Sundry: I worked alone without any help.

1 Buffon's Needle on a Grids

(a) $\mathbb{P}[\text{no intersection at } \theta] = 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$

Note that a random throw of the needle is completely specified by 3 random variables:
(1) the horizontal distance X between the midpoint of the needle and the closest vertical line;
(2) the vertical distance Y between the midpoint of the needle and the closest horizontal line;
(3) the angle θ between the needle and the horizontal lines.

Since we assume a perfectly random throw, so we may assume that the position of the center of the needle and its orientation are independent and uniformly distributed (i.e. X, Y, θ are i.i.d.). Then, since the r.v.s X and Y range between 0 and θ is fixed, so their joint distribution has density $f(x, y)$ that is uniform over the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Since this square has area $\frac{1}{4}$, so the density should be:

$$f(x, y, \theta) = 4 \qquad \text{for } (x, y) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$$

$$\text{and} \qquad f(x, y, \theta) = 0 \qquad \text{otherwise}$$

Sanity Check:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, \theta)\ dxdy = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 4\ dxdy = 1$$

Now let E denote the event that the needle does NOT intersect a line. By elementary geometry the vertical distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\sin\theta$, and the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will NOT intersect any grid lines if and only if $(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)$.

Therefore, with our density function and bounds, so we have that:

$$\mathbb{P}[E] = \mathbb{P}[(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)] = \int_{\frac{1}{2}\sin\theta}^{\infty}\int_{\frac{1}{2}\cos\theta}^{\infty} f(x, y, \theta)\ dxdy$$

$$\Longrightarrow \mathbb{P}[E] = \int_{\frac{1}{2}\sin\theta}^{\frac{1}{2}}\int_{\frac{1}{2}\cos\theta}^{\frac{1}{2}} 4\ dxdy = 4 \cdot (\frac{1}{2} - \frac{1}{2}\cos\theta)(\frac{1}{2} - \frac{1}{2}\sin\theta) = 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$$

(b) $\mathbb{P}[\text{intersection}] = \frac{3}{\pi}$

Using a similar argument, we have that the r.v.s X and Y range between 0 and $\frac{1}{2}$, while θ ranges between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since we assume a perfectly random throw, so we may assume that the position of the center of the needle and its orientation are independent and uniformly distributed (i.e. X, Y, θ are i.i.d.), and thus, their joint distribution has density $f(x, y, \theta)$ that is uniform over the cube $[0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since this cube has volume $\frac{\pi}{4}$, so the density should be:

$$f(x, y, \theta) = \frac{4}{\pi} \qquad \text{for } (x, y, \theta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\text{and} \qquad f(x, y, \theta) = 0 \qquad \text{otherwise}$$

Sanity Check:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,\theta)\ dxdyd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = 1$$

Now let E_2 denote the event that the needle does NOT intersect a line. By elementary geometry the vertical distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\sin\theta$, and the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will NOT intersect any grid lines if and only if $(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)$.

Thus, with our density function and bounds, so we have that:

$$\mathbb{P}[E_2] = \mathbb{P}[(X > \frac{1}{2}\cos\theta) \wedge (Y > \frac{1}{2}\sin\theta)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{1}{2}\sin\theta}^{\infty}\int_{\frac{1}{2}\cos\theta}^{\infty} f(x,y,\theta)\ dxdyd\theta$$

$$\implies \mathbb{P}[E_2] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{1}{2}\sin\theta}^{\frac{1}{2}}\int_{\frac{1}{2}\cos\theta}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{\pi}\cdot(\frac{1}{2}-\frac{1}{2}\cos\theta)(\frac{1}{2}-\frac{1}{2}\sin\theta)\ d\theta$$

$$\implies \mathbb{P}[E_2] = 2\int_{0}^{\frac{\pi}{2}} \frac{1}{\pi}\cdot(1-\sin\theta-\cos\theta+\sin\theta\cos\theta)\ d\theta = \frac{2}{\pi}\cdot\int_{0}^{\frac{\pi}{2}} 1-\sin\theta-\cos\theta+\frac{1}{2}\sin(2\theta)\ d\theta$$

$$\implies \mathbb{P}[E_2] = \frac{2}{\pi}\cdot\left(\theta+\cos\theta-\sin\theta-\frac{1}{4}\cos(2\theta)\right)\Big|_{0}^{\frac{\pi}{2}} = \frac{}{\pi}\cdot\left((\frac{\pi}{2}+0-1+\frac{1}{4})-(0+1-0-\frac{1}{4})\right) = \frac{\pi-3}{\pi}$$

Therefore, we have that the probability that the needle intersects a grid line is:

$$\mathbb{P}[\text{intersection}] = \mathbb{P}[\overline{E_2}] = 1-\mathbb{P}[E_2] = 1-\frac{\pi-3}{\pi} = \frac{3}{\pi}$$

(c) $\mathbb{E}[X] = \frac{4}{\pi}$

Using indicator variables, we have that $X = H + V$, where H is the r.v. with $H = 1$ if the needle intersects a horizontal gridline, and 0 otherwise; V is the r.v. with $V = 1$ if the needle intersects a vertical gridline, and 0 otherwise.

Now, using linearity of expectation, we have that $\mathbb{E}[X] = \mathbb{E}[H] + \mathbb{E}[V]$. Consider $\mathbb{E}[H]$ first:

Using a similar setup as part (b), we have that the horizontal distance of the endpoint of the needle from its midpoint is $\frac{1}{2}\cos\theta$, so the needle will intersect a horizontal gridline if and only if $(x <= \frac{1}{2}\cos\theta)$.

Thus, we have that:

$$\mathbb{E}[H] = 1\cdot\mathbb{P}[H=1] + 0\cdot\mathbb{P}[H=0] = \mathbb{P}[H=1] = 2\int_{0}^{\frac{\pi}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{1}{2}\cos\theta} f(x,y,\theta)\ dxdyd\theta$$

With our desnsity function and constraints, we can rewrite the integral as:

$$\mathbb{E}[H] = 2\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}\cos\theta} \frac{4}{\pi}\ dxdyd\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta}{\pi}\ d\theta = \frac{2}{\pi}\sin\theta\Big|_{0n}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

Similarly, we would have that:

$$\mathbb{E}[V] = 2\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{1}{2}\sin\theta}\int_{0}^{\frac{1}{2}} \frac{4}{\pi}\ dxdyd\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{\pi}\ d\theta = -\frac{2}{\pi}\cos\theta\Big|_{0}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

Therefore, we can conclude that:

$$\mathbb{E}[X] = \mathbb{E}[H] + \mathbb{E}[V] = \frac{4}{\pi}$$

(d) $\mathbb{P}[X = 1] = \frac{2}{\pi}$

Since we have that the only possible numbers of a needle intersecting the gridlines are 0, 1 and 2, and that we have from part (b) that $\mathbb{P}[\text{intersection}] = \frac{2}{\pi}$, which gives us that

$$\mathbb{P}[X = 1] + \mathbb{P}[X = 2] = \frac{3}{\pi}$$

Thus, we have that $\mathbb{P}[X = 2] = \frac{3}{\pi} - \mathbb{P}[X = 1]$. Then, from part (c), we have that

$$\mathbb{E}[X] = \frac{4}{\pi}$$

which can be rewritten as having $\mathbb{E}[X] = 0 \cdot \mathbb{P}[X = 0] + 1 \cdot \mathbb{P}[X = 1] + 2 \cdot \mathbb{P}[X = 2] = \mathbb{P}[X = 1] + 2 \cdot (\frac{3}{\pi} - \mathbb{P}[X = 1]) = \frac{6}{\pi} - \mathbb{P}[X = 1] = \frac{4}{\pi}$

Thus, we can calculate that:

$$\mathbb{P}[X = 1] = \frac{2}{\pi}$$

(e) $\mathbb{E}[Z] = \frac{4}{\pi}$

let Z be the random variable representing the number of times such an equilateral triangle intersects the gridlines. We can "split" the triangle into three length-$\frac{1}{3}$ unit needles and get $Z = I_1 + I_2 + I_3$, where I_i is the number of times the i^{th} segment of the triangle intersects the gridlines. Thus, using linearity of expectation and the fact that each unit needle is identical (i.e. each of the $\mathbb{E}[I_i]$'s are equal), so we have:

$$\mathbb{E}[Z] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3] = \mathbb{E}[\text{unit length needle intersection}]$$

Now, using our result from part (c), we have that the expectation of the number of times a needle intersects the gridlines is: $\mathbb{E}[I_1] = \frac{4}{\pi}$. Therefore,

$$\mathbb{E}[Z] = \frac{4}{\pi}$$

2 Variance of the Minimum of Uniform Random Variables

(a) $\text{var}(Y) = \frac{n}{(n+1)^2\,(n+2)}$

Given that $n \in \mathbb{Z}^+,\ X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$, and that $Y = \min\{X_1, \ldots, X_n\}$, so using result from HW 12 Problem 6(a), we have that:

$$\mathbb{E}(Y) = \frac{1}{n+1}$$

Now, since $0 \leq Y \leq 1$, so we also have that $0 \leq Y^2 \leq 1$ and thus, we can use the tail sum formula to obtain:

$$\mathbb{E}(Y^2) = \int_0^\infty \mathbb{P}(Y^2 > y)\,dy = \int_0^1 \mathbb{P}(Y^2 > y)\,dy$$

Now, since $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$, and $Y = \min\{X_1, X_2, \ldots, X_n\}$, so we have that $Y^2 = \min\{X_1^2, X_2^2, \ldots, X_n^2\}$. Thus, for any $y \in [0,1]$, we can calculate an expression for $\mathbb{P}(Y^2 > y)$ as: $\mathbb{P}(Y^2 > y) = \mathbb{P}(\min\{X_1^2, \ldots, X_n^2\} > y) = \mathbb{P}(X_1^2 > y, \ldots, X_n^2 > y) = \mathbb{P}(X_1^2 > y) \cdot \cdots \cdot \mathbb{P}(X_n^2 > y)$ where for any $i \in \{1, \ldots, n\}$, $\mathbb{P}(X_i^2 > y) = \mathbb{P}(X_i > \sqrt{y}) = \frac{1-\sqrt{y}}{1-0} = 1 - \sqrt{y}$. Thus,

$$\mathbb{P}(Y^2 > y) = (1-\sqrt{y}) \cdot \cdots \cdot (1-\sqrt{y}) = (1-\sqrt{y})^n$$

Therefore, we can now calculate the expectation of Y, which is:

$$\mathbb{E}(Y^2) = \int_0^1 \mathbb{P}(Y^2 > y)\,dy = \int_0^1 (1-\sqrt{y})^n\,dy \tag{1}$$

Now, we first use the substitution $u = 1 - \sqrt{y}$, and so: $du = -\frac{1}{2}y^{-\frac{1}{2}}\,dy$, which can be transformed into: $dy = -2\sqrt{y}\,du = (2u-2)\,du$. Also, the bounds of the integrals changes with $u = 0$ as $y = 1$, and $u = 1$ as $y = 0$. Thus, we can continue to evaluate Eq. (1) as:

$$\mathbb{E}(Y^2) = \int_1^0 u^n \cdot (2u-2)\,du = \int_1^0 2u^{n+1} - 2u^n\,du = \left(\frac{2}{n+2}u^{n+2} - \frac{2}{n+1}u^{n+1}\right)\Big|_1^0$$

$$\implies \mathbb{E}(Y^2) = 0 - \left(\frac{2}{n+2} - \frac{2}{n+1}\right) = \frac{2}{(n+1)(n+2)}$$

Therefore, we can calculate the variance of Y by definition:

$$\text{var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{2}{(n+1)(n+2)} - \left(\frac{1}{n+1}\right)^2 = \frac{n}{(n+1)^2\,(n+2)}$$

3 Erasures, Bounds, and Probabilities

(1) $p \leq 2 \cdot 10^{-7}$

We're given that Alice is sending 1000 bits to Bob, the probability that a bit gets erased is p, the erasure of each bit is independent of the others, and that Alice is using a scheme that can tolerate up to one-fifth of the bits being erased.

Let X be the r.v. indicating the number of bits being lost, so $0 \leq X \leq 1000$, and thus, as we identify $X \sim \text{Bin}(1000, p)$, so by Theorem ???:

$$\mathbb{E}[X] = 1000p$$

Thus, we can use the Markov's inequality since X is non-negative with a finite mean

$$\mathbb{P}[X \geq 200] \leq \frac{\mathbb{E}[X]}{200} = 5p$$

Then, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[X \geq 200] \leq 10^{-6}$. Therefore, we would have

$$\mathbb{P}[X \geq 200] \leq 5p \leq 10^{-6}$$

1 which gives us an upper bound of p:

$$p \leq 2 \cdot 10^{-7}$$

(2) $p \leq 4.00 \cdot 10^{-5}$

Similarly to part (a) and using its results, with $X \sim \text{Bin}(1000, p)$, so we have that:

$$\mu = \mathbb{E}[X] = 1000p$$

$$\text{var}(X) = 1000\, p(1-p)$$

Now, we have that:

$$\mathbb{P}[X \geq 200] \;=\; \mathbb{P}[X - \mu \geq 200 - \mu] \;\leq\; \mathbb{P}\big[|X - \mu| \geq |200 - \mu|\big]$$

Thus, using Chebyshev's Inequality, we could set up another equation:

$$\mathbb{P}[X \geq 200] \;\leq\; \mathbb{P}\big[|X - \mu| \geq |200 - \mu|\big] \;\leq\; \frac{\text{var}(X)}{(|200 - \mu|)^2} \;=\; \frac{1000\, p(1-p)}{(200 - 1000p)^2}$$

Again, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[X \geq 200] \leq 10^{-6}$. Therefore, we would have

$$\mathbb{P}[X \geq 200] \;\leq\; \frac{1000\, p(1-p)}{(200 - 1000p)^2} \;\leq\; 10^{-6}$$

Well, using a calculator, we have an upper bound of p:

$$p \leq 4.00 \cdot 10^{-5}$$

(3) $p \leq 0.1468$

Let $X = X_1 + X_2 + \cdots + X_1000$, where each $X_i = 1$ if the i^{th} bit gets erased and 0 otherwise. We're told that all the X_i's are i.i.d. random variables, and they have common finite expectation:

$$\mu = \mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$$

and finite variance:

$$\sigma^2 = \mathrm{var}(X_I) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = p - p^2$$

which also gives us that $\sigma = \sqrt{p - p^2}$.

Now, let $S_n = \sum_{i=1}^{n} X_i$, then for very large n (i.e. when $n = 1000$), the Central Limit Theorem gives us that:

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{200-1000\mu}{\sigma\sqrt{1000}}} e^{-x^2/2}\, dx \tag{1}$$

Now, since we wish to have the probability of a communications breakdown being at most 10^{-6}, i.e. $\mathbb{P}[S_{1000} \geq 200] \leq 10^{-6}$, which is equivalent to $\mathbb{P}[S_{1000} < 200] = \mathbb{P}[\overline{S_{1000} \geq 200}] \geq 1 - 10^{-6}$. Then, since

$$\mathbb{P}[S_{1000} < 200] \;=\; \mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} < \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \;\leq\; \mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}]$$

Thus, for $\mathbb{P}[S_{1000} < 200] \geq 1 - 10^{-6}$, we need

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \;\geq\; 1 - 10^{-6}$$

which, by Eq. (1) and our setup of μ and σ, is (approximately) equivalent to having:

$$\mathbb{P}[\frac{S_{1000} - 1000\mu}{\sigma\sqrt{1000}} \leq \frac{200 - 1000\mu}{\sigma\sqrt{1000}}] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{200-1000p}{\sqrt{1000(p-p^2)}}} e^{-x^2/2}\, dx \;\geq\; 1 - 10^{-6}$$

$$\implies \int_{-\infty}^{\frac{200-1000p}{\sqrt{1000(p-p^2)}}} e^{-x^2/2}\, dx \;\geq\; (1 - 10^{-6}) \cdot \sqrt{2\pi}$$

Again, using a calculator, and defining $a = \frac{200-1000p}{\sqrt{1000(p-p^2)}}$ for simplicity in format, so we have an inequality that looks like...

$$\mathrm{erf}(\frac{a}{\sqrt{2}}) \geq \frac{499,999}{500,000}$$

which gives us an approximate bound:

$$\frac{200 - 1000p}{\sqrt{1000(p - p^2)}} = a \geq 4.75243$$

Thus, using the calculator, we have an approximate upper bound for p (with CLT) as:

$$p \leq 0.1468$$

4 Sampling a Gaussian With Uniform

(a) Direct Proof

By definition, since $U_1 \sim Uniform(0,1)$, so U_1 has PDF:

$$f_{U_1}(x) = 1 \qquad \text{if } 0 \le x \le 1$$

$$f_{U_1}(x) = 0 \qquad \text{otherwise}$$

Then, since we have that for $RHS = -\ln U_1$, its CDF is:

$$F_{RHS}(x) = \mathbb{P}[-\ln U_1 \le x] = \mathbb{P}[U_1 \ge e^{-x}] = 1 - \mathbb{P}[U_1 < e^{-x}] = 1 - F_{U_1}(e^{-x})$$

Thus, we have the PDF of the RHS as:

$$f_{RHS}(x) = \frac{d}{dx}F_{RHS}(x) = \frac{d}{dx}(1 - F_{U_1}(e^{-x})) = e^{-x} \cdot f_{U_1}(e^{-x})$$

Then, since we have f_{U_1} above, so we can conclude that:

$$f_{RHS}(x) = e^{-x} \qquad \text{if } x \ge 0$$

$$f_{RHS}(x) = 0 \qquad \text{otherwise}$$

Also, by definition, the LHS, which is Expo(1), has PDF:

$$f_{LHS}(x) = e^{-x} \qquad \text{if } x \ge 0$$

$$f_{LHS}(x) = 0 \qquad \text{otherwise}$$

Therefore, $f_{RHS}(x) = f_{LHS}(x)$, which means that $-\ln U_1$ and $Expo(1)$ have the same PDF, and thus, they have the same distribution, i.e. $-\ln U_1 \sim Expo(1)$, as desired.

Q.E.D.

(b) Direct Proof

Given that $N_1, N_2 \sim \mathcal{N}(0,1)$ where N_1, N_2 are independent, so we have that they have the same PDF with:

$$f_{N_1}(x) = f_{N_2}(x) = \frac{1}{\sqrt{2\pi \cdot 1}}\, e^{-(x-0)^2/(2\cdot 1)} = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Since N_1, N_2 are independent, so by Theorem 20.1, we have that the joint density of N_1, N_2 is:

$$f_N(u,v) = \frac{1}{2\pi}\, e^{-(u^2+v^2)/2}$$

Then, since we have that for $RHS = N_1^2 + N_2^2$, its CDF can be written as:

$$F_{RHS}(x) = \mathbb{P}[N_1^2 + N_2^2 \le x] = \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{x-v^2}}^{\sqrt{x-v^2}} f_N(u,v)\, dudv$$

$$\text{(By the hint)} \Longrightarrow F_{RHS}(x) = \int_0^{2\pi} \int_0^{\sqrt{x}} r \cdot f_N(r\cos\theta, r\sin\theta)\, drd\theta = \int_0^{2\pi} \int_0^{\sqrt{x}} r \cdot \frac{1}{2\pi}\, e^{-r^2/2}\, drd\theta$$

$$\Longrightarrow F_{RHS}(x) = \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\sqrt{x}}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} 1 - e^{-x/2}\, d\theta = 1 - e^{-x/2}$$

Thus, we have that the (general) PDF of the RHS, $N_1^2 + N_2^2$, is:

$$f_{RHS}(x) = \frac{d}{dx} F_{RHS}(x) = \frac{d}{dx}(1 - e^{-x/2}) = \frac{1}{2}\, e^{-x/2}$$

Yet, notice that since $N_1^2 + N_2^2$ is the sum of two squares, so $N_1^2 + N_2^2 \geq 0$, i.e. for any $x < 0$, the probability/density of $N_1^2 + N_2^2 = x$ is 0, which gives us the final PDF:

$$f_{RHS}(x) = \frac{1}{2}\, e^{-x/2} \qquad \text{if } x \geq 0$$

$$f_{RHS}(x) = 0 \qquad \text{otherwise}$$

Then, by definition, the LHS, which is Expo(1/2), has PDF:

$$f_{LHS}(x) = \frac{1}{2}\, e^{-x/2} \qquad \text{if } x \geq 0$$

$$f_{LHS}(x) = 0 \qquad \text{otherwise}$$

Therefore, $f_{RHS}(x) = f_{LHS}(x)$, which means that $N_1^2 + N_2^2$ and $Expo(1/2)$ have the same PDF, and thus, they have the same distribution, i.e. $N_1^2 + N_2^2 \sim Expo(1/2)$, as desired.

Q.E.D.

(c) ???

First, using the (extension of) results from parts (a) and (b), we have that for $U_1 \sim Uniform(0, 1)$, we can generate an exponential r.v. and the sum of the square of the normal r.v. ($N_1, N_2 \sim \mathcal{N}(0, 1)$ where they're independent) with:

$$-\frac{1}{2} \ln U_1 \sim Expo(1/2) \sim N_1^2 + N_2^2$$

Then, to sample the angle, we would use $U_2 \sim Uniform(0, 1)$ by multiplying it with 2π, which would give us $Uniform(0, 2\pi)$, and thus, we have that the angle is uniform.

Combining these two ($-\frac{1}{2} \ln U_1$ and $2\pi U_2$) would give us a pair of r.v. $\sim \mathcal{N}(0, 1)$, and we can pick either one to complete the transformation.

5 Markov Chain Terminology

(a) Irreducible: $a \neq 0 \wedge b \neq 0$; Reducible: $a = 0 \vee b = 0$

For the given Markov chain to be irreducible, it has to be able to transform between the two states, which means that the constraint is:

$$a \neq 0 \ \wedge \ b \neq 0$$

Since the term reducible Markov chains is the complement of irreducible Markov chains, so we have that for the Markov chain to be reducible, we have:

$$a = 0 \ \vee \ b = 0$$

(b) Direct Proof

Given this case with $a = b = 1$, so the Markov chain can go from state 0 to state 0 in n steps for all $n \in \{2, 4, 6, 8, \dots\}$, so we have

$$d(0) = gcd\{2, 4, \cdots\} = 2 \neq 1$$

Thus, by definition of periodicity, we have that the given Markov chain is periodic.

Q.E.D.

(c) Direct Proof

Given this case with $a, b \in (0, 1)$, so the Markov chain can go from state 0 to state 0 in any of the $n \geq 1$ steps, i.e. $n = \{1, 2, 3, \dots\}$, so we have

$$d(0) = gcd\{1, 2, 3, \cdots\} = 1$$

Similarly, we have that $d(1) = 1$. Therefore,

$$d(i) = 1 \ \forall \ i \in \mathscr{X}$$

which implies, by definition, that the given Markov chain is aperiodic.

Q.E.D.

(d) $\mathbf{P} = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$

We have that the transition probability matrix for the given Markov chain is:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e) $\pi_0 = \frac{a}{a+b}, \ \pi_1 = \frac{b}{a+b}$

We have that the balance equations can be set up with $\pi\mathbf{P} = \pi$, and so

$$[\pi_0, \pi_1] = [\pi_0, \pi_1] \cdot \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

which gives us two linear equations, and then since there's also the condition that the components of π sum up to one, so we have 3 linear equations in total:

$$\pi_0 = \pi_0(1-b) + \pi_1 a$$

$$\pi_1 = \pi_0 b + \pi_1(1-a)$$

$$\pi_0 + \pi_1 = 1$$

We could then solve them as:

$$\pi_0 = \frac{a}{a+b}$$

$$\pi_1 = \frac{b}{a+b}$$

6 Analyze a Markov Chain

(a) Direct Proof

Given this case with $a, b \in (0,1)$, so the Markov chain can go from state 0 to state 0 in any of the $n \geq 1$ steps, i.e. $n = \{1, 2, 3, \dots\}$, so we have

$$d(0) = gcd\{1, 2, 3, \cdots\} = 1$$

Similarly, we have that $d(1) = 1$ and $d(2) = 2$. Therefore,

$$d(i) = 1 \; \forall \; i \in \mathscr{X}$$

which implies, by definition, that the given Markov chain is aperiodic.

Q.E.D.

(b) $a(1-b)(1-a)a = a^2(1-a)(1-b)$

Given that $X(0) = 0$, so

$$\mathbb{P}[X(1) = 1 \mid X(0) = 0] = a$$

and similarly

$$\mathbb{P}[X(2) = 0 \mid X(1) = 1] = 1 - b$$

$$\mathbb{P}[X(3) = 0 | X(2) = 0] = 1 - a$$

$$\mathbb{P}[X(4) = 1 \mid X(3) = 0] = a$$

Thus, we have that $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0] =$ $\mathbb{P}[X(1) = 1 \mid X(0) = 0] \cdot \mathbb{P}[X(2) = 0 \mid X(1) = 1] \cdot \mathbb{P}[X(3) = 0 | X(2) = 0] \cdot \mathbb{P}[X(4) = 1 \mid X(3) = 0]$

$$\Longrightarrow \mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0] = a(1-b)(1-a)a = a^2(1-a)(1-b)$$

(c) $\pi = (\frac{1-b}{ab+a-b+1}, \frac{a}{ab+a-b+1}, \frac{ab}{ab+a-b+1})$

First, we calculate that the transition probability matrix for the given Markov chain is:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1-a & a & 0 \\ 1-b & 0 & b \\ 0 & 1 & 0 \end{bmatrix}$$

Then, we have that the balance equations can be set up with $\pi \mathbf{P} = \pi$, and so

$$[\pi_0, \pi_1, \pi_2] = [\pi_0, \pi_1, \pi_2] \cdot \begin{bmatrix} 1-a & a & 0 \\ 1-b & 0 & b \\ 0 & 1 & 0 \end{bmatrix}$$

Then, since the components of π sum up to one, so we have 4 linear equations in total:

$$(1-a)\pi_0 + (1-b)\pi_1 + 0 \cdot \pi_2 = \pi_0$$

$$a \cdot \pi_0 + 0 \cdot \pi_1 + 1 \cdot \pi_2 = \pi_1$$

$$0 \cdot \pi_0 + b \cdot \pi_1 + 0 \cdot \pi_2 = \pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

We could then solve them to get the invariant distribution as:

$$\pi_0 = \frac{1-b}{ab+a-b+1}$$

$$\pi_1 = \frac{a}{ab+a-b+1}$$

$$\pi_2 = \frac{ab}{ab+a-b+1}$$

(d) $\frac{ab-b+1}{ab}$

Since $X(0) = 1$, so $\mathbb{P}[X(1) = 2|X(0) = 1] = b, \mathbb{P}[X(1) = 1|X(0) = 1] = 0$, and $\mathbb{P}[X(1) = 0|X(0) = 1] = 1 - b$. Now, since we want the expectation of the number of steps until we transit to state 2 for the first time, so we need to further examine the condition of $X(1) = 0$.

Since for $i \in \mathbb{N}$, we have $\mathbb{P}[X(i+1) = 0|X(i) = 0] = 1 - a$, $\mathbb{P}[X(i+1) = 1|X(i) = 0] = a$ and $\mathbb{P}[X(i+1) = 2|X(i) = 0] = 0$, so we have that
$\mathbb{E}[T_2|X(0) = 1] =$
$\mathbb{P}[X(1) = 0|X(0) = 0] \cdot \mathbb{E}[X(1) = 0|X(0) = 0] + \mathbb{P}[X(1) = 1|X(0) = 0] \cdot \mathbb{E}[X(1) = 1|X(0) = 0] + \mathbb{P}[X(1) = 2|X(0) = 0] \cdot \mathbb{E}[X(1) = 2|X(0) = 0] =$
$(1-a) \cdot (1 + \mathbb{E}[T_2 \mid X(0) = 1]) + a \cdot (1 + \mathbb{E}[T_2 \mid X(0) = 1])$

which can be simplified a bit to:

$$\mathbb{E}[T_2 \mid X(0) = 1] = \mathbb{E}[T_2 \mid X(0) = 1] + \frac{1}{a}$$

Thus, we can combine this result with our previous setup to get:

$$\mathbb{E}[T_2 \mid X(0) = 1]) = b \cdot 1 + 0 + (1-b) \cdot \big(\mathbb{E}[T_2 \mid X(0) = 1] + \frac{1}{a}\big)$$

$$\implies \mathbb{E}[T_2 \mid X(0) = 1]) = \frac{ab-b+1}{ab}$$

7 Boba in a Straw

(a) $\tau = (1-p)(1+\tau) + p \cdot \big(2p + (1-p)(2+\tau)\big)$

With the given setup, action happens every second, and the probability of a new boba entering the bottom component is p, and we can set up an equation for hitting time with the concept of Markov chain as:

$$\tau = (1-p)(1+\tau) + p \cdot \big(2p + (1-p)(2+\tau)\big)$$

(b) $\tau = (1-p)(1+\tau) + p \cdot \Big(1 + 3p + (1-p) \cdot \big(1 + 3p + (1-p) \cdot \tau\big)\Big)$

With the given new setup, action still happens every second but might take longer, and the probability of a new boba entering the bottom component is still $\mathbb{P}[B] = p$, and we can set up an equation (based on the information) for hitting time with the concept of Markov chain as:

$$\tau = (1-p)(1+\tau) + p \cdot \Big(1 + p \cdot 3 + (1-p) \cdot \big(1 + p \cdot 3 + (1-p) \cdot \tau\big)\Big)$$

(c) $10p$

Define our Markov chain to be the pair of numbers demonstrating whether of boba exist in each component of the straw, with the first number indicating the top component and the second number indicating the bottom component, so there are four possible states:

(0) State 0: (0,0) i.e. empty straw
(1) State 1: (0,1) i.e. boba in bottom only
(2) State 2: (1,0) i.e. boba in top only
(3) State 3: (1,1) i.e. full straw

With the probability of a new boba entering the bottom component being p, so we first write out the transition probability matrix for the given Markov chain condition as:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{bmatrix}$$

Then, we have that the balance equations can be set up with $\pi\mathbf{P} = \pi$, and so

$$[\pi_0, \pi_1, \pi_2, \pi_3] = [\pi_0, \pi_1, \pi_2, \pi_3] \cdot \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{bmatrix}$$

Then, since the components of π sum up to one, so we have 5 linear equations in total:

$$(1-p)\pi_0 + (1-p)\pi_2 = \pi_0$$

$$p\pi_0 + p\pi_2 = \pi_1$$

$$(1-p)\pi_1 + (1-p)\pi_3 = \pi_2$$

$$p\pi_1 + p\pi_3 = \pi_3$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

We could then solve them to get the invariant distribution as:

$$\pi_0 = (1-p)^2$$

$$\pi_1 = p - p^2$$

$$\pi_2 = p - p^2$$

$$\pi_3 = p^2$$

Thus, the long-run average rate of Jonathan's boba consumption (i.e. expectation of bobas in the top component) is:

$$0 \cdot \pi_0 + 0 \cdot \pi_1 + 1 \cdot \pi_2 + 1 \cdot \pi_3 = p$$

Therefore, with each boba being roughly 10 calories, so the long-run average rate of Jonathan's calorie consumption is $10p$.

(d) $2p$

Using our results from part (c), we have solved the invariant distribution of my Markov chain, which gives us that the long-run average number of boba which can be found inside the straw is:

$$0 \cdot \pi_0 + 1 \cdot \pi_1 + 1 \cdot \pi_2 + 2 \cdot \pi_3 = 2p$$