4 Darts

(a) $1 - e^{-4}$

Since we have that the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x) = e^{-x}$ and that the board's radius is 4 units, so:

$$\mathbb{P}[\text{within board}] = \mathbb{P}[0 \leq x \leq 4] = \int_0^4 f_X(x)\,dx = \int_0^4 e^{-x}\,dx = -e^{-x}\,\Big|_0^4 = 1 - e^{-4}$$

(b) $\frac{1-e^{-1}}{1-e^{-4}}$

We first calculate the probability Alvin made it within 1 unit from the center, which is:

$$\mathbb{P}[1 \text{ unit}] = \mathbb{P}[0 \leq x \leq 1] = \int_0^1 f_X(x)\,dx = \int_0^1 e^{-x}\,dx = -e^{-x}\,\Big|_0^1 = 1 - e^{-1}$$

Then, since $\mathbb{P}[1 \text{ unit} \cap \text{within board}] = \mathbb{P}[1 \text{ unit}] = 1 - e^{-1}$, and by definition of conditional probability, so we have that:

$$\mathbb{P}[1 \text{ unit} \mid \text{within board}] = \frac{\mathbb{P}[1 \text{ unit} \cap \text{within board}]}{\mathbb{P}[\text{within board}]} = \frac{1 - e^{-1}}{1 - e^{-4}}$$

(c) $4 - e^{-1} - e^{-2} - e^{-3} - e^{-4}$

Let $S = \lfloor 5 - x \rfloor$ denote the score of Alvin. First, by definition of our score and shooting, the value of S can only fall into these values: $5, 4, 3, 2, 1, 0$. Then, using given information, we can calculate that:

$$\mathbb{P}[S = 5] = \mathbb{P}[0 \text{ unit}] = \mathbb{P}[0 \leq x \leq 0] = \int_0^0 f_X(x)\,dx = 0$$

$$\mathbb{P}[S = 4] = \mathbb{P}[\text{between 0 and 1 unit}] = \mathbb{P}[0 \leq x \leq 1] = \int_0^1 f_X(x)\,dx = 1 - e^{-1}$$

$$\mathbb{P}[S = 3] = \mathbb{P}[\text{between 1 and 2 unit}] = \mathbb{P}[1 \leq x \leq 2] = \int_1^2 f_X(x)\,dx = -e^{-x}\,\Big|_1^2 = e^{-1} - e^{-2}$$

$$\mathbb{P}[S = 2] = \mathbb{P}[\text{between 2 and 3 unit}] = \mathbb{P}[2 \leq x \leq 3] = \int_2^3 f_X(x)\,dx = -e^{-x}\,\Big|_2^3 = e^{-2} - e^{-3}$$

$$\mathbb{P}[S = 1] = \mathbb{P}[\text{between 3 and 4 unit}] = \mathbb{P}[3 \leq x \leq 4] = \int_3^4 f_X(x)\,dx = -e^{-x}\,\Big|_3^4 = e^{-3} - e^{-4}$$

$$\mathbb{P}[S = 0] = \mathbb{P}[\overline{\text{within board}}] = 1 - (1 - e^{-4}) = e^{-4}$$

Thus, Alvin's expected score after one throw is:

$$\mathbb{E}[S] = \int_{-\infty}^{\infty} x f(x)\,dx = \sum_{i=0}^{5} i \cdot \mathbb{P}[S = i]$$

$$\implies \mathbb{E}[S] = 0 \cdot e^{-4} + 1 \cdot (e^{-3} - e^{-4}) + 2 \cdot (e^{-2} - e^{-3}) + 3 \cdot (e^{-1} - e^{-2}) + 4 \cdot (1 - e^{-1}) + 5 \cdot 0$$

$$\implies \mathbb{E}[S] = 4 - e^{-1} - e^{-2} - e^{-3} - e^{-4}$$