3 Fizzbuzz

(a) $\frac{8}{15}n$

If $15 \mid n$, then there will be exactly $\frac{n}{3}$ multiples of 3 from 1 to n, $\frac{n}{5}$ multiples of 5 from 1 to n, and $\frac{n}{15}$ multiples of 15 from 1 to n.

Let A_1 be the event that picking an integer between 1 and n is a multiple of 3, and let A_2 be the event that picking an integer between 1 and n is a multiple of 5, so $A_1 \cup A_2$ is the event that picking an integer between 1 and n that is a multiple of 15. Thus, we have $\mathbb{P}[A_1] = \frac{\frac{n}{3}}{n} = \frac{1}{3}, \mathbb{P}[A_2] = \frac{\frac{n}{5}}{n} = \frac{1}{5}$, and $\mathbb{P}[A_1 \cup A_2] = \frac{\frac{n}{15}}{n} = \frac{1}{15}$.

Thus, the probability of randomly choosing an integer that printed words (i.e. multiple of 3 or 5) is $\mathbb{P}[\text{word}] = \mathbb{P}[U_{i=1}^{2} A_i] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cup A_2] = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$, which means that $\mathbb{P}[\text{integer}] = \mathbb{P}[\overline{\text{word}}] = 1 - \mathbb{P}[\text{word}] = \frac{8}{15}$.

Since the size of the sample space $|\Omega| = n$, so the size of the sample space, where the event is that the printed line contains integer, is $|\omega| = |\Omega| \cdot \mathbb{P}[\text{integer}] = \frac{8}{15}n$.

Thus, if n is a multiple of 15, then $\frac{8}{15}n$-many printed lines will contain an integer.

(b) Direct Proof

We proceed by a direct proof. Since the only prime factors of n are $p_1, p_2, \ldots, p_k$, and they're distinct, so we could eliminate the prime factors with a similar procedure/idea from part (a).

Using the Principle of Inclusion-Exclusion (Theorem 14.2), so the probability of randomly picking a line that contains words (not coprime with n) is that

$$\mathbb{P}[\cup_{j=1}^{k} A_j] = \sum_{a_1=1}^{k} \mathbb{P}[A_{a_1}] - \sum_{a_1<a_2} \mathbb{P}[A_{a_1} \cap A_{a_2}] + \sum_{a_1<a_2<a_3} \mathbb{P}[A_{a_1} \cap A_{a_2} \cap A_{a_3}] - \ldots + (-1)^{n-1} \mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_k]$$

$$= \sum_{a_1=1}^{k} \frac{1}{p_{a_1}} - \sum_{a_1<a_2} \frac{1}{p_{a_1} p_{a_2}} + \sum_{a_1<a_2<a_3} \frac{1}{p_{a_1} p_{a_2} p_{a_3}} - \ldots + (-1)^{n-1} \frac{1}{p_1 p_2 \ldots p_k}$$

Thus, the probability of randomly picking a line that contains an integer (i.e. picking a number that's coprime with n) is:

$$\mathbb{P}[integer] = 1 - \mathbb{P}[\cup_{j=1}^{k} A_j] = \prod_{j=1}^{k} (1 - \frac{1}{p_j})$$

.

Therefore, $\mathbb{P}[integer]$ is the same as the probability of randomly picking a number that's coprime with n, i.e. $\frac{\phi(n)}{n}$, which means that $\frac{\phi(n)}{n} = \prod_{j=1}^{k} (1 - \frac{1}{p_j})$, as desired.

Q.E.D.