2 Will I Get My Package?

(a) $\mathbb{E}(X) = \frac{1}{2}$

We can make use of Theorem 15.1. Let X denote the number of customers who receive their own packages unopened, so $X = I_1 + I_2 + \cdots + I_n$ where $I_i = 0$ if the i^{th} customer received his/her own package unopened.

Since $\mathbb{E}[I_i] = 0 \cdot \mathbb{P}[I_i = 0] + 1 \cdot \mathbb{P}[I_i = 1] = \mathbb{P}[I_i = 1] = \frac{1}{n} \cdot \frac{1}{2} = \frac{1}{2n}$, so

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \cdots + \mathbb{E}[I_n] = n \cdot \frac{1}{2n} = \frac{1}{2}$$

(b) $\text{var}(X) = \frac{1}{2}$

Here, we have that $\mathbb{E}[X] = \frac{1}{2}$, so we need to calculate $\mathbb{E}[X^2]$, which we have:

$$\mathbb{E}[X^2] = \sum_{i=1}^{n} \mathbb{E}[I_i^2] + 2 \sum_{i<j} \mathbb{E}[I_i I_j]$$

Since I_i are all indicator variables, so again $\mathbb{E}[I_i^2] = \mathbb{E}[I_i = 1] = \mathbb{P}[I_i = 1] = \frac{1}{2n}$. Now, due to the properties of indicator variables, so $\mathbb{E}[I_i I_j]$ can be simplified as:

$$\mathbb{E}[I_i I_j] = \mathbb{P}[I_i I_j = 1] = \mathbb{P}[I_i = 1 \wedge I_j = 1] = \mathbb{P}[\text{both i,j are fixed points}] = \frac{1}{2n \cdot 2(n-1)}$$

Thus, $\mathbb{E}[X^2] = n \cdot \frac{1}{2n} + 2\binom{n}{2} \frac{1}{2n \cdot 2(n-1)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Thus, using Theorem 16.1, we have that:

$$var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{3}{4} - (\frac{1}{2})^2 = \frac{1}{2}$$