Sundry: I worked alone without any help.

1 Buffon's Needle on a Grids

(a) $\mathbb{P}[\text{no intersection at } \theta] = 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$

Note that a random throw of the needle is completely specified by 3 random variables:
(1) the horizontal distance X between the midpoint of the needle and the closest vertical line;
(2) the vertical distance Y between the midpoint of the needle and the closest horizontal line;
(3) the angle θ between the needle and the horizontal lines.

Since we assume a perfectly random throw, so we may assume that the position of the center of the needle and its orientation are independent and uniformly distributed (i.e. X, Y, θ are i.i.d.). Then, since the r.v.s X and Y range between 0 and θ is fixed, so their joint distribution has density $f(x, y)$ that is uniform over the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Since this square has area $\frac{1}{4}$, so the density should be:

$$f(x, y, \theta) = 4 \qquad \text{for } (x, y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$$

$$\text{and} \qquad f(x, y, \theta) = 0 \qquad \text{otherwise}$$

Sanity Check:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, \theta)\ dxdy = \int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} 4\ dxdy = 1$$

Now let E denote the event that the needle does NOT intersect a line. By elementary geometry the vertical distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\sin\theta$, and the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will NOT intersect any grid lines if and only if $(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)$.

Therefore, with our density function and bounds, so we have that:

$$\mathbb{P}[E] = \mathbb{P}[(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)] = \int_{\frac{1}{2}\sin\theta}^{\infty}\int_{\frac{1}{2}\cos\theta}^{\infty} f(x, y, \theta)\ dxdy$$

$$\Longrightarrow \mathbb{P}[E] = \int_{\frac{1}{2}\sin\theta}^{\frac{1}{2}}\int_{\frac{1}{2}\cos\theta}^{\frac{1}{2}} 4\ dxdy = 4 \cdot (\frac{1}{2} - \frac{1}{2}\cos\theta)(\frac{1}{2} - \frac{1}{2}\sin\theta) = 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$$

(b) $\mathbb{P}[\text{intersection}] = \frac{3}{\pi}$

Using a similar argument, we have that the r.v.s X and Y range between 0 and $\frac{1}{2}$, while θ ranges between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since we assume a perfectly random throw, so we may assume that the position of the center of the needle and its orientation are independent and uniformly distributed (i.e. X, Y, θ are i.i.d.), and thus, their joint distribution has density $f(x, y, \theta)$ that is uniform over the cube $[0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since this cube has volume $\frac{\pi}{4}$, so the density should be:

$$f(x, y, \theta) = \frac{4}{\pi} \qquad \text{for } (x, y, \theta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\text{and} \qquad f(x, y, \theta) = 0 \qquad \text{otherwise}$$

Sanity Check:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,\theta)\ dxdyd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = 1$$

Now let E_2 denote the event that the needle does NOT intersect a line. By elementary geometry the vertical distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\sin\theta$, and the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will NOT intersect any grid lines if and only if $(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)$.

Thus, with our density function and bounds, so we have that:

$$\mathbb{P}[E_2] = \mathbb{P}[(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{1}{2}\sin\theta}^{\infty}\int_{\frac{1}{2}\cos\theta}^{\infty} f(x,y,\theta)\ dxdyd\theta$$

$$\implies \mathbb{P}[E_2] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{1}{2}\sin\theta}^{\frac{1}{2}}\int_{\frac{1}{2}\cos\theta}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{\pi}\cdot(\frac{1}{2}-\frac{1}{2}\cos\theta)(\frac{1}{2}-\frac{1}{2}\sin\theta)\ d\theta$$

$$\implies \mathbb{P}[E_2] = 2\int_{0}^{\frac{\pi}{2}} \frac{1}{\pi}\cdot(1-\sin\theta-\cos\theta+\sin\theta\cos\theta)\ d\theta = \frac{2}{\pi}\cdot\int_{0}^{\frac{\pi}{2}} 1-\sin\theta-\cos\theta+\frac{1}{2}\sin(2\theta)\ d\theta$$

$$\implies \mathbb{P}[E_2] = \frac{2}{\pi}\cdot\left(\theta+\cos\theta-\sin\theta-\frac{1}{4}\cos(2\theta)\right)\Big|_0^{\frac{\pi}{2}} = \frac{}{\pi}\cdot\left((\frac{\pi}{2}+0-1+\frac{1}{4})-(0+1-0-\frac{1}{4})\right) = \frac{\pi-3}{\pi}$$

Therefore, we have that the probability that the needle intersects a grid line is:

$$\mathbb{P}[\text{intersection}] = \mathbb{P}[\overline{E_2}] = 1-\mathbb{P}[E_2] = 1-\frac{\pi-3}{\pi} = \frac{3}{\pi}$$

(c) $\mathbb{E}[X] = \frac{4}{\pi}$

Using indicator variables, we have that $X = H + V$, where H is the r.v. with $H = 1$ if the needle intersects a horizontal gridline, and 0 otherwise; V is the r.v. with $V = 1$ if the needle intersects a vertical gridline, and 0 otherwise.

Now, using linearity of expectation, we have that $\mathbb{E}[X] = \mathbb{E}[H] + \mathbb{E}[V]$. Consider $\mathbb{E}[H]$ first:

Using a similar setup as part (b), we have that the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will intersect a horizontal gridline if and only if $(x <= \frac{1}{2}\cos\theta)$.

Thus, we have that:

$$\mathbb{E}[H] = 1\cdot\mathbb{P}[H=1] + 0\cdot\mathbb{P}[H=0] = \mathbb{P}[H=1] = 2\int_{0}^{\frac{\pi}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{1}{2}\cos\theta} f(x,y,\theta)\ dxdyd\theta$$

With our desnsity function and constraints, we can rewrite the integral as:

$$\mathbb{E}[H] = 2\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}\cos\theta} \frac{4}{\pi}\ dxdyd\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta}{\pi}\ d\theta = \frac{2}{\pi}\sin\theta\Big|_{0n}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

Similarly, we would have that:

$$\mathbb{E}[V] = 2\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}\sin\theta}\int_{0}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{\pi}\ d\theta = -\frac{2}{\pi}\cos\theta\Big|_{0}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

Therefore, we can conclude that:

$$\mathbb{E}[X] = \mathbb{E}[H] + \mathbb{E}[V] = \frac{4}{\pi}$$

(d) $\mathbb{P}[X = 1] = \frac{2}{\pi}$

Since we have that the only possible numbers of a needle intersecting the gridlines are 0, 1 and 2, and that we have from part (b) that $\mathbb{P}[\text{intersection}] = \frac{2}{\pi}$, which gives us that

$$\mathbb{P}[X = 1] + \mathbb{P}[X = 2] = \frac{3}{\pi}$$

Thus, we have that $\mathbb{P}[X = 2] = \frac{3}{\pi} - \mathbb{P}[X = 1]$. Then, from part (c), we have that

$$\mathbb{E}[X] = \frac{4}{\pi}$$

which can be rewritten as having $\mathbb{E}[X] = 0 \cdot \mathbb{P}[X = 0] + 1 \cdot \mathbb{P}[X = 1] + 2 \cdot \mathbb{P}[X = 2] = \mathbb{P}[X = 1] + 2 \cdot (\frac{3}{\pi} - \mathbb{P}[X = 1]) = \frac{6}{\pi} - \mathbb{P}[X = 1] = \frac{4}{\pi}$

Thus, we can calculate that:

$$\mathbb{P}[X = 1] = \frac{2}{\pi}$$

(e) $\mathbb{E}[Z] = \frac{4}{\pi}$

let Z be the random variable representing the number of times such an equilateral triangle intersects the gridlines. We can "split" the triangle into three length-$\frac{1}{3}$ unit needles and get $Z = I_1 + I_2 + I_3$, where I_i is the number of times the i^{th} segment of the triangle intersects the gridlines. Thus, using linearity of expectation and the fact that each unit needle is identical (i.e. each of the $\mathbb{E}[I_i]$'s are equal), so we have:

$$\mathbb{E}[Z] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3] = \mathbb{E}[\text{unit length needle intersection}]$$

Now, using our result from part (c), we have that the expectation of the number of times a needle intersects the gridlines is: $\mathbb{E}[I_1] = \frac{4}{\pi}$. Therefore,

$$\mathbb{E}[Z] = \frac{4}{\pi}$$