Linear-Quadratic Control

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ENGR108
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November 28, 2021

Outline

Linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots$$

- n-vector x_t is *state* at time t
- m-vector u_t is *input* at time t
- $n \times n$ matrix A is *dynamics matrix*
- $n \times m$ matrix B is *input matrix*
- sequence $x_1, x_2, \ldots$ is called *state trajectory*

Simulation

- given x_1, $u_1, u_2, \ldots$ find $x_2, x_3, \ldots$
- can be done by recursion: for $t = 1, 2, \ldots$,

$$x_{t+1} = Ax_t + Bu_t$$

Vehicle example

consider a vehicle moving in a plane:

- sample position and velocity at times $\tau = 0, h, 2h, \ldots$
- 2-vectors p_t and v_t are position and velocity at time ht
- 2-vector u_t gives applied force on the vehicle time ht
- friction force is $-\eta v_t$
- vehicle has mass m
- for small h,

$$m\frac{v_{t+1} - v_t}{h} \approx -\eta v_t + u_t, \qquad \frac{p_{t+1} - p_t}{h} \approx v_t$$

- we use approximate state update

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t, \qquad p_{t+1} = p_t + hv_t$$

- vehicle state is 4-vector $x_t = (p_t, v_t)$
- dynamics recursion is

$$x_{t+1} = Ax_t + Bu_t,$$

where

$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ h/m & 0 \\ 0 & h/m \end{bmatrix}$$

Outline

Control

- x_1 is given
- choose $u_1, u_2, \ldots, u_{T-1}$ to achieve some goals, *e.g.*,
  - terminal state should have some fixed value: $x_T = x^{\text{des}}$
  - $u_1, u_2, \ldots, u_{T-1}$ should be small, say measured as

    $$\|u_1\|^2 + \cdots + \|u_{T-1}\|^2$$

    (sometimes called 'energy')
- many control problems are linearly constrained least-squares problems

Minimum-energy state transfer

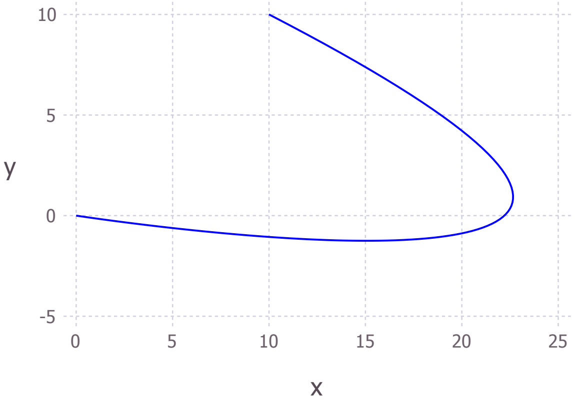
- given initial state x_1 and desired final state x^{des}
- choose $u_1, \ldots, u_{T-1}$ to minimize 'energy'

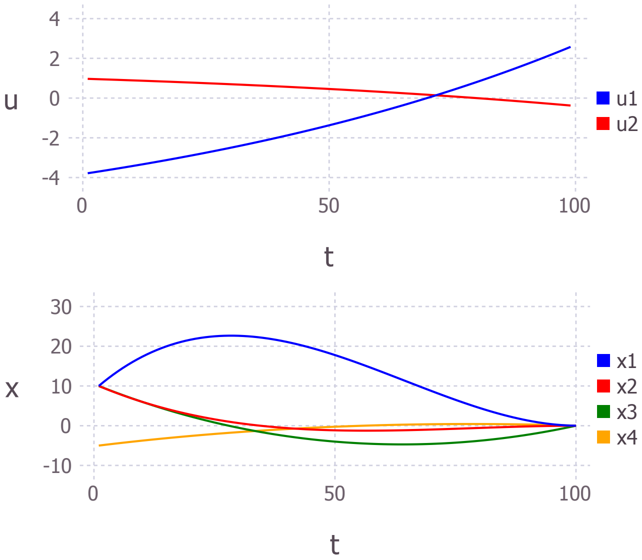
$$
\begin{array}{ll}
\text{minimize} & \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\
\text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\
 & x_T = x^{\mathrm{des}}
\end{array}
$$

  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$
- roughly speaking: find minimum energy inputs that steer the state to given target state over T periods

State transfer example

vehicle model with $T = 100,\ x_1 = (10, 10, 10, -5),\ x^{\mathrm{des}} = 0$




4
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20
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0
-10
x
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50
100
t
x1
x2
x3
x4

Outline

Output tracking

- $y_t = Cx_t$ is output (*e.g.*, position)
- y_t should follow a desired trajectory, *i.e.*, sum square *tracking error*

$$\|y_2 - y_2^{\text{des}}\|^2 + \cdots + \|y_T - y_T^{\text{des}}\|^2$$

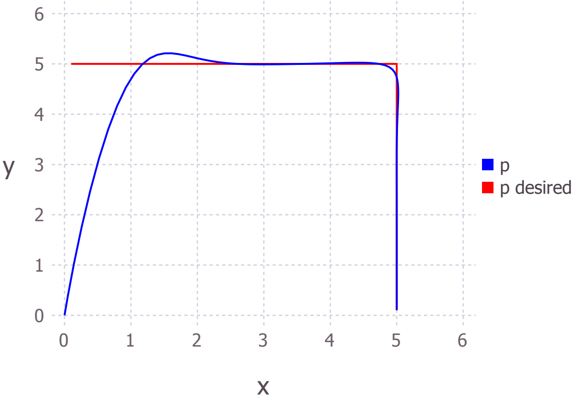
  should be small
- the output tracking problem is

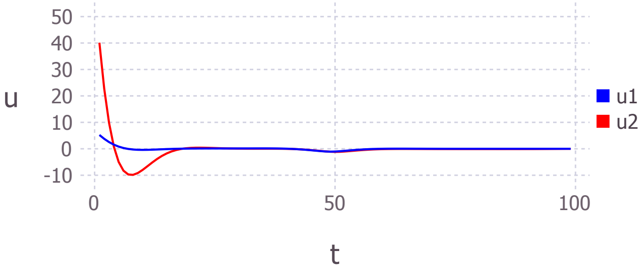
$$\begin{array}{ll} \text{minimize} & \sum_{t=2}^{T} \|y_t - y_t^{\text{des}}\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & y_t = Cx_t, \quad t = 1, \ldots, T-1 \end{array}$$

  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, y_2, \ldots, y_T$
- parameter $\rho > 0$ trades off control 'energy' and tracking error

Output tracking example

vehicle model with $T = 100,\ \rho = 0.1,\ x_1 = 0,\ y_t = p_t$ (position tracking)





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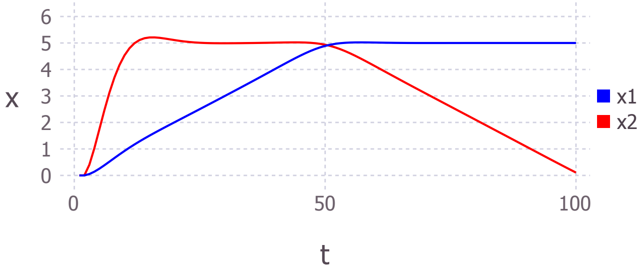

6
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100
t
x1
x2

Waypoints

- using output, can specify *waypoints*
- specify output (position) $w^{(k)}$ at time t_k at K total places

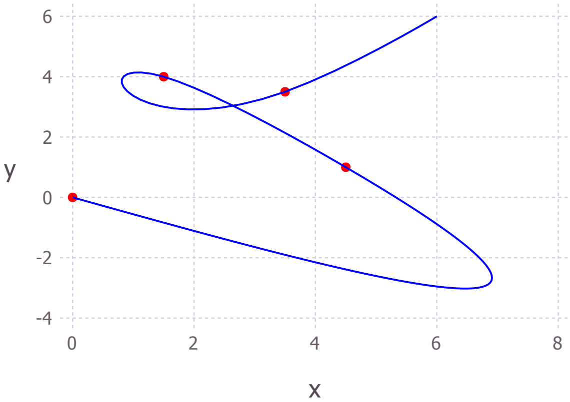
$$\begin{array}{ll} \text{minimize} & \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & Cx_{t_k} = w^{(k)}, \quad k = 1, \ldots, K \end{array}$$

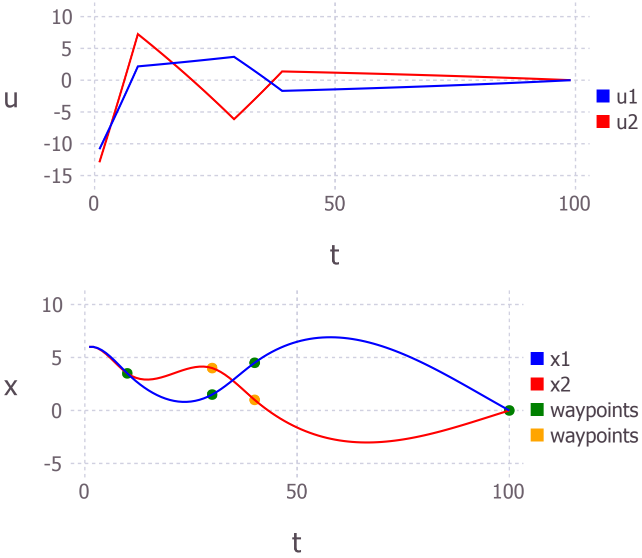
variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$

Waypoints example

- vehicle model
- $T = 100,\ x_1 = (10, 10, 20, 0),\ x^{\mathrm{des}} = 0$
- $K = 4,\ t_1 = 10,\ t_2 = 30,\ t_3 = 40,\ t_4 = 80$
- $w^{(1)} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix},\ w^{(2)} = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix},\ w^{(3)} = \begin{bmatrix} 4.5 \\ 1 \end{bmatrix},\ w^{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Waypoints example

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x
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100
t
x1
x2
waypoints
waypoints

Outline

Rendezvous

- we control two vehicles with dynamics

$$x_{t+1} = Ax_t + Bu_t, \quad z_{t+1} = Az_t + Bv_t$$

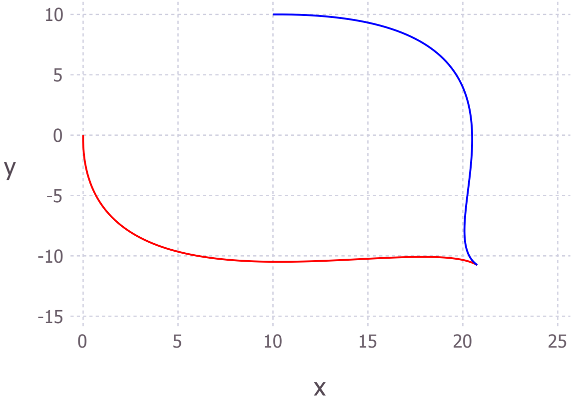
- final relative state constraint $x_T = z_T$
- formulate as state transfer problem:

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T-1}(\|u_t\|^2 + \|v_t\|^2) \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1, \\ & z_{t+1} = Az_t + Bv_t, \quad t = 1, \ldots, T-1, \\ & x_T = z_T \end{array}$$

variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$, $z_2, \ldots, z_T, v_1, \ldots, v_{T-1}$

Rendezvous example

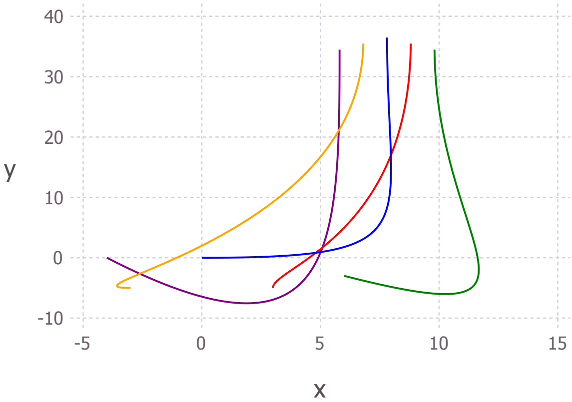
$x_1 = (0, 0, 0, -5)$, $z_1 = (10, 10, 5, 0)$



Formation

- generalize rendezvous example to several vehicles
- final position for each vehicle defined relative to others (*e.g.*, relative to a 'leader')
- leader has a final velocity constraint

Formation example



Outline

Linear quadratic regulator

- minimize energy while driving state to the origin:

$$\begin{array}{ll} \text{minimize} & \sum_{t=2}^{T} \|x_t\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \end{array}$$

  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$
- $\sum_{t=2}^{T} \|x_t\|^2$ is (sum square) *regulation*
- $x = 0$ is some desired (equilibrium, target) state
- parameter $\rho > 0$ trades off regulation versus control 'energy'

- LQR problem is a linearly constrained least-squares problem:

$$\begin{array}{ll} \text{minimize} & \|Fz\|^2 \\ \text{subject to} & Gz = d \end{array}$$

- variable z is $(x_2, \ldots, x_T, u_1, \ldots, u_{T-1})$
- F, G depend on A, B, ρ; d depends (linearly) on x_1
- solution is $\hat{z} = Hd$ for some H
- optimal first input $\hat{u}_1$ is a linear function of x_1, *i.e.*,

$$\hat{u}_1 = Kx_1$$

  for some $m \times n$ matrix K (called *LQR gain matrix*)
- finding K involves taking correct 'slice' of inverse KKT matrix
- entries of K depend on horizon T, and converge as T grows large

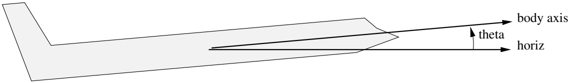
State feedback control

- find K for LQR problem (with large T)
- for each t,
  - measure state x_t
  - implement control $u_t = Kx_t$
- with $u_t = Kx_t$ is called *state feedback control policy*
- combine with (‘*open-loop dynamics*’) $x_{t+1} = Ax_t + Bu_t$ to get *closed-loop dynamics*

$$x_{t+1} = (A + BK)x_t$$

- we can simulate open- and closed-loop dynamics to compare

Example: longitudinal flight control



variables are (small) deviations from operating point or *trim conditions*;

state is $x_t = (w_t, v_t, \theta_t, q_t)$:

- w_t: velocity of aircraft along body axis
- v_t: velocity of aircraft perpendicular to body axis (down is positive)
- θ_t: angle between body axis and horizontal (up is positive)
- $q_t = \dot{\theta}_t$: angular velocity of aircraft (pitch rate)

input is $u_t = (e_t, f_t)$:

- e_t: elevator angle ($e_t > 0$ is down)
- f_t: thrust

Linearized dynamics

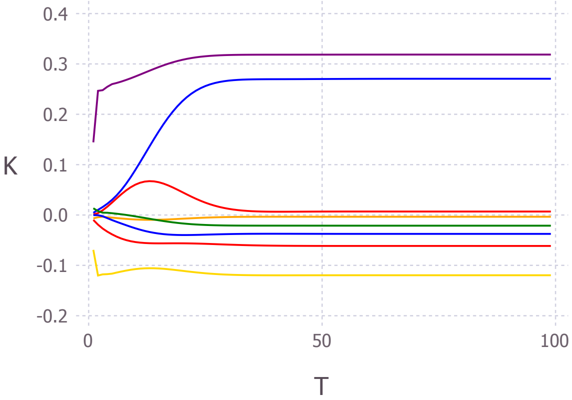
for 747, level flight, 40000 ft, 774 ft/sec, dynamics are
$x_{t+1} = Ax_t + Bu_t$, where

$$A = \begin{bmatrix} .99 & .03 & -.02 & -.32 \\ .01 & .47 & 4.7 & .00 \\ .02 & -.06 & .40 & -.00 \\ .01 & -.04 & .72 & .99 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.01 & 0.99 \\ -3.44 & 1.66 \\ -0.83 & 0.44 \\ -0.47 & 0.25 \end{bmatrix}$$

- units: ft, sec, crad ($= 0.01$rad $\approx 0.57°$)
- discretization is 1 sec

LQR gain

gain matrix K converged for $T \approx 30$



LQR for 747 model

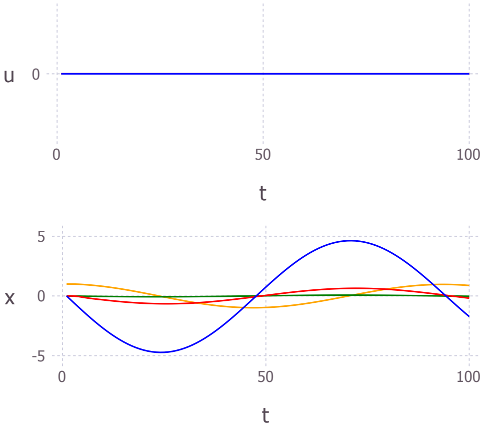
- LQR gain matrix (for $T = 100$, $\rho = 100$) is:

$$K = \begin{bmatrix} -.038 & .021 & .319 & -.270 \\ -.061 & -.004 & -.120 & .007 \end{bmatrix}$$

- *e.g.*, $K_{14} = -.27$ is gain from pitch rate ($(x_t)_4$) to elevator angle ($(u_t)_1$)

747 simulation

$u_t = 0$ ('open loop')



747 simulation

$u_t = K x_t$ ('closed loop')