I worked alone without getting any help, except asking questions on Piazza and reading the Notes of this course.

1 Hit or Miss?

(a) Incorrect.

The proof only shows that the proposition is true for all $n \in \mathbb{Z}^+$. However, since the restriction on n is that $n \in \mathbb{R}$, and $n > 0$, only proving that the proposition works for all positive integers is not enough.

Additionally, for a counterexample, consider $n = 0.5$, so n is positive and $n \in \mathbb{R}$. Yet, $n^2 = 0.25 < 0.5 = n$.

Q.E.D.

(b) Correct.

(c) Incorrect.

In order for the principle of mathematical induction to apply, the Inductive Step must show that for all $n, P(n) \implies P(n+1)$, and it claims that $n+1$ can be written as "$a+b$ where $0 < a, b \leq n$." Yet, it only proved one base case where $n = 0$, and the proof breaks when $n = 1$, so for $n = 0 + 1 = 1, n$ couldn't be written as the sum of two positive integers that follows the pattern. Moreover, 1 can't be written as the sum of two positive integers in the first place.

Additionally, for a counterexample, consider $n = 1$, so n is a nonnegative integer. Yet, $2n = 2 \neq 0$.

2 A Coin Game

Proof. We proceed by induction on n.

Base case ($n = 1$): Nothing could be split, so the score is always 0, and we have $0 = \frac{1*(1-1)}{2} = 0$.
Base case ($n = 2$): There is only one splitting strategy: splitting 2 coins into two stacks of 1.
So, the score is always $1 * 1 = 1$, and we have $1 = \frac{2*(2-1)}{2} = 1$.
Thus, the base cases are correct.

Inductive Hypothesis: Assume that the claim, the score is always $\frac{n(n-1)}{2}$, is true for all $1 \leq n \leq k$ for $k \geq 2$.

Inductive Step: We prove the claim for $n = k+1 \geq 3$. Since there is only one stack initially, assume the first turn split it into x and $(n-x)$ coins, with $1 \leq x \leq n-1$, so the score of the first turn is $x(n-x)$ and $x, n-x \leq n-1 = k$. Thus, the Inductive Hypothesis implies that the stack with x coins would eventually score $\frac{x(x-1)}{2}$, and the stack with $n-x$ coins would eventually score $\frac{(n-x)(n-x-1)}{2}$. Therefore, the total score would be: $x(n-x) + \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2} = xn - x^2 + \frac{(x^2-x)+(n^2-nx-n-nx+x^2+x)}{2} =$ $= xn - x^2 + \frac{2x^2+n^2-2nx-n}{2} = xn - x^2 + x^2 - nx + \frac{n^2-n}{2} = \frac{n(n-1)}{2}$, which means that the score is always $\frac{n(n-1)}{2}$.

Thus, by the principle of mathematical induction, the claim holds.

Q.E.D.

3 Grid Induction

Proof. We proceed by induction on n, where $n = i + j$, and we call it the "distance" between Pacman and $(0, 0)$.

Base case ($n = 0$): Considering the restraints, so $i = j = 0$, so Pacman ends at $(0, 0)$
Thus, the base case is correct.

Inductive Hypothesis: Assume that, for arbitrary $n = k \geq 0$, the claim, Pacman would reach $(0, 0)$ in finite time, is true.

Inductive Step: We prove the claim for $n = k+1 \geq 1$. Let Pacman be at position $(i_1, j_1), i_1 + j_1 = k+1$. Since Pacman only has two options, either walk one step down or walk one step to the left, which means that his position after one unit time is either $(i_1, j_1 - 1)$ or $(i_1 - 1, j_1)$. Moreover, Pacman's constraints tell us that he has to stay in the first quadrant, which means that at any time, let his location be (i^*, j^*), then $i^*, j^* >= 0$. So, after one unit time, his "distance" is always $i_1 + j_1 - 1 = k + 1 - 1 = k$. Thus, the Inductive Hypothesis implies that he'll reach $(0, 0)$ from here within finite time. Therefore, Pacman would reach $(0, 0)$ in finite time for $n = k + 1$.

Thus, by the principle of mathematical induction, the claim holds.

Q.E.D.

4 Stable Marriage

(a)

Stages	Women	Men
1	1	**A**, B, C
	2	
	3	**D**
	4	
2	1	**A**
	2	
	3	**B**, C, D
	4	
3	1	**D**, A
	2	**C**
	3	**B**
	4	
4	1	**D**
	2	**A**
	3	**B**
	4	**C**

Thus, the algorithm outputs the stable pairing: {(A, 2), (B, 3), (C, 4), (D, 1)}.

(b)

Under the traditional Stable Matching algorithm, using Theorem 4.2, which states that "the pairing output by the Stable Marriage algorithm is male optimal," and using my proof and results from Problem 5, which gives that "no two men can have the same optimal partner," so we have that the pairing P produced by the traditional algorithm is unique, and is male optimal.

Now, consider a new algorithm with the relaxed rules. We claim that it is also male optimal.

Suppose for sake of contradiction that the pairing is not male optimal. We first define that the k^{th} round of proposal for a man means that this is his k^{th} proposal. Then, there exists a round of proposal during which some man was rejected by his optimal woman; let round k be the first such round. On this round of proposal, let man M be rejected by his optimal partner, W, who chose man M^* instead. Yet, by the definition of optimal partner, so there must exist a stable pairing T in which M and W are paired together. Suppose T looks like this: $\{..., (M, W), ..., (M^*, W'), ...\}$. We will argue that (M^*, W) is a rogue couple in T, thus contradicting stability.

First, by our assumption, W prefers M to M^* since she chose M^* over M. Moreover, since the k^{th} round is the first round when some man got rejected by his optimal woman, so by the k^{th} round, M^* hasn't been rejected by his optimal woman, which means that M^* prefers W to his optimal partner. Thus, by definition of optimal partner for a man, M^* prefers W to W' (his partner in the stable pairing T).

Therefore, (M^*, W) will form a rogue couple in T, and so T is not stable. Thus, we have a contradiction, implying that the pairing by this new algorithm (relaxed rules for men) is still male optimal.

Again, using my proof and results from Problem 5, which gives that "no two men can have the same optimal partner," so we have that the pairing P^* produced by the new algorithm is unique, and is male optimal, which implies that pairing P is the same as P^*. Thus, the modification will not change what pairing the algorithm outputs.

Q.E.D.

5 Optimal Partners

Proof by Contradiction.

Proof. We proceed by contradiction. Assume that the proposition is false, which means that for some two different men, M and M^*, they have the same optimal partner, W.

Due to the strict ordering of preferences, WLOG, let W prefer M to M^*. Then, by the definition of optimal partner, so there must exist a stable pairing T in which M^* and W are paired together. Suppose T looks like this: $\{..., (M^*, W), ..., (M, W'), ...\}$. We will argue that (M, W) is a rogue couple in T, thus contradicting stability.

First, by our assumption, W prefers M to M^*. Moreover, since W is M's optimal partner, by definition of optimal partner for a man, so M prefers W to W' (his partner in the stable pairing T).

Therefore, (M, W) will form a rogue couple in T, and so T is not stable. Thus, we have a contradiction, implying that no two men can have the same optimal partner.

Q.E.D.

6 Examples or It's Impossible

For simplicity, numbers and letters are used to label the men and women.

(a) Possible.

Consider the case:

Men	Preferences	Women	Preferences
A	$1 > 2 > 3$	1	B > C > A
B	$2 > 3 > 1$	2	C > A > B
C	$3 > 1 > 2$	3	A > B > C

Since every man proposes to a different woman on the first day, the algorithm ends immediately.
Thus, the algorithm outputs the stable pairing: {(A, 1), (B, 2), (C, 3)}, so every man gets his first choice.

(b) Possible.

Consider the case:

Men	Preferences	Women	Preferences
A	$2 > 1 > 3$	1	A > B > C
B	$3 > 2 > 1$	2	B > A > C
C	$2 > 3 > 1$	3	C > A > B

On the first day, both A and C proposes to 2, and B proposes to 3, so A is on a string with 2, and B is on a string with 3. C moves on and proposes to his second choice, 3, who would put him on a string and reject B. Then, B moves on to his second choice, 2, who would put him on a string and reject A. After that, A moves on to his second choice, 1, who would put him on a string, and this leads to the end of the algorithm.
Thus, the algorithm outputs the stable pairing: {(A, 1), (B, 2), (C, 3)}, so every woman gets her first choice, even though her first choice does not prefer her the most.

(c) Possible.

Consider the case:

Men	Preferences	Women	Preferences
A	$1 > 2 > 3$	1	B > C > A
B	$2 > 3 > 1$	2	C > A > B
C	$3 > 1 > 2$	3	A > B > C

Since every man proposes to a different woman on the first day, the algorithm ends immediately.
Thus, the algorithm outputs the stable pairing: {(A, 1), (B, 2), (C, 3)}, so every woman gets her last choice.

(d) Impossible.

Proof by Contradiction.

Assume, for a contradiction, that this situation is possible, so there must exist a stable pairing T in which every man is paired with his last choice. Suppose the algorithm terminates on k^{th} day, and WLOG, let M be (one of) the men who got rejected on $(k-1)^{th}$ day. Let W' be the woman who rejected him on that day and chose M' instead, and let W be his last choice, whom he will propose to and get accepted on the k^{th} day by assumption. Since the k^{th} day is the day when the algorithm terminates, so W must have no one on her string on the k^{th} day; or else if she has M^* on her string, and accepts M's proposal, then M^* would be proposing to another woman on the $(k+1)^{th}$ day, which contradicts our assumption. Let R be the assertion that W must have no one on her string on the k^{th} day. Similarly, when M' propose to W' again on the k^{th} day again, they would still be paired together, and thus, (M', W') would be a pair in the final stable pairing T.

However, this situation is impossible. Since by assumption, every man gets his last choice, so W' is the last choice of M', which means that M' is bound to have proposed to W before the $(k-1)^{th}$ day. Thus, by Lemma 4.2, the Improvement Lemma, W would have someone on a string that she likes at least as much as M' on the k^{th} day, which implies $\neg R$.

We conclude that $R \wedge \neg R$ holds; thus, we have a contradiction, as desired. Therefore, a situation where every man gets his last choice is impossible.

Q.E.D.

(e) Possible.

Consider the case:

Men	Preferences	Women	Preferences
A	$1 > 2 > 3$	1	A > C > B
B	$3 > 2 > 1$	2	B > C > A
C	$3 > 1 > 2$	3	A > C > B

On the first day, A is on a string with 1, and B is on a string with 3. C moves on and proposes to his second choice, 1, who would reject him as she prefers A to C, so C moves on and proposes to his last choice, 2, who would put him on the string, which leads to the end of the algorithm.

Thus, the algorithm outputs the stable pairing: {(A, 1), (B, 3), (C, 2)}, and C, the man who is second on every woman's list, gets Woman 2, who is his last choice.