I worked with Jessica (jexicagjr@berkeley.edu), mainly on Q3.

# 1 Family Planning

(a)

The sample space is $(G, C) = \{(1,1), (1,2), (1,3), (0,3)\}$, where we can calculate the probability of each sample to be:

$\mathbb{P}[(1,1)] = \frac{1}{2}$ since it just represents the probability of their first child being a girl.

$\mathbb{P}[(1,2)] = \frac{1}{4}$ i.e. the probability of first child being a boy and second child being a girl.

$\mathbb{P}[(1,3)] = \frac{1}{8}$ with similar logic.

$\mathbb{P}[(0,3)] = \frac{1}{8}$ with similar logic.

(b)

| | $C=1$ | $C=2$ | $C=3$ |
|---|---|---|---|
| $G=0$ | 0 | 0 | $\frac{1}{8}$ |
| $G=1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

(c)

| | |
|---|---|
| $\mathbb{P}(G=0)$ | $\frac{1}{8}$ |
| $\mathbb{P}(G=1)$ | $\frac{7}{8}$ |

| $\mathbb{P}(C=1)$ | $\mathbb{P}(C=2)$ | $\mathbb{P}(C=3)$ |
|---|---|---|
| $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

The probability of the Browns having 0 girls is equivalent to them having 3 boys in a row, which is $\mathbb{P}(G=0) = (\frac{1}{2})^3 = \frac{1}{8}$, so we have $\mathbb{P}(G=1) = \mathbb{P}(\overline{G=0}) = \frac{7}{8}$.

Results confirmed since we could calculate the probability of them having 1 child, 2 children, 3 children, respectively could be done in a similar way to get: $\mathbb{P}(C=1) = \frac{1}{2}\mathbb{P}(C=2) = \frac{1}{4}\mathbb{P}(C=3) = \frac{1}{4}$, which confirms our result.

(d) No, they aren't.

Consider the case when the Browns have 0 girls and 3 children in total, so we have $\mathbb{P}(G=0, C=3) = \frac{1}{8}$. On the other hand, $\mathbb{P}(G=0)\mathbb{P}(C=3) = \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}$, which gives that $\mathbb{P}(G=0, C=3) \neq \mathbb{P}(G=0)\mathbb{P}(C=3)$, which implies that $G$ and $C$ aren't independent.

(e) $\mathbb{E}[G] = \frac{7}{8}, \mathbb{E}[C] = \frac{7}{4}$

We can calculate that:

$$\mathbb{E}(G) = \frac{1}{8} \cdot 0 + \frac{7}{8} \cdot 1 = \frac{7}{8}$$

$$\mathbb{E}(C) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 = \frac{7}{4}$$

# 2 Will I Get My Package?

(a) $\mathbb{E}(X) = \frac{1}{2}$

We can make use of Theorem 15.1. Let $X$ denote the number of customers who receive their own packages unopened, so $X = I_1 + I_2 + \cdots + I_n$ where $I_i = 0$ if the $i^{th}$ customer received his/her own package unopened.

Since $\mathbb{E}[I_i] = 0 \cdot \mathbb{P}[I_i = 0] + 1 \cdot \mathbb{P}[I_i = 1] = \mathbb{P}[I_i = 1] = \frac{1}{n} \cdot \frac{1}{2} = \frac{1}{2n}$, so

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \cdots + \mathbb{E}[I_n] = n \cdot \frac{1}{2n} = \frac{1}{2}$$

(b) $\mathrm{var}(X) = \frac{1}{2}$

Here, we have that $\mathbb{E}[X] = \frac{1}{2}$, so we need to calculate $\mathbb{E}[X^2]$, which we have:

$$\mathbb{E}[X^2] = \sum_{i=1}^{n} \mathbb{E}[I_i^2] + 2\sum_{i<j} \mathbb{E}[I_i I_j]$$

Since $I_i$ are all indicator variables, so again $\mathbb{E}[I_i^2] = \mathbb{E}[I_i = 1] = \mathbb{P}[I_i = 1] = \frac{1}{2n}$. Now, due to the properties of indicator variables, so $\mathbb{E}[I_i I_j]$ can be simplified as:

$$\mathbb{E}[I_i I_j] = \mathbb{P}[I_i I_j = 1] = \mathbb{P}[I_i = 1 \wedge I_j = 1] = \mathbb{P}[\text{both i,j are fixed points}] = \frac{1}{2n \cdot 2(n-1)}$$

Thus, $\mathbb{E}[X^2] = n \cdot \frac{1}{2n} + 2\binom{n}{2}\frac{1}{2n \cdot 2(n-1)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Thus, using Theorem 16.1, we have that:

$$var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{3}{4} - (\frac{1}{2})^2 = \frac{1}{2}$$

# 3 Double-Check Your Intuition Again

(a) (i) $\text{cov}(X+Y, X-Y) = 0$

By definition of covariance, so $\text{cov}(X+Y, X-Y) = \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y] = \mathbb{E}[X^2 - Y^2] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y]$

Since $X$ and $Y$ are independent, so $\mathbb{E}[X^2 - Y^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2], \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, and $\mathbb{E}[X-Y] = \mathbb{E}[X] - \mathbb{E}[Y]$, which gives us that: $\text{cov}(X+Y, X-Y) = \mathbb{E}[X^2 - Y^2] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y] = \mathbb{E}[X^2] - \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y]) \cdot (\mathbb{E}[X] - \mathbb{E}[Y]) = \mathbb{E}[X^2] - \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 - \mathbb{E}[Y]^2) = 0$

(ii) Proof by Contradiction

Suppose, for a contradiction, that $X+Y$ and $X-Y$ are independent, then by definition, $\mathbb{P}[X+Y = a, X-Y = b] = \mathbb{P}[X+Y = a] \cdot \mathbb{P}[X-Y = b] \quad \forall a, b$

Yet, consider the case when $X+Y = 2, X-Y = 0$. Since $X, Y \geq 1$, so $X = Y = 1$, so $X - Y = 0$, which means that

$$\mathbb{P}[X+Y = 2, X-Y = 0] = \mathbb{P}[X+Y = 2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

On the other hand,

$$\mathbb{P}[X-Y = 0] = \frac{6}{36} = \frac{1}{6}$$

So, we have that:

$$\mathbb{P}[X+Y = 2] \cdot \mathbb{P}[X-Y = 0] = \frac{1}{36} \cdot \frac{1}{6} = \frac{1}{216} \neq \mathbb{P}[X+Y = 2, X-Y = 0]$$

Thus, this gives the contradiction, which implies that $X+Y$ and $X-Y$ are not independent.

Q.E.D.

(b) Yes

Since by definition and given information, we have that

$$var(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = 0$$

Now, for any $a \in (X - \mathbb{E}[X])^2$, we have that $a \geq 0$. Thus, let $\mathscr{A}$ be the set of all values $(X - \mathbb{E}[X])^2$ can take on, so $\mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{a \in \mathscr{A}} a \cdot \mathbb{P}[a] \geq 0$, and the equivalence is reached only if $a = 0 \ \forall \ a \in \mathscr{A}$, which implies that $(x - \mathbb{E}[X])^2 = 0$ for all possible $x$ that can be taken by $X$, which gives $x = \mathbb{E}[X]$, and thus implies that $X$ is a constant.

(c) No

We proceed by providing a counterexample. Consider random variable $X$ where $\mathbb{P}[X = 0] = 1/2, \mathbb{P}[X = 1] = 1/2$ and constant $c = 2$.

We have that:

$\mathbb{E}[X] = 0 \cdot 1/2 + 1 \cdot 1/2 = \frac{1}{2}$, $\mathbb{E}[X^2] = 0 \cdot 1/2 + 1^2 \cdot 1/2 = \frac{1}{2}$

$\mathbb{E}[cX] = 0 \cdot 1/2 + 2 \cdot 1/2 = 1$, $\mathbb{E}[(cX)^2] = 0 \cdot 1/2 + 2^2 \cdot 1/2 = 2$

which gives that $\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{4}$, so $c{\cdot}\text{var}(X) = 2 \cdot \frac{1}{4} = \frac{1}{2}$, and $\text{var}(cX) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2 = 2 - 1 = 1$.

Thus, in this case, $c{\cdot}\text{var}(X) \neq \text{var}(cX)$, which gives the counterexample.

(d) No

We proceed by providing a counterexample. Consider the example from part (a), let $A = X+Y, B = X - Y$, we know that $\text{cov}(A, B) = 0$, and then we can calculate $\sigma(A) = \sigma(B) = \sqrt{\frac{35}{12}}$ using results from Note 16. Thus, we have that $\text{Corr}(A, B) = \frac{\text{cov}(A,B)}{\sigma(A)\sigma(B)} = 0$, but $A, B$ are not independent.

(e) Yes

Givent that $\text{Corr}(X, Y) = 0$, so $\frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)} = \frac{\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]}{\sigma(X)\sigma(Y)} = 0$, which implies that $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$, or equivalently, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Thus, $\text{var}(X + Y) = \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - (\mathbb{E}[X]^2 + \mathbb{E}[Y]^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) = (\mathbb{E}[X^2] - \mathbb{E}[X]^2) + (\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) + 2((\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) = Var(X) + Var(Y) + 2((\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y])$. Now, since we have that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, so $\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)$, as desired.

(f) Yes

Given r.v. $X$ and $Y$, let $\mathscr{A}, \mathscr{B}$ denote the set of all values $X, Y$ can take on, respectively. Thus, we have that $\mathbb{E}[max(X, Y)min(X, Y)] = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot max(a, b)min(a, b)$.

Now, we have that $max(a, b)min(a, b) = ab$ since exactly one of the situations must be true: (1) $a \geq b$ or (2) $a < b$. In Case (1), $max(a, b)min(a, b) = ab$; In Case (2), $max(a, b)min(a, b) = ba = ab$.

Thus, $\mathbb{E}[max(X, Y)min(X, Y)] = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot max(a, b)min(a, b) = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot ab = \mathbb{E}[XY]$, as desired.

(g) No

Consider independent r.v. $X, Y$ where $\mathbb{P}[X = 1] = \mathbb{P}[X = 3] = \mathbb{P}[Y = 2] = \mathbb{P}[Y = 4] = \frac{1}{2}$.

Now, we can see that r.v. $max(X, Y), min(X, Y)$ can be calculated as $\mathbb{P}[max(X, Y) = 1] = 0, \mathbb{P}[max(X, Y) = 2] = \mathbb{P}[max(X, Y) = 3] = \frac{1}{4}, \mathbb{P}[max(X, Y) = 4] = \frac{1}{2}$ and $\mathbb{P}[min(X, Y) = 1] = \frac{1}{2}, \mathbb{P}[min(X, Y) = 2] = \mathbb{P}[min(X, Y) = 3] = \frac{1}{4}, \mathbb{P}[min(X, Y) = 4] = 0$.

Thus, $\mathbb{E}[max(X, Y)] = \frac{13}{4}$ and $\mathbb{E}[min(X, Y)] = \frac{7}{4}$, while using results from part (f) we have $\mathbb{E}[max(X, Y)min(X, Y)] = \mathbb{E}[XY] = \frac{1}{4} \cdot (2 + 4 + 6 + 12) = 6$, so $\text{Corr}(max(X, Y), min(X, Y)) = \frac{cov(max(X,Y),min(X,Y))}{\sigma(max(X,Y))\sigma(min(X,Y))} = \frac{5}{16}$ while $\text{Corr}(X, Y) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)} = 0$.

Since Corr($max(X,Y), min(X,Y)$) $\neq$Corr($X,Y$), so this is a counterexample.