5 Exponential Practice

(a) $f_Y(y) = \lambda^2 e^{-\lambda y} y$ for $y \geq 0$; and 0 otherwise

Since $X_1 \sim \text{Exp}(\lambda)$ and $\lambda > 0$, so by definition X_1 has probability density function $f_{X_1}(x) = \lambda e^{-\lambda x}$ if $x \geq 0$ and 0 otherwise, and also that it has CDF $F_{X_1}(x) = \mathbb{P}[X_1 \leq x] = 1 - e^{-\lambda x}$ for $x \geq 0$. Similarly, X_2 has the same PDF and CDF.

Now, since $Y = X_1 + X_2$, X_1, X_2 are independent, and that all values of X_1, X_2 has to be non-negative (since the probability of them having negative values is 0), so we can calculate the cumulative distribution function of Y as: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[(X_1 + X_2) \leq y] = \int_{-\infty}^{y} \mathbb{P}[X_1 \leq y - x \mid X_2 = x] \cdot f_{X_2}(x)\, dx = \int_{-\infty}^{y} \mathbb{P}[X_1 \leq y - x] \cdot f_{X_2}(x)\, dx = \int_{-\infty}^{y} (1 - e^{-\lambda(y-x)}) \cdot \lambda e^{-\lambda x}\, dx = \int_{-\infty}^{y} \lambda e^{-\lambda x}\, dx - \int_{-\infty}^{y} \lambda e^{\lambda x - \lambda y - \lambda x}\, dx = \int_0^y \lambda e^{-\lambda x}\, dx - \int_0^y \lambda e^{-\lambda y}\, dx = -e^{-\lambda x}\Big|_0^y - (\lambda e^{-\lambda y} x)\Big|_0^y = 1 - e^{-\lambda y} - \lambda e^{-\lambda y} y$

Since the density of Y is $f_Y(y) = \frac{dF_Y(y)}{dy}$ by definition, so for $y \geq 0$,

$$f_Y(y) = \lambda e^{-\lambda y} - \lambda(-y\lambda e^{-\lambda y} + e^{-\lambda y}) = \lambda^2 e^{-\lambda y} y$$

and 0 otherwise ($y < 0$).

(b) $\frac{x}{t}$

By given information and the definition of conditional probability, so we have that the CDF is:

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t)}{\mathbb{P}(X_1 + X_2 = t)}$$

Now, since X_1, X_2 can only take non-negative values again, and using the hint to condition on the event $\{X_1 + X_2 \in [t, t+\epsilon]\}$ where $\epsilon > 0$ and is small instead of $\{X_1 + X_2 = t\}$, so we have that: $\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t) = \mathbb{P}(X_1 \leq x \cap (X_1 + X_2) \in [t, t+\epsilon]) = \int_0^x \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m)\, dm\, dn = \int_0^x \int_{t-n}^{t-n+\epsilon} (\lambda e^{-\lambda n}) \cdot (\lambda e^{-\lambda m})\, dm\, dn = \int_0^x \lambda e^{-\lambda n} \left(-e^{-\lambda m}\Big|_{t-n}^{t-n+\epsilon} \right) dn = \int_0^x \lambda e^{-\lambda t}(1 - e^{\lambda \epsilon})\, dn = \lambda e^{-\lambda t}(1 - e^{-\lambda \epsilon}) x$

Similarly, $\mathbb{P}(X_1 + X_2 = t) = \int_0^t \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m)\, dm\, dn = \lambda e^{-\lambda t}(1 - e^{-\lambda \epsilon}) t$

Thus, the CDF is

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\epsilon \lambda \cdot e^{-\lambda t} x}{\epsilon \lambda \cdot e^{-\lambda t} t} = \frac{x}{t}$$

Again, using the hint and steps from part (a) where $Y = X_1 + X_2$ and that $f_Y(y) = \lambda^2 e^{-\lambda y} y$ for $y \geq 0$, so we can approximate $\mathbb{P}(X_1 + X_2 = t)$ to be:

$$\mathbb{P}(X_1 + X_2 \in [t, t+\epsilon]) = \mathbb{P}(X_1 + X_2 \leq t + \epsilon) - \mathbb{P}(X_1 + X_2 \leq t) \approx \epsilon \cdot F_Y(t) = \epsilon \lambda^2 e^{-\lambda t} t$$

Thus, the CDF is

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\epsilon \lambda \cdot e^{-\lambda t} x}{\epsilon \lambda^2 e^{-\lambda t} t} = \frac{x}{\lambda t}$$

$$\int_0^x \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m)\, dm\, dn =$$