

Sundry: I worked alone without any help.

## 1 Safeway Monopoly Cards

### Direct Proof

We proceed by a direct proof. This is very similar to the coupon collector's problem.

As usual, we start by writing  $X = I_1 + I_2 + \cdots + I_n$  where  $I_i$  is the number of times we visit Safeway while trying to get the  $i^{th}$  new card (starting immediately after we have gotten the  $(i-1)^{th}$  new card). With this definition,  $I_1$  is trivial: no matter what happens, we always get a new card the first time (since we have none to start with). So  $\mathbb{P}[I_1 = 1] = 1$ , and thus  $\mathbb{E}[I_1] = 1$ .

Then,  $I_2$  has the geometric distribution with parameter  $p = \frac{n-1}{n}$ , and thus, using Theorem 19.2, we have  $\mathbb{E}[I_2] = \frac{1}{p} = \frac{n}{n-1}$ . Similarly, for any  $i = 1, 2, \dots, n$ ,  $I_i$  has the geometric distribution with parameter  $p = \frac{n-i+1}{n}$ , and hence,  $\mathbb{E}[I_i] = \frac{n}{n-i+1}$  by Theorem 19.2.

Now, we could apply the linearity of expectation to get:

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \cdots + \mathbb{E}[I_n] = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n \sum_{i=1}^n \frac{1}{i}$$

Similarly, since all the events  $I_i$  are mutually independent, so the variance of their sum is the sum of their variance by Theorem 16.3, and thus we have:  $\text{var}(X) = \text{var}(I_1 + I_2 + \cdots + I_n) = \text{var}(I_1) + \text{var}(I_2) + \cdots + \text{var}(I_n)$ , where for any  $I_i \sim \text{Geo}(\frac{n-i+1}{n})$ , by Theorem 19.3 we have  $\text{var}(I_i) = \frac{1 - \frac{n-i+1}{n}}{(\frac{n-i+1}{n})^2} = \frac{n(i-1)}{(n-i+1)^2}$ . Thus,

$$\begin{aligned} \text{var}(X) &= \text{var}(I_1) + \text{var}(I_2) + \cdots + \text{var}(I_n) = \frac{n \cdot 0}{n^2} + \frac{n \cdot 1}{(n-1)^2} + \frac{n \cdot 2}{(n-2)^2} + \cdots + \frac{n(n-1)}{1^2} \\ &\implies \text{var}(X) = n \cdot \sum_{i=1}^n \frac{i-1}{(n-(i-1))^2} \\ &\implies \text{var}(X) = n \cdot \sum_{i=1}^n \frac{n-i}{(n-(n-i))^2} = n \cdot \sum_{i=1}^n \frac{n-i}{i^2} \\ &\implies \text{var}(X) = n \cdot \left( \sum_{i=1}^n \frac{n}{i^2} - \sum_{i=1}^n \frac{i}{i^2} \right) \\ &\implies \text{var}(X) = n^2 \cdot \sum_{i=1}^n \frac{1}{i^2} - n \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Since  $\mathbb{E}[X] = n \sum_{i=1}^n \frac{1}{i}$ , so we have that:

$$\text{var}(X) = n^2 \cdot \sum_{i=1}^n \frac{1}{i^2} - \mathbb{E}[X]^2$$

which is the desired result. Q.E.D.