5 Markov Chain Terminology

(a) Irreducible: $a \neq 0 \wedge b \neq 0$; Reducible: $a = 0 \vee b = 0$

For the given Markov chain to be irreducible, it has to be able to transform between the two states, which means that the constraint is:

$$a \neq 0 \ \wedge \ b \neq 0$$

Since the term reducible Markov chains is the complement of irreducible Markov chains, so we have that for the Markov chain to be reducible, we have:

$$a = 0 \ \vee \ b = 0$$

(b) Direct Proof

Given this case with $a = b = 1$, so the Markov chain can go from state 0 to state 0 in n steps for all $n \in \{2, 4, 6, 8, \ldots\}$, so we have

$$d(0) = gcd\{2, 4, \cdots\} = 2 \neq 1$$

Thus, by definition of periodicity, we have that the given Markov chain is periodic.

Q.E.D.

(c) Direct Proof

Given this case with $a, b \in (0, 1)$, so the Markov chain can go from state 0 to state 0 in any of the $n \geq 1$ steps, i.e. $n = \{1, 2, 3, \ldots\}$, so we have

$$d(0) = gcd\{1, 2, 3, \cdots\} = 1$$

Similarly, we have that $d(1) = 1$. Therefore,

$$d(i) = 1 \ \forall \ i \in \mathscr{X}$$

which implies, by definition, that the given Markov chain is aperiodic.

Q.E.D.

(d) $\mathbf{P} = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$

We have that the transition probability matrix for the given Markov chain is:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e) $\pi_0 = \frac{a}{a+b}$, $\pi_1 = \frac{b}{a+b}$

We have that the balance equations can be set up with $\pi\mathbf{P} = \pi$, and so

$$[\pi_0, \pi_1] = [\pi_0, \pi_1] \cdot \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

which gives us two linear equations, and then since there's also the condition that the components of π sum up to one, so we have 3 linear equations in total:

$$\pi_0 = \pi_0(1-b) + \pi_1 a$$

$$\pi_1 = \pi_0 b + \pi_1(1-a)$$

$$\pi_0 + \pi_1 = 1$$

We could then solve them as:

$$\pi_0 = \frac{a}{a+b}$$

$$\pi_1 = \frac{b}{a+b}$$