The Vandermonde matrix

Suppose we want to find a polynomial

$$P_n = a_0 + a_1 x + \ldots + a_n x^n$$

that “fits” the data points (x_i, y_i) for $i = 0, \ldots, n$.

- For $n = 3$, write out the equations $P(x_i) = y_i$ for $i = 0, 1, 2, 3$. Convince yourself that the resulting system of equations is linear in the a_i.
- Write the system of equations for general n in matrix-vector form. The matrix you found is called the Vandermonde matrix.
- Write a pseudo-code of a function that takes as an input the data (x_i, y_i) for $i = 0, \ldots, n$ and returns the coefficients $\{a_i\}_{i=0}^n$ such that P_n, as defined above, fits the data.
- Count the number of FLOPS in your function. Be careful to include the correct FLOP count for solving the linear system and for forming the Vandermonde matrix.
- Now implement the function and test it with the input data

  $$x_i = i,\ y_i = e^{x_i},\ i = 0, \ldots, n$$

  for different values of n between 4 and 15.
- What happens to the condition number of the Vandermonde matrix as n increases? Up to what number of input data can you expect to find a good aproximation of the coefficients a_i?

Discussion

It would seem that the Vandermonde matrix can become ill conditioned for interpolation of high polynomial order. Can you think of a way to mitigate this problem?