Final Practice

1 Diversify Your Hand

You are dealt 13 cards from a standard 52 card deck. Let X be the number of distinct values in your hand (The 13 possible values are Ace, 2, 3, 4, ..., Jack, Queen, King). For instance, the hand (A, A, A, 2, 3, 4, 4, 5, 7, 9, 10, J, J) has 9 distinct values.

Calculate $E[X]$.

Solution:

Let X_i be the indicator of the ith value appearing in your hand. Then, $X = X_1 + X_2 + ... + X_13$ (Let 13 correspond to K, 12 correspond to Q, 11 correspond to J). By linearity of expectation then, $E[X] = \sum_{i=1}^{13} E[X_i]$. We can calculate $\mathbb{P}[X_i = 1]$ by taking the complement, $1 - Pr[X_i = 0]$, or 1 minus the probability that the card does not appear in your hand. This is $1 - \frac{\binom{48}{13}}{\binom{52}{13}}$. Then, $E[X] = 13\mathbb{P}[X_1 = 1] = 13(1 - \frac{\binom{48}{13}}{\binom{52}{13}})$.

To calculate variance, since the indicators are not independent, we have to use the formula $E[X^2] = \sum_{i=j} E[X_i^2] + \sum_{i\neq j} E[X_iX_j]$.

2 Combining Distributions

(a) Let $X \sim Pois(\lambda), Y \sim Pois(\mu)$ be independent. Prove that $X + Y \sim Pois(\lambda + \mu)$.

Hint: Recall the binomial theorem, which states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

(b) Let X and Y be defined as in the previous part. Prove that the distribution of X conditional on $X+Y$ is a binomial distribution, e.g. that $X|X+Y$ is binomial. What are the parameters of the binomial distribution?

Hint: Your result from the previous part will be helpful.

Solution:

(a) If we want $X+Y$ to take on the value k, we can have X take on any value from 0 to k as long as $Y=k-X$. Since the events of these happening are disjoint, we can write

$$\mathbb{P}[X+Y=k]=\sum_{x=0}^{k}\mathbb{P}[X=x\cap Y=k-x]$$

Since X and Y are independent, this becomes

$$\begin{aligned}\mathbb{P}[X+Y=k]&=\sum_{x=0}^{k}\mathbb{P}[X=x]\cdot\mathbb{P}[Y=k-x]\\&=\sum_{x=0}^{k}\frac{\lambda^{x}e^{-\lambda}}{x!}\cdot\frac{\mu^{k-x}e^{-\mu}}{(k-x)!}\\&=e^{-\lambda-\mu}\sum_{x=0}^{k}\frac{\lambda^{x}\mu^{k-x}}{x!(x-k)!}\end{aligned}$$

If we then factor out a $\frac{1}{k!}$ from each term of this summation, we get

$$\begin{aligned}\mathbb{P}[X+Y=k]&=\frac{e^{-(\lambda+\mu)}}{k!}\sum_{x=0}^{k}\lambda^{x}\mu^{k-x}\frac{k!}{x!(k-x)!}\\&=\frac{e^{-(\lambda+\mu)}}{k!}\sum_{x=0}^{k}\lambda^{x}\mu^{k-x}\binom{k}{x}\end{aligned}$$

Applying the binomial theorem, we can replace the summation by $(\lambda+\mu)^{k}$, so we have

$$\mathbb{P}[X+Y=k]=\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{k}}{k!}$$

Thus, the PMF of $X+Y$ matches that of a Poisson random variable with parameter $\lambda+\mu$. Since the PMF uniquely defines the distribution, this means that $X+Y\sim Pois(\lambda+\mu)$.

(b)

$$\begin{aligned}P(X=k|X+Y=n)&=\frac{P(X=k\cap X+Y=n)}{P(X+Y=n)}\\&=\frac{\frac{\lambda^{k}e^{-k}}{k!}\times\frac{\mu^{n-k}e^{-\mu}}{(n-k)!}}{\frac{(\lambda+\mu)^{n}e^{-(\lambda+\mu)}}{n!}}\\&=\frac{n!}{k!(n-k)!}\times\frac{e^{-\lambda}e^{-\mu}}{e^{-(\lambda+\mu)}}\times\frac{\lambda^{k}\mu^{n-k}}{(\lambda+\mu)^{n}}\\&=\binom{n}{k}(\frac{\lambda}{\lambda+\mu})^{k}(\frac{\mu}{\lambda+\mu})^{n-k}\\&=\binom{n}{k}(\frac{\lambda}{\lambda+\mu})^{k}(1-\frac{\lambda}{\lambda+\mu})^{n-k}\end{aligned}$$

Recall that if $X=k$ and $X+Y=n$, then $Y=n-k$.

3 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

Solution:

(a) Integrate:

$$\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = c\int_1^2 x^{-2}\,\mathrm{d}x = -cx^{-1}\Big|_{x=1}^2 = -c\left(\frac{1}{2}-1\right) = \frac{c}{2},$$

so $c = 2$.

(b) To find $\mathbb{P}(A)$,

$$\mathbb{P}(A) = \int_{1.5}^2 f_X(x)\,\mathrm{d}x = 2\int_{1.5}^2 x^{-2}\,\mathrm{d}x = -2x^{-1}\Big|_{x=1.5}^2 = -2\left(\frac{1}{2}-\frac{2}{3}\right) = \frac{1}{3}.$$

The conditional PDF is thus

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(A)} = 6x^{-2}, \qquad x \in [1.5, 2].$$

4 Bus Arrivals

Buses arrive at a bus stop according to a Poisson Arrival Process with $\lambda = 1$ bus per 20 minutes starting time 0.

(a) What's the probability there are no arrivals between times 1 and 3?

(b) What's the probability that the fourth bus arrives within 5 minutes of the third?

(c) You arrive at the bus stop at time s. What is the expected amount of time you wait?

Solution:

(a) This interval has length 2, so the number of buses arriving in this interval is distributed according to $\text{Poisson}(\lambda \cdot 2) = \text{Poisson}(2/20)$. We want the probability that this r.v. is 0, so

$$Pr[N([1,3]) = 0] = \frac{\lambda^0 e^{-0.1}}{0!} = e^{-0.1}$$

(b) This is the probability $W_4 < 5$. Recall that $W_4 \sim \text{Exp}(\lambda)$.

$$\mathbb{P}[W_4 < 5] = 1 - e^{-5\lambda} = 1 - e^{-1/4}$$

(c) If we recall, W_i is an exponential r.v. and exponentials have the memoryless property, meaning

$$\mathbb{P}[W_i > t + s | W_i > s] = \mathbb{P}[W_i > t].$$

Since we've arrived some time s after the last bus arrived, the time we wait has the same distribution as if we arrived immediately after the last bus arrived!

So we expect to wait 20 minutes.

5 Vegas

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

1. Given the results of your experiment, how should you estimate p?
2. How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

Solution:

1. We want to construct an estimate $\hat{p}$ such that $\mathbb{E}[\hat{p}] = p$. Then, if we have a large enough sample, we'd expect to get a good estimate of p. Let X_i be the indicator that the ith person's coin flips to a heads. What we observe is the fraction of people whose coin is heads. In other words, we measure $q = \frac{1}{n}\sum_{i=1}^{n} X_i$. How can we use this observation to construct $\hat{p}$?

   First,
   $$\mathbb{E}[q] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mathbb{E}[X_i] = p \cdot 1 + (1-p) \cdot \frac{1}{2},$$
   where the last equality follows from total probability. Solving for p, we find that
   $$p = 2\,\mathbb{E}[q] - 1 = \mathbb{E}[2q - 1].$$
   Thus, our estimator $\hat{p}$ should be $2q - 1$.

2. We want to find n such that $P[|\hat{p}-p| \leq 0.05] > 0.95$. Another way to state this is that we want
$$P[|\hat{p}-p| > 0.05] \leq 0.05.$$
Notice that $\mathbb{E}[\hat{p}] = p$ by construction, so we can immediately apply Chebyshev's inequality on $\hat{p}$. What we get is:
$$P[|\hat{p}-p| > 0.05] \leq \frac{\mathrm{var}[\hat{p}]}{0.05^2} \leq 0.05$$
So, we want n such that $\mathrm{var}[\hat{p}] \leq 0.05^3$.
$$\mathrm{var}[\hat{p}] = \mathrm{var}[2q-1] = 4\,\mathrm{var}[q] = \frac{4}{n^2}\,\mathrm{var}\left[\sum_{i=1}^{n} X_i\right] = \frac{4}{n}\,\mathrm{var}[X_1].$$
But X_i is an indicator (Bernoulli variable), so its variance is bounded by $\frac{1}{4}$. Therefore we have
$$\mathrm{var}[\hat{p}] \leq \frac{4}{n}\frac{1}{4} = \frac{1}{n}.$$
So, we choose n such that $\frac{1}{n} \leq 0.05^3$, so $n \geq \frac{1}{0.05^3} = 8000$.

6 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Solution:

We can model this problem as a Markov chain with the following states:

- S: Start state, which we are only in before flipping any coins.
- H: We see a head, which means no streak of tails currently exists.
- T: We've seen exactly one tail in a row so far.
- TT: We've seen exactly two tails in a row so far.
- TTT: We've accomplished our goal of seeing three tails in a row and stop flipping.

We can write the first step equations and solve for $\gamma(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\begin{aligned}
\gamma(S) &= .5\gamma(T) + .5\gamma(H) \\
\gamma(H) &= 1 + .5\gamma(H) + .5\gamma(T) \\
\gamma(T) &= .5\gamma(TT) + .5\gamma(H) \\
\gamma(TT) &= .5\gamma(H) + .5\gamma(TTT) \\
\gamma(TTT) &= 0
\end{aligned}$$

From the second equation, we see that

$$.5\gamma(H) = 1 + .5\gamma(T)$$

and can substitute that into equation 3 to get

$$.5\gamma(T) = .5\gamma(TT) + 1.$$

Substituting this into equation 4, we can deduce that $\gamma(TT) = 4$. This allows us to conclude that $\gamma(T) = 6$, $\gamma(H) = 8$, and $\gamma(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.