3 Bipartite Graph

Proof. To prove this, we must establish two directions: if, and only if.

Only if. We give a proof by contradiction for the forward direction, i.e., if a graph G is bipartite, then it has no tours of odd length. Assume, for a contradiction, that for a bipartite graph G_1, there is a tour T of odd length.

By definition of partite, consider the 2 disjoint sets of vertices of G, say L and R, such that no 2 vertices in the same set have an edge between them. WLOG, let our tour T begin in a vertex $v_0 \in R$, and let our i^{th} edge be $\{v_{i-1}, v_i\}$. We claim that for any odd-numbered edge, let it be the $(2n+1)^{th}$ edge $\{v_{2n}, v_{2n+1}\}$, $n \in \mathbb{N}$, then vertex $v_{2n+1} \in L$.

We proceed by induction on n.

Base case (n = 0): Consider edge $\{v_0, v_1\}$. Since there must not be an edge between 2 vertices in the same set (as stated above by definition), so $v_1 \in L$.

Thus, the base case is correct.

Inductive Hypothesis: Assume that the claim, for the $(2n+1)^{th}$ edge $\{v_{2n}, v_{2n+1}\}$ and $n \in \mathbb{N}$, $v_{2n+1} \in L$, is true for $n = k \in \mathbb{N}$.

Inductive Step: We prove the claim for $n = k+1 \geq 1$. Since $v_{2k+1} \in L$, and by definition of bipartite, so edge v_{2k+1}, v_{2k+2} connects two vertices in different sets, which implies that $v_{2k+2} \in R$. Thus, similarly, since v_{2k+2}, v_{2k+3} is an edge, so $v_{2k+3} \in L$, which gives us that $v_{2(k+1)+1} \in L$.

Thus, by the principle of mathematical induction, the claim holds.

Thus, the last vertex of the tour T would always be in the set R, while the first vertex of the tour T, by assumption, is in the set L, which implies that T is not a tour.

Therefore, there isn't a tour T of odd length in a bipartite graph, so we have that if G is bipartite, then G has no tours of odd length.

If. We give a Proof by Cases by providing a bipartition on the set of vertices of G where G is a graph with no tours of odd length, and we'll show that G is bipartite.

We proceed by cases. Let us divide our proof into three cases, exactly one of which must be true: (1) G has no tours; or (2) G has tours, but no tours of odd length.

Case (1): With the assumption that G has no tours, consider any path in G, so it doesn't have repeating vertices (or else it would result in a sub-path that is a tour). Similarly, G have no cycles. Thus, G only have walks. Let $W_1, W_2, ..., W_k$ be the edge disjoint walks of G. Let's start with W_1. Let the sequence of vertices that W_1 traverse be $v_{1,1}, v_{1,2}, ..., v_{1,w1}$. Let L, R be the two sets we will be putting our vertices into, and we define them such that no 2 vertices in the same set have an edge between them. First, let $v_{1,1} \in L$, and then alternate R, L to put in the vertices W_1 traverses. In general, $v_{1,2i-1} \in L, v_{1,2j} \in R$ where $i, j \in \mathbb{Z}^+$. Since all the edges in W_1 are of the form $\{v_{1,k}, v_{1,k+1}\}, k \in \mathbb{Z}^+$, so this assignment of vertices would not violate our assumption regarding L and R.

Then, consider W_2. Again, Let the sequence of vertices that W_2 traverse be $v_{2,1}, v_{2,2}, ..., v_{2,w2}$. There are two situations: either no vertex in $W-2$ have already been assigned into L or R, or there are vertices in $W-2$ that have already been assigned into L or R.

If none of the vertices in $W-2$ have already been assigned into L or R, then we repeat the process for vertices in W_2 just as we did for W_1.

Else if some vertices in $W-2$ have already been assigned into L or R. We claim that there is at most 1 such vertex. We proceed by contradiction. Assume that there are two vertices u, v that have already been assigned into L or R. Then, there is a sub-walk of W_2 from u to v, and similarly, there is a sub-walk of W_1 from u to v, which contradicts our assumption that there should be no tours in G. Thus, there is at most 1 vertex that have been assigned already. WLOG, assume this vertex $v_{2,i} \in L$. So, similar to how we assigned vertices of W_1, we would assign all vertices that are adjacent to $v_{2,i}$ into R, then assign all vertices that are adjacent to them into L again, and continue this process. This would give us a bipartition that does not violate the assumption.

Now, consider W_3. Let the sequence of vertices that W_3 traverse be $v_{3,1}, v_{3,2}, ..., v_{3,w2}$. For situations like having zero or one vertices that have already been assigned, we will utilize a similar strategy as discussed above. Yet, there will only be one exception: suppose both $v_{3,i} \in W_i$ and $v_{3,j} \in W_j$ have already been assigned. We have proved that they could not have been assigned in the same walk W_k, or else it would lead to a tour, and thus contradicting our assumption. Start with $v_{3,i}$, and WLOG, let $v_{3,i} \in L$. Label its neighbors into R, and repeat this process. When we reach v_{3_j}, if our new assignment is the same as its previous assignment, then the algorithm would lead to a successful bipartition; on the other hand, if our new assignment is different as its previous assignment, then we would flip the assignment of the entire walk W_j. This would lead to a successful bipartition as v_{3_j} would now have the same assignment as its previous assignment, and the labeled W_j would still be successfully bipartitioned. Continue this process for all walks $W_4, ..., W_k$, and we will have successfully bipartitioned all the vertices of G.

Thus, we could always find a bipartition if G has no tours, which implies that if G has no tours, then G is bipartite.

Case (2): With the assumption that G has tours and no tours of odd length, we have that all the tours $T_1, T_2, ..., T_t$ are of even length, which means that for any arbitrary tour $T_x, 1 \le x \le t$, and let the vertices T_x traverse be $v_{x,1}, v_{x,2}, ..., v_{x,wx}, v_{x,1}$ (since a tour is closed). Since T_x is of even length, so there are an even number of edges in T_x, denoted as $\{v_{x,1}, v_{x,2}\}, \{v_{x,2}, v_{x,3} ..., \{v_{x,wx-1}, v_{x,wx}\}, \{v_{x,wx}, v_{x,1}\}$. Thus, this implies that wx is even, which means that we could assign the vertices of T_x into disjoint sets L and R just like how we assigned the vertices of walks in Case (1). Thus, similar to our proof in Case (1), we have that if G only has no tours of odd length, then G is still bipartite.

Therefore, we have that if G has no tours of odd length, then G is bipartite.

Therefore, we conclude that a graph is bipartite if and only if it has no tours of odd length.

Q.E.D.