1 Double-Check Your Intuition

(a) (i)

Since $X \in \text{Bin}(5, \frac{1}{4})$, so using the formula given, we have that $\mathbb{P}[X = i] = \binom{5}{i} \cdot (\frac{1}{4})^i (1 - \frac{1}{4})^{5-i}$ for $i \in \{0, 1, 2, 3, 4, 5\}$, i.e. $\mathbb{P}[X = i] = f(i)$ where $f = \binom{5}{i}(\frac{1}{4})^i(1-\frac{1}{4})^{5-i}$ and its domain is $i \in \{0, 1, 2, 3, 4, 5\}$

Then, since we define $Y = 5 - X$, so $\mathbb{P}[Y = 5 - j] = \binom{5}{j} \cdot (\frac{1}{4})^j (1 - \frac{1}{4})^{5-j}$ for $j = 0, 1, 2, 3, 4, 5$, so we have that $\mathbb{P}[Y = i] = \binom{5}{5-i} \cdot (\frac{1}{4})^{5-i}(1 - \frac{1}{4})^i$ for $i \in \{0, 1, 2, 3, 4, 5\}$, i.e. $\mathbb{P}[Y = i] = g(i)$ where $g = \binom{5}{5-i}(\frac{1}{4})^{5-i}(1 - \frac{1}{4})^i$ and its domain is also $i \in \{0, 1, 2, 3, 4, 5\}$

(ii) $\mathbb{E}[Z^2] = \frac{91}{6}$

Using given information, we know that $Z \in \{1, 2, 3, 4, 5, 6\}$, and also $\mathbb{P}[Z = 1] = \mathbb{P}[Z = 2] = \mathbb{P}[Z = 3] = \mathbb{P}[Z = 4] = \mathbb{P}[Z = 5] = \mathbb{P}[Z = 6] = \frac{1}{6}$. Thus, we can conclude that $Z^2 \in \{1, 4, 9, 16, 25, 36\}$, and so $\mathscr{A} = \{1, 4, 9, 16, 25, 36\}$, with $\mathbb{P}[Z = 1] = \mathbb{P}[Z = 4] = \mathbb{P}[Z = 9] = \mathbb{P}[Z = 16] = \mathbb{P}[Z = 25] = \mathbb{P}[Z = 36] = \frac{1}{6}$.

Thus, $\mathbb{E}[Z^2] = \sum_{a \in \mathscr{A}} a \cdot \mathbb{P}[Z^2 = a] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}$

(b) True

We proceed by a direct proof. Since $\sum_{i \in \mathbb{Z}} \mathbb{P}[A = i] = 1$ by definition of probability and we also have that for any $i \in \mathbb{Z}$, $\mathbb{P}[A = i] \geq 0$, which implies that there exists some $k \in \mathbb{Z}$ such that $\mathbb{P}[A = k] > 0$

Then, since $A = B$ is equivalent to $A = i$ and $B = i$ for all $i \in \mathbb{Z}$, so $\mathbb{P}[A = B] = \sum_{i \in \mathbb{Z}} \mathbb{P}[A = i] \cdot \mathbb{P}[B = i] = \sum_{i \in \mathbb{Z}} (\mathbb{P}[A = i])^2$ using given information, and thus $\mathbb{P}[A = B] = \sum_{i \in \mathbb{Z}} (\mathbb{P}[A = i])^2 \geq (\mathbb{P}[A = k])^2 > 0$, as desired. Q.E.D.

(c) False

We proceed by providing a counterexample. Let C be a random variable denoting the result of a die roll (so $1 \leq C \leq 6$uniformly at random). Using our result from part (a.ii), so $\mathbb{E}[C^2] = \frac{91}{6}$.

Now, using the example of a single die incidence provided in Note 15, so $\mathbb{E}[C] = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}$, which means that $\mathbb{E}[C]^2 = (\frac{7}{2})^2 = \frac{49}{4} \neq \frac{91}{6} = \mathbb{E}[C]$, which gives the counterexample.

(d) False

We proceed by providing a counterexample. Let X be a random variable on a sample space $\{1, 10^6\}$ such that $\mathbb{P}[X = 1] = 99.9\%, \mathbb{P}[X = 10^6] = 0.1\%$; then let Y be a random variable on a sample space $\{2\}$ such that $\mathbb{P}[Y = 2] = 1$.

Now, $\mathbb{E}[X] = 1 \cdot 99.9\% + 10^6 \cdot 0.1\% = 1000.999$ and $\mathbb{E}[Y] = 2 \cdot 1 = 2$, so it satisfies the condition that $\mathbb{E}[X] > 100\, \mathbb{E}[Y]$. But, consider the probability of $X > Y$. Since $Y = 2$ is constant, so only when $X = 10^6$ does $X > Y$ hold, which means that $\mathbb{P}(X > Y) = \mathbb{P}[X = 10^6] = 0.1\% < 1/100$, which gives the countradiction and counterexample.

(e) False

We proceed by providing a counterexample. Let X be a random variable (taking positive values) on a sample space $\{1, 2\}$ such that $\mathbb{P}[X = 1] = \mathbb{P}[X = 2] = 0.5$; then let Y be a random variable (taking positive values) on a sample space $\{1\}$ such that $\mathbb{P}[Y = 1] = 1$.

Thus, we can easily determine that $\mathbb{P}[\frac{X}{X+Y} = \frac{1}{2}] = \mathbb{P}[\frac{X}{X+Y} = \frac{2}{3}] = 0.5$, which gives us that $\mathbb{E}[\frac{X}{X+Y}] = \frac{1}{2} \cdot 0.5 + \frac{2}{3} \cdot 0.5 = \frac{7}{12}$. Then, we also have that $\mathbb{P}[X + Y = 2] = \mathbb{P}[X + Y = 3] = 0.5 = \mathbb{P}[X = 1] = \mathbb{P}[X = 2]$, so we have $\mathbb{E}[X + Y] = 2 \cdot 0.5 + 3 \cdot 0.5 = 2.5$ and $\mathbb{E}[X] = 1.5$.

Thus, $\frac{\mathbb{E}[X]}{\mathbb{E}[X+Y]} = \frac{1.5}{2.5} = \frac{3}{5} \neq \frac{7}{12} = \mathbb{E}[\frac{X}{X+Y}]$, which gives the counterexample.

(f) True

Since A, B, C are events such that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$, then by Definition 14.4, so they're mutually independent. Q.E.D.

(g) False (No, an event A is not never independent with itself.)

Consider the sample space be flipping a fair coin, and the event A be that the coin lands and stays $45°$ from the ground (assuming a universe where a coin always lands on either heads or tails and doesn't behave like this), then we have that $\mathbb{P}(A) = 0$. Now, we can easily determine that $\mathbb{P}(A \cap A) = 0 = \mathbb{P}(A) \times \mathbb{P}(A)$, and since they're in the same probability space, so by Definition 14.3, the events A, A are indeed independent.

(h) True

Using given information, we have that $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A), \mathbb{P}(\overline{B}) = 1 - \mathbb{P}(B)$, and that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$. Thus, $\mathbb{P}(\overline{A}) \times \mathbb{P}(\overline{B}) = (1 - \mathbb{P}(A)) \cdot (1 - \mathbb{P}(B)) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B)$.

Then, using the Theorem 14.2 (Inclusion-Exclusion), so we have that $\mathbb{P}(\overline{A}) \times \mathbb{P}(\overline{B}) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) = 1 - (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) = 1 - \mathbb{P}(A \cup B) = \mathbb{P}(\overline{A} \cap \overline{B})$ where the last equality results from the definition of union and intersections of sets. Since $\mathbb{P}(\overline{A} \cap \overline{B}) = \mathbb{P}(\overline{A}) \times \mathbb{P}(\overline{B})$, so by Definition 14.3, we have that $\overline{A}, \overline{B}$ are also independent. Q.E.D.