1 Quick Computes

(a) 5

Since 11 is a prime and $1 \leq 3 \leq 11 - 1 = 10$, which means that $3 \in \{1, 2, ..., 11 - 1\}$, so using Fermat's Little Theorem, we have that $3^10 \equiv 1 \pmod{11}$.

Thus, $3^{33} = 3^{10\cdot 3+3} = (3^{10})^3 \cdot 3^3 \equiv 1^3 \cdot 27 = 27 \equiv 5 \pmod{11}$

(b) 5

Since $10001 = 17 \cdot 588 + 5$, so $10001^{10001} \equiv 5^{10001} \pmod{17}$. Again, since 17 is prime and $5 \in \{1, 2, ..., 17 - 1\}$, so using Fermat's Little Theorem, we have that $5^16 \equiv 1 \pmod{17}$.

Thus, $10001^{10001} \equiv 5^{10001} \pmod{17} = 5^{16\cdot 625+1} = (5^{16})^{625} \cdot 5^1 \equiv 1^{625} \cdot 5 \equiv 5 \pmod{17}$

(c) 1

Since $10 = 7\cdot 1+3, 20 = 7\cdot 2+6, 30 = 7\cdot 4+2, 40 = 7\cdot 5+5$, so $10^{10}+20^{20}+30^{30}+40^{40} \equiv 3^{10}+6^{20}+2^{30}+5^{40} \pmod{7}$

Then again, since 7 is prime and $3, 6, 2, 5 \in \{1, 2, ..., 7 - 1\}$, so using Fermat's Little Theorem, we have that $3^6 \equiv 6^6 \equiv 2^6 \equiv 5^6 \equiv 1 \pmod{7}$

Thus, $10^{10} + 20^{20} + 30^{30} + 40^{40} \equiv 3^{10} + 6^{20} + 2^{30} + 5^{40} \equiv 3^6 \cdot 3^4 + (6^6)^3 \cdot 6^2 + (2^6)^5 + (5^6)^6 \cdot 5^4 \equiv 1 \cdot 81 + 1^3 \cdot 36 + 1^5 + 1^6 \cdot 625 = 81 + 36 + 1 + 625 = 743 = 7 \cdot 106 + 1 \cdot 1 \pmod{7}$