1 Counting, Counting, and More Counting

(a) $\frac{(n+k)!}{n!k!}$

First, assume that all the numbers are distinct, which gives $(n+k)!$ arrangements. Then, we're multi-counting the arrangments of n 1s $n!$-many times, and similarly, we're multi-counting the arrangments of k 0s $k!$-many times. Thus, there are $\frac{(n+k)!}{n!k!}$ many ways to arrange them into sequence.

(b) $\binom{52}{13}$; $\binom{48}{13}$; $\binom{48}{9}$; $\binom{13}{6}\binom{39}{7}$

Choosing a 13-card hand from 52 cards, where order does not matter, means that we have $\binom{52}{13}$ different 13-card bridge hands.

Regarding choosing a 13-card bridge hand that contain no aces, this means we can only choose from $(52-4=)$ 48 cards, so similar logic leads us to having $\binom{48}{13}$ different 13-card bridge hands that contain no aces.

Regarding choosing a 13-card bridge hand that contain all four aces, this means that we're choosing the rest of $(13-4=)$ 9 cards in the hand from the rest of the $(52-4=)$ 48 cards, so again, similar logic leads us to having $\binom{48}{9}$ different 13-card bridge hands that contain all four aces.

Regarding choosing a 13-card bridge hand that contain exactly 6 spades means that we're divided into two steps: (1) Choose 6 spades from 13 spades, and (2) Choose the rest of $(13-6=)$ 7 cards in the hand from the rest of $(52-13=)$ 39 cards of the deck. For step (1), we have $\binom{13}{6}$ choices, and for step (2), we have $\binom{39}{7}$ choices. Since we're using the First Rule of Counting, so there is a total of $\binom{13}{6}\binom{39}{7}$ different 13-card bridge hands that contain exactly 6 spades.

(c) $\frac{104!}{2^{52}}$

Since we're mixing two identical decks of 52 cards, so this is similar to part (a), where we have 2 of each of the 52 cards (2 of Spade 1, 2 of Spade 2, etc.), which means that we can first consider all 104 to be distinct, which gives 104! different order of the stack in this assumption. Then, since we have 2 copies of 52 cards, so we're repetitively counting this stacking order by $(2!)^{52} = 2^{52}$ times. Thus, there are $\frac{104!}{2^{52}}$ ways to order this stack of 104 cards.

(d) 2^{98}

Since there is a total of 2^{99}-many 99-bit strings, and each digit is either 0 or 1. Then, since there are 99 bits, so in any case, either the number of 1's is greater than the number of 0's, or vice versa, and the possibility should be the same, i.e. in half of all cases, there should be more 1's than 0's, so there are $\frac{2^{99}}{2} = 2^{98}$

(e) 5040, 120, 210, 1260

Since all 7 letters of FLORIDA are different, so there are $7! = 5040$ different anagrams of FLORIDA

Since in ALASKA, there are 6 total letters of 4 different types, with 3 A's, and 1 each of L, S, K. Again, first assume that all six letter are distinct, so there are 6! different anagrams. Then, since the 3 A's are exactly the same, so we're counting repetitively any sequence by 3! times, so there are

$\frac{6!}{3!} = \frac{720}{6} = 120$ different anagrams of ALASKA

Since in ALABAMA, there are 7 total letters of 4 different types, with 4 A's, and 1 each of L, S, K. Again, first assume that all seven letter are distinct, so there are 7! different anagrams. Then, since the 4 A's are exactly the same, so we're counting repetitively any sequence by 4! times, so there are $\frac{7!}{4!} = \frac{5040}{24} = 210$ different anagrams of ALASKA

Since in MONTANA, there are 7 total letters of 5 different types, with 2 A's, 2 N's and 1 each of M, O, T. Again, first assume that all seven letter are distinct, so there are 7! different anagrams. Then, since the 2 A's are exactly the same, so we're counting repetitively any sequence by 2! times; similarly, the 2 N's means that we're counting repetitively by 2! times. So, there are $\frac{7!}{2!2!} = \frac{5040}{2 \cdot 2} = 1260$ different anagrams of MONTANA

(f) 120; 360

Problem (1): Given that C is the left neighbor of E, so in any sequence, CE can be combined as one. We could reassign the combination of CE as letter M, which means that we are just arranging five letters, ABDFM, so there are $5! = 120$ different anagrams.

Problem (2): Given that C is on the left of E, so there are several cases: (1) C is the left-most letter (anagram looks like Cxxxxx); (2) anagram looks like xCxxxx; (3) anagram looks like xxCxxx; (4) anagram looks like xxxCxx; or (5) anagram looks like xxxxCx.

Case (1): We can arrange the rest of the 5 letters in any order, so there are $5! = 120$ different such anagrams.

Case (2): There are 4 possible positions for letter E, and the rest 4 letters (ABDF) can be arranged in any order, so there are $4 \cdot 4! = 96$ different such anagrams.

Case (3): There are 3 possible positions for letter E, and the rest 4 letters (ABDF) can be arranged in any order, so there are $3 \cdot 4! = 72$ different such anagrams.

Case (4): There are 2 possible positions for letter E, and the rest 4 letters (ABDF) can be arranged in any order, so there are $2 \cdot 4! = 48$ different such anagrams.

Case (5): There is only 1 possible positions for letter E, and the rest 4 letters (ABDF) can be arranged in any order, so there are $1 \cdot 4! = 24$ different such anagrams.

Thus, there's a total of $120 + 96 + 72 + 48 + 24 = 360$ different such anagrams.

(g) 27^9

Since all 9 balls and 27 bins are distinguishable, so each of the 9 balls have 27 possibilities, so there are 9^{27} different ways to distribute them.

(h) 28

Since the 9 balls are identical and the 7 bins are distinguishable, and since we are distributing balls such that no bin is empty, so we can first distribute 7 balls into 7 bins to get rid off the restrictions. In other words, we can next distribute the rest of 2 balls into any bins in any order, and there are two cases, exactly one must be true: (1) 2 balls in the same bin; or (2) 2 balls in two different bins.

Case (1): There are 7 different ways as any of the 7 bins could have the extra two balls.
Case (2): There are $\binom{7}{2} = 21$ different ways as we are just picking out 2 bins from the 7 distinguishable bins.

Thus, in total, there are $7 + 21 = 28$ different ways.

(i) $\binom{35}{9}$

Basically, we're sampling with replacement where the order doesn't matter, so we have $\binom{27+9-1}{9} = \binom{35}{9}$ choices.

(j) $\frac{20!}{2^{10}}$

We're first choosing 2 students from the 20, and then another 2 from the rest $20 - 2 = 18$, and so on. Thus, there are $\binom{20}{2} \cdot \binom{18}{2} \cdot \binom{16}{2} \cdots \binom{4}{2} \cdot \binom{2}{2} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdots 2 \cdot 1}{(2 \cdot 1)^{10}} = \frac{20!}{2^{10}}$ ways to do the pairing.

(k) $\binom{n+k-1}{k-1}$ if $n \in \mathbb{N}$; 0 if $n < 0$ or $n \notin \mathbb{Z}$

Since each x is non-negative integer, so we can divide the the situation into two possibilities: (1) $n < 0$, then there will be 0 solutions; and (2) $n \geq 0$, i.e. $n \in \mathbb{N}$, which I'll discuss below.

First, we can add k to both sides, which will result in a form like this: $(x_0 + 1) + (x_1 + 1) + \cdots + (x_k + 1) = n + k$. Now, we can say that all the elements, i.e. $x_i + 1$'s are all positive integers. Thus, we're now dividing $n + k$ into k positive integers, which means that we can should be picking $k - 1$ locations of inserting "dividers" in $n + k - 1$ position, so there are $\binom{n+k-1}{k-1}$ solutions if any exists.

(l) $n - 1$ if $n \geq 2, n \in \mathbb{Z}$; 0 if $n < 2$ or $n \notin \mathbb{Z}$

Again, we consider two cases, since $x_0, x_1 \in \mathbb{Z}^+$, so $x_0 + x_1 \geq 1 + 1 = 2$, so if $n < 2$, then there will be no solutions.

Else if $n \geq 2$, we'll utilize a similar logic as in part (k), which is that we'll be inserting 1 "divider" in the $n - 1$ possible positions, so there are $\binom{n-1}{1} = n - 1$ solutions if any exists.

(m) $\binom{n-1}{k-1}$ if $n \geq k, n \in \mathbb{Z}$; 0 if $n < k$ or $n \notin \mathbb{Z}$

Again, we consider two cases, since for each x, we have $x \in \mathbb{Z}^+$, so $x_0 + x_1 + \cdots x_k \geq k$, so if $n < k$, then there will be no solutions.

Else if $n \geq k$, we'll utilize a similar logic as in part (k), which is that we'll be inserting $k - 1$ "divider" in the $n - 1$ possible positions, so there are $\binom{n-1}{k-1}$ solutions if any exists.

2 Binomial Beads

(a) $\binom{n}{k}$

First we'll make all the beads unique, which means that there are $n!$ unique keychains by such definition, and then we consider all the duplicates as the blue/golden beads are all the same, which gives us $\frac{n!}{k!(n-k!)} = \binom{n}{k}$ unique keychains.

(b) $x^k y^{n-k}$

By definition given on the question, the price of a keychain with exactly k blue beads and thus $n-k$ gold beads is: $x^k y^{n-k}$

(c) $\sum\limits_{0}^{n} \binom{n}{k} x^k y^{n-k}$

Using our results from parts (a) and (b), his total revenue is:

$$\sum_{0}^{n} \binom{n}{k} x^k y^{n-k}$$

(d)

On the one hand $(x+y)^n$ is the sum of all different combination of the product of choosing a total of n numbers from x and y, sampling with replacement.

On the other hand, $\sum\limits_{0}^{n} \binom{n}{k} x^k y^{n-k}$ is the sum of: the product of 0 x's and n y's, the product of 1 x's and $(n-1)$ y's, the product of 2 x's and $(n-2)$ y's, $\ldots$, the product of $(n-1)$ x's and 1 y's, and the product of n x's and 0 y's. Thus, $\sum\limits_{0}^{n} \binom{n}{k} x^k y^{n-k}$ also represents the sum of all the different combinations of the product of choosing a total of n numbers from x and y, sampling with replacement, which gives the connection between that and $(x+y)^n$.

3 Minesweeper

(a) (i) $\frac{5}{32}$

The event space of revealing a mine on first click is that the rest $8^2 - 1 = 63$ squares only have $10 - 1 = 9$ mines, and the entire sample space is having 10 mines randomly in 64 squares, so$\mathbb{P}_{\text{mine}} = \frac{\text{Having 9 mines in the rest 63 squares}}{\text{Having 10 mines in 64 squares}} = \frac{\binom{63}{9}}{\binom{64}{10}} = \frac{\frac{63!}{9!54!}}{\frac{64!}{10!54!}} = \frac{63! \cdot 10!}{64! \cdot 9!} = \frac{10}{64} = \frac{5}{32}$

Alternatively, the probability should be equivalent to randomly picking one of the 10 squares with a mine in a total 64 squares, so $\mathbb{P} = \frac{10}{64} = \frac{5}{32}$

(ii) $\frac{\binom{55}{10}}{\binom{64}{10}}$

The probability of revealing a blank space means the 9 squares (with this one as the center) does not contain any mines; in other words, the rest $64 - 9 = 55$ squares contain all 10 mines, so $\mathbb{P}_{\text{blank}} = \frac{\text{Having 10 mines in the rest 55 squares}}{\text{Having 10 mines in the 64 squares}} = \frac{\binom{55}{10}}{\binom{64}{10}}$, which could be further simplied, but will be omitted here.

(iii) $\frac{\binom{55}{10-k}\binom{8}{k}}{\binom{64}{10}}$ if $k \in [1, 8]$; and 0 otherwise

Similarly, the probability of revealing a blank space means that the 9 squares (with this one as the center) contains exactly k mines, in other words, the rest $64 - 9 = 55$ squares contain the rest of the $(10-k)$ mines, AND that the 8 squares surrounding the square we picked have exactly k mines. Thus, we can consider two cases: (1) $k \in [1, 8]$, or (2) all other values of k. Case (1) will be discussed below; and Case (2), by definition of our game, is not possible, and thus have a probability $\mathbb{P} = 0$.

Case (1):

$$\mathbb{P}_k = \frac{\text{Having } (10-k) \text{ mines in the rest 55 squares AND having } k \text{ mines in the 8 surrounding squares}}{\text{Having 10 mines in the 64 squares}} = \frac{\binom{55}{10-k}\binom{8}{k}}{\binom{64}{10}}$$

(b) If $k = 1$, then pick a square next to the first pick; if $k \in [2, 8]$, then pick another square.

Here, $k \in [1, 8]$. Since the first square you picked revealed the number k, so there are exactly k mines in the 8 surrounding squares and $(10 - k)$ mines in the rest $64 - 9 = 55$ squares. We'll discuss the probability of two choices next.

Case (1): Picking a square adjacent to the first pick. So, the probability of picking a mine in this step is just $\frac{k}{8}$.

Case (2): Picking a different square (not the surrounding 8). So, the probability of picking a mine in this step is just $\frac{10-k}{55}$.

We compare the two choices. We notice that only when $k = 1$ do we have $\frac{k}{8} = \frac{1}{8} < \frac{9}{55} = \frac{10-k}{55}$; and in all other cases (i.e. $k \in [2, 8]$), with $k \geq 2$, so we have $\frac{10-k}{55} \leq \frac{10-2}{55} < \frac{1}{4} = \frac{2}{8} \leq \frac{k}{8}$.

Thus, I should pick a square next to the first pick if $k = 1$, and I should pick a different square otherwise, i.e. if $k \in [2, 8]$.

(c) $\mathbb{P} = \frac{\binom{52}{6}}{\binom{55}{9}\cdot 2}$

First, given that the first square we picked reveals 1, which means that in the 8 surrounding squares, there's one and only one mine.

Now, we label the first square we picked as (0,0), and utilize the Cartesian coordinate system, i.e. our second square is (1,0). We divide the problem into two cases, exactly one of them must be true: (1) the mine indicated by our first pick (0,0) is in one of the four squares at $(-1,1),(-1,0),(-1,-1),(1,0)$; or (2) the mine indicated is not in these three positions.

Case (1): In this case, our second pick could not reveal the number 4, because a mine at $(-1,1),(-1,0)$ or $(-1,-1)$ means that there's no mine at $(0,1),(1,1),(0,-1),(1,-1)$, and thus, there is a maximum of 3 mines around our second pick, so it couldn't reveal the number 4. Also, a mine at $(1,0)$ means that our second pick will hit the mine, and thus not reveal a number.

Case (2): This case has 4 possibilities itself, with the mine at $(0,1),(0,-1),(1,1),(1,-1)$. Then, for our second pick to reveal 4, as discussed in Case (1) also, all three squares to the right of our second pick must be mines, i.e. $(2,1),(2,0),(2,-1)$ must all be mines, and that the rest $64-12=52$ squares contain exactly $10-4=6$ mines. Thus, the total number of possible events for our second pick to be 4 with our first pick being 1 is # of 6 mines in the rest 52 squares times the 4 possible arangements of mines in the 12 focused squares, so $|\omega| = \binom{52}{6}\cdot 4$.

Now, our $|\omega| = \binom{52}{6}\cdot 4$, but our $|\Omega|$ is limited to the situations where our first pick reveals number 1, which is # 9 mines in the rest $64-9=55$ squares times the 8 possibilities of the mine in the 8 surrounding squares, so $|\Omega| = \binom{55}{9}\cdot 8$.

Thus,

$$\mathbb{P} = \frac{|\omega|}{|\Omega|} = \frac{\binom{52}{6}\cdot 4}{\binom{55}{9}\cdot 8} = \frac{\binom{52}{6}}{\binom{55}{9}\cdot 2}$$

4 Playing Strategically

(a) $\frac{3}{7}$

Let's call each shot a round. With Bob shooting first, he can only win the duel against Eve on $(2k+1)^{th}$ rounds, where $k \in \mathbb{N}$. Then, given that Bob has accuracy $\frac{1}{3}$ and Eve has accuracy $\frac{2}{3}$, so we have:

$k = 0$: Bob hits Eve directly, $\mathbb{P}_0 = \frac{1}{3}$
$k = 1$: Bob doesn't hit Eve first, Eve doesn't hit Bob either on round 2, and then Bob hits Eve, so $\mathbb{P}_1 = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$
$k = 2$: Similarly, Bob and Eve doesn't hit each other for two shots each, and then Bob hits Eve, so $\mathbb{P}_2 = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = (\frac{2}{3} \cdot \frac{1}{3})^2 \cdot \frac{1}{3}$
...

We can conlude a pattern that for any $k \in \mathbb{N}$, the probability of Bob winning on the $(2k+1)^{th}$ round is $(\frac{2}{3} \cdot \frac{1}{3})^k \cdot \frac{1}{3} = (\frac{2}{9})^k \cdot \frac{1}{3}$, so using the formula of the sum of geometric series, we have that the total probability of Bob winning a duel against Eve is:

$$P[E_1] = \sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} (\frac{2}{9})^k \cdot \frac{1}{3} = \frac{\frac{1}{3}}{1 - \frac{2}{9}} = \frac{\frac{1}{3}}{\frac{7}{9}} = \frac{3}{7}$$

(b) $\frac{1}{7}$

Let's call each shot a round. With Bob shooting second, he can only win the duel against Eve on $(2k+2)^{th}$ rounds, where $k \in \mathbb{N}$. Then, given that Bob has accuracy $\frac{1}{3}$ and Eve has accuracy $\frac{2}{3}$, so we have:

$k = 0$: Eve doesn't hit Bob, and Bob hits Eve directly, $\mathbb{P}_0 = \frac{1}{3} \cdot \frac{1}{3}$
$k = 2$: Eve doesn't hit Bob, Bob doesn't hit Eve, Eve doesn't hit Bob again on round 1, and then Bob hits Eve, so $\mathbb{P}_1 = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$
$k = 2$: Similarly, Bob and Eve doesn't hit each other for two shots each, and then Eve doesn't hit Bob still, and Bob hits Eve, so $\mathbb{P}_2 = (\frac{1}{3} \cdot \frac{2}{3})^2 \cdot \frac{1}{3} \cdot \frac{1}{3}$
...

We can conlude a pattern that for any $k \in \mathbb{N}$, the probability of Bob winning on the $(2k+2)^{th}$ round is $(\frac{1}{3} \cdot \frac{2}{3})^k \cdot \frac{1}{3} \cdot \frac{1}{3} = (\frac{2}{9})^k \cdot \frac{1}{9}$, so using the formula of the sum of geometric series, we have that the total probability of Bob winning a duel against Eve is:

$$P[E_2] = \sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} (\frac{2}{9})^k \cdot \frac{1}{9} = \frac{\frac{1}{9}}{1 - \frac{2}{9}} = \frac{\frac{1}{9}}{\frac{7}{9}} = \frac{1}{7}$$

(c) $\mathbb{P}[C_1] = \frac{1}{3}$; $\mathbb{P}[C_2] = 0$

Since Carol has an accuracy of 1, so as soon as Carol gets the chance to shoot at Bob, then Bob would loose.

Situation (1): Bob shoots first. Since Bob and Carol are taking turns to shoot, with Bob shooting first, so Bob could win the duel if and only if he hits Carol on his first shot, which has the probability of $\frac{1}{3}$ (If he misses, then Carol gets te chance to shoot, and would win the duel on her shot with 100% accuracy). Thus, $\mathbb{P}[C_1] = \frac{1}{3}$.

Situation (2): Bob shoots second. The analyzation of this situation is easy because with Carol (who never misses) shooting first, she will hit Bob on her first shot and win the duel. Thus, Bob has no chance of winning this duel if he shoots second, which is equivalent to having a probability $\mathbb{P}[C_2] = 0$

(d) Direct Proof

Because Eve and Carol are playing rationally, so they should shoot at each other if given the chance since they have the highest accuracies. We can first calculate the probabilities of a duel between Eve and Carol. Notice that the two events, Eve winning and Carol winning, forms a partition, so $\mathbb{P}_{Eve} + \mathbb{P}[Carol] = 1$.

If Eve shoots second (i.e. Carol shoots first), then Carol will hit Eve directly and win, so Carol's winning probability is $\mathbb{P}[Carol] = 1$, so Eve's winning probability is $\mathbb{P}[Eve] = 1 - 1 = 0$. Else if Eve shoots first, similar to our argument for part (c), Eve's winning probability is now $\mathbb{P}[Eve] = \frac{2}{3}$, so $\mathbb{P}[Carol] = 1 - \frac{2}{3} = \frac{1}{3}$.

Now, we take Bob into consideration. There's 3 options or possibilities for Bob, exactly one of which must be true, and they form a partition: (1) Bob hits Eve; (2) Bob hits Carol; or (3) Bob doesn't hit anyone.

Case (1): Since Bob hits Eve in the first round, so we're left with Bob and Carol in a duel, with Bob shooting second. As calculated in part (c), so Bob has a winning probability of $\mathbb{P}[Bob_1] = \mathbb{P}[C_2] = 0$.

Case (2): Since Bob hits Carol in the first round, so we're left with Bob and Eve in a duel, with Bob shooting second. As calculated in part (b), so Bob has a winning probability of $\mathbb{P}[Bob_2] = \mathbb{P}[B_2] = \frac{1}{7}$.

Case (3): Since Bob doesn't hit anyone, and that Eve and Carol are playing rationally, so they should shoot at each other first, so they're essentially playing a duel against each other, with Eve shooting first, so Eve has a winning possibility $\mathbb{P}[Eve] = \frac{2}{3}$, and Carol has a winning probability $\mathbb{P}[Carol] = \frac{1}{3}$. Since Eve winning and Carol winning forms a partition, so exactly one of the following two cases must be true: (3.a) Eve wins the duel; or (3.b) Carol wins the duel.

Case (3.a): Since Eve wins, so we're left with a duel between Bob and Eve, with Bob shooting first, using the result from part (a), so Bob has a winning probability of $\frac{3}{7}$, which means in this 3-person shooting game under this situation, Bob has a total winning probability of $\mathbb{P}_1 = \frac{2}{3} \cdot \frac{3}{7} = \frac{2}{7}$.

Case (3.b): Since Carol wins, so we're left with a duel between Bob and Carol, with Bob shooting first, using the result from part (c), so Bob has a winning probability of $\frac{1}{3}$. So, in this 3-person shooting game under this situation, Bob has a total winning probability of $\mathbb{P}_2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Thus, the total probability of Bob winning in this case is: $\mathbb{P}[Bob_3] = \mathbb{P}_1 + \mathbb{P}_2 = \frac{2}{7} + \frac{1}{9} = \frac{25}{63}$

Therefore, we have that $\mathbb{P}[Bob_1] = 0, \mathbb{P}[Bob_2] = \frac{1}{7}, \mathbb{P}[Bob_3] = \frac{25}{63}$, and so $\mathbb{P}[Bob_3] > \mathbb{P}[Bob_2] > \mathbb{P}[Bob_1]$, which means that Bob has the greatest probability of winning if he doesn't hit anyone, so Bob's best course of action is to shoot into the air (i.e., intentionally miss).

5 Weathermen

(a) 60.9%

We represent snowing as “S”, not snowing as “NS”, Tom predicting snow as “TS”, Tom predicting no snowing as “TNS”. So, using the given information, we have that the probability of actually snowing in New York is $\mathbb{P}[\mathrm{S}] = 0.1$, and $\mathbb{P}[TS|S] = 0.7, \mathbb{P}[TNS|NS] = 0.95$.

Since days that snow and days that don't snow form a partition, and the days that Tom predicts to snow and the days that Tom predicts to not snow form another partition, so we have that

$$\mathbb{P}[\mathrm{NS}] = 1 - \mathbb{P}[\mathrm{S}] = 0.9$$

$$\mathbb{P}[TNS|S] = \mathbb{P}[\overline{TS}|S] = 1 - \mathbb{P}[TS|S] = 0.3$$

$$\mathbb{P}[TS|NS] = \mathbb{P}[\overline{TNS}|NS] = 1 - \mathbb{P}[TNS|NS] = 0.05$$

.

So, $\mathbb{P}[TS] = \mathbb{P}[TS|S]\mathbb{P}[S] + \mathbb{P}[TS|\overline{S}](1 - \mathbb{P}[S]) = \mathbb{P}[TS|S]\mathbb{P}[S] + \mathbb{P}[TS|NS](1 - \mathbb{P}[S]) = 0.7 \cdot 0.1 + 0.05 \cdot 0.9 = 0.115$

Thus, this problem is asking for the probability $\mathbb{P}[S|TS] = \frac{\mathbb{P}[TS|S] \cdot \mathbb{P}[S]}{\mathbb{P}[TS]} = \frac{0.7 \cdot 0.1}{0.115} = 0.609 = 60.9\%$, which means that if Tom says that it is going to snow, then the probability it will actually snow is 60.9%.

(b) 92.5%

We first have that $\mathbb{P}[TNS] = \mathbb{P}[\overline{TS}] = 1 - \mathbb{P}[TS] = 1 - 0.115 = 0.885$. Then, using the same strategy, we could calculate Tom's accuracy when he predicts no snow:

$$\mathbb{P}[NS|TNS] = \frac{\mathbb{P}[TNS|NS] \cdot \mathbb{P}[NS]}{\mathbb{P}[TNS]} = \frac{0.95 \cdot 0.9}{0.885} = 0.966$$

Thus, $\mathbb{P}[\mathrm{Tom}] = \mathbb{P}[S|TS] \cdot \mathbb{P}[TS] + \mathbb{P}[S|\overline{TS}](1 - \mathbb{P}[TS]) = \mathbb{P}[S|TS] \cdot \mathbb{P}[TS] + \mathbb{P}[S|TNS](1 - \mathbb{P}[TS]) = 0.609 \cdot 0.115 + 0.966 \cdot 0.885 = 0.925 = 92.5\%$, which means that Tom's overall accuracy is 92.5%.

(c) Example

Consider this situation for Jerry: Let Alaska be snowy 90% of the days, and that she predicts snow correctly 75% of the actual snowy days, and she predicts no snow (sunny) correctly 100% of the actual no snow (sunny) days.

So for Jerry, $\mathbb{P}[S] = 0.9, \mathbb{P}[JS|S] = 0.75 > \mathbb{P}[TS|S] = 0.7, \mathbb{P}[JNS|NS] = 1 > \mathbb{P}[TNS|NS] = 0.95$, which means that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny days. Yet, let's now calculate Jerry's overall accuracy.

Using similar logic as we did in parts (a) and (b), so

$$\mathbb{P}[NS] = \mathbb{P}[\overline{S}] = 1 - \mathbb{P}[S] = 0.1$$

$$\mathbb{P}[JNS|S] = \mathbb{P}[\overline{JS}|S] = 1 - \mathbb{P}[JS|S] = 1 - 0.75 = 0.25$$

$$\mathbb{P}[JS|NS] = \mathbb{P}[\overline{JS}|NS] = 1 - \mathbb{P}[JNS|NS] = 1 - 1 = 0$$

Thus, we have that: $\mathbb{P}[JS] = \mathbb{P}[JS|S] \cdot \mathbb{P}[S] + \mathbb{P}[JS|NS] \cdot \mathbb{P}[NS] = 0.75 \cdot 0.9 + 0 \cdot 0.1 = 0.675$, which means that when Jerry says it's going to snow, the probability that it will actually snow is $\mathbb{P}[S|JS] = \frac{\mathbb{P}[JS|S] \cdot \mathbb{P}[S]}{\mathbb{P}[JS]} = \frac{0.75 \cdot 0.9}{0.675} = 1$.

Similarly, with $\mathbb{P}[JNS] = 1 - \mathbb{P}[JS] = 0.325$, so when Jerry says it's going to not snow, the probability that it will actually not snow is $\mathbb{P}[NS|JNS] = \frac{\mathbb{P}[JNS|NS] \cdot \mathbb{P}[NS]}{\mathbb{P}[JNS]} = \frac{1 \cdot 0.1}{0.325} = 0.308$.

Thus, Jerry's overall accuracy is

$$\mathbb{P}[\text{Jerry}] = \mathbb{P}[S|JS] \cdot \mathbb{P}[JS] + \mathbb{P}[NS|JNS] \cdot \mathbb{P}[JNS] = 1 \cdot 0.675 + 0.308 \cdot 0.325 = 0.775 = 77.5\%$$

Here, we could see that $\mathbb{P}[\text{Jerry}] < \mathbb{P}[\text{Tom}]$ even though $\mathbb{P}[JS|S] > \mathbb{P}[TS|S]$ and $\mathbb{P}[JNS|NS] > \mathbb{P}[TNS|NS]$, which gives an examples to the situation described in the problem.

More generally, since both weatherman predicts no snow much better than snow days, so a weatherman in a region that snows a lot (i.e. Alaska compared to New York) will have much lower overall accuracy (given the weight of abundant snowy days) even if she/he is better at predicting snow on snow days and no snow on sunny days.