FE RM

- Security pricing, Portfolio selection, Risk management
- Defn No-Arbitrage
  For a contract : pay p @ $t=0$, receive c_k @ $t=k$, $k \in \mathbb{Z}^+$ (c_k can be negative)
  * Weak No-Arbitrage : $c_k \geq 0\ \forall k \implies p \geq 0$
  * Strong No-Arbitrage : $c_k \geq 0\ \forall k$ and $c_\ell > 0$ for some $\ell \implies p > 0$
- Simple vs Compound interest
  $A(1+nr)$ vs $A(1+r)^n$
  Continuous compounding : $\lim_{k\to\infty} A(1+\frac{r}{k})^{kn} = A \cdot e^{rn}$
  Bond : for half-year bonds, yield to maturity λ (= rate at which price P = PV of coupon payments) follows

$$P = \frac{F}{(1+\lambda/2)^{2T}} + \sum_{k=1}^{2T} \frac{c}{(1+\lambda/2)^k}$$

  Lower quality → lower price → higher λ (YTM) - crude measure however
- Thm Linear Pricing : if price of CF c_a is p_a and price of CF c_b is p_b, then price of CF that pays $c = c_a + c_b$ must be $p_a + p_b$
- Floating interest rate : break down a floating r bond into simpler CFs by constructing a deterministic CF

Price of contract that pays $r_{k-1}F$ at time k

Goal: Construct a portfolio that has a **deterministic** cash flow

- The price of a deterministic cash flow at time $t=0$ is given by the NPV

	$t=0$	$t=k-1$	$t=k$
Buy contract	$-p_k$		$r_{k-1}F$
Borrow α over $[0,k-1]$	α	$-\alpha(1+r_0)^{k-1}$	
Borrow $\alpha(1+r_0)^{k-1}$ over $[k-1,k]$		$\alpha(1+r_0)^{k-1}$	$-\alpha(1+r_0)^{k-1}(1+r_{k-1})$
Lend α from $[0,k]$	$-\alpha$		$\alpha(1+r_0)^k$

unknown!

Cash flow at time k

$$\begin{aligned} c_k &= r_{k-1}F - \alpha(1+r_0)^{k-1}(1+r_{k-1}) + \alpha(1+r_0)^k \\ &= \underbrace{\left(F - \alpha(1+r_0)^{k-1}\right) r_{k-1}}_{\text{random}} + \underbrace{\alpha r_0 (1+r_0)^{k-1}}_{\text{deterministic}} \end{aligned}$$

Set $\alpha = \frac{F}{(1+r_0)^{(k-1)}}$. Then the random term is 0.

Net cash flow is now deterministic ... $c_k = \alpha r_0 (1+r_0)^{k-1} = F r_0$

Thus, the price of the portfolio above is $= p_k - \alpha + \alpha = \frac{F \cdot r_0}{(1+r_0)^k}$

$$\implies P_f = \frac{F}{(1+r_0)^n} + \sum_{k=1}^{n} p_k = \cdots = F$$

Price of floating rate bond is always the face value!

- Term structure of r, i.e. depends on duration

  Spot rates s_t = interest rate for a loan maturing in t years

  A in year $t \to \text{PV} = \frac{A}{(1+s_t)^t}$

  Discount rate $d(0,t) = \frac{1}{(1+s_t)^t}$

  Forward rate f_{uv} = interest rate quoted today for lending from year u to v ($v \geq u$)

$$(1+s_v)^v = (1+s_u)^u \cdot (1+f_{uv})^{v-u} \implies f_{uv} = \Big(\frac{(1+s_v)^v}{(1+s_u)^u}\Big)^{\frac{1}{v-u}} - 1$$

  Relation b/w spot and forward rates

$$(1+s_t)^t = \prod_{k=0}^{t-1}(1+f_{k,k+1})$$

- Defn Forward Contract : gives the buyer the right and obligation to purchase a specified amount of an asset, at a specified time T, at a specified price F (called the forward price) set at time $t=0$

  f_t = value/price at time t of a long position in the forward contract

  Value at time T : $f_T = S_T - F$ where S_T is price of asset at T

  Forward price F is set s.t. at $t=0$ value/price $f_0 = 0$

Portfolio: Buy contract, **short** sell the underlying and lend S_0 up to time T

Cash flow	$t=0$	$t=T$
Buy contract	$f_0 = 0$	$f_T = S_T - F$
Short sell asset and buy back at time T	$+S_0$	$-S_T$
Lend S_0 up to T	$-S_0$	$S_0/d(0,T)$
Net cash flow	0	$S_0/d(0,T) - F$

The portfolio has a deterministic cash flow at time T and the cost $= 0$.

$$0 = \Big(\frac{S_0}{d(0,T)} - F\Big)d(0,T) \implies F = \frac{S_0}{d(0,T)}$$

  $F > S_0$ due to cost of carry

  - $f_t = (F_t - F_0)\cdot d(t,T)$

- Defn Swap : contracts that transform one kind of cash flow into another

  Leverage strength

  $\to$ Pricing interest rate swaps (fixed rate vs floating - based on LIBOR)

  CFs at $t = 1, ..., T$

  A (long) receives Nr_{t-1}, pays NX; B (short) receives NX, pays Nr_{t-1} where X is the fixed rate.

  Set value of swap $V_A = N(1-d(0,T)) - NX\sum_{t=1}^{T} d(0,t) = 0$ gives:

$$X = \frac{1-d(0,T)}{\sum_{t=1}^{T} d(0,t)}$$

- Futures

  Need Martingale pricing formalism

  Deterministic r : forward price = futures price

  At maturity, futures price F_T = price of underlying S_T

- Defn European call/put option : gives the buyer the right but not the obligation to purchase/sell 1 unit of underlying at specified price K (strike price) at a specified time T (expiration).

  Defn American call/put option : gives ... right but not obligation to purchase/sell 1 unit ... at any time until a specified T (expiration).

  \- EU call option payoff $= \max\{S_T - K, 0\}$, i.e. nonlinear

  Intrinsic value of a EU call option at time $t \leq T = \max\{S_t - K, 0\}$

  $\rightarrow$ in the money if $S_t > K$; at the money if $S_t = K$; out of the money if $S_t < K$

  \- Notation

  Price of EU call/put with strike K and expiration T : $c_E(t; K, T)$ and $p_E(t; K, T)$

  Price of American call/put with strike K and expiration T : $c_A(t; K, T)$ and $p_A(t; K, T)$

  European put-call parity at time t for non-dividend paying stock : $p_E(t; K, T) + S_t = c_E(t; K, T) + K \cdot d(t, T)$

  With dividend (PV of all dividend until maturity $= D$), $p_E(t; K, T) + S_t - D = c_E(t; K, T) + K \cdot d(t, T)$

  \- Never optimal to exercise an American call on a non-dividend paying stock early!

  Since $c_A(t; K, T) \geq c_E(t; K, T) \geq \max\{S_t - Kd(t, T), 0\} > \max\{S_t - K, 0\}$

  Not always true for put options.

- Options Pricing

  Assume stock price dynamics in Binomial (each step Bernoulli(p) goes up by u portion or down by d with $ud = 1$)

  Assume a risk-free asset (or cash account) is available, i.e. \$1 becomes $(1 + r)^t$ after t periods

  Utility function $u(\cdot)$ should be monotonic and concave, e.g. log