2 Counting Tools

(a) Countable

We divide the problem into two cases, where exactly one must be true: (1) A, B are both finite; or (2) A, B are not both finite.

Case (1): Given that A, B are both finite, so we can enumerate the elements of A as $a_0, a_1, a_2, ..., a_m$ and the elements of B as $b_0, b_1, ..., b_n$. Thus, there is a total of $(m+1)(n+1)$ different elements of $A \times B$, which means that $A \times B$ is finite, and thus it's countable by definition.

Case (2): Given that A, B are not both finite (we include the cases where exactly one of them is finite, as it will still be a bijection between $A \times B$ and $\mathbb{N}$ when we do the spiral/diagonal enumeration).

Since A is countable, so we can enumerate the elements of A like this: $a_0, a_1, a_2, ...$ Similarly, the elements of the countable set B can be enumerated as $b_0, b_1, b_2, ...$, so we can write $A \times B$ as:

$$\begin{array}{cccc} (a_0, b_0) & (a_1, b_0) & (a_2, b_0) & \cdots \\ (a_0, b_1) & (a_1, b_1) & (a_2, b_1) & \cdots \\ (a_0, b_2) & (a_1, b_2) & (a_2, b_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Thus, we can enumerate the elements of $A \times B$, i.e. create an injection from $A \times B$ to $\mathbb{N}$, by counting the elements of $A \times B$ in a spiral/diagonal way like this (following the lines and arrows):

$$\begin{array}{ccccc} & \overset{0}{(a_0, b_0)} & \overset{1}{(a_1, b_0)} & \overset{5}{(a_2, b_0)} & \cdots \\ 2 & (a_0, b_1) & \overset{4}{(a_1, b_1)} & (a_2, b_1) & \cdots \\ 3 & (a_0, b_2) & (a_1, b_2) & (a_2, b_2) & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{array}$$

Thus, there is an injection $f : A \times B \to \mathbb{N}$ as no two elements lie in the same position and by the fact that this mapping certainly maps every element of $A \times B$ to a natural number, because every such element appears somewhere (exactly once) in the grid, and the spiral hits every point in the grid.

On the other hand, due to our counting strategy (no double-counting) as well as the fact that each element of $A \times B$ is unique, so there's also an injection $g : \mathbb{N} \to A \times B$ just by following our counting strategy. Thus, using the Cantor-Bernstein theorem and our Note, so there's a **bijection** $h : A \times B \to \mathbb{N}$, which by definition, shows that $A \times B$ is countable.

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(b) Countable

We divide the problem into two cases, where exactly one must be true: (1) A, B_i are all finite; or (2) A, B_i are not all finite.

Case (1): Given that A, B_i are all finite, so we can enumerate the elements of A as $a_0, a_1, a_2, ..., a_m$. Thus, $\cup_{i\in A} B_i = B_{a_0} \cup B_{a_1} \cup B_{a_2} \cup ... \cup B_{a_m}$. Then, we can enumerate of each B_{a_j} that has $n_j \in \mathbb{N}$ elements as $b_{j,0}, b_{j,1}, ..., b_{j,n_j-1}$. Thus, there's a total of $\sum_{i=0}^{m} n_i$ elements in $\cup_{i\in A} B_i$, and this is a finite number. Thus, $\cup_{i\in A} B_i$ is finite, and thus it's countable by definition.

Case (2): Given that A, B are not both finite (again, we include the case where exactly one of them are finite, as it will still be a bijection between $\cup_{i\in A} B_i$ and $\mathbb{N}$ when we do the spiral/diagonal enumeration).

Since A is countable, so we can enumerate the elements of A like this: $a_0, a_1, a_2, ...$ Similarly, the elements of any of the countable set B_{a_j} can be enumerated as $b_{j,0}, b_{j,1}, b_{j,2}, ...$ So we can write $\cup_{i\in A} B_i = (B_{a_0} \cup B_{a_1} \cup B_{a_2} \cup ...)$ as:

$$\begin{array}{ccccc}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\
b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\
b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\
b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Thus, we can enumerate the elements of $\cup_{i\in A} B_i$, i.e. create an injection from $\cup_{i\in A} B_i$ to $\mathbb{N}$, by counting the elements of $\cup_{i\in A} B_i$ in a spiral/diagonal way like this (following the lines and arrows):

$$\begin{array}{cccccc}
 & 0 & 1 & 5 & 6 & \\
 & b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\
2 & b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\
3 & b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\
4 & b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Thus, there is an injection $f : \cup_{i\in A} B_i \to \mathbb{N}$ as no two elements lie in the same position and by the fact that this mapping certainly maps every element of $\cup_{i\in A} B_i$ to a natural number, because every such element appears somewhere (exactly once) in the grid, and the spiral hits every point in the grid.

On the other hand, due to our counting strategy (no double-counting) as well as the fact that each element of $\cup_{i\in A} B_i$ is unique, so there's also an injection $g : \mathbb{N} \to \cup_{i\in A} B_i$ just by following our counting strategy. Thus, using the Cantor-Bernstein theorem and our Note, so there's a **bijection** $h : \cup_{i\in A} B_i \to \mathbb{N}$, which by definition, shows that $\cup_{i\in A} B_i$ is countable.

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(c) Uncountable

We use the Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. We can represent a function $f \in \mathscr{F}$ as an infinite sequence $(f(0), f(1), ...)$, where the i^{th} element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$ where we list the functions f of $\mathscr{F}$ in increasing order (i.e., Let $x < y, x, y \in \mathbb{N}$, then for the x^{th} and y^{th} enumerated function, f_x and f_y, we have that $f_x(i) \leq f_y(i)\ \forall i \in \mathbb{N}$):

$$\begin{array}{rcl} 0 & \iff & (f_0(0), f_0(1), f_0(2), f_0(3), ...) \\ 1 & \iff & (f_1(0), f_1(1), f_1(2), f_1(3), ...) \\ 2 & \iff & (f_2(0), f_2(1), f_2(2), f_2(3), ...) \\ 3 & \iff & (f_3(0), f_3(1), f_3(2), f_3(3), ...) \\ & \vdots & \end{array}$$

Consider the function $g : \mathbb{N} \to \mathbb{N}$ where $g(i) = f_i(i) + 1\ \forall i \in \mathbb{N}$. We claim that g is a valid non-decreasing function from $\mathbb{N}$ to $\mathbb{N}$ that is not in our list of functions f. We provide a short proof by contradiction for this claim.

Suppose, for a contradiction, that g is in our list of functions, then let g be the n^{th} function, i.e. $g = f_n$. So, $g(\cdot) = f_n(\cdot)$. However, $g(\cdot)$ and $f_n(\cdot)$ differ in the n^{th} number, since $g(n) = f_n(n) + 1 \neq f_n(n)$, which implies that $g \neq f_n$, which gives the contradiction.

Thus, by the Cantor's Diagonalization Proof, we have that the set of all functions f from $\mathbb{N}$ to $\mathbb{N}$, such that f is non-decreasing, is uncountable.

Q.E.D.

(d) Countable

Let $\mathscr{F}$ be the set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. We can represent a function $f \in \mathscr{F}$ as an infinite sequence $(f(0), f(1), ...)$, where the i^{th} element is $f(i)$. Consider the following representation of all elements (functions) of $\mathscr{F}$:

$$\mathscr{F} = \cup_{i_1 \in \mathbb{N}}(i_1, ...) = (0, ...) \cup (1, ...) \cup (2, ...)$$

i.e. (one) infinite sequence that starts with 0 $\cup$ infinite sequences that starts with 1 $\cup$ infinite sequences that starts with 2 $\cup$ ...

Then, any infinite sequence that starts with k could be enumerated/represented as:

$$\cup_{i_2}(k, i_2, ...) = (k, 0, ...), (k, 1, ...), (k, 2, ...), ...$$

i.e. infinite sequence that starts with $k, 0 \cup$ infinite sequence that starts with $k, 1 \cup$ infinite sequence that starts with $k, 2 \cup$ ...

Continuing the argument, we could enumerate each possibility of the current element by representing them as a union of infinite sequences that "identifies" the next unidentified (yet) element.

Since each step of enumeration where we "identify" one more element in the infinite sequence is a union of $\mathbb{N}$, which is countable, sets of countable elements by recursion, so each "subset" of the enumeration is countable by the results obtained in 2(b). Therefore, $\mathscr{F}$, the set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing, is also countable as it's also the union of countable sets ($\mathbb{N}$) of countable elements...

Q.E.D.

(e) Countable

Let $\mathscr{F}$ be the set of all bijective functions f from $\mathbb{N}$ to $\mathbb{N}$. Again, We can represent a function $f \in \mathscr{F}$ as an infinite sequence $(f(0), f(1), ...)$, where the i^{th} element is $f(i)$. Consider the following listing of all elements (functions) of $\mathscr{F}$:

$$0 \iff (0, 1, 2, 3, 4, ...)$$

$$1 \iff (1, 0, 2, 3, 4, ...)$$

$$\begin{aligned}
2 &\iff (0, 2, 1, 3, 4, ...) \\
3 &\iff (1, 2, 0, 3, 4, ...) \\
4 &\iff (2, 0, 1, 3, 4, ...) \\
5 &\iff (2, 1, 0, 3, 4, ...)
\end{aligned}$$

$$\begin{aligned}
6 &\iff (0, 1, 3, 2, 4, ...) \\
7 &\iff (0, 2, 3, 1, 4, ...) \\
8 &\iff (0, 3, 1, 2, 4, ...) \\
9 &\iff (0, 3, 2, 1, 4, ...) \\
10 &\iff (1, 0, 3, 2, 4, ...) \\
11 &\iff (1, 2, 3, 0, 4, ...) \\
12 &\iff (1, 3, 0, 2, 4, ...) \\
13 &\iff (1, 3, 2, 0, 4, ...) \\
14 &\iff (2, 0, 3, 1, 4, ...) \\
15 &\iff (2, 1, 3, 0, 4, ...) \\
16 &\iff (2, 3, 0, 1, 4, ...) \\
17 &\iff (2, 3, 1, 0, 4, ...) \\
18 &\iff (3, 0, 1, 2, 4, ...) \\
19 &\iff (3, 0, 2, 1, 4, ...) \\
20 &\iff (3, 1, 0, 2, 4, ...) \\
21 &\iff (3, 1, 2, 0, 4, ...) \\
22 &\iff (3, 2, 0, 1, 4, ...) \\
23 &\iff (3, 2, 1, 0, 4, ...) \\
&\vdots
\end{aligned}$$

Thus, we are listing out the elements of $\mathscr{F}$ in an "increasing" order, as we would list out all possible permutations of the first k numbers (while leaving $f(n) = n$ for all $n > k, n \in \mathbb{N}$) before having the $(k+1)^{th}$ number being "switched out of its place", as demonstrated by the enumerating technique above.

Thus, we can conclude that this mapping certainly maps every element of $\mathscr{F}$ to a natural number, which implies that there is an injection $f : \mathscr{F} \to \mathbb{N}$ as no two elements lie in the same position. Therefore, there is a bijection between $\mathscr{F}$ and some subset $C \subseteq \mathbb{N}$ of $\mathbb{N}$, which by definition, means that $\mathscr{F}$, the set of all bijective functions f from $\mathbb{N}$ to $\mathbb{N}$, is countable.

Q.E.D.