Computer Science Mentors
Fall 2018

CSM 70 Mock Exam

CSM is an independent organization separate from course staff and makes no promises about the degree to which this mock exam will reflect the true exam. There are **178** points on this exam.

1 True or False? (2 Points Each, 28 points total)

Circle either True or False in each of the problems below.

1. True or False: $P \land \neg P \implies Q$
   True. False implies everything.

2. True or False: $(\forall x)(\forall y)(P(x,y) \land \neg Q(x,y)) \equiv \neg(\exists x)(\forall y)(P(x,y) \implies Q(x,y))$
   False. Follow the implications.

3. True or False: $(\forall x)(\exists y)(P(x) \implies Q(y)) \equiv (\exists x)(\exists y)(\neg Q(x) \implies \neg P(y))$
   True. Don't get confused by the change of variables.

4. True or False: If in a stable marriage algorithm terminates after the first day no matter if the men or women propose first, then every male has a unique preference list.
   True. Each man just has to have a unique first preference, which means they have different preference lists.

5. $\forall n, (n-1)^{-1} \equiv n-1 \mod n$.
   True. $(n-1)(n-1) \equiv n^2 - 2n + 1 \equiv 1 \mod n$

6. If n is even, then $\binom{n}{2} \equiv \frac{n}{2} \mod n$
   True. You can see this easily by computing a few examples. A more rigorous proof would be that $\binom{n}{2} \equiv \frac{(n-1)(n)}{2} \equiv \frac{n^2}{2} - \frac{n}{2}$. Note that n is even, so n^2 not only divides 2, it divides 4. Therefore we know that $\frac{n^2}{2} \equiv 0 \mod n$, and the result easily follows.

7. True or False: If a graph G contains a subgraph that is a clique of k vertices, then it also contains a subgraph that is a tree of $k+1$ vertices.
   False. One counterexample: If a graph has k vertices, then it is impossible.

8. True or False: If two events A and B are independent then A^c and B^c are dependent.
   False. They are actually independent: $P(A^c \cap B^c = 1 - (P(A) + P(B) - P(A \cap B)) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$ as desired.

9. True or False: There are uncountably many programs that output 7.
   False. Each program is a bitstring of finite length. All sets of finite size objects are countable (iterate through all size 1 objects, then all size 2 objects, and so on). So there are a countable number of bitstrings, and the programs that output 7 are a subset of all programs, so there are countably many.

10. True or False: X takes on values between -2 and 10 and has an expected value of 4. Using Markov's we say that $P(X \geq 6) = \frac{4}{6}$
    False. We can't use Markov's without a transformation because it takes negative values.

11. True or False: If A and B are events such that $P(A \cup B) = P(A) + P(B) = 1$, then $P(C) = P(A \cap C) + P(B \cap C)$.
True. This is the law of total probability.

12. True or False: $E[\text{Geom}(\frac{1}{kn})] = E[\text{Binomial}(k, p)]$
True. Expectation of geometric is $\frac{1}{\text{p}} = kn$, and expectation of binomial is pn

13. True or False: Say that we know that every day 4 bills will be pushed to the Senate, distributed like a Poisson Distribution. We can use the CLT to approximate the number of bills that are pushed in an entire year.
True.

14. True or False: If a self loop exists in an irreducible Markov chain, then it is aperiodic.
True. If there is a self loop then we can always use it to create paths of lengths that have GCD 1.

15. True or False: If random variables X and Y are independent, then $\text{var}(Y) = \text{var}(X - Y) - \text{var}(X)$
True. $Var(X - Y) - Var(X) = Var(X) + (-1)^2 Var(Y) - Var(X) = Var(Y)$

16. True or False: If random variables X and Y represent two fair dice rolls, then by symmetry $P(X > Y) = \frac{1}{2}$.
False. There is a $\frac{1}{6}$ chance the dice rolls are equal, meaning that by symmetry $S > Y$ w/ probability $\frac{5}{12}$

2 Misc (34 points)

1. What is 11^{61} (mod 12) (3 pts)?
-1. See that $11 \equiv -1 \mod 12$, and it follows easily.

2. How many distinct polynomials of degree at most 3 are there modulo 53, such that the value at the points x = 1, 2, 3, 4 lie in the set $\{6, 7, 8, 9, 10\}$ (3 pts)?
5^4 because 4 points uniquely identify a polynomial of at most degree 3, so each of the 4 points can take on value $\in \{6, 7, 8, 9, 10\}$

3. How many ways can 3 people park their cars in 5 parking spots?
$5 * 4 * 3 = 60$, first person has 5 options, next has 4, last has 3.

4. How many ways can 3 people park their cars in 5 parking spots, given that there must bust be at least a space between each car (3 pts)?
$3! = 6$. There is actually only 1 orientation that achieves the requirement (1st, 3rd, and 5th slot) and there are 3! different ways to arrange the 3 cars in those slots.

5. How many numbers between 1 and 111111011_2 can be divided by 8 (3 pts)? (Hint: if a number is divisible by 8, what must the last 3 digits of its base 2 representation be? Don't express the number in base 10)
$2^6 - 1$. Note that for a number to be divisible by 8, the last 3 digits of its base 3 representation must be 0. It doesn't matter what the rest of the digits are, and each other digit has 2 options: 0, or 1. So we we have 2^6. But we don't count 000000000_2 because it is less than 1, so we have $2^6 - 1$.

6. What is the probability that a poker hand (5 cards drawn from a deck of cards) has a 3-pair (a triple)? (3 pts) No need to simplify.

Choose the number/face and which 3 of the 4 cards, then the other two cards to fill out the hand: $\frac{\binom{13}{1}\binom{4}{3}\binom{48}{3}}{\binom{52}{5}}$

7. m balls are thrown into n bins. Each bin is then randomly dumped into one of k bags. What is the expected number of balls in a bag (3 pts)?
$\frac{m}{k}$: each ball is equally likely to end up in each bag.

8. **This question was removed because it didn't make sense.** Suppose $n = 2$. What is the probability no balls end up in a bag (3 pts)?

9. You are sending packets through a channel that has non-deterministic behavior. Specifically, it will corrupt 2 packets with probability .8, and 4 packets with probability 0.2. If your message is of length 10, what is the minimum number of packets that you must send, in expectation, to be able to decode the message 95% of the time (5 pts)? (Hint: the channel is probabilistic, so you should consider a probabilistic encoding procedure; i.e, with probability p do one thing and with $1-p$ do another.)
First, if the channel corrupts 2 packets, then we will have to send 14 packets to decode fully. If the channel corrupts 4 packets, then we will have to send 18 packets to decode fully. If we send 14 packets and the channel corrupts 4, then we won't be able to recover if we send 18 and channel corrupts 2 then we're good. Say that we decide to send 14 packets with probability p. The probability that we can't decode the message is if we chose to send 14 packets, and the channel corrupts 4 packets, or $0.2 \cdot p$. We need this quantity to be less than 0.05, so we have $.02p \leq 0.05$. Solving we find that $p = 0.25$. This means that on average we will send $0.25 \cdot 14 + 0.75 \cdot 18 = 17$ packets.

10. Say that a random variable X takes on values from 2 to 10, with mean 5. Use Markov's bound to bound the probability that $X \leq 3$. (5 pts)
Remember that the general form of Markov's inequality is $P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$. However, we want to bound the probability that X less than some quanitity. So we apply the transformation $Y = -X$: now $P(X \leq 3) = P(Y \geq -3)$. However, remember that Markov's only works for non-negative random variables. So we say $Y = -X + 10$, guaranteeing that Y is also nonnegative. Y has mean 5. Now see $P(X \leq 3) = P(Y \geq 7) \leq \frac{5}{7}$. On the whole this is a pretty crap bound!

11. Suppose X and Y are random variables which take on values $\mod n$. Suppose Y is uniformly distributed. Find the distribution of random variable $Z = X + Y \mod n$ (5 pts)
To find the distribution of Z, we can look at $P(Z = k)$

$$P(Z=k) = \sum_{i=0}^{n-1} P(Z=k, Y=i) = \sum_{i=0}^{n-1} P(X=k-j, Y=i) = \sum_{i=0}^{n-1} P(X=k-j) \cdot P(Y=i)$$

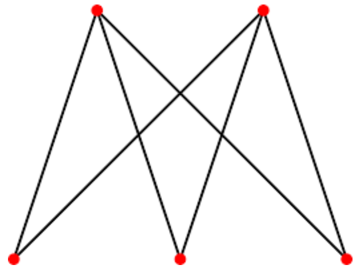
$$= \frac{1}{n} \sum_{i=0}^{n-1} P(X=k-j) = \frac{1}{n}$$

Therefore, we see that Z is uniformly distributed as well

3 Graphs

A complete bipartite graph (represented by $K_{m,n}$ is a graph with two sets vertices, where every vertex is connected to every vertex in the other set and no vertices in its own. Set

1. Draw the graph $K_{3,2}$



2. How many vertices does $K_{m,n}$ have? How many edges?

   $m+n$ vertices, mn edges

3. Under what conditions is there an eulerian tour of the graph?

   There is an eulerian tour if both m and n are even..

4 Proving Useful Properties (15 points)

For these questions, to prove that two variables are distributed identically, it is enough to show that their CDFs or PDFs are equal. For reference, $f_{exp(\lambda)}(x) = \lambda e^{-\lambda x}$

1. Prove that $\min(X,Y) \sim \exp(\lambda+\mu)$ where $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$ and are independent (5 points).
   Call $Z \sim \min(X,Y)$ Let's calculate the CDF

   $$P(Z \leq z) = 1 - P(Z > z) = 1 - P(X > z \wedge Y > z) = 1 - P(X > z)P(Y > z)$$

   By independence

   $$= 1 - e^{-\lambda z}e^{-\mu z} = 1 - e^{-(\lambda+\mu)z}$$

   Which is the CDF of a an exponential with parameter $\lambda+\mu$

2. Prove the memoryless property i.e.

   $$P(X \geq x+k | X \geq k) = P(X \geq x)$$

   什么鬼？？？

   where X is an exponential random variable (5 points). Let's look at the CDF of X

   $$P(X \leq x) = P(\sigma Z + \mu \leq x) = P(Z \leq \frac{x-\mu}{\sigma})$$

5 Probabilistic Stable Marriage

Assume that we have n men and n women, with the men labeled $M_1, \ldots, M_n$, and the women labeled $W_1, \ldots W_n$. We consider all possible stable marriage instances of this problem.

1. Say that $n = 3$. Come up with a stable marriage instance where the algorithm will terminate on the first day, regardless whether or not the men or women are proposing.

   We give an example:

Men's preferences:

M_1	W_1	W_2	W_3
M_2	W_2	W_1	W_3
M_3	W_3	W_1	W_2

Women's preferences:

W_1	M_1	M_2	M_3
W_2	M_2	M_1	M_3
W_3	M_3	M_1	M_2

2. Come up with a general condition on the stable marriage instance if there are n men and women such that the algorithm will terminate after the first day, no matter if the men or women propose first.
The algorithm terminates on the first day if and only if all the proposers have different first choices. So the algorithm terminates on the first day regardless of who proposes if and only if all the men have unique first choices, and all the women have unique first choices.

3. How many stable marriage instances are there for n men and n women, with the men labeled $M_1, \ldots, M_n$, and the women labeled $W_1, \ldots W_n$?
Each man has an ordered list of every woman, and vice versa. For a given man, are $n!$ ways to order the women. Each instance is determined by $2n$ preferences (lists of another gender), so there are $n!^{2n}$ total.

4. Assume that we pick a stable marriage instance uniformly at random. What is the probability that the algorithm will terminate after the first day on this instance, regardless of if the men or women propose first? (*Hint: use part (b)*)
We counted the total number of stable marriage instances as $n!^{2n}$. As they are chosen uniform at random, we only need to count the number of instances that will terminate on the first day regardless of who proposes.

   By part (b), this is the number of instances where all the men have unique first preferences and all the women have unique first preferences. There are $n!^2$ ways to give all the men and women unique first preferences, and we multiply by $(n-1)!^{2n}$ for how many ways there are to fill out the rest of the table. We get:

$$\frac{n!^2(n-1)!^{2n}}{n!^{2n}} = \frac{(n!)^2}{n^{2n}}$$

6 Fun with Circles

We pick three points along the circumference of a unit circle centered at the origin. Interpret these points as cutting the circle into three arcs. Find the expected length of the arc that contains the point (1, 0).

We pick three points along the circumference of a unit circle centered at the origin. Interpret these points as cutting the circle into three arcs.

1. Imagine an ant begins walking along the circle, counter-clockwise, from the point (1, 0). Let L_1 be the length that the ant must walk before reaching the end of the arc it is on. Find an expression for $\mathbb{P}(L_1 > x)$.

We want to compute $\mathbb{P}(L > x)$. This only happens when all three points chosen are beyond a distance x from (0, 1) in the counter-clockwise direction along the circle. So we have

$$\mathbb{P}(L > x) = (1 - \frac{x}{2\pi})^3$$

2. Now imagine the ant walks along the circle starting from (1,0) clockwise. Let L_2 be how far the ant must walk before reaching the end of its arc. Find an expression for $P(L_2 > x)$. Does this answer differ from the previous part? If so, what is it?

   We want to compute $\mathbb{P}(L > x)$. This only happens when all three points chosen are beyond a distance x from (0, 1) in the clockwise direction along the circle. So we have

$$\mathbb{P}(L > x) = (1 - \frac{x}{2\pi})^3$$

3. Find $\mathbb{E}(L_1)$ Remember the Tail Sum Formula: $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)dx$ for any non-negative random variable X.
   Using the tail-sum formula,

$$\mathbb{E}(L) = \int_0^{2\pi} \mathbb{P}(L > x)\,dx = \int_0^{2\pi} (1 - \frac{x}{2\pi})^3 dx = \frac{\pi}{2}$$

4. Find the expected length of the arc that contains the point $(1,0)$. (Hint: How does this quantity relate to the quantities from the previous two parts?)
   We can essentially think of the arc that will contain this point as being the sum of the length of the average length of the arcs in both the clockwise and the counter clockwise directions, so we get $2 * \frac{pi}{2} = \pi$. This may seem extremely counter-intuitive (why isn't the answer $\frac{2\pi}{3}$, by symmetry? The reasoning is that a specific point is more likely to be on a larger arc than a smaller one.

7 Moving Particle (25 points)

A particle sits on the real number line, starting at the origin (0). At each timestep, we flip a fair coin and move the particle as follows:

- If we see heads, we move the particle one unit to the left
- If we see tails, we move the particle one unit to the right

Let X_n be the position of the particle at time-step n, and assume $X_0 = 0$. Say that 100 time steps have passed.

1. Give an expression for P(X_{100} = 0) (5 points).
   We will return to the center position if the coin comes up heads exactly 50 times and tails exactly 50 times. This is a binomial distribution, with $n = 100, p = 0.5$. $P(X_{100} = 0) = \binom{100}{50} 0.5^{100}$

2. Given that X_{10} is positive, write an expression for the probability that $X_{10} = 4$. The expression will not be pretty (5 points).

   The probability that $P(X_i > 0) = \frac{1-P(X_i=0)}{2} = \frac{1-\binom{10}{5}0.5^{10}}{2}$. $P(X_{10} = 4)$ is the probability that we get 7 tails and 3 heads, which is $\binom{10}{3}0.5^{10}$. Note that if $X_{10} = 4$,, this guarantees that X_{10} is positive. So with Bayes' rule we can say that $P(X_{10} = 4|P(X_{10} > 0) = \frac{\binom{10}{3}0.5^{10}}{\frac{1-\binom{10}{5}0.5^{10}}{2}}$

3. Given that $X_{10} = 4$, what is the probability that the 3rd flip was a heads? (5 points) As we discussed in the last part, for X_{10} to be 4, we need to have flipped 3 heads and 7 tails. Each individual sequence of flips is equally likely, since the flips are independent. Therefore, by symmetry we can say that the probability that the $3rd$ flip was a heads is $\frac{3}{10}$

4. Use the CLT to give an approximation/ bound on the probability that the particle is within 10 units of the center (after 100 time steps). You can receive partial credit if you give a weaker bound, by for example using Chebyshevs. Hint: remember the 68-95-99.7 rule: For $Z \sim \mathcal{N}(0,1)$, $P(|Z| \leq 1) = 0.68, P(|Z| \leq 2) = 0.95, P(|Z| \leq 3) = .997$ (10 points).
   Let H represent the number of heads we got. We want $P(|H-50| \leq 5)$ Remember, since 100 is a pretty large number of trials, we can approximate H with a normally distributed H', with the same mean (50) and variance $(100(0.5)(1-0.5) = 25)$. Remember, however, we only have access to the CDF of the standard normal, so we have to convert H' to another variable Z that is distributed like the standard normal. To do this, we say $Z = \frac{H-50}{\sqrt{25}}$. Now we can write:

   $$P(|H-50| \leq 5) \approx P(|H'-50| \leq 5) = P(|Z| \leq 1) = 0.68$$

   as per the hint.

8 Chains

Customers arrive in a store according to Poisson distribution every day with mean λ. Assume that arrivals on different days are independent. We start counting the total number of arrivals to the store on day 1. Let $X(n)$ be the total number of customers that we have seen arrive at the store by day n.

1. Find the probability that $X(1)$ is even.

   (*Hint*: $e^{\lambda} + e^{-\lambda} = (1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \ldots) + (1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \ldots)$)

   p is the probability that there are an even number of arrivals on day 1. The arrivals in this interval follow a *Poisson*(λ) distribution so we have

   $$p = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \frac{e^{-\lambda}}{2} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \right) = \frac{e^{-\lambda}}{2} \left(e^{\lambda} + e^{-\lambda} \right)$$

   Which can be simplified to give

   $$p = \frac{1 + e^{-2\lambda}}{2}$$

2. Let $Y(n) = X(n) \mod 2$. As we allow n to get arbitrarily large, what is the long term probability that $Y(n) = 0$? Does this probability converge to something? Justify your answer. You may refer to the probability found in the previous question as p.

   $X(n)$ is a discrete time Markov chain over states $\{0,1\}$. There is a self loop on each state with probability p and $P(1,0) = P(0,1) = 1 - p$. This is an irreducible aperiodic DTMC and hence the distribution over the states converges to a unique stationary distribution, which is $(1/2, 1/2)$ by symmetry.

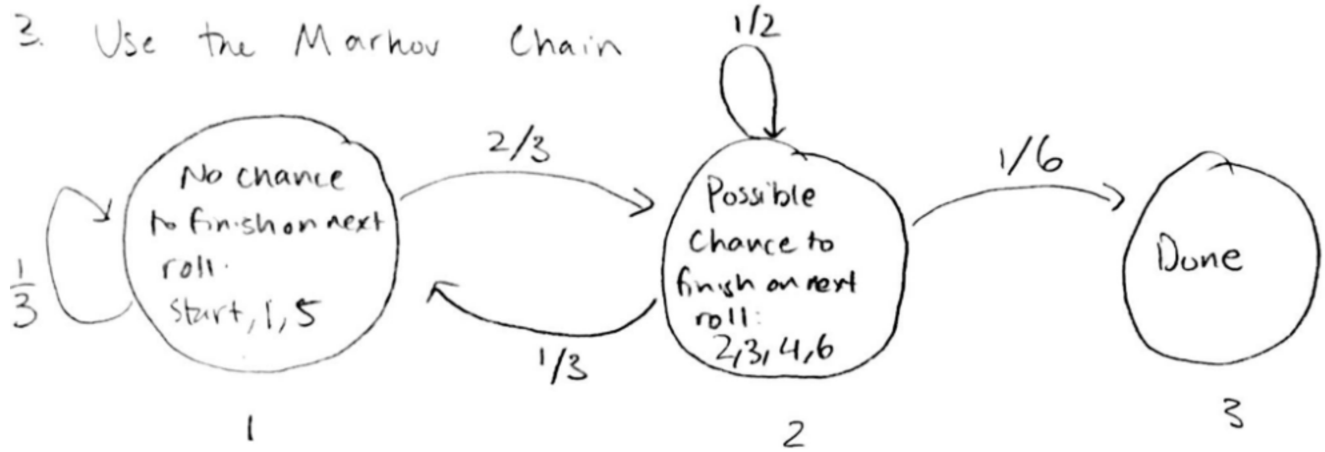
9 Product Chain (12 points)

A dice is rolled until the product of the last 2 numbers rolled is 12.

1. Using just **3** states, draw a Markov chain to represent this scenario. You can get partial credit for using more states. (8 points)

2. Solve for the expected amount rolls needed (4 points)

3. Use the Markov Chain



$$\beta(1) = \frac{1}{3}\beta(1) + \frac{2}{3}\beta(2)$$

$$\beta(2) = \frac{1}{3}\beta(1) + \frac{1}{2}\beta(2) + \frac{1}{6}\beta(3)$$

$$\beta(3) = 0$$

Solving, we find $\boxed{\beta(1) = \frac{21}{2}}$

Note: in the aboec graphic, the equations should be:

$$\beta(1) = 1 + \frac{1}{3}\beta(1) + \frac{2}{3}\beta(2)$$

$$\beta(1) = 1 + \frac{1}{3}\beta(1) + \frac{1}{2}\beta(2) + \frac{1}{6}\beta(3)$$

$$\beta(3) = 0$$

We add the extra 1 for the extra step we need to take from the states in the markov chain that do not model the "done" state.