CS 70 Discrete Mathematics and Probability Theory

Summer 2014 James Cook

Final Exam

Friday August 15, 2014, 5:10pm-8:10pm.

Instructions:

- Do not turn over this page until the proctor tells you to.
- Don't write any answers on the backs of pages (we won't be scanning those). There is an extra page at the end in case you run out of space.
- The exam has 24 pages (the last two are mostly blank).

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________

(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

Important note:

In any multi-part problem, you can refer to numbers from the previous parts, even if you couldn't solve them! Use A for the answer to part (a), B for the answer to part (b), etc.
For example, if you know that the answer to (b) should be 5 times the answer to (a), write $5A$ as your answer to (b).

Important Note

Have you read the important note on the front of the exam?

True/False

1. [Law of the excluded middle](10 pts.) **For each of the following statements, circle T if it is true and F otherwise. You do not need to justify or explain your answers.**

T **F** $(\forall n \in \mathbf{N})\left((2n)! = \binom{2n}{n}(n!)^2\right)$

Solution: True. We can prove this algebraically: $\binom{2n}{n}(n!)^2 = \frac{(2n)!}{n!(2n-n)!}(n!)^2 = (2n)!$**.**
Alternatively, we can prove this using a combinatorial proof. Both sides count the number of permutations of the numbers from 1 **to** $2n$**. This is clearly equal to the left-hand side,** $(2n)!$**. for the right-hand side, construct a permutation as follows: first, choose which** n **items will come first (so the remaining** n **items go after). There are** $\binom{2n}{n}$ **ways to do this. Then choose how to order those** n **items (**$n!$ **ways) and choose how to order the remaining** n **items (also** $n!$ **ways).**

T **F** **Q and** $\mathbf{N}^3$ **have the same cardinality.** (Note: $\mathbf{N}^3$ is the set of triples of natural numbers, e.g. $(5,0,17) \in \mathbf{N}^3$.)

Solution: True. As seen in class, $\mathbf{N}^2$ **and N have the same cardinality. Call this bijection** $f(a,b) : \mathbf{N}^2 \to \mathbf{N}$**. We can use this twice to build a bijection between** $\mathbf{N}^3$ **and N: given** $(a,b,c) \in \mathbf{N}^3$**, first find the number** $d = f(a,b)$ **corresponding to** $(a,b) \in \mathbf{N}^2$**, then take** $f(d,c) \in \mathbf{N}$**. In other words, we've built a bijection** $g : \mathbf{N}^3 \to \mathbf{N}$ **by** $g(a,b,c) = f(f(a,b),c)$**.**
We also saw in class that Q and N have the same cardinality. So we can combine our bijection between $\mathbf{N}^3$ **and N with our bijection between Q and N to get a bijection between** $\mathbf{N}^3$ **and Q.**

T **F** **If you throw** n^2 **balls into** n^3 **bins uniformly at random, the probability that at least one bin will have more than one ball will be less than** 0.1 **(for large enough** n**).**

Solution: False. In Note 13, we learned that for $m = n^3$ **bins, once we throw** $\sqrt{(2\ln 2)m}$ **balls, our probability of a collision reaches** $\frac{1}{2}$**. But** $\sqrt{(2\ln 2)m} = \sqrt{(2\ln 2)n^3} < n^2$ **for large enough** n**.**

Consider the following instance of the stable marriage problem.

Woman	Preferences				Man	Preferences		
A	**1**	**2**	**3**		**1**	**B**	**A**	**C**
B	**1**	**2**	**3**		**2**	**C**	**B**	**A**
C	**3**	**1**	**2**		**3**	**B**	**A**	**C**

T F In the above stable marriage instance, (1, B), (2, A), (3, C) is a stable pairing.

Solution: True. (There are no rogue couples.)

T F In the above stable marriage instance, 1 is A's optimal man.

Solution: False. There is no stable pairing that puts 1 and A together, since $(1,B)$ would be a rogue couple in any such pairing.

Short Answer

2. [U mod?](4 pts.) **Find an integer** x **such that** $8x \equiv 20 \pmod{22}$**.**
(Warning: $\gcd(22,8) \neq 1$.)

Solution: Since $\gcd(22,8) \neq 1$, there is no number $8^{-1} \pmod{22}$. But we can still use the extended euclidean algorithm to find numbers a and b such that $22a + 8b = \gcd(22,8)$:

$$\begin{array}{r|c|c}
\gcd(22,8) & 6 = 22 - 2\cdot 8 & 2 = 1\cdot 8 - 1\cdot(22 - 2\cdot 8) = -1\cdot 22 + \boxed{3}\cdot 8 \\
= \gcd(8,6) & 2 = 8 - 6 & 2 = 0\cdot 6 + 1\cdot(8-6) = 1\cdot 8 - 1\cdot 6 \\
= \gcd(6,2) & 0 = 6 - 3\cdot 2 & 2 = 1\cdot 2 + 0\cdot(6 - 3\cdot 2) = 0\cdot 6 + 1\cdot 2 \\
= \gcd(2,0) & & 2 = 1\cdot 2 + 0\cdot 0
\end{array}$$

So, we discover that $\gcd(22,8) = 2$, and also that $2 = -1\cdot 22 + 3\cdot 8$, so $8\cdot 3 \equiv 2 \pmod{22}$.

So $8\cdot 30 \equiv 8\cdot 8 \equiv 20 \pmod{22}$: take $x = 8$.

There are other solutions: for example, $x = -3$ and $x = 30$ both satisfy $8x \equiv 20 \pmod{22}$.

Note: It turns out that x is a solution if and only if $x \equiv 8 \pmod{11}$. To see why, try searching for information about the Chinese Remainder Theorem. In this case, the theorem says that $8x \equiv 20 \pmod{22}$ iff $8x \equiv 0 \pmod{2}$ and $8x \equiv 9 \pmod{11}$. The first equation is always true (8 times anything is even), and the second equation means the same thing as $x \equiv 8^{-1}\cdot 9 \pmod{11}$.

3. [Come Together](4 pts.) **The code to open your secret club's treasure chest is 15. Your club has 4 other members, named John, Paul, George and Ringo. What information can you give each of them so that any 3 of them can discover that the secret is 15, but if only 2 of them share their information, they cannot discover the secret? Fill in the boxes below.**

(If you need any random numbers, feel free to just use 0 if it makes your calculations easier.)

Scratch work:

Answer:

Tell all of them: "$P(x)$ is a polynomial of degree at most $\boxed{}$. The secret is $P(\boxed{})$."

Tell John: $P(\boxed{}) \equiv \boxed{} \pmod{\boxed{}}$.

Tell Paul: $\boxed{}$

Tell George: $\boxed{}$

Tell Ringo: $\boxed{}$

Solution: According to the secret-sharing scheme described in class, we should choose $P(x)$ to be a polynomial of degree at most 2 such that $P(0) = 15$. We can work modulo 17, or any other prime number bigger than 15. We should choose $P(x) = 15 + a_1x + a_2x^2$, where a_1 and a_2, are random numbers between 0 and 16. To make calculations easier, we'll choose 0 for every random number, so $P(x) = 15$.

Tell all of them: "$P(x)$ is a polynomial of degree at most $\boxed{2}$. The secret is $P(\boxed{0})$."

Tell John: $P(\boxed{1}) \equiv \boxed{15} \pmod{\boxed{17}}$.

Tell Paul: $\boxed{P(2) \equiv 15 \pmod{17}}$.

Tell George: $\boxed{P(3) \equiv 15 \pmod{17}}$.

Tell Ringo: $\boxed{P(4) \equiv 15 \pmod{17}}$.

Short Answer (continued)

4. [Connect the dots](3 pts.) **Find a polynomial $p(x)$ such that $p(1) \equiv 0 \pmod 7$, $p(4) \equiv 0 \pmod 7$ and $p(5) \equiv 1 \pmod 7$. You do not need to simplify your answer.**

(Hint: This shouldn't take very long.)

Solution: The Δ polynomials in Lagrange interpolation are designed to do exactly this. So use:

$$p(x) = \Delta(x) = \frac{(x-1)(x-4)}{(5-1)(5-4)}.$$

Common mistake: Many students went ahead and simplified their polynomial, even though we said that they didn't need to. This is fine, as long as you simplify correctly: the answer is $p(x) = 2x^2 + 4x + 1$. However, for quite a few students, simplifying led to arithmetic errors and losing unnecessary points. Be lazy when you can! Also, I notice that some of you didn't take advantage of our hint or realize what it meant until it was too late :(

5. [What do you expect?](4 pts.) **Suppose X is a random variable with the following distribution:** $\Pr[X = 1] = \frac{1}{12}$**,** $\Pr[X = 2] = \frac{1}{6}$**,** $\Pr[X = 3] = \frac{3}{4}$**. What is the expected value of $\frac{1}{X^2}$?**

Solution: $\frac{1}{X}$ is 1, $\frac{1}{4}$ and $\frac{1}{9}$ with probabilities $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{3}{4}$, respectively. So $\mathrm{E}(\frac{1}{X}) = \frac{1}{12} \cdot 1 + \frac{1}{6} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{9} = \boxed{\frac{5}{24}}$.

Common mistakes:

- Trying to use linearity of expectation or indicator variables when they're not necessary.
- Thinking $\frac{1}{E(X^2)} = E(\frac{1}{X^2})$, or $\frac{1}{E(X)^2}$, or something similar. In general, $E(X^a) \neq E(X)^a$.

Short Answer (continued)

6. [He is a Cook after all…](4 pts.) **James is baking cakes. Every cake he bakes is rated one of E, O or T (for Excellent, just Okay, or Terrible). We record the sequence of ratings his cakes get.**

The cakes are rated randomly: E with probability $\frac{1}{2}$, O with probability $\frac{1}{4}$ and T with probability $\frac{1}{4}$. The ratings are independent. (For example, if he bakes 7 cakes, one possible sequence is EOETOEE, which occurs with probability $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = 2^{-10}$.)

If James bakes 20 cakes, what is the probability that he gets 7 Es, 4 Os and 9 Ts?

Solution: There are $\binom{20}{7}$ ways to choose where to put the Es. In the 13 remaining spaces, there are $\binom{13}{4}$ ways to choose where to put the Os, and then the Ts must go in the nine remaining spaces. So the number of sequences with 7 Es, 4 Os and 9 Ts is $\binom{20}{7}\binom{13}{4}$.

Each such sequence has probability $(\frac{1}{2})^7(\frac{1}{4})^4(\frac{1}{4})^9$, so the probability is $\binom{20}{7}\binom{13}{4}(\frac{1}{2})^7(\frac{1}{4})^4(\frac{1}{4})^9$.

Common Mistakes: There were a variety of common errors, but the most common mistake was not realizing that the the three events "James gets 7 Es", "James gets 4 Es" and "James gets 9 Ts" can't be considered independently (because they weren't independent). A second mistake was forgetting that there were many permutations that could allow the ratings to be assigned to cakes. Finally, a common error was dividing by 3^{20}. Because the sample space was not uniform, a counting approach was not appropriate for this problem (also, some students both divided by 3^{20} as well as taking into account the listed probabilities, which doesn't much sense).

7. [Dicey stuff](3 pts.) **Suppose you roll a standard fair die 100 times. Let X be the sum of the numbers that appear over the 100 rolls. Give a bound on the probability that X is between 300 and 400.**

Solution: Notice that we need to use Chebychev here

Common Mistakes: Many people tried to use Markov's inequality here, and almost everyone that tried this used it incorrectly. By far, the most common issue was using Markov to get $P[X \geq 400] \leq 7/8$ and $P[X \geq 300] \leq 7/6$, then saying $P[300 \leq X \leq 400] = P[X \geq 300] - (1 - P[X \geq 400]) \leq 7/8$, or something similar. Note that that *equality* does hold, but the *inequality* doesn't! When you say $P[X] \leq A$, then it is not true that $1 - P[X] \leq A$, its the opposite, so the result is unbounded. Another common mistake was to use $P[300 \leq X \leq 400] = P[X \geq 300 \wedge X \leq 400] = P[X \geq 300]P[X \leq 400]$. This would only hold if those two events were independent, which they are not. Another Markov based common mistake was assuming that it's still true if you just flip both signs. It's not.

Chebychev states that

$$\Pr[|X - \mathrm{E}[X]| \geq a] \leq \mathrm{Var}(X)/a^2$$

Common Mistake: Forgetting the absolute value, and then dividing by 2 to make up for it, forgetting to square the a in the denominator, or writing equality instead of $\leq$.

To use Chebychev we need to know the mean and variance of X. Let X_i denote the result of the ith roll, so that $X = \sum_i X_i$. We know that X_i is distributed $Unif(6)$,

$$\begin{aligned}
\mathrm{E}[X] =& \mathrm{E}[\sum_i X_i] \\
=& \sum_i \mathbf{E}[X_i] \\
=& 100\,\mathrm{E}[X_i] \\
=& 100 \sum_{j=0}^{6} (1/6) \times j \\
=& 350
\end{aligned}$$

Common Mistake: Several people used the wrong distribution for the die, the most common being $Binom(100, 1/6)$.

Similarly for the variance,

$$\begin{aligned}
\mathrm{Var}(X) =& \mathrm{Var}(\sum_i X_i) \\
=& \sum_i \mathrm{Var}(X_i) \\
=& 100\,\mathrm{Var}(X_i) \\
=& 100(\mathrm{E}[X_i^2] - 3.5^2) \\
=& 100((\sum_{j=0}^{6} (1/6) \times j^2) - 3.5^2) \\
=& 100(35/12)
\end{aligned}$$

where the second line follows from linearity of variance for independent rolls.

Common Mistake: Many people didn't use linearity of variance, and tried to compute it from the definition. This is much harder and the vast majority of the people that tried it made an error somewhere along the way, although a few got it. Another common problem was claiming $\text{Var}(X) = \text{Var}(100 * X_i) = 100^2 \text{Var}(X_i)$. That is only true for a constant multiplying a single random variable. When there is a sum of different independent random variables you need to use linearity of variance.

Now from Chebychev we know that $\Pr[|X - 350| \geq 50] \leq \text{Var}(X)/(50^2) = \frac{100(35/12)}{50^2} = \frac{7}{60}$. Now notice that this is the probability we get *outside* of 300 and 400, to get the probability of getting inside, we subtract this from 1 to get,

$$\Pr[300 < X < 400] \geq 1 - 7/60 = 53/60 \tag{1}$$

Common Mistakes: Many people forgot to subtract the final quantity from 1, and thus gave the bound on being outside instead of inside. Among people that did subtract from 1, many people forgot to change the direction of the bound because of the minus sign. Perhaps the most common mistake of all though was stating this final bound as an equality, or simply saying "the bound equals." A bound cannot equal a single number, it has to have a direction as well.

Note: We were intentionally vague about the direction of the bound and the technique needed to find it. We wanted you to realize that the best way to bound this was by Chebychev, and that you would have to give a lower bound. Also, there was no need to simplify the final answer, full credit was given for answers containing several fractions or for answers that were correct until an error was made in the simplification of the fraction.

Short Answer (continued)

8. [Probably Fun?](4 pts.) **Show that if A and B are events, then $\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1$.**

Solution: Proof: The inclusion/exclusion principle tells us that $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Since $1 \geq \Pr[A \cup B]$, we can conclude $1 \geq \Pr[A] + \Pr[B] - \Pr[A \cap B]$, which can be re-arranged as $\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1$. ■

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You won't believe what these numbers sum to!

9. (8 pts.) **Prove that for every $n \in \mathbf{N}$,**

$$\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}.$$

Solution: We'll prove this by induction on n.

Base case: $n = 0$. Here, the left and right sides are both 1.

Induction step. Assume $\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1} \\
&= \frac{3^{n+1}-1}{2} + 3^{n+1} \\
&= \frac{3^{n+1} + 2 \cdot 3^{n+1} - 1}{2} \\
&= \frac{3^{n+2}-1}{2}.
\end{aligned}$$

Variation on a Theme

10. (10 pts.) **Ziva has a set of 10 magical dice named $D_1, D_2, \ldots, D_{10}$. When she rolls all 10 dice, the number 1 always appears at least once. Notice that there are $6^{10} - 5^{10}$ possible outcomes when she rolls the dice. Every one of these outcomes appears with probability $\frac{1}{6^{10} - 5^{10}}$ (so the dice follow the uniform distribution).**

Leave your answers below as unevaluated expressions like $\frac{5!}{6^{100}}$.

(a) (2 pts.) What is the probability that $D_1 = 1$? (The answer is not $\frac{1}{6}$.)

(b) (2 pts.) Let X be the number of dice that show 1. What is $\mathrm{E}(X)$?

(c) (6 pts.) What is $\mathrm{Var}(X)$?

Solution:

(a) There are 6^9 ways that the first die can be 1 (and all of those ways satisfy the constraint that there is at least one 1). So the probability is $\frac{6^9}{6^{10} - 5^{10}}$.

(b) Let X_i be the indicator random variable for the event that the ith die is 1. Then $\mathrm{E}(X_i) = \frac{6^9}{6^{10} - 5^{10}}$, so $\mathrm{E}(X) = \mathrm{E}(X_1 + \cdots + X_{10}) = 10A$, where A is the answer to part (a).

Common mistakes:

- Many students said that $\mathrm{E}(X) = 1 + \frac{9}{6}$ or something similar, with the reasoning that one die must be 1, and then the remaining 9 each have a one in six chance of being 1 since the condition is already satisfied. Unfortunately this argument does not work. For example, there could be more than one choice for the first die. To use linearity of expectation to get the $\frac{9}{6}$, you would need to split up the random variable X as $1 + X_1 + \cdots + X_9$, but it's not clear what those nine random variables should represent.
- Some students computed $\mathrm{E}[X]$ using the formula $\mathrm{E}[X] = \sum_a a \Pr[X = a]$. This is somewhat error-prone and leads to a much less simple answer. One mistake many students made in this case is to compute $\Pr[X = a]$ as $\frac{6^{10-a}}{6^{10} - 5^{10}}$, forgetting that it should be multiplied by $\binom{10}{a}$, the number of choices for which a dice should show 1.

(c) First, the probability that the first and second dice (or any particular two dice) are both one is $\dfrac{6^8}{6^{10}-5^{10}}$, using the same logic as in part (a). So

$$\begin{aligned}
&\mathrm{E}(X^2)\\
=&\mathrm{E}((X_1+\cdots+X_{10})^2)\\
=&\sum_{i=1}^{10}\mathrm{E}(X_i^2)+\sum_{i\neq j}\mathrm{E}(X_iX_j)\\
=&10\cdot\frac{6^9}{6^{10}-5^{10}}+90\frac{6^8}{6^{10}-5^{10}}\\
=&\frac{10\cdot 6^9+90\cdot 6^8}{6^{10}-5^{10}}
\end{aligned}$$

So $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2=\frac{10\cdot 6^9+90\cdot 6^8}{6^{10}-5^{10}}-B^2$ (where B is the answer to part (b)).

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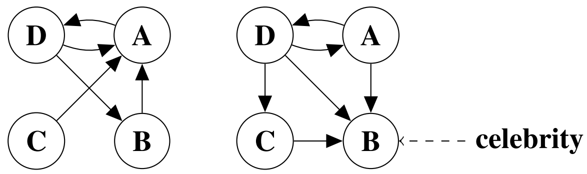
Party at Ajay's House

11. (23 pts.) **Ajay's thinking of having a party at his house. He has invited n guests ($n \geq 2$). To keep track of which guests know which other guests, he's drawn a directed graph G with n nodes (one per guest), with an edge from guest x to guest y whenever x knows y.**

A guest x is called a *celebrity* if every other guest knows x, but x does not know any other guest. For example, below are two parties. The party on the left has no celebrities. In the other party, B is a celebrity.



(a) (4 pts.) If the party has a celebrity, can G have an Eulerian cycle (one that visits every edge once)? Can it have a Hamiltonian cycle (one that visits every node once)? Justify your answers.

For the remaining questions, suppose that G is generated randomly as follows. For every two guests x and y, x knows y with probability p, and all such events are independent. (Examples: The probability that x knows y and y also knows x is p^2. The probability that w knows x and y but doesn't know z is $p^2(1-p)$.)

(b) (4 pts.) What is the probability that every guest knows every other guest?

Continued on the next page!

Party at Ajay's House (continued)

(c) (3 pts.) Let's pick a particular guest u. What's the probability that u is a celebrity?

(d) (4 pts.) Let's pick a second guest v. What's the probability that u knows v given that u is not a celebrity?

(e) (4 pts.) What's the probability that one of u and v is a celebrity? *(Hint: a party cannot have more than one celebrity.)*

(f) (4 pts.) What's the probability that there are no celebrities?

Solution: *Overall common mistake:* Many students didn't take our advice to write the later answers in terms of previous parts! This led to some points being lost unnecessarily.

(a) G cannot have an Eulerian cycle because the celebrity node's indegree and outdegree are not equal. G cannot have a Hamiltonian cycle because once a cycle visits the celebrity node, it cannot leave, since there are no outgoing edges.

Common mistake: By definition, a cycle must end where it starts. Many students didn't realize that a Hamiltonian cycle can't end at a celebrity, because it would also have to start there.

(b) There are $2\binom{n}{2}$ possible edges, so the probability is $p^{2\binom{n}{2}}$.

(c) Every possible ingoing edge to u must be present, and each one is present with probability p. Every possible outgoing edge must be absent, and each one is absent with probability $(1-p)$. All these events are independent, and there are $n-1$ possible incoming and $n-1$ possible outgoing edges, so the probability is $p^{n-1}(1-p)^{n-1}$.

(d) Let A be the event that u is a celebrity and B be the event that u knows v. Notice that A and B are disjoint events (u can't be a celebrity if u knows v) and so $\overline{A}\cap B = B$. So $\Pr[B|\overline{A}] = \Pr[\overline{A}\cap B]/\Pr[\overline{A}] = \Pr[B]/(1-\Pr[A]) = p/(1-C)$ (where C is the answer to part (c)).

(e) Let A be the event that u is a celebrity and B be the event that v is a celebrity. A and B are disjoint events, so $\Pr[A+B] = \Pr[A]+\Pr[B] = 2C$.

Common mistakes:

- The hint says that a party couldn't have multiple celebrities. A surprising number of students still tried to incorporate the event that both u and v are celebrities as part of their answer.
- Many students tried including a term like $\Pr(u \text{ is a celebrity} \mid v \text{ is not a celebrity})$, and we suggest students who did so look carefully at their equations, since the most straight-forward way of approaching this problem shouldn't be introducing such a term at all.
- Note that the fact that a party can't have more than one celebrity means that the events that u and v are *disjoint*, which means that they are *not independent*.

(f) Using the same logic as part (e), the probability that someone is a celebrity is nC, so the probability that there is no celebrity is $1-nC$.

Common mistake: A common mistake was to multiply the probabilities that people aren't celebrities together (those events are dependent, so can't be multiplied), instead of extending part e to n people. $(1-C)^n$, for example, was a common error.

Your Bassic Geometric Distribution

12. (9 pts.) **Hao has a computer with two speakers. Every day, each speaker fails with probability p independently. Let L and R be the number of days the left and right speakers last until failing. So $L \sim$ Geom(p), $R \sim$ Geom(p), and L and R are independent random variables.** (For example, if both speakers fail on the first day, then $L = R = 1$.)

In the below questions, simplify your answers as much as possible.

(a) (3 pts.) For an integer $k \geq 1$, what is $\Pr[L \geq k]$? (In other words, the probability that the left speaker fails on day k or later.)

(b) (3 pts.) Let X be the number of days until both speakers have failed: in other words, X is the larger of L and R.

For an integer $k \geq 1$, what is $\Pr[X < k]$?

(c) (3 pts.) For an integer $k \geq 1$, what is $\Pr[X = k]$?

Solution:

(a) This is the event that the left speaker doesn't fail on the first $k-1$ days, so the probability is $(1-p)^{k-1}$.

(b) Note that $X < k$ if and only if $L < k \wedge R < k$. So, using the fact that L and R are independent, $\Pr[X < k] = \Pr[L < k \cap R < k] = \Pr[L < k]\Pr[R < k] = (1 - \Pr[L \geq k])(1 - \Pr[R \geq k]) = (1-A)^2$, where A is the answer from part A.

Common mistakes: Many students computed $1 - A^2$ instead of $(1-A)^2$. Some students made a mistake in calculating the case when one speaker fails not both.

(c) Note that $X = k$ and $X < k$ are disjoint events whose union is $X < k+1$. So $\Pr[X = k] = \Pr[X < k+1] - \Pr[X < k] = (1-(1-p)^k)^2 - (1-(1-p)^{k-1})^2$.

Binomial Process

13. (16 pts.) **Three wise men, named W_1, W_2 and W_3, are flipping coins independently at each timestep until the end of time. Specifically, W_1 flips a coin with bias p_1 at every timestep, W_2 flips his coin with bias p_2 every timestep, W_3 flips his coin with bias p_3 every timestep, and all coin flips are independent.** (The *bias* of a coin is the probability that it comes up heads when you flip it.) **You are a lonely wanderer who comes upon the three wise men and starts observing them.**

Here is an example outcome:

Timestep	1	2	3	4	5	6	⋯
Wise Man W_1	T	H	T	T	T	T	⋯
Wise Man W_2	T	T	T	T	T	T	⋯
Wise Man W_3	H	H	T	H	T	T	⋯

(a) (2 pts) Let X be the number of timesteps it takes for Wise Man W_1 to flip his first heads. (In the example outcome above, $X = 2$.) **Give the probability distribution of X, along with any parameters necessary to describe it.**

(b) (3 pts.) Let B be the probability that, on a given timestep, at least one of the coins comes up heads. Find B.

Continued on the next page!

Binomial Process (continued)

(c) (2 pts.) If you look at the first 20 timesteps, let Y be the number of those timesteps where only tails were flipped. Give the probability distribution of Y, along with any parameters necessary to describe it.

(d) (3 pts.) On a particular timestep, you notice one wise man flipped heads, and the other two flipped tails. Given that, what's the probability that Wise Man W_1 was the one who flipped heads?

Continued on the next page!

Binomial Process (continued)

(e) (3 pts.) Let Z be the number of timesteps it takes to see 2 timesteps in a row with no heads (including those two time steps). (In the example outcome, $Z = 6$.) **Find** $\mathrm{E}(Z)$**.**

(f) (3 pts.) Let R be the event that the first time heads is flipped, it was Wise Man W_1 that flipped heads, and neither of the other wise men flip heads on that time step. (In the example outcome, R does not happen, since Wise Man W_3 flips heads first.) **Find** $\Pr[R]$**.**

Note: This problem corresponds to a very real thing called a Binomial process, which you just derived several of the key properties of! The all-too-important continuous analog is called a Poisson process (see EE 126), which is used to handle queues, hits on a website, traffic, and many other real-life problems!

Solution:

(a) $X \sim \mathrm{Geom}(p_1)$

(b) $B = \Pr[\text{at least one coin is heads}] = 1 - \Pr[\text{all three coins are tails}]$. Since the coin flips are independent, $\Pr[\text{all three coins are tails}] = \Pr[\text{first coin is tails}] \cdot \Pr[\text{second coin is tails}] \cdot \Pr[\text{ third coin is tails }]$, so $B = 1 - (1-p_1)(1-p_2)(1-p_3)$

(c) $Y \sim \mathrm{Bin}(20, 1-B)$

(d) Let I be the event that one wise man flipped heads and the other two flipped tails, and J be the event that Wise Man 1 flipped heads and the other two flipped tails. The question is asking for $\Pr[J|I]$. Notice that $J \cap I = I$, since $J \subseteq I$.

$$\Pr[J|I] = \frac{\Pr[J \cap I]}{\Pr[I]} = \frac{\Pr[J]}{\Pr[I]}$$

Since the results of the three coin flips are independent, $\Pr[I] = p_1(1-p_2)(1-p_3)$.

There are three ways event I can happen: Wise Man 1 can flip the heads, or it can be Wise Man 2 or Wise Man 3. All three of these events are independent, so $\Pr[I] = p_1(1-p_2)(1-p_3) + p_2(1-p_1)(1-p_3) + p_3(1-p_2)(1-p_1)$.

So

$$\Pr[J|I] = \frac{p_1(1-p_2)(1-p_3)}{p_1(1-p_2)(1-p_3) + p_2(1-p_1)(1-p_3) + p_3(1-p_2)(1-p_1)}.$$

(e) We'll solve this problem using one of the techniques we used in lecture to compute the expectation of a geometric random variable, and that we also used to solve Question 1 on Homework 7.

We'll make use of the probability B that in a given timestep, at least one of the coins comes up heads.

We'll divide into cases based on the first two flips.

- Let E_1 be the event that the first two flips are heads. $\Pr[E_1] = (1-B)^2$, and given event E, $Z = 2$: we can write this as $\mathrm{E}[Z|E_1] = 2$.
- Let E_2 be the event that the first flip is all tails but there is a heads in the second flip. Then $\Pr[E_2] = (1-B)B$, and $\mathrm{E}[Z|E_2] = 2 + \mathrm{E}[Z]$, since we need to start over waiting for two flips with all tails.
- Let E_3 be the event that the first flip has a heads. Then $\Pr[E_3] = B$, and $\mathrm{E}[Z|E_3] = 1 + \mathrm{E}[Z]$, since we have to start over waiting for two flips in a row with all tails.

The three events E_1, E_2 and E_3 partition the whole sample space (that is, every outcome falls in exactly one of the three events). So

$$\begin{aligned}\mathrm{E}[Z] =& \mathrm{E}[Z|E_1]\Pr[E_1] + \mathrm{E}[Z|E_2]\Pr[E_2] + \mathrm{E}[Z|E_3]\Pr[E_3] \\ =& 2(1-B)^2 + (2+\mathrm{E}[Z])(1-B)B + (1+\mathrm{E}[Z])B.\end{aligned}$$

Now we'll solve for $\mathrm{E}[Z]$. Collecting all $\mathrm{E}[Z]$ terms to one side, we have

$$\mathrm{E}[Z](1-(1-B)B-B) = 2(1-B)^2 2(1-B)B + B,$$

so

$$\mathrm{E}(Z) = \frac{2(1-B)^2 + 2(1-B)B + B}{1-B(2-B)}.$$

(f) Let S be the event that heads is flipped at least once at the first timestep. Let's examine separately the cases S and $\overline{S}$.

- $\Pr[R \cap S]$. $R \cap S$ is the event that in the first timestep, Wise Man 1 flips heads and neither of the other two flips heads. This is the same as the event J from part In our solution to part (d); the probability is $p_1(1-p_2)(1-p_3)$.
- $\Pr[R \cap \overline{S}]$. If heads is not flipped on the first timestep, then we start the experiment from timestep 2, still waiting for the first heads. This is the same as the original experiment, so $\Pr[R|\overline{S}] = \Pr[R]$, so $\Pr[R \cap \overline{S}] = \Pr[R|\overline{S}]\Pr[\overline{S}] = \Pr[R](1-B)$, where B is the value we computed in part (b).

So

$$\Pr[R] = \Pr[R \cap S] + \Pr[R \cap \overline{S}] = p_1(1-p_2)(1-p_3) + \Pr[R](1-B).$$

Solving for $\Pr[R]$, we get

$$\Pr[R] = \frac{p_1(1-p_2)(1-p_3)}{B}.$$

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