Sundry: I worked alone without any help.

1 Buffon's Needle on a Grids

(a) $\mathbb{P}$[no intersection at θ] =?

We proceed by a direct proof. This is very similar to the coupon collector's problem.

As usual, we start by writing $X = I_1 + I_2 + \cdots + I_n$ where I_i is the number of times we visit Safeway while trying to get the i^{th} new card (starting immediately after we have gotten the $(i-1)^{th}$ new card). With this definition, I_1 is trivial: no matter what happens, we always get a new card the first time (since we have none to start with). So $\mathbb{P}[I_1 = 1] = 1$, and thus $\mathbb{E}[I_1] = 1$.

Then, I_2 has the geometric distribution with parameter $p = \frac{n-1}{n}$, and thus, using Theorem 19.2, we have $\mathbb{E}[I_2] = \frac{1}{p} = \frac{n}{n-1}$. Similarly, for any $i = 1, 2, \ldots, n$, I_i has the geometric distribution with parameter $p = \frac{n-i+1}{n}$, and hence, $\mathbb{E}[I_i] = \frac{n}{n-i+1}$ by Theorem 19.2.

Now, we could apply the linearity of expectation to get:

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_n] = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n\sum_{i=1}^{n} \frac{1}{i}$$

Similarly, since all the events I_i are mutually independent, so the variance of their sum is the sum of their variance by Theorem 16.3, and thus we have: $\text{var}(X) = \text{var}(I_1 + I_2 + \cdots + I_n) = \text{var}(I_1) + \text{var}(I_2) + \cdots + \text{var}(I_n)$, where for any $I_i \sim \text{Geo}(\frac{n-i+1}{n})$, by Theorem 19.3 we have $\text{var}(I_i) = \frac{1 - \frac{n-i+1}{n}}{(\frac{n-i+1}{n})^2} = \frac{n(i-1)}{(n-i+1)^2}$. Thus,

$$\text{var}(X) = \text{var}(I_1) + \text{var}(I_2) + \cdots + \text{var}(I_n) = \frac{n \cdot 0}{n^2} + \frac{n \cdot 1}{(n-1)^2} + \frac{n \cdot 2}{(n-2)^2} + \cdots + \frac{n(n-1)}{1^2}$$

$$\implies \text{var}(X) = n \cdot \sum_{i=1}^{n} \frac{i-1}{(n-(i-1))^2}$$

$$\implies \text{var}(X) = n \cdot \sum_{i=1}^{n} \frac{n-i}{(n-(n-i))^2} = n \cdot \sum_{i=1}^{n} \frac{n-i}{i^2}$$

$$\implies \text{var}(X) = n \cdot (\sum_{i=1}^{n} \frac{n}{i^2} - \sum_{i=1}^{n} \frac{i}{i^2})$$

$$\implies \text{var}(X) = n^2 \cdot \sum_{i=1}^{n} \frac{1}{i^2} - n\sum_{i=1}^{n} \frac{1}{i}$$

Since $\mathbb{E}[X] = n\sum\limits_{i=1}^{n} \frac{1}{i}$, so we have that:

$$\text{var}(X) = n^2 \cdot \sum_{i=1}^{n} i^{-2} - \mathbb{E}[X]$$

which is the desired result. Q.E.D.