2 Eulerian Tour and Eulerian Walk

(a) No, there isn't.

Consider vertex 3, which has a degree of 3, so it's not even degree, which means that the graph G is not even degree. Using Theorem 5.1, an undirected graph G has an Eulerian tour if and only if G is even degree and connected, so there isn't an Eulerian tour in the graph.

(b) Yes, there is.

Consider a walk W: {3, 4}, {4, 2}, {2, 3}, {3, 1}, {1, 4}, {4, 8}, {8, 1}, {1, 2}, {2, 6}, {6, 1}, {1, 7}, {7, 8}, {8, 6}, {6, 7}.

Since W is a walk that uses each edge exactly once, so by definition, W is an Eulerian walk in the graph.

(c) The condition is that there is exactly zero or exactly two vertices with odd degree, and G is connected.

Proof. In other words, there is an Eulerian walk in an undirected graph G if and only if there is exactly zero or exactly two vertices with odd degree, and G is connected. To prove this, we must establish two directions: if, and only if.

Only if. We give a proof by cases for this situation, i.e., given the assumption that G is connected, and that there is exactly zero or exactly two vertices with odd degree, so there would be two cases: either (1) G has exactly zero odd vertices, or (2) G has exactly two odd vertices.

Case 1: Let G have exactly zero odd vertices, which means that G is even degree, then using Theorem 5.1, G has an Eulerian tour. Since by definition, an Eulerian tour is an Eulerian walk, so G has an Eulerian walk.

Case 2: Let u, v be the two odd vertices. Just like our proof for Theorem 5.1, we can use a recursive algorithm for finding an Eulerian walk.

Start with u_0, we claim a walk W that doesn't use an edge twice, would always get stuck at v_0. Just like our proof for the claim in Theorem 5.1, after we leave u_0 initially, every vertex in G except for v_0 has even degree, which means that whenever we enter a vertex $v^* \neq v_0$, there is at least one edge we haven't used that's incident to v^*, which means that the walk wouldn't be stuck at v^*. Thus, the walk would always get stuck at v_0. This walk would not always be an Eulerian walk. However, after the initial walk, the "unused" degrees of all vertices in G is even, so similar to our proof in Theorem 5.1, a walk that starts at an arbitrary vertex v and doesn't use an edge twice would be a cycle that gets stuck at v. Thus, again, similar to our proof in Theorem 5.1, we could always find a set of **edge disjoint** tours, $T_1, T_2, ..., T_k, k \geq 1$, along with our initial walk W from u_0 to v_0 that covers all the edges in G.

Then, since G is connected by assumption, so again, similar to our proof in Theorem 5.1, we could splice together W and $T_1, T_2, ..., T_k$ so that it forms an Eulerian walk.

If. We give a direct proof for the forward direction, i.e., if an undirected graph G has an Eulerian walk, then G is connected and has exactly 0 or exactly 2 odd degree vertices.

Assume that G has an Eulerian walk W. By definition of Eulerian tours, so every vertex must have an edge adjacent to it, which implies that G is connected.

Then, let W traverse the vertices in this way: $v_0, v_1, ..., v_w$. Excluding the first and last vertices, for any vertex $v_i, 0 < v < w$, the edges $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ can be paired up. So every time a vertex is reached in the middle of W, there would always be two edges adjacent to it, which implies that all the vertices (except v_0, v_w) are even vertices. Now, we divide the situation into two cases, and exactly one of which must be true: (1) $v_0 = v_w$; or (2) $v_0 \neq v_w$.

Case (1): If $v_0 = v_w$, then by definition, W is a tour, which means that W is an Eulerian tour. Using Theorem 5.1, we have that G is even, which implies that there is 0 odd vertices in G.

Case (2): If $v_0 \neq v_w$, then consider v_0. Suppose it has appeared $k, k \in \mathbb{N}$ times in the walk besides being the initial vertex, then using our proof above, so it has a degree of $2k + 1$, which is an odd number, so v_0 is an odd vertex. Similarly, v_w is also an odd vertex, and they're the only two odd vertices in G, which implies that G has exactly two odd vertices.

Thus, we have proved that if G has an Eulerian walk, then G is connected, and G has exactly zero or exactly two odd vertices.

Therefore, the condition is that there is exactly zero or exactly two vertices with odd degree, and G is connected.

Q.E.D.