Title: Exact Solution to the Forced Burgers Equation
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Full Paper (Section 4.2)

INTRODUCTION: The forced Burgers equation is a partial differential equation (PDE) widely used in engineering and physics to model simple turbulent flows with mild nonlinearity:

$$\frac{\partial u(x,t)}{\partial t} + au(x,t)\frac{\partial u(x,t)}{\partial x} = f(x,t) + b\frac{\partial^2 u(x,t)}{\partial x^2}$$

PROBLEM: The forced Burgers equation, despite its significance, has known exact solutions only for simple, predictable forms of $f(x,t)$, while numerical solutions (though pragmatically valuable) are insufficient to dig the deep footings of mathematical understanding needed for affordable, scalable innovation at minimal risk. The primary roadblock is the lack of effective PDE solution methods for arbitrary variable $f(x,t)$.

SOLUTION: By exploiting a connection to the Schrödinger equation, an explicit, closed-form solution to the forced Burgers equation was found for arbitrary $f(x,t)$ and arbitrary initial/boundary data, removing the decades-old restrictions on the form of $f(x,t)$, opening the door to exact analytical treatment:

$$u(x,t) = \frac{\partial}{\partial x}\left(\left(\frac{2b^3}{a^3}\int f(x,t)dx\left(\int_0^L \frac{\ln^2\left(\int_0^{x'} u_0(x'')dx''\right)}{L\int_0^{x'} f_0(x'')dx''}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi x'}{L}\right)\exp\left(-\frac{bk^2\pi^2 t}{L^2}\right)dx' + \int_0^t\int_0^L \frac{2}{L}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi x'}{L}\right)\exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)dx'd\tau + \int_0^t \frac{\pi b\ln^2\left(u_l(\tau)\right)}{L^2\int f(x,\tau)dx|_{x=0}}\sum_{k=1}^{\infty}k\sin\left(\frac{k\pi x}{L}\right)\cdot \exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)d\tau - \int_0^t \frac{\pi b\ln^2\left(u_u(\tau)\right)}{L^2\int f(x,\tau)dx|_{x=L}}\sum_{k=1}^{\infty}(-1)^k\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)d\tau\right)\right)^{1/2}\right)\cdot \exp\left(\left(\frac{2b^3}{a^3}\int f(x,t)dx\left(\int_0^L \frac{\ln^2\left(\int_0^{x'} u_0(x'')dx''\right)}{L\int_0^{x'} f_0(x'')dx''}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi x'}{L}\right)\exp\left(-\frac{bk^2\pi^2 t}{L^2}\right)dx' + \int_0^t\int_0^L \frac{2}{L}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi x'}{L}\right)\exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)dx'd\tau + \int_0^t \frac{\pi b\ln^2\left(u_l(\tau)\right)}{L^2\int f(x,\tau)dx|_{x=0}}\sum_{k=1}^{\infty}k\sin\left(\frac{k\pi x}{L}\right)\cdot \exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)d\tau - \int_0^t \frac{\pi b\ln^2\left(u_u(\tau)\right)}{L^2\int f(x,\tau)dx|_{x=L}}\sum_{k=1}^{\infty}(-1)^k\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{bk^2\pi^2(t-\tau)}{L^2}\right)d\tau\right)\right)^{1/2}\right)$$

METHOD: When applying the Cole-Hopf transformation to the forced Burgers equation resulted in a Schrödinger equation, an exact and general solution to the Schrödinger equation was invoked to complete the Burgers solution, which was itself discovered through modifying the Cole-Hopf transformation in the same paper. The full paper may be accessed at the provided QR code while all notation is defined as follows:

Quantity	**Notation**	**SI Units**	**Customary Units**
Space coordinate	x	m	ft
Time coordinate	t	s	s
Absolute spatial boundary position	L	m	ft
Nonlinear throttling coefficient	$a = \text{constant}$	Dimensionless	Dimensionless
Kinematic viscosity	$b = \text{constant}$	$\text{m}^2 \cdot \text{s}$	$\text{ft}^2 \cdot \text{s}$
Velocity	$u(x,t)$	$\text{m} \cdot \text{s}^{-1}$	$\text{ft} \cdot \text{s}^{-1}$
Velocity initial condition	$u_0(x)$	$\text{m} \cdot \text{s}^{-1}$	$\text{ft} \cdot \text{s}^{-1}$
Upper velocity boundary condition	$u_l(x,t)$	$\text{m} \cdot \text{s}^{-1}$	$\text{ft} \cdot \text{s}^{-1}$
Lower velocity boundary condition	$u_u(x,t)$	$\text{m} \cdot \text{s}^{-1}$	$\text{ft} \cdot \text{s}^{-1}$
Forcing term	$f(x,t)$	$\text{m} \cdot \text{s}^{-2}$	$\text{ft} \cdot \text{s}^{-2}$
Forcing term initial condition	$f_0(x)$	$\text{m} \cdot \text{s}^{-2}$	$\text{ft} \cdot \text{s}^{-2}$