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1. (a) $x \wedge (x \Rightarrow y) \wedge (\neg y)$ Contradiction.

x	y	$x \Rightarrow y$	$\neg y$	$x \wedge (x \Rightarrow y) \wedge (\neg y)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T.	F

(b). $x \Longrightarrow (x \vee y)$ Tautology

x	y	$x \vee y$	$x \Rightarrow (x \vee y)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(c) $(x \vee y) \vee (x \vee \neg y)$ Tautology

x	y	$\neg y$	$x \vee y$	$x \vee \neg y$	$(x \vee y) \vee (x \vee \neg y)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	T	T

(d) $(x \Rightarrow y) \vee (x \Rightarrow \neg y)$ Tautology

x	y	$\neg y$	$x \Rightarrow y$	$x \Rightarrow \neg y$	$(x \Rightarrow y) \vee (x \Rightarrow \neg y)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	T	T
F	F	T	T	T	T

1\. (e) $(x \vee y) \wedge (\neg(x \wedge y))$. Neither

x	y	$x \vee y$	$x \wedge y$	$\neg(x \wedge y)$	$(x \vee y) \wedge (\neg(x \wedge y))$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

(f) $(x \Rightarrow y) \wedge (\neg x \Rightarrow y) \wedge (\neg y)$. Contradiction.

x	y	$\neg x$	$\neg y$	$x \Rightarrow y$	$\neg x \Rightarrow y$	$(x \Rightarrow y) \wedge (\neg x \Rightarrow y) \wedge (\neg y)$
T	T	F	F	T	T	F
T	F	F	T	F	T	F
F	T	T	F	T	T	F
F	F	T	T	T	F	F

2 Miscellaneous Logic

(a)

(i) Possibly true.

false example:

Let $G(x,y) : y = x + 2$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
However, since $3 + 2 = 5 \neq 4$, so $G(3,4)$ is false.

true example:

Let $G(x,y) : y = x + 1$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
And, since $3 + 2 = 4$, so $G(3,4)$ is true.

(ii) Possibly true.

false example:

Let $G(x,y) : y = x + 2$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
However, consider $x = 0$.
Since $0 + 2 = 2 \neq 3$, so $(\forall x \in \mathbb{R})\ G(x,3)$ is false.

true example:

Let $G(x,y) : y = 3$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
Since the statement given indicates that y is always 3,
So, $(\forall x \in \mathbb{R})\ G(x,3)$ is true.

(iii) Certainly true.

Since the statement given, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ G(x,y)$, is true, and since $3 \in \mathbb{R}$, so there must exist a $y \in \mathbb{R}$ such that $G(3,y)$ is true. Thus, $\exists y\ G(3,y)$ is a true statement.

(iv) Certainly false.

Since the statement given, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ G(x,y)$, is true, and since $3 \in \mathbb{R}$, so there must exist a $y \in \mathbb{R}$ such that $G(3,y)$ is true, which means that $\forall y\ \neg G(3,y)$ is a false statement.

(v) Possibly true.

false example:

Let $G(x,y) : y = 3$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
So $\forall x \in \mathbb{R},\ y = 3$, which means that there's no x such that $y = 4$.
So $\exists x\ G(x,4)$ is false.

true example:

Let $G(x,y) : y = x + 2$, so $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ is true for $G(x,y)$.
Consider $x = 2, x \in \mathbb{R}$.
Since $2\ +\ 2 = 4$, so $\exists x\ G(x,4)$ is true.

(b)

$$\Big(X \vee Y \vee Z\Big) \wedge \Big(\neg\,(X \wedge Y) \vee (Y \wedge Z) \vee (Z \wedge X)\Big)$$

3 Propositional Practice

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$

True.
Consider $x = \pi$. $\pi \in \mathbb{R}$, and $\pi \notin \mathbb{Q}$, so the proposition is true.

(b) $(\forall x \in \mathbb{Z})\ \Big(\big((x \in \mathbb{N}) \vee (x < 0)\big) \wedge \big(\neg\big((x \in \mathbb{N}) \wedge (x < 0)\big)\big)\Big)$

True.
Let $x \in \mathbb{Z}$, so $x >= 0$ or $x < 0$, but not both.
If $x >= 0$, then x is a natural number; if $x < 0$, then x is negative; x can't be both.
Thus, the proposition is true.

(c) $(\forall x \in \mathbb{N})\ \big((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x))\big)$

True.
Let $x \in \mathbb{N}$, $x = 6 * k$, so $k \in \mathbb{N}$
So $x = 2 * (3k)$ where $3k \in \mathbb{N}$, which means that $2 \mid x$
So $((2 \mid x) \vee (3 \mid x))$ is true, which means that the proposition is true.

(d) All real numbers are complex numbers.

True.
Let $x \in \mathbb{R}$, so $x = x + 0 * i$, and since $x, 0 \in \mathbb{R}$,
So by definition of complex numbers, x is a complex number.

(e) If an integer is divisible by 2 or is divisble by 3, then it is divisible by 6.

False.
Consider $x = 2$, so x is an integer.
Since x is divisible by 2, so it is divisible by 2 or by 3.
However, there's no such integer a such that $2 * a = 6$
So by definition, x is not divisble by 6, so the proposition is false.

(f) If a natural number is greater than 7, then it can be expressed as the sum of two natural numbers.

True.
Let $x \in \mathbb{N}, x > 7$
Consider $a = 0, b = x$, so $a, b \in \mathbb{N}$
Thus, since $a + b = 0 + x = x$, so the proposition is true.

4 Proof by?

(a)

We proceed by contradiction. Assume that the proposition is false, which means that for some $x, y \in \mathbb{Z}, (10 \nmid xy)$, and that $((10 \mid x)$ or $(10 \mid y))$. Let our assertion R state that $10 \nmid xy$.

Without loss of generality, let $10 \mid x$, so let $x = 10k,\ k \in \mathbb{Z}$. So $xy = 10ky = 10(ky)$ where $ky \in \mathbb{Z}$, which by definition, means that $10 \mid xy$. This implies $\neg R$.

We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired.

Q.E.D.

I used Proof by Contradiction.

(b)

Prove. The contrapositive proposition is: $(\forall x, y \in \mathbb{Z})\ ((10 \mid x) \vee (10 \mid y)) \implies (10 \mid xy)$.

As shown in part (a) above, if $((10 \mid x) \vee (10 \mid y))$, then $10 \mid xy$. Thus, the contrapositive is true.

Q.E.D.

(c)

Disprove. The converse proposition is: $(\forall x, y \in \mathbb{Z})\ ((10 \nmid x) \wedge (10 \nmid y)) \implies (10 \nmid xy)$.

Consider $x = 2, y = 5$, so $x, y \in \mathbb{Z}$. Since there's no such integer m, n such that $10*m = 2$ or $10*n = 5$, so by definition, x, y is not divisible by 10. So $((10 \nmid x) \wedge (10 \nmid y))$ is true.

Now, $xy = 2*5 = 10 = 1*10$ where $1 \in \mathbb{Z}$, so by definition, $10 \mid xy$. So $(10 \nmid xy)$ is false.//[.1cm] So we have $True \implies False$, which shows that the converse is false.

Q.E.D.

5 Proof or Disprove

(a) Prove. Direct Proof

Let $n \in \mathbb{N}$ be an odd number, so let $n = 2k+1, k \in \mathbb{N}$.

So $n^2 + 2n = (2k+1)^2 + 2 * (2k+1) = 4k^2 + 4k + 1 + 4k + 2 = 4k^2 + 8k + 3 = 2 * (2k^2 + 4k + 1) + 1$
Since $k \in \mathbb{N}$, so $(2k^2 + 4k + 1) \in \mathbb{N}$, so $n^2 + 2n$ is odd.

Thus, the proposition is true.

Q.E.D.

(b) Prove. Proof by Cases

Let $x, y \in \mathbb{R}$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (1) $x >= y$; or (2) $x < y$.

Case (1): Since $x >= y$, so $|x - y| = x - y$ and $\min(x, y) = y$.

So $(x + y - |x - y|)/2 = (x + y - x + y)/2 = (2y)/2 = y = \min(x, y)$

Case (2): Since $x < y$, so $|x - y| = -x + y$, and $\min(x, y) = x$

So $(x + y - |x - y|)/2 = (x + y + x - y)/2 = (2x)/2 = x = \min(x, y)$

Thus, $\min(x, y) = (x + y - |x - y|)/2$

Q.E.D.

(c) Prove. Proof by Contradiction

We proceed by contradiction. Assume that the proposition is false, which means that for some $a, b \in \mathbb{R}, (a + b <= 10)$, and that $((a <= 7)$ or $(b <= 3))$ is false. Let our assertion R state that $(a + b <= 10)$.

Since $((a <= 7)$ or $(b <= 3))$ is false, so $(a > 7)$ and $(b > 3)$. So $a + b > 7 + 3 > 10$.

This implies $\neg R$. We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired.

Thus, the proposition is true.

Q.E.D.

(d) Prove. Proof by Contradiction

We proceed by contradiction. Assume that the proposition is false, which means that for some $r \in \mathbb{R}$, r is irrational and $r+1$ is rational. Let our assertion R state that r is irrational. Since $r+1$ is rational, by definition, let $r + 1 = \frac{p}{q}$ such that $p, q \in \mathbb{Z}$. So $r = r + 1 - 1 = \frac{p}{q} - 1 = \frac{p-q}{q}$. Since $p - q, q \in \mathbb{Z}$, so by definition, r is rational.

This implies $\neg R$. We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired.

Thus, the proposition is true.

Q.E.D.

(e) Disprove.

Consider $n = 6 \in \mathbb{Z}^+$.

So $10n^2 = 10 * 6^2 = 360$, and $n! = 6! = 720$.

Since $360 < 720$, so $10n^2 > n!$ is false for $n = 6 \in \mathbb{Z}^+$.

Thus, the proposition is false.

Q.E.D.

6 Preserving Set Operations

(a)

We would show two parts: (1) $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$; and (2) $(f^-1(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$.

Part (1): For any element $e \in f^{-1}(A \cup B)$, by definition of inverse images, so $f(e) \in A \cup B$, so $f(e) \in A$ or $f(e) \in B$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (i) $f(e) \in A$; or (ii) $f(e) \notin A$.

Case (i): Since $f(e) \in A$, so by definition, $e \in f^{-1}(A)$, so $e \in (f^{-1}(A) \cup f^{-1}(B))$

So $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$

Case (ii): Since $f(e) \notin A$, and since $f(e) \in A$ or $f(e) \in B$, so $f(e) \in B$.

So by definition, $e \in f^{-1}(B)$, so $e \in (f^{-1}(A) \cup f^{-1}(B))$

So $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$

Thus, we have that $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \cup f^{-1}(B))$, so $e \in f^{-1}(A)$ or $e \in f^{-1}(B)$. We proceed by cases. Let us divide our proof into two cases, exactly one of which must be true: (i) $e \in f^{-1}(A)$; or (ii) $e \notin f^{-1}(A)$.

Case (i): Since $e \in f^{-1}(A)$, so by definition, $f(e) \in A$, so $f(e) \in A \cup B$.

So by definition, $e \in f^{-1}(A \cup B)$.

Case (ii): Since $e \notin f^{-1}(A)$, and since $e \in f^{-1}(A)$ or $e \in f^{-1}(B)$, so $e \in f^{-1}(B)$.

So by definition, $f(e) \in B$, so $f(e) \in A \cup B$.

So by definition, $e \in f^{-1}(A \cup B)$.

Thus, we have that $(f^{-1}(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$.

Since $f^{-1}(A \cup B) \subseteq (f^{-1}(A) \cup f^{-1}(B))$ and $(f^{-1}(A) \cup f^{-1}(B)) \subseteq f^{-1}(A \cup B)$, so we have that $f^{-1}(A \cup B) = (f^{-1}(A) \cup f^{-1}(B))$.

Q.E.D.

(b)

We would show two parts: (1) $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$; and (2) $(f^-1(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$.

Part (1): For any element $e \in f^{-1}(A \cap B)$, by definition of inverse images, so $f(e) \in A \cap B$, so $f(e) \in A$ and $f(e) \in B$. Since $f(e) \in A$, by definition, so $e \in f^{-1}(A)$. Similarly, $e \in f^{-1}(B)$. So we have $e \in (f^{-1}(A) \cap f^{-1}(B))$, which implies that $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \cap f^{-1}(B))$, so $e \in f^{-1}(A)$ and $e \in f^{-1}(B)$. Since $e \in f^{-1}(A)$, so by definition of inverse images, $f(e) \in A$. Similarly, $f(e) \in B$. So $f(e) \in A \cap B$. So by definition, $e \in f^{-1}(A \cap B)$, which implies that $(f^-1(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$.

Since $f^{-1}(A \cap B) \subseteq (f^{-1}(A) \cap f^{-1}(B))$ and $(f^{-1}(A) \cap f^{-1}(B)) \subseteq f^{-1}(A \cap B)$, so we have that $f^{-1}(A \cap B) = (f^{-1}(A) \cap f^{-1}(B))$.

Q.E.D.

(c)

We would show two parts: (1) $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$; and (2) $(f^-1(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$.

Part (1): For any element $e \in f^{-1}(A \setminus B)$, by definition of inverse images, so $f(e) \in A \setminus B$, so $f(e) \in A$ and $f(e) \notin B$. Since $f(e) \in A$, by definition, so $e \in f^{-1}(A)$. Similarly, $e \notin f^{-1}(B)$. So we have $e \in (f^{-1}(A) \setminus f^{-1}(B))$, which implies that $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$.

Part (2): For any element $e \in (f^{-1}(A) \setminus f^{-1}(B))$, so $e \in f^{-1}(A)$ and $e \notin f^{-1}(B)$. Since $e \in f^{-1}(A)$, so by definition of inverse images, $f(e) \in A$. Similarly, $f(e) \notin B$. So $f(e) \in A \setminus B$. So by definition, $e \in f^{-1}(A \setminus B)$, which implies that $(f^-1(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$.

Since $f^{-1}(A \setminus B) \subseteq (f^{-1}(A) \setminus f^{-1}(B))$ and $(f^{-1}(A) \setminus f^{-1}(B)) \subseteq f^{-1}(A \setminus B)$, so we have that $f^{-1}(A \setminus B) = (f^{-1}(A) \setminus f^{-1}(B))$.

Q.E.D.

(d)

We would show two parts: (1) $f(A \cup B) \subseteq (f(A) \cup f(B))$; and (2) $(f(A) \cup f(B)) \subseteq f(A \cup B)$.

Part (1): Consider any element $e \in f(A \cup B)$. By definition of images, so there exists some $x \in A \cup B$ such that $e = f(x)$. WLOG, let $x \in A$. So by definition, $e \in f(A)$, so $e \in f(A) \cup f(B)$, which implies that $f(A \cup B) \subseteq (f(A) \cup f(B))$.

Part (2): Consider any element $e \in f(A) \cup f(B)$. WLOG, let $e \in f(A)$. By definition, so $\exists x \in A$ such that $e = f(x)$. Since $x \in A$, so $x \in A \cup B$, so by definition, $e \in f(A \cup B)$, which implies that $(f(A) \cup f(B)) \subseteq f(A \cup B)$.

Thus, we have that $f(A \cup B) = (f(A) \cup f(B))$.

Q.E.D.

(e)

For any element $e \in f(A \cap B)$, by definition of images, so there exists some $x \in A \cap B$ such that $e = f(x)$, so $x \in A$ and $x \in B$. Since $e = f(x)$ and $x \in A$, again by definition, we have $e \in f(A)$. Similarly, $e \in f(B)$, so $e \in f(A) \cap f(B)$, which implies that $f(A \cap B) \in f(A) \cap f(B)$.

An example where the equality does not hold:

Consider $f(x) = x^2, A = \{0, 2\}, B = \{0, -2\}$.

So $A \cap B = \{0\}$. By definition of images, we have $f(A \cap B) = \{0\}, f(A) = \{0, 4\}, f(B) = \{0, 4\}$, so $f(A) \cap f(B) = \{0, 4\}$, which gives that $f(A \cap B) \neq f(A) \cap f(B)$.

Q.E.D.

(f)

For any element $e \in f(A) \setminus f(B)$, so $e \in f(A)$ and $e \notin f(B)$. By definition of images, there exists some $x \in A$ such that $e = f(x)$. Similarly, there's no such $y \in B$ such that $e = f(y)$, which implies that $x \notin B$, which means that $x \in A \setminus B$. And since $e = f(x)$, so by definition, $e \in f(A \setminus B)$, which implies that $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

An example where the equality does not hold:

Consider $f(x) = x^2, A = \{0, 2\}, B = \{-2\}$.

So $A \setminus B = \{0, 2\}$. By definition of images, we have $f(A \setminus B) = \{0, 4\}, f(A) = \{0, 4\}, f(B) = \{4\}$, so $f(A) \setminus f(B) = \{0\}$, which gives that $f(A \setminus B) \neq f(A) \setminus f(B)$.

Q.E.D.