6 Analyze a Markov Chain

(a) Direct Proof

Given this case with $a, b \in (0,1)$, so the Markov chain can go from state 0 to state 0 in any of the $n \geq 1$ steps, i.e. $n = \{1, 2, 3, \dots\}$, so we have

$$d(0) = gcd\{1, 2, 3, \cdots\} = 1$$

Similarly, we have that $d(1) = 1$ and $d(2) = 2$. Therefore,

$$d(i) = 1 \; \forall \; i \in \mathscr{X}$$

which implies, by definition, that the given Markov chain is aperiodic.

Q.E.D.

(b) $a(1-b)(1-a)a = a^2(1-a)(1-b)$

Given that $X(0) = 0$, so

$$\mathbb{P}[X(1) = 1 \mid X(0) = 0] = a$$

and similarly

$$\mathbb{P}[X(2) = 0 \mid X(1) = 1] = 1 - b$$

$$\mathbb{P}[X(3) = 0 | X(2) = 0] = 1 - a$$

$$\mathbb{P}[X(4) = 1 \mid X(3) = 0] = a$$

Thus, we have that $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0] =$
$\mathbb{P}[X(1) = 1 \mid X(0) = 0] \cdot \mathbb{P}[X(2) = 0 \mid X(1) = 1] \cdot \mathbb{P}[X(3) = 0 | X(2) = 0] \cdot \mathbb{P}[X(4) = 1 \mid X(3) = 0]$

$$\implies \mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0] = a(1-b)(1-a)a = a^2(1-a)(1-b)$$

(c) $\pi = (\frac{1-b}{ab+a-b+1}, \frac{a}{ab+a-b+1}, \frac{ab}{ab+a-b+1})$

First, we calculate that the transition probability matrix for the given Markov chain is:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1-a & a & 0 \\ 1-b & 0 & b \\ 0 & 1 & 0 \end{bmatrix}$$

Then, we have that the balance equations can be set up with $\pi \mathbf{P} = \pi$, and so

$$[\pi_0, \pi_1, \pi_2] = [\pi_0, \pi_1, \pi_2] \cdot \begin{bmatrix} 1-a & a & 0 \\ 1-b & 0 & b \\ 0 & 1 & 0 \end{bmatrix}$$

Then, since the components of π sum up to one, so we have 4 linear equations in total:

$$(1-a)\pi_0 + (1-b)\pi_1 + 0 \cdot \pi_2 = \pi_0$$

$$a \cdot \pi_0 + 0 \cdot \pi_1 + 1 \cdot \pi_2 = \pi_1$$

$$0 \cdot \pi_0 + b \cdot \pi_1 + 0 \cdot \pi_2 = \pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

We could then solve them to get the invariant distribution as:

$$\pi_0 = \frac{1-b}{ab+a-b+1}$$

$$\pi_1 = \frac{a}{ab+a-b+1}$$

$$\pi_2 = \frac{ab}{ab+a-b+1}$$

(d) $\frac{ab-b+1}{ab}$

Since $X(0) = 1$, so $\mathbb{P}[X(1) = 2|X(0) = 1] = b, \mathbb{P}[X(1) = 1|X(0) = 1] = 0$, and $\mathbb{P}[X(1) = 0|X(0) = 1] = 1 - b$. Now, since we want the expectation of the number of steps until we transit to state 2 for the first time, so we need to further examine the condition of $X(1) = 0$.

Since for $i \in \mathbb{N}$, we have $\mathbb{P}[X(i+1) = 0|X(i) = 0] = 1 - a$, $\mathbb{P}[X(i+1) = 1|X(i) = 0] = a$ and $\mathbb{P}[X(i+1) = 2|X(i) = 0] = 0$, so we have that
$\mathbb{E}[T_2|X(0) = 1] =$
$\mathbb{P}[X(1) = 0|X(0) = 0] \cdot \mathbb{E}[X(1) = 0|X(0) = 0] + \mathbb{P}[X(1) = 1|X(0) = 0] \cdot \mathbb{E}[X(1) = 1|X(0) = 0] + \mathbb{P}[X(1) = 2|X(0) = 0] \cdot \mathbb{E}[X(1) = 2|X(0) = 0] =$
$(1-a) \cdot (1 + \mathbb{E}[T_2 \mid X(0) = 1]) + a \cdot (1 + \mathbb{E}[T_2 \mid X(0) = 1])$

which can be simplified a bit to:

$$\mathbb{E}[T_2 \mid X(0) = 1] = \mathbb{E}[T_2 \mid X(0) = 1] + \frac{1}{a}$$

Thus, we can combine this result with our previous setup to get:

$$\mathbb{E}[T_2 \mid X(0) = 1]) = b \cdot 1 + 0 + (1-b) \cdot \big(\mathbb{E}[T_2 \mid X(0) = 1] + \frac{1}{a}\big)$$

$$\implies \mathbb{E}[T_2 \mid X(0) = 1]) = \frac{ab-b+1}{ab}$$