3 Erasures, Bounds, and Probabilities

(1) $p \leq 2 \cdot 10^{-7}$

We're given that Alice is sending 1000 bits to Bob, the probability that a bit gets erased is p, the erasure of each bit is independent of the others, and that Alice is using a scheme that can tolerate up to one-fifth of the bits being erased.

Let X be the r.v. indicating the number of bits being lost, so $0 \leq X \leq 1000$, and thus, as we identify $X \sim \text{Bin}(1000, p)$, so by Theorem ???:

$$\mathbb{E}[X] = 1000p$$

Thus, we can use the Markov's inequality since X is non-negative with a finite mean

$$\mathbb{P}[X \geq 200] \leq \frac{\mathbb{E}[X]}{200} = 5p$$

Then, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[X \geq 200] \leq 10^{-6}$. Therefore, we would have

$$\mathbb{P}[X \geq 200] \leq 5p \leq 10^{-6}$$

1 which gives us an upper bound of p:

$$p \leq 2 \cdot 10^{-7}$$

(2) $p \leq 4.00 \cdot 10^{-5}$

Similarly to part (a) and using its results, with $X \sim \text{Bin}(1000, p)$, so we have that:

$$\mu = \mathbb{E}[X] = 1000p$$

$$\text{var}(X) = 1000\, p(1-p)$$

Now, we have that:

$$\mathbb{P}[X \geq 200] \ = \ \mathbb{P}[X - \mu \geq 200 - \mu] \ \leq \ \mathbb{P}\big[|X - \mu| \geq |200 - \mu|\big]$$

Thus, using Chebyshev's Inequality, we could set up another equation:

$$\mathbb{P}[X \geq 200] \ \leq \ \mathbb{P}\big[|X - \mu| \geq |200 - \mu|\big] \ \leq \ \frac{\text{var}(X)}{(|200 - \mu|)^2} \ = \ \frac{1000\, p(1-p)}{(200 - 1000p)^2}$$

Again, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[X \geq 200] \leq 10^{-6}$. Therefore, we would have

$$\mathbb{P}[X \geq 200] \ \leq \ \frac{1000\, p(1-p)}{(200 - 1000p)^2} \ \leq \ 10^{-6}$$

Well, using a calculator, we have an upper bound of p:

$$p \leq 4.00 \cdot 10^{-5}$$

(3) $p \leq 0.1468$

Let $X = X_1 + X_2 + \cdots + X_1000$, where each $X_i = 1$ if the i^{th} bit gets erased and 0 otherwise. We're told that all the X_i's are i.i.d. random variables, and they have common finite expectation:

$$\mu = \mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$$

and finite variance:

$$\sigma^2 = \text{var}(X_I) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = p - p^2$$

which also gives us that $\sigma = \sqrt{p - p^2}$.

Now, let $S_n = \sum\limits_{i=1}^{n} X_i$, then for very large n (i.e. when $n = 1000$), the Central Limit Theorem gives us that:

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{200-1000\mu}{\sigma\sqrt{1000}}} e^{-x^2/2}\, dx \tag{1}$$

Now, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[S_{1000} \geq 200] \leq 10^{-6}$, which is equivalent to $\mathbb{P}[S_{1000} < 200] = \mathbb{P}[\overline{S_{1000} \geq 200}] \geq 1 - 10^{-6}$. Then, since

$$\mathbb{P}[S_{1000} < 200] \;=\; \mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} < \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \;\leq\; \mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}]$$

Thus, for $\mathbb{P}[S_{1000} < 200] \geq 1 - 10^{-6}$, we need

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \;\geq\; 1 - 10^{-6}$$

which, by Eq. (1) and our setup of μ and σ, is (approximately) equivalent to having:

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{200-1000p}{\sqrt{1000(p-p^2)}}} e^{-x^2/2}\, dx \;\geq\; 1 - 10^{-6}$$

$$\implies \int_{-\infty}^{\frac{200-1000p}{\sqrt{1000(p-p^2)}}} e^{-x^2/2}\, dx \;\geq\; (1 - 10^{-6}) \cdot \sqrt{2\pi}$$

Again, using a calculator, and defining $a = \frac{200-1000p}{\sqrt{1000(p-p^2)}}$ for simplicity in format, so we have an inequality that looks like...

$$\text{erf}(\frac{a}{\sqrt{2}}) \geq \frac{499,999}{500,000}$$

which gives us an approximate bound:

$$\frac{200 - 1000p}{\sqrt{1000(p - p^2)}} = a \geq 4.75243$$

Thus, using the calculator, we have an approximate upper bound for p (with CLT) as:

$$p \leq 0.1468$$