Title: Exact Arbitrary-Potential Solution to Schrödinger Equation
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Full Paper

INTRODUCTION: The Schrödinger equation is a partial differential equation (PDE) used in aerospace and defense technology to model wave propagation, signal interference, atomic modeling in composite/alloy fabrication, and error correction in quantum computers:

$$i\hbar\frac{\partial\Psi\left(\mathbf{r},t\right)}{\partial t}=V\left(\mathbf{r},t\right)\Psi\left(\mathbf{r},t\right)-\frac{\hbar^2}{2m}\nabla^2\Psi\left(\mathbf{r},t\right)$$

PROBLEM: The Schrödinger equation, despite its lucrative application, has known exact solutions only for cases too trivial for widespread application, while numerical solutions (though pragmatically valuable) are insufficient to provide that deep mathematical understanding which spurs those innovative ideas which back not only applications of theory to defense problems, but the logical clarity and transparency needed to do so. The primary roadblock is the lack of effective PDE solution methods for arbitrary variable $V(\mathbf{r},t)$.

SOLUTION: By adapting a well-known PDE method in fluid mechanics (namely the Cole-Hopf transformation) to suit the Schrödinger equation, an exact and general solution was discovered for arbitrary variable $V(\mathbf{r},t)$, arbitrary initial and boundary data, arbitrary domain, and arbitrary number of space dimensions:

$$\Psi\left(\mathbf{r},t\right)=\exp\left(\pm\left(V(\mathbf{r},t)\int_{\Omega(t)}\frac{\ln^2\left(\Psi_0(\mathbf{r}')\right)}{V_0(\mathbf{r}')}G(\mathbf{r},\mathbf{r}',t)d\mathbf{r}'-\frac{2V(\mathbf{r},t)}{i\hbar}\int_0^t\int_{\Omega(t)}G(\mathbf{r},\mathbf{r}',t-\tau)d\mathbf{r}'d\tau+\right.\right.$$
$$\left.\left.\frac{i\hbar V(\mathbf{r},t)}{2m}\int_0^t\oint_{\partial\Omega(t)}\frac{\ln^2\left(\Psi_\Omega(\mathbf{r}',\tau)\right)}{V_\Omega(\mathbf{r}',\tau)}\frac{\partial G\left(\mathbf{r},\mathbf{r}',t-\tau\right)}{\partial\hat{N}_{\mathbf{r}'}}d\mathbf{r}'d\tau\right)^{1/2}\right)\cdot$$
$$\left(\left(\int_{\Omega(t)}\left|\exp\left(\left(V(\mathbf{r},t)\int_{\Omega(t)}\frac{\ln^2\left(\Psi_0(\mathbf{r}')\right)}{V_0(\mathbf{r}')}G(\mathbf{r},\mathbf{r}',t)d\mathbf{r}'-\frac{2V(\mathbf{r},t)}{i\hbar}\int_0^t\int_{\Omega(t)}G(\mathbf{r},\mathbf{r}',t-\tau)d\mathbf{r}'d\tau+\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\frac{i\hbar V(\mathbf{r},t)}{2m}\int_0^t\oint_{\partial\Omega(t)}\frac{\ln^2\left(\Psi_\Omega(\mathbf{r}',\tau)\right)}{V_\Omega(\mathbf{r}',\tau)}\frac{\partial G\left(\mathbf{r},\mathbf{r}',t-\tau\right)}{\partial\hat{N}_{\mathbf{r}'}}d\mathbf{r}'d\tau\right)^{1/2}\right)\right|^2d\mathbf{r}\right)^{-1/2}\right)$$

METHOD: The modified Cole-Hopf transformation enabled solution by converting the Schrödinger equation into a solvable complex-valued forced heat equation. Inversion of the conversion formula enabled reconciliation of heat equation initial/boundary conditions with those of the Schrödinger equation. The full paper may be accessed at the provided QR code while all notation is defined as follows:

Quantity	**Notation**	**SI Units**	**Customary Units**
Number of dimensions	n	Dimensionless	Dimensionless
Position vector	$\mathbf{r}$	m	ft
Position vector "dummy variable"	$\mathbf{r}'$	m	ft
Time	t	s	s
Domain	$\Omega\subseteq\mathbb{R}^n=\Omega(t)$	m^n	ft^n
Reduced Planck constant	$\hbar=\text{constant}$	$1.055\times10^{-34}\,\mathrm{J}\cdot\mathrm{s}$	$7.778\times10^{-34}\,\mathrm{ft}\cdot\mathrm{lbf}\cdot\mathrm{s}$
Particle mass	$m=\text{constant}$	kg	slug
Wave function	$\Psi(\mathbf{r},t)$	$\mathrm{m}^{-n/2}$	$\mathrm{ft}^{-n/2}$
Wave function initial condition	$\Psi_0(\mathbf{r})$	$\mathrm{m}^{-n/2}$	$\mathrm{ft}^{-n/2}$
Wave function boundary condition	$\Psi_{\Omega(t)}(\mathbf{r},t)$	$\mathrm{m}^{-n/2}$	$\mathrm{ft}^{-n/2}$
Potential	$V(\mathbf{r},t)$	J	$\mathrm{ft}\cdot\mathrm{lbf}$
Potential initial condition	$V_0(\mathbf{r})$	J	$\mathrm{ft}\cdot\mathrm{lbf}$
Potential boundary condition	$V_{\Omega(t)}(\mathbf{r},t)$	J	$\mathrm{ft}\cdot\mathrm{lbf}$
Green's function of the heat equation	$G(\mathbf{r},\mathbf{r}',t)$	m^{-n}	ft^{-n}