PCA

- For $A \in \mathbb{R}^{n \times m}$, mean-centered $\widetilde{A}$
- Covariance matrix $S := \frac{1}{n}\widetilde{A}^T\widetilde{A}$; can always be computed, unlike R, correlation matrix
- PCA: Diagonalize $S = P\Lambda P^T$ (Spectral Theorem tells us the real, symmetric S can always be diagonalized.)
  - The columns of P are the "principal components", and forms an orthonormal m-dimensional basis
  - Since all eigenvals of $\widetilde{A}^T\widetilde{A}$ are nonnegative, so entries of $\Lambda \geq 0$. WLOG, order the eigenvals in Λ in descending values.
  - The weight of each principal component $\vec{p}_i$ is $\sqrt{\lambda_i} = \sigma_i$
  - The first principal component $\vec{p}_1$ (i.e. has the largest eigenvalue) corresponds to the direction that maximizes the variance of the data, when projected onto it. (Sim. for k-dim subspace, i.e. first k principal components)

    Pf: Let $\widetilde{A} = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$, then for any unit vector $\vec{v}$, we have $proj_{\vec{v}}\vec{a}_i = \vec{a}_i^T\vec{v}$, so stacking the projections, we have $\widetilde{A}\vec{v}$, and the variance of these projections is: $||\widetilde{A}\vec{v}||$, equiv. to maximizing $||\widetilde{A}\vec{v}||^2$, ...
  - The correlation between the data projected onto any pair of principal components is 0
- PCA-SVD (bi-directional):
  - Consider SVD of $\widetilde{A} = U\Sigma V^T$, so $S = V(\frac{1}{n}\Sigma^T\Sigma)V^T$ since U, V are orthonormal ($U^TU = I$)
  - Thus, the columns of V are the principal components, and eigenvals vs. singular vals has relationship: $\lambda_i = \frac{1}{n}\sigma_i^2$ for $1 \leq i \leq m$

Interpolation

- Vectorize a DT signal, if the start-time is not specified, then assume the signals start at $t = 0$.
- Zero-order hold (ZOH) : applying the DT control $u(k) = y$, i.e. apply the constant CT input $u(t) = y$ over the interval $k\Delta \le t \le (k+1)\Delta$ to the CT system.
- Interpolation by basis functions $\phi(t)'s$
  - $y(t) = \sum_{k=0}^{N-1} y_d(k) \cdot \phi(t - k\Delta)$ where $\phi(t)$ satisfies:
    1. $\phi(0) = 1$
    2. $\phi(k\Delta) = 0$ for all $k \neq 0$
  - ZOH : box-shaped ϕ
  - PWL (piecewise linear) : hat-shaped (connect the dots directly)
  - Sinc : $\phi(t) = sinc(t/\Delta) = \frac{\sin(\pi t/\Delta)}{\pi t/\Delta}$ is differentiable and band-limited (connect the dots with sinusoid)
- Interpolation by global polynomials
  - $y(t) = \sum_{i=0}^{N-1} a_i t^i$ where a_i are chosen s.t. $y(k\Delta) = y_d(k)$
  - Solve N unknowns-N linear equations of $\vec{y} = V\vec{a}$:

$$\begin{bmatrix} y_d(0) \\ y_d(1) \\ y_d(2) \\ \vdots \\ y_d(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \Delta & \Delta^2 & \cdots & \Delta^{N-1} \\ 1 & 2\Delta & (2\Delta)^2 & \cdots & (2\Delta)^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (N-1)\Delta & \Big((N-1)\Delta\Big)^2 & \cdots & \Big((N-1)\Delta\Big)^{N-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

    Due to the Vandermonde structure of the matrix, so $det(V) = (N-1)!\Delta \neq 0$, and the non-zero determinant gives that $\vec{y} = V\vec{a}$ has a unique solution, so the polynomial interpolation of a DT signal is unique.
  - Lagrange interpolation : use a different set of basis $\{L_0(x), L_1(x), \cdots, L_{n-1}(x)\}$, which has the property that $L_i(x_j) = 1$ if $i = j$, and $= 0$ otherwise.

$$L_i(x) = \prod_{j=0, j\neq i}^{n-1} \frac{x - x_j}{x_i - x_j}$$

  - Poor fit near the edges

DFT, Fourier Series

- $\omega_N = e^{j2\pi/N}$, DFT basis vectors: $u_k[n] = \omega_N^{-kn}$ for $k, n = 0, \cdots, N-1$, i.e. $\vec{u}_k = \begin{bmatrix} 1 & \omega_N^{-k} & \cdots & \omega_N^{-(N-1)k} \end{bmatrix}^T$
- DFT Matrix: $F_N^T = F_N$, full rank with $F_N^{-1} = \frac{1}{N} F_N^*$, columns are orthogonal with norm $\sqrt{N}$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \cdots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \cdots & \omega_N^{-(N-1)^2} \end{bmatrix}$$

- Given an N-timestep signal $\vec{y}$ (time domain), then its DFT is: $\vec{Y} = F_N \vec{y}$ (frequency domain), with $Y_i = \sum_{j=0}^{N-1} \omega^{-ij} y_j$ and $y_i = \frac{1}{N} \cdot \sum_{j=0}^{N-1} \omega^{ij} Y_j$
- Disc: $\vec{y} \in \mathbb{R} \iff Y_0 \in \mathbb{R}$ and $Y_i = \overline{Y_{N-i}}$ for $i = 1, \ldots, N-1$
- HW: For $x[n] = cos(\frac{2\pi}{N} pn)$ for some N, p and $n = 0, \cdots, N-1$, then its DFT X has: $X[k] = \frac{N}{2}$ for $k = p, N-p$, and $= 0$ otherwise.
  (Copy intuition from HW 12 Q4)
- $x(t) = A_0 + \sum_{k=1}^{M} A_k \cos\left(2\pi(kf_0)t + \theta_k\right)$ where f_0 is the fundamental frequency, so phasor $= A_0 + \sum_{k=1}^{M} \frac{A_k}{2} e^{j\theta_M}$

  1. Taking $N = 2M+1$ samples at timepoints $k\Delta$ where $k = 0, \cdots, (N-1)$ and $\Delta = \frac{T}{N}$; T is the smallest time period over which, all of the above signals are periodic. (This N ensures no overlapping terms, i.e. aliasing)
  2. First, define $\vec{u}_k^T = \begin{bmatrix} 1 & \omega_N^{-k} & \cdots & \omega_N^{-(N-1)k} \end{bmatrix}$ and $\vec{X}_k = F_N \vec{x}_k$
  3. Thus, $\vec{x}_k = \frac{A_k}{2}(e^{j\theta_k} \overline{\vec{u}_k} + e^{j\theta_k} \vec{u}_k)$, and its DFT (can be used to determine A_k, θ_k for each k):

  $$\vec{X}_k = F_N \vec{x}_k = \begin{bmatrix} \vec{u}_0^T \\ \vdots \\ \vec{u}_{N-1}^T \end{bmatrix} \cdot \frac{A_k}{2}(e^{j\theta_k} \overline{\vec{u}_k} + e^{j\theta_k} \vec{u}_k) = \frac{NA_k}{2}(e^{j\theta_k} \vec{e}_k + e^{j\theta_k} \vec{e}_{N-k})$$

  where $\vec{e}_a$ has 0 everywhere except at component a where $= 1$. (cf. HW result above)
  4. For $\vec{X}_k$, the magnitude of its k^{th} coefficient (nonzero entry) is $\frac{NA_k}{2}$; k^{th} coefficient's phase is θ_k
  5. Ex) Since $\vec{u}_0 = [1, \cdots, 1]$, so $A_0 = \sum_{i=0}^{N-1} 1 \cdot \vec{x}[i] = \sum_{i=0}^{N-1} \vec{x}[i]$

- Periodic $\iff$ $x(t+T) = x(t)$ for all t and setting fundamental frequency $f_0 = \frac{1}{T}$ and $N = \frac{T}{\Delta}$ where T is the chosen period (total length of the sample)
- A function applied to a T-periodic sinusoid will produce a (more complicated) T-periodic waveform. E.g. a cosine with period T is $x(t) = \cos(\frac{2\pi}{T} t)$

- h^{th} harmonic: $A_h \cos(2\pi h f_0 t + \phi_h)$, sampling at N points, so k^{th} component is: $\vec{x}[k] = A_h \cos(2\pi h f_0 k \Delta + \phi_h) = A_h \cos(\frac{2\pi h k}{N} + \phi_h)$

- Fourier Series repr: $x(t) = \sum_{i=0}^{+\infty} B_i \cos(2\pi i f_0 t + \theta_i) = \sum_{k=-\infty}^{+\infty} A_k e^{j2\pi f_0 k t}$ where $B, \theta \in \mathbb{R}$, $A \in \mathbb{C}$. The coefficients can be calculated, for any $k \in \mathbb{Z}$:

$$A_k = \frac{1}{T} \cdot \int_0^T e^{-j2\pi f_0 k t} x(t)\, dt$$

- Truncate to sum of $N = 2M + 1$ sinusoids, so $x(t) = \sum_{k=-M}^{M} X_k \cdot e^{j\frac{2\pi}{T}kt}$

- Difference: DFT repr discrete waveform as a summation of discrete sinusoids; Fourier Series repr continuous waveforms.

- $x(t) = C$ can be defined to have any T and f_0, when we represent it using sinusoids (Fourier Series), it must be represented with a sinusoid of $f = 0$. Therefore, usually $f = 0$ and $T = \infty$.

- Ex) $F_N \cdot \vec{e}_p = \vec{u}_p$ where $\vec{e}$ is the unit (basis) vector and $\vec{u}$ is the DFT basis vector

- Ex) DFT sampling matching of N samples
  - For pure harmonics (sinusoids), if the signal completes k periods during the discrete sequence, then it (only) has nonzero k^{th} and $(N-k)^{th}$ components in DFT sequence
  - For unit impulse, $F_N \cdot \vec{e}_k = \vec{u}_k$, so each magnitudes of the coefficient is 1
  - # of samples is the same, i.e. # vector pointers the same for time and frequency domains

Extra Sanity Checks

- SVD-based approach have the spirit of minimalism, i.e. min energy/norm etc.
- Hermitian: basically transpose, except we do complex conjugate
- Complex power: $(Me^{j\theta})^k = M^k e^{j\theta k}$
- Hermitian : takes complex conj. and transpose, i.e. for $A = \begin{bmatrix} 1 & j \\ 2 & 3j \end{bmatrix}$, then $A^* = A^H = \begin{bmatrix} 1 & 2 \\ -j & -3j \end{bmatrix}$
- Projection: $proj_{\vec{b}}\vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{b}, \vec{b} \rangle} \cdot \vec{b}$
- Inner product
  - Real: commutative and symmetric, $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b}$,
  - Complex: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^* \vec{b}$, so conjugate symmetric, but neither symmetric nor commutative
- Roots of Unity: For $z^N = 1$, then $z = e^{j\frac{2\pi}{N}k}$ for $k = 0, \cdots, N-1$, and define
$$\omega_N = e^{j2\pi/N}$$
- For valid $a \geq 1$, (DFT basis vector) $\vec{u}_a = \overline{\vec{u}_{N-a}}$, so $\overline{F_N} = \begin{bmatrix} \overline{\vec{u}_0} & \overline{\vec{u}_1} & \cdots & \overline{\vec{u}_{N-1}} \end{bmatrix} = \begin{bmatrix} \vec{u}_0 & \vec{u}_{N-1} \cdots & \vec{u}_1 \end{bmatrix}$, similar result follows for its transpose, F_N^*
- Least Squares solution to $H\vec{x} = \vec{y}$ for $H = U\Sigma V^T$ is $(V\widetilde{\Sigma}U^T)\vec{y}$ where $\widetilde{\Sigma}$ is the psuedo-inverse that has entries as inverse of nonzero entries in Σ (with $\widetilde{\Sigma}\Sigma = I$), implying that it's almost identical to min-norm solution (care for dimensions)
- After eigendecomposition, $\frac{1}{\lambda}$ are the poles of the transfer function.
- Least Squares estimate: $\vec{\hat{s}} = (D^T D)^{-1} D^T \vec{y}$
- DFT $F_N \cdot$ time domain signal = frequency (DFT) signal
- S needs to satisfy: (1) symmetry and (2) for any i, j, $S_{ii}^2 \geq |S_{ij}|$
- Orthogonal: inner product = 0 (Complex $\langle \vec{v}_i, \vec{v}_j \rangle = \vec{v}_i^* \vec{v}_j$)
- DFT $\vec{y} = F_N^{-1}\vec{Y} = \frac{1}{N} F_N^* \vec{Y} = \frac{1}{N} \overline{F_N} \vec{X}$
- $x_t = \sin(\frac{2\pi t}{N})$ and $t = 0, \ldots, N-1$, so $\vec{X}$ is? We have $x(t) = \sin(\frac{2\pi t}{N}) = \cos(\frac{2\pi t}{N} - \frac{\pi}{2})$, so $\vec{X} = [0, \frac{N}{2}e^{-j\pi/2}, 0, \cdots, 0, \frac{N}{2}e^{+j\pi/2}]^T$
- Tools and Topics
  - Diagonalization (transient analysis) $A = V\Lambda V^{-1}$
  - Phasors (sinusoids)
  - Orthonormalization
  - Upper Triangularization (Schur)
  - CCF, i.e. Controllable Canonical Form $\rightarrow$ eigenvalue placement (closed-loop)
  - SVD-PCA (open-loop, pseudo-inverse $\rightarrow$ min energy control) N.B. Spectral + complex defn of inner products etc.
  - DFT (interpolation)