2 Counting Tools

(a) Countable

We divide the problem into two cases, where exactly one must be true: (1) A, B are both finite; or (2) A, B are not both finite.

Case (1): Given that A, B are both finite, so we can enumerate the elements of A as $a_0, a_1, a_2, ..., a_m$ and the elements of B as $b_0, b_1, ..., b_n$. Thus, there is a total of $(m+1)(n+1)$ different elements of $A \times B$, which means that $A \times B$ is finite, and thus it's countable by definition.

Case (2): Given that A, B are not both finite (we include the cases where exactly one of them is finite, as it will still be a bijection between $A \times B$ and $\mathbb{N}$ when we do the spiral/diagonal enumeration).

Since A is countable, so we can enumerate the elements of A like this: $a_0, a_1, a_2, ...$ Similarly, the elements of the countable set B can be enumerated as $b_0, b_1, b_2, ...$, so we can write $A \times B$ as:

$$\begin{array}{cccc} (a_0, b_0) & (a_1, b_0) & (a_2, b_0) & \cdots \\ (a_0, b_1) & (a_1, b_1) & (a_2, b_1) & \cdots \\ (a_0, b_2) & (a_1, b_2) & (a_2, b_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Thus, we can enumerate the elements of $A \times B$, i.e. create an injection from $A \times B$ to $\mathbb{N}$, by counting the elements of $A \times B$ in a spiral/diagonal way like this (following the lines and arrows):

$$\begin{array}{ccccc} & 0 & 1 & 5 & \\ & (a_0, b_0) & (a_1, b_0) & (a_2, b_0) & \cdots \\ 2 & (a_0, b_1) & \overset{4}{(a_1, b_1)} & (a_2, b_1) & \cdots \\ 3 & (a_0, b_2) & (a_1, b_2) & (a_2, b_2) & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{array}$$

Thus, there is an injection $f : A \times B \to \mathbb{N}$ as no two elements lie in the same position and by the fact that this mapping certainly maps every element of $A \times B$ to a natural number, because every such element appears somewhere (exactly once) in the grid, and the spiral hits every point in the grid.

On the other hand, due to our counting strategy (no double-counting) as well as the fact that each element of $A \times B$ is unique, so there's also an injection $g : \mathbb{N} \to A \times B$ just by following our counting strategy. Thus, using the Cantor-Bernstein theorem and our Note, so there's a **bijection** $h : A \times B \to \mathbb{N}$, which by definition, shows that $A \times B$ is countable.

(b) Countable

We divide the problem into two cases, where exactly one must be true: (1) A, B_i are all finite; or (2) A, B_i are not all finite.

Case (1): Given that A, B_i are all finite, so we can enumerate the elements of A as $a_0, a_1, a_2, ..., a_m$. Thus, $\cup_{i\in A} B_i = B_{a_0} \cup B_{a_1} \cup B_{a_2} \cup ... \cup B_{a_m}$. Then, we can enumerate of each B_{a_j} that has $n_j \in \mathbb{N}$ elements as $b_{j,0}, b_{j,1}, ..., b_{j,n_j-1}$. Thus, there's a total of $\sum_{i=0}^{m} n_i$ elements in $\cup_{i\in A} B_i$, and this is a finite number. Thus, $\cup_{i\in A} B_i$ is finite, and thus it's countable by definition.

Case (2): Given that A, B are not both finite (again, we include the case where exactly one of them are finite, as it will still be a bijection between $\cup_{i\in A} B_i$ and $\mathbb{N}$ when we do the spiral/diagonal enumeration).

Since A is countable, so we can enumerate the elements of A like this: $a_0, a_1, a_2, ...$ Similarly, the elements of any of the countable set B_{a_j} can be enumerated as $b_{j,0}, b_{j,1}, b_{j,2}, ...$ So we can write $\cup_{i\in A} B_i = (B_{a_0} \cup B_{a_1} \cup B_{a_2} \cup ...)$ as:

$$\begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Thus, we can enumerate the elements of $\cup_{i\in A} B_i$, i.e. create an injection from $\cup_{i\in A} B_i$ to $\mathbb{N}$, by counting the elements of $\cup_{i\in A} B_i$ in a spiral/diagonal way like this (following the lines and arrows):

$$\begin{array}{cccccc} & 0 & 1 & 5 & 6 & \\ & b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\ 2 & b_{1,0} & {}^{4}b_{1,1} & {}^{7}b_{1,2} & b_{1,3} & \cdots \\ 3 & b_{2,0} & {}^{8}b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\ 9 & b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Thus, there is an injection $f : \cup_{i\in A} B_i \to \mathbb{N}$ as no two elements lie in the same position and by the fact that this mapping certainly maps every element of $\cup_{i\in A} B_i$ to a natural number, because every such element appears somewhere (exactly once) in the grid, and the spiral hits every point in the grid.

On the other hand, due to our counting strategy (no double-counting) as well as the fact that each element of $\cup_{i\in A} B_i$ is unique, so there's also an injection $g : \mathbb{N} \to \cup_{i\in A} B_i$ just by following our counting strategy. Thus, using the Cantor-Bernstein theorem and our Note, so there's a **bijection** $h : \cup_{i\in A} B_i \to \mathbb{N}$, which by definition, shows that $\cup_{i\in A} B_i$ is countable.