Counting ∞

Are there more blue dots or red dots?

Did you count all of the dots?

How did you know the answer?

Today: We count to ∞ and beyond.

Review: Bijections

A function $f : A \to B$ is:

- **one-to-one** (an **injection**) if $f(x) = f(y)$ implies $x = y$. Or, $x \neq y$ implies $f(x) \neq f(y)$. Distinct inputs, distinct outputs.
- **onto** (a **surjection**) if for each $y \in B$, there is an $x \in A$ with $f(x) = y$. Every element in B is hit.

Then, f is a **bijection** if it is both an injection and a surjection.

A bijection "rearranges" the elements of A to form B.

Counting Infinite Sets

How did we know that there were the same number of dots of each color, *without counting*?

You found a bijection between the blue dots and red dots!

To count infinities, we will take the definition of "same size" to be "there exists a bijection between the sets".

Countability

What does it mean for us to "count" the elements of a set?

Our model for counting: $\mathbb{N} = \{0, 1, 2, \ldots\}$.

A set A is called **countable** if there exists a bijection between A and a subset of $\mathbb{N}$.

- Any finite set is countable. Consider the set Odin's notable children $= \{\text{Hela}, \text{Thor}, \text{Loki}\}$.

  $$f(\text{Hela}) = 0, \qquad f(\text{Thor}) = 1, \qquad f(\text{Loki}) = 2.$$

  Then, f : Odin's notable children $\to \{0, 1, 2\}$ is a bijection.
- $\mathbb{N}$ itself is countable.
- If A is countable and infinite, then we say it is **countably infinite**.
- What else is countable?

Hilbert's Hotel I

Consider an infinite hotel, one room for each $n \in \mathbb{N}$. The rooms are all filled by guests.

A new guest arrives. Can we accommodate the new guest?

For each $n \in \mathbb{N}$, move the guest in room n to $n+1$. Then place the new guest in room 0.

In other words, we found a bijection $f : \mathbb{N} \cup \{-1\} \to \mathbb{N}$.

$$f(-1) = 0, \qquad f(n) = n+1 \text{ for } n \in \mathbb{N}.$$

Adding one more element to $\mathbb{N}$ does not change its size.

Hilbert's Hotel II

Now suppose that a new bus of passengers arrives. There is a new guest n for each positive integer n.

Can we still accommodate the guests?

For each $n \in \mathbb{N}$, move guest in room n to room $2n$. Put the ith new guest into the ith odd-numbered room.

We found a bijection $f : \mathbb{Z} \to \mathbb{N}$.

$$f(n) = 2n \text{ for } n \in \mathbb{N}, \qquad f(-n) = 2n - 1 \text{ for positive } n.$$

Adding a countably infinite number of elements to $\mathbb{N}$ does not change its size.

Proving the Bijection Formally

Recall: If A and B are *finite* and have the same size, then if $f : A \to B$ is injective *or* surjective, then it is both.

This is not true for infinite sets, so we must check *both* injectivity and surjectivity.

$$f(n) = 2n \text{ for } n \in \mathbb{N}, \qquad f(-n) = 2n-1 \text{ for positive } n.$$

Proof that f is bijective.

- One-to-one: Assume $f(x) = f(y)$. Prove $x = y$.
- If $f(x) = f(y)$ are odd, then $-2x-1 = -2y-1$. So, $x = y$.
- If $f(x) = f(y)$ are even, then $2x = 2y$. So, $x = y$.
- Onto: Consider any $n \in \mathbb{N}$. Either n is even or odd.
- If n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Then, $f(k) = n$.
- If n is odd, then $n = 2k-1$ for some positive k. Then $f(-k) = n$. □

Countably Infinite Sets

Here are some countably infinite sets.

- $\mathbb{N}$. $\mathbb{N} \cup \{-1\}$. $\mathbb{Z}$.
- The set of even numbers. The set of odd numbers.
- The set of prime numbers.

Why is the set of prime numbers countably infinite? It is infinite (we proved this). But we can *list* them.

$$2, 3, 5, 7, 11, \ldots$$

The list is *exhaustive.* Every prime number shows up in the list.

An exhaustive list is equivalent to a bijection.

$$f(2) = 0,\ f(3) = 1,\ f(5) = 2,\ f(7) = 3,\ f(11) = 4, \ldots$$

A set whose elements can be listed is countable.

Be Careful

Is the following a listing of $\mathbb{Z}$?

$$0, 1, 2, 3, \ldots, -1, -2, -3, \ldots$$

Where does the element -1 show up in the list?

To give a listing of a set A, every element of A must show up at some *finite index* in the list.

- In the example above, we never "reach" the element -1.

Here is a valid listing of $\mathbb{Z}$:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Be careful with "…" in the middle of your listing.

Hilbert's Hotel III

Now a countably infinite number of buses arrive, each bus containing a countably infinite number of passengers.

Can we accomodate the guests?

First, "make room for ∞" (send guest n to room $2n$ as before).

Label each bus with a prime number p. Label each person in the bus with a positive integer.

Send the ith person in bus p to the p^i-th odd numbered room.

- Bus 2's passengers get sent to: $2 \cdot 2^1 - 1$, $2 \cdot 2^2 - 1$, $2 \cdot 2^3 - 1, \ldots$
- Bus 3's passengers get sent to: $2 \cdot 3^1 - 1$, $2 \cdot 3^2 - 1$, $2 \cdot 3^3 - 1, \ldots$

Adding a countably infinite number of countable infinities to $\mathbb{N}$ does not change its size.

Since for each bus, the passenger's number would not repeat (strictly increasing); considering any two bus, the prime factorization (neglecting the -1) would be different, and thus, resulting in a unique seat for every person of every bus.

The Formal Injection

We found an injection

$$f : \{\text{prime numbers}\} \times \{1, 2, 3, \ldots\} \to \{\text{odd numbers}\}$$

given by $f(p, i) = p^i$-th odd number.

Since $\{\text{prime numbers}\}$, $\{1, 2, 3, \ldots\}$, and $\{\text{odd numbers}\}$ all have the same size as $\mathbb{N}$, this is the same as finding an injection

$$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

Why? There are bijections

$$f_1 : \mathbb{N} \to \{\text{prime numbers}\},$$
$$f_2 : \mathbb{N} \to \{1, 2, 3, \ldots\},$$
$$f_3 : \mathbb{N} \to \{\text{odd numbers}\},$$

so we get an injection $g(m, n) = f_3^{-1}(f(f_1(m), f_2(n)))$.

Bijections Compose

Fact: If $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f$.

Proof.

- If $g(f(x)) = g(f(y))$, then g is one-to-one so $f(x) = f(y)$.
- Since f is one-to-one, then $x = y$. So $g \circ f$ is one-to-one.
- If $c \in C$, then there is a $b \in B$ such that $g(b) = c$ (since g is onto).
- There is an $a \in A$ such that $f(a) = b$ (since f is onto).
- So, $g(f(a)) = g(b) = c$. So $g \circ f$ is onto. □

Bijections compose.

Exercise: If there are bijections $f : A \to A'$ and $g : B \to B'$, then $h(a, b) = (f(a), g(b))$ is a bijection $A \times B \to A' \times B'$.

Bijections Compose

Fact: If $f : A \to B$ is a bijection, and there are bijections $f_1 : A \to A'$ and $f_2 : B \to B'$, then there is a bijection $g : A' \to B'$.

Proof.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow f_1 & & \downarrow f_2 \\ A' & \xrightarrow{g} & B' \end{array}$$

Define $g = f_2 \circ f \circ f_1^{-1}$. The composition of bijections is a bijection. □

To show that A has the same size as $\mathbb{N}$, we can show that A has the same size as A', where A' has the same size as $\mathbb{N}$.

Is $\mathbb{Q}$ Countable?

Is $\mathbb{Q}$ countable?

- We found an injection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. So, $\mathbb{N} \times \mathbb{N}$ is countable.
- Since $\mathbb{Z}$ has the same size as $\mathbb{N}$, then $\mathbb{Z} \times \mathbb{Z}$ is countable.
- Every rational number $q \in \mathbb{Q}$ can be written as $q = a/b$, where $a, b \in \mathbb{Z}$, $b > 0$, and a/b is in lowest terms.
- This defines an *injection* $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$.
- An injection implies that $\mathbb{Q}$ is "smaller" than $\mathbb{N} \times \mathbb{N}$, so $\mathbb{Q}$ is countable.
- On the other hand, $\mathbb{Q}$ is infinite, so $\mathbb{Q}$ is countably infinite.

Principle: To show that a set A is countable, we only need to find an injection from A into a countable set.

Interleaving Argument

Suppose that A is a countable alphabet. Consider the set of all *finite* strings whose symbols come from A.

A is countable.

Proof.

- List the alphabet $A = \{a_1, a_2, a_3, \dots\}$.
- Step 0: List the empty string.
- Step 1: List all strings of length ≤ 1 using symbols from $\{a_1\}$. a_1.
- Step 2: List all strings of length ≤ 2 using symbols from $\{a_1, a_2\}$. a_1, a_2, a_1a_1, a_1a_2, a_2a_1, a_2a_2.
- Step 3: List all strings of length ≤ 3 using symbols from $\{a_1, a_2, a_3\}$.
- Continue forever. This exhaustively lists the members of the set. □

Polynomials with Rational Coefficients

Consider the set of polynomials with rational coefficients. Is this set countable?

For a polynomial, e.g., $P(x) = (2/3)x^4 - 2x^2 + (1/10)x + 9$, think of it as a string: $(2/3, 0, -2, 1/10, 9)$.

The alphabet is $\mathbb{Q}$, countably infinite.

Each polynomial is a finite-length string from the alphabet.

The polynomials with rational coefficients are countable.

Combine the fact that we proved |Q is countable and the Interleaving Argument.

Is $\mathbb{R}$ Countable?

Is $\mathbb{R}$ countable? First, let us study the closed unit interval $[0, 1]$.

Each element of $[0, 1]$ can be represented as a infinite-length decimal string.

- For example, take the element 0.37. This can also be represented as 0.36999....

Suppose we had a list of all numbers in $[0, 1]$.

$$\begin{array}{cccccc} 0 & . & 9 & 9 & 1 & \dots \\ 0 & . & 0 & 2 & 3 & \dots \\ 0 & . & 2 & 8 & 9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

If we change the numbers on the diagonal, $0.929\dots$, we get a number which is *not* in the list.

- Change all 8s to 1s and change all other numbers to 8s.

Cantor's Diagonalization Argument

- Assume we could list all numbers in $[0, 1]$.
- Form a new number in $[0, 1]$ by changing each number in the diagonal.
- This number cannot be the ith element of the list because it differs in the ith digit.
- We found an element not in our original list!
- So, $[0, 1]$ is **uncountable**.

What happens when we try to apply the diagonalization argument to $\mathbb{N}$?

- We get a number with infinitely many digits.
- This is not a natural number! Not a contradiction.

The Size of $\mathbb{R}$ v.s. $[0,1]$

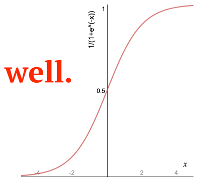
Are $[0,1]$ and $\mathbb{R}$ the same size? Bijection?

Cantor-Schröder-Bernstein Theorem: If there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $A \to B$.

It suffices to find an injection both ways.

- $[0,1] \to \mathbb{R}$: Map $x \mapsto x$.
- $\mathbb{R} \to [0,1]$: Try $x \mapsto (1 + \exp(-x))^{-1}$.

Why exp(-x)?
exp(x) works as well.



What Is Not Countable?

Recall: Given a set S, the **power set** $\mathscr{P}(S)$ of S is the set of all subsets of S.

If $|S| = n$, then $|\mathscr{P}(S)| = 2^n$.

Is the size (cardinality) of the power set of S larger than the size of S when S is infinite?

Example of a function $f : \{0,1,2\} \to \mathscr{P}(\{0,1,2\})$:

$$f(0) = \{1,2\}, \qquad f(1) = \{1\}, \qquad f(2) = \varnothing.$$

The Power Set Is Large

Theorem: There is no bijection $S \to \mathscr{P}(S)$.

Proof.

- Consider any $f : S \to \mathscr{P}(S)$. We will show that f is not a bijection.
- We will define a set $A \subseteq S$ so that nothing maps to A, i.e., $f(x) \neq A$ for all x.
- Consider the set $A \subseteq S$, defined by $A = \{x \in S : x \notin f(x)\}$.
- Case 1: If $x \in f(x)$, then $x \notin A$. So, $f(x) \neq A$.
- Case 2: If $x \notin f(x)$, then $x \in A$. So $f(x) \neq A$.
- Conclusion: No x gets mapped to A. So f cannot be surjective. □

Comparison with Interleaving Argument

$\mathbb{N}$ and $\mathscr{P}(\mathbb{N})$ are not the same size.

Interleaving argument: The set of finite-length strings with symbols from $\mathbb{N}$ is countable.

$\mathscr{P}(\mathbb{N})$ can be thought of as the set of *infinite-length* strings with symbols from $\{0,1\}$.

- For $S \subseteq \mathbb{N}$, if $i \in S$, then the ith bit of the string is 1.
- Example: $\{2,3,4\} \equiv (0,0,1,1,1,0,0,0,\ldots)$

In fact, since the numbers in $[0,1]$ can be written as infinite-length bit strings (binary expansion), there is a bijection

$$f : [0,1] \to \mathscr{P}(\mathbb{N}).$$

Proof 1 that P(N) is not countable?

The Power Set Is Large, Again

Suppose S is countable, $S = \{s_0, s_1, s_2, s_3, \ldots\}$.

Consider the table:

x	is s_0 in $f(x)$?	is s_1 in $f(x)$?	is s_2 in $f(x)$?	$\cdots$
s_0	N	N	N	$\cdots$
s_1	Y	Y	N	$\cdots$
s_2	N	Y	N	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Is every element of $\mathscr{P}(S)$ listed? Consider the set formed by "flipping the diagonal": $\{s_0, s_2, \ldots\} = \{x \in S : x \notin f(x)\}$.
This set is not listed.

The previous proof is also a proof by diagonalization!

Cardinal Numbers

The power set of a set S has strictly larger cardinality than S.

This means that $\mathscr{P}(\mathbb{R})$ has even larger cardinality than $\mathbb{R}$! And then there is $\mathscr{P}(\mathscr{P}(\mathbb{R}))$...

The size of sets is measured by cardinal numbers.

- Each natural number is a cardinal number.
- The size of $\mathbb{N}$ is a cardinal number (countably infinite).
- $\mathbb{R}$ has the "cardinality of the continuum".
- There are even larger cardinal numbers!

Are there cardinalities between $\mathbb{N}$ and $\mathbb{R}$? (**Continuum Hypothesis**) Not provable/disprovable from our axioms!

Summary

- A set is countable if there is an injection into $\mathbb{N}$.
- Countable sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, prime numbers, finite-length strings from a countable alphabet.
- Cantor introduced a diagonalization argument. We proved that $[0, 1]$ is uncountable.
- Cantor-Schröder-Bernstein Theorem: If there is an injection both ways, there is a bijection.
- The power set is strictly larger than the original set!