2 A Coin Game

Proof. We proceed by induction on n.

Base case ($n = 1$): Nothing could be split, so the score is always 0, and we have $0 = \frac{1*(1-1)}{2} = 0$.
Base case ($n = 2$): There is only one splitting strategy: splitting 2 coins into two stacks of 1.
So, the score is always $1 * 1 = 1$, and we have $1 = \frac{2*(2-1)}{2} = 1$.
Thus, the base cases are correct.

Inductive Hypothesis: Assume that the claim, the score is always $\frac{n(n-1)}{2}$, is true for all $1 \leq n \leq k$ for $k \geq 2$.

Inductive Step: We prove the claim for $n = k+1 \geq 3$. Since there is only one stack initially, assume the first turn split it into x and $(n-x)$ coins, with $1 \leq x \leq n-1$, so the score of the first turn is $x(n-x)$ and $x, n-x \leq n-1 = k$. Thus, the Inductive Hypothesis implies that the stack with x coins would eventually score $\frac{x(x-1)}{2}$, and the stack with $n-x$ coins would eventually score $\frac{(n-x)(n-x-1)}{2}$. Therefore, the total score would be: $x(n-x) + \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2} = xn - x^2 + \frac{(x^2-x)+(n^2-nx-n-nx+x^2+x)}{2} =$ $= xn - x^2 + \frac{2x^2+n^2-2nx-n}{2} = xn - x^2 + x^2 - nx + \frac{n^2-n}{2} = \frac{n(n-1)}{2}$, which means that the score is always $\frac{n(n-1)}{2}$.

Thus, by the principle of mathematical induction, the claim holds.

Q.E.D.