1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(k) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(m) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6}\binom{39}{7}$ ways to make up the hand.

(c) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

(d) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$. Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = (1/2)\cdot 2^{99} = 2^{98}$.

Answer 2: Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = (1/2) \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(e) This is the number of ways of rearranging 7 distinct letters and is $7!$.

In this 6 letter word, the letter A is repeated 3 times while the other letters appear once. Hence, the number $6!$ overcounts the number of different anagrams by a factor of $3!$ (which is the number of ways of permuting the 3 A's among themselves). Hence, there are $6!/3!$ different anagrams.

In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Hence, there are $7!/4!$ anagrams.

In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $7!/(2!)^2$ different anagrams.

(f) (1) We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is $5!$. (2) Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \bigcup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = 6!/2$.

(g) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.

(h) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins.

From class (see notes 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k=2$ and $n=7$, we get $\binom{8}{2}$ ways to do this.
EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k=9$ and $n=27$, so there are $\binom{35}{9}$ ways.

(j) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20!/(10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(k) $\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

(l) $n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1,2,\ldots,n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(m) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Binomial Beads

(a) Alistair is making school spirit keychains, which consist of a sequence of n beads on a string. He has blue beads and gold beads. How many unique keychains can he make with exactly $k \le n$ blue beads?

(b) Alistair decides to sell his keychains! He decides on the following pricing scheme:

- Blue beads have a value of x
- Gold beads have a value of y
- The price of a keychain is the product of the values of all of its beads.

What is the price of a keychain with exactly $k \le n$ blue beads?

(c) Alistair decides to make exactly one of every possible unique keychain. If he sells every keychain he creates, how much revenue will he make? Use parts (a) and (b), and leave your answer in summation form.

(d) Draw a connection between part (c) and the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Hint: How do you calculate the product (x + y)(x + y)?

Solution:

(a) $\binom{n}{k}$. We choose k locations for the blue beads to go.

(b) $x^k y^{n-k}$.

(c) Alistair can place $0-n$ blue beads on a string. Using the above parts, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(d) The expansion of the LHS results in the sum of many terms, with each term representing the product of a string of x's and y's. We can think of the LHS as summing up the prices of all the keychains individually. The RHS, as calculated in part (c), is also the sum of the prices of each unique bracelet, but grouped by the number of "blue beads".

3 Minesweeper

Minesweeper is a game that takes place on a grid of squares. When you click a square, it disappears to reveal either an integer $\in [1,8]$, a mine, or a blank space. If it reveals a mine, you instantly lose. If it reveals a number, that number refers to the number of mines adjacent to that square (including diagonally adjacent). If it reveals a blank space, there were 0 mines adjacent to it.

You are playing on a 8x8 board with 10 mines randomly distributed across the board. In your first move, you click a square near the center of the board.

(a) What is the probability that the square reveals...

  i. a mine?

  ii. a blank space?

  iii. the number k?

(b) The first square you picked revealed the number k. For your next move, you want to minimize the probability of picking a mine. Should you pick a square adjacent to your first pick, or a different square? Your answer should depend on the value of k.

(c) Your first move resulted in the number 1. You pick the square to the right for your next move. What is the probability that this square reveals the number 4?

Solution:

(a) i. There are 10 mines and 64 squares, so the probability of a square being a mine is $\frac{10}{64}$

  ii. This is the probability that the picked square and its 8 adjacent squares are not mines. Then, we calculate the probability that all 10 mines are among the other 55 squares. $\frac{\binom{55}{10}}{\binom{64}{10}}$

  iii. $\frac{\binom{8}{k}\binom{55}{10-k}}{\binom{64}{10}}$. We choose locations for the k adjacent mines and locations for the remaining $10-k$ mines. The denominator is the total number of possible arrangements of mines.

(b) The probability of picking a mine if you click an adjacent square is $\frac{k}{8}$. The probability of picking a mine if you click a different square is $\frac{10-k}{55}$. You should pick an adjacent square if $\frac{k}{8} \leq \frac{10-k}{55}$. This occurs only when $k = 1$.

(c) The square to the right will share 4 neighbors with the original square. In order to reveal the number 4, one of the mutual neighbors must be a mine. The three new neighbors must also be mines.

The probability that one of the mutual neighbors is a mine is $\frac{1}{2}$. Given that one of the mutual neighbors is a mine, the probability that the three new neighbors are also all mines is $\frac{\binom{52}{6}}{\binom{55}{9}}$ The probability that both these events occur must then be:

$$\frac{1}{2} \times \frac{\binom{52}{6}}{\binom{55}{9}}$$

4 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. The last person standing wins the game. What is Bob's best course of action regarding his first shot?

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

(c) Compute the probability of the same events for a duel of Bob against Carol.

(d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)

Solution:

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.

Observe that:

$$\begin{aligned}
\mathbb{P}[E_1] &= \mathbb{P}[\text{Bob hits Eve }] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[\text{Eve misses Bob}]\mathbb{P}[E_1] \\
&= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}\mathbb{P}[E_1]
\end{aligned}$$

Thus, $\mathbb{P}[E_1] = \frac{3}{7}$.

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

Observe that:

$$\begin{aligned}
\mathbb{P}[E_2] &= \mathbb{P}[\text{Eve misses Bob}]\left(\mathbb{P}[\text{Bob hits Eve}] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[E_2]\right) \\
&= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\mathbb{P}[E_2]\right)
\end{aligned}$$

Thus, $\mathbb{P}[E_2] = \frac{1}{7}$.

(c) Compute the probability of the same events for a duel of Bob against Carol alone.

The probability of the event E_3 that Bob, with shot, survives against Carol is:

$$\begin{aligned}\mathbb{P}[E_3] &= \mathbb{P}[\text{Bob hits Carol}] + \mathbb{P}[\text{Bob misses Carol}]\mathbb{P}[\text{Carol misses Bob}]\mathbb{P}[E_3] \\ &= \frac{1}{3} + \frac{2}{3}\cdot 0 \\ &= \frac{1}{3}\ .\end{aligned}$$

The probability of the event E_4 that Bob, without shot, survives against Carol is:

$$\mathbb{P}[E_4] \le \mathbb{P}[\text{Carol misses}] = 0\ .$$

(d) To maximize their chances each player prefers to be left with a weaker opponent. This means that Eve would not shoot at Bob in preference to Carol, and Carol will not shoot at Bob in preference to Eve. Therefore if Bob misses, he will not be shot at until either Eve or Carol lose and he will either be left standing with Eve or Carol, with or without the shot.

So Bob is best off not shooting anyone since the advantage he gains by having the first shot exceeds any possible benefit of facing Eve rather than Carol. He should shoot into the air.

5 Weathermen

Tom is a weatherman in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.

(a) If Tom says that it is going to snow, what is the probability it will actually snow?

(b) What is Tom's overall accuracy?

(c) Tom's friend Jerry is a weatherman in Alaska. Jerry claims that she is a better weatherman than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny days. How is this possible?

Hint: what is the weather like in Alaska?

Solution:

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned} P(S|T) &= \frac{P(S\cap T)}{P(T)} \\ &= \frac{P(S\cap T)}{P(S\cap T)+P(\overline{S}\cap T)} \\ &= \frac{.1\times .7}{.1\times .7+.9\times .05} \end{aligned}$$

(b)

$$\begin{aligned} P(\text{Tom is correct}) &= P(S\cap T)+P(\overline{S}\cap\overline{T}) \\ &= .1\times .7+.9\times .95 \end{aligned}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better weatherman if the weather is different.

For example, let's assume that it snows 50% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Bob's even though she is better at predicting both categories!

For more info on this kind of phenomena, check out Simpson's Paradox!