Finite Elements: examples 3

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1. Let L be a linear functional on a Hilbert space. Prove that L is continuous if and only if L is bounded.
   Solution: *If L is bounded, then there exists a constant $C > 0$ such that*
   $$|L(v)| \leq C\|v\|, \quad \forall v \in V.$$
   Then, for $\epsilon > 0$,
   $$\|L(u + \epsilon v) - L(u)\| = \epsilon\|L(v)\| \leq \epsilon C\|v\|,$$
   i.e. *L is continuous. If L is continuous, then there exists $\delta > 0$ such that*
   $$\|L(u)\| = \|L(u) - L(0)\| \leq 1, \quad \forall \|u\| < \delta.$$
   Then,
   $$\|L(u)\| = \left\|\frac{\|v\|}{\delta} L\left(\delta\frac{v}{\|v\|}\right)\right\| = \frac{\|v\|}{\delta}\left\|L\left(\delta\frac{v}{\|v\|}\right)\right\| \leq \frac{\|v\|}{\delta},$$
   so L is bounded with bounding constant $1/\delta$.

2. Consider the variational problem with bilinear form
   $$a(u, v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x.$$
   Prove that $a(\cdot, \cdot)$ is continuous and coercive on $H^1([0, 1])$.
   Solution: *For continuity,*
   $$\begin{aligned}|a(u, v)| &\leq |(u, v)_{H^1}| + \left|\int_0^1 u'v\,\mathrm{d}\,x\right| \\ &\leq \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L_2}, \\ &\leq 2\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$
   For coercivity,
   $$\begin{aligned}a(v, v) &= \int_0^1 (v')^2 + v'v + v^2\,\mathrm{d}\,x, \\ &= \frac{1}{2}\int_0^1 (v' + v)^2\,\mathrm{d}\,x + \frac{1}{2}\int_0^1 (v')^2 + v^2\,\mathrm{d}\,x, \\ &\geq \frac{1}{2}\|v\|_{H^1}.\end{aligned}$$

3. For the differential equation $-u'' + ku' + u = f$, find a value for k such that $a(v, v) = 0$ but $v \neq 0$ for some $v \in H^1([0, 1])$.
   Solution: *The bilinear form is*
   $$a(u, v) = \int_0^1 u'v' + u'v + uv\,\mathrm{d}\,x.$$
   We need to find a function for which the $\int kv'v\,\mathrm{d}\,x$ is sufficiently negative to cancel out the other two positive terms. For example, if $v = (x - 1)$, then $v' = 1$, and
   $$a(v, v) = \int_0^1 1 + k(1 - x) + (1 - x)^2\,\mathrm{d}\,x = 1 - k/2 + 1/3 = 0,$$
   if $k = 2 + 2/3$.

4. Let $a(\cdot,\cdot)$ be the inner product for a Hilbert space V. For $F \in V'$, and U an arbitrary (closed) subspace U of V, show that the following two statements are equivalent:

   (a) $u \in U$ satisfies $a(u,v) = F(v)\ \forall v \in U$.

   (b) u minimises $\frac{1}{2}a(v,v) - F(v)$ over $v \in U$.

.

Solution: *Assume that u solves problem (a) above. Define*

$$J(v) = \frac{1}{2}a(v,v) - l(v).$$

Then

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}a(v,v) - l(v) - \frac{1}{2}a(u,u) + l(u), \\ &= \frac{1}{2}a(v,v) - \frac{1}{2}a(u,u) - l(v-u), \end{aligned}$$

using linearity of l. Since u is a solution, we have $l(v-u) = a(u, v-u)$, and hence

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}a(v,v) - \frac{1}{2}a(u,u) - a(u,v-u), \\ &= \frac{1}{2}\left(a(v,v) - 2a(u,v) + a(u,u)\right), \\ &= \frac{1}{2}a(v-u, v-u), \\ &= \|v-u\|_V^2 \geq 0, \end{aligned}$$

using the bilinearity of $a(\cdot,\cdot)$ and the definition of $\|\cdot\|_V$ from $a(\cdot,\cdot)$. Hence, $J(v) \geq J(u)$ for all $v \in V$. This means that u is a minimiser of J over V. To check that u is the unique minimiser, assume the converse, so that $\tilde{u}$ also minimises J. Then, $J(v) \geq J(\tilde{u})$ for all $v \in V$. In particular, $J(u) \geq J(\tilde{u})$. Since u is a minimiser, we also have that $J(\tilde{u}) \geq J(u)$. Hence, $J(u) = J(v)$. From the calculation above,

$$0 = J(u) - J(\tilde{u}) = \frac{1}{2}\|u - \tilde{u}\|_H \implies u = \tilde{u}.$$

On the other hand, let u be a minimiser of problem (b). Then,

$$\lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon} = 0, \quad \forall v \in V.$$

We have

$$\begin{aligned} J(u+\epsilon v) &= \frac{1}{2}a(u+\epsilon v, u+\epsilon v) - l(u+\epsilon v), \\ &= \frac{1}{2}\big(a(u,u) + 2\epsilon a(u,v) + \epsilon^2 a(v,v)\big) - l(u) - \epsilon l(v), \\ &= \frac{1}{2}a(u,u) - l(u) + \epsilon\left(a(u,v) - l(v)\right) + \epsilon^2 a(v,v), \end{aligned}$$

so

$$\lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon} = a(u,v) - l(v), \quad \forall v \in V.$$

Hence, u solves problem (a).

5. Let

$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x,$$

with

$$V = \{v \in H^1([0,1]) : v(0) = v(1) = 0\}.$$

Prove that

$$a(v,v) = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}\,x := \|v\|_{H^1}^2, \quad \forall v \in V.$$

Solution: *$v'v = (v^2)'/2$. So,*

$$\int_0^1 v'v \,\mathrm{d}\,x = \int_0^1 \frac{1}{2}(v^2)' \,\mathrm{d}\,x = \left[\frac{1}{2}v^2\right]_0^1 = 0,$$

by the boundary conditions. Then,

$$a(v,v) = \int_0^1 \left((v')^2 + v'v + v^2\right) \mathrm{d}\,x = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}\,x.$$

6. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a variational formulation of the problem

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma(x)\frac{\partial u}{\partial x_i}\right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Solution: *Multiply by test function v and integrate by parts, to obtain*

$$\begin{aligned}\int_\Omega \nabla v \cdot (\sigma \nabla u) \,\mathrm{d}\,x &= \int_\Omega vf \,\mathrm{d}\,x + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{=0} \sigma v \,\mathrm{d}\,x, \\ &= \int_\Omega vf \,\mathrm{d}\,x.\end{aligned}$$

Hence, the problem becomes to find $u \in H^1(\Omega)$, such that

$$a(u,v) = (f,v), \quad \forall v \in H^1(\Omega),$$

where

$$a(u,v) = \int_\Omega \nabla v \cdot (\sigma \nabla u) \,\mathrm{d}\,x.$$

(b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show that a finite element discretisation of this problem based on Lagrange elements has a unique solution, and give the rate of convergence to zero with h of the H^1 norm of the error.

Solution: *We need to check coercivity and continuity in $H^1(\Omega)$. For continuity,*

$$\begin{aligned}|a(u,v)| &= \left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\sigma\frac{\partial v}{\partial x_i} \,\mathrm{d}\,x\right|, \\ &\le b\left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \,\mathrm{d}\,x\right|, \\ &\le b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^2}, \\ &\le b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\sum_j \left\|\frac{\partial v}{\partial x_j}\right\|_{L^2}, \\ &\le b\left(\|u\|_{L^2}^2 + \sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2\right)^{1/2}\left(\|v\|_{L^2}^2 + \sum_j \left\|\frac{\partial v}{\partial x_j}\right\|_{L^2}^2\right)^{1/2}, \\ &= b\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

For coercivity (use result from lectures),

$$
\begin{aligned}
\|v\|^2_{H^1\Omega} &\leq \left(1 + C_\Omega^2\right) |v|^2_{H^1(\Omega)}, \\
&= \left(1 + C_\Omega^2\right) \int_\Omega \nabla v \cdot \nabla v \, \mathrm{d}\, x, \\
&= \left(1 + C_\Omega^2\right) \int_\Omega \frac{1}{\sigma} \sigma \nabla v \cdot \nabla v \, \mathrm{d}\, x, \\
&\leq \left(1 + C_\Omega^2\right) \frac{1}{a} \int_\Omega \sigma \nabla v \cdot \nabla v \, \mathrm{d}\, x = \frac{1}{a} \left(1 + C_\Omega^2\right) a(v, v).
\end{aligned}
$$

Hence, $a(\cdot, \cdot)$ is a symmetric bilinear form that is continous and coercive on $H^1(\Omega)$, and hence a unique solution exists. Further, the degree-p Lagrange elements is a subspace of $H^1(\Omega)$ and hence a unique finite element approximation solution exists as well. From Céa's Lemma, we have

$$
\begin{aligned}
\|u - u_h\|^2_{H^1} &\leq \frac{ba}{1 + C_\Omega^2} \min_{v \in V_h} \|v - u\|_{H^1} \\
&\leq \frac{ba}{1 + C_\Omega^2} \|u - \mathcal{I}_h u\|_{H^1} \\
&\leq \frac{ba}{1 + C_\Omega^2} c_0 h^p \|u\|_{H^2},
\end{aligned}
$$

using the approximation theory result from lectures.

7. Find a variational formulation for the Poisson equation

$$
-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,
$$

for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under which a finite element discretisation of this problem based on Lagrange elements has a unique solution.
Solution: *We write $u = u^H + u^g$, where $u^H = 0$ on $\partial\Omega$, and $u^g = g$ on $\partial\Omega$. Then,*

$$
a(u^H, v) = (f, v) - a(g, v), \quad \forall v \in V,
$$

where

$$
a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\, x,
$$

and

$$
V = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.
$$

We have already checked coercivity and continuity of a in lectures, so we just need to check continuity of $L(v)$ given by

$$
L(v) = (f, v) - a(g, v).
$$

We have

$$
|L(v)| \leq \|f\|_{L^2} \|v\|_{L^2} + \|g\|_{H^1} \|v\|_{H^1} = \left(\|f\|_{L^2} + \|g\|_{H^1}\right) \|v\|_{H^1},
$$

so it is continuous as required.

8. Find a variational formulation for the Poisson equation

$$
-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,
$$

for a function r defined on $\partial\Omega$.
Solution: *Multiplication by test function and integration by parts gives*

$$
\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}\, x = \int_\Omega f v \, \mathrm{d}\, x.
$$

Application of the boundary condition gives

$$
\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x + \int_{\partial\Omega} u v \, \mathrm{d}\, x = \int_\Omega f v \, \mathrm{d}\, x + \int_{\partial\Omega} r v \, \mathrm{d}\, S.
$$

Hence, we obtain a variational formulation with

$$
\begin{aligned}
a(u,v) &= \int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\,x + \int_{\partial\Omega} uv \,\mathrm{d}\,x, \\
L(v) &= \int_\Omega fv \,\mathrm{d}\,x + \int_{\partial\Omega} rv \,\mathrm{d}\,S.
\end{aligned}
$$