Computer Science Mentors
Fall 2018

CSM 70 Mock Exam

CSM is an independent organization separate from course staff and makes no promises about the degree to which this mock exam will reflect the true exam. There are **182** points on this exam.

1 True or False? (2 Points Each, 32 points total)

Circle either True or False in each of the problems below.

1. True of False: $P \wedge \neg P \implies Q$
2. True of False: $(\forall x)(\forall y)(P(x,y) \wedge \neg Q(x,y)) \equiv \neg(\exists x)(\forall y)(P(x,y) \implies Q(x,y))$
3. True of False: $(\forall x)(\exists y)(P(x) \implies Q(y)) \equiv (\exists x)(\exists y)(\neg Q(x) \implies \neg P(x))$
4. True of False: If in a stable marriage algorithm terminates after the first day no matter if the men or women propose first, then every male has a unique preference list.
5. $\forall n, (n-1)^{-1} \equiv n-1 \mod n$
6. If n is even, then $\binom{n}{2} \equiv \frac{n}{2} \mod n$
7. True of False: If a graph G contains a subgraph that is a clique of k vertices, then it also contains a subgraph that is a tree of $k+1$ vertices.
8. True of False: If two events A and B are independent then A^c and B^c are dependent.
9. True of False: There are uncountably many programs that output 7.
10. True of False: X takes on values between -2 and 10 and has $E[X] = 4$. Using Markov's we say that $P(X \geq 6) = \frac{4}{6}$
11. True or False: If A and B are events such that $P(A \cup B) = P(A) + P(B) = 1$, then $P(C) = P(A \cap C) + P(B \cap C)$.
12. True of False: $E[\text{Geom}(\frac{1}{pn})] = E[\text{Binomial}(n,p)]$
13. True of False: Say that we know that every day 4 bills will be pushed to the senate, distributed like a Poisson Distribution. We can use the CLT to approximate the number of bills that are pushed in an entire year.
14. True or False: If a self loop exists in an irreducible markov chain, then it is aperiodic.
15. True or False: If random variables X and Y are independent, then $\text{var}(Y) = \text{var}(X-Y) - \text{var}(X)$
16. True or False: If random variables X and Y represent two fair dice rolls, then by symmetry $P(X > Y) = \frac{1}{2}$.

2 Misc (39 points)

1. What is $11^{61} \pmod{12}$ (3 pts)?

2. How many distinct polynomials of degree at most 3 are there modulo 53, such that the value at the points x = 1, 2, 3, 4 lie in the set $\{6, 7, 8, 9, 10\}$ (3 pts)?

3. How many ways can 3 people park their cars in 5 parking spots?

4. How many ways can 3 people park their cars in 5 parking spots, given that there must bust be at least a space between each car (3 pts)?

5. How many numbers between 1 and 111111011_2 can be divided by 8 (3 pts)? (Hint: if a number is divisible by 8, what must the last 3 digits of its base 2 representation be? Don't express the number in base 10)

6. What is the probability that a poker hand (5 cards drawn from a deck of cards) has a 3-pair (a triple) (3 pts)?

7. m balls are thrown into n bins. Each bin is then randomly dumped into one of k bags. What is the expected number of balls in a bag (3 pts)?

8. Suppose $n = 2$. What is the probability no balls end up in a bag (3 pts)?

9. You are sending packets through a channel that has non-deterministic behavior. Specifically, it will corrupt 2 packets with probability .8, and 4 packets with probability 0.2. If your message is of length 10, what is the minimum number of packets that you must send, in expectation, to be able to decode the message 95% of the time (5 pts)? (Hint: the channel is probabilistic, so you should consider a probabilistic encoding procedure; i.e, with probability p do one thing and with $1 - p$ do another.)

10. Say that a random variable X takes on values from 2 to 10, with mean 5. Use Markov's bound to bound the probability that $X \leq 3$. (5 pts)

11. Suppose X and Y are random variables which take on values $\mod n$. Suppose Y is uniformly distributed. Find the distribution of random variable $Z = X + Y \mod n$ (5 pts)

3 Graphs (10 points)

A complete bipartite graph (represented by $K_{m,n}$ is a graph with two sets vertices, where every vertex is connected to every vertex in the other set and no vertices in its own. Set

1. Draw the graph $K_{3,2}$ (2 points)

2. How many vertices does $K_{m,n}$ have? How many edges (3 points)?

3. Under what conditions is there an eulerian tour of the graph (5 points)?

4 Proving Useful Properties (10 points)

For these questions, to prove that two variables are distributed identically, it is enough to show that their CDFs or PDFs are equal. For reference, $f_{exp(\lambda)}(x) = \lambda e^{-\lambda x}$

1. Prove that $\min(X,Y) \sim \exp(\lambda + \mu)$ where $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$ and are independent (5 points).

2. Prove the memoryless property i.e.

$$P(X \geq x + k | X \geq k) = P(X \geq x)$$

where X is an exponential random variable (5 points).

5 Probabilistic Stable Marriage (18 points)

Assume that we have n men and n women, with the men labeled $M_1, \ldots, M_n$, and the women labeled $W_1, \ldots W_n$. We consider all possible stable marriage instances of this problem.

1. Say that $n = 3$. Come up with a stable marriage instance where the algorithm will terminate on the first day, regardless whether or not the men or women are proposing. (3 points)

2. Come up with a general condition on the stable marriage instance if there are n men and women such that the algorithm will terminate after the first day, no matter if the men or women propose first. (5 points)

3. How many stable marriage instances are there for n men and n women, with the men labeled $M_1, \ldots, M_n$, and the women labeled $W_1, \ldots W_n$? (5 points)

4. Assume that we pick a stable marriage instance uniformly at random. What is the probability that the algorithm will terminate after the first day on this instance, regardless of if the men or women propose first? (*Hint: use part (b)*) (5 points)

6 Fun with Circles (22 points)

We pick three points along the circumference of a unit circle centered at the origin. Interpret these points as cutting the circle into three arc.

1. Imagine an ant begins walking along the circle, counter-clockwise, from the point (1, 0). Let L_1 be the length that the ant must walk before reaching the end of the arc it is on. Find an expression for $\mathbb{P}(L_1 > x)$ (7 points).

2. Now imagine the ant walks along the circle starting from (1,0) clockwise. Let L_2 be how far the ant must walk before reaching the end of its arc. Find an expression for $P(L_2 > x)$. Does this answer differ from the previous part? If so, what is it (5 points)?

3. Find $\mathbb{E}(L_1)$ Remember the Tail Sum Formula: $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)dx$ for any non-negative random variable X (5 points).

4. Find the expected length of the arc that contains the point $(1,0)$. (Hint: How does this quantity relate to the quantities from the previous three parts?) (5 points)

7 Moving Particle (25 points)

A particle sits on the real number line, starting at the origin (0). At each timestep, we flip a fair coin and move the particle as follows:

- If we see heads, we move the particle one unit to the left
- If we see tails, we move the particle one unit to the right

Let X_n be the position of the particle at time-step n, and assume $X_0 = 0$. Say that 100 time steps have passed.

1. Give an expression for P(X_{100} = 0) (5 points).

2. Given that X_{10} is positive, write an expression for the probability that $X_{10} = 4$. The expression will not be pretty (5 points)

3. Given that $X_{10} = 4$, what is the probability that the 3rd flip was a heads? (5 points)

4. Use the CLT to give an expression for a bound on the probability that the particle is within 10 units of the center. You can receive partial credit if you give a weaker bound, by for example using Chebyshevs. (Hint: remember the 68-95-99.7 rule: For $Z \sim \mathcal{N}(0,1)$, $P(|Z| \leq 1) = 0.68, P(|Z| \leq 2) = 0.95, P(|Z| \leq 3) = .997$) (10 points).

8 Chains (14 points)

Customers arrive in a store according to Poisson distribution every day with mean λ. Assume that arrivals on different days are independent. We start counting the total number of arrivals to the store on day 1. Let $X(n)$ be the total number of customers that we have seen arrive at the store by day n Hint: What is the name of this problem?.

1. Find the probability that $X(1)$ is even (7 points).
   (*Hint*: $e^{\lambda}+e^{-\lambda}=(1+\lambda+\frac{\lambda^2}{2}+\frac{\lambda^3}{3!}+...)+(1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+...)$)

2. Let $Y(n)=X(n) \mod 2$. As we allow n to get arbitrarily large, what is the long term probability that $Y(n)=0$? Does this probability converge to something? Justify your answer. You may refer to the probability found in the previous question as p (7 points) .

9 Product Chain (12 points)

A dice is rolled until the product of the last 2 numbers rolled is 12.

1. Using just **3** states, draw a Markov chain to represent this scenario. You can get partial credit for using more states. (8 points) .

2. Solve for the expected amount rolls needed (4 points)