Exercise Sheet 5: Data Science Methods

Technische Universität München

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Instructions

You can hand in your solutions during the next tutorial on 30/01/25 or submit them earlier at the group office 1269 in the Physics Department.

1 Exercise: Conjugate Distributions

1.1 Let the likelihood for one observation x be $p\left(x|\mu,\sigma\right) \sim \mathcal{N}\left(\mu,\sigma\right)$, for given σ. Furthermore, choose a normally distributed prior $p\left(\mu\right) \sim \mathcal{N}\left(\mu_0,\sigma_0\right)$, where μ_0 and σ_0 are known. Compute the posterior distribution $p\left(\mu|x,\sigma\right)$.

1.2 Now generalize your result for multiple observations $\vec{x} = (x_1,\ldots,x_n)$ (i.i.d.) instead of only one observation x, i.e. determine the posterior $p\left(\mu|\vec{x},\sigma\right)$.

2 Exercise: Jeffreys Prior

The **Fisher information matrix** for a vector of parameters $\vec{\theta} = (\theta_1,\theta_2,\ldots,\theta_k)$ is defined as:

$$\mathcal{I}_{ij}(\vec{\theta}) = \mathbb{E}\left[\frac{\partial \log f\left(x \mid \vec{\theta}\right)}{\partial\theta_i}\frac{\partial \log f\left(x \mid \vec{\theta}\right)}{\partial\theta_j}\right],$$

where $f\left(x \mid \vec{\theta}\right)$ is the likelihood function.

The **Jeffreys prior** for $\vec{\theta}$ is then defined as:

$$\pi\left(\vec{\theta}\right) \propto \sqrt{\det \mathcal{I}\left(\vec{\theta}\right)}.$$

Now, determine the Jeffreys prior, assuming that the likelihood is given by a normal distribution with unknown mean μ and variance σ^2.

3 Exercise: Maximum a Posteriori (MAP) Estimator

The Maximum a Posteriori (MAP) estimator finds the parameter value that maximizes the posterior probability.

1.1 Argue that for a uniformly distributed prior, the MAP estimator is equivalent to the MLE in classical statistics.

1.2 Consider again the situation of Exercise 1 with multiple observations $\vec{x} = (x_1,\ldots,x_n)$. Determine the MAP estimator and compare it to the well-known MLE from the frequentist approach.