Finite Elements: examples 1

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1. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise linear finite element space known as P1. A finite element discretisation of the 1D Poisson equation

$$-\frac{\partial^2 p}{\partial x^2} = f,$$

defines the numerical solution $u \in P1$ such that

$$a(u, v) = F[v], \quad \forall v \in P1,$$

where

$$a(u, v) = \int_0^1 u'v' \,\mathrm{d}\,x, \quad F[v] = \int_0^1 fv \,\mathrm{d}\,x.$$

Compute the entries of the matrix

$$K_{ij} = a(\phi_i, \phi_j),$$

and the right-hand side vector

$$F_i = (\phi_i, f_I),$$

where $f_I \in S$ is the interpolant of f.
For an equispaced mesh with $h_i = x_i - x_{i-1} := h$, write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

2. Using the methodology of the introductory lecture, develpo a finite element discretisation for the following ODEs,

   (a)
$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$

   (b)
$$-u'' + u = f, \quad u'(0) = 0,\, u'(1) = \alpha.$$

   (c)
$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

3. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise quadratic finite element space known as P2, defined by the following:

   (a) $S \subset C^0([0, 1])$.
   (b) For $v \in S$, $v|_{[x_{j-1},x_j]}$ is a quadratic function of x.
   (c) $v(0) = 0$.

   Find a nodal basis for S, using the nodes for P1 plus nodes at element midpoints $(x_j + x_{j+1})/2$. Evaluate the matrix K for this basis.

4. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.

5. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.

6. Let K be a rectangle, Q_2 be the space of biquadratic polynomials, and let $\mathcal{N}$ be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that $\mathcal{N}$ determines the finite element.

7. For K being the unit square, determine the nodal basis for the element in the previous question.