Finite Elements: examples 2

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1. Consider the variational problem with bilinear form
$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x,$$
corresponding to the differential equation
$$-u'' + u' + u = f.$$
Prove that $a(\cdot,\cdot)$ is continuous and coercive on a C^0 finite element space V defined on $[0,1]$, equipped with the H^1 inner product.

2. For the differential equation $-u'' + ku' + u = f$, formulate a C^0 finite element discretisation with bilinear form $a(u,v)$. Find a value for k such that $a(v,v) = 0$ but $v \neq 0$ for some $v \in V$.

3. Let $a(\cdot,\cdot)$ be the inner product for a finite element space U. For $F \in U'$, show that the following two statements are equivalent:
   (a) $u \in U$ satisfies $a(u,v) = F(v)\ \forall v \in U$.
   (b) u uniquely minimises $\frac{1}{2}a(v,v) - F(v)$ over $v \in U$.

   Use this to conclude that the Poisson and Helmholtz finite element discretisations can be formulated as minimisation problems. .

4. Let
$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x.$$
Let V be a C^0 finite element space on $[0,1]$ and let $\mathring{V}$ be the subspace of functions that vanish at $x=0$ and $x=1$. Prove that
$$a(v,v) = \int_0^1 \left((v')^2 + v^2\right)\mathrm{d}\,x := \|v\|_{H^1}^2, \quad \forall v \in \mathring{V}.$$
Hence conclude that the bilinear form is coercive on $\mathring{V}$.

5. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a variational formulation of the problem
$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma(x)\frac{\partial u}{\partial x_i}\right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$
   (b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show that $a(\cdot,\cdot)$ is continuity and coercive with respect to the H^1 norm.

6. Find a C^0 finite element formulation for the Poisson equation
$$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$
for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under the discretisation has a unique solution.

7. Find a C^0 finite element formulation for the Poisson equation
$$-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,$$
for a function r defined on $\partial\Omega$. Derive conditions under which this discretisation has a unique solution.