Geodesic Patterns for free-form Architecture

MPDA'18 Master Thesis —— UPC-ETSAV

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Abstract

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1 Introduction

- With the development of advanced 3D CAD tools in the last decades, architects are now able to incorporate complex freeform surfaces into their designs.

- This new capabilities have given birth to new architectural possibilities and challenges.
- In an architectural scale, this smooth freeform surfaces cannot be built as designed. They must be *rationalized*
- *Rationalization* is the simplification of a complex surface into smaller parts that fit some specified requirements.

- The *rationalization* of a freeform surfaces implies a difference, or loss of precision, between the reference and the rationalized surface.
- This difference must be small in order for the rationalization to closely approximate the original.
- Some requirements are related to aesthetics, such as the precision, the desired panel shape (quad, hex, triangular, strips…) or the aspect ratio,
- while others are closely tied to the constructability of the rationalized surface, such as panel planarity, curvature allowed by material (when planar panels are not required)
- Examples of surface Rationalization are abundant in today's current architecture (Zaha Hadid, Ghery, Foster, Toyo-Ito…)
- In this *chapter* we will concentrate on the rationalization of free-form surfaces in straight *developable* panels that are rectangular or mostly rectangular in shape, and which can be bent on their weak axis to seamlessly cover the desired surface.
- Covering surfaces with long rectangular, or quasi-rectangular panels of equal width, is a topic widely explored in traditional boat building, and has also been recently applied in several architectural projects, such as Toyo Ito's Minna No Mori, or the interior cladding at Frank Ghery's.
- We will also explore current methods to convert a freeform surface into *developable* patches (the concept of developability is further explained in the following chapters).
- Rationalizing a shape into *developable* patches is one of the main features of architect Frank Ghery's practice.
- Also closely related to architectural membrane patterning, which is usually done by some variation of the parallel transport method explained in Sec. 4.1.

2 Background

1. Burj Khalifa interior panelling (Meredith and Kotronis 2013)
2. Ghery's architecture in general uses same width metal sheets to cover entire buildings, although I am not shure if that is not Orlando's subject either…
3. Denis Shelden thesis on constructability of gherys architecture (Shelden 2002)
4. MAYBE?? Include non-optimized builidng examples to demonstrate the method's usefulness.
5. Looking for other built examples or previous/further research on the subject.

2.1 Computer programs using this technique

There are no programs that develop this technique "out of the box", but it it based on simple algorithms and can be easily reproduced in any of the latest 3D modeling programs that allow any form of scripting (visual or otherwise) to generate and manipulate 3D geometries. Some examples of this might be:

1. Rhino+Grasshopper
2. Revit + Dynamo
3. IOGram (Currently in beta, it's the 'new kid on the block' of parametric design)
4. Houdini
5. 3DMax

There also exist some powerful geometry processing libraries that can help with the task of computing geodesic curves, distances & fields (which are widely used in this chapter) and other libraries oriented to general scientific and mathematical computing, which are usefull when numerical optimization is needed during the process. Some of those libraries are:

1. LibiGL(C++ with Python bindings)
2. CGal (C++)
3. OpenMesh (C++ with Python bindings)
4. NumPy (Python Computing Library)
5. SciPy (Python Scientific Computing Library)

2.2 Geodesic curves

In differential geometry, a *geodesic curve* is the generalization of a straight line into curved spaces (see fig. 1).

Also, in the presence of an *affine connection*, a geodesic is defined to be a curve whose tangent vectors remain parallel if they are transported along it. We will explore the notion of vector *parallel transport* in the following sections.

For triangle meshes, shortest polylines cross edges at ***equal angles***.

Finding the truly shortest geodesic paths requires the computation of distance fields (see Carmo 2016; Kimmel and Sethian 1998)

2.2.1 Algorithmic ways of generating geodesics

The computation of geodesics on smooth surfaces is aclassical topic, and can be reduced to two different solutions, depending on the initial conditions of the problem, you can basically find two tipes of problems (Deng 2011):

Initial value problem given a point $p \in S$ and a vector $v \in T_pS$, find a geodesic which is incident with p, such that the tangent vector fo g at p is v.

Boundary value problem given two points $p_1, p_2 \in S$ find a geodesic g connecting p_1 and p_2.

Both problems have different ways of being solved either numerically, graphically or by the means of simulations. The initial value problem can be solved using the concept of *straightest geodesics* (Polthier

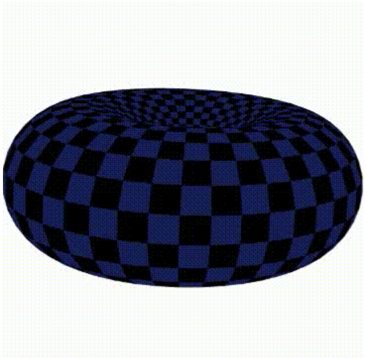

Fig. 1: If an insect is placed on a surface and continually walks "forward", by definition it will trace out a geodesic (image taken from Wikipedia).

and Schmies 1998), whereas the boundary value problem has a very close relation with the computation of *shortest paths* between two points on a surface.

It is important to note that, during this chapter, all surfaces are discretized as triangular meshes (V,E,F) of sufficient precision.
What would that precision be?? %?? distance to reference surf??

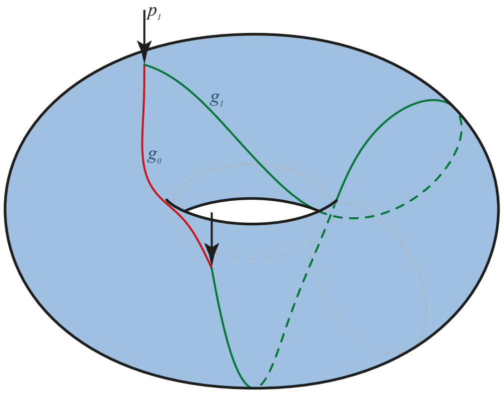

Fig. 2: The concept of 'shortest geodesics': curves g_0(red) and g_1(green) are both geodesic cuves of a torus, although g_1 is more than double the length of g_0.

2.3 Developable surfaces

This is very well explained in p.170 of Denis Shelden thesis (Gerard's suggestion). Explanation is inspired by that section.

It is also important to introduce the concept of the *developable surface*, a special kind of surface that have substantial and variable normal curvature while guaranteeing zero gaussian curvature, and as such, this surfaces can be *unrolled* into a flat plane with no deformation of the surface. This surfaces have been an extremely important design element in the practice of known architect Frank Ghery and explained in (Shelden 2002).

There are several ways of generating a developable surface using a curve in space.

1. Select the starting point of the curve
2. Obtain the perpendicular frame of the curve at the specified parameter.
3. Deconstruct the frame into it's X, Y & Z vectors.

4. Select one of the vectors at all the sample points
5. Draw a line using the selected vector at the start point of the curve.
6. XXXXXX

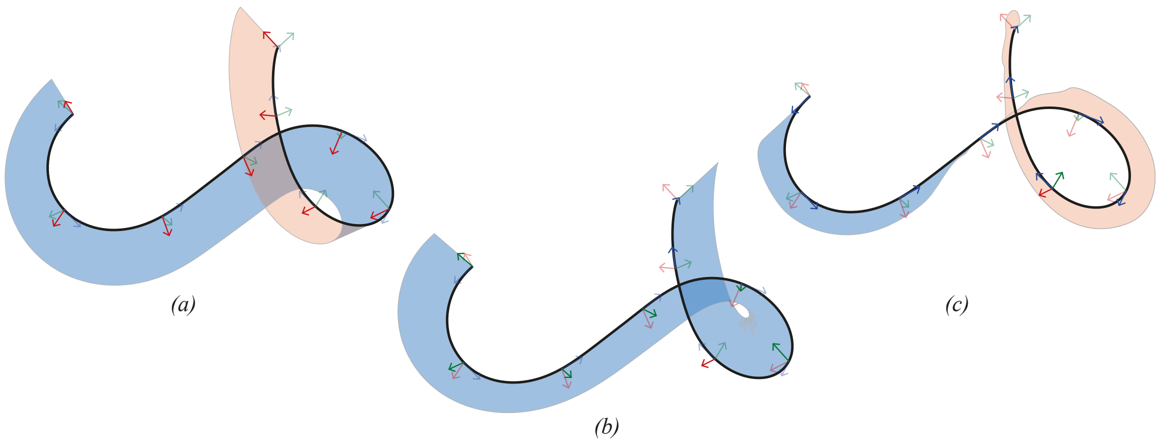

Fig. 3: Developable surface generated using curve g_1 of fig. 2. (a) using the X component of the curve's perp frame; (b) using the Y component & (c) using the Z component (tangent of the curve).

3 Construction technique

3.1 Geometric Properties

We will first introduce the properties we aim for when generating geodesic panels:

3.1.1 Geodesic property

- Long Thin panels that bend about their weak axis
- Zero geodesic curvature
- Represent the shortest path between two points on a surface

3.1.2 Constant width property

- Panels whose original, unfolded shape is a rectangle.
- The only way this can happen is if the entire surface is developable.
- For all other surfaces:
  - Assuming no gaps between panels
  - Panels will not be exactly rectangular when unfolded
  - ***Requirement:*** Geodesic curves that guide the panels must have approximately constant distance from thier neighbourhood curves.

3.1.3 Developable (or 'pure-bending') property

- Bending panels on surfaces changes the distances in points only by a small amount so,
- A certain amount of twiting is also present in this aplications.
  Some methods in this chapter do not take into account this property.

3.2 Benefits discussion

Problem 1 Look for a system of geodesic curves that covers a freeform surface in a way that:

1. They have approximate constant distance with it's neighbours.
2. This curves will serve as guiding curves for the panels.
3. The panels are to cover the surface with ***no overlap*** and ***only small gaps***

Problem 2 Look for a system of geodesic curves in a freeform surface which:

1. Serve as the boundaries of wooden panels.
2. The panel's deveopment is ***nearly straight***.
3. Those panels cover the surface with ***no gaps***

3.3 Quality Assesment

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3.4 Panel modeling

In this section, we will discuss several ways to generate panels from curves lying on a given surface:

3.4.1 Tangent-developable method

The notion of ***Conjugate tangents*** on smooth surfaces must be defined:

- Strictly related to the ***Dupin Indicatrix***
- In negatively curved areas, the Dupin Indicatrix is an hyperbola whose asymptotic directions (A1, A2)
- Any parallelogram tangentialy circumscribed to the indicatrix defines two conjugate tangents **T** and **U**.
- The asymptotic directions of the dupin indicatrix are the diagonals of any such parallelogram.

Fig. 4: Tangent developable method for panels

Initial algorithm is as follows:

For all geodesics s_i in a given pattern:

1. Compute the *tangent developable surfaces* $\rightarrow \Psi_i$
2. Trim Ψ_i along the intersection curves with their respective neighbours.
3. Unfold the trimed Ψ_i, obtaining the panels in flat state.

Unfortunately, this method needs to be refined in order to work in practice because:

1. The rulings of tangent developables may behave in weird ways
2. The intersection of the neighbouring Ψ_i's is often *ill-defined*.

Therefore, the procedure was modified in the following way:

1. Compute the *tangent developable surfaces* Ψ_i for all surfaces $s_i \rightarrow i =$ even numbers
2. Delete all rulings where the angle enclose with the tangent α is smaller than a certain threshold (i.e. 20º).
3. Fill the holes in the rulings by interpolation (???)
4. On each ruling:
    1. Determine points $A_i(x)$ and $B_i(x)$ which are the closest to geodesics s_{i-1} and s_{i+1}. This serves for trimming the surface Ψ_i.
5. Optimize globaly the positions of points $A_i(x)$ and $B_i(x)$ such that
    1. Trim curves are *smooth*
    2. $A_i(x)$ and $B_i(x)$ are close to geodesics s_{i-1} and s_{i+1}
    3. The ruling segments $A_i(x)B_i(x)$ lies close to the *original surface* Φ

This adjustments to the algorithm allow for the modeling of panels that meet the requirements of developability and approximately constant width, although it must be noted computation times increase, as double the ammount of geodesic curves need to be generated, and subsequently, the desired width function needs to be half the desired width of the panels. Fig. 5 shows the result of computing such panels and the subsequent gaps between the generated panels; these gaps need to be kept within a certain width in order to produce a succesfull watertight panelization of the original surface. Furthermore, material restrictions such as bending or torsion where not taken into account during the construction of the panels.

Fig. 5: Panels computed using the using the tangent developable method.

3.4.2 The Bi-Normal Method

The second method for defining panels, once an appropriate system of geodesics has been found on Φ, works directly with the geodesic curves.

> Assume that a point $P(t)$ traverses a geodesic s with unit speed, where t is the time parameter. For each time t there is:
>
> - a velocity vector $T(t)$
> - the normal vector $N(t)$
> - a third vector $B(t)$, *the binormal vector*
>
> This makes $T.N.B$ a ***moving orthogonal right-handed frame***

The surface Φ is represented as a triangle mesh and s is given as a polyline. For each geodesic, the associated surface is constructed according to Fig. 6. Points $L(t)$ and $R(t)$ represent the border of the panel, whose distance from $P(t)$ is half the panel width.

3.4.3 Method Comparison

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4 Algorithmic strategies

In this section, we will introduce the different existent methods for generating geodesic curve patterns that will be used to model the final panels, using methods indicated in Sec. 3.4.

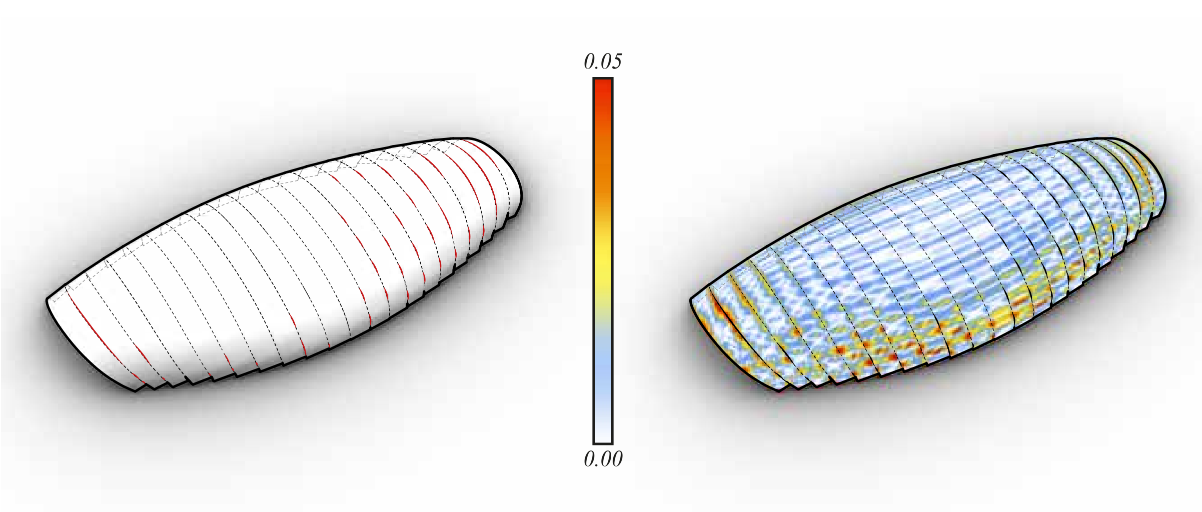


Fig. 6: Binormal Method for panels & T.N.B. frame. On the left, the computed panels with the corresponding panel gaps highlighted in red. On the right, panels coloured by distance to reference mesh.

4.1 Parallel Transport Method

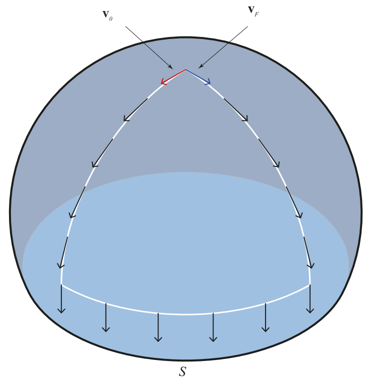


Fig. 7: Parallel transport of a vector on a 'piece-wise geodesic' path on a sphere.

This method, described in (Pottmann et al. 2010), allows for the generation of a system of geodesic curves where either the maximum distance or the minimum distance between adjacent points ocurrs at a prescribed location.

In differential geometry, the concept of *parallel transport* (see fig. 8) of a vector $\boldsymbol{V}$ along a curve $\boldsymbol{S}$ contained in a surface means moving that vector along $\boldsymbol{S}$ such that:

1. It remains tangent to the surface
2. It changes as little as possible in direction
3. It is a known fact that the length of the vector remains unchanged

4.1.1 Procedure

Following this procedure, *extremal distances between adjacent geodesics occur near the chosen curve.* Meaning:

1. For surfaces of **positive curvature**, the parallel transport method will yield a 1-geodesic pattern on which the *maximum distance* between curves will be W.
2. On the other hand, for surfaces of **negative curvature**, the method will yield a 1-geodesic pattern with W being the *closest (or minimum)* distance between them.

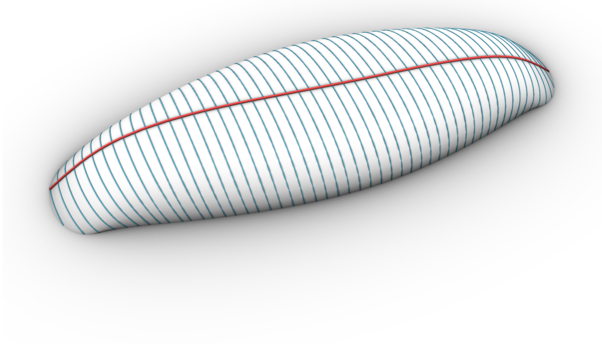


Fig. 8: Example of parallel transport method. Generatrix geodesic g (red) and geodesics $g^{\perp}$ generated from a parallel transported vector (blue) computed given a point and a vector $\mathbf{v}$ tangent to the surface, in both positive and negative directions.

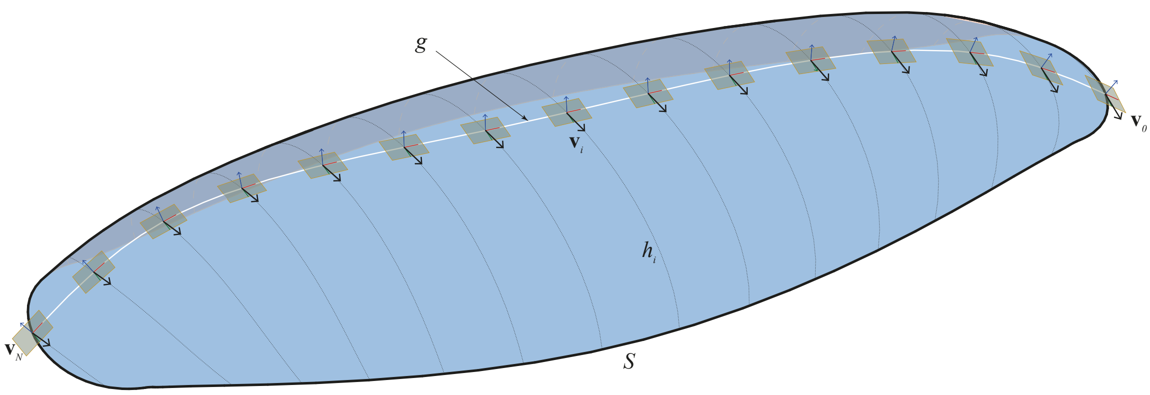


Fig. 9: Parallel transport along a curve g lying on surface S is equivalent to projecting $\mathbf{v}_{i-1}$ onto the tangent plane on p_i and subsequently normalizing $\mathbf{v}_i$.

Algorithm 1 Geodesic patterns by parallel transport

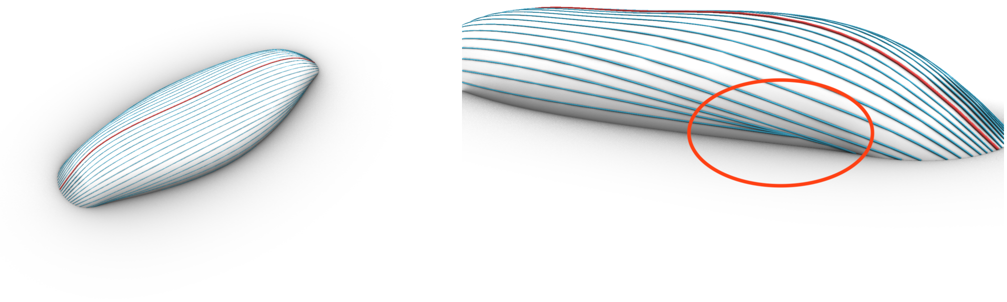
Input: A surface Φ, represented as a triangular mesh (V,E,F)
Output: Set of geodesic curves $g_i, \quad where \quad i = 0, ..., M$

1: Place a geodesic curve g_x along S such that it divides the surface completely in 2.
2: Divide the curve into N equally spaced points p with distance W.
3: Place an vector $\mathbf{v}$ onto p_0
4: Parallel transport that vector along g_x as described in [@fig:parTransProc].
5: **for all** points p_i where i = 0,...,M **do**
6: Generate geodesic curve $+g_i$ and $-g_i$ using vector $\mathbf{v}_i$ and $-\mathbf{v}_i$ respectively.
7: Join $+g_i$ and $-g_i$ together to obtaing g_i
8: Add g_i to output.
9: **end for**

The placement of the first geodesic curve and the selection of the initial vector are not trivial tasks. For surfaces with high variations of surfaces, the results might be unpredictable and, as such, this method is only suitable for surfaces with nearly constant curvature. Other solutions might involve cutting the surface into patches of nearly-constant curvature, and applying the *parallel transport method* independently on each patch.

4.2 Evolution Method

Two main concepts are covered in this section, both proposed by (Pottmann et al. 2010): the first, what is called the *evolution method*, and a second method based on *piecewise-geodesic* vector fields.



(a) (b)

Fig. 10: Surface covered by a 1-geodesic pattern using the evolution method without introducint greakpoints. Fig. 12a shows an overview of the result; while fig. 12b highlights the intersection point of several geodesic curves. This problem will be adressed by introducing the concept of 'piece-wise' geodesic curves; which are curves that are not geodesics, but are composed of segments of several connected geodesic curves.

4.2.1 The *evolution method*

As depicted in: Starting from a source geodesic somewhere in the surface:

- Evolve a pattern of geodesics iteratively computing 'next' geodesics.
- 'Next' geodesics must fullfil the condition of being at approximately constant distance from its predecesor.
- If the deviation from its predecesor is too great, breakpoints are introduced and continued as a *'piecewise geodesic'*.
- 'Next geodesics' are computed using Jacobi Fields

4.2.2 Distances between geodesics

1. No straight forward solution.
    1. Only for rotational surfaces (surfaces with evenly distriuted meridian curves).
2. **But** a first-order aproximation of this distance can be approximated:

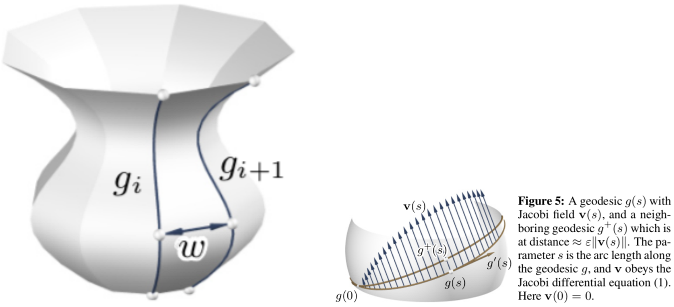
Starting at time $t = 0$ with a geodesic curve $g(s)$, parametrized by arc-length s, and let it move within the surface.
A snapshot at time $t = \varepsilon$ yields a geodesic g^+ near g.

$$g^+(s) = g(s) + \varepsilon \mathbf{v}(s) + \varepsilon^2(\dots) \tag{1}$$

The derivative vector field $\mathbf{v}$ is called a *Jacobi field*. We may asume it is orthogonal to $g(s)$ and it is expressed in terms of the geodesic tangent vector g' as:

$$\mathbf{v}(s) = w(s) \cdot R_{\pi/2}(g'(s)), \quad \text{where } w'' + Kw = 0 \tag{2}$$

Since the distance between infinitesimally close geodesics are governed by Eq. 2, that equation also goberns the width of a strip bounded by two geodesics at a small finite distance. { } Using this principle, you can develop strips whose width $w(s)$ fulfills the Jacobi equation $w(s) = \alpha \cosh(s\sqrt{|K|})$[1] for some value $K < 0$.
Gluing them together will result in a surface of approximate Gaussian curvature.



(a) (b)

Fig. 11: Geodesic distances on sphere

4.2.3 Obtaining the 'next' geodesic

Input: A freeform surface S, a desired width W and a starting geodesic curve g_0
Output: The 'next' computed geodesic on S
Pseudocode: Given a valid S, W and g_0:

1. Sample the curve g_0 at uniformly distritbuted arc-length parameters x_i.
2. Compute a a set of geodesics $h_i \perp g_0$ startingt points $p(x_i)$ on g_0.
3. Select a width function $\omega(s)$ that is closest to a desired target width function $W(s)$(Width function can return a fixed value or a computed value using curvature. It could be further improved accomodate other measures).
4. Select a Jacobi field $\mathbf{v}(s)$ using $\omega(s)$ as stated in Eq. 2.
5. 'Walk' the lengh of the vector $\mathbf{v}(s)$ on h_i to obtain χ_i. These points approximate position of the *next geodesic* "g^+_i".
6. The previous step can have a very big error margin, therefore we must compute g_i^+:
   1. Select a point χ_i and name it χ_0.
   2. We can compute the next geodesic by minimizing the error between the width function $\omega(s)$ and the *signed distance* of the computed g_i^+ to g_0.
7. Return computed geodesic g_i^+

Any given surface can be completely covered in this maner by recursivly computing the next geodesic using the previous, given some margin of error.

4.3 Level-Set Method

We can also compute geodesic curve patterns on a surface by computing a scalar function on each vertex of a mesh that minimizes a combination of error measurements $F_{min} = F_k + \lambda F_{nabla} + F_w$.

[1] **Question:** What is α in this formula? Missing image

(a) (b)

Fig. 12: Surface covered by a 1-geodesic pattern using the evolution method. Fig. 12a shows ***normal*** implementation; while fig. 12b implements the piece-wise geodesic concept. Both images use the same parameters

4.4 Geodesic Webs

We can also expand the Leve-Set method explained in the previous section from one familiy of curves to several families of interconnected quasi-geodesic curves by simply computing several sets of scalar functions on each vertex of the surface.

5 Shape Optimization

5.1 Geodesic Vector-fields

The level-set method described in Sec. 4.3 is not suitable for arbitrary surfaces, and therefore it must be adapted to achieve the desired result. In this section, we introduce the concept of *Geodesic Vector-fields* to divide the surface into patches that could be easily covered by equal width panels using the level-set method.

5.2 Dynamic shape optimization (Kennan Crane. . .)

Another option is to modify the original surface to make it *developable*. This algorithm was first presented by (Stein, Grinspun, and Crane 2018) and it allows to convert any given triangle mesh into a developable approximation by minimizing a specified energy applied to each edge of the mesh and subsequently subdividing in order to achieve a smooth developable aproximation to the reference surface.

6 Mathematical background

References

Arsan, Güler Gürpinar, and Abdülkadir Özdeger. 2015. "Bianchi Surfaces Whose Asymptotic Lines Are Geodesic Parallels." *Adv. Geom.* 15 (1). https://doi.org/10.1515/advgeom-2014-0020.

Azencot, Omri, Maks Ovsjanikov, Frédéric Chazal, and Mirela Ben-Chen. 2015. "Discrete Derivatives of Vector Fields on Surfaces–An Operator Approach." *ACM Transactions on Graphics (TOG)* 34 (3). ACM: 29.

Carmo, Manfredo P do. 2016. *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*. Courier Dover Publications.

Chen, Jindong, and Yijie Han. 1996. "Shortest Paths on a Polyhedron, Part I: Computing Shortest Paths." *Int. J. Comput. Geom. Appl.* 06 (02). https://doi.org/10.1142/s0218195996000095.

Cheng, Peng, Chunyan Miao, Yong-Jin Liu, Changhe Tu, and Ying He. 2016. "Solving the Initial Value Problem of Discrete Geodesics." *Computer-Aided Design* 70. Elsevier: 144–52. http://dx.doi.org/10.1016/j.cad.2015.07.012.

Deng, Bailin. 2011. "Special Curve Patterns for Freeform Architecture." Edited by Helmut Pottmanm and Niloy J. Mitra. http://resolver.obvsg.at/urn:nbn:at:at-ubtuw:1-40363.

Deng, Bailin, Helmut Pottmann, and Wallner Johannes. n.d. "Functional Webs for Freeform Architecture." *Computer Graphics Forum* 30 (5): 1369–78. doi:10.1111/j.1467-8659.2011.02011.x.

Eigensatz, Michael, Martin Kilian, Alexander Schiftner, Niloy J Mitra, Helmut Pottmann, and Mark Pauly. 2010. "Paneling Architectural Freeform Surfaces." *ACM Transactions on Graphics (TOG)*. doi:10.1145/1833349.1778782.

Jakob, Wenzel, Marco Tarini, Daniele Panozzo, and Olga Sorkine-Hornung. 2015. "Instant Field-Aligned Meshes." *ACM Trans. Graph.* 34 (6). Citeseer: 189–1.

Jia, Yan-Bin. 2017. "Curves on a Surface."

Kahlert, Joe, Matt Olson, and Hao Zhang. 2010. "Width-Bounded Geodesic Strips for Surface Tiling." *Vis. Comput.* 27 (1). https://doi.org/10.1007/s00371-010-0513-3.

Kensek, Karen, Judith Leuppi, and Douglas Noble. 2000. "Plank Lines of Ribbed Timber Shell Structures." In *Proceedings of the 22nd Annual Conference of the Association for Computer-Aided Design in Architecture, Acadia 2000: Eternity, Infinity and Virtuality in Architecture*, 261–66.

Kimmel, R, and J A Sethian. 1998. "Computing Geodesic Paths on Manifolds." *Proc. Natl. Acad. Sci. U. S. A.* 95 (15). https://doi.org/10.1073/pnas.95.15.8431.

Meredith, Neil, and James Kotronis. 2013. "Self-Detailing and Self-Documenting Systems for Wood Fabrication: The Burj Khalifa." In *Advances in Architectural Geometry 2012*, edited by Lars Hesselgren, Shrikant Sharma, Johannes Wallner, Niccolo Baldassini, Philippe Bompas, and Jacques Raynaud, 185–98. Vienna: Springer Vienna. https://doi.org/10.1007/978-3-7091-1251-9_14.

Polthier, Konrad, and Markus Schmies. 1998. "Straightest Geodesics on Polyhedral Surfaces." In *Mathematical Visualization*, 135–50. https://doi.org/10.1145/1185657.1185664.

Pottmann, Helmut. 2010. "Architectural Geometry as Design Knowledge." *Architectural Design* 80 (4). Wiley Online Library: 72–77. https://doi.org/10.1002/ad.1109.

Pottmann, Helmut, Michael Eigensatz, Amir Vaxman, and Johannes Wallner. 2015. "Architectural Geometry." *Computers and Graphics* 47: 145–64. doi:10.1016/j.cag.2014.11.002.

Pottmann, Helmut, Qixing Huang, Bailin Deng, Alexander Schiftner, Martin Kilian, Leonidas Guibas, and Johannes Wallner. 2010. "Geodesic Patterns." In *ACM SIGGRAPH 2010 Papers on - SIGGRAPH '10*. https://doi.org/10.1145/1833349.1778780.

Pottmann, Helmut, Alexander Schiftner, and Johannes Wallner. 2008. "Geometry of Architectural Freeform Structures." In *Symposium on Solid and Physical Modeling*, 9. https://doi.org/10.1145/1364901.1364903.

Rose, Kenneth, Alla Sheffer, Jamie Wither, Marie-Paule Cani, and Boris Thibert. 2007. "Developable Surfaces from Arbitrary Sketched Boundaries." In *SGP '07 - 5th Eurographics Symposium on Geometry Processing*, edited by Alexander G. Belyaev and Michael Garland. SGP '07 Proceedings of the Fifth Eurographics Symposium on Geometry Processing. Eurographics Association. http://dl.acm.org/citation.cfm?id=1281991.1282014.

Rusinkiewicz, Szymon. 2004. "Estimating Curvatures and Their Derivatives on Triangle Meshes." In *Null*, 486–93. IEEE.

Shelden, Dennis Robert. 2002. "Digital Surface Representation and the Constructibility of Gehry's Architecture." PhD thesis, Massachusetts Institute of Technology. http://hdl.handle.net/1721.1/16899.

Stein, Oded, Eitan Grinspun, and Keenan Crane. 2018. "Developability of Triangle Meshes." *ACM Trans. Graph.* 37 (4). New York, NY, USA: ACM.

Surazhsky, Vitaly, Tatiana Surazhsky, Danil Kirsanov, Steven J Gortler, and Hugues Hoppe. 2005. "Fast Exact and Approximate Geodesics on Meshes." *ACM Trans. Graph.* 24 (3). https://doi.org/10.1145/1186822.1073228.

Tong, Yiying, Santiago Lombeyda, Anil N Hirani, and Mathieu Desbrun. 2003. "Discrete Multiscale Vector Field Decomposition." In *ACM Transactions on Graphics (Tog)*, 22:445–52. 3. ACM.

Wallner, Johannes, Alexander Schiftner, Martin Kilian, Simon Flöry, Mathias Höbinger, Bailin Deng, Qixing Huang, and Helmut Pottmann. 2010. "Tiling Freeform Shapes with Straight Panels: Algorithmic Methods." In *Advances in Architectural Geometry 2010*, edited by Cristiano Ceccato, Lars Hesselgren, Mark Pauly, Helmut Pottmann, and Johannes Wallner. Vienna: Springer Vienna. https://doi.org/10.1007/978-3-7091-0309-8_5.

Weinand, Yves, and Claudio Pirazzi. 2006. "Geodesic Lines on Free-Form Surfaces - Optimized Grids for Timber Rib Shells." *World Conference in Timber Engineering WCTE*. https://infoscience.epfl.ch/record/118623.