Updated to 2019 Syllabus

CIE A-LEVEL MATHS 9709 (P3)

FORMULAE AND SOLVED QUESTIONS FOR PURE 3 (P3)

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1. Algebra

1.1 The Modulus Function

- No line with a modulus ever goes under the x-axis
- Any line that does go below the x-axis, when modulated is reflected above it

$$|a \text{ x } b| = |a| \text{x} \, |b|$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$
$$|x^2| = |x|^2 = x^2$$
$$|x| = |a| \Leftrightarrow x^2 = a^2$$
$$\sqrt{x^2} = |x|$$

1.2 Polynomials

- To find unknowns in a given identity
  - Substitute suitable values of x

OR

  - Equalize given coefficients of like powers of x
- **Factor theorem:** If $(x - t)$ is a factor of the function $p(x)$ then $p(t) = 0$
- **Remainder theorem:** If the function $f(x)$ is divided by $(x - t)$ then the remainder: $R = f(t)$

DIVIDEND = DIVISOR × QUOTIENT + REMAINDER

1.3 Binomial Series

Expanding $(1 + x)^n$ where $|x| < 1$

$$1 + \frac{n}{1}x + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^3 + \cdots$$

- **Factor case:** if constant is not 1, pull out a factor from brackets to make it 1 & use general equation. Do not forget the indices.
- **Substitution case:** if bracket contains more than one x term (e.g. $(2 - x + x^2)$) then make the last part u, expand and then substitute back in.
- **Finding the limit of x in expansion:** E.g. $(1 + ax)^n$, limit can be found by substituting ax between the modulus sign in $|x| < 1$ and altering it to have only x in the modulus

1.4 Partial Fractions

$$\frac{ax + b}{(px + q)(rx + s)} \equiv \frac{A}{px + q} + \frac{B}{rx + s}$$

- Multiply $(px + q)$, substitute $x = -\frac{q}{p}$ and find A
- Multiply $(rx + s)$, substitute $x = -\frac{s}{r}$ and find B

$$\frac{ax^2 + bx + c}{(px + q)(rx + s)^2} \equiv \frac{A}{px + q} + \frac{B}{rx + s} + \frac{C}{(rx + s)^2}$$

- Multiply $(px + q)$, substitute $x = -\frac{q}{p}$ and find A
- Multiply $(rx + s)^2$, substitute $x = -\frac{s}{r}$ and find C
- Substitute any constant e.g. $x = 0$ and find B

$$\frac{ax^2 + bx + c}{(px + q)(rx^2 + s)} \equiv \frac{A}{px + q} + \frac{Bx + C}{rx^2 + s}$$

- Multiply $(px + q)$, substitute $x = -\frac{q}{p}$ and find A
- Take $\frac{A}{px+q}$ to the other side, subtract and simplify.
- Linear eqn. left at top is equal to $Bx + C$

- **Improper fraction case:** if numerator has x to the degree of power equivalent or greater than the denominator then another constant is present. This can be found by dividing denominator by numerator and using remainder

{S12-P33} **Question 8:**

Express the following in partial fractions:

$$\frac{4x^2 - 7x - 1}{(x + 1)(2x - 3)}$$

Solution:

Expand the brackets

$$\frac{4x^2 - 7x - 1}{2x^2 - x - 3}$$

Greatest power of x same in numerator and denominator, thus is an improper fraction case

Making into proper fraction:

$$\begin{array}{r|l} & \quad\; 2 \\ \hline 2x^2 - x - 3 & 4x^2 - 7x - 1 \\ & 4x^2 - 2x - 6 \\ \hline & \quad\; -5x + 5 \end{array}$$

This is written as:

$$2 + \frac{5 - 5x}{(x + 1)(2x - 3)}$$

Now proceed with normal case for the fraction:

$$\frac{A}{x + 1} + \frac{B}{2x - 3} = \frac{5 - 5x}{(x + 1)(2x - 3)}$$

$$A(2x-3)+B(x+1)=5-5x$$

When $x=-1$

$$-5A=5+5$$
$$A=-2$$

When $x=\frac{3}{2}$

$$\frac{5}{2}B=5-\frac{15}{2}$$
$$B=-1$$

Thus the partial fraction is:

$$2+\frac{-2}{x+1}+\frac{-1}{2x-3}$$

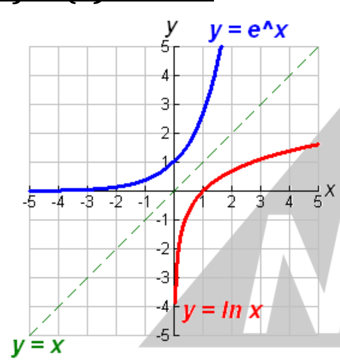
2. Logarithmic & Exponential Functions

$$y=a^x \Leftrightarrow \log_a y = x$$

$$\log_a 1 = 0 \qquad \log_a a = 1$$

$$\log_a b^n \equiv n\log_a b$$
$$\log_a b + \log_a c \equiv \log_a bc$$
$$\log_a b - \log_a c \equiv \log_a \frac{b}{c}$$
$$\log_a b \equiv \frac{\log b}{\log a}$$
$$\log_a b \equiv \frac{1}{\log_b a}$$

2.1 Graphs of ln(x) and e^x



3. Trigonometry

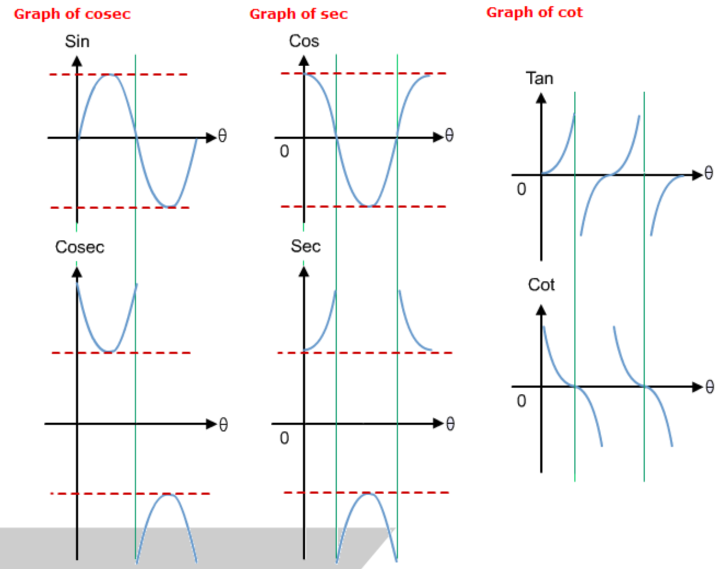
3.1 Ratios

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \qquad \sec\theta=\frac{1}{\cos\theta}$$
$$\operatorname{cosec}\theta=\frac{1}{\sin\theta} \qquad \cot\theta=\frac{\cos\theta}{\sin\theta}$$

3.2 Identities

$$(\cos\theta)^2+(\sin\theta)^2\equiv 1$$
$$1+(\tan\theta)^2\equiv(\sec\theta)^2$$
$$(\cot\theta)^2+1\equiv(\operatorname{cosec}\theta)^2$$

3.3 Graphs



3.4 Double Angle Identities

$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv (\cos A)^2-(\sin A)^2 \equiv 2(\cos A)^2-1$$
$$\equiv 1-2(\sin A)^2$$
$$\tan 2A \equiv \frac{2\tan A}{1-(\tan A)^2}$$

3.5 Addition Identities

$$\sin(A\pm B)\equiv \sin A\cos B \pm \cos A\sin B$$
$$\cos(A\pm B)\equiv \cos A\cos B \mp \sin A\sin B$$
$$\tan(A\pm B)\equiv \frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

3.6 Changing Forms

$$a\sin x\pm b\cos x \Leftrightarrow R\sin(x\pm\alpha)$$
$$a\cos x\pm b\sin x \Leftrightarrow R\cos(x\mp\alpha)$$

Where $R=\sqrt{a^2+b^2}$ and

$R\cos\alpha=a,\ R\sin\alpha=b$ with $0<\alpha<\frac{1}{2}\pi$

{S13-P33} **Question 9:**

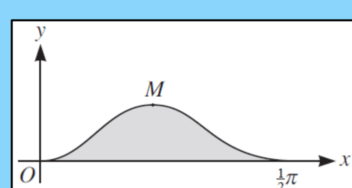


Diagram shows curve, $y=\sin^2 2x\cos x$, for $0\le x\ge\frac{\pi}{2}$, and M is maximum point. Find the x coordinate of M.

Solution:

Use product rule to differentiate:

$$u = \sin^2 2x \qquad v = \cos x$$
$$u' = 4\sin 2x \cos 2x \qquad v' = -\sin x$$
$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = (4\sin 2x \cos 2x)(\cos x) + (\sin^2 2x)(-\sin x)$$
$$\frac{dy}{dx} = 4\sin 2x \cos 2x \cos x - \sin^2 2x \sin x$$

Use following identities:

$$\cos 2x = 2\cos^2 x - 1$$
$$\sin 2x = 2\sin x \cos x$$
$$\sin^2 x = 1 - \cos^2 x$$

Equating to 0:

$$\frac{dy}{dx} = 0$$
$$\therefore 4\sin 2x \cos 2x \cos x - \sin^2 2x \sin x = 0$$
$$4\sin 2x \cos 2x \cos x = \sin^2 2x \sin x$$

Cancel $\sin 2x$ on both sides

$$4\cos 2x \cos x = \sin 2x \sin x$$

Substitute identities

$$4(2\cos^2 x - 1)\cos x = (2\sin x \cos x)\sin x$$

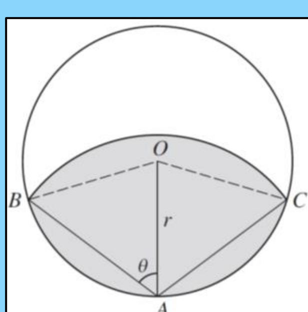
Cancel $\cos x$ and constant 2 from both sides

$$4\cos^2 x - 2 = \sin^2 x$$

Use identity

$$4\cos^2 x - 2 = 1 - \cos^2 x$$
$$5\cos^2 x = 3$$
$$\cos^2 x = \frac{3}{5}$$
$$\cos x = 0.7746$$
$$x = \cos^{-1}(0.7746)$$
$$x = 0.6847 \approx 0.685$$

{W13-P31} **Question 6:**



A is a point on circumference of a circle center O, radius r. A circular arc, center A meets circumference at B & C. Angle OAB is θ radians. The area of the shaded region is equal to half the area of the circle.

Show that:

$$\cos 2\theta = \frac{2\sin 2\theta - r}{4\theta}$$

Solution:

First express area of sector $OBAC$

$$Sector\ Area = \frac{1}{2}\theta r^2$$
$$OBAC = \frac{1}{2}(2\pi - 4\theta)r^2 = (\pi - 2\theta)r^2$$

Now express area of sector ABC

$$ABC = \frac{1}{2}(2\theta)(Length\ of\ BA)^2$$

Express BA using sine rule

$$BA = \frac{r\sin(\pi - 2\theta)}{\sin\theta}$$

Use double angle rules to simplify this expression

$$BA = \frac{r\sin 2\theta}{\sin\theta}$$
$$= \frac{2r\sin\theta\cos\theta}{\sin\theta}$$
$$= 2r\cos\theta$$

Substitute back into initial equation

$$ABC = \frac{1}{2}(2\theta)(2r\cos\theta)^2$$
$$ABC = 4\theta r^2\cos^2\theta$$

Now express area of kite $ABOC$

$$ABOC = 2 \times Area\ of\ Triangle$$
$$ABOC = 2\times\frac{1}{2}r^2\sin(\pi - 2\theta)$$
$$= r^2\sin(\pi - 2\theta)$$

Finally, the expression of shaded region equated to half of circle

$$4r^2\theta\cos^2\theta + r^2(\pi - 2\theta) - r^2\sin(\pi - 2\theta) = \frac{1}{2}\pi r^2$$

Cancel our r^2 on both sides for all terms

$$4\theta\cos^2\theta + \pi - 2\theta - (\sin\pi\cos 2\theta + \sin 2\theta\cos\pi) = \frac{1}{2}\pi$$

Some things in the double angle cancel out

$$4\theta\cos^2\theta + \pi - 2\theta - \sin 2\theta = \frac{1}{2}\pi$$

Use identity here

$$4\theta\left(\frac{\cos 2\theta + 1}{2}\right) + \pi - \sin 2\theta - 2\theta = \frac{1}{2}\pi$$
$$4\theta\cos 2\theta + 4\theta + 2\pi - 2\sin 2\theta - 4\theta = \pi$$

Clean up

$$4\theta\cos 2\theta + 2\pi - 2\sin 2\theta = \pi$$
$$4\theta\cos 2\theta = 2\sin 2\theta - \pi$$
$$\cos 2\theta = \frac{2\sin 2\theta - \pi}{4\theta}$$

4. Differentiation

4.1 Basic Derivatives

x^n	nx^{n-1}
e^u	$\frac{du}{dx}e^u$
$\ln u$	$\frac{du/dx}{u}$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$

4.2 Chain, Product and Quotient Rule

- **Chain Rule:**

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

- **Product Rule:**

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

- **Quotient Rule:**

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

4.3 Parametric Equations

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

- In a parametric equation x and y are given in terms of t and you must use the above rule to find the derivative

4.4 Implicit Functions

- These represent circles or lines with circular curves, on a Cartesian plane
- Difficult to rearrange in form $y=\therefore$ differentiate as is
- Differentiate x terms as usual
- For y terms, differentiate the same as you would x but multiply with $\frac{dy}{dx}$
- Then make $\frac{dy}{dx}$ the subject of formula for derivative

5. Integration

5.1 Basic Integrals

ax^n	$a\frac{x^{n+1}}{(n+1)}+c$
e^{ax+b}	$\frac{1}{a}e^{ax+b}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}$

- Use trigonometrical relationships to facilitate complex trigonometric integrals
- Integrate by decomposing into partial fractions

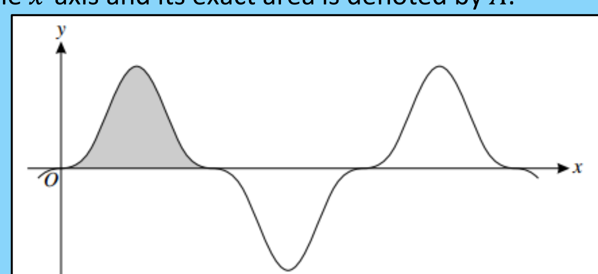
5.2 Integration by u-Substitution

$$\int f(x)\ dx=\int f(x)\frac{dx}{du}\ du$$

- **Make x equal to something:** when differentiated, multiply the substituted form directly
- **Make u equal to something:** when differentiated, multiply the substituted form with its reciprocal
- With definite integrals, change limits in terms of u

{W12-P33} **Question 7:**

The diagram shows part of curve $y=\sin^3 2x\cos^3 2x$. The shaded region shown is bounded by the curve and the x-axis and its exact area is denoted by A.



Use the substitution $u=\sin 2x$ in a suitable integral to find the value of A

Solution:

To find the limit, you are trying to the find the points at which $y=0$

$\sin x=0$ at $x=0,\pi,2\pi$ $\qquad$ $\cos x=0$ at $x=\frac{\pi}{2},\frac{3\pi}{4}$

Choose the two closest to 0 because the shaded area has gone through $y=0$ only twice

$$\therefore 0 \text{ and } \frac{\pi}{2}$$

Since it is $\sin 2x$ and $\cos 2x$, divide both by 2

$$\therefore \text{Limits are } 0 \text{ and } \frac{\pi}{4}$$

Integrate by u substitution, let:

$$u=\sin 2x \qquad \frac{du}{dx}=2\cos 2x \qquad \frac{dx}{du}=\frac{1}{2\cos 2x}$$

$$\sin^3 2x\cos^3 2x\equiv(\sin 2x)^3(\cos^2 2x)\cos 2x$$

$$\equiv\left(\sin^3 2x\times(1-\sin^2 2x)\right)\cos 2x$$

$$\equiv (\sin^3 2x - \sin^5 x)\cos 2x \times \frac{1}{2\cos 2x}$$
$$\equiv \frac{1}{2}(u^3 - u^5)$$

Now integrate:

$$\frac{1}{2}\int (u^3 - u^5) = \frac{1}{2}\left(\frac{u^4}{4} - \frac{u^6}{6}\right)$$

The limits are $x = 0$ and $x = \frac{\pi}{4}$. In terms of u,

$$u = \sin 2(0) = 0 \text{ and } u = \sin 2\left(\frac{\pi}{4}\right) = 1$$

Substitute limits

$$\frac{1}{2}\left(\frac{1^4}{4} - \frac{1^6}{6}\right) - \frac{1}{2}\left(\frac{0^4}{4} - \frac{0^6}{6}\right) = \frac{1}{24}$$

5.3 Integrating $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + k$$

{S10-P32} **Question 10:**

By splitting into partial fractions, show that:

$$\int_1^2 \frac{2x^3 - 1}{x^2(2x-1)}dx = \frac{3}{2} + \frac{1}{2}\ln\left(\frac{16}{27}\right)$$

Solution:

Write as partial fractions

$$\int_1^2 \frac{2x^3 - 1}{x^2(2x-1)}dx \equiv \int_1^2 1 + \frac{2}{x} + \frac{1}{x^2} + \frac{3}{2x-1}dx$$
$$\equiv x + 2\ln x - x^{-1} - \frac{3}{2}\ln|2x - 1|$$

Substitute the limits

$$2 + 2\ln 2 - \frac{1}{2} - \frac{3}{2}\ln 3 - 1 - 2\ln 1 + 1 + \frac{3}{2}\ln 1$$
$$\frac{3}{2} + \frac{1}{2}\ln 16 + \frac{1}{2}\ln\frac{1}{3^3} \equiv \frac{3}{2} + \frac{1}{2}\ln\frac{16}{27}$$

5.4 Integrating By Parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

For a definite integral:

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

What to make u:

L	A	T	E
Logs	Algebra	Trig	e

{W13-P31} **Question 3:**

Find the exact value of

$$\int_1^4 \frac{\ln x}{\sqrt{x}}dx$$

Solution:

Convert to index form:

$$\frac{\ln x}{\sqrt{x}} = x^{\frac{1}{2}}\ln x$$

Integrate by parts, let:

$$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-\frac{1}{2}} \qquad v = 2x^{\frac{1}{2}}$$
$$\therefore \ln x\, 2x^{\frac{1}{2}} - \int 2x^{\frac{1}{2}} \times x^{-1} \equiv 2\sqrt{x}\ln x - \int 2x^{-\frac{1}{2}}$$
$$\equiv 2\sqrt{x}\ln x - 4\sqrt{x}$$

Substitute limits

$$= 4\ln 4 - 4$$

5.5 Integrating Powers of Sine or Cosine

To integrate $\sin x$ or $\cos x$ with a power:

- If power is odd, pull out a $\sin x$ or $\cos x$ and use Pythagorean identities and double angle identities
- If power is even, use the following identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

5.6 Integrating $\cos^m x \sin^n x$

If m or n are odd and even, then:

- Factor out one power from **odd trig function**
- Use Pythagorean identities to transform remaining **even trig function** into the **odd trig function**
- Let u equal to **odd trig function** and integrate

If m and n are both even, then:

- Replace all even powers using the double angle identities and integrate

If m and n are both odd, then:

- Choose one of the trig. functions & factor out one power
- Use Pythagorean identity to transform remaining even power of chosen trig function into other trig. function

If either m or n or both = 1, then:

- Let u equal to the trig function whose power doesn't equal 1 then integrate
- If both are 1, then let u equal either

{W09-P31} **Question 5:**

(i) Prove the identity

$$\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$$

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta \, d\theta$$

Solution:

Part (i)

Use double angle identities

$$\cos 4\theta - 4\cos 2\theta + 3$$
$$\equiv 1 - 2\sin^2 2\theta - 4(1 - 2\sin^2\theta) + 3$$

Open everything and clean

$$\equiv 1 - 2\sin^2 2\theta - 4 + 8\sin^2\theta + 3$$
$$\equiv 1 - 2(\sin 2\theta)^2 - 4 + 8\sin^2\theta + 3$$
$$\equiv 1 - 2(2\sin\theta\cos\theta)^2 - 4 + 8\sin^2\theta + 3$$
$$\equiv 1 - 2(4\sin^2\theta\cos^2\theta) - 4 + 8\sin^2\theta + 3$$
$$\equiv 1 - 2(4\sin^2\theta\,(1 - \sin^2\theta)) - 4 + 8\sin^2\theta + 3$$
$$\equiv 1 - 8\sin^2\theta + 8\sin^4\theta - 4 + 8\sin^2\theta + 3$$
$$\equiv 8\sin^4\theta$$

Part (ii)

Use identity from (part i):

$$\frac{1}{8}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos 4\theta - 4\cos 2\theta + 3$$

$$\equiv \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin\theta + 3\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

Substitute limits

$$\equiv \frac{1}{32}\left(2\pi - \sqrt{3}\right)$$

{W12-P32} **Question 5:**

(i) By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$

(ii) Show that $\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$

(iii) Deduce that:

$$\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x\tan x$$

(iv) Hence show that:

$$\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}\left(8\sqrt{2} - \pi\right)$$

Solution:

Part (i)

Change to index form:

$$\frac{1}{\cos x} = \cos^{-1} x$$

Differentiate by chain rule:

$$\frac{dy}{dx} = -1(\cos x)^{-2}\times(-\sin x)$$

$$-1(\cos x)^{-2}\times(-\sin x) \equiv \frac{\sin x}{\cos^2 x} \equiv \frac{\sin x}{\cos x}\times\frac{1}{\cos x}$$

$$\frac{\sin x}{\cos x}\times\frac{1}{\cos x} \equiv \sec x\tan x$$

Part (ii)

Multiply numerator and denominator by $\sec x + \tan x$

$$\frac{\sec x + \tan x}{(\sec x - \tan x)(\sec x + \tan x)} \equiv \frac{\sec x + \tan x}{\sec^2 x - \tan^2 x}$$

$$\frac{\sec x + \tan x}{\sec^2 x - \tan^2 x} \equiv \frac{\sec x + \tan x}{1} \equiv \sec x + \tan x$$

Part (iii)

Substitute identity from (part ii)

$$\frac{1}{(\sec x - \tan x)^2} \equiv (\sec x + \tan x)^2$$

Open out brackets

$$(\sec x + \tan x)^2$$
$$\equiv \sec^2 x + 2\sec x\tan x + \tan^2 x$$
$$\equiv \sec^2 x + 2\sec x\tan x + \sec^2 x - 1$$
$$\equiv 2\sec^2 x + 2\sec x\tan x - 1$$
$$\equiv 2\sec^2 x - 1 + 2\sec x\tan x$$

Part (iv)

$$\int \frac{1}{(\sec x - \tan x)^2}\,dx$$
$$\equiv \int 2\sec^2 x - 1 + 2\sec x\tan x\,dx$$
$$\equiv 2\int \sec^2 x - \int 1 + 2\int \sec^2 x\tan^2 x$$

Using differential from part i:

$$\equiv 2\tan x - x + 2\sec x$$

Substitute boundaries:

$$= \frac{1}{4}\left(8\sqrt{2} - \pi\right)$$

5.5 Trapezium Rule

$$Area = \frac{Width\ of\ 1st\ Strip}{2}\times[1st\ height + Last\ height + 2(sum\ of\ h\ middle)]$$

$$Width\ of\ 1st\ Strip = \frac{b-a}{no.\ of\ intervals} \qquad \text{for} \quad \int_a^b dx$$

6. Solving Equations Numerically

6.1 Approximation

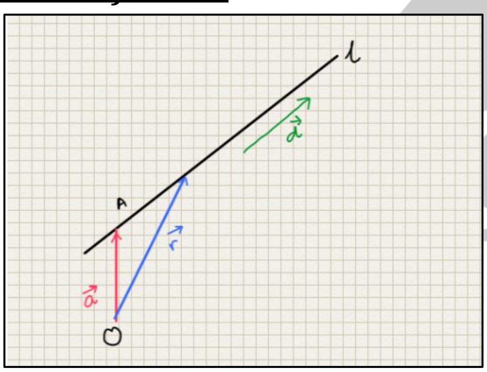
- To find root of a graph, find point where graph passes through x-axis ∴ look for a sign change
- Carry out decimal search
  - Substitute values between where a sign change has occurred
  - Closer to zero, greater accuracy

6.2 Iteration

- To solve equation $f(x) = 0$, you can rearrange $f(x)$ into a form $x = \cdots$
- This function represents a sequence that starts at x_0, moving to x_r
- Substitute a value for x_0 and put back into function getting x_1 and so on.
- As you increase r, value becomes more accurate
- Sometimes iteration don't work, these functions pare called divergent, and you must rearrange formula for x in another way
- For a successful iterative function, you need a convergent sequence

7. Vectors

7.1 Equation of a Line



- The column vector form:

$$r = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

- The linear vector form:

$$r = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + t(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$$

- The parametric form:

$$x = 1 + t, y = 2 + t, z = -2 + 3t$$

- The cartesian form; rearrange parametric

$$\frac{x-1}{1} = \frac{y-3}{1} = \frac{z+2}{3}$$

7.2 Parallel, Skew or Intersects

For the two lines:

$$\overrightarrow{OA} = \tilde{\mathbf{a}} + s\tilde{\mathbf{c}} \qquad \overrightarrow{OB} = \tilde{\mathbf{b}} + t\tilde{\mathbf{d}}$$

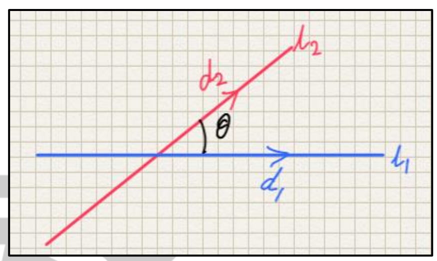
- **Parallel:**
  - For the lines to be parallel $\tilde{\mathbf{c}}$ must equal $\tilde{\mathbf{d}}$ or be in some ratio to it e.g. $1:2$
- **Intersects:**
  - Make $\overrightarrow{OA} = \overrightarrow{OB}$
  - If simultaneous works then intersects
  - If unknowns cancel then no intersection
- **Skew:**
  - First check whether line parallel or not
  - If not, then make $\overrightarrow{OA} = \overrightarrow{OB}$
  - Carry out simultaneous
  - When a pair does not produce same answers as another, then lines are skew

7.3 Angle between Two Lines

- Use dot product rule on the two direction vectors:

$$\frac{a.b}{|a||b|} = \cos\theta$$

- Note: a and b must be moving away from the point at which they intersect



7.4 Finding the Equation of a Line

- Given 2 points:
  - Find the direction vector using
    e.g. $AB = OB - OA$
  - Place either of the points as a given vector
- To check if a point lies on a line, check if constant of the direction vector is the same for x, y and z components

7.5 ⊥ Distance from a Line to a Point

- **AKA:** shortest distance from the point to the line
- Find vector for the point, B, on the line

Vector equation of the line: $\tilde{\mathbf{r}} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

$$\therefore \overrightarrow{OB} = \begin{pmatrix} 1+t \\ 3+t \\ 3t-2 \end{pmatrix}$$

- A is the point given

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 1+t-2 \\ 3+t-3 \\ 3t-2-4 \end{pmatrix} = \begin{pmatrix} t-1 \\ t \\ 3t-6 \end{pmatrix}$$

- Use Dot product of AB and the direction vector

$$\overrightarrow{AB}.\mathbf{d} = \cos 90$$

$$\begin{pmatrix} t-1 \\ t \\ 3t-6 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$1(t-1) + 1(t) + 3(3t-6) = 0$$

$$11t - 19 = 0$$

$$t = \frac{19}{11}$$

- Substitute t into equation to get foot
- Use Pythagoras' Theorem to find distance

{S08-P3} **Question:**

The points A and B have position vectors, relative to the origin O, given by

$OA = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $OB = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The line l has vector equation

$$\mathbf{r} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$$

(i) Show that l does not intersect the line passing through A and B.

(ii) The point P lies on l and is such that angle PAB is equal to $60°$. Given that the position vector of P is $(1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P

Solution:

Part (i)

Firstly, we must find the equation of line AB

$$AB = OB - OA$$

$$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

Equating the two lines

$$\begin{pmatrix} 1+s \\ 2-s \\ 3 \end{pmatrix} = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$$

Equation 1: $1 + s = 1 - 2t$ so $s = -2t$

Equation 2: $2 - s = 5 + t$

Substitute 1 into 2:

$$2 + 2t = 5 + t$$

$$\therefore t = 3 \text{ and then } s = -6$$

Equation 3:

$$3 = 2 - t$$

Substitute the value of t

$$3 = 2 - 3 \text{ so } 3 = -1$$

This is incorrect therefore lines don't intersect

Part (ii)

Angle PAB is formed by the intersection of the lines AP and AB

$$P = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$$

$$AP = OP - OA$$

$$AP = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2t \\ 3+t \\ -1-t \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Now use the dot product rule to form an eqn.

$$\frac{|AP.AB|}{|AP||AB|}; \frac{-3t-3}{\sqrt{6t^2+8t+10}\times\sqrt{2}} = \cos 60$$

$$-3t - 3 = \frac{1}{2}\sqrt{6t^2+8t+10}\times\sqrt{2}$$

$$36t^2 + 72t + 36 = 12t^2 + 16t + 20$$

$$24t^2 + 56t + 16 = 0$$

$$t = -\tfrac{1}{3} \text{ or } t = -2$$

{W11-P31} **Question:**

With respect to the origin O, the position vectors of two points A and B are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and

$\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B, and $\overrightarrow{AP} = \lambda\overrightarrow{AB}$

(i) $\overrightarrow{OP} = (1+2\lambda)\mathbf{i} + (2+2\lambda)\mathbf{j} + (2-2\lambda)\,\mathbf{k}$

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ, find the value of λ for which OP bisects the angle AOB.

Solution:

Part (i)

$$\overrightarrow{AP} = \lambda\overrightarrow{AB} = \lambda(OB - OA)$$

$$= \lambda\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

$$\therefore AP = \begin{pmatrix} 2\lambda \\ 2\lambda \\ -2\lambda \end{pmatrix}$$

$$OP = OA + \begin{pmatrix} 2\lambda \\ 2\lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 2\lambda \\ -2\lambda \end{pmatrix}$$

Part (ii)

Interpreting the question gives the information that AOP is equal to BOP $\therefore$ $\cos AOP$ is equal to $\cos BOP$. Now you can equate the two dot product equations

$$\cos AOP = \frac{OA.OP}{|OA||OP|} = \frac{9+2\lambda}{3\sqrt{9+4\lambda+12\lambda^2}}$$

$$\cos BOP = \frac{OB.OP}{|OB||OP|} = \frac{11+14\lambda}{5\sqrt{9+4\lambda+12\lambda^2}}$$

$$\frac{9+2\lambda}{3\sqrt{9+4\lambda+12\lambda^2}} = \frac{11+14\lambda}{5\sqrt{9+4\lambda+12\lambda^2}}$$

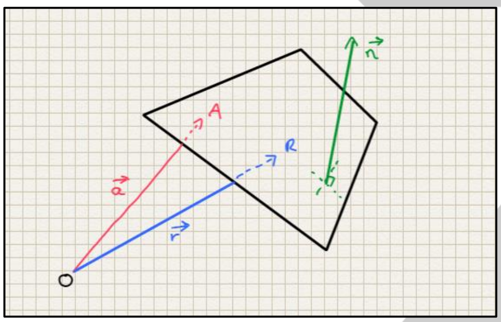
Cancel out the denominator to give you

$$\frac{9+2\lambda}{3} = \frac{11+14\lambda}{5}$$

$$45 + 10\lambda = 33 + 42\lambda$$

$$12 = 32\lambda \text{ and } \therefore \lambda = \tfrac{3}{8}$$

7.6 Equation of a Plane



- Scalar product form:

$$\tilde{\mathbf{r}}.\begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} = -13$$

The vector after $\tilde{\mathbf{r}}$ is the normal to the plane

- The components of the normal vector of the plane are the coefficients of x, y and z in the Cartesian form. You must substitute a point to find d
- Cartesian form:

$$4x + 5y + z = 13$$

7.7 Cross Product Rule

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} mr - nq \\ np - lr \\ lq - mp \end{pmatrix}$$

7.8 Finding the Equation of a Plane

- Given 3 points on a plane:
  - $A(1,2,-1), B(2,1,0), C(-1,3,2)$
  - Use this equation: $\tilde{r}.\tilde{n} = \tilde{a}.\tilde{n}$
  - $\tilde{r}$ is what we want to find
  - $\tilde{n}$ is the cross product of 2 vectors parallel to the plane
  - If we use $\widetilde{AB}$ and $\widetilde{AC}$ then $\tilde{a} = OA$
  - $\therefore \tilde{n} = AB \times AC = \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix}$
  - Substitute point A to get $\tilde{a}.\tilde{n}$
  - $\therefore \tilde{r}.\begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} = -13$
- Given a point and a line on the plane:
  - $A(1,2,3)$ and $\tilde{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
  - Make 2 points on the line
  - Substitute different values for s
  - Repeat 3 point process
- Given 2 lines on a plane:
  - Find a point on one line
  - Find 2 points on the other line
  - Repeat 3 point process

7.9 A Line and a Plane

- If a **line lies on a plane** then any two points on the line ($t = 0$ and $t = 1$)should satisfy the plane equation – substitute and see if equation works
- If a **line is parallel to plane,** the dot product of the direction vector and normal of the plane is zero

7.10 Finding the Point of Intersection between Line and Plane

- Form Cartesian equation for line
- Form Cartesian equation for plane
- Solve for x, y and z

{S13-P32} **Question:**

The points A and B have position vectors $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ respectively. The plane p has equation $x + y = 5$

(i) Find position vector of the point of intersection of the line through A and B and the plane p.

(ii) A second plane q has an equation of the form $x + by + cz = d$. The plane q contains the line AB, and the acute angle between the planes p and q is $60°$. Find the equation of q.

Solution:

Part (i)

$$AB = OB - OA = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

The equation of the line $AB = OA + AB$

$$= \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+3\lambda \\ \lambda - 3 \\ 2 - \lambda \end{pmatrix}$$

Substitute values into plane equation

$$x + y = 5 \implies 2 + 3\lambda + \lambda - 3 = 5$$

$$4\lambda - 1 = 5 \implies \lambda = \frac{3}{2}$$

Substitute lambda back into general line equation

$$\begin{pmatrix} 2 + (3\times 1.5) \\ 1.5 - 3 \\ 2 - 1.5 \end{pmatrix} = \begin{pmatrix} 6.5 \\ -1.5 \\ 0.5 \end{pmatrix}$$

Part (ii)

Using the fact that line AB lies on the plane, the direction vector of AB is perpendicular to the plane. Remember there is no coefficient for x which means that it is equal to 1.

$$\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ b \\ c \end{pmatrix} = 0$$

$$3 + b - c = 0 \qquad \text{so } c = 3 + b$$

Using the fact that the plane p and q intersect at an angle of 60°

$$\frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ b \\ c \end{pmatrix}}{\sqrt{2} \times \sqrt{1 + b^2 + c^2}} = \cos 60 = \frac{1}{2}$$

$$2 + 2b = \sqrt{2 + 2b^2 + 2c^2}$$

$$4b^2 + 8b + 4 = 2b^2 + 2c^2 + 2$$

Substitute the first equation into c

$$2b^2 + 8b + 2 - 18 - 12b - 2b^2 = 0$$

$$-4b - 16 = 0 \qquad b = -4 \text{ and } c = -1$$

We have found the normal to the plane, now we must find d

$$x - 4y - z = 0$$

Substitute the point A into the equation because the point lies on it

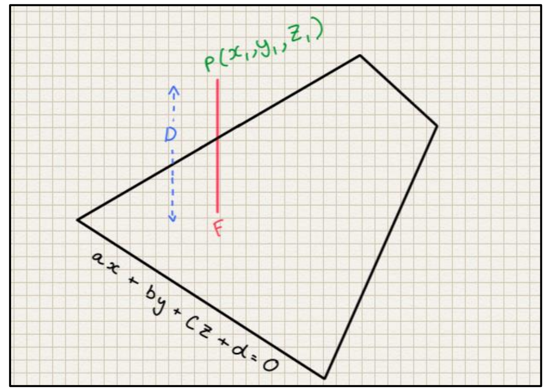
$$(2) - 4(-3) - 2 = d \qquad d = 12$$

$$x - 4y - z = 12$$

7.11 Finding Line of Intersection of Two Non-Parallel Planes

- The direction vector of this line is $\widetilde{\mathbf{n_1}} \times \widetilde{\mathbf{n_2}}$
- $\widetilde{\mathbf{n_1}}$ is the normal of the first plane
- $\widetilde{\mathbf{n_2}}$ is the normal of the second plane

7.12 ⊥ Distance from a Point to a Plane



$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- Point F is the foot of the perpendicular

{S12-P32} **Question:**

Two planes, m and n, have equations $x + 2y - 2z = 1$ and $2x - 2y + z = 7$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

(i) Show that l is parallel to m

(ii) A point P lies on l such that its perpendicular distances from m and n are equal. Find the position vectors of the two possible positions for P and calculate the distance between them.

Solution:

Part (i)

If m is parallel to l, then the direction vector of l would be perpendicular to the normal of m ∴ their dot product is equal to zero

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0$$

Part (ii)

Any point on l would have the value

$$\begin{pmatrix} 1 + 2\lambda \\ 1 + \lambda \\ 2\lambda - 1 \end{pmatrix}$$

Using the distance formula of a point to a plane, find the perpendicular distance of the general point on l from the plane m and n

$$D_m = \left|\frac{4}{3}\right| \qquad \text{and} \qquad D_n = \left|\frac{-8+4\lambda}{3}\right|$$

Equate them as they equal the same distance

$$\left|\frac{4}{3}\right| = \left|\frac{-8+4\lambda}{3}\right| \implies |4| = |-8 + 4\lambda|$$

Remove modulus sign by taking into consideration the positive and negative

$$4 = -8 + 4\lambda \qquad \text{and} \qquad -4 = -8 + 4\lambda$$

$$\lambda = 3 \text{ and } \lambda = 1$$

Substitute lambda values back into vector general line l equation to get the two points P_1 and P_2

$$P_1 = \begin{pmatrix}7\\4\\5\end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

Use Pythagoras's Theorem to find the distance

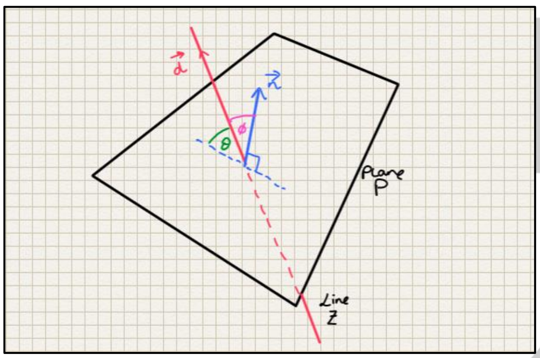
$$\sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$$

7.13 Angle between Two Planes

$$\cos\theta = \frac{\widetilde{\mathbf{n_1}}.\widetilde{\mathbf{n_2}}}{|\widetilde{\mathbf{n_1}}||\widetilde{\mathbf{n_2}}|}$$

- The $\tilde{\mathbf{n}}$'s here represent the normals of each plane
- Ignore any negative signs

7.14 Angle between a Line and a Plane

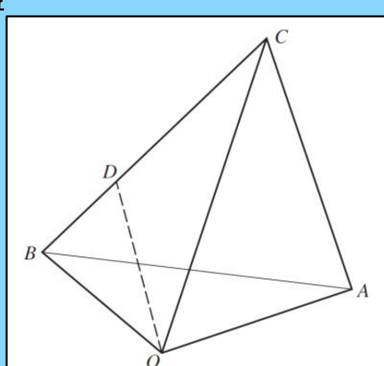


- First find $\emptyset$:

$$\cos\emptyset = \frac{\tilde{\mathbf{n}}.\tilde{\mathbf{d}}}{|\tilde{\mathbf{n}}||\tilde{\mathbf{d}}|}$$

- $\theta = 90 - \emptyset$
- θ is the angle between the line and the plane

{W13-P32} **Question:**



The diagram shows three points A, B and C whose position vectors with respect to the origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix}2\\-1\\2\end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix}0\\3\\1\end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix}3\\0\\4\end{pmatrix}.$$

The point D lies on BC, between B and C, and is such that $CD = 2DB$.

(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$

(ii) Find the position vector of D.

(iii) Show that the length of the perpendicular from A to OD is $\frac{1}{3}\sqrt{65}$

Solution:

Part (i)

First find two vectors on the plane e.g. AB and AC

$$AB = OB - OA = \begin{pmatrix}-2\\4\\-1\end{pmatrix} \text{ and } AC = OC - OA = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

Find the common perpendicular of the two

$$\begin{pmatrix}-2\\4\\-1\end{pmatrix} \times \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}9\\3\\-6\end{pmatrix}$$

We have now found the normal to the plane and now must find d

$$9x + 3y - 6z = d$$

Substitute a point that lies on the plane e.g. A

$$9(2) + 3(-1) - 6(2) = d \qquad d = 3$$

$$9x + 3y - 6z = 3$$

Part (ii)

(i) $CD = 2DB$

$OD - OC = 2OB - 2OD$

$$OD = \frac{1}{3}(2OB + OC) = \begin{pmatrix}3\\6\\6\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

Part (iii)

Finding a perpendicular from A to OD; find the equation of the line OA

$$OD = \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$$

A point Q lies on OD and is perpendicular to A. First we must find the vector AQ

$$AQ = OQ - OA = \begin{pmatrix}\lambda - 2\\2\lambda + 1\\2\lambda - 2\end{pmatrix}$$

Dot product of the point AQ and the direction vector of OD is equal to zero as it is perpendicular

$$\begin{pmatrix}\lambda - 2\\2\lambda + 1\\2\lambda - 2\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0$$

$$9\lambda = 4 \therefore \lambda = \frac{4}{9}$$

Substitute back into general equation of OD to find Q

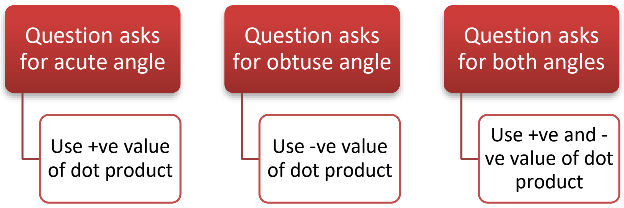
$$Q = \left(\frac{4}{9}, \frac{8}{9}, \frac{8}{9}\right)$$

To find the shortest distance, use Pythagoras theorem to find the distance from point A to Q

$$\sqrt{\left(\frac{14}{9}\right)^2 + \left(-\frac{17}{9}\right)^2 + \left(\frac{10}{9}\right)^2} = \sqrt{\frac{65}{9}} = \frac{1}{3}\sqrt{65}$$

7.15 Angles

- When using dot product rule to fine an angle,



8. Complex Numbers

8.1 The Basics

$$i^2 = -1$$

- General form for all complex numbers:

$$a + bi$$

- From this we say:

$$Re(a+bi) = a \qquad \& \qquad Im(a+bi) = b$$

- **Conjugates:**
  - The complex number z and its conjugate z^*

$$z = a + bi \qquad \& \qquad z^* = a - bi$$

- **Arithmetic:**
  - **Addition** and **Subtraction**: add and subtract real and imaginary parts with each other
  - **Multiplication:** carry out algebraic expansion, if i^2 present convert to -1
  - **Division:** rationalize denominator by multiplying conjugate pair
  - **Equivalence:** equate coefficients

8.2 Quadratic

- Use the quadratic formula:
  - $b^2 - 4ac$ is a negative value
  - Pull out a negative and replace with i^2
  - Simplify to general form
- Use sum of 2 squares: consider the example

Example:

Solve: $z^2 + 4z + 13 = 0$

Solution:

Convert to completed square form:

$$(z+2)^2 + 9 = 0$$

Utilize i^2 as -1 to make it difference of 2 squares:

$$(z+2)^2 - 9i^2 = 0$$

Proceed with general difference of 2 squares method:

$$(z+2+3i)(z+2-3i) = 0$$
$$z = -2 + 3i \quad and \quad z = -2 - 3i$$

8.3 Square Roots

Example:

Find square roots of: $4 + 3i$

Solution:

We can say that:

$$\sqrt{4+3i} = a + bi$$

Square both sides

$$a^2 - b^2 + 2abi = 4 + 3i$$

Equate real and imaginary parts

$$a^2 - b^2 = 4 \qquad 2ab = 3$$

Solve simultaneous equation:

$$a = \frac{3\sqrt{2}}{2} \qquad b = \frac{\sqrt{2}}{2}$$

$$\therefore \sqrt{4+3i} = \frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \qquad or \qquad -\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

8.4 Argand Diagram

For the complex number $z = a + bi$

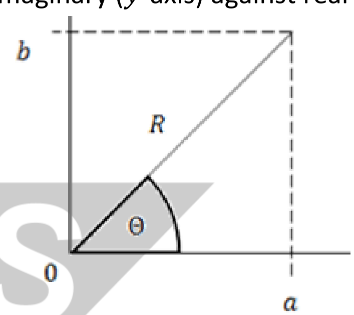
- Its magnitude is defined as the following:

$$|z| = \sqrt{a^2 + b^2}$$

- Its argument is defined as the following:

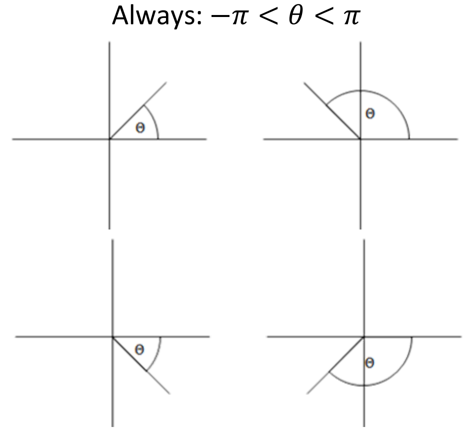
$$\arg z = \tan^{-1}\frac{b}{a}$$

- Simply plot imaginary (y-axis) against real (x-axis):



Arguments:

Always: $-\pi < \theta < \pi$

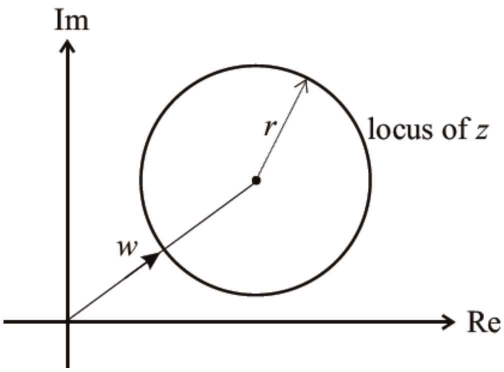


- The position of z^* is a reflection in the x-axis of z

8.5 Locus

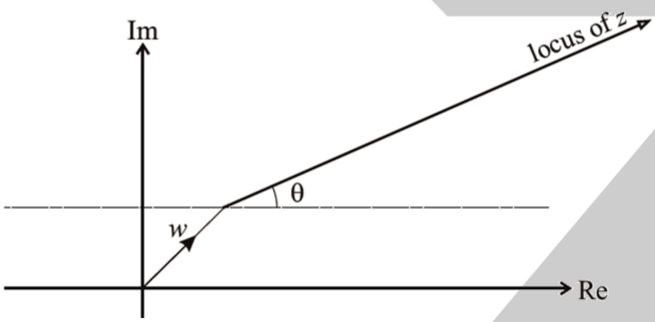
$$|z - w| = r$$

The locus of a point z such that $|z - w| = r$, is a circle with its centre at w and with radius r.



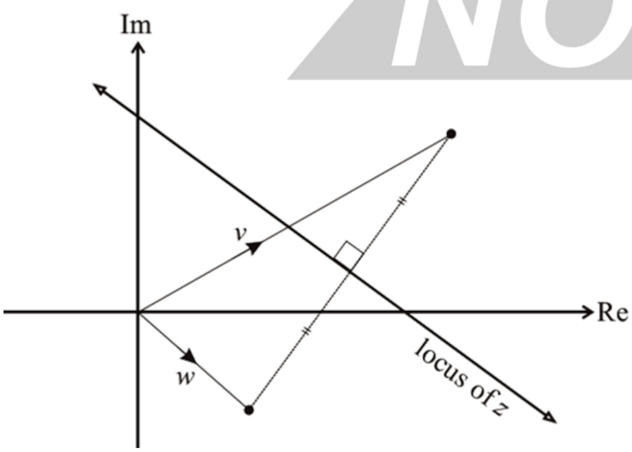
$$\arg(z - w) = \theta$$

The locus of a point z such that $\arg(z - w) = \theta$ is a ray from w, making an angle θ with the positive real axis.



$$|z - w| = |z - v|$$

The locus of a point z such that $|z - w| = |z - v|$ is the perpendicular bisector of the line joining w and v



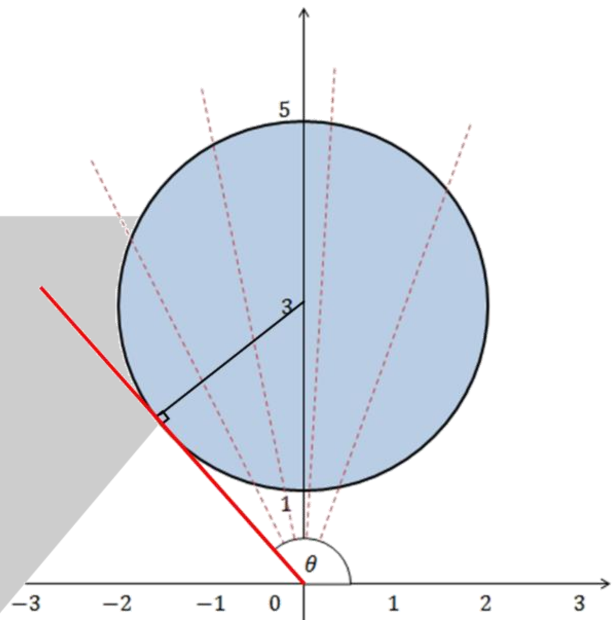
{W11-P31} **Question 10:**

On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3i| \leq 2$. Find the greatest value of $\arg z$ for points in this region.

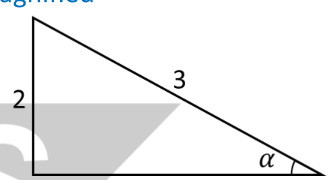
Solution:

The part shaded in blue is the answer.

To find the greatest value of $\arg z$ within this region we must use the tangent at point on the circle which has the greatest value of θ from the horizontal (red line)



The triangle magnified



$$\sin\alpha = \frac{2}{3}$$

$$\alpha = 0.730$$

$$\theta = \alpha + \frac{\pi}{2} = 0.730 + \frac{\pi}{2} = 2.30$$

{W11-P31} **Question 10:**

i. On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities

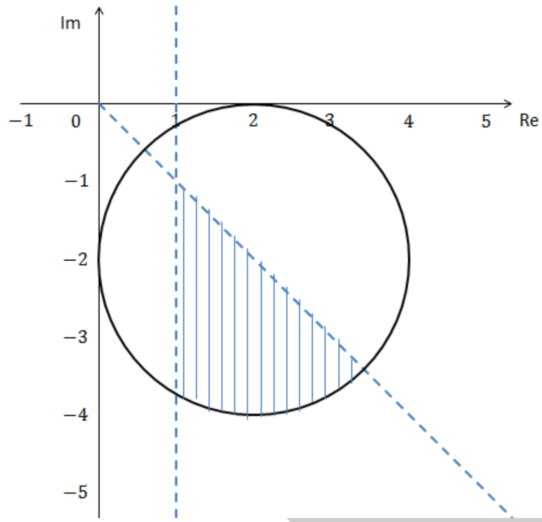
$|z - 2 + 2i| \leq 2, \arg z \leq -\frac{1}{4}\pi$ and $Re\ z \geq 1$,

ii. Calculate the greatest possible value of $Re\ z$ for points lying in the shaded region.

Solution:

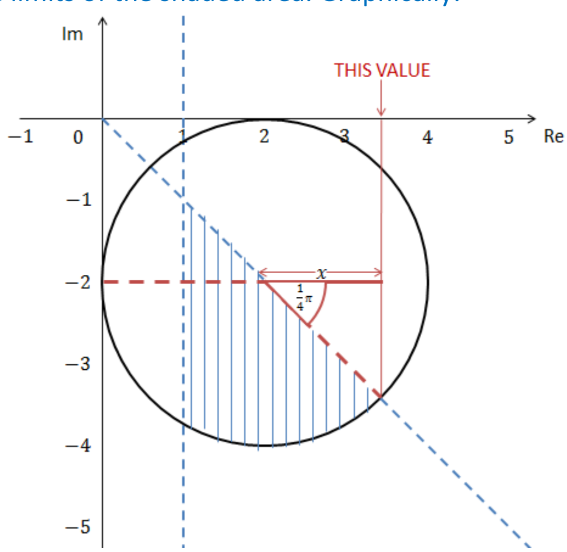
Part (i)

Argand diagram:



Part (ii)

The greatest value for the real part of z would be the one which is furthest right on the Re axis but within the limits of the shaded area. Graphically:



Now using circle and Pythagoras theorems we can find the value of x:

$$x = 2\times\cos\frac{1}{4}\pi$$
$$x = \sqrt{2}$$

$\therefore$ greatest value of $Re\ z = 2 + \sqrt{2}$

8.6 Polar Form

- For a complex number z with magnitude R and argument θ:

$$z = R(\cos\theta + i\sin\theta) = Re^{i\theta}$$
$$\therefore \cos\theta + i\sin\theta = e^{i\theta}$$

Polar Form to General Form:

Example:

Convert from polar to general, $z = 4e^{\frac{\pi}{4}i}$

Solution:

$$R = 4 \qquad \arg z = \frac{\pi}{4}$$
$$\therefore z = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$z = 4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$
$$z = 2\sqrt{2} + (2\sqrt{2})i$$

General Form to Polar Form:

Example:

Convert from general to polar, $z = 2\sqrt{2} + (2\sqrt{2})i$

Solution:

$$z = 2\sqrt{2} + (2\sqrt{2})i$$
$$R = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$
$$\theta = \tan^{-1}\frac{2\sqrt{2}}{2\sqrt{2}} = \frac{\pi}{4}$$
$$\therefore 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 4e^{\frac{\pi}{4}i}$$

8.7 Multiplication and Division in Polar Form

- To find **product** of two complex numbers in polar form:
  - Multiply their magnitudes
  - Add their arguments

$$z_1z_2 = |z_1||z_2|(\arg z_1 + \arg z_2)$$

Example:

Find z_1z_2 in polar form given,

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad z_2 = 4\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Solution:

$$z_1z_2 = (2\times4)\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{8}\right)\right)$$
$$z_1z_2 = 8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

- To find **quotient** of two complex numbers in polar form:
  - Divide their magnitudes
  - Subtract their arguments

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\arg z_1 - \arg z_2)$$

Example:
Find $\frac{z_1}{z_2}$ in polar form given,

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad z_2 = 4\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Solution:

$$\frac{z_1}{z_2} = \left(\frac{2}{4}\right)\left(\cos\left(\frac{\pi}{4} - \frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{8}\right)\right)$$

$$\frac{z_1}{z_2} = \frac{1}{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

8.8 De Moivre's Theorem

$$z^n = R^n(\cos n\theta + i\sin n\theta) = R^n e^{in\theta}$$

9. Differential Equations

- Form a differential equation using the information given
  - If something is proportional, add constant of proportionality k
  - If rate is decreasing, add a negative sign
- Separate variables, bring dx and dt on opposite sides
- Integrate both sides to form an equation
- Add arbitrary constant
- Use conditions given to find c and/or k

{W10-P33} **Question 9:**
A biologist is investigating the spread of a weed in a particular region. At time t weeks, the area covered by the weed is Am^2. The biologist claims that rate of increase of A is proportional to $\sqrt{2A - 5}$.

i. Write down a differential equation given info
ii. At start of investigation, area covered by weed was $7m^2$. 10 weeks later, area covered $= 27m^2$ Find the area covered 20 weeks after the start of the investigation.

Solution:

Part (i)

$$\frac{dA}{dt} \propto \sqrt{2A - 5} = k\sqrt{2A - 5}$$

Part (ii)
Proceed to form an equation in A and t:

$$\frac{dA}{dt} = k\sqrt{2A - 5}$$

Separate variables

$$\frac{1}{\sqrt{2A - 5}}dA = kdt$$

Integrate both side

$$kt + c = (2A - 5)^{\frac{1}{2}}$$

When $t = 0$:

$$A = 7 \quad \therefore \quad c = 3$$

$$kt + 3 = (2A - 5)^{\frac{1}{2}}$$

When $t = 10$:

$$10k + 3 = (2(27) - 5)^{\frac{1}{2}}$$

$$10k = \sqrt{49} - 3$$

$$k = 0.4$$

Now substitute 20 as t and then find A:

$$0.4(20) + 3 = (2A - 5)^{\frac{1}{2}}$$

$$11 = (2A - 5)^{\frac{1}{2}}$$

$$121 = 2A - 5$$

$$A = 63m^2$$

{S13-P31} **Question 10:**
Liquid is flowing into a small tank which has a leak. Initially the tank is empty and, t minutes later, the volume of liquid in the tank is $V\ cm^3$. The liquid is flowing into the tank at a constant rate of 80 cm^3 per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to $kV\ cm^3$ per minute where k is a positive constant.

i. Write down a differential equation describing this situation and solve it to show that:

$$V = \frac{1}{k}\left(80 - 80e^{-kt}\right)$$

ii. $V = 500$ when $t = 15$, show:

$$k = \frac{4 - 4e^{-15k}}{25}$$

Find k using iterations, initially $k = 0.1$

iii. Work out volume of liquid at $t = 20$ and state what happens to volume after a long time.

Solution:

Part (i)
Represent the given information as a derivative:

$$\frac{dV}{dt} = 80 - kV$$

Proceed to solve the differential equation:

$$\frac{dt}{dV} = \frac{1}{80 - kV}$$

$$dt = \frac{1}{80 - kV}dV$$

$$\int (1)dt = \int \frac{1}{80 - kV}dV$$

$$t + c = -\frac{1}{k}\ln|80 - kV|$$

Use the given information; when $t = 0, V = 0$:

$$\therefore c = -\frac{1}{k}\ln(80)$$

Substitute back into equation:

$$t - \frac{1}{k}\ln(80) = -\frac{1}{k}\ln|80 - kV|$$

$$t = \frac{1}{k}\ln(80) - \frac{1}{k}\ln|80 - kV|$$

$$t = \frac{1}{k}\ln\left(\frac{80}{80-kV}\right)$$
$$kt = \ln\left(\frac{80}{80-kV}\right)$$
$$e^{kt} = \frac{80}{80-kV}$$
$$80 - kV = \frac{80}{e^{kt}}$$
$$kV = 80 - 80e^{-kt}$$
$$V = \frac{1}{k}\left(80 - 80e^{-kt}\right)$$

Part (ii)

You did the mishwaar iterations and found:

$$\therefore k = 0.14\ (2d.p.)$$

Part (iii)

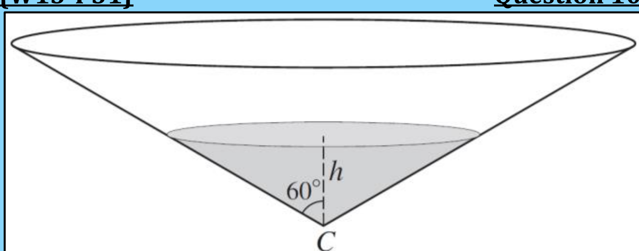
Simply substitute into the equation's t:

$$V = \frac{1}{0.14}\left(80 - 80e^{-0.14(20)}\right) = 537\ cm^3$$

The volume of liquid in the tank after a long time **approaches** the max volume:

$$V = \frac{1}{0.14}(80) = 571\ cm^3$$

{W13-P31} **Question 10:**



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

i. Show that h and t satisfy a differential equation of the form:
$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}}$$
Where A is a positive constant.

ii. Solve differential equation given in part i and obtain an expression for t in terms of h and H.

Solution:

Part (i)

First represent info they give us as an equation:

$$V = \frac{1}{3}\pi r^2 h$$
$$r = \tan 60 \times h = h\sqrt{3}$$
$$\therefore V = \frac{1}{3}\pi\left(h\sqrt{3}\right)^2 h = \pi h^3$$
$$\frac{dV}{dh} = 3\pi h^2$$
$$\frac{dV}{dt} \propto -\sqrt{h} = -kh^{\frac{1}{2}}$$

Find the rate of change of h:

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$$
$$\frac{dh}{dt} = \frac{-kh^{\frac{1}{2}}}{3\pi h^2} = -\frac{k}{3\pi}h^{-\frac{3}{2}}$$

Part (ii)

$$dt = \frac{1}{-Ah^{-\frac{3}{2}}}dh$$
$$\int A\,dt = \int \frac{1}{-h^{-\frac{3}{2}}}dh$$
$$At + c = -\frac{2}{5}h^{\frac{5}{2}}$$

Use given information to find unknowns; when $t = 0$:

$$-A(0) + c = \frac{2}{5}(H)^{\frac{5}{2}} \quad \therefore c = \frac{2}{5}H^{\frac{5}{2}}$$

When $t = 60$:

$$-A(60) + c = 0$$
$$c = 60A$$
$$A = \frac{1}{150}H^{\frac{5}{2}}$$

Thus the initial equation becomes:

$$-\frac{1}{150}H^{\frac{5}{2}}t + \frac{2}{5}H^{\frac{5}{2}} = \frac{2}{5}h^{\frac{5}{2}}$$
$$H^{\frac{5}{2}}\left(-\frac{t}{150} + \frac{2}{5}\right) = \frac{2}{5}h^{\frac{5}{2}}$$
$$-\frac{t}{150} + \frac{2}{5} = \frac{2h^{\frac{5}{2}}}{5H^{\frac{5}{2}}}$$
$$\frac{t}{150} = \frac{2}{5} - \frac{2h^{\frac{5}{2}}}{5H^{\frac{5}{2}}}$$
$$t = 150\left(\frac{2}{5} - \frac{2h^{\frac{5}{2}}}{5H^{\frac{5}{2}}}\right) = 60 - 60h^{\frac{5}{2}}H^{-\frac{5}{2}}$$
$$t = 60\left(1 - \left(\frac{h}{H}\right)^{\frac{5}{2}}\right)$$

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