MOS Devices, RLC

- Inverting op-amp(?)
- Capacitors and Inductors
  - $I_C(t) = C\,\frac{dV_C(t)}{dt}$, $V_L(t) = L\,\frac{dI_L(t)}{dt}$ with C in F and L in H
  - $E = \frac{1}{2}CV^2 = \frac{1}{2}LI^2$
  - $|Z_C| = \frac{1}{2\pi Cf} = \frac{1}{\omega C}$, $|Z_L| = 2\pi Lf = \omega L$
- In an NMOS, the source terminal is the terminal at a lower voltage, while in an PMOS, the source terminal is the terminal at a higher voltage.
- Transient Analysis and Phasor Analysis
- When charged, each capacitor will store the energy $\frac{1}{2}CV^2$, and so the net energy dissipated per cycle per transistor is (Cap + Resis) $\Delta E = CV^2$, so the power consumed by a capacitor being switched at a frequency f is $P = CV^2 f$
- Series: $Z_{eq} \approx \max(Z_1, Z_2)$; Parallel: $Z_{eq} \approx \min(Z_1, Z_2)$

Figure 3: NMOS Transistor

Figure 4: PMOS Transistor

l together to perform boolean algebra. For exam
sents a NOT gate.

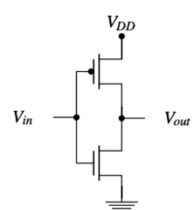


Figure 5: CMOS Inverter

- We can build an inverter with just a resistor + NMOS, but a cap at gate will load the output of the next transistor. (HW1 P6)
- Power (in EE 16B) is dissipated in a CMOS circuit only when there is switching
- NMOS devices turn on with high voltage and off with low voltage
- For a given choice of inputs in a CMOS circuit, we canNOT create a low resistance path from ground to high voltage

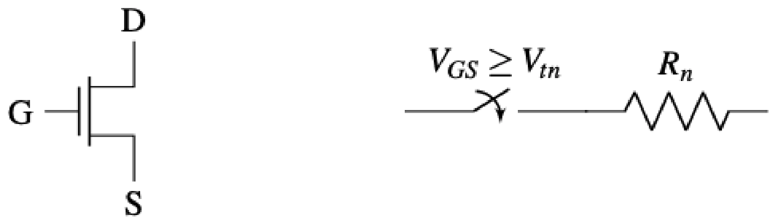


Figure 6: NMOS Transistor Resistor-switch model

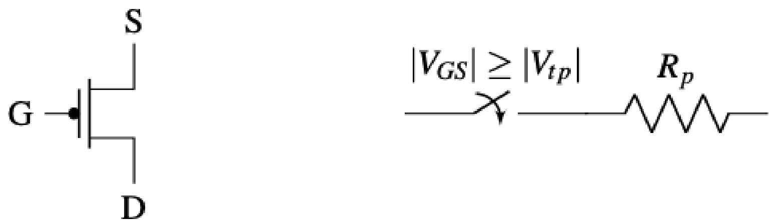
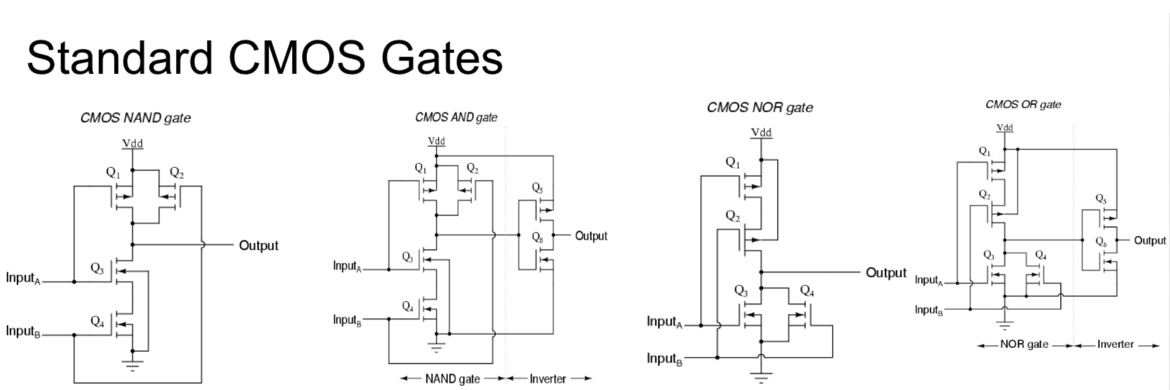


Figure 7: PMOS Transistor Resistor-switch model



- Diode: $I_D = I_S(e^{V_D/V_{th}} - 1)$. If $V_D < 0$, we strengthen the electric field, and thus less current (negative but effectively very close to 0), and vice versa.

Complex, Phasors

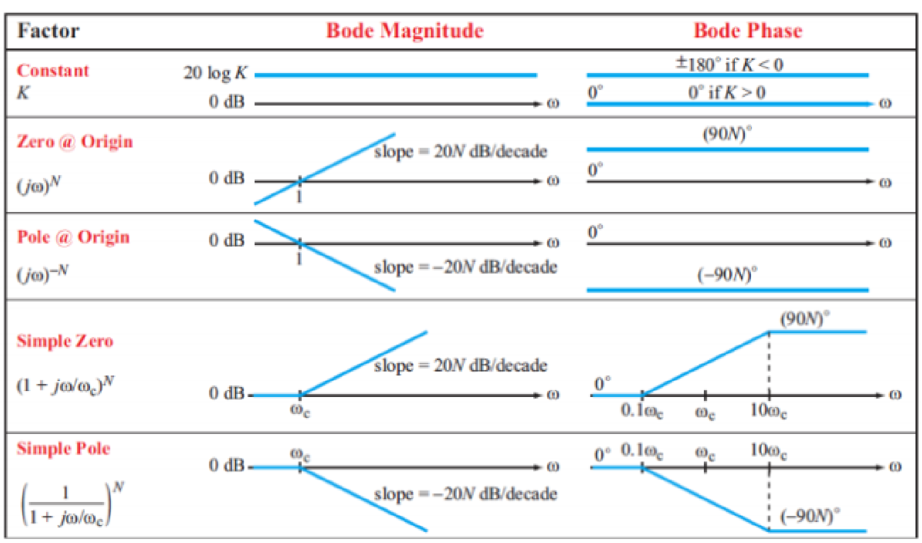
- Let $z = x + jy$, then $z = |z|e^{j\theta} = |z|\angle\theta$ where $|z| = \sqrt{x^2 + y^2}$ and $\theta = atan2(y, x)$
- $-ae^{j\theta} = ae^{j(\theta-\pi)}$
- Complex Conjugate $\overline{z} = x - jy = |z|e^{-j\theta}$; $\overline{z^n} = (\overline{z})^n$
- Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$
- $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}, \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$
- Multiplication $z_1 z_2 = |z_1||z_2|\, e^{j(\theta_1+\theta_2)}$, i.e. Rotating CCW and scaling
- Power $z^n = |z|^n\, e^{jn\theta}$, and $z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}}\, e^{j\theta/2}$
- For $k \in \mathbb{R}, z \in \mathbb{C}$, then $\angle z^k = k\angle z$, $\angle kz = \angle z$, $\angle(\frac{z_1}{z_2}) = \angle z_1 - \angle z_2$
- Phasors can only be used if all signals are sinusoids (can change into cos). Let $v(t) = V_0\cos(\omega t + \phi_v)$, then $\widetilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$
- $\sin(x) = \cos(x - \frac{\pi}{2})$
- Impedance $\xrightarrow{\triangle} \frac{\widetilde{V}}{\widetilde{I}}$
- $Z_C = \frac{1}{j\omega C}$, $Z_L = j\omega L$
- The general solution for $z^n = 1$ is $z = e^{j2\pi k/n}$ for $0 \leq k < n$.
- $|\frac{a}{b+j\omega C}| = \frac{a}{\sqrt{b^2+\omega^2C^2}}$. For example, if $H(\omega) = \frac{1}{1+j\omega RC}$, then
  - $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$
  - $H(\omega) = \frac{1-j\omega RC}{1+\omega^2R^2C^2} \implies \angle H(\omega) = \theta(\omega) = -\tan^{-1}(\omega RC)$
  - $H(\omega) = |H(\omega)|e^{j\theta(\omega)}$
- E.g. Let $v(t) = V_0\sin(\omega t) = \frac{V_0}{2j}e^{j\omega t} +$ c.c. term, so $\widetilde{V} = \frac{V_0}{2j}$
- E.g. Changing back, let $x(t) = Me^{j(\omega t+\theta)} + Me^{-j(\omega t+\theta)} = 2M\cos(\omega t + \theta)$

Filters, Transfer, Bode Plots

- Tranfers and ω_p for RC, LR, CR, RL, LC, CL, $\omega_n = \frac{1}{\sqrt{LC}}$ (Reality models LC as RLC) with quality $Q = \frac{1}{\omega_n RC}$
- Bode Plot Algorithm – Given a frequency response $H(\omega)$
  1. Break $H(\omega)$ into a product of poles and zeros. Appropriately divide terms to reduce $H(\omega)$ into one of the given forms.
  2. Draw out the Bode plot for each pole and zero in the product above.
  3. Add the resulting plots to get the final Bode plot.
- 20 dB $\cdot$ log slope = decibels/decade – 20 db = factor of 10^1, 40 db = factor of 10^2
- For magnitude, each zero contributes +1 slope, and each pole contributes −1 slope $\forall\ \omega \geq \omega_p$
- For phase, zeros: 0 if $\omega \ll z$, +90 if $\omega \gg z$, and +45 when $\omega = z$; poles: 0 if $\omega \ll p$, −90 if $\omega \gg p$, and −45 when $\omega = p$
- Write the Diff. Eqn. system in the following form: $dx(t)/dt = Ax(t)$
  Find the eigenvals/vecs of A. If A has distinct eigenvalues, the general solution is: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2} v_2$

$$
\begin{aligned}
H(\omega) = \frac{z(\omega)}{p(\omega)} &= \frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}} \left(\frac{(j\omega)^n \alpha_n + (j\omega)^{n-1}\alpha_{n-1} + \cdots + j\omega\alpha_1 + \alpha_0}{(j\omega)^m \beta_m + (j\omega)^{m-1}\beta_{m-1} + \cdots + j\omega\beta_1 + \beta_0} \right) \\
&= K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)}
\end{aligned}
$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles", and N_{z0}, N_{p0} are the number of zeros and poles at $\omega = 0$



Robotics and Control

- In order for there to be no oscillation, the eigenvalues of the system must all $\in \mathbb{R}$.
- For $\frac{dx(t)}{dt} = a \cdot x(t)$ and initial cond. $x(0) = k$, then $x(t) = k \cdot e^{at}$
- For $\vec{x}$ and eigen-basis $\vec{\tilde{x}}$, $\frac{d\vec{x}}{dt} = A\vec{x}$, V is the eigenspace for A, $\vec{\tilde{x}} = V^{-1}\vec{x}, \vec{x} = V\vec{\tilde{x}}$, and $\frac{d\vec{\tilde{x}}}{dt} = \widetilde{A}\vec{\tilde{x}}$ with $\widetilde{A} = V^{-1}AV$ and should usually $= \Lambda$, i.e. eigenvals on diagonal, and 0 everywhere else.
  Say $\vec{\tilde{x}}(0) = [x_1\ x_2]^T$ and $\widetilde{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $\vec{\tilde{x}}(t) = \begin{bmatrix} x_1 e^{a_1 t} + x_2 e^{a_2 t} \\ x_1 e^{a_3 t} + x_2 e^{a_4 t} \end{bmatrix} = \begin{bmatrix} x_1 e^{\lambda_1 t} \\ x_2 e^{\lambda_2 t} \end{bmatrix}$
  (Ignore the two constant terms if typically $a_2 = a_3 = 0$)
- Shorter Approach: The solution for $\vec{x}(t)$ will be of the form: $\vec{x}(t) = \sum_k K_{i,k}\, e^{\lambda_k t}$ where the $\lambda_k' s$ are eigenvals, and $K_{i,k}' s$ are constants derived from initial conds. Then do guess and solve for these constants. Justify using uniqueness proved in HW.
- Let $\frac{dx(t)}{dt} = \lambda x(t) + u(t)$ with initial cond $x(0) = x_0$, then we can first guess that $u(t) = e^{st}$ and $x(t) = \alpha e^{\lambda t} + \beta e^{st}$
  Gets that if $s \neq \lambda$, then $x(t) = (x_0 - \frac{1}{s-\lambda})e^{\lambda t} + \frac{1}{s-\lambda} e^{st}$
- Controllability (Below are not in scope, but... who knows)
  - Given $\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i) + \vec{w}(i)$ where the $u's$ are inputs and $w's$ are disturbances
  - Checking: We need $span(B, AB, A^2B, \ldots, A^{n-1}B) = \mathbb{R}^n$
  - Candidate test: If at step k, we didn't get a new dimension, then we won't ever get a new dim.
- Observability
  - We have observations $\vec{y}(i) = C\vec{x}(i)$, if $[C, CA, CA^2, \ldots, CA^{n-1}]^T = \mathbb{R}^n$, then we can infer all past states.
  - Least Squares for $A\vec{x} \approx \vec{b}:\ \hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$
  - Problem might occur if system doesn't depend on all n states (then our A^TA, or D^TD, won't be invertible)
- Stability (Defn): We'll say (linear systems) are <u>stable</u> if all its eigenvals $|\lambda_i(A)| < 1$, and unstable if $\exists\ \lambda_i$ s.t. $|\lambda_i(A)| \geq 1$

Extra Sanity Checks

- Tools: Phsors, Nodal Analysis, KCL/KVL, $I_c = C\frac{dV_c}{dt}$, $V_L = L\frac{dI_L}{dt}$; Diode in reverse bias : $I_D = 0$
- Phase: Change into $\cos(\cdot)$ form (unless otherwise stated) for simplicity (sin works as well)
- Least Squares Solution: $\vec{\tilde{p}} = (D^T D)^{-1} D^T \vec{s} \rightarrow$ Problem: $D^T D$ might not be invertible, i.e. columns are not linearly indep., which might be caused if two (several) inputs are too similar, or if $\vec{x} = k\vec{u}$ etc.
- Always remember that Bode Plots (BP) are an approximation, and a BP has two plots − both a magnitude plot and a phase plot.
- Phasors can only be used if all are sinusoids.
- $H(\omega)$ will scale the input signal by a factor of $M(\omega) = |H(\omega)|$, and shift its phase by an angle $\theta = \angle H(\omega)$, i.e. for $\widetilde{V}_{in}(t) = V_0$, then $V_o(t) = V_0|H(\omega)| \cdot \cos(\omega t + \theta)$
- For solving Diff. Eqs, always guess $x(t) = \alpha e^{\lambda t}$. With inputs, guess it as $x(t) = \alpha e^{\lambda t} + \beta$ and expand OR guess input to be $u(t) = e^{st}$ and consider whether $s = t$ (two cases). For systems or second derivative, guess $x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$ where $\lambda' s$ are eigenvalues of the state matrix A
- Corner frequency ω_c happens at magnitude $\frac{1}{\sqrt{2}}$ (cutoff f is just the corner f in LPF)
- Critically damped for the RLC case refers to when there is only one eigenvalue despite having two state variables. This means that the $\pm$ term in the quadratic formula has to be zero.
- Cap acts as an open circuit for low frequencies; Inductors act as wires in steady state.
- To ignore loading effects (of filters), we can either use a buffer or try to make the difference in impedance very high (typically 100 times R, and 1/100 times C) so that loading could effectively be ignore.
- Maximum approximation error for $|H(\omega)|$ is $\sqrt{2} \approx 3dB$
- Whenever we face an additional constant or input term, we can always make a change of variable (∼transient analysis) to retrun to regular.
- Let A be an $n \times n$ (orthorgonal) matrix with eigenvals $\lambda_0 > \lambda_1 > \cdots > \lambda_n > 0$ and corresponding eigenvecs $\vec{u_i}' s$, and let some vector $\vec{x} = \sum_{i=0}^{n-1} \beta_i \vec{u}_i$, then $\vec{x}^{(k+1)} = \vec{x} - \sum_{i=0}^{k} \vec{u}_i \vec{u}_i^T \vec{x}$ has all projections as 0 for λ_0 to λ_k, and can be computed with some method to get the next $\vec{u}_{k+1}$
- $E = \int_{t_0}^{t_1} P\, dt = \int_{t_0}^{t_1} IV\, dt$
- E.g. Phase: a negative sign at DC gain contributes to -180^o to the phase at 0 rad/s (not 10^0), and a zero at 0 rad/s contributes 90^o at 0 rad/s, so the initial phase is -90^o.
- E.g. If we have two poles, and we want phase shift to be less than $-2k^o$ at ω, then each must contribute $\leq -k^o$, i.e. $\angle H(\omega/\omega_p) = -\tan^{-1}(\cdot) = -k^o$ and calculate ω_p from that