Sundry: I worked alone without any help.

1 Safeway Monopoly Cards

Direct Proof

We proceed by a direct proof. This is very similar to the coupon collector's problem.

As usual, we start by writing $X = I_1 + I_2 + \cdots + I_n$ where I_i is the number of times we visit Safeway while trying to get the i^{th} new card (starting immediately after we have gotten the $(i-1)^{th}$ new card). With this definition, I_1 is trivial: no matter what happens, we always get a new card the first time (since we have none to start with). So $\mathbb{P}[I_1 = 1] = 1$, and thus $\mathbb{E}[I_1] = 1$.

Then, I_2 has the geometric distribution with parameter $p = \frac{n-1}{n}$, and thus, using Theorem 19.2, we have $\mathbb{E}[I_2] = \frac{1}{p} = \frac{n}{n-1}$. Similarly, for any $i = 1, 2, \ldots, n$, I_i has the geometric distribution with parameter $p = \frac{n-i+1}{n}$, and hence, $\mathbb{E}[I_i] = \frac{n}{n-i+1}$ by Theorem 19.2.

Now, we could apply the linearity of expectation to get:

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_n] = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n\sum_{i=1}^{n}\frac{1}{i}$$

Similarly, since all the events I_i are mutually independent, so the variance of their sum is the sum of their variance by Theorem 16.3, and thus we have: $\mathrm{var}(X) = \mathrm{var}(I_1 + I_2 + \cdots + I_n) = \mathrm{var}(I_1) + \mathrm{var}(I_2) + \cdots + \mathrm{var}(I_n)$, where for any $I_i \sim \mathrm{Geo}(\frac{n-i+1}{n})$, by Theorem 19.3 we have $\mathrm{var}(I_i) = \frac{1-\frac{n-i+1}{n}}{(\frac{n-i+1}{n})^2} = \frac{n(i-1)}{(n-i+1)^2}$. Thus,

$$\mathrm{var}(X) = \mathrm{var}(I_1) + \mathrm{var}(I_2) + \cdots + \mathrm{var}(I_n) = \frac{n \cdot 0}{n^2} + \frac{n \cdot 1}{(n-1)^2} + \frac{n \cdot 2}{(n-2)^2} + \cdots + \frac{n(n-1)}{1^2}$$

$$\implies \mathrm{var}(X) = n \cdot \sum_{i=1}^{n} \frac{i-1}{(n-(i-1))^2}$$

$$\implies \mathrm{var}(X) = n \cdot \sum_{i=1}^{n} \frac{n-i}{(n-(n-i))^2} = n \cdot \sum_{i=1}^{n} \frac{n-i}{i^2}$$

$$\implies \mathrm{var}(X) = n \cdot (\sum_{i=1}^{n} \frac{n}{i^2} - \sum_{i=1}^{n} \frac{i}{i^2})$$

$$\implies \mathrm{var}(X) = n^2 \cdot \sum_{i=1}^{n} \frac{1}{i^2} - n\sum_{i=1}^{n}\frac{1}{i}$$

Since $\mathbb{E}[X] = n\sum\limits_{i=1}^{n}\frac{1}{i}$, so we have that:

$$\mathrm{var}(X) = n^2 \cdot \sum_{i=1}^{n} i^{-2} - \mathbb{E}[X]$$

which is the desired result. Q.E.D.

2 Geometric Distribution

(a) Direct Proof

Proof. We proceed by a direct proof. First, since $X_1 \sim \text{Geo}(p_1), X_2 \sim \text{Geo}(p_2)$, so $\mathbb{P}[X_1 = i] = (1-p_1)^{i-1} \cdot p_1, \mathbb{P}[X_2 = i] = (1-p_2)^{i-1} p_2$ for any $i \in \mathbb{Z}^+$ by definition.

Now, for any $i \in \mathbb{Z}^+$, i.e. $i = 1, 2, \ldots$, by the Principal of Inclusion-Exclusion, we have that the probability of at least one of the machines failing at the i^{th} day is (also by the mutually exclusive property of the two machines and the sum of infinite geometric series): $\mathbb{P}[X = i] = \mathbb{P}[X_1 = i, X_2 \geq i] + \mathbb{P}[X_1 \geq i, X_2 = i] - \mathbb{P}[X_1 = i, X_2 = i] = \mathbb{P}[X_1 = i]\mathbb{P}[X_2 \geq i] + \mathbb{P}[X_1 \geq i]\mathbb{P}[X_2 = i] - \mathbb{P}[X_1 = i]\mathbb{P}[X_2 = i]$. Since we have:

$$\mathbb{P}[X_1 \geq i] = \sum_{k=i}^{\infty} \mathbb{P}[X_1 = k] = \sum_{k=i}^{\infty} (1-p_1)^{k-1} p_1 = p_1 \cdot \frac{(1-p_1)^{i-1}}{1-(1-p_1)} = (1-p_1)^{i-1}$$

And similarly,

$$\mathbb{P}[X_2 \geq i] = (1-p_2)^{i-1}$$

Thus, for $i = 1, 2, 3, \ldots$, we have:

$$\mathbb{P}[X = i] = (1-p_1)^{i-1} p_1 \cdot (1-p_2)^{i-1} + (1-p_1)^{i-1} \cdot (1-p_2)^{i-1} p_2 - (1-p_1)^{i-1} p_1 \cdot (1-p_2)^{i-1} p_2$$

$$\implies \mathbb{P}[X = i] = (1-p_1)^{i-1}(1-p_2)^{i-1} \cdot (p_1 + p_2 - p_1 p_2)$$

$$\implies \mathbb{P}[X = i] = (1-(p_1 + p_2 - p_1 p_2))^{i-1} \cdot (p_1 + p_2 - p_1 p_2)$$

which implies that X is the geometric distribution with parameter $p_1 + p_2 - p_1 p_2$, as desired.

Q.E.D.

(b) $\frac{1}{2-p_1-p_2+p_1p_2}$

The probability that the first technician is the first one to find a faulty machine is equivalent to the probability of having the first failure on either machine happen after an even number of runs (starting from 0). Since we proved in part (a) that $X \sim \text{Geo}(p_1+p_2-p_1p_2)$, denoting parameter $p = p_1+p_2-p_1p_2$ so we have that: $\mathbb{P} = \mathbb{P}[\text{first technician finds failure first}] = \mathbb{P}[\text{first failure after odd number of runs}] = \sum_{i=0}^{\infty} \mathbb{P}[X = 2i] = \sum_{i=0}^{\infty} (1-p)^{2i} \cdot p = p \cdot \sum_{i=0}^{\infty} (1-p)^{2i}$.

Since the summation represents an infinite geometric series, so with $p = p_1 + p_2 - p_1 p_2$ and that $p \neq 0$, so we can take its sum and simplify that:

$$\mathbb{P} = p \cdot \frac{1}{1-(1-p)^2} = p \cdot \frac{1}{2p - p^2} = \frac{1}{2-p} = \frac{1}{2 - p_1 - p_2 + p_1 p_2}$$

Thus, the probability that the first technician is the first one to find a faulty machine is $\frac{1}{2-p_1-p_2+p_1p_2}$.

3 Geometric and Poisson

(a) $e^{-\lambda p}$

Since $X \sim \text{Geo}(p), Y \sim \text{Poisson}(\lambda)$, so we have that $\mathbb{P}(X > Y) = \sum_{i=0}^{\infty} \mathbb{P}[Y = i]\mathbb{P}[X > i]$. Now, for any $i \in \mathbb{N}$, we have that

$$\mathbb{P}[Y = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

and with $p \neq 0$, we have

$$\mathbb{P}[X > i] = \sum_{j=i+1}^{\infty} \mathbb{P}[X = j] = \sum_{j=i+1}^{\infty} (1-p)^{j-1} p = \frac{(1-p)^i p}{1-(1-p)} = (1-p)^i$$

Then, using the Taylor series expansion of e^x, we have that:

$$\mathbb{P}(X > Y) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \cdot (1-p)^i = e^{-\lambda} \cdot \sum_{i=o}^{\infty} \frac{(\lambda(1-p))^i}{i!} = e^{-\lambda} \cdot e^{\lambda(1-p)}$$

Thus,

$$\mathbb{P}(X > Y) = e^{-\lambda+\lambda-\lambda p} = e^{-\lambda p}$$

(b) 1

Since Z is defined as $Z = \max(X, Y)$, so by definition, we have that $\forall i,\ Z \geq X$. Therefore, $\mathbb{P}(Z \geq X) = 1$.

(c) $1 - e^{-\lambda p}$

Again, since $Z = \max(X, Y)$, so we have that $\mathbb{P}(Z \leq Y) = \mathbb{P}(Z = Y) = \mathbb{P}(X \leq Y) = \mathbb{P}(\overline{X > Y}) = 1 - \mathbb{P}(X > Y)$. Using our result from part (a), so:

$$\mathbb{P}(Z \leq Y) = 1 - \mathbb{P}(X > Y) = 1 - e^{-\lambda p}$$

4 Darts

(a) $1 - e^{-4}$

Since we have that the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x) = e^{-x}$ and that the board's radius is 4 units, so:

$$\mathbb{P}[\text{within board}] = \mathbb{P}[0 \le x \le 4] = \int_0^4 f_X(x)\,dx = \int_0^4 e^{-x}\,dx = -e^{-x}\,\Big|_0^4 = 1 - e^{-4}$$

(b) $\frac{1-e^{-1}}{1-e^{-4}}$

We first calculate the probability Alvin made it within 1 unit from the center, which is:

$$\mathbb{P}[1 \text{ unit}] = \mathbb{P}[0 \le x \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 e^{-x}\,dx = -e^{-x}\,\Big|_0^1 = 1 - e^{-1}$$

Then, since $\mathbb{P}[1 \text{ unit} \cap \text{within board}] = \mathbb{P}[1 \text{ unit}] = 1 - e^{-1}$, and by definition of conditional probability, so we have that:

$$\mathbb{P}[1 \text{ unit} \mid \text{within board}] = \frac{\mathbb{P}[1 \text{ unit} \cap \text{within board}]}{\mathbb{P}[\text{within board}]} = \frac{1-e^{-1}}{1-e^{-4}}$$

(c) $4 - e^{-1} - e^{-2} - e^{-3} - e^{-4}$

Let $S = \lfloor 5 - x \rfloor$ denote the score of Alvin. First, by definition of our score and shooting, the value of S can only fall into these values: $5, 4, 3, 2, 1, 0$. Then, using given information, we can calculate that:

$$\mathbb{P}[S = 5] = \mathbb{P}[0 \text{ unit}] = \mathbb{P}[0 \le x \le 0] = \int_0^0 f_X(x)\,dx = 0$$

$$\mathbb{P}[S = 4] = \mathbb{P}[\text{between } 0 \text{ and } 1 \text{ unit}] = \mathbb{P}[0 \le x \le 1] = \int_0^1 f_X(x)\,dx = 1 - e^{-1}$$

$$\mathbb{P}[S = 3] = \mathbb{P}[\text{between } 1 \text{ and } 2 \text{ unit}] = \mathbb{P}[1 \le x \le 2] = \int_1^2 f_X(x)\,dx = -e^{-x}\,\Big|_1^2 = e^{-1} - e^{-2}$$

$$\mathbb{P}[S = 2] = \mathbb{P}[\text{between } 2 \text{ and } 3 \text{ unit}] = \mathbb{P}[2 \le x \le 3] = \int_2^3 f_X(x)\,dx = -e^{-x}\,\Big|_2^3 = e^{-2} - e^{-3}$$

$$\mathbb{P}[S = 1] = \mathbb{P}[\text{between } 3 \text{ and } 4 \text{ unit}] = \mathbb{P}[3 \le x \le 4] = \int_3^4 f_X(x)\,dx = -e^{-x}\,\Big|_3^4 = e^{-3} - e^{-4}$$

$$\mathbb{P}[S = 0] = \mathbb{P}[\overline{\text{within board}}] = 1 - (1 - e^{-4}) = e^{-4}$$

Thus, Alvin's expected score after one throw is:

$$\mathbb{E}[S] = \int_{-\infty}^{\infty} x f(x)\,dx = \sum_{i=0}^{5} i \cdot \mathbb{P}[S = i]$$

$$\implies \mathbb{E}[S] = 0 \cdot e^{-4} + 1 \cdot (e^{-3} - e^{-4}) + 2 \cdot (e^{-2} - e^{-3}) + 3 \cdot (e^{-1} - e^{-2}) + 4 \cdot (1 - e^{-1}) + 5 \cdot 0$$

$$\implies \mathbb{E}[S] = 4 - e^{-1} - e^{-2} - e^{-3} - e^{-4}$$

5 Exponential Practice

(a) $f_Y(y) = \lambda^2 e^{-\lambda y} y$ for $y \geq 0$; and 0 otherwise

Since $X_1 \sim \text{Exp}(\lambda)$ and $\lambda > 0$, so by definition X_1 has probability density function $f_{X_1}(x) = \lambda e^{-\lambda x}$ if $x \geq 0$ and 0 otherwise, and also that it has CDF $F_{X_1}(x) = \mathbb{P}[X_1 \leq x] = 1 - e^{-\lambda x}$ for $x \geq 0$. Similarly, X_2 has the same PDF and CDF.

Now, since $Y = X_1 + X_2$, X_1, X_2 are independent, and that all values of X_1, X_2 has to be non-negative (since the probability of them having negative values is 0), so we can calculate the cumulative distribution function of Y as: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[(X_1 + X_2) \leq y] = \int_{-\infty}^{y} \mathbb{P}[X_1 \leq y - x \mid X_2 = x] \cdot f_{X_2}(x)\,dx = \int_{-\infty}^{y} \mathbb{P}[X_1 \leq y - x] \cdot f_{X_2}(x)\,dx = \int_{-\infty}^{y} (1 - e^{-\lambda(y-x)}) \cdot \lambda e^{-\lambda x}\,dx = \int_{-\infty}^{y} \lambda e^{-\lambda x}\,dx - \int_{-\infty}^{y} \lambda e^{\lambda x - \lambda y - \lambda x}\,dx = \int_{0}^{y} \lambda e^{-\lambda x}\,dx - \int_{0}^{y} \lambda e^{-\lambda y}\,dx = -e^{-\lambda x}\Big|_0^y - (\lambda e^{-\lambda y} x)\Big|_0^y = 1 - e^{-\lambda y} - \lambda e^{-\lambda y} y$

Since the density of Y is $f_Y(y) = \frac{dF_Y(y)}{dy}$ by definition, so for $y \geq 0$,

$$f_Y(y) = \lambda e^{-\lambda y} - \lambda(-y\lambda e^{-\lambda y} + e^{-\lambda y}) = \lambda^2 e^{-\lambda y} y$$

and 0 otherwise ($y < 0$).

(b) $\frac{x}{t}$

By given information and the definition of conditional probability, so we have that the CDF is:

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t)}{\mathbb{P}(X_1 + X_2 = t)}$$

Now, since X_1, X_2 can only take non-negative values again, and using the hint to condition on the event $\{X_1 + X_2 \in [t, t + \epsilon]\}$ where $\epsilon > 0$ and is small instead of $\{X_1 + X_2 = t\}$, so we have that: $\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t) = \mathbb{P}(X_1 \leq x \cap (X_1 + X_2) \in [t, t + \epsilon]) = \int_0^x \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m)\,dm\,dn = \int_0^x \int_{t-n}^{t-n+\epsilon} (\lambda e^{-\lambda n}) \cdot (\lambda e^{-\lambda m})\,dm\,dn = \int_0^x \lambda e^{-\lambda n} \Big(-e^{-\lambda m}\Big|_{t-n}^{t-n+\epsilon} \Big)\,dn = \int_0^x \lambda e^{-\lambda t}(1 - e^{\lambda \epsilon})\,dn = \lambda e^{-\lambda t}(1 - e^{-\lambda \epsilon})x$

Similarly, $\mathbb{P}(X_1 + X_2 = t) = \int_0^t \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m)\,dm\,dn = \lambda e^{-\lambda t}(1 - e^{-\lambda \epsilon})t$

Thus, the CDF is

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\epsilon\lambda \cdot e^{-\lambda t} x}{\epsilon\lambda \cdot e^{-\lambda t} t} = \frac{x}{t}$$

6 Uniform Means

(a) $\frac{1}{n+1}$

Given that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed uniform random variables on the interval $[0,1]$ (where $n \in \mathbb{Z}^+$), and that $Y = \min\{X_1, X_2, \ldots, X_n\}$, so we have $0 \le Y \le 1$, and thus, we can use the tail sum formula to obtain:

$$\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y > y)\,dy = \int_0^1 \mathbb{P}(Y > y)\,dy$$

Now, since $X_1, X_2, \ldots, X_n$ are uniform i.i.ds, and $Y = \min\{X_1, X_2, \ldots, X_n\}$, so for any $y \in [0,1]$, we can calculate an expression for $\mathbb{P}(Y > y)$ as: $\mathbb{P}(Y > y) = \mathbb{P}(\min\{X_1, X_2, \ldots, X_n\} > y) = \mathbb{P}(X_1 > y, X_2 > y, \ldots, X_n > y) = \mathbb{P}(X_1 > y) \cdot \mathbb{P}(X_2 > y) \cdot \cdots \cdot \mathbb{P}(X_n > y)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i > y) = \frac{1-y}{1-0} = 1 - y$. Thus,

$$\mathbb{P}(Y > y) = (1-y) \cdot \cdots \cdot (1-y) = (1-y)^n$$

Therefore, we can now calculate the expectation of Y, which is:

$$\mathbb{E}(Y) = \int_0^1 \mathbb{P}(Y > y)\,dy = \int_0^1 (1-y)^n\,dy = -\frac{(1-y)^{n+1}}{n+1}\Big|_0^1 = -0 + \frac{1}{n+1} = \frac{1}{n+1}$$

(b) $\frac{n}{n+1}$

Given that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed uniform random variables on the interval $[0,1]$ (where $n \in \mathbb{Z}^+$), and that $Z = \max\{X_1, X_2, \ldots, X_n\}$, so we have $0 \le Z \le 1$, and thus, we can use the formula for Cumulative Distribution Function to obtain:

$$\mathbb{E}(Z) = 1 - \int_{-\infty}^1 \mathbb{P}(Z \le z)\,dz = 1 - \int_0^1 \mathbb{P}(Z \le z)\,dz$$

Now, since $X_1, X_2, \ldots, X_n$ are uniform i.i.ds, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, so for any $z \in [0,1]$, we can calculate an expression for $\mathbb{P}(Z \le z)$ as: $\mathbb{P}(Z \le z) = \mathbb{P}(\max\{X_1, X_2, \ldots, X_n\} \le z) = \mathbb{P}(X_1 \le z, X_2 \le z, \ldots, X_n \le z) = \mathbb{P}(X_1 \le z) \cdot \mathbb{P}(X_2 \le z) \cdot \cdots \cdot \mathbb{P}(X_n \le z)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i \le z) = \frac{z-0}{1-0} = z$. Thus,

$$\mathbb{P}(Z \le z) = z \cdot \cdots \cdot z = z^n$$

Therefore, we can now calculate the expectation of Z, which is:

$$\mathbb{E}(Z) = 1 - \int_0^1 \mathbb{P}(Z \le z)\,dz = 1 - \int_0^1 z^n\,dz = 1 - \left(\frac{z^{n+1}}{n+1}\Big|_0^1\right) = 1 - (\frac{1}{n+1} - 0) = \frac{n}{n+1}$$