4 Berlekamp-Welch for General Errors

(a) $\deg(E(x)) = 1$, $\deg(Q(x)) = 3$.

Since there's only one error, so $k = 1$ here, so the degree of $E(x) = k = 1$

Now, since Hector wants to send a length $n = 3$ message, so the degree of $Q(x) = n + k - 1 = 3$

Using the given relation, so we can write:

$E(x) = x - e_1$
$Q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

Then, since $Q(i) = r_iE(i)$ for $0 \leq i < 5$, and with $E(x) = x - e_i = x + b_0$, so we have that:

$Q(0) = r_0E(0)$, which gives: $a_0 = 3(0 + b_0) = 3b_0$
$Q(1) = r_1E(1)$, which gives: $a_3 + a_2 + a_1 + a_0 = 7(1 + b_0) = 7 + 7b_0$
$Q(2) = r_2E(2)$, which gives: $8a_3 + 4a_2 + 2a_1 + a_0 = 0(2 + b_0) = 0$
$Q(3) = r_3E(3)$, which gives: $27a_3 + 9a_2 + 3a_1 + a_0 = 2(3 + b_0) = 6 + 2b_0$
$Q(4) = r_4E(4)$, which gives: $64a_3 + 16a_2 + 4a_1 + a_0 = 10(4 + b_0) = 40 + 10b_0$

(b) $Q(x) = 3x^3 + 6x^2 + 5x + 8, E(x) = x - 1$; error located at position 1.

Suppose we're working over $GF(11)$, and with the system of equations derived above, we can translate them into:

$a_0 + 8b_0 = 0$
$a_3 + a_2 + a_1 + a_0 + 4b_0 = 7$
$8a_3 + 4a_2 + 2a_1 + a_0 = 0$
$5a_3 + 9a_2 + 3a_1 + a_0 + 9b_0 = 6$
$9a_3 + 5a_2 + 4a_1 + a_0 + b_0 = 7$

Since $2 \cdot 6 = 12 = 11 + 1$, so $2^{-1} \equiv 6 \pmod{11}$. Thus, solving this system of linear equations (with the help of Jupyter Notebook) would yield:

$a_3 = -\frac{5}{2} \equiv -5 \cdot 6 = -3 \cdot 11 + 3 \equiv 3 \pmod{11}$,
$a_2 = \frac{1}{2} \equiv 1 \cdot 6 = 6 \pmod{11}$,
$a_1 = 5 \pmod{11}$,
$a_0 = 8 \pmod{11}$,
$b_0 = -1 \pmod{11}$.

Thus, $Q(x) = 3x^3 + 6x^2 + 5x + 8$ and $E(x) = x - 1$; the location of this error is at position $e_1 = -b_0 = 1$.

(c) $P(x) = 3x^2 + 9x + 3$; message = "DEA".

Then, since we're working over $GF(11)$, with $Q(x) = 3x^3 + 6x^2 + 5x + 8$ and $E(x) = x - 1$, so we can calculate $P(x) = \frac{Q(x)}{E(x)} = \frac{3x^3+6x^2+5x+8}{x-1} = 3x^2 + 9x + 3$.

Since we noticed that the first character was corrupted as $e_1 = 1$, so we calculate $P(1) = 3+9+3 = 4 =$ "E", which means that