2 Variance of the Minimum of Uniform Random Variables

(a) $\mathrm{var}(Y) = \frac{n}{(n+1)^2\,(n+2)}$

Given that $n \in \mathbb{Z}^+$, $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$, and that $Y = \min\{X_1, \ldots, X_n\}$, so using result from HW 12 Problem 6(a), we have that:

$$\mathbb{E}(Y) = \frac{1}{n+1}$$

Now, since $0 \leq Y \leq 1$, so we also have that $0 \leq Y^2 \leq 1$ and thus, we can use the tail sum formula to obtain:

$$\mathbb{E}(Y^2) = \int_0^\infty \mathbb{P}(Y^2 > y)\, dy = \int_0^1 \mathbb{P}(Y^2 > y)\, dy$$

Now, since $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$, and $Y = \min\{X_1, X_2, \ldots, X_n\}$, so we have that $Y^2 = \min\{X_1^2, X_2^2, \ldots, X_n^2\}$. Thus, for any $y \in [0,1]$, we can calculate an expression for $\mathbb{P}(Y^2 > y)$ as: $\mathbb{P}(Y^2 > y) = \mathbb{P}(\min\{X_1^2, \ldots, X_n^2\} > y) = \mathbb{P}(X_1^2 > y, \ldots, X_n^2 > y) = \mathbb{P}(X_1^2 > y) \cdot \cdots \cdot \mathbb{P}(X_n^2 > y)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i^2 > y) = \mathbb{P}(X_i > \sqrt{y}) = \frac{1-\sqrt{y}}{1-0} = 1 - \sqrt{y}$. Thus,

$$\mathbb{P}(Y^2 > y) = (1-\sqrt{y}) \cdot \cdots \cdot (1-\sqrt{y}) = (1-\sqrt{y})^n$$

Therefore, we can now calculate the expectation of Y, which is:

$$\mathbb{E}(Y^2) = \int_0^1 \mathbb{P}(Y^2 > y)\, dy = \int_0^1 (1-\sqrt{y})^n\, dy \tag{1}$$

Now, we first use the substitution $u = 1 - \sqrt{y}$, and so: $du = -\frac{1}{2}y^{-\frac{1}{2}}\, dy$, which can be transformed into: $dy = -2\sqrt{y}\, du = (2u-2)\, du$. Also, the bounds of the integrals changes with $u = 0$ as $y = 1$, and $u = 1$ as $y = 0$. Thus, we can continue to evaluate Eq. (1) as:

$$\mathbb{E}(Y^2) = \int_1^0 u^n \cdot (2u-2)\, du = \int_1^0 2u^{n+1} - 2u^n\, du = \left(\frac{2}{n+2}u^{n+2} - \frac{2}{n+1}u^{n+1}\right)\Big|_1^0$$

$$\implies \mathbb{E}(Y^2) = 0 - \left(\frac{2}{n+2} - \frac{2}{n+1}\right) = \frac{2}{(n+1)(n+2)}$$

Therefore, we can calculate the variance of Y by definition:

$$\mathrm{var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{2}{(n+1)(n+2)} - \left(\frac{1}{n+1}\right)^2 = \frac{n}{(n+1)^2\,(n+2)}$$