CS 70	Discrete Mathematics and Probability Theory	
Fall 2018	Alistair Sinclair and Yun Song	DIS 07A

1 Counting Mappings

(a) Let $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$. How many distinct functions f are there from X to Y?

(b) How many of these functions f are injective?

(c) For functions f as in part (a), consider the (ordered) lists $\left(|f^{-1}(\{1\})|, |f^{-1}(\{2\})|, \ldots, |f^{-1}(\{m\})|\right)$. How many distinct such lists are there? How many are there if we only consider surjective functions?

Solution:

(a) For each $x \in X$ there are m choices, and there are n elements in X. Thus, we are making a series of n choices each of which gives us m options, for a total of m^n different possible mappings.

(b) If $m < n$, there cannot be any injective mappings, which follows from the pigeonhole principle. Otherwise, there are m choices for $f(1)$, and given this choice, there are $m-1$ choices for $f(2)$, and so on. Thus, the total number of injective mappings is $m(m-1)(m-2)...(m-n+1) = \frac{m!}{(m-n)!}$.

(c) We can think such lists as configurations of n indistinguishable balls thrown into m distinguishable bins: For each element in $y \in Y$ (the bins), the list keeps track of how many elements from X (the balls) were assigned to y by f. But we know that there are exactly $\binom{n+m-1}{m-1}$ ways of assigning n balls to m bins. If f is surjective, then each bin must contain at least one ball, so we are left with $n-m$ balls that we need to distribute. Hence we end up with $\binom{n-1}{m-1}$ possibilities, if $n > m$ and 0 otherwise (for there must be at least as many balls as bins to fill each bin with at least one ball, or in other words, the domain must contain at least as many elements as the range for f to be surjective).

2 Counting on Graphs

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct cycles are there in a complete graph with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

Solution:

(a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$.

(b) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

(c) The number of vertices k in a cycle is at least 3 and at most n. Without accounting for duplicates, there are $n!/(n-k)!$ cycles. Due to inversions and rotations, the number of cycles equivalent to any given cycle is $2k$. Hence the total number of distinct cycles is

$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}.$$

(d) Without considering symmetries there are $6!$ ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

3 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(b) $\binom{n}{i} = \binom{n}{n-i}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$.

Solution:

(a) For each i on the LHS, we can think of selecting a team of i members out of a pool of n players, and susequently choosing a captain out of the i team members. The RHS does the same by first choosing the captain out of the n players, and then a subset of the remaining $n-1$ players to constitute the team.

(b) Choosing i players out of n to play on a team is the same as choosing $n-i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

(c) Assume we have n women and n men. Using part (b) we can rewrite the LHS as $\sum_{i=1}^{n} i\binom{n}{i}\binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n-i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n-1$ players from all remaining men and women to form the team.