4 Hypercubes

(b) Direct Proof

Proof. We give a direct proof by providing a bipartition on the set of vertices of G where G is an n-dimensional hypercube with $n \geq 1$. We claim that this assignment of vertices would create a valid bipartition: let L, R be two disjoint sets of vertices of G. Let $s_0 = 0^n$ (the n-bit string that is entirely comprised of 0) be in L. Then, for any $\{0,1\}^n$ string s_1 that has an even number of different bit position compared to s_0, we would have $s_1 \in L$; for any $\{0,1\}^n$ string s_2 that has an odd number of different bit position compared to s_0, we would have $s_2 \in R$.

Since by definition of hypercubes, two vertices x and y are connected by edge $\{x, y\}$ if and only if x and y differ in exactly one bit position, and since in our assignment of vertices, WLOG, consider the set L. For any two vertices $u, v \in L$, we have that u and v would always differ in an even number of bit positions, which implies that u and v would not differ in exactly one bit position, and thus, they wouldn't be connected by an edge. Similarly, for any two vertices in R, they wouldn't be connected by an edge. Thus, this assignment of vertices would create vertex disjoint sets L and R such that no 2 vertices in the same set have an edge between them.

Therefore, for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Q.E.D.