Systems of Equations and Gaussian Elimination

- Gaussian elimination:
  - No solution: a row of 0s sum to $\neq 0$ (Priority)
  - Unique solution: For n variables, we have n pivots
  - Infinite solutions: Fewer pivots than variables

Linear Transformations and Linear Dependence

- !!! Care: dimensions in multiplication
- Def. A standard unit vector, $\vec{e_1}, \vec{e_2}$ etc., is a vector with all components equal to 0 except for one element, which is equal to 1.
- Def. $\mathbf{A}_{ij}$: i^{th} row and j^{th} column of $\mathbf{A}$
- Def. Linear Dependence: A set of vectors $\vec{v_1}, ..., \vec{v_n}$ is linearly dependent
  - $\iff$ There exist scalars $\alpha_1, ..., \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i \vec{v_i} = \alpha_1 \vec{v_1} + ... + \alpha_n \vec{v_n} = \vec{0}$ and not all α_i's are equal to zero. (Easier starting point mathematically.)
  - $\iff$ There exist scalars $\alpha_1, ..., \alpha_n$ and an index i such that $\vec{v_i} = \sum_{j \neq i} \alpha_j \vec{v_j}$. (In words, one of the vectors could be written as a linear combination of the rest of the vectors.)
- Def. A set of vectors $\vec{v_1}, ..., \vec{v_n}$ is linearly independent $\iff$ $\alpha_1 \vec{v_1} + ... + \alpha_n \vec{v_n} = \vec{0}$ implies $\alpha_1 = ... = \alpha_n = 0$
- Theorem 3.1: If the system of linear equations $A\vec{x} = \vec{b}$ has infinite number of solutions, then the columns of A are linearly dependent.
- Theorem 3.2 ($\approx$ converse): If the columns of A in system $A\vec{x} = \vec{b}$ are linearly dependent, then the system does not have a unique solution (either no solution or infinite solutions).
- Theorem 3.3: If the system of linear equations $A\vec{x} = \vec{b}$ has an infinite number of solutions and the number of rows in $A \geq$ the number of columns (A is a square or a tall matrix), then the rows of A are linearly dependent.
- (Proved in HW): Let $n \in \mathbb{Z}^+$, and let $\{\vec{v_1}, ..., \vec{v_k}\}$ be a set of k linearly dependent vectos in $\mathbb{R}^n$. Then, for any nxn matrix A, the set $\{A\vec{v}_1, ..., A\vec{v}_k\}$ is a set of linearly dependent vectors.
- !!! Rotation matrices R would rotate any vector by angle θ in the counterclockwise direction.

$$\mathbf{R} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

  - All rotation matrices in n dimensions can be decomposed into a series of rotations in 2 dimensions. So all rotations are products of the basic rotation matrix R generalized to a larger nxn matrix with 1's in the dimensions that aren't being rotated.

- Reflection matrices: R_1 reflects across x-axis, R_2 reflects across y-axis, R_3 reflects across $y = x$, R_4 reflects across $y = -x$..

$$\mathbf{R_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathbf{R_2} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{R_3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{R_4} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

- $\vec{v_1}, \vec{v_2}$, and $\vec{v_1} + \vec{v_2}$ are all solutions to the system of linear equation $A\vec{x} = \vec{b}$. Prove that $\vec{b} = \vec{0}$.
- Common mistake: Two linearly independent vectors with 3 elements in them do not span $\mathbb{R}^2$.

State Transition Matrices and Inverses

- Calculations of solutions, inverses, null spaces: Check by actually multiplying out the answer with the original matrix.
- Proved in lecture: The left and right inverses are identical.
- Proved in Notes: If A is an invertible matrix, then its inverse must be unique.
- Theorem 6.1: A matrix M is invertible $\iff$ its rows are linearly independent.
- Theorem 6.2: A matrix M is invertible $\iff$ its columns are linearly independent.
- Invertible Matrix Theorem: A is invertible
  - $\iff$ $\text{Null}(A) = \vec{0}$
  - $\iff$ The columns of A are linearly independent (Th. 6.2)
  - $\iff$ The equation $A\vec{x} = \vec{0}$, has a unique solution, which is $\vec{x} = \vec{0}$
  - $\iff$ For each column vector $\vec{b} \in \mathbb{R}^n, A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.
  - $\iff$ A does not have an eigenvalue $\lambda = 0$ (Can use directly?)
  - $\iff$ The determinant of $\det(A) \neq 0$
  - $\iff$ The rank of A is equal to its dimension
- $(AB)^{-1} = B^{-1}A^{-1}$ since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
- If a state-transition matrix has a non-trivial nullspace, then it's non-invertible, so the information about previous states isn't preserved.

Vector Spaces

- The Basis for $\mathbb{R}^n$ must be exactly n linearly independent vectors in $\mathbb{R}^n$. (Because Basis is defined as a **minimal**, spanning set of vectors for a given vector space)
- Proof requires:
  - Closure 1 - scaling
  - Closure 2 - additivity
  - Contains $\vec{0}$
  - Subset of $\mathbb{R}^n$

Eigenvalues and PageRank (Diagonalization)

- (Proved in HW:) If an invertible matrix $\mathbf{A}$ has an eigenvalue λ, then $\mathbf{A}^{-1}$ has the eigenvalue $\frac{1}{\lambda}$.
- (Proved in HW:) If $\mathbf{A}$ has an eigenvalue λ, then $\mathbf{A}^T$ also has the eigenvalue λ. Because of the fact that $\det(\mathbf{A} - \lambda I) = \det(\mathbf{A}^T - \lambda I)$
- Every eigenvector can only correspond to one eigenvalue.
- A matrix with only real entries can have complex eigenvalues. $\big[[0, -1],\ [1, 0]\big]$

- The zero matrix only has 1 distinct eigenvalue. -> A diagonal nxn does NOT necessarily have n distinct eigenvalues.
- (In Notes) An nxn matrix is diagonalizable if it has n linearly independent eigenvectors, (Copy down detailed diagonalization procedure in P38 of review guide); but if An nxn matrix is invertible then it doesn't necessarily have n linearly independent eigenvectors,
- Steady State: An eigenvalue of 1 does not mean there is always a steady state. We also want the other eigenvalues to < 1 in magnitude so that as n goes to infinity, λ^n goes to 0.
- Regarding determinant,
  - $\det(I_n) = 1$
  - $\det(A^T) = \det(A)$
  - $\det(A^{-1}) = \frac{1}{det(A)} = \det(A)^{-1}$
  - $\det(AB) = \det(A)\cdot\det(B)$ for square matrices A, B of the same size
  - $\det(cA) = c^n\cdot\det(A)$ for an nxn matrix
- (Proved In Notes) If two nxn diagonalizable matrices A and B have the same eigenvectors, then their matrix multiplication is commutative, i.e. $AB = BA$
- If A and B are nxn matrices that share the same n distinct eigenspaces, then A and B commute, that is, $AB = BA$.

Extra Sanity Checks

- $AA^{-1} = A^{-1}A = I$
- For matrix A, if $\lambda, \vec{v_1}$ is a eigenpair, then $A\vec{v_1} = \lambda\vec{v_1}$
- Non-trivial translation is not a linear transformation.
- For a state-transition matrix to converge, it has to have all $|\lambda| \leq 1$
- In Notes: If A is nxn, then the dimension of its column space + dimension of its nullspace = n
- Diagonalization: $A = P\Lambda P^{-1}$ where Λ is a diagonal matrix of its eigenvalues, and P is a matrix whose column vectors are the corresponding eigenvectors.
- Proofs are generally not expected to be "hardcore"
  - Past exams also usually have "direct" proofs rather than something like contradiction, etc.
  - Oftentimes you can do it in a few lines with some explanation
  - If your proof becomes a bit too unwieldy, try to start over with a different strategy
  - Start with the fundamental equations about what you know. Try to slowly involve the things you care about showing