I worked alone without getting any help, except asking questions on Piazza and reading the Notes of this course.

1 Bijective or not?

(a)

(i) Yes

Proof (one-to-one): Suppose $f(x) = f(y)$, then $10^{-5}x = 10^{-5}y$. Since $10^{-5} \neq 0$, so we can divide both sides by $10^{-5} \neq 0$, which gives us that $x = y$. So, $f(x) = f(y) \implies x = y$, so $f : \mathbb{R} \to \mathbb{R}$ is injective.

Proof (onto): If $y \in \mathbb{R}$, then $f(10^5 y) = 10^{-5}10^5 y = y$. With $y \in \mathbb{R}$, so $10^5 y \in \mathbb{R}$, which means that y has a pre-image. Thus every $y \in \mathbb{R}$ has a pre-image, so $f : \mathbb{R} \to \mathbb{R}$ is onto.

Thus, $f : \mathbb{R} \to \mathbb{R}$ is both one-to-one and onto, so it is a bijection.

(ii) No

We proceed by providing a counterexample to show that $f : \mathbb{Z} \cup \{\pi\} \to \mathbb{R}$ is not onto, which implies that it is not a bijection.

Consider $y = 0.123456$, where $y \in \mathbb{R}$, so y is in the range. Suppose, for a contradiction, that some $x_y \in \mathbb{Z} \cup \{\pi\}$ such that $f(x_y) = y$. Let A be the assertion that $x_y \in \mathbb{Z} \cup \{\pi\}$. So, $10^{-5}x_y = y = 0.123456$. So, multiply both sides 10^5, $10^5 10^{-5} x_y = 10^5 \cdot 0.123456$, which gives us that $x_y = 12345.6$. However, we know that $x_y \notin \mathbb{Z}$ and that $x_y \notin \pi$, so we have $x_y \notin \mathbb{Z} \cup \{\pi\}$, which implies that $\neg A$. So, $A \wedge \neg A$ holds, which gives the contradiction.

Thus, $f : \mathbb{Z} \cup \{\pi\} \to \mathbb{R}$ is not onto, which implies that it is not a bijection.

(b)

(i) No

Consider $p = 7$, and then consider $x_1 = 2, x_2 = 3$, which are both in the domain of $f : \mathbb{N} \setminus \{0\} \to \{0, ..., p\}$ with $2, 3 \in \mathbb{N} \setminus \{0\}$. However, with $x_1 = 2, x_2 = 3$, so $f(x_1) = p = 7 = 2 \cdot 3 + 1 \equiv 1 \pmod 2$ and $f(x_2) = p = 7 = 2 \cdot 3 + 1 \equiv 1 \pmod 3$, which means that $f(x_1) = f(x_2)$ while $x_1 \neq x_2$.

Thus, $f : \mathbb{N} \setminus \{0\} \to \{0, ..., p\}$ is not one-to-one by definition, so it's not a bijection.

(ii) Yes

First, since $p > 2$ is prime, so p is an odd number, and so $(p+1)/2, (p-1)/2 \in \mathbb{Z}$. Then, for any arbitrary x in the domain of $f : \{(p+1)/2, ..., p\} \to \{0, ..., (p-1)/2\}$, we have $(p+1)/2 \leq x \leq p$. So, $0 \leq (p - x) \leq \frac{p-1}{2} < x$, and so $p - x \in \{0, ..., (p-1)/2\}$, which means that $p - x$ is the only solution. Thus, $f(x) = p \bmod x = p - x$.

Proof (one-to-one): Suppose $f(x_1) = f(x_1)$, then using our deduction above, so $p - x_1 = p - x_2$. Add $(-p + x_1 + x_2)$ to both sides and we have that $x_2 = x_1$. So $f(x_1) = f(x_2) \implies x_1 = x_2$, so $f : \{(p+1)/2, ..., p\} \to \{0, ..., (p-1)/2\}$ is injective.

Proof (onto): Now, if $y \in \{0, ..., (p-1)/2\}$, then by our deductions above again, so $f(p - y) = p - (p - y) = y$. With $0 \leq y \leq \frac{p-1}{2}$, so $\frac{p+1}{2} \leq (p - y) \leq p$, which means that $(p - y) \in \{(p+1)/2, ..., p\}$; in other words, $(p - y)$ is in the domain of $f : \{(p+1)/2, ..., p\} \to \{0, ..., (p-1)/2\}$, which means that y has a pre-image. Thus, every $y \in \{0, ..., (p-1)/2\}$ has a pre-image, so $f : \{(p+1)/2, ..., p\} \to \{0, ..., (p-1)/2\}$ is onto.

Thus, $f : \{(p+1)/2, ..., p\} \to \{0, ..., (p-1)/2\}$ is both one-to-one and onto, so it is a bijection.

(c) No

Since the domain D is defined as $D = \{0, ..., n\}$, so its cardinality is $|D| = n + 1$, which is finite. Then, since the range is $\mathscr{P}(D)$, using Note 10, so its cardinality is $|\mathscr{P}(D)| = 2^{|D|} = 2^{n+1} > n+1 = |D|$ for all $n \in \mathbb{N}$. We will do a short proof by induction for the claim that $2^{n+1} > n + 1 \ \forall n \in \mathbb{N}$.

Base case ($n = 0$): $2^1 = 2 > 1$, so the base case is correct.
Induction Hypothesis: For $n = k \geq 0$, $2^{k+1} > k + 1$
Inductive Step: Consider $n = k + 1 \geq 1$, so $2^{n+1} = 2^{k+2} = 2 \cdot 2^{k+1}$. Then, using our induction hypothesis, so $2^{k+2} = 2 \cdot 2^{k+1} > 2 \cdot (k + 1) = 2k + 2 \geq k + 2 = (k + 1) + 1$, as desired.
Thus, by the principal of mathematical induction, we have $2^{n+1} > n + 1 \ \forall n \in \mathbb{N}$.

Thus, the cardinality of the domain of $f : D \to \mathscr{P}(D)$ is strictly smaller than its range, which means that $f : D \to \mathscr{P}(D)$ cannot be surjective. We'll insert a small proof by contradiction to for this claim.

Let R be the assertion that $|\mathscr{P}(D)| > |D|$. Assume that $f : D \to \mathscr{P}(D)$ is surjective, then there must exist a function $g : \mathscr{P}(D) \to D$ that's injective, which indicates that $|\mathscr{P}(D)| \leq |D|$, which implies $\neg R$, so $R \wedge \neg R$ holds, which raises the contradiction.

Therefore, $f : D \to \mathscr{P}(D)$ is not surjective, and thus, it cannot be bijective.

(d) Yes

Since $X = 1234567890$, so X does not have any repeating digits. Then, since X' is obtained by randomly shuffling X, so X' have the same set of digits as X, and X' does not have any repeating digits.

Proof (one-to-one): Suppose $f(x) = f(y)$ with $x, y \in \{0, ..., 9\}$, then the $(x+1)^{th}$ digit of X' is the same as the $(y + 1)^{th}$ digit of X'. Since we have shown earlier that X' does not have any repeating digits, so this means that $x + 1 = y + 1$, adn so we have $x = y$. So, $f(x) = f(y) \implies x = y$, so $f : \{0, ..., 9\} \to \{0, ..., 9\}$ is injective.

Proof (onto): If $y \in \{0, ..., 9\}$ is in the range of f, then y is a digit of the original X, and thus, y is a digit of X' by assumption of X'. Suppose y is the k^{th} digit of X', so we have that $k \in \mathbb{Z}, 1 \leq k \leq 10$. Consider $f(k - 1)$, with $0 \leq (k - 1) \leq 9$, which means that $k - 1 \in \{0, ..., 9\}$ is in the domain. Then, we also have that $f(k - 1) =$ the $(k - 1 + 1)^{th} = k^{th}$ digit of X', which by our assumption, is equal to y. So, $f(k - 1) = y$, which means that y has a pre-image. Thus every $y \in \{0, ..., 9\}$ has a pre-image, so $f : \{0, ..., 9\} \to \{0, ..., 9\}$ is onto.

Thus, $f : \{0, ..., 9\} \to \{0, ..., 9\}$ is both one-to-one and onto, so it is a bijection.