Finite Elements: examples 2

Colin Cotter

April 12, 2017

1. Consider the variational problem with bilinear form

$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x,$$

corresponding to the differential equation

$$-u'' + u' + u = f.$$

Prove that $a(\cdot,\cdot)$ is continuous and coercive on a C^0 finite element space V defined on $[0,1]$, equipped with the H^1 inner product.
Solution: *For continuity,*

$$\begin{aligned}|a(u,v)| &\leq |(u,v)_{H^1}| + \left|\int_0^1 u'v\,\mathrm{d}\,x\right| \\ &\leq \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L_2}, \\ &\leq 2\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

For coercivity,

$$\begin{aligned}a(v,v) &= \int_0^1 (v')^2 + v'v + v^2\,\mathrm{d}\,x, \\ &= \frac{1}{2}\int_0^1 (v'+v)^2\,\mathrm{d}\,x + \frac{1}{2}\int_0^1 (v')^2 + v^2\,\mathrm{d}\,x, \\ &\geq \frac{1}{2}\|v\|_{H^1}.\end{aligned}$$

2. For the differential equation $-u'' + ku' + u = f$, formulate a C^0 finite element discretisation with bilinear form $a(u,v)$. Find a value for k such that $a(v,v) = 0$ but $v \neq 0$ for some $v \in V$.
Solution: *The bilinear form is*

$$a(u,v) = \int_0^1 u'v' + ku'v + uv\,\mathrm{d}\,x.$$

We need to find a function for which the $\int kv'v\,\mathrm{d}\,x$ is sufficiently negative to cancel out the other two positive terms. For example, if $v = (x-1)$, which is in all C^0 finite element spaces on $[0,1]$, then $v' = 1$, and

$$a(v,v) = \int_0^1 1 + k(x-1) + (1-x)^2\,\mathrm{d}\,x = 1 - k/2 + 1/3 = 0,$$

if $k = 2 + 2/3$.

3. Let $a(\cdot,\cdot)$ be the inner product for a finite element space U. For $F \in U'$, show that the following two statements are equivalent:

   (a) $u \in U$ satisfies $a(u,v) = F(v)\ \forall v \in U$.

   (b) u uniquely minimises $\frac{1}{2}a(v,v) - F(v)$ over $v \in U$.

Use this to conclude that the Poisson and Helmholtz finite element discretisations can be formulated as minimisation problems. .

Solution: *Assume that u solves problem (a) above. Define*

$$J(v) = \frac{1}{2}a(v,v) - l(v).$$

Then

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}a(v,v) - l(v) - \frac{1}{2}a(u,u) + l(u), \\ &= \frac{1}{2}a(v,v) - \frac{1}{2}a(u,u) - l(v-u), \end{aligned}$$

using linearity of l. Since u is a solution, we have $l(v-u) = a(u, v-u)$, and hence

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}a(v,v) - \frac{1}{2}a(u,u) - a(u,v-u), \\ &= \frac{1}{2}\left(a(v,v) - 2a(u,v) + a(u,u)\right), \\ &= \frac{1}{2}a(v-u,v-u), \\ &= \|v-u\|_U^2 \geq 0, \end{aligned}$$

using the bilinearity of $a(\cdot,\cdot)$ and the definition of $\|\cdot\|_U$ from $a(\cdot,\cdot)$. Hence, $J(v) \geq J(u)$ for all $v \in U$. This means that u is a minimiser of J over U. To check that u is the unique minimiser, assume the converse, so that $\tilde{u}$ also minimises J. Then, $J(v) \geq J(\tilde{u})$ for all $v \in U$. In particular, $J(u) \geq J(\tilde{u})$. Since u is a minimiser, we also have that $J(\tilde{u}) \geq J(u)$. Hence, $J(u) = J(v)$. From the calculation above,

$$0 = J(u) - J(\tilde{u}) = \frac{1}{2}\|u - \tilde{u}\|_H \implies u = \tilde{u}.$$

On the other hand, let u be a minimiser of problem (b). Then,

$$\lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon} = 0, \quad \forall v \in U.$$

We have

$$\begin{aligned} J(u+\epsilon v) &= \frac{1}{2}a(u+\epsilon v, u+\epsilon v) - l(u+\epsilon v), \\ &= \frac{1}{2}\big(a(u,u) + 2\epsilon a(u,v) + \epsilon^2 a(v,v)\big) - l(u) - \epsilon l(v), \\ &= \frac{1}{2}a(u,u) - l(u) + \epsilon\left(a(u,v) - l(v)\right) + \epsilon^2 a(v,v), \end{aligned}$$

so

$$\lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon} = a(u,v) - l(v), \quad \forall v \in U.$$

Hence, u solves problem (a).

4. Let

$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}\,x.$$

Let V be a C^0 finite element space on $[0,1]$ and let $\mathring{V}$ be the subspace of functions that vanish at $x=0$ and $x=1$. Prove that

$$a(v,v) = \int_0^1 \left((v')^2 + v^2\right)\mathrm{d}\,x := \|v\|_{H^1}^2, \quad \forall v \in \mathring{V}.$$

Hence conclude that the bilinear form is coercive on $\mathring{V}$.

Solution: *First we note that the fundamental theorem of calculus holds for C^0 finite element functions, by subdividing the integral into cells or partial cells as usual. Second we note that $v'v = (v^2)'/2$. So,*

$$\int_0^1 v'v\,\mathrm{d}\,x = \int_0^1 \frac{1}{2}(v^2)'\,\mathrm{d}\,x = \left[\frac{1}{2}v^2\right]_0^1 = 0,$$

by the boundary conditions. Then,

$$a(v,v) = \int_0^1 \left((v')^2 + v'v + v^2\right) \mathrm{d}\, x = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}\, x.$$

5. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a finite element formulation of the problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sigma(x)\frac{\partial u}{\partial x_i}\right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Solution: *Let V be a C^0 finite element space defined on Ω. Multiply by test function $v \in V$ and integrate by parts, to obtain*

$$\begin{aligned}\int_\Omega \nabla v \cdot (\sigma \nabla u)\, \mathrm{d}\, x &= \int_\Omega v f \,\mathrm{d}\, x + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{=0} \sigma v \,\mathrm{d}\, x, \\ &= \int_\Omega v f \,\mathrm{d}\, x.\end{aligned}$$

Hence, the finite element discretisation requires to find $u_h \in V$, such that

$$a(u_h, v) = (f, v), \quad \forall v \in V,$$

where

$$a(u,v) = \int_\Omega \nabla v \cdot (\sigma \nabla u)\, \mathrm{d}\, x.$$

(b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show continuity and coercive for your formulation with respect to the H^1 norm.
Solution: *For continuity,*

$$\begin{aligned}|a(u,v)| &= \left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\sigma\frac{\partial v}{\partial x_i}\,\mathrm{d}\, x\right|, \\ &\le b\left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}\, x\right|, \\ &\le b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^2}, \\ &\le b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\sum_j\left\|\frac{\partial v}{\partial x_j}\right\|_{L^2}, \\ &\le b\left(\|u\|_{L^2}^2 + \sum_i\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2\right)^{1/2}\left(\|v\|_{L^2}^2 + \sum_j\left\|\frac{\partial v}{\partial x_j}\right\|_{L^2}^2\right)^{1/2}, \\ &= b\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

For coercivity (use result from lectures),

$$\begin{aligned}\|v\|_{H^1\Omega}^2 &\le \left(1 + C_\Omega^2\right)|v|_{H^1(\Omega)}^2, \\ &= \left(1 + C_\Omega^2\right)\int_\Omega \nabla v\cdot\nabla v\,\mathrm{d}\, x, \\ &= \left(1 + C_\Omega^2\right)\int_\Omega \frac{1}{\sigma}\sigma\nabla v\cdot\nabla v\,\mathrm{d}\, x, \\ &\le \left(1 + C_\Omega^2\right)\frac{1}{a}\int_\Omega \sigma\nabla v\cdot\nabla v\,\mathrm{d}\, x = \frac{1}{a}\left(1 + C_\Omega^2\right)a(v,v).\end{aligned}$$

Hence, $a(\cdot,\cdot)$ is a symmetric bilinear form that is continous and coercive on $H^1(\Omega)$, and hence a unique solution exists.

6. Find a C^0 finite element formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$

for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under the discretisation has a unique solution.

Solution: *Choosing a C^0 finite element space V, we define $\mathring{V}$ as the subspace of functions vanishing on $\partial\Omega$ as above. We write $u = u^H + u^g$, where $u^H \in \mathring{V}$, and $u^g = g$ on $\partial\Omega$. Then,*

$$a(u^H, v) = (f, v) - a(g, v), \quad \forall v \in \mathring{V},$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\, x.$$

We have already checked coercivity and continuity of a in lectures, so we just need to check continuity of $L(v)$ given by

$$L(v) = (f, v) - a(g, v).$$

We have

$$|L(v)| \leq \|f\|_{L^2}\|v\|_{L^2} + \|g\|_{H^1}\|v\|_{H^1} = (\|f\|_{L^2} + \|g\|_{H^1})\,\|v\|_{H^1},$$

so it is continuous as required.

7. Find a C^0 finite element formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,$$

for a function r defined on $\partial\Omega$. Derive conditions under which this discretisation has a unique solution.

Solution: *Multiplication by test function and integration by parts gives*

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}\, x = \int_\Omega f v \, \mathrm{d}\, x.$$

Application of the boundary condition gives

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x + \int_{\partial\Omega} u v \, \mathrm{d}\, x = \int_\Omega f v \, \mathrm{d}\, x + \int_{\partial\Omega} r v \, \mathrm{d}\, S.$$

Hence, we obtain a variational formulation with

$$a(u, v) = \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\, x + \int_{\partial\Omega} u v \, \mathrm{d}\, S,$$
$$L(v) = \int_\Omega f v \, \mathrm{d}\, x + \int_{\partial\Omega} r v \, \mathrm{d}\, S.$$

Continuity follows from

$$\begin{aligned}\int_{\partial\Omega} u v \, \mathrm{d}\, S &\leq \|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \\ &\leq c^2 \|u\|_{H^1}\|v\|_{H^1},\end{aligned}$$

where $c > 0$ is the constant in the trace theorem. Hence,

$$a(u, v) \leq (1 + c^2)\|u\|_{H^1}\|v\|_{H^1}.$$

Continuity of $L(v)$ follows similarly.

To check coercivity, we initially follow the proof for Poisson with Dirichlet boundary conditions

$$\begin{aligned}\|v\|^2_{H^1(\Omega)} &\leq \|v - \bar{v}\|_{L^2(\Omega)} + \|\bar{v}\|_{L^2(\Omega)}, \\ &\leq C|v|_{H^1(\Omega)} + C_1(|\int_{\partial\Omega} v \, \mathrm{d}\, S| + |\int_{\partial\Omega} v - \bar{v} \, \mathrm{d}\, S|), \\ &\leq C_2|v|_{H^1(\Omega)} + C_3|v|_{L^2(\partial\Omega)},\end{aligned}$$

so there exists $C_4 > 0$ with

$$\|v\|^2_{H^1(\Omega)} \leq C_4 a(v, v),$$

as required.