CSM CS70 Fall 2018 Mock Midterm 1

CSM 70 Mentors

September 23rd 2018, 6-9PM Soda 306 (HP Auditorium)

1. **T/F (2 pts each)**
   (a) $\neg P \vee Q \equiv \neg Q \implies \neg P$

   (b) $\neg(P \wedge Q) \equiv P \implies \neg Q$

   (c) $\forall x \exists y, \neg(P(x) \vee Q(x,y)) \implies \exists x, \neg P(x)$

   (d) $\forall n, P(n) \implies P(n+1) \equiv \forall n, P(n)$

   (e) $\forall x \in S, [P(x) \wedge Q(x)] \equiv [\exists x \in S, P(x)] \vee [\exists x \in S, Q(x)]$

   (f) If 2 men ever propose to the same W, then it must be that W is the same rank for 2 different men.

   (g) In stable marriage, if a man M is the last preference for every woman then he cannot get his best choice ($n \geq 2$).

(h) A set of n men and women go through stable marriage as normal. After the pairings terminate, someone hits their head and reverses all of their preferences. Is it possible that, despite this person's complete switch in preferences, the matching is still stable if...

1. the person is a woman?

2. the person is a man?

(i) In a stable matching instance with 10 men and women, it is possible for 5 men to end up with women who are in the bottom half of their preference lists.

(j) If a graph has average degree at least 2, then it must contain a cycle.

(k) A graph with k edges has $\geq |V| - k$ connected components

(l) Given a tree where $S = \{v \in V \mid \deg(v) > 2\}$ and $T = \{v \in V \mid \deg(v) = 1\}$, $|S| < |T|$

(m) The maximum degree of any planar graph is 10

(n) There exists a graph such that no 2 vertices have the same degree. ($n \geq 2$)

(o) A graph where every set of 3 vertices has at least 2 edges between them is connected. ($n \geq 3$)

(p) There are graphs with an odd number of odd-degree vertices.

(q) If $\gcd(x,\, y) = c$, $\gcd(x,\, z) = d$, and $x \geq y, z$, then $\gcd(y, z) \leq c, d$.

(r) If $\gcd(x, y) = d$ and $\gcd(y, z) = c$, then $\gcd(x, z) \geq \gcd(c, d)$

(s) $a^{pq-1} \equiv 1 \mod pq$ if p and q are prime, $p \neq q$, and a is not a multiple of p or q.

(t) Let φ be the Euler totient function, i.e. $\varphi(n) = |\{1 \leq k \leq n \colon gcd(k, n) = 1\}|$. For all m and n relatively prime, we have $\varphi(m \cdot n) = \varphi(m)\varphi(n)$.

2. **Short Answer (4 points each)**

   (a) What are all the possible solutions to the expression, $((X \wedge Y) \rightarrow Z) \wedge ((Y \wedge Z) \rightarrow X) \wedge ((Z \wedge X) \rightarrow Y)$?

   (b) A planar graph has 12 faces, all of which are pentagons. How many edges are there in this graph?

   (c) A double elimination bracket is a tournament style where if you lose a round, you are sent to compete in a "loser's" bracket; only after losing twice are you completely eliminated from the tournament. We can make a graph out a double elimination tournament by making each team a vertex and connecting two vertices if they play a match against each other. If two teams play multiple matches, there can be multiple edges between the same two vertices (so this is not a simple graph, but rather a type of graph known as a multigraph). What is the maximum number of edges in this multigraph? ok

(d) To separate a hypercube of dimension n into 2 hypercubes of dimension $n-1$, how many edges do you have to cut?

(e) You have a hypercube of dimension n. Create a new graph by doing the following: For every 4-cycle in the hypercube, add a vertex. Connect two vertices if their corresponding 4-cycles share an edge.

1. How many vertices are in this new graph?

2. How many edges are in the graph?

(f) On a 12-hour clock, what time will it be 2^{30} hours after midnight?

(g) A graph has 12 nodes labeled 0....11. Edges are added to the graph in the following way: Starting at vertex 0, for each vertex i, an fedge is drawn to all x s.t $3x + i \equiv 2 \mod 12$. How many edges are present in this graph?

(h) How many values of x are there such that $150 \equiv x \mod 2x$?

(i) Find an expression for $5^{2(7^n - 6^n)} \pmod{26}$

(j) Let p be a prime number. Find $(p-1)! \pmod{p}$

3. **Proofs (5 points each)**

   (a) Prove that $(1+x)^n \geq 1+xn$, for $x \geq 0$

   (b) Let $b_1 = 1$ and $b_{n+1} = 5b^3$. Prove that $b_n \leq 5^{3^n}$. Hint: You will need to strengthen the inductive hypothesis.

   (c) Prove that if you direct the edges of a (connected) tree, then there must be at least one vertex with zero in degree, and one vertex with zero out degree.

(d) Prove that $n^3 \not\equiv 1 \mod 5 \implies n \not\equiv 1 \mod 5$

(e) Prove that if p is a prime number, then $x^2 \equiv p \mod p^2$ has no solutions.

(f) Show that a connected undirected graph with at most two vertices of odd degree has an Eulerian walk.

4. **Stable Marriage (5 points each)**

Suppose there are n men, n women, and n pet dogs that we want to group into trios of one man, woman, and dog each. The men and women have preference lists for each other as in the usual stable marriage problem. In addition, the women and dogs have preference lists for each other. However, each man has a set of dogs he she hates. We want to find a way to group the men, women, and dogs into trios of one man, one woman, and one dog each such that the following stability criteria hold:

1. No woman and dog not in the same trio prefer each other to another dog and woman in their respective trios
2. Each dog is in a trio with the best woman he can get in any trio satisfying condition 1
3. No man and woman not in the same trio prefer each other to their respective woman and man in another trio
4. No man is in a trio with a dog that she hates

(a) What can we do to ensure there are no rogue woman, dog pairings and each dog gets his optimal woman?

(b) Building on part (a), how can we devise an algorithm that outputs a stable matching if one exists, or terminates if it doesn't?

5. **Buddy System (2/2/2/5 points)**

A k-regular graph is a graph where every vertex has degree k.
A k-connected graph is a connected graph where at least k vertices must be removed to disconnect the graph. So, a 1-connected graph is just a connected graph.

(a) Suppose a graph is k connected. True or False: every vertex in the graph has degree $\geq k$. (hint: ask yourself: which is the "stronger" property?

(b) True or false: for every value of k. there is a k-regular graph that is k-connected.

(c) What is the fewest number of edges in a 2-connected graph with $|V|$ vertices?

(d) Prove that if a graph is 2-connected, then every vertex is a part of a cycle (a vertex i is "part of" a cycle if edges $(j, i), (i, k)$ occur in the cycle (in order) for some j, k).