**1. TRUE or FALSE?: 2pts each**

For each of the questions below, answer TRUE or FALSE.

**Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!**

1. $(\neg P \implies R) \land (\neg P \implies \neg R) \equiv P$

   ◯ True

   ◯ False

2. $\forall x \in \mathbb{N}, (P(x) \land (\exists y \in \mathbb{N}, Q(x,y)) \equiv \forall y \in \mathbb{N}, \exists x \in \mathbb{N}, P(x) \land Q(x,y)$.

   ◯ True

   ◯ False

3. $(\neg P(0) \land \forall n \in \mathbb{N}, (P(n) \implies P(n-1))) \equiv \forall n \in \mathbb{N}, \neg P(n)$

   ◯ True

   ◯ False

4. $\forall x, ((P(x) \implies Q(x)) \land Q(x)) \equiv \forall x, P(x)$

   ◯ True

   ◯ False

5. $P \lor Q \equiv \neg P \implies Q$

   ◯ True

   ◯ False

**For the following two parts, assume that $Q(x,y)$ and $P(x)$ are predicates when $x, y \in \mathbb{N}$.**

6. "$\forall x \in \mathbb{N}, P(x)$" is a proposition.

   ◯ True

   ◯ False

7. "$\forall x \in \mathbb{N}, P(x) \land Q(x,y)$" is a proposition.

   ◯ True

   ◯ False

8. In a stable marriage instance where there is a man at the bottom of each woman's preference list, the man is paired with his least favorite woman in every stable pairing.

   ◯ True

   ◯ False

9. In a stable marriage instance where there is a man at the top of each woman's preference list, the man is paired with his favorite woman in every stable pairing.

○ True

○ False

10. Say I take a walk in any connected graph, making sure I only follow edges I haven't followed before, and keep walking until there are no edges I haven't followed incident to the the vertex I am in. If in the course of this walk I never see the same vertex twice then the degree of the vertex I get stuck at has degree exactly one.

○ True

○ False

11. For the stable roommates problem with $2n$ people, there is a stable pairing if every preference list has the form $\{p_1 > \cdots > p_{2n}\}$, that is, $i < j \implies p_i > p_j$.

○ True

○ False

12. A graph with $k$ edges and $n$ vertices has a vertex of degree at least $2k/n$.

○ True

○ False

13. If we remove one edge from $K_n$, the resulting graph can be vertex colored with $n-1$ colors.

○ True

○ False

14. For a graph with average degree $k$, more than half of the vertices must have degree at most $k$.

○ True

○ False

15. If one adds an edge to a graph $G$ and vertex colors it with 3 colors, then the original graph $G$ is 3-colorable.

○ True

○ False

16. Any graph where $|E| \leq 3|V| - 6$ is planar.

○ True

○ False

17. For every connected, undirected graph there is a tour that uses every edge at least once and at most twice.

◯ True

◯ False

18. Any connected graph with average degree strictly less than 2 is a tree.

◯ True

◯ False

19. Any walk in a hypercube that always traverses unused edges must return to the starting vertex before getting stuck at a vertex where all the edges have been used.

◯ True

◯ False

20. Any complete graph has a Hamiltonian tour. (Recall that a Hamiltonian tour is a cycle that visits every **vertex** exactly once.)

◯ True

◯ False

21. Any graph where every triple of vertices is a triangle, i.e., for vertices $u,v,w \in V$, $(u,v),(v,w),(w,u) \in E$, is a complete graph.

◯ True

◯ False

22. If $\gcd(x,y) = d$ and $gcd(y,z) = c$, then $gcd(x,z) \geq gcd(c,d)$.

◯ True

◯ False

23. The map $f(x) = ax \pmod{m}$ is a bijection from and to $\{0,\ldots,m-1\}$ when $gcd(a,m) = 1$. (A function $f : A \to B$ is a bijection from $A$ to $B$ if it is it is one-to-one, $f(x) \neq f(x')$ for $x \neq x'$, and for $y \in B$, there is a $x \in A$, where $f(x) = y$.)

◯ True

◯ False

**2. An expression or number: 3 points each. Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!**

1. What is the minimum number of colors that are required to vertex color a graph that is a simple cycle of length $k$ for $k \geq 2$ ? (**Answer is an expression possibly involving $k$.**)

2. If $\gcd(x, y) = d$ then the smallest common multiple of $x$ and $y$ is?

3. Let $n$ be the largest number that $K_n$ is a planar graph. **How many faces** does any planar drawing of this graph have? (**Answer is an expression or number.**)

4. Consider a planar graph $G$ with $e$ edges where each face is a triangle. How many vertices does it have? (**Answer is an expression or number.**)

5. How **many solutions** to $5x = 25 \pmod{27}$? (**Answer is a number and recall that we are working modulo 27 so the maximum number of solutions is 27.**)

6. What day of the week is 10 years from today? (Note that there are two leap years between now and then.)

7. If we divide an $n$ dimensional hypercube into two disjoint $n-1$ dimensional hypercubes, how many edges in the $n$ dimensional hypercube have an endpoints in two different hypercubes. **In other words, how many edges "cross" between the subcubes.** (**Answer is an expression.**)

8. Consider the graph $G$ formed with vertex set $V = \{0, \ldots, m-1\}$, and edge set $E = \{(x,y) : y = x + g \pmod{m}\}$ with $g \in \{0, \ldots, m/2 - 1\}$. What is the maximum length cycle in this graph? (**Answer is an expression. It may involve $g$ and $m$.** )

9. Recall that any graph where $e > 3v - 6$ is non-planar. Given an example of a graph where $e \leq 3v - 6$ that is non-planar.

10. Consider an $n$ vertex planar graph with no degree one or two vertices. What is the minimum number of edges in such a graph? (**Answer is an expression perhaps involving $n$.**)

11. Give an example of a stable roommates instance with at least two stable pairings.

12. Show how in any 2 by 2 instance of a stable marriage problem, the women can collaborate to produce a women optimal pairing. (Note: the men must follow the TMA algorithm, but the women could reject men who propose in an arbitrary fashion.) **Answer is short argument.**

**3. Expression Proofs: 6/6/6**

1. Prove: $n^2 \not\equiv 1 \pmod 7 \implies n \not\equiv 1 \pmod 7$

2. Prove: $\forall n \in \mathbb{N}, n \geq 2 \implies (1-1/4)(1-1/9)\cdots(1-1/n^2) = \frac{n+1}{2n}$.

3. Use induction to prove that $1 + \frac{1}{2} + \cdots (\frac{1}{2})^n \leq 2$? (Hint: strengthen the statement.)

**4. Graphs and modular: 5/5/5 points.**

1. The integer $a$ is a quadratic residue of $n$ if $\gcd(a,n) = 1$ and $x^2 \equiv a \pmod{n}$ has a solution. Prove that if $p$ is a prime number, $p \neq 2$, then there are $(p-1)/2$ quadratic residues of $p$ among $\{1,\ldots,p-1\}$.

2. Consider a directed graph where every pair of vertices $u$ and $v$ are connected by a single directed arc either from $u$ to $v$ or from $v$ to $u$. Show that every vertex has a directed path of length at most two to **the vertex with maximum in-degree.** Note that this is quite similar to a homework problem but asks for a more specific answer. (Hint: Our solution doesn't require induction.)

3. Consider a simple $n$-vertex graph that contains a path, $v_1, \ldots, v_n$, of length $n$ and where the degrees of $v_1$ and $v_n$ are at least $n/2$. Show the graph has a Hamiltonian cycle. (Recall a Hamiltonian cycle is one that visits each vertex exactly once.)

**5. Reverse Preference Stable Marriage: 10 points.**

A witch bewitches all the men just before a run of TMA so they propose in reverse order of their preference lists (equivalent to the men giving reversed lists as inputs to TMA). The spell wears off after the algorithm is run and a pairing is obtained. (Note : ONLY MALES reverse their preference lists and after the spell wears off the men remember their original preferences)

1. Find the output pairings for the following two instances:

| **Instance 1:** | **Instance 2:** |
|---|---|
| **Men** | **Men** |
| A:1,2 | A:1,2 |
| B:1,2 | B:2,1 |
| **Women** | **Women** |
| 1:B,A | 1:A,B |
| 2:A,B | 2:B,A |

Answers only

2. Prove that the pairing is either unstable or all the women got their first choice male. (**Hint:** Consider how to translate the notion of stability for the TMA pairing in this case. There is no couple $(w^*, m)$ such that $w^*$ has $m$ higher than her partner $m^*$ and $m$ has $w^*$ higher on his reversed list than his partner $w$. )

**6. Cycles, pairings, and circle of worry: 10 points.**

1. Argue that any directed simple graph where every vertex has out-degree at least one has a directed cycle.

2. Consider the graph formed with vertices corresponding to the men and women in a stable marriage instance and edges according to two different stable pairings, $S$ and $S'$. If a pair is in both pairings only include a single edge in $G$, which ensures it is a simple graph. Argue that there is a cycle of length strictly greater than 2.

3. Define a man $m$ as **feeling threatened by** another man $m'$ with respect to a pairing $S$ if $(m,w)$ is in $S$ and $w$ likes $m'$ better than $m$. We define the **male feeling threatened graph** for a stable pairing $S$ as the **directed** graph whose vertices are men and with a directed arc for each pair $(m,m')$ where $m$ is feeling threatened by $m'$. Show that the male feeling threatened graph for the male optimal pairing has a cycle if there is more than one pairing. (Hint: using the previous parts may be helpful.)