Question 1: True/False

b)i) This statement says “For all *x* cats, there exists a *y* cat such that x’s eyes are green **and** …” Not every cat has green eyes.

d)iii) The graph is planar. Flip the positions of the two vertices in the middle. The resulting graph after flipping their positions only has one crossing, which can be removed more easily. (Also, this graph is $K_{3, 3}$ with one edge removed, not $K_{3, 3}$.)

Question 2: Short Answers

a) Some responses did not distribute correctly and ended up as $y/3 + 2$

b) Arithmetic errors while executing Euclid’s algorithm ended up with 1

c) Getting 1 after not noticing the extra factor of 37

d) Incorrectly solving for m instead of x in the equation $11m = 5x - 7$

e) Some common errors:

- Incorrectly computing $7^{-1} \bmod 143$ instead of $7^{-1} \bmod 120$
- Sign errors (answering 17 instead of -17)

f) i) Answers that were off by 1 in the denominator ($1/(q - 1)$ instead of $1 / q$)

f) ii) Some common errors:

- Assuming an ordering of the roots answers involving the factor $q! / (q-k)!$ Instead of ${q \choose k}$
- Using q^k in the denominator instead of $q^{k + 1}$
- Not accounting for distinctness of the roots by using the balls and bins formula
- Using the wrong factor when selecting the k roots (Not using ${q \choose k}$)

g) Common sign errors answering with -3 instead of 3 etc

h) Common errors:

- Assuming $\frac{1}{6}$ of 100 packets are corrupted instead of $\frac{1}{6}$ of total packets sent (Answering 134 instead of 150)
- Solving for the case of erasures instead of corruptions (Answering 120 instead of 150)

i) Answers not running P in any way do not receive credit. Note that calling TestHalt on P does not run it. Also, you cannot assume TestHalt exists.

j) Not choosing suits for each card - each card can take on one of 2 suits.

l)i) Choosing 3 edges, i.e. doing ((n choose 2) choose 3) instead of choosing 3 vertices. One cannot guarantee that the three edges are part of the same (potential) triangle.

l)ii) Very little partial credit is given for this question - the responses that receive partial are clearly outlined in the rubric. Very good job to those who got this!

m) Leaving expectations or indicators in the response - this received no credit.

r) Answers involving an integral over the normal pdf received no credit. Also note that when normalizing the variance, you divide terms by the square root of the variance (i.e. standard deviation), and when "factoring" out a constant in a variance calculation, you square the constant.

s) The only answer that got partial credit was one sixth, since that is the probability of the complement of the event we're interested in.

u) When taking the derivative of -(1 - x)^2 many students got -2(1 - x) but probably forgot to use the chain rule on the -x on the inside to receive the correct answer.
Another common error was failing to consider the valid domain of the pdf. The pdf is defined as 2(1 - x) only when x is between 0 and 1. This makes sense because the CDF is piecewise, and thus the pdf should be as well.

v) Some students left answers in terms of pi. These received no credit. No credit was given to solutions that were not valid probability distributions (that is, solutions for which pi_1 and pi_2 didn't add up to 1, or for which pi_1 or pi_2 could be negative for some choice of a and b).

w)

- In the recurrence, many students added 1, or mixed up the terms in an otherwise correct answer. These answers received partial credit.
    - Additionally, someone wrote a recurrence-like statement without changing the value of i or j - these answers were not given any credit.

Question 3: Trees
3a)

- Many students had as base case a tree with two vertices (and a single edge). The base case should be a single vertex, because the induction may create single-vertex trees.
    - Note that a single-vertex tree can trivially be colored with one color, and hence with two colors. This is the only base case you need. We did not penalize for a 2-vertex base case (though we did include it as a 0pt rubric item).
- Many students were confused about the structure of the induction: you must start with an arbitrary tree with *n* vertices, and remove an edge (or a vertex) to create smaller trees, to which you can apply the induction hypothesis.

    - Many instead started with a tree of (n - 1) vertices and added a vertex to it. This is a classic case of doing induction “backwards.”
- When an edge e is removed, it creates two disjoint subtrees, each of which can be 2-colored by the induction hypothesis. When you add e back in, you need to be sure that its endpoints have different colors: many people either forgot to do this, or just said that we can assume e connects vertices of different colors in the two trees. Note that we don’t get to choose the endpoints of e: they are what they are.
    - Instead, we need to switch the colors in one of the subtrees (if necessary) to ensure the endpoints of e get different colors.
- Many students tried to use the fact that every tree has at least one leaf, so that they could remove a leaf edge rather than an arbitrary edge: this gives you one tree with n-1 vertices and the other with 1 vertex. This approach is fine, except that you need to be explicit about the fact you are assuming the existence of a leaf, and that this is the edge you are removing; you also need to be careful not to do the induction “backwards” (see above).
- Some students gave a non-inductive argument based on leveling the tree. This received at most 2pts, because the question explicitly asked for an induction proof and that is what we were testing.

3b) There were two ways to obtain full credit (1) Argue that the LHS and RHS both counted the **total degree** of the graph, or (2) Prove that the LHS double counted the amount of edges in the tree.

Both approaches required recognizing (1) A tree has (n - 1) edges, and (2) The total degree of the graph is equal to 2|E|. Common errors were:

- Proving the RHS was equal to the total degree, but not connecting it to the LHS. This was given partial credit.
    - We realize that this step may be trivial, but without explicitly stating that the RHS represented the total degree of the graph, the proof is incomplete!
- Some sort of argument involving induction. This is potentially a valid approach, but it’s tricky to get the inductive step right.
    - In particular, many answers were prone to build up error.
- Simply stating something along the lines of “the LHS double counts the edges of T”, without further justification were incorrect.

3c)

- Many students plugged in n = 6 and tried to reason from there, but this doesn’t suffice as proof for all n >= 6.
- An inductive proof doesn’t work here.
- Many arithmetic errors led students on the wrong path of the proof. We were lenient on arithmetic errors only if the structure and direction of the proof remained unchanged.
- Many students subtracted n/2 +2 from 2(n - 1) and got 3n/2 instead of 3n/2 - 4.

Question 4: Wilson's Theorem

4a) Almost everyone received either full or no credit on this question. Full credit was rewarded for correctly applying the correct definition of mod. A lot of you got full credit, awesome job! There were still common errors:

- Using an incorrect definition of mod: the correct definition is $a = b$ (mod n) iff $n \mid (a - b)$ iff $a - b = kn$ for some integer k. Common incorrect definitions confused mod with extended Euclid: for instance, there were several $ac - b = kn$ or $ax + by = kn$.
- Writing $q = kn$ instead of $n = kq$; the statement is no longer correct.
- We received many proofs that stated the result as a fact, or that went directly from $a = b$ (mod kq) to $a = b$ (mod q).
- Some valid solutions work with fractional quantities; but if a fraction-based solution omits key facts (e.g. certain fractional quantities are integers) they did not receive credit.

4b) Most full credit answers followed the solutions. Common errors include:

- Saying n has a factor, or $(n - 1)!$ has no multiplicative inverse without proper explanation of why it helps prove the statement did not receive credit.
  - While there are valid proofs that apply these facts, they require many more steps and justification. It is difficult to differentiate between an answer that is just stating facts vs. an answer actually working towards such a proof. We only gave credit when the relationship between inverses and the statement $(n - 1)!$ =\= -1 (mod n) were **made clear** (e.g. we use the fact that -1 has an inverse or is its own inverse).
- Some solutions had the right idea in applying Part (a) but stated something logically incorrect, such as: if $(n - 1)! = 0$ (mod q) then $(n - 1)! = 0$ (mod n). These solutions got 1.5 points due to the logical error.
  - Even if other parts of the solution look good, one thing we are testing is accurately reasoning about negations and contrapositives!
- Some stated $(n - 1)! = 0$ (mod n) from only noting a single factor q of n. We can't conclude this from one factor, as q =/= 0 (mod n). These responses received no credit.
- Some answers remedied the above error by finding $1 < p < q < (n - 1)$ so that $pq = n$, and $(n - 1)! = 0$ mod n. However, this is also insufficient.
  - If n is the square of a prime number, we can't always find such p and q. Solutions like this that were explained received 1.5 points. Consider $n = 4$ for a counterexample.
- Some students wanted to cite the FLT proof. However, for FLT, we assume we are working mod a prime, and nothing is said about what happens mod a composite, so simply citing the lecture notes does not suffice.
- Applying part (a) and instead work with the expression $(q - 1)! = -1$ (mod q), which is not given by part (a), received no credit.
- There were quite a few incorrect applications of Part (a). Part (a) says that $a = b$ (mod n) implies $a = b$ (mod q) when q is a divisor of n, not that a (mod n) = a (mod q). Several answers incorrectly tried to argue that $(n - 1)$ =/= $(q - 1)$ despite both being equivalent to -1 (mod n).

4c) Full credit required understanding the correct equation or modular congruence to solve and using primality to complete the proof.

- Many students showed that $x = 1$, $n - 1$ were their own inverses but missed that the point of the question was to show that no other values of x were their own inverses.
- One common error was a result of circular logic. Students would claim that $(n - k)^2 = n^2 - 2k + k^2 \equiv k^2 \pmod{n}$ so $k^2 \equiv 1 \pmod{n}$ which could only be true if k was 1, -1 (mod n) … but showing that $k^2 \equiv 1 \pmod{n}$ implies k is 1, -1 is the actual question!
- Another common error was to try to argue from a "unique inverse" direction. This approach was flawed because the inverse of x could still be x as long as the inverse of y was not x for any other y.
- Many students believed that if $\gcd(n, x) = 1$, then x could not be its own inverse mod n. Being coprime only says that there exists an inverse for x, it does not say that x cannot be its own inverse (e.g. $\gcd(n, 1) = 1$).
- Many students correctly set up the equation $(x - 1)(x + 1) = kn$ but had an incorrect proof for why $x = 1$, $n - 1$.
    - Some students said that k "must" be 0 without a proof (this is always dangerous). Of course this is false because when $x = n - 1$, then $k = n - 2$.
    - Some students claimed that there could only be 2 roots for this polynomial because it had degree 2. That is only true for each fixed value of k. For some values of k, $x = 1$, $n - 1$ are not valid solutions to the quadratic. The point is that one needs to show no other values of k besides 0 and $n - 2$ give integer solutions of x that are between 1 and $n - 1$.
- Many students did not demonstrate an understanding of why primality is essential to this problem. The claim that $(x - 1)(x + 1) = 0 \pmod{n}$ has at most 2 roots is not true for general n.

4d) For full credit, it was necessary to describe how part (c) shows that $(n - 2)! \equiv 1 \pmod{n}$, e.g. by pairing up the elements of $\{2, 3, \ldots, n - 2\}$. We did not give full credit to students who did not describe a pairing or an equivalent construction.

- Some students think that $k(n - k) \equiv -1 \pmod{n}$ for all values of k (not just $k = 1$). Of course, this is false as one can see from $k = 2$ (-4 is not equivalent to -1 (mod n) in general). No credit was awarded for solutions that relied on this misconception.
- Some students tried to argue that each number x in $\{2, 3, \ldots, n - 2\}$ has a unique inverse (mod n), via a gcd argument, so their product should be 1 (mod n). Unfortunately these students missed the point of part (c) and didn't realize that it was vital for x to not be equal to its inverse. If $x^2 \equiv 1 \pmod{n}$, then pairing up the elements of $\{2, 3, \ldots, n - 2\}$ doesn't work.
- We gave very little partial credit to students who (correctly) stated that $(n - 2)! \equiv 1$ (mod n) unless they explained how part (c) gave them a pairing argument to evaluate the product. This is because we are asking students to prove $(n - 1)! \equiv -1 \pmod{n}$ so it

is clear that (n - 2)! \equiv 1 (mod n). The crux of the problem is explaining *why* (n - 2)! \equiv 1 (mod n).

Question 5:

a)

- Many binomial coefficients did not account for not resampling the same balls, i.e. products like (10 choose 5)*(10 choose 3)*(10 choose 2)
- Denominator is derived from stars-and-bars rather than (1/3)
- Incorrect (16 choose 10) to account for choosing red balls

b)

- Many students forgot one of the binomial coefficients
- Denominator is derived from stars-and-bars rather than (1/3)

c) Below are some common mistakes

- Unnecessary factors of ½ in numerator or denominator of P[E | Urn 1]
- Omitting the ½ factors in either the numerator alone or the denominator alone when setting up Bayes Rule. This resulted in only 1 point for the Bayes Rule part.
- Omitting factors for P[E | Urn 1]
    - Although this was common, if the Bayes Rule was set up correctly and consistently (i.e. P[E | Urn 2] was computed with the same assumptions that P[E | Urn 1] was) we still awarded full credit for Bayes.

Question 6:

6e)

- Many students left the limit as Phi(\delta \sqrt{n/2}), but we had to give this 0 points for consistency with the rubric. If they had let n grow, this would be zero.
- Many explanations tried to tie S_n to 4n, but LLN or CLT allow us to make statements about S_n / n, or about the normalized average. To get the third point students had to use LLN/CLT correctly on an expression related to the average (or use part (d) or Chebyshev's inequality).

Question 7:

a)

- Many students treated the number of of uncorrupted packages as the product of (1-p) and the total number of packages received. This is not true: It might very well be that all or none of the received packages get lost.
- Showing that the number of uncorrupted packages has expectation and variance (1-p)lambda is not sufficient to show that its Poisson distributed; many other distributions have same mean and variance.
- Rephrasing the question by mentioning Poisson thinning or describing the corruption process in different words did not receive any points.

- Many students mistook X_0 for the number of packets received one month ago. These two variables have identical distributions, but they are not the same, and this matters when conditioning on the number of packages received.
- The big sum involving binomial and Poisson probabilities was often not explicitly justified through the law of total probability.

b)

- Many students were able to identify the Poisson distribution but could not get the parameter correctly.
- Several students wrote Poisson$[(1-p)\lambda]$ instead of Poisson$[(1-p)^n \lambda]$

c)

- Several students made an error in either the lower or the upper bound of summation in the rate $\sum_{k=0}^n (1-p)^k \lambda$
- Several students took the upper bound of summation to be ∞ rather than n.

d)

- Using LLN incorrectly to conclude probability is 1 (“By LLN, the probability that X_0 is ‘far’ from the mean λ tends to 0”)
    - This reasoning is not correct, because LLN requires scaling by 1/lambda rather than its square root.
- Re-expressing the probability in terms of sums of Poisson probabilities doesn’t constitute a valid answer.
- Any expression that still involves a lambda cannot be correct: lambda is sent to infinity!