

Geodesic Patterns for free-form Architecture

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Abstract

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1 Introduction

With the development of advanced 3D CAD tools in the last decades, architects are now able to incorporate complex freeform surfaces into their designs. This new capabilities have given birth to new architectural possibilities and, more importantly, new needs and challenges for the industry that can often be solved by the means of *Differential Geometry* (Carmo 2016) and, more specifically, *Discrete Differential Geometry* (Keenan Crane 2013), which studies the discrete counterparts of *Differential Geometry*. This new fusion between architecture and mathematics gave birth to the field known as *Architectural Geometry* (Pottmann, Schiftner, and Wallner 2008; Pottmann 2010), which has been further developed during the past years.

Given the scale an architectural scale, this smooth freeform surfaces cannot be built as designed in 3D CAD applications and, therefore, must be divided in smaller, simpler parts that could be easily manufacturable. This process is called *Rationalization*, and is not a trivial task; since the *Rationalization* of a surface implies a loss of precision in the resulting shape, it must be done in such a way that minimizes this loss. The selected shape of the generated panels also plays a key role when approaching this problems, since it is clear the final appearance of the design will vary greatly depending whether the panels are triangular, quad, hexagonal or long strips. Constructibility of the selected design (panel planarity, curvature, error margins) is of great concern in rationalization methods for architectural purposes, as opposed to related Computer Graphics methods.

In this publication we center on the generation of *Geodesic Panels on Freeform Surfaces*, meaning, seamlessly covering any given surface with a pattern of *thin, long, straight panels*, which must be *developable* and rectangular (or quasi-rectangular) in shape and must approximate the surface by being bent only on their weak axis. We will introduce the basic geometric properties behind this concept and the algorithmic implementation of several of the methods. We will also explore new methods for optimizing any given shape towards *developability*.

2 Background

Examples of the application of *Architectural Geometry* are abundant in today’s architecture. Architect Frank Ghery is known for his work ground-breaking work on the design of architectural shapes based purely on *developable* patches (Stein, Grinspun, and Crane 2018), which can be easily covered by a variety of panel patterns.

Recent projects by Zaha Hadid, Foster, Toyo-Ito... XXXXXX.

Covering surfaces with long rectangular, or quasi-rectangular panels of equal width, is a topic widely explored in traditional boat building, and has also been recently applied in several architectural projects, such as Toyo Ito's Minna No Mori (Japan), or the interior cladding at Frank Ghery's Burj Khalifa (Dubai) (Meredith and Kotronis 2013).

Some recently finished buildings would have also benefitted from the optimization of paneling and/or beam layout like... XXXXXX.

Looking for other built examples or previous/further research on the subject.

3 Geometric properties

3.1 Geodesic curves

In differential geometry, a *geodesic curve* is the generalization of a straight line into curved spaces (see fig. 1).

Also, in the presence of an *affine connection*, a geodesic is defined to be a curve whose tangent vectors remain parallel if they are transported along it. We will explore the notion of vector *parallel transport* in the following sections.

For triangle meshes, shortest polylines cross edges at *equal angles*.

Finding the truly shortest geodesic paths requires the computation of distance fields (see Carmo 2016; Kimmel and Sethian 1998)

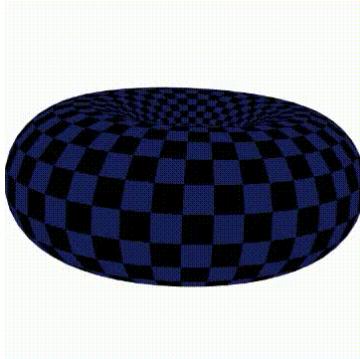


Figure 1: If an insect is placed on a surface and continually walks “forward”, by definition it will trace out a geodesic (image taken from [Wikipedia](#)).

3.1.1 Algorithmic ways of generating geodesics

The computation of geodesics on smooth surfaces is a classical topic, and can be reduced to two different solutions, depending on the initial conditions of the problem, you can basically find two types of problems (Deng 2011):

Initial value problem given a point $p \in S$ and a vector $v \in T_p S$, find a geodesic which is incident with p , such that the tangent vector to g at p is v .

Boundary value problem given two points $p_1, p_2 \in S$ find a geodesic g connecting p_1 and p_2 .

Both problems have different ways of being solved either numerically, graphically or by the means of simulations. The initial value problem can be solved using the concept of *straightest geodesics* (Polthier and Schmies 1998), whereas the boundary value problem has a very close relation with the computation of *shortest paths* between two points on a surface.

It is important to note that, during this chapter, all surfaces are discretized as triangular meshes (V,E,F) of sufficient precision.

What would that precision be?? %?? distance to reference surf??

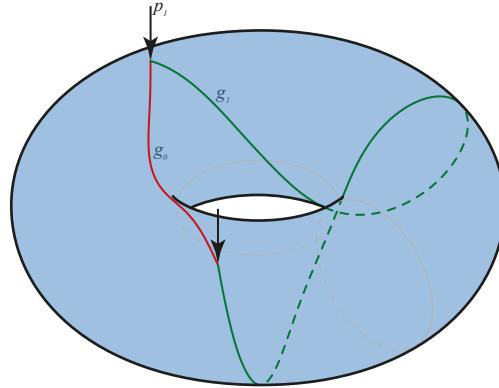


Figure 2: The concept of ‘shortest geodesics’: curves g_0 (red) and g_1 (green) are both geodesic curves of a torus, although g_1 is more than double the length of g_0 .

3.2 Developable surfaces

This is very well explained in p.170 of Denis Shelden thesis (Gerard’s suggestion). Explanation is inspired by that section.

It is also important to introduce the concept of the *developable surface*, a special kind of surface that have substantial and variable normal curvature while guaranteeing zero gaussian curvature, and as such, this surfaces can be *unrolled* into a flat plane with no deformation of the surface. This surfaces have been an extremely important design element in the practice of known architect Frank Ghery and explained in (Shelden 2002).

There are several ways of generating a developable surface using a curve in space.

1. Select the starting point of the curve
2. Obtain the perpendicular frame of the curve at the specified parameter.
3. Deconstruct the frame into it’s X, Y & Z vectors.
4. Select one of the vectors at all the sample points
5. Draw a line using the selected vector at the start point of the curve.
6. XXXXXX

3.3 Properties to aim for {#sec:properties to aim for}

We will first introduce the properties we aim for when generating geodesic panels:

3.3.1 Geodesic property

- Long Thin panels that bend about their weak axis
- Zero geodesic curvature
- Represent the shortest path between two points on a surface

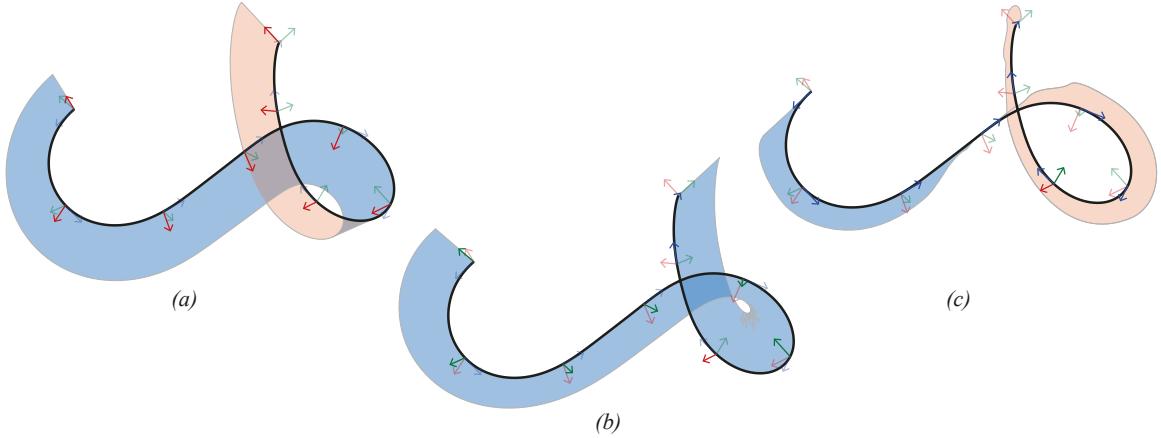


Figure 3: Developable surface generated using curve g_1 of fig. 2. (a) using the X component of the curve's perp frame; (b) using the Y component & (c) using the Z component (tangent of the curve).

3.3.2 Constant width property

- Panels whose original, unfolded shape is a rectangle.
- The only way this can happen is if the entire surface is developable.
- For all other surfaces:
 - Assuming no gaps between panels
 - Panels will not be exactly rectangular when unfolded
 - **Requirement:** Geodesic curves that guide the panels must have approximately constant distance from their neighbourhood curves.

3.3.3 Developable (or ‘pure-bending’) property

- Bending panels on surfaces changes the distances in points only by a small amount so,
- A certain amount of twisting is also present in these applications.
Some methods in this chapter do not take into account this property.

3.3.4 Objectives

Problem 1 Look for a system of geodesic curves that covers a freeform surface in a way that:

1. They have approximate constant distance with its neighbours.
2. These curves will serve as guiding curves for the panels.
3. The panels are to cover the surface with **no overlap** and **only small gaps**

Problem 2 Look for a system of geodesic curves in a freeform surface which:

1. Serve as the boundaries of wooden panels.
2. The panel's development is **nearly straight**.
3. Those panels cover the surface with **no gaps**

4 Algorithmic strategies

In this section, we will introduce the different existent methods for generating geodesic curve patterns that will be used to model the final panels, using methods indicated in Sec. 4.5.

4.1 Parallel Transport Method

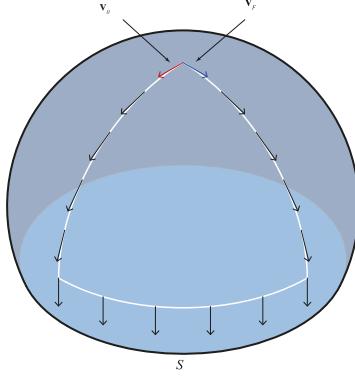


Figure 4: Parallel transport of a vector on a ‘piece-wise geodesic’ path on a sphere.

This method, described in (Pottmann et al. 2010), allows for the generation of a system of geodesic curves where either the maximum distance or the minimum distance between adjacent points occurs at a prescribed location.

In differential geometry, the concept of *parallel transport* (see fig. 5) of a vector \mathbf{V} along a curve S contained in a surface means moving that vector along S such that:

1. It remains tangent to the surface
2. It changes as little as possible in direction
3. It is a known fact that the length of the vector remains unchanged

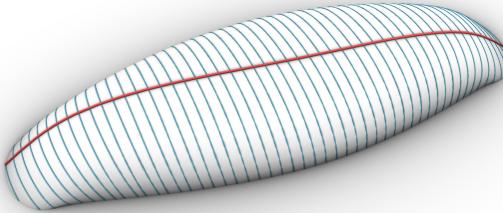


Figure 5: Example of parallel transport method. Generatrix geodesic g (red) and geodesics g^\perp generated from a parallel transported vector (blue) computed given a point and a vector \mathbf{v} tangent to the surface, in both positive and negative directions.

4.1.1 Procedure

Following this procedure, *extremal distances between adjacent geodesics occur near the chosen curve*. Meaning:

1. For surfaces of **positive curvature**, the parallel transport method will yield a 1-geodesic pattern on which the *maximum distance* between curves will be W .
2. On the other hand, for surfaces of **negative curvature**, the method will yield a 1-geodesic pattern with W being the *closest (or minimum)* distance between them.

The placement of the first geodesic curve and the selection of the initial vector are not trivial tasks. For surfaces with high variations of surfaces, the results might be unpredictable and, as such, this method is

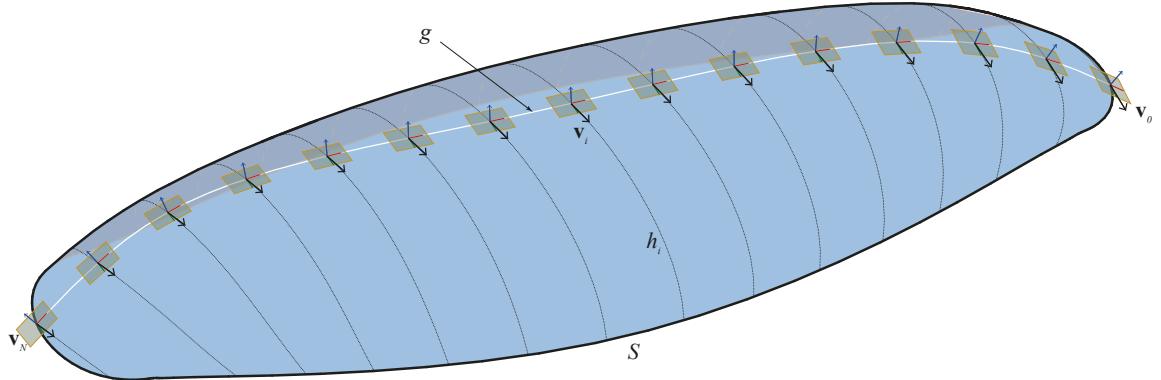


Figure 6: Parallel transport along a curve g lying on surface S is equivalent to projecting \mathbf{v}_{i-1} onto the tangent plane on p_i and subsequently normalizing \mathbf{v}_i .

Algorithm 1 Geodesic patterns by parallel transport

Input: A surface Φ , represented as a triangular mesh (V, E, F)

Output: Set of geodesic curves g_i , where $i = 0, \dots, M$

- 1: Place a geodesic curve g_x along S such that it divides the surface completely in 2.
 - 2: Divide the curve into N equally spaced points p with distance W .
 - 3: Place an vector \mathbf{v} onto p_0
 - 4: Parallel transport that vector along g_x as described in [@fig:parTransProc].
 - 5: **for all** points p_i where $i = 0, \dots, M$ **do**
 - 6: Generate geodesic curve $+g_i$ and $-g_i$ using vector \mathbf{v}_i and $-\mathbf{v}_i$ respectively.
 - 7: Join $+g_i$ and $-g_i$ together to obtain g_i
 - 8: Add g_i to output.
 - 9: **end for**
-

only suitable for surfaces with nearly constant curvature. Other solutions might involve cutting the surface into patches of nearly-constant curvature, and applying the *parallel transport method* independently on each patch.

4.2 Evolution Method

Two main concepts are covered in this section, both proposed by (Pottmann et al. 2010): the first, what is called the *evolution method*, and a second method based on *piecewise-geodesic* vector fields.

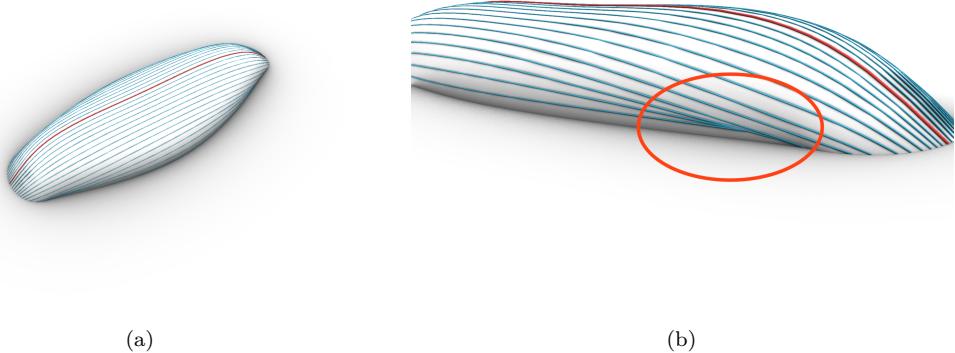


Figure 7: Surface covered by a 1-geodesic pattern using the evolution method without introducing breakpoints. Fig. 9a shows an overview of the result; while fig. 9b highlights the intersection point of several geodesic curves. This problem will be addressed by introducing the concept of ‘piece-wise’ geodesic curves; which are curves that are not geodesics, but are composed of segments of several connected geodesic curves.

4.2.1 The *evolution method*

As depicted in: Starting from a source geodesic somewhere in the surface:

- Evolve a pattern of geodesics iteratively computing ‘next’ geodesics.
- ‘Next’ geodesics must fulfill the condition of being at approximately constant distance from its predecessor.
- If the deviation from its predecessor is too great, breakpoints are introduced and continued as a ‘*piecewise geodesic*’.
- ‘Next geodesics’ are computed using Jacobi Fields

4.2.2 Distances between geodesics

1. No straight forward solution.
 1. Only for rotational surfaces (surfaces with evenly distributed meridian curves).
 2. **But** a first-order approximation of this distance can be approximated:

Starting at time $t = 0$ with a geodesic curve $g(s)$, parametrized by arc-length s , and let it move within the surface.

A snapshot at time $t = \varepsilon$ yields a geodesic g^+ near g .

$$g^+(s) = g(s) + \varepsilon \mathbf{v}(s) + \varepsilon^2(\dots) \quad (1)$$

The derivative vector field \mathbf{v} is called a *Jacobi field*. We may assume it is orthogonal to $g(s)$ and it is expressed in terms of the geodesic tangent vector g' as:

$$\mathbf{v}(s) = w(s) \cdot R_{\pi/2}(g'(s)), \quad \text{where } w'' + Kw = 0 \quad (2)$$

Since the distance between infinitesimally close geodesics are governed by Eq. 2, that equation also governs the width of a strip bounded by two geodesics at a small finite distance. { } Using this principle, you can develop strips whose width $w(s)$ fulfills the Jacobi equation $w(s) = \alpha \cosh(s\sqrt{|K|})^1$ for some value $K < 0$.

Gluing them together will result in a surface of approximate Gaussian curvature.

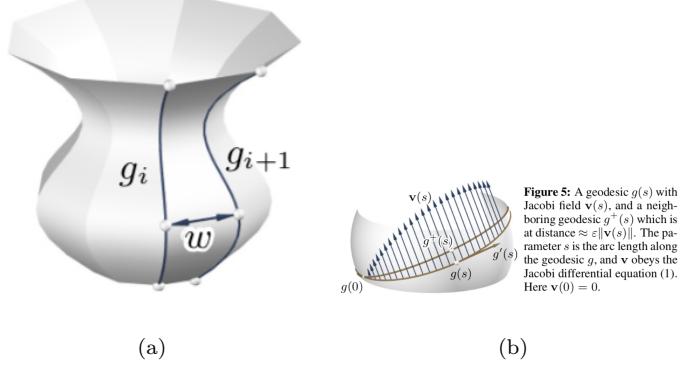


Figure 8: Geodesic distances on sphere

4.2.3 Obtaining the ‘next’ geodesic

Input: A freeform surface S , a desired width W and a starting geodesic curve g_0

Output: The ‘next’ computed geodesic on S

Pseudocode: Given a valid S , W and g_0 :

1. Sample the curve g_0 at uniformly distributed arc-length parameters x_i .
2. Compute a set of geodesics $h_i \perp g_0$ starting points $p(x_i)$ on g_0 .
3. Select a width function $\omega(s)$ that is closest to a desired target width function $W(s)$ (Width function can return a fixed value or a computed value using curvature. It could be further improved to accommodate other measures).
4. Select a Jacobi field $\mathbf{v}(s)$ using $\omega(s)$ as stated in Eq. 2.
5. ‘Walk’ the length of the vector $\mathbf{v}(s)$ on h_i to obtain χ_i . These points approximate position of the next geodesic “ g^+_i ”.
6. The previous step can have a very big error margin, therefore we must compute g_i^+ :
 1. Select a point χ_i and name it χ_0 .
 2. We can compute the next geodesic by minimizing the error between the width function $\omega(s)$ and the signed distance of the computed g_i^+ to g_0 .
7. Return computed geodesic g_i^+

Any given surface can be completely covered in this manner by recursively computing the next geodesic using the previous, given some margin of error.

4.3 Level-Set Method

We can also compute geodesic curve patterns on a surface by computing a scalar function on each vertex of a mesh that minimizes a combination of error measurements $F_{min} = F_k + \lambda F_{nabla} + F_w$.

¹**Question:** What is α in this formula? Missing image

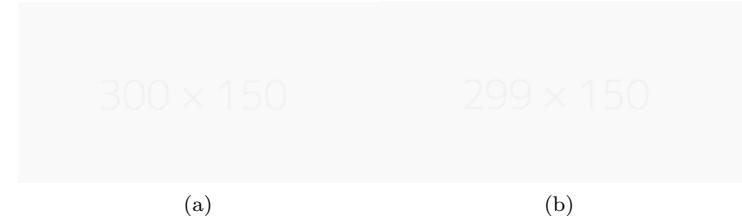


Figure 9: Surface covered by a 1-geodesic pattern using the evolution method. Fig. 9a shows *normal* implementation; while fig. 9b implements the piece-wise geodesic concept. Both images use the same parameters

4.4 Geodesic Webs

We can also expand the Level-Set method explained in the previous section from one family of curves to several families of interconnected quasi-geodesic curves by simply computing several sets of scalar functions on each vertex of the surface.

4.5 Panel modeling

In this section, we will discuss several ways to generate panels from curves lying on a given surface:

4.5.1 Tangent-developable method

The notion of *Conjugate tangents* on smooth surfaces must be defined:

- Strictly related to the *Dupin Indicatrix*
- In negatively curved areas, the Dupin Indicatrix is an hyperbola whose asymptotic directions (A1, A2)
- Any parallelogram tangentially circumscribed to the indicatrix defines two conjugate tangents \mathbf{T} and \mathbf{U} .
- The asymptotic directions of the dupin indicatrix are the diagonals of any such parallelogram.



Figure 10: Tangent developable method for panels

Initial algorithm is as follows:

For all geodesics s_i in a given pattern:

1. Compute the *tangent developable surfaces* $\rightarrow \Psi_i$
2. Trim Ψ_i along the intersection curves with their respective neighbours.
3. Unfold the trimmed Ψ_i , obtaining the panels in flat state.

Unfortunately, this method needs to be refined in order to work in practice because:

1. The rulings of tangent developables may behave in weird ways
2. The intersection of the neighboring Ψ_i 's is often *ill-defined*.

Therefore, the procedure was modified in the following way:

1. Compute the *tangent developable surfaces* Ψ_i for all surfaces $s_i \rightarrow i = \text{even numbers}$
2. Delete all rulings where the angle enclose with the tangent α is smaller than a certain threshold (i.e. 20°).
3. Fill the holes in the rulings by interpolation (???)
4. On each ruling:
 1. Determine points $A_i(x)$ and $B_i(x)$ which are the closest to geodesics s_{i-1} and s_{i+1} .
This serves for trimming the surface Ψ_i .
5. Optimize globally the positions of points $A_i(x)$ and $B_i(x)$ such that
 1. Trim curves are *smooth*
 2. $A_i(x)$ and $B_i(x)$ are close to geodesics s_{i-1} and s_{i+1}
 3. The ruling segments $A_i(x)B_i(x)$ lies close to the *original surface* Φ

This adjustments to the algorithm allow for the modeling of panels that meet the requirements of developability and approximately constant width, although it must be noted computation times increase, as double the amount of geodesic curves need to be generated, and subsequently, the desired width function needs to be half the desired width of the panels. Fig. 11 shows the result of computing such panels and the subsequent gaps between the generated panels; these gaps need to be kept within a certain width in order to produce a successful watertight panelization of the original surface. Furthermore, material restrictions such as bending or torsion where not taken into account during the construction of the panels.



Figure 11: Panels computed using the using the tangent developable method.

4.5.2 The Bi-Normal Method

The second method for defining panels, once an appropriate system of geodesics has been found on Φ , works directly with the geodesic curves.

Assume that a point $P(t)$ traverses a geodesic s with unit speed, where t is the time parameter.
For each time t there is:

- a velocity vector $T(t)$
- the normal vector $N(t)$
- a third vector $B(t)$, the *binormal vector*

This makes $T.N.B$ a *moving orthogonal right-handed frame*

The surface Φ is represented as a triangle mesh and s is given as a polyline. For each geodesic, the associated surface is constructed according to Fig. 12. Points $L(t)$ and $R(t)$ represent the border of the panel, whose distance from $P(t)$ is half the panel width.

4.5.3 Method Comparison

XXXXXX

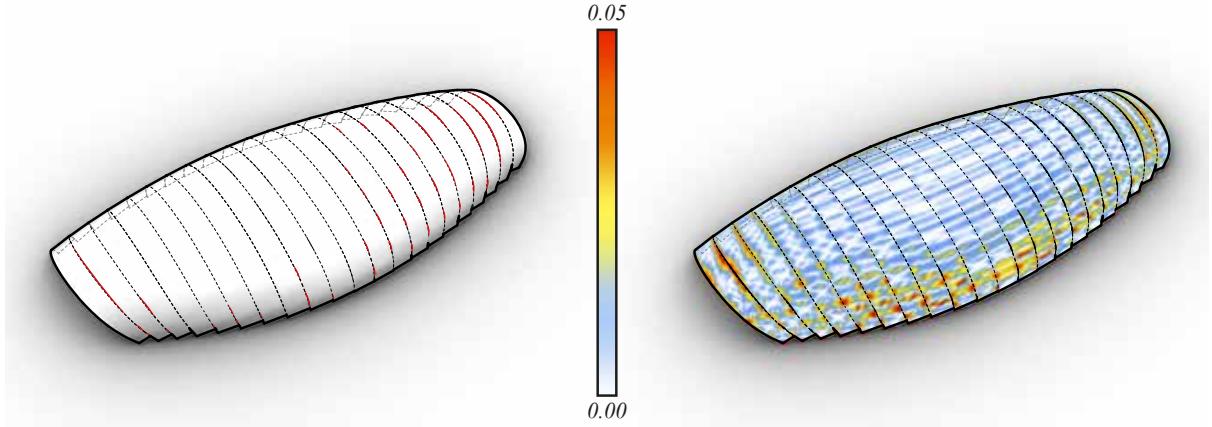


Figure 12: Binormal Method for panels & T.N.B. frame. On the left, the computed panels with the corresponding panel gaps highlighted in red. On the right, panels colored by distance to reference mesh.

4.6 Shape Optimization

4.6.1 Geodesic Vector-fields

The level-set method described in Sec. 4.3 is not suitable for arbitrary surfaces, and therefore it must be adapted to achieve the desired result. In this section, we introduce the concept of *Geodesic Vector-fields* (Pottmann et al. 2010) to divide the surface into patches that could be easily covered by equal width panels using the level-set method.

4.6.2 Dynamic shape optimization (Kennan Crane...)

Another option is to modify the original surface to make it *developable*. This algorithm was first presented by (Stein, Grinspun, and Crane 2018) and it allows to convert any given triangle mesh into a developable approximation by minimizing a specified energy applied to each edge of the mesh and subsequently subdividing in order to achieve a smooth developable approximation to the reference surface.

5 Quality Assessment

5.1 Panel gaps

5.2 Stress and strain in panels

The following section investigates the behavior of a rectangular strip of elastic material when it is bent to the shape of a ruled surface Ψ in such way that:

The central line m of the strip follows the ‘middle geodesic’ s in Ψ

This applies to both methods defining panels.fig. 13

5.2.1 Formulas

$$\rho = 1/\sqrt{K}, \quad (3)$$

$$\varepsilon = \frac{1}{2}(d/2\rho)^2 + \dots \quad (4)$$

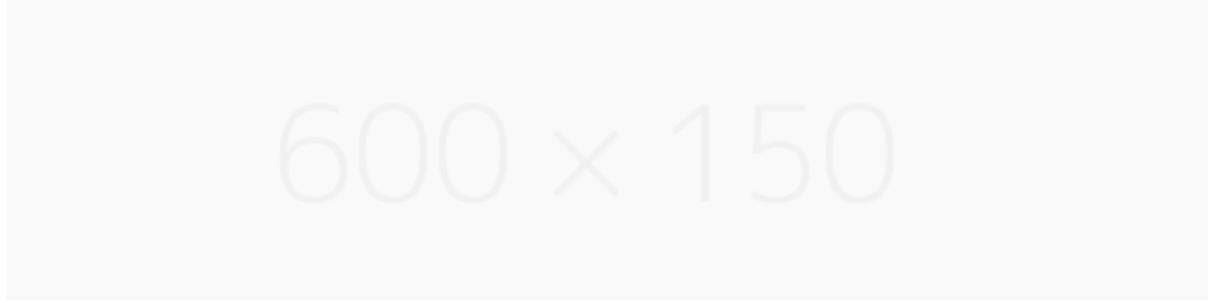


Figure 13: Stress in panels

$$d/2\rho \leq C, \quad \text{with} \quad C = \sqrt{\sigma_{max}/E}, \quad (5)$$

where σ_{max} is the maximum admissible stress and E is Young's modulus.

The value of C is a material constant which yields an *upper bound* $\sigma_{max} = 2\rho_{min}C$ for the maximum strip width.

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