Piecewise and non-uniform interpolation

Consider the function

$$f(x) = \arctan(5x),\ x \in [-2, 2]$$

and suppose you need to find a polynomial approximation.

(**a**) Now write a script that generates N equally spaced knots on the domain and computes the (free) cubic spline interpolant $S_N(x)$. Plot the interpolant together with f for $N = 4, 8, 16, 32$. Also, plot the error $|f(x) - S_N(x)|$. Is the spline interpolation on these knots useful for approximating f on this domain?

(**b**) Finally, write a script that interpolates f on the nodes

$$x_k = 2\cos(kh),\ h = \pi/N,\ k = 0, \ldots, N$$

and call the resulting interpolant R_N. Plot R_N together with f for $N = 4, 8, 16, 32$. Also, plot the error $|f(x) - R_N(x)|$. Is interpolation on these nodes useful for approximating f on this domain?

Discussion: We have already seen polynomial interpolation on equidistant nodes fail for a function like this one (namely the Runge function). In this tutorial, you have demonstrated two possible solutions: a non-uniform grid with most points near the boundaries of the domain of interpolation and piecewise interpolation. Which solution do you think is best when using interpolation for what purpose?