2 Markov's Inequality and Chebyshev's Inequality

(a) True (Direct Proof)

We proceed by a direct proof. Using Theorem 16.1, we have that $\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, which gives us that:

$$\mathbb{E}[X^2] = \text{var(X)} + \mathbb{E}[X]^2 = 9 + 2^2 = 13$$

This gives the value we desire. Q.E.D.

(b) True (Proof by Contradiction)

Assume, for a contradiction, that $\mathbb{P}[X \leq 1] > 8/9$. Let R denote the assertion that $\mathbb{E}[X] = 2$. Since the value of X is never greater than 10, so we have that

$$\mathbb{E}[X] < 1 \cdot 8/9 + 10 \cdot (1 - 8/9) = 2$$

In other words, the expectation of X would never reach 2, which implies $\neg R$. So $R \wedge \neg R$ holds, which gives the contradiction. Q.E.D.

(c) True (Direct Proof)

Givent that $\mu = \mathbb{E}[X] = 2$ and $\text{var}(X) = 9$, so using Theorem 18.3 (Chebyshev's Inequality), we have that:

$$\mathbb{P}[|X - \mu| \geq 4] \ \leq \ \frac{\text{var}(X)}{4^2}$$

which gives that $\mathbb{P}[|X - 2| \geq 4] \leq \frac{9}{16}$. Then, since $\mathbb{P}[|X - 2| \geq 4] = \mathbb{P}[X \geq 6] + \mathbb{P}[X \leq -2]$, and since $\mathbb{P}[X \leq -2]$ is nonnegative, so we can conclude that:

$$\mathbb{P}[X \geq 6] \ \leq \ \mathbb{P}[|X - 2| \geq 4] \ \leq \ \frac{9}{16}$$

This gives the result we desire. Q.E.D.

(d) False (Counterexample)

Consider random variable X where $\mathbb{P}[X = 6] = \frac{11}{32}, \mathbb{P}[X = -10] = \frac{1}{160}, \mathbb{P}[X = 0] = \frac{13}{20}$.

Here, we can first verify that all values X can take on is not greater than 10. Then, $\mathbb{E}[X] = 6 \cdot \frac{11}{32} + (-10) \cdot \frac{1}{160} + 0 \cdot \frac{13}{20} = 2$ as desired, and $\mathbb{E}[X^2] = 6^2 \cdot \frac{11}{32} + (-10)^2 \cdot \frac{1}{160} + 0^2 \cdot \frac{13}{20} = 13$, which gives that $\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 13 - 2^2 = 9$ as desired. In other words, this is a random variable that satisfies all constraints. Yet,

$$\mathbb{E}[X \geq 6] \geq \mathbb{E}[X = 6] = \frac{11}{32} > \frac{9}{32}$$

Thus, this is a valid counterexample. Q.E.D.