CSM CS70 Fall 2018 Mock Midterm 1 Solutions

CSM 70 Mentors

September 23rd 2018, 6-9PM Soda 306 (HP Auditorium)

1. **T/F (2 pts each)**

   (a) $\neg P \vee Q \equiv \neg Q \implies \neg P$
   True. By writing out the truth table for both logical propositions we find they are equivalent.

   (b) $\neg(P \wedge Q) \equiv P \implies \neg Q$
   True. By writing out the truth table for both logical propositions we find they are equivalent.

   (c) $\forall x \exists y, \neg(P(x) \vee Q(x,y)) \implies \exists x, \neg P(x)$
   True. Note that for the left hand side to be true both $\forall x \exists y, (x)$ and $\forall x \exists y, (x,y)$ must be true. Since (x) for all x there must exist at least one x

   (d) $\forall n, P(n) \implies P(n+1) \equiv \forall n, P(n)$
   False no base case.

   (e) $\forall x \in S, [P(x) \wedge Q(x)] \equiv [\exists x \in S, P(x)] \vee [\exists x \in S, Q(x)]$
   False. Consider when $P(x)$ is true when x is odd and $Q(x)$ is true when x is even.

   (f) If 2 men ever propose to the same W, then it must be that W is the same rank for 2 different men.
   False.

   (g) In stable marriage, if a man M is the last preference for every woman then he cannot get his best choice ($n \geq 2$).
   False. consider a stable matching now add M and another woman who is his best choice. Have all other men rank his best choice last.

   (h) A set of n men and women go through stable marriage as normal. After the pairings terminate, someone hits their head and reverses all of their preferences. Is it possible that, despite this person's complete switch in preferences, the matching is still stable if...

      1. the person is a woman?
      True consider if this woman is the last choice for all men.
      2. the person is a man?
      True. Say the man is at the bottom of the preference list for each woman. By definition, there cannot be a rogue couple because every woman prefers her current match to him.

   (i) In a stable matching instance with 10 men and women, it is possible for 5 men to end up with women who are in the bottom half of their preference lists.
   True consider if everyone has the same preference list then by symmetry half of the men must have paired with women in the bottom half of their preference list.

   (j) If a graph has average degree at least 2, then it must contain a cycle.
   True. If a graph has average degree at least 2 then there must be at least n edges in the graph where n is the number of vertices. Since a tree by definition has n vertices and $n-1$ edges and adding any edge would produce a cycle, then this graph must have a cycle.

   (k) A graph with k edges has $\geq |V| - k$ connected components
   True. To have fewer connected components you want to connect as many vertices as possible with as few edges. The most efficient usage is to connect the vertices in a line and k edges can reduce $k+1$ vertices to 1 connected component. Thus there are $|V| - k - 1$ left unconnected and the one that was connected in a line making a total of $|V| - k$ connected components.

(l) Given a tree where $S = \{v \in V \mid \deg(v) \geq 2\}$ and $T = \{v \in V \mid \deg(v) = 1\}$, $|S| < |T|$
False Consider a line of 4 vertices there are 2 nodes with degree 2 and 2 nodes with degree 1.

(m) The maximum degree of any planar graph is 10
False. Consider a two vertex one edge graph. The maximum degree of this graph is one yet it is planar.

(n) There exists a graph such that no 2 vertices have the same degree. $(n \geq 2)$ Take a graph with n vertices and assume no two vertices have the same degree. We know the maximum degree of any vertex is at most $n-1$. In order to assign every vertex a different degree, we give each vertex a unique degree value from the following set: $\{n-1, n-2, \ldots, 1, 0\}$. However it is not possible to have a graph with both a degree $n-1$ vertex and a degree 0 vertex.

(o) A graph where every set of 3 vertices has at least 2 edges between them is connected. $(n \geq 3)$ Take three vertices and insert two edges between them- because we're dealing with simple graphs these three vertices must be connected, meaning there exists a simple path between any two of these vertices.

(p) There are graphs with an odd number of odd-degree vertices.
False. If we have an odd number of odd-degree vertices, the sum of our degrees must be odd. This is impossible as the sum of the degrees is twice the number of edges.

(q) If $\gcd(x, y) = c$, $\gcd(x, z) = d$, and $x \geq y, z$, then $\gcd(y, z) \leq c, d$.
False. Consider $x = 17, y = 16, z = 8$.

(r) If $\gcd(x, y) = d$ and $\gcd(y, z) = c$, then $\gcd(x, z) \geq \gcd(c, d)$
True. Say $gcd(c, d) = a$. We know then know that $x|a$, $y|a$, $z|a$. So $gcd(x, z)$ must be at least a

(s) $a^{pq-1} \equiv 1 \mod pq$ if p and q are prime, $p \neq q$, and a is not a multiple of p or q.
False. pq can actually be composite, but you must modify the proof to FLT to only consider the numbers that are coprime to both p and q, of which there are $(p-1)(q-1)$.

(t) Let φ be the Euler totient function, i.e. $\varphi(n) = |\{1 \leq k \leq n \colon gcd(k, n) = 1\}|$. For all m and n relatively prime, we have $\varphi(m \cdot n) = \varphi(m)\varphi(n)$.
True by Chinese Remainder Theorem.

2. **Short Answer (4 points each)**

   (a) What are all the possible solutions to the expression, $((X \wedge Y) \to Z) \wedge ((Y \wedge Z) \to X) \wedge ((Z \wedge X) \to Y)$?

   $(X, Y, Z) \in \{(T, T, T), (F, F, F), (T, F, F), (F, T, F), (F, F, T)\}$

   (b) A planar graph has 12 faces, all of which are pentagons. How many edges are there in this graph?

   12 * 5 counts each edge twice. So number of edges is 30.

   (c) A double elimination bracket is a tournament style where if you lose a round, you are sent to compete in a "loser's" bracket; only after losing twice are you completely eliminated from the tournament. We can make a graph out a double elimination tournament by making each team a vertex and connecting two vertices if they play a match against each other. If two teams play multiple matches, there can be multiple edges between the same two vertices (so this is not a simple graph, but rather a type of graph known as a multigraph). What is the maximum number of edges in this multigraph?
   The number of edges is the number of matches played. Everybody has to lose twice except for the champion who can lose at most once, and in a single match there's only 1 loser. Therefore there must be $2n - 1$ matches.

   (d) To separate a hypercube of dimension n into 2 hypercubes of dimension $n - 1$, how many edges do you have to cut?

   2^{n-1}

(e) You have a hypercube of dimension n. Create a new graph by doing the following: For every 4-cycle in the hypercube, add a vertex. Connect two vertices if their corresponding 4-cycles share an edge.

1. How many vertices are in this new graph?

   $2^n n(n-1)$. Given a vertex of the form $\{0,1\}^n$, construct a 4 cycle by picking 2 bit places i and j. Swap i to get to a new vertex v_2 and then swap j to get to v_3, then return to v_1 by swapping the bits again. There are $n(n-1)$ ways of choosing i and j, and there are 2^n vertices.

2. How many edges are in the graph?

   Consider any edge $v_i - v_j$. We show that this edge is adjacent to $n-1$ 4 cycles. As we said above, we can construct a 4 cycle by choosing a starting vertex, picking a first bit k, then picking a second bit m. If we choose v_i as the starting point, then the edge $v_i \to v_j$ refers to one bit position. Now, picking another bit position gives us a cycle that includes this edge, so there are $n-1$ such choices. Therefore, each edge is adjacent to $(n-1)$ 4-cycles.
   So, each 4 cycle is adjacent to $4(n-1)$ 4 cycles. We saw earlier that there are $2^n n(n-1)$ vertices in the new graph, and here we have shown that they each have degree $4(n-1)$. So, sum of degrees is $2^{n+2}n(n-1)^2$. So the number of edges is $2^{n+1}n(n-1)^2$

(f) On a 12-hour clock, what time will it be 2^{30} hours after midnight?

$$2^2 \equiv 4 \mod 12$$
$$2^4 \equiv 4 \mod 12$$

Notice that all even powers of 2 would be 4, so $2^{30} \mod 12 = 4$ ‘

(g) A graph has 12 nodes labeled 0....11. Edges are added to the graph in the following way: Starting at vertex 0, for each vertex i, an edge is drawn to all x s.t $3x + i \equiv 2 \mod 12$. How many edges are present in this graph?

(h) How many values of x are there such that $150 \equiv x \mod 2x$?

We have $150 - x = 2xk$ for $k \in \mathbb{Z}$. Thus $150 = (2k+1)x$. Factorize 150 into $150 = 2 \times 3 \times 5 \times 5$. Any x such that 150 can be written as x multiply by an odd number would work. Notice that x is determined once $2k+1$ is determined. Thus, possible values of $2k+1$ are $1, 3, 5, 15, 25, 75$, so the possible number of x is 6.

(i) Find an expression for $5^{2(7^n - 6^n)} \pmod{26}$

$$25^{7^n - 6^n} \equiv (-1)^{7^n - 6^n} \mod 26$$

7^n is odd and 6^n is even, so $7^n - 6^n$ is odd.
Thus $(-1)^{7^n - 6^n} \equiv -1 \mod 26$ for $n > 0$, and $1 \mod 26$ for $n = 0$

(j) Let p be a prime number. Find $(p-1)! \pmod{p}$
We expand $(p-1)! \mod p$ to be $1 * 2 * 3 * ... * (p-1) \mod p$
Because p is prime, every number from 1 to $p-1$ has a unique multiplicative inverse.
All the numbers 2 through $p-2$ will "pair up" with their own inverses and cancel out to become 1.
However, we must consider that the inverses of the numbers 2 through $p-2$ are not themselves, or else the claim will not hold.
We show that the only numbers in which the inverse are their own inverses are 1 and -1
If an integer k is its own inverse, then $k * k \equiv 1 \mod p$
Therefore, $k^2 - 1 \equiv 0 \mod p$ and we see that $p | k^2 - 1 \implies p | (k+1)(k-1)$
Therefore, $p | (k+1)$ or $p | (k-1)$. This can only happen if $k = -1$ or $k = 1$ since p is prime.
Thus, the numbers 2 through $p-2$ will map to another number that is their inverse.
The inverses are all unique, and therefore, $2 * 3 * ... * (p-2) \equiv 1 \mod p$
Therefore, $(p-1)! \mod p \equiv (p-1) \mod p$

3. **Proofs (5 points each)**

(a) Prove that $(1+x)^n \geq 1+xn$, for $x \geq 0$ We wil proceed by induction on n.

- Base Case:
  $n = 0 : (1+x)^0 = 1 \geq 1 + x \cdot 0 = 1$
  $n = 1 : (1+x)^1 = 1 + x \geq 1 + x$
  $n = 2 : (1+x)^2 = x^2 + 2x + 1 \geq 1 + 2x$
- Inductive Hypothesis: Assume the claim holds for some $n = k$, that is $(1+x)^k \geq 1 + xk$, for $x \geq 0$
- Inductive Step: Show the claim is true for $n = k+1$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\
&\geq (1+xk) \cdot (1+x) && \text{By the I.H.} \\
&= 1 + xk + x + x^2k \\
&= 1 + x(k+1) + x^2k \\
&\geq 1 + x(k+1) && x^2k \geq 0
\end{aligned}$$

(b) Let $b_1 = 1$ and $b_{n+1} = 5b_n^3$. Prove that $b_n \leq 5^{3^n}$. Hint: You will need to strengthen the inductive hypothesis.

We will prove the stronger statement $b_n \leq 5^{3^n - 1}$

- Base Case: $n = 1$
- Inductive Hypothesis: Assume true for $n = k$, that is $b_k \leq 5^{3^k - 1}$
- Inductive Step: Show true for $n = k+1$.

$$\begin{aligned}
b_{k+1} &= 5b_k^3 && \text{By definition} \\
&\leq 5(5^{3^k - 1})^3 && \text{By the I.H.} \\
&= 5(5^{3^{k+1} - 3}) \\
&= 5^{3^{k+1} - 2} \\
&\leq 5^{3^{k+1} - 1}
\end{aligned}$$

(c) Prove that if you direct the edges of a (connected) tree, then there must be at least one vertex with zero in degree, and one vertex with zero out degree.

A tree of n vertices has $n-1$ edges. Therefore, if we direct the edges, it must have total out-degree of $n-1$ as well.

If we look at the average out-degree, it must then be $\frac{n-1}{n} < 1$

Because not everyone can be above average, there must be at least 1 vertex with 0 out-degree.

Similarly, the tree must have in-degree of $\frac{n-1}{n}$ as well, and thus the average in-degree for every vertex is $\frac{n-1}{n} < 1$. Because not everyone can be above average, again, there must exist at least one vertex with 0 in-degree.

(d) Prove that $n^3 \not\equiv 1 \mod 5 \implies n \not\equiv 1 \mod 5$
We proceed with proof by contraposition.
$n \equiv 1 \mod 5 \implies n^3 \equiv 1 \mod 5$
If $n \equiv 1 \mod 5$, then $n^3 = 1^3 \mod 5$ and the claim holds true.

(e) Prove that if p is a prime number, then $x^2 \equiv p \mod p^2$ has no solutions.
We rewrite the equation to be: $x^2 - p \equiv 0 \mod p^2$
Therefore, we know that $x^2 - p = kp^2$ for some integer k
We divide both sides by p to get:
$\frac{x^2}{p} - 1 = kp$
This means that $\frac{x^2}{p}$ is an integer.
If p is a factor of x^2, then p must appear in the prime factorization of x^2.
We take a look at the prime factorization of x
Let $x = p_0^{\alpha_0} * p_1 * (\alpha_1) * ... * p_n^{\alpha_n}$ be the prime factorization of x.
Because x^2 is a squared number, all of the prime factors of x will appear twice in the prime factorization of x^2
Then we have $x^2 = p_0^{2\alpha_0} * p_1^{\alpha_1} * ... * p_n^{2\alpha_n}$
Therefore, we see that p^2 must also be a factor of x^2 as since we assumed that p divided x^2, then it must be one of the prime numbers in the prime factorization of x^2, and because x^2 is a squared number, each prime is raised to an even power, so p^2 is a factor of x^2.
However, if this happens, then $x^2 \equiv 0 \mod p^2$ rather than $p \mod p^2$ and we reach a contradiction as we originally the expression was:
$x^2 \equiv p \mod p^2$
Therefore, the original equation $x^2 \equiv p \mod p^2$ must not have any solutions.

(f) Show that a connected undirected graph with at most two vertices of odd degree has an Eulerian walk.

Let's proceed with proof by cases.

- 0 vertices of odd degree. If there are 0 vertices of odd degree, then every vertex must have even degree. By Euler's theorem, there exists a Eulerian tour on the graph, which by definition is a Eulerian Walk.
- 1 vertex of odd degree. If there exists exactly 1 vertex of odd-degree, then the sum of degrees must be odd. This is not possible as we know that the sum of the degrees must equal to $2|E|$ which must be even. $|E|$ is the number of edges in the graph.
- If there are exactly 2 vertices of odd-degree, let's call them u and v. If we add a vertex w to the Graph, and connect it to only u and v. We thus have added edges $(u, w), (w, v)$ to the Graph. In the new graph, every vertex must have even degree as we have added one degree to both u and v and w as degree 2. Therefore, by Euler's theorem there must exist a Eulerian Tour on the graph. If we then delete vertex w and its adjacent edges from the graph, the remaining part of the tour must be a Eulerian Walk as the tour must have visited all of the other edges in the graph exactly once.

4. **Stable Marriage (5 points each)**

Suppose there are n men, n women, and n pet dogs that we want to group into trios of one man, woman, and dog each. The men and women have preference lists for each other as in the usual stable marriage problem. In addition, the women and dogs have preference lists for each other. However, each man has a set of dogs he she hates. We want to find a way to group the men, women, and dogs into trios of one man, one woman, and one dog each such that the following stability criteria hold:

1. No woman and dog not in the same trio prefer each other to another dog and woman in their respective trios
2. Each dog is in a trio with the best woman he can get in any trio satisfying condition 1
3. No man and woman not in the same trio prefer each other to their respective woman and man in another trio
4. No man is in a trio with a dog that she hates

(a) What can we do to ensure there are no rogue woman, dog pairings and each dog gets his optimal woman?
Solution: We can run stable marriage with the "dogs" as men and the women as women. This will output a dog-optimal, stable pairing.

(b) Building on part (a), how can we devise an algorithm that outputs a stable matching if one exists, or terminates if it doesn't?
Solution: Assume that we ran the algorithm from part a. For the men's preference lists, move all of the women that are paired with dogs they hate to the bottom of their preference lists, while maintaining their order. For example, Say Male M had preference list $1, 2, 3, 4, 5$ but hates the dog that women 1 and 3 were paired with. His preference list is $2, 4, 5, 1, 3$ now. Try running the stable marriage women, except with the women proposing and the men accepting. If a stable pairing is reached before any man is matched with a dog that he hates, then one exists, otherwise, it doesnt.

5. **Buddy System (2/2/2/5 points)**

A k-regular graph is a graph where every vertex has degree k.
A k-connected graph is a connected graph where at least k vertices must be removed to disconnect the graph. So, a 1-connected graph is just a connected graph.

(a) Suppose a graph is k connected. True or False: every vertex in the graph has degree $\geq k$. (hint: ask yourself: which is the "stronger" property?

Solution True. To disconnect a k-connected graph, we must remove at least k vertices. This means that no vertex can be connected to fewer than k other vertices, and consequently, that every vertex has degree $\geq$ k.

(b) True or false: for every value of k. there is a k-regular graph that is k-connected.

Solution True. For each value of k, we can create a graph with $k + 1$ vertices, each connected to the other k vertices. Each vertex will have degree k and k edges must be removed to disconnect the graph.

(c) What is the fewest number of edges in a 2-connected graph with $|V|$ vertices?

Solution The fewest number of edges is $|V|$ as each vertex must have degree ≥ 2.

(d) Prove that if a graph is 2-connected, then every vertex is a part of a cycle (a vertex i is "part of" a cycle if edges $(j, i), (i, k)$ occur in the cycle (in order) for some j, k).

Solution Pick an arbitrary vertex v. Now pick one of its neighbours u, and pick a final arbitrary vertex $z \neq v \neq u$. Now, since the graph is 2-connected, removing u leaves the graph connected. So there is a path P_1 from $v \to z$ that does not pass through u. Similarly, realize that there exists a path $u \to z$ that does not pass through v. So $v \to u \to z$ is a path P_2 from $v \to z$ that is distinct from P_1. Therefore, the union of these two paths, after removing overlapping edges, must leave a cycle that contains v.