Finite Elements: examples 1

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1. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise linear finite element space known as P1. Compute the entries of the matrix

$$K_{ij} = a(\phi_i, \phi_j),$$

and the right-hand side vector

$$F_i = (\phi_i, f_I),$$

where $f_I \in S$ is the interpolant of f.

Solution: *The local shape functions on the reference element interval* $[0, 1]$ *are*

$$\psi_0(x) = 1 - x, \quad \psi_1(x) = x,$$

and so

$$\psi_0'(x) = -1, \quad \psi_1'(x) = 1.$$

For $i = j$ *we have*

$$\begin{aligned}
K_{ii} &= \int_0^1 \phi_i'(x)\phi_i'(x)\,\mathrm{d}\,x, && (1)\\
&= \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} \phi_i'(x)\phi_i'(x)\,\mathrm{d}\,x, && (2)\\
&= \frac{1}{x_i - x_{i-1}}\int_0^1 (\psi_1'(x'))^2\,\mathrm{d}\,x' + \frac{1}{x_{i+1} - x_i}\int_0^1 (\psi_0'(x'))^2\,\mathrm{d}\,x', && (3)\\
&= \frac{1}{h_{i+1}} + \frac{1}{h_i}, && (4)
\end{aligned}$$

where we have used the change of variables

$$x = x_{i-1} + x'(x_i - x_{i-1}).$$

The first term is removed for $i = n$*. For* $i = j - 1$ *we have*

$$\begin{aligned}
K_{i,i+1} &= \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} \phi_i'(x)\phi_{i+1}'(x)\,\mathrm{d}\,x, && (5)\\
&= \frac{1}{x_{i+1} - x_i}\int_0^1 \psi_0'(x')\psi_1'(x)\,\mathrm{d}\,x', && (6)\\
&= -\frac{1}{h_{i+1}}. && (7)
\end{aligned}$$

Similarly, for $i = j + 1$*, we have*

$$K_{i,i-1} = -\frac{1}{h_{i-1}}.$$

Otherwise, $K_{ij} = 0$. *Similarly,*

$$F_i = \int_0^1 \phi_i(x) f_I(x) \,\mathrm{d}\, x, \tag{8}$$

$$= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \phi_i(x) f_I(x) \,\mathrm{d}\, x, \tag{9}$$

$$= h_i \int_0^1 \psi_1(x')(f_{i-1} + x'(f_i - f_{i-1})) \,\mathrm{d}\, x' + h_{i+1} \int_0^1 \psi_0(x')(f_i + x'(f_{i+1} - f_i)) \,\mathrm{d}\, x' \tag{10}$$

$$= h_i \left(\frac{f_{i-1}}{2} + \frac{f_i - f_{i-1}}{3} \right) + h_{i+1} \left(\frac{f_i}{2} + \frac{f_{i+1} - f_i}{3} \right). \tag{11}$$

For an equispaced mesh with $h_i = x_i - x_{i-1} := h$, write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

Solution: *For* $h_i := h$, *we have*

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f_{i-1}}{6} + \frac{f_i}{2} + \frac{f_{i+1}}{6},$$

for $i < n$. *This is the centred difference formula for the second derivative with an averaged value of* f *on the right-hand side. The classical centred difference formula is*

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i.$$

2. Give a weak formulation for the following ODEs,

(a)

$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$

Solution: *Let* $b(u, v)$ *be*

$$b(u, v) = \int_0^1 u'v' + uv \,\mathrm{d}\, x.$$

Define V *by*

$$V = \{u \in L^2([0, 1]) : b(u, u) < \infty\}.$$

Then, find $u \in V$ *such that*

$$b(u, v) = (f, v), \quad \forall v \in V.$$

(b)

$$-u'' + u = f, \quad u'(0) = 0,\, u'(1) = \alpha.$$

Solution: *Under same definitions, find* $u \in V$ *such that*

$$b(u, v) = (f, v) - v(1)\alpha, \quad \forall v \in V.$$

(c)

$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

Solution: *Let* $a(u, v)$ *be*

$$a(u, v) = \int_0^1 u'v' \,\mathrm{d}\, x.$$

Define V *by*

$$V = \{u \in L^2([0, 1]) : a(u, u) < \infty\}.$$

Then, find $u \in V$ *such that*

$$a(u, v) = (f, v), \quad \forall v \in V.$$

What is wrong with the formulation of the last ODE?
Solution: *The problem with this formulation is that addition on an arbitrary constant to the solution* u *produces another solution, so the problem is not well-posed.*

3. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise quadratic finite element space known as P2, defined by the following:

   (a) $S \subset C^0([0, 1])$.

   (b) For $v \in S$, $v|_{[x_{j-1},x_j]}$ is a quadratic function of x.

   (c) $v(0) = 0$.

   Find a nodal basis for S, using the nodes for P1 plus nodes at element midpoints $(x_j + x_{j+1})/2$.
   Solution: *In the reference element* $[0, 1]$*, the nodal points are* $(z_0, z_1, z_2) = (0, 0.5, 1)$*. The nodal basis in the reference element satisfies*

$$\psi_i(z_j) = \delta_{ij}.$$

We get

$$\psi_i(x') = \frac{\prod_{j\neq i}(x' - x_j)}{\prod_{j\neq i}(x_i - x_j)},$$

hence,

$$\psi_0(x') = \frac{(x' - 0.5)(x' - 1)}{0.5 \times 1} = 2x'^2 - 3x' + 1, \tag{12}$$
$$\psi_1(x') = \frac{x'(x' - 1)}{-0.5 \times 0.5} = -4x'^2 + 4x', \tag{13}$$
$$\psi_2(x') = \frac{x'(x' - 0.5)}{0.5} = 2x'^2 - x'. \tag{14}$$

Derivatives are

$$\psi_0'(x') = 4x' - 3, \tag{15}$$
$$\psi_1'(x') = -8x' + 4, \tag{16}$$
$$\psi_2'(x') = 4x' - 1. \tag{17}$$

Evaluate the matrix K for this basis.
Solution: *The local matrix is*

$$\hat{K}_{ij} = \frac{1}{h}\int_0^1 \psi_i'(x)\phi_j'(x)\,\mathrm{d}\,x,$$

so we have

$$\hat{K}_{00} = \frac{1}{h}\int_0^1 (4x' - 3)^2\,\mathrm{d}\,x = \frac{7}{3h},$$
$$\hat{K}_{11} = \frac{1}{h}\int_0^1 (-8x' + 4)^2\,\mathrm{d}\,x = \frac{16}{3h},$$
$$\hat{K}_{22} = \frac{1}{h}\int_0^1 (4x' - 1)^2\,\mathrm{d}\,x = \frac{7}{3h},$$
$$\hat{K}_{01} = \hat{K}_{10} = \frac{1}{h}\int_0^1 (4x'^2 - 3)(-8x'^2 + 4)\,\mathrm{d}\,x = -\frac{8}{3h},$$
$$\hat{K}_{02} = \hat{K}_{20} = \frac{1}{h}\int_0^1 (4x'^2 - 3)(4x'^2 - 1)\,\mathrm{d}\,x = \frac{1}{3h},$$
$$\hat{K}_{12} = \hat{K}_{21} = \frac{1}{h}\int_0^1 (-8x'^2 + 4)(4x'^2 - 1)\,\mathrm{d}\,x = \frac{-8}{3h}.$$

The local matrix is

$$\hat{K} = \begin{pmatrix} \frac{7}{3h} & -\frac{8}{3h} & \frac{1}{3h} \\ -\frac{8}{3h} & \frac{16}{3h} & \frac{-8}{3h} \\ \frac{1}{3h} & \frac{-8}{3h} & \frac{7}{3h} \end{pmatrix}.$$

Hence, if we adopt a global node numbering such that odd numbered nodes are located at subinterval midpoints, and even numbered nodes are located at subinterval boundaries, then for odd i,

$$K_{ii} = \frac{16}{3h_{i/2}}, \quad K_{i,i+1} = K_{i+1,i} = K_{i,i-1} = K_{i-1,i} = \frac{-8}{3h_{i/2}}.$$

For even i,

$$\begin{aligned} K_{ii} &= \frac{7}{3}\left(\frac{1}{h_{i/2}} + \frac{1}{h_{i/2+1}}\right), \\ K_{i,i-1} &= -\frac{8}{3}\frac{1}{h_{i/2}}, \\ K_{i,i-2} &= \frac{1}{3}\frac{1}{h_{i/2}}, \\ K_{i,i+1} &= -\frac{8}{3}\frac{1}{h_{i/2+1}}, \\ K_{i,i+2} &= \frac{1}{3}\frac{1}{h_{i/2+1}}. \end{aligned}$$

4. Under the same assumptions as Theorem 1.8, prove that

$$\|u - u_I\| \leq Ch^2\|u''\|.$$

(hint: make use of the fact that $u(0) = 0$ to write u in terms of u'.)
Solution: *We aim to show that*

$$\int_{x_{j-1}}^{x_j} (u - u_I)(x)^2 \,\mathrm{d}\,x \leq c(x_j - x_{j-1})^4 \int_{x_{j-1}}^{x_j} u''(x)^2 \,\mathrm{d}\,x,$$

since summing over elements will give the result. After a change of variables to the unit interval, this is equivalent to

$$\int_0^1 w(x)\,\mathrm{d}\,x \leq c\int_0^1 w''(x)\,\mathrm{d}\,x,$$

for $w(0) = w(1) = 0$. To show this, we write

$$w(x) = \int_0^x w'(t)\,\mathrm{d}\,t,$$

and therefore, from Rolle's theorem, there exists $0 \leq \xi \leq 1$ such that

$$w(x) = \int_0^x \int_\xi^y w''(t)\,\mathrm{d}\,t\,\mathrm{d}\,y.$$

Then, by Schwartz's inequality,

$$\begin{aligned} |w(x)| &\leq \int_0^x \left|\int_\xi^y 1\,\mathrm{d}\,t\right|^{1/2} \left|\int_\xi^y w''(t)^2\,\mathrm{d}\,t\right|^{1/2} \mathrm{d}\,y, \\ &\leq \int_0^x |y-\xi|^{1/2} \left|\int_0^1 w''(t)^2\,\mathrm{d}\,t\right|^{1/2} \mathrm{d}\,y, \\ &\leq \frac{x}{2}\left|\int_0^1 w''(t)^2\,\mathrm{d}\,t\right|^{1/2}. \end{aligned}$$

Squaring and integrating gives

$$\int_0^1 w(x)^2\,\mathrm{d}\,x \leq \frac{1}{4}\int_0^1 w''(x)^2\,\mathrm{d}\,x,$$

and we obtain the required result with $c = 1/4$.

5. Under the same assumptions as Theorem 1.8, prove the following version of *Sobolev's inequality*:

$$\|v\|_{\max}^2 \le Ca(v,v), \quad \forall v \in V \cap C^1([0,1]).$$

Give a value for C.
Solution: *Since $v \in V$, we have $v(0) = 0$. Since $v \in C^1([0,1])$, we have*

$$v(x) = \int_0^x v'(y)\,\mathrm{d}\,y.$$

From the Schwartz inequality, we have

$$\begin{aligned} |v(x)| &\le \left|\int_0^x 1\,\mathrm{d}\,x\right|^{1/2} \left|\int_0^x (v'(x))^2\,\mathrm{d}\,x\right|^{1/2}, \\ &\le x^{1/2} \left|\int_0^1 (v'(x))^2\,\mathrm{d}\,x\right|^{1/2}, \end{aligned}$$

and so

$$\max_{0\le x\le 1} |v(x)| \le c \left|\int_0^x (v'(x))^2\,\mathrm{d}\,x\right|^{1/2},$$

with $c = 1$.

6. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.
Solution: *Let P be a cubic polynomial on the triangle K such that $N_i(P) = 0$, $i = 1,2,3,4,5,6$. Let L_1, L_2 and L_3 be the three edges of the triangle opposite vertices z_1, z_2 and z_3 respectively. Restricting P to L_1, we find that $P(z_0) = P'(z_0) = P(z_1) = P'(z_1)$ where $'$ indicates differentiation in the direction of L_1. Therefore $P|_{L_1}$ has double roots at z_0 and z_1, and so $P|_{L_1} = 0$. Similar argument for the other three edges means that $P = cL_1L_2L_3$ for constant c. But, P also vanishes at the midpoint z_4, so $c = 0$.*

7. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.
Solution:

$$\begin{aligned} \mathcal{I}_K(f+\alpha g) &= \sum_{i=1}^n \phi_i(x)(f+\alpha g)(x_i), \\ &= \sum_{i=1}^n \phi_i(x) f(x_i) + \alpha \sum_{i=1}^n \phi_i(x) g(x_i), \\ &= \mathcal{I}_K f + \alpha \mathcal{I}_K g. \end{aligned}$$

8. Let K be a rectangle, Q_2 be the space of biquadratic polynomials, and let $\mathcal{N}$ be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that $\mathcal{N}$ determines the finite element.
Solution: *Let z_i, $i = 1,2,3,4$ be the rectangle vertices, z_i, $i = 5,6,7,8$ be the edge midpoints, and z_9 be the rectangle centre. Let L_1, L_2, L_3 and L_4 be the linear functions determining the top, bottom, left and right rectangle edges respectively. Then L_1 and L_2 are functions of x only, whilst L_3 and L_4 are functions of y only. If P is biquadratic, then P restricted to L_1 is quadratic, and vanishes at three points on L_1. Therefore, $P|_{L_1} = 0$. The same reasoning for the other three edges means that we have*

$$P = cL_1L_2L_3L_4,$$

where c must be constant since $L_1L_2L_3L_4$ is biquadratic. Since P also vanishes at the rectangle centre, we conclude that $P = 0$ as required.

9. For K being the unit square, determine the nodal basis for the element in the previous question.
Solution: *The local basis functions are determined as products*

$$\Phi_{3i+j}(x,y) = \phi_i(x)\phi_j(y), \quad 0 \le i,j \le 2,$$

where $\phi_i(x)$ is the 1D quadratic Lagrange basis function.