4 Sampling a Gaussian With Uniform

(a) Direct Proof

By definition, since $U_1 \sim Uniform(0,1)$, so U_1 has PDF:

$$f_{U_1}(x) = 1 \quad \text{if } 0 \leq x \leq 1$$

$$f_{U_1}(x) = 0 \quad \text{otherwise}$$

Then, since we have that for $RHS = -\ln U_1$, its CDF is:

$$F_{RHS}(x) = \mathbb{P}[-\ln U_1 \leq x] = \mathbb{P}[U_1 \geq e^{-x}] = 1 - \mathbb{P}[U_1 < e^{-x}] = 1 - F_{U_1}(e^{-x})$$

Thus, we have the PDF of the RHS as:

$$f_{RHS}(x) = \frac{d}{dx}F_{RHS}(x) = \frac{d}{dx}(1 - F_{U_1}(e^{-x})) = e^{-x} \cdot f_{U_1}(e^{-x})$$

Then, since we have f_{U_1} above, so we can conclude that:

$$f_{RHS}(x) = e^{-x} \quad \text{if } x \geq 0$$

$$f_{RHS}(x) = 0 \quad \text{otherwise}$$

Also, by definition, the LHS, which is Expo(1), has PDF:

$$f_{LHS}(x) = e^{-x} \quad \text{if } x \geq 0$$

$$f_{LHS}(x) = 0 \quad \text{otherwise}$$

Therefore, $f_{RHS}(x) = f_{LHS}(x)$, which means that $-\ln U_1$ and $Expo(1)$ have the same PDF, and thus, they have the same distribution, i.e. $-\ln U_1 \sim Expo(1)$, as desired.

Q.E.D.

(b) Direct Proof

Given that $N_1, N_2 \sim \mathcal{N}(0,1)$ where N_1, N_2 are independent, so we have that they have the same PDF with:

$$f_{N_1}(x) = f_{N_2}(x) = \frac{1}{\sqrt{2\pi \cdot 1}}\, e^{-(x-0)^2/(2\cdot 1)} = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Since N_1, N_2 are independent, so by Theorem 20.1, we have that the joint density of N_1, N_2 is:

$$f_N(u,v) = \frac{1}{2\pi}\, e^{-(u^2+v^2)/2}$$

Then, since we have that for $RHS = N_1^2 + N_2^2$, its CDF can be written as:

$$F_{RHS}(x) = \mathbb{P}[N_1^2 + N_2^2 \leq x] = \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{x-v^2}}^{\sqrt{x-v^2}} f_N(u,v)\, du dv$$

$$\text{(By the hint)} \implies F_{RHS}(x) = \int_0^{2\pi} \int_0^{\sqrt{x}} r \cdot f_N(r\cos\theta, r\sin\theta)\, dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{x}} r \cdot \frac{1}{2\pi}\, e^{-r^2/2}\, dr d\theta$$

$$\implies F_{RHS}(x) = \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\sqrt{x}}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} 1 - e^{-x/2}\, d\theta = 1 - e^{-x/2}$$

Thus, we have that the (general) PDF of the RHS, $N_1^2 + N_2^2$, is:

$$f_{RHS}(x) = \frac{d}{dx} F_{RHS}(x) = \frac{d}{dx}(1 - e^{-x/2}) = \frac{1}{2}\, e^{-x/2}$$

Yet, notice that since $N_1^2 + N_2^2$ is the sum of two squares, so $N_1^2 + N_2^2 \geq 0$, i.e. for any $x < 0$, the probability/density of $N_1^2 + N_2^2 = x$ is 0, which gives us the final PDF:

$$f_{RHS}(x) = \frac{1}{2}\, e^{-x/2} \qquad \text{if } x \geq 0$$

$$f_{RHS}(x) = 0 \qquad \text{otherwise}$$

Then, by definition, the LHS, which is Expo(1/2), has PDF:

$$f_{LHS}(x) = \frac{1}{2}\, e^{-x/2} \qquad \text{if } x \geq 0$$

$$f_{LHS}(x) = 0 \qquad \text{otherwise}$$

Therefore, $f_{RHS}(x) = f_{LHS}(x)$, which means that $N_1^2 + N_2^2$ and $Expo(1/2)$ have the same PDF, and thus, they have the same distribution, i.e. $N_1^2 + N_2^2 \sim Expo(1/2)$, as desired.

Q.E.D.

(c) ???

First, using the (extension of) results from parts (a) and (b), we have that for $U_1 \sim Uniform(0, 1)$, we can generate an exponential r.v. and the sum of the square of the normal r.v. ($N_1, N_2 \sim \mathcal{N}(0, 1)$ where they're independent) with:

$$-\frac{1}{2} \ln U_1 \sim Expo(1/2) \sim N_1^2 + N_2^2$$

Then, to sample the angle, we would use $U_2 \sim Uniform(0, 1)$ by multiplying it with 2π, which would give us $Uniform(0, 2\pi)$, and thus, we have that the angle is uniform.

Combining these two ($-\frac{1}{2} \ln U_1$ and $2\pi U_2$) would give us a pair of r.v. $\sim \mathcal{N}(0, 1)$, and we can pick either one to complete the transformation.