I worked alone without getting any help, except asking questions on Piazza and reading the Notes of this course.

1 Short Answer: Graphs

(a) 3

Since there is no loop in a tree by definition, so a degree 3 node n would be connecting to 3 other nodes, n_1, n_2, n_3, such that no two nodes among n_1, n_2, n_3 would be connected after n is removed (or else there would be a loop). Thus, n_1, n_2, n_3 are in three different parts, and since a tree is connected by definition, so the 3 parts n_1, n_2, n_3 are in, respectively, must be the only 3 connected components.

(b) 7

For a n-vertex tree, G, by definition, G is connected and has $n-1$ edges. So, after Bob's and Alice's movements, there would be $n-1+10-5=n+4$ edges in G. On the other hand, since the resulting graph has three connected components, C_1, C_2, C_3 and assume each component has n_1, n_2, n_3 vertices, respectively. WLOG, consider component C_1, which has n_1 vertices. Since we want to remove all cycles in the resulting graph (with minimum removal), by the fourth equivalent definition of a tree, the optimal option is to turn C_1 into a tree, which would make it have n_1-1 edges. Similarly, the final C_2 and C_3 should have n_2-1 and n_3-1 edges, respectively.

Then, since we are considering graph G, so the number of vertices remains the same, which means that $n_1+n_2+n_3=n$. Thus, the total number edges after removing all cycles would be $n_1-1+n_2-1+n_3-1=n_1+n_2+n_3-3=n-3$ edges. Thus, this implies that we need to remove $(n+4)-(n-3)=7$ edges.

(c) False

Consider $n=3$ as a counterexample. For K_3, the complete graph on 3 vertices forms a triangle, so K_3 has 3 edges. On the other hand, using Lemma 5.1, the number of edges in a 3-dimensional hypercube is $3*2^{3-1}=3*2^2=12$. Since when $n=3, 3<12$, so the proposition is false.

(d) $\frac{n-1}{2}$

For a complete graph with n vertices, by definition, every pair of vertices is connected, so we have that the degree of every vertex is $n-1$, and since there are n vertices, with each edge connecting 2 vertices (or contributing 2 degrees), so there are $\frac{n(n-1)}{2}$ edges in the graph. On the other hand, by definition of a Hamiltonian cycle, since each vertex appears exactly once and that there is a final edge connecting the first and last same vertex, so there would be $n-1+1=n$ edges in each Hamiltonian cycle. Then, desiring the least number of Hamiltonian cycles, we expect them to all be edge-disjoint. Therefore, x, the number of Hamiltonian cycles needed to cover the complete graph is at least $\frac{\frac{n(n-1)}{2}}{n}=\frac{n(n-1)}{2}$.

(e) A set of **two** Hamiltonian cycles, H_1 and H_2.

$$H_1=\Big\{(0,1),(1,2),(2,3),(3,4),(4,0)\Big\}$$

$$H_2=\Big\{(0,2),(2,4),(4,1),(1,3),(3,0)\Big\}$$