Finite Elements: examples 3

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1. Let L be a linear functional on a Hilbert space. Prove that L is continuous if and only if L is bounded.

2. Consider the variational problem with bilinear form
$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x.$$
Prove that $a(\cdot,\cdot)$ is continuous and coercive on $H^1([0,1])$.

3. For the differential equation $-u'' + ku' + u = f$, find a value for k such that $a(v,v) = 0$ but $v \neq 0$ for some $v \in H^1([0,1])$.

4. Let $a(\cdot,\cdot)$ be the inner product for a Hilbert space V. For $F \in V'$, and U an arbitrary (closed) subspace U of V, show that the following two statements are equivalent:
   (a) $u \in U$ satisfies $a(u,v) = F(v)\ \forall v \in U$.
   (b) u minimises $\frac{1}{2}a(v,v) - F(v)$ over $v \in U$.

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5. Let
$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x,$$
with
$$V = \{v \in H^1([0,1]) : v(0) = v(1) = 0\}.$$
Prove that
$$a(v,v) = \int_0^1 \left((v')^2 + v^2\right)\mathrm{d}\,x := \|v\|^2_{H^1}, \quad \forall v \in V.$$

6. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a variational formulation of the problem
$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma(x)\frac{\partial u}{\partial x_i}\right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$
   (b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show that a finite element discretisation of this problem based on Lagrange elements has a unique solution, and give the rate of convergence to zero with h of the H^1 norm of the error.

7. Find a variational formulation for the Poisson equation
$$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$
for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under which a finite element discretisation of this problem based on Lagrange elements has a unique solution.

8. Find a variational formulation for the Poisson equation
$$-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,$$
for a function r defined on $\partial\Omega$.