### FE RM

• Security pricing, Portfolio selection, Risk management

• Defn No-Arbitrage

For a contract: pay p @ t = 0, receive  $c_k @ t = k$ ,  $k \in \mathbb{Z}^+$  ( $c_k$  can be negative)

\* Weak No-Arbitrage :  $c_k \geq 0 \ \forall k \implies p \geq 0$ 

\* Strong No-Arbitrage :  $c_k \ge 0 \ \forall k$  and  $c_\ell > 0$  for some  $\ell \implies p > 0$ 

• Simple vs Compound interest

$$A(1+nr)$$
 vs  $A(1+r)^n$ 

Continuous compounding:  $\lim_{k \to \infty} A(1 + \frac{r}{k})^{kn} = A \cdot e^{rn}$ 

Bond: for half-year bonds, yield to maturity  $\lambda$  (= rate at which price P = PV of coupon payments) follows

$$P = \frac{F}{(1 + \lambda/2)^{2T}} + \sum_{k=1}^{2T} \frac{c}{(1 + \lambda/2)^k}$$

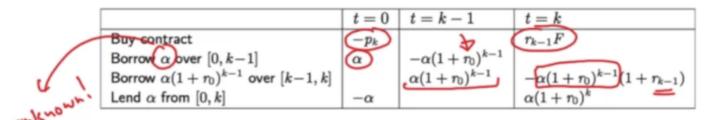
Lower quality  $\rightarrow$  lower price  $\rightarrow$  higher  $\lambda$  (YTM) - crude measure however

- Thm Linear Pricing: if price of CF  $c_a$  is  $p_a$  and price of CF  $c_b$  is  $p_b$ , then price of CF that pays  $c = c_a + c_b$  must be  $p_a + p_b$
- $\bullet$  Floating interest rate : break down a floating r bond into simpler CFs by constructing a deterministic CF

# Price of contract that pays $r_{k-1}F$ at time k

Goal: Construct a portfolio that has a deterministic cash flow

ullet The price of a deterministic cash flow at time t=0 is given by the NPV



Cash flow at time k

$$c_{k} = r_{k-1}F \alpha(1+r_{0})^{k-1}(1+r_{k-1}) + \alpha(1+r_{0})^{k}$$

$$= \underbrace{\left(F - \alpha(1+r_{0})^{k-1}\right)r_{k-1}}_{\text{random}} + \underbrace{\alpha r_{0}(1+r_{0})^{k-1}}_{\text{deterministic}}$$

Set  $\alpha = \frac{F}{(1+r_0)^{(k-1)}}$ . Then the random term is 0.

Net cash flow is now deterministic ...  $c_k = \alpha r_0 (1 + r_0)^{k-1} = Fr_0$ 

Thus, the price of the portfolio above is  $= p_k - \alpha + \alpha = \frac{F \cdot r_0}{(1+r_0)^k}$ 

$$\implies P_f = \frac{F}{(1+r_0)^n} + \sum_{k=1}^n p_k = \dots = F$$

Price of floating rate bond is always the face value!

 $\bullet$  Term structure of r, i.e. depends on duration

Spot rates  $s_t = \text{interest rate for a loan maturing in } t \text{ years}$ 

A in year 
$$t \to PV = \frac{A}{(1+s_t)^t}$$

Discount rate  $d(0,t) = \frac{1}{(1+s_t)^t}$ 

Forward rate  $f_{uv}$  = interest rate quoted today for lending from year u to v ( $v \ge u$ )

$$(1+s_v)^v = (1+s_u)^u \cdot (1+f_{uv})^{v-u} \implies f_{uv} = \left(\frac{(1+s_v)^v}{(1+s_u)^u}\right)^{\frac{1}{v-u}} - 1$$

Relation b/w spot and forward rates

$$(1+s_t)^t = \prod_{k=0}^{t-1} (1+f_{k,k+1})$$

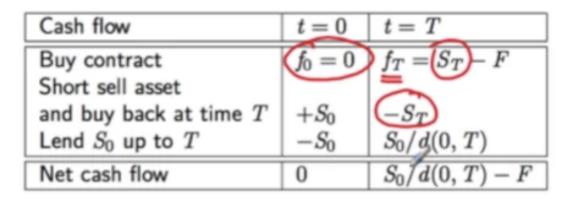
• Defn Forward Contract: gives the buyer the right and obligation to purchase a specified amount of an asset, at a specified  $\overline{\text{time }}T$ , at a specified price F (called the forward price) set at time t=0

 $f_t = \text{value/price}$  at time t of a long position in the forward contract

Value at time  $T: f_T = S_T - F$  where  $S_T$  is price of asset at T

Forward price F is set s.t. at t = 0 value/price  $f_0 = 0$ 

## Portfolio: Buy contract, short sell the underlying and lend $S_0$ up to time T



### The portfolio has a deterministic cash flow at time T and the cost = 0.

$$0 = \left(\frac{S_0}{d(0,T)} - F\right) d(0,T) \implies F = \frac{S_0}{d(0,T)}$$

 $F > S_0$  due to cost of carry

$$- f_t = (F_t - F_0) \cdot d(t, T)$$

• Defn Swap : contracts that transform one kind of cash flow into another

Leverage strength

→ Pricing interest rate swaps (fixed rate vs floating - based on LIBOR)

CFs at t = 1, ..., T

A (long) receives  $Nr_{t-1}$ , pays NX; B (short) receives NX, pays  $Nr_{t-1}$  where X is the fixed rate.

Set value of swap  $V_A = N(1 - d(0,T)) - NX \sum_{t=1}^{T} d(0,t) = 0$  gives:

$$X = \frac{1 - d(0, T)}{\sum_{t=1}^{T} d(0, t)}$$

#### • Futures

Need Martingale pricing formalism

Deterministic r: forward price = futures price

At maturity, futures price  $F_T$  = price of underlying  $S_T$ 

• <u>Defn</u> European call/put option: gives the buyer the right but not the obligation to purchase/sell 1 unit of underlying at specified price K (strike price) at a specified time T (expiration).

<u>Defn</u> American call/put option: gives ... right but not obligation to purchase/sell 1 unit ... at any time until a specified  $\overline{T}$  (expiration).

- EU call option payoff =  $\max\{S_T - K, 0\}$ , i.e. nonlinear

Intrinsic value of a EU call option at time  $t \leq T = \max\{S_t - K, 0\}$ 

- $\rightarrow$  in the money if  $S_t > K$ ; at the money if  $S_t = K$ ; out of the money if  $S_t < K$
- Notation

Price of EU call/put with strike K and expiration  $T: c_E(t; K, T)$  and  $p_E(t; K, T)$ 

Price of American call/put with strike K and expiration  $T: c_A(t; K, T)$  and  $p_A(t; K, T)$ 

European put-call parity at time t for non-dividend paying stock :  $p_E(t; K, T) + S_t = c_E(t; K, T) + K \cdot d(t, T)$ 

With dividend (PV of all dividend until maturity = D),  $p_E(t; K, T) + S_t - D = c_E(t; K, T) + K \cdot d(t, T)$ 

- Never optimal to exercise an American call on a non-dividend paying stock early!

Since  $c_A(t; K, T) \ge c_E(t; K, T) \ge \max\{S_t - Kd(t, T), 0\} > \max\{S_t - K, 0\}$ 

Not always true for put options.

#### • Options Pricing

Assume stock price dynamics in Binomial (each step Bernoulli(p) goes up by u portion or down by d with ud = 1)

Assume a risk-free asset (or cash account) is available, i.e. \$1 becomes  $(1+r)^t$  after t periods

Utility function  $u(\cdot)$  should be monotonic and concave, e.g. log