1 Propositional Logic Language

For each of the following sentences, use the notation introduced in class to convert the sentence into propositional logic. Then write the statement's negation in propositional logic.

(a) The cube of a negative integer is negative.

(b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

(d) For any two distinct real numbers, we can find a rational number in between them.

Solution:

(a) We can rephrase the sentence as "if $n < 0$, then $n^3 < 0$", which can be written as

$$(\forall n \in \mathbb{Z})((n < 0) \implies (n^3 < 0))$$

or equivalently as

$$(\forall n \in \mathbb{Z})((n \geq 0) \vee (n^3 < 0)).$$

The latter is easier to negate, and its negation is given by

$$(\exists n \in \mathbb{Z})((n < 0) \wedge (n^3 \geq 0))$$

(b) The sentence is

$$(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10).$$

The negation is

$$(\exists x, y \in \mathbb{Z})(x^2 - y^2 = 10)$$

(c) Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\,((p(x) = 0) \wedge ((\forall y \in \mathbb{R})(p(y) = 0) \implies (x = y)))\,.$$

Its negation is given by

$$(\forall x \in \mathbb{R})\,((p(x) \neq 0) \vee ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y)))\,.$$

This can be equivalently expressed as

$$(\forall x \in \mathbb{R})((p(x) = 0) \implies ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y))).$$

(d) The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{R})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})\,((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))).$$

Then the negation is

$$(\exists x, y \in \mathbb{R})(x \neq y) \wedge ((\forall z \in \mathbb{Q})((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

2 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$.

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedant is true and the consequence is false, thus the entire implication statement is false.

(d) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note that the two statements are not equivalent as the converse of this is statement **??**, which is false.

3 Logic

Decide whether each of the following is true or false and justify your answer:

(a) $\forall x\, \big(P(x) \wedge Q(x)\big) \;\equiv\; \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x\, \big(P(x) \vee Q(x)\big) \;\equiv\; \forall x\, P(x) \vee \forall x\, Q(x)$

(c) $\exists x\, \big(P(x) \vee Q(x)\big) \;\equiv\; \exists x\, P(x) \vee \exists x\, Q(x)$

(d) $\exists x\, \big(P(x) \wedge Q(x)\big) \;\equiv\; \exists x\, P(x) \wedge \exists x\, Q(x)$

Solution:

(a) **True**.

Assume that the LHS is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the RHS is true. Now assume the RHS. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the LHS is true.

(b) **False**. If $P(1)$ is true, $Q(1)$ is false, $P(2)$ is false and $Q(2)$ is true, the left-hand side will be true, but the right-hand side will be false.

(c) **True**

Assuming that the LHS is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the RHS is true. To prove the other direction, assume the LHS is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the RHS is false.

(d) **False**. If $P(1)$ is true and $P(x)$ is false for all other x, and $Q(2)$ is true and $Q(x)$ is false for all other x, the right hand side would be true. However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true.