

## 5 Exponential Practice

(a)  $f_Y(y) = \lambda^2 e^{-\lambda y} y$  for  $y \geq 0$ ; and 0 otherwise

Since  $X_1 \sim \text{Exp}(\lambda)$  and  $\lambda > 0$ , so by definition  $X_1$  has probability density function  $f_{X_1}(x) = \lambda e^{-\lambda x}$  if  $x \geq 0$  and 0 otherwise, and also that it has CDF  $F_{X_1}(x) = \mathbb{P}[X_1 \leq x] = 1 - e^{-\lambda x}$  for  $x \geq 0$ . Similarly,  $X_2$  has the same PDF and CDF.

Now, since  $Y = X_1 + X_2$ ,  $X_1, X_2$  are independent, and that all values of  $X_1, X_2$  has to be non-negative (since the probability of them having negative values is 0), so we can calculate the cumulative distribution function of  $Y$  as:  $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[(X_1 + X_2) \leq y] = \int_{-\infty}^y \mathbb{P}[X_1 \leq y - x \mid X_2 = x] \cdot f_{X_2}(x) dx = \int_{-\infty}^y \mathbb{P}[X_1 \leq y - x] \cdot f_{X_2}(x) dx = \int_{-\infty}^y (1 - e^{-\lambda(y-x)}) \cdot \lambda e^{-\lambda x} dx = \int_{-\infty}^y \lambda e^{-\lambda x} dx - \int_{-\infty}^y \lambda e^{\lambda x - \lambda y - \lambda x} dx = \int_0^y \lambda e^{-\lambda x} dx - \int_0^y \lambda e^{-\lambda y} dx = -e^{-\lambda x} \Big|_0^y - (\lambda e^{-\lambda y} x) \Big|_0^y = 1 - e^{-\lambda y} - \lambda e^{-\lambda y} y$

Since the density of  $Y$  is  $f_Y(y) = \frac{dF_Y(y)}{dy}$  by definition, so for  $y \geq 0$ ,

$$f_Y(y) = \lambda e^{-\lambda y} - \lambda(-y\lambda e^{-\lambda y} + e^{-\lambda y}) = \lambda^2 e^{-\lambda y} y$$

and 0 otherwise ( $y < 0$ ).

(b)  $\frac{x}{t}$

By given information and the definition of conditional probability, so we have that the CDF is:

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t)}{\mathbb{P}(X_1 + X_2 = t)}$$

Now, since  $X_1, X_2$  can only take non-negative values again, and using the hint to condition on the event  $\{X_1 + X_2 \in [t, t + \epsilon]\}$  where  $\epsilon > 0$  and is small instead of  $\{X_1 + X_2 = t\}$ , so we have that:  $\mathbb{P}(X_1 \leq x \cap X_1 + X_2 = t) = \mathbb{P}(X_1 \leq x \cap (X_1 + X_2) \in [t, t + \epsilon]) = \int_0^x \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m) dm dn = \int_0^x \int_{t-n}^{t-n+\epsilon} (\lambda e^{-\lambda n}) \cdot (\lambda e^{-\lambda m}) dm dn = \int_0^x \lambda e^{-\lambda n} \left( -e^{-\lambda m} \Big|_{t-n}^{t-n+\epsilon} \right) dn = \int_0^x \lambda e^{-\lambda t} (1 - e^{-\lambda \epsilon}) dn = \lambda e^{-\lambda t} (1 - e^{-\lambda \epsilon}) x$

Similarly,  $\mathbb{P}(X_1 + X_2 = t) = \int_0^t \int_{t-n}^{t-n+\epsilon} f_{X_1}(n) \cdot f_{X_2}(m) dm dn = \lambda e^{-\lambda t} (1 - e^{-\lambda \epsilon}) t$

Thus, the CDF is

$$\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t) = \frac{\epsilon \lambda \cdot e^{-\lambda t} x}{\epsilon \lambda \cdot e^{-\lambda t} t} = \frac{x}{t}$$