

Finite Elements: examples 2

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1. Consider the variational problem with bilinear form

$$a(u, v) = \int_0^1 (u'v' + u'v + uv) \, dx,$$

corresponding to the differential equation

$$-u'' + u' + u = f.$$

Prove that $a(\cdot, \cdot)$ is continuous and coercive on a C^0 finite element space V defined on $[0, 1]$, equipped with the H^1 inner product.

2. For the differential equation $-u'' + ku' + u = f$, formulate a C^0 finite element discretisation with bilinear form $a(u, v)$. Find a value for k such that $a(v, v) = 0$ but $v \neq 0$ for some $v \in V$.
3. Let $a(\cdot, \cdot)$ be the inner product for a finite element space U . For $F \in U'$, show that the following two statements are equivalent:
 - (a) $u \in U$ satisfies $a(u, v) = F(v) \, \forall v \in U$.
 - (b) u uniquely minimises $\frac{1}{2}a(v, v) - F(v)$ over $v \in U$.

Use this to conclude that the Poisson and Helmholtz finite element discretisations can be formulated as minimisation problems. .

4. Let

$$a(u, v) = \int_0^1 (u'v' + u'v + uv) \, dx.$$

Let V be a C^0 finite element space on $[0, 1]$ and let \mathring{V} be the subspace of functions that vanish at $x = 0$ and $x = 1$. Prove that

$$a(v, v) = \int_0^1 ((v')^2 + v^2) \, dx := \|v\|_{H^1}^2, \quad \forall v \in \mathring{V}.$$

Hence conclude that the bilinear form is coercive on \mathring{V} .

5. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a finite element formulation of the problem

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sigma(x) \frac{\partial u}{\partial x_i} \right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

- (b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show continuity and coercive for your formulation with respect to the H^1 norm.

6. Find a C^0 finite element formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$

for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under the discretisation has a unique solution.

7. Find a C^0 finite element formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,$$

for a function r defined on $\partial\Omega$. Derive conditions under which this discretisation has a unique solution.