3 Double-Check Your Intuition Again

(a) (i) $\text{cov}(X+Y, X-Y) = 0$

By definition of covariance, so $\text{cov}(X+Y, X-Y) = \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y] = \mathbb{E}[X^2 - Y^2] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y]$

Since X and Y are independent, so $\mathbb{E}[X^2 - Y^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2], \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, and $\mathbb{E}[X-Y] = \mathbb{E}[X] - \mathbb{E}[Y]$, which gives us that: $\text{cov}(X+Y, X-Y) = \mathbb{E}[X^2 - Y^2] - \mathbb{E}[X+Y] \cdot \mathbb{E}[X-Y] = \mathbb{E}[X^2] - \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y]) \cdot (\mathbb{E}[X] - \mathbb{E}[Y]) = \mathbb{E}[X^2] - \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 - \mathbb{E}[Y]^2) = 0$

(ii) Proof by Contradiction

Suppose, for a contradiction, that $X+Y$ and $X-Y$ are independent, then by definition, $\mathbb{P}[X+Y = a, X-Y = b] = \mathbb{P}[X+Y = a] \cdot \mathbb{P}[X-Y = b] \quad \forall a, b$

Yet, consider the case when $X+Y = 2, X-Y = 0$. Since $X, Y \geq 1$, so $X = Y = 1$, so $X - Y = 0$, which means that

$$\mathbb{P}[X+Y = 2, X-Y = 0] = \mathbb{P}[X+Y = 2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

On the other hand,

$$\mathbb{P}[X-Y = 0] = \frac{6}{36} = \frac{1}{6}$$

So, we have that:

$$\mathbb{P}[X+Y = 2] \cdot \mathbb{P}[X-Y = 0] = \frac{1}{36} \cdot \frac{1}{6} = \frac{1}{216} \neq \mathbb{P}[X+Y = 2, X-Y = 0]$$

Thus, this gives the contradiction, which implies that $X+Y$ and $X-Y$ are not independent.

Q.E.D.

(b) Yes

Since by definition and given information, we have that

$$var(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = 0$$

Now, for any $a \in (X - \mathbb{E}[X])^2$, we have that $a \geq 0$. Thus, let $\mathscr{A}$ be the set of all values $(X - \mathbb{E}[X])^2$ can take on, so $\mathbb{E}[(X - \mathbb{E}[X])^2] = \sum\limits_{a \in \mathscr{A}} a \cdot \mathbb{P}[a] \geq 0$, and the equivalence is reached only if $a = 0 \; \forall \; a \in \mathscr{A}$, which implies that $(x - \mathbb{E}[X])^2 = 0$ for all possible x that can be taken by X, which gives $x = \mathbb{E}[X]$, and thus implies that X is a constant.

(c) No

We proceed by providing a counterexample. Consider random variable X where $\mathbb{P}[X = 0] = 1/2, \mathbb{P}[X = 1] = 1/2$ and constant $c = 2$.

We have that:

$\mathbb{E}[X] = 0 \cdot 1/2 + 1 \cdot 1/2 = \frac{1}{2}$, $\mathbb{E}[X^2] = 0 \cdot 1/2 + 1^2 \cdot 1/2 = \frac{1}{2}$

$\mathbb{E}[cX] = 0 \cdot 1/2 + 2 \cdot 1/2 = 1$, $\mathbb{E}[(cX)^2] = 0 \cdot 1/2 + 2^2 \cdot 1/2 = 2$

which gives that $\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{4}$, so $c{\cdot}\text{var}(X) = 2 \cdot \frac{1}{4} = \frac{1}{2}$, and $\text{var}(cX) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2 = 2 - 1 = 1$.

Thus, in this case, $c{\cdot}\text{var}(X) \neq \text{var}(cX)$, which gives the counterexample.

(d) No

We proceed by providing a counterexample. Consider the example from part (a), let $A = X+Y, B = X - Y$, we know that $\text{cov}(A,B) = 0$, and then we can calculate $\sigma(A) = \sigma(B) = \sqrt{\frac{35}{12}}$ using results from Note 16. Thus, we have that $\text{Corr}(A,B) = \frac{\text{cov}(A,B)}{\sigma(A)\sigma(B)} = 0$, but A, B are not independent.

(e) Yes

Givent that $\text{Corr}(X,Y) = 0$, so $\frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sigma(X)\sigma(Y)} = 0$, which implies that $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$, or equivalently, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Thus, $\text{var}(X+Y) = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - (\mathbb{E}[X]^2 + \mathbb{E}[Y]^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) = (\mathbb{E}[X^2] - \mathbb{E}[X]^2) + (\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) + 2((\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) = Var(X) + Var(Y) + 2((\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])$. Now, since we have that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, so $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$, as desired.

(f) Yes

Given r.v. X and Y, let $\mathscr{A}, \mathscr{B}$ denote the set of all values X, Y can take on, respectively. Thus, we have that $\mathbb{E}[max(X,Y)min(X,Y)] = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot max(a,b)min(a,b)$.

Now, we have that $max(a,b)min(a,b) = ab$ since exactly one of the situations must be true: (1) $a \geq b$ or (2) $a < b$. In Case (1), $max(a,b)min(a,b) = ab$; In Case (2), $max(a,b)min(a,b) = ba = ab$.

Thus, $\mathbb{E}[max(X,Y)min(X,Y)] = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot max(a,b)min(a,b) = \sum_{a\in\mathscr{A}, b\in\mathscr{B}} \mathbb{P}[X = a, Y = b] \cdot ab = \mathbb{E}[XY]$, as desired.

(g) No

Consider independent r.v. X, Y where $\mathbb{P}[X = 1] = \mathbb{P}[X = 3] = \mathbb{P}[Y = 2] = \mathbb{P}[Y = 4] = \frac{1}{2}$.

Now, we can see that r.v. $max(X,Y), min(X,Y)$ can be calculated as $\mathbb{P}[max(X,Y) = 1] = 0, \mathbb{P}[max(X,Y) = 2] = \mathbb{P}[max(X,Y) = 3] = \frac{1}{4}, \mathbb{P}[max(X,Y) = 4] = \frac{1}{2}$ and $\mathbb{P}[min(X,Y) = 1] = \frac{1}{2}, \mathbb{P}[min(X,Y) = 2] = \mathbb{P}[min(X,Y) = 3] = \frac{1}{4}, \mathbb{P}[min(X,Y) = 4] = 0$.

Thus, $\mathbb{E}[max(X,Y)] = \frac{13}{4}$ and $\mathbb{E}[min(X,Y)] = \frac{7}{4}$, while using results from part (f) we have $\mathbb{E}[max(X,Y)min(X,Y)] = \mathbb{E}[XY] = \frac{1}{4} \cdot (2 + 4 + 6 + 12) = 6$, so $\text{Corr}(max(X,Y), min(X,Y)) = \frac{cov(max(X,Y), min(X,Y))}{\sigma(max(X,Y))\sigma(min(X,Y))} = \frac{5}{16}$ while $\text{Corr}(X,Y) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)} = 0$.

Since Corr$(max(X,Y), min(X,Y)) \neq$Corr$(X,Y)$, so this is a counterexample.