Finite Elements: examples 2

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1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation $\mathcal{T}$ of a domain Ω. Show that functions in V do not have weak derivatives in general.

2. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\, x \, \mathrm{d}\, y = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0) \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta,$$

where $\bar{u} = u \circ g$, ξ and η are the coordinates on K, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ.

3. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\, x \, \mathrm{d}\, y \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta.$$

Hence show that

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\, x \, \mathrm{d}\, y \leq Ch^2 \int_\Delta \left| \frac{\partial^2 u}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta.$$

4. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_{\mathcal{T}}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \mathrm{d}\, x \, \mathrm{d}\, y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial xy} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 \mathrm{d}\, x \, \mathrm{d}\, y.$$

5. Show that

$$D^\beta Q_B^k f = Q_B^{k-|\beta|} D^\beta f,$$

where Q_B^l is the degree l averaged Taylor polynomial of f, and D^β is the β-th derivative where β is a multi-index.