

(e)

For any element $e \in f(A \cap B)$, by definition of images, so there exists some $x \in A \cap B$ such that $e = f(x)$, so $x \in A$ and $x \in B$. Since $e = f(x)$ and $x \in A$, again by definition, we have $e \in f(A)$. Similarly, $e \in f(B)$, so $e \in f(A) \cap f(B)$, which implies that $f(A \cap B) \in f(A) \cap f(B)$.

An example where the equality does not hold:

Consider $f(x) = x^2, A = \{0, 2\}, B = \{0, -2\}$.

So $A \cap B = \{0\}$. By definition of images, we have $f(A \cap B) = \{0\}, f(A) = \{0, 4\}, f(B) = \{0, 4\}$, so $f(A) \cap f(B) = \{0, 4\}$, which gives that $f(A \cap B) \neq f(A) \cap f(B)$.

Q.E.D.

(f)

For any element $e \in f(A) \setminus f(B)$, so $e \in f(A)$ and $e \notin f(B)$. By definition of images, there exists some $x \in A$ such that $e = f(x)$. Similarly, there's no such $y \in B$ such that $e = f(y)$, which implies that $x \notin B$, which means that $x \in A \setminus B$. And since $e = f(x)$, so by definition, $e \in f(A \setminus B)$, which implies that $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

An example where the equality does not hold:

Consider $f(x) = x^2, A = \{0, 2\}, B = \{-2\}$.

So $A \setminus B = \{0, 2\}$. By definition of images, we have $f(A \setminus B) = \{0, 4\}, f(A) = \{0, 4\}, f(B) = \{4\}$, so $f(A) \setminus f(B) = \{0\}$, which gives that $f(A \setminus B) \neq f(A) \setminus f(B)$.

Q.E.D.