Finite Elements

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March 19, 2018

- Consider a triangulation $\mathcal{T}$ with recursively refined triangulations $\mathcal{T}_h$ and corresponding finite element spaces V_h.
- Given stable finite element variational problems, we have a sequence of solutions u_h as $h \to 0$, satisfying the h-independent bound

$$\|u_h\|_{H^1(\Omega)} \leq C.$$

  What are these solutions converging to?
- We need to find a Hilbert space that contains all V_h as $h \to 0$, that extends the H^1 norm to the closure of finite element functions.

Our first task is to define a derivative that works for all finite element functions, without reference to a mesh. This requires some preliminary definitions.

Definition 1 (Compact support on Ω)

A function u has compact support on Ω if there exists $\epsilon > 0$ such that $u(x) = 0$ when $\min_{y\in\partial\Omega} |x - y| < \epsilon$.

Definition 2 ($C_0^\infty(\Omega)$)

We denote by $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$ corresponding to functions that have compact support on Ω.

First we will define a space containing the generalised derivative.

Definition 3 (L^1_{loc})

For triangles $K \subset \mathrm{int}(\Omega)$, we define

$$\|u\|_{L^1(K)} = \int_K |u| \,\mathrm{d}x,$$

and

$$L^1_K = \left\{ u : \|u\|_{L^1(K)} < \infty \right\}.$$

Then

$$L^1_{loc} = \left\{ f : f \in L^1(K) \quad \forall K \subset \mathrm{int}(\Omega) \right\}.$$

Definition 4 (Weak derivative)

The weak derivative $D_w^\alpha f \in L^1_{loc}(\Omega)$ of a function $f \in L^1_{loc}(\Omega)$ is defined by

$$\int_\Omega \phi D_w^\alpha f \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega D^\alpha \phi f \,\mathrm{d}x, \quad \forall \phi \in C_0^\infty(\Omega).$$

Proposition 5

Let V be a C^0 finite element space. Then, for $u \in V$, the finite element (first) derivative of u is equal to the weak (first) derivative of u.

Similar results hold for C^1 finite element space with second derivatives and so on.

Proof.

Taking any $\phi \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
\int_\Omega \phi \frac{\partial}{\partial x_i}|_{FE} u \,\mathrm{d}x &= \sum_K \int_K \phi \frac{\partial u}{\partial x_i} \,\mathrm{d}x, \\
&= \sum_K \left(-\int_K \frac{\partial \phi}{\partial x_i} u \,\mathrm{d}x + \int_{\partial K} \phi n_i u \,\mathrm{d}S \right), \\
&= -\sum_K \int_K \frac{\partial \phi}{\partial x_i} u \,\mathrm{d}x = -\int_\Omega \frac{\partial \phi}{\partial x_i} u \,\mathrm{d}x,
\end{aligned}
$$

as required. □

Proposition 6

Then, for $u \in C^{|\alpha|}(\Omega)$, the usual "strong" derivative D^α of u is equal to the weak derivative D_w^α of u.

Proof.

Exercise. [very similar to previous proof] $\square$

Due to these equivalences, we do not need to distinguish between strong, weak and finite element derivatives. All derivatives are assumed to be weak from now on.

We are now in a position to define a space that contains all C^0 finite element spaces.

Definition 7 (Sobolev spaces)

$H^1(\Omega)$ is the function space defined by

$$H^1(\Omega) = \left\{ u \in L^1_{loc} : \|u\|_{H^1(\Omega)} < \infty \right\}.$$

More generally, $H^k(\Omega)$ is the function space defined by

$$H^k(\Omega) = \left\{ u \in L^1_{loc} : \|u\|_{H^k(\Omega)} < \infty \right\}.$$

We conventionally write $H^0 = L^2$.

Proposition 8 (H^k spaces are Hilbert spaces)

The space $H^k(\Omega)$ is closed.

Proof

Let $\{u_i\}$ be a Cauchy sequence in H^k. Then $\{D^\alpha u_i\}$ is a Cauchy sequence in $L^2(\Omega)$ (which is closed), so $\exists v^\alpha \in L^2(\Omega)$ such that $D^\alpha u_i \to v^\alpha$ for $|\alpha| \leq k$. If $w_j \to w$ in $L^2(\Omega)$, then for $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega (w_j - w)\phi \, \mathrm{d}x \leq \|w_j - w\|_{L^2(\Omega)} \|\phi\|_{L^\infty} \to 0. \qquad (1)$$

Proof (Cont.)

We use (1) to get

$$\int_\Omega v^\alpha \phi \, \mathrm{d}x = \lim_{i\to\infty} \int_\Omega \phi D^\alpha u_i \, \mathrm{d}x,$$
$$= \lim_{i\to\infty} (-1)^{|\alpha|} \int_\Omega u_i D^\alpha \phi \, \mathrm{d}x,$$
$$= (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, \mathrm{d}x,$$

i.e. v^α is the weak derivative of u as required. □

We quote the following much deeper results without proof.

Theorem 9 ($H = W$)

Let Ω be any open set. Then $H^k(\Omega) \cap C^\infty(\Omega)$ is dense in $H^k(\Omega)$.

The interpretation is that for any function $u \in H^k(\Omega)$, we can find a sequence of C^∞ functions u_i converging to u. This is very useful as we can compute many things using C^∞ functions and take the limit.

Theorem 10 (Sobolev's inequality)

Let Ω be a d-dimensional domain with Lipschitz boundary, let k be an integer with $k > d/2$. Then there exists a constant C such that

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \leq C\|u\|_{H^k(\Omega)}.$$

Further, there is a continuous function in the $L^\infty(\Omega)$ equivalence class of u.

The interpretation is that if $u \in H^k$ then there is a continuous function u_0 such that the set of points where $u \neq u_0$ has zero area/volume.

We can now consider linear variational problems defined on H^k spaces, by taking a bilinear form $b(u, v)$ and linear form $F(v)$, seeking $u \in H^k$ (for chosen H^k) such that

$$b(u, v) = F(v), \quad \forall v \in H^k.$$

Since H^k is a Hilbert space, the Lax-Milgram theorem the existence of a unique solution to an H^k linear variational problem.

Proposition 11 (Well-posedness for (modified) Helmholtz))

The Helmholtz variational problem on H^1 satisfies the conditions of the Lax-Milgram theorem.

Proof.

The proof for C^0 finite element spaces extends immediately to H^1. $\square$

Functions in H^k make boundary conditions hard to interpret since they are not guaranteed to have defined values on the boundary. We make the following definition.

Definition 12 (Trace of H^1 functions)

Let $u \in H^1(\Omega)$ and choose $u_i \in C^\infty(\Omega)$ such that $u_i \to u$. We define the trace $u|_{\partial\Omega}$ on $\partial\Omega$ as the limit of the restriction of u_i to $\partial\Omega$.

Proposition 13 (Trace theorem for H^1 functions)

Let $u \in H^1(\Omega)$ for a polygonal domain Ω. Then the trace $u|_{\partial\Omega}$ satisfies

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}. \tag{2}$$

Proof.

Adapt the proof for C^0 finite element functions, choosing $u \in C^\infty(\Omega)$, and pass to the limit in $H^1(\Omega)$. $\square$

Interpretation: if $u \in H^1(\Omega)$ then $u|_{\partial\Omega} \in L^2(\partial\Omega)$.

Proposition 14

Let $u \in H^2(\Omega)$, $v \in H^1(\Omega)$. Then

$$\int_\Omega (-\nabla^2 u) v \, \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}S.$$

Proof.

First note that $u \in H^2(\Omega) \implies \nabla u \in (H^1(\Omega))^d$. Then

$$\|v\nabla u\|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \|\nabla u\|_{H^1(\Omega)} \implies v\nabla u \in H^1(\Omega).$$

Then, take $v_i \in C^\infty(\Omega)$ and $u_i \in C^\infty(\Omega)$ converging to v and u, respectively, and $v_i \nabla u_i \in C^\infty(\Omega)$ converges to $v\nabla u$. These satisfy the equation; we obtain the result by passing to the limit. $\square$

Proposition 15

For $f \in L^2$, let $u \in H^2(\Omega)$ solve

$$u - \nabla^2 u = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

in the L^2 sense, i.e. $\|u - \nabla^2 u - f\|_{L^2} = 0$. Then u solves the variational form of the Helmholtz equation.

Proof.

$u \in H^2 \implies \|u\|_{H^2} < \infty \implies \|u\|_{H^1} < \infty \implies u \in H^1$.
Multiplying by test function $v \in H^1$, and using the previous proposition gives

$$\int_\Omega uv + \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x, \quad \forall v \in H^1(\Omega),$$

as required. $\square$

Proposition 16

Let $f \in L^2(\Omega)$ and suppose that $u \in H^2(\Omega)$ solves the variational Helmholtz equation on a polygonal domain Ω. Then u solves the strong form Helmholtz equation with zero Neumann boundary conditions.

Proof

Using integration by parts for $u \in H^2$, $v \in C_0^\infty(\Omega) \in H^1$, we have

$$\int_\Omega (u - \nabla^2 u - f)v \, \mathrm{d}x = \int_\Omega uv + \nabla u \cdot \nabla v - vf \, \mathrm{d}x = 0.$$

It is a standard result that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ and therefore we can choose a sequence of v converging to $u - \nabla^2 u - f$ and we obtain $\|u - \nabla^2 u - f\|_{L^2(\Omega)} = 0$.

Proof (Cont.)

On the other hand,

$$0 = \int_\Omega uv + \nabla u \cdot \nabla v - fv \, \mathrm{d}x = \int_\Omega uv + \nabla u \cdot \nabla v - (u - \nabla^2 u) v \, \mathrm{d}x$$
$$= \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}S.$$

We can find arbitrary $v \in L_2(\partial\Omega)$, hence $\|\frac{\partial u}{\partial n}\|_{L^2(\partial\Omega)} = 0$. $\square$

Definition 17 (Galerkin approximation)

For a finite element space $V_h \subset V = H^k(\Omega)$, the Galerkin approximation of the H^k variational problem above seeks to find $u_h \in V_h$ such that

$$b(u_h, v) = F(v), \quad \forall v \in V_h.$$

What is the size of the error $u - u_h$?

Theorem 18 (Céa)

Let $V_h \subset V$, and let u solve a linear variational problem on V, whilst u_h solves the equivalent Galerkin approximation on V_h. Then

$$\|u - u_h\|_V \leq \frac{M}{\gamma} \min_{v \in V_h} \|u - v\|_V,$$

where M and γ are the continuity and coercivity constants of $b(u, v)$, respectively.

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Proof

We have

$$b(u, v) = F(v) \quad \forall v \in V,$$
$$b(u_h, v) = F(v) \quad \forall v \in V_h.$$

Choosing $v \in V_h \subset V$ means we can use it in both equations, and subtraction and linearity lead to the "Galerkin orthogonality" condition

$$b(u - u_h, v) = 0, \quad \forall v \in V_h.$$

Proof (Cont.)

Then, for all $v \in V_h$,

$$
\begin{aligned}
\gamma\|u-u_h\|_V^2 &\leq b(u-u_h, u-u_h), \\
&= b(u-u_h, u-v) + \underbrace{b(u-u_h, v-u_h)}_{=0}, \\
&\leq M\|u-u_h\|_V\|u-v\|_V.
\end{aligned}
$$

Minimising over all v completes the proof. $\square$

$$\|u - u_h\|_V \leq \frac{M}{\gamma} \min_{v \in V_h} \|u - v\|_V.$$

- The interpretation is that the error is proportional to the minimal error in approximating u in V_h.
- We have an estimate for approximating u in V_h if $u \in C^2(\Omega)$. What if $u \in H^2(\Omega)$ instead?
- If $u \in H^2(\Omega)$, then u does not necessarily have a degree 2 Taylor polynomial, since derivatives are not defined at arbitrary points. We need a more general definition.

Definition 19 (Averaged Taylor polynomial)

Let $\Omega \subset \mathbb{R}^n$ be a domain with diameter d, that is star-shaped with respect to a ball B with radius ϵ, contained within Ω. For $f \in H^{k+1}(\Omega)$ the averaged Taylor polynomial $Q_{k,B}f \in \mathcal{P}_k$ is defined as

$$Q_{k,B}f(x) = \frac{1}{|B|}\int_B T^k f(y,x)\,\mathrm{d}y,$$

where $T^k f$ is the Taylor polynomial of degree k of f,

$$T^k f(y,x) = \sum_{|\alpha|\leq k} D^\alpha f(y)\frac{(x-y)^\alpha}{\alpha!},$$

evaluated using weak derivatives.

This definition makes sense since the Taylor polynomial coefficients are in $L^1_{loc}(\Omega)$ and thus their integrals over B are defined.

Proposition 20

Let $\Omega \subset \mathbb{R}^n$ be a domain with diameter d, that is star-shaped with respect to a ball B with radius ϵ, contained within Ω. There exists a constant $C(k,n)$ such that for $0 \leq |\beta| \leq k+1$,

$$\|D^\beta(f - Q_{k,B}f)\|_{L^2} \leq C\frac{|\Omega|^{1/2}}{|B|^{1/2}} d^{k+1-|\beta|}\|\nabla^{k+1}f\|_{L^2(\Omega)}.$$

Proof

Assume $f \in C^\infty(\Omega)$, pass to the limit. Remainder theorem gives

$$f(x) - T_k f(y,x) =$$
$$(k+1)\sum_{|\alpha|=k+1} \frac{(x-y)^\alpha}{\alpha!}\int_0^1 D^\alpha f(ty+(1-t)x)t^k \,\mathrm{d}t.$$

Integration over y in B and dividing by $|B|$ gives

$$f(x) - Q_{k,B}f(x) = \frac{k+1}{|B|}\sum_{|\alpha|=k+1}\int_B \frac{(x-y)^\alpha}{\alpha!}\times$$
$$\int_0^1 D^\alpha f(ty+(1-t)x)t^k \,\mathrm{d}t\,\mathrm{d}y.$$

Proof (Cont.)

Then

$$\int_\Omega |f(x) - Q_{k,B}f(x)|^2 \,\mathrm{d}x$$

$$\leq C\frac{d^{2(k+1)}}{|B|^2}\sum_{|\alpha|=k+1}\int_\Omega\left(\int_B\int_0^1 |D^\alpha f(ty+(1-t)x)|t^k \,\mathrm{d}t\,\mathrm{d}y\right)^2 \mathrm{d}x,$$

$$\leq C_0\frac{d^{2(k+1)}}{|B|^2}\sum_{|\alpha|=k+1}\int_\Omega\int_B\int_0^1 |D^\alpha f(ty+(1-t)x)|^2 \,\mathrm{d}t\,\mathrm{d}y$$

$$\times\int_B\int_0^1 t^{2k}\,\mathrm{d}t\,\mathrm{d}y\ \mathrm{d}x.$$

Proof (Cont.)

Then

$$\int_\Omega |f(x) - Q_{k,B}f(x)|^2 \,\mathrm{d}x$$
$$\leq C_1 \frac{d^{2(k+1)}}{|B|^2} \sum_{|\alpha|=k+1} \int_\Omega \int_B \int_0^1 |D^\alpha f(ty + (1-t)x)|^2 \,\mathrm{d}t \,\mathrm{d}y \,\mathrm{d}x.$$

For each α term we can split the t integral into $[0, 1/2]$ and $[1/2, 1]$. Call these terms I and II.

Proof (Cont.)

Denote by g_α the extension by zero of $D^\alpha f$ to $\mathbb{R}^n$. Then

$$\begin{aligned}
I =& \int_B \int_0^{1/2} \int_{\mathbb{R}^n} |g_\alpha(ty + (1-t)x)|^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y, \\
=& \int_B \int_0^{1/2} \int_{\mathbb{R}^n} |g_\alpha((1-t)x)|^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y, \\
=& \int_B \int_0^{1/2} \int_{\mathbb{R}^n} |g_\alpha(z)|^2 (1-t)^{-n} \, \mathrm{d}z \, \mathrm{d}t \, \mathrm{d}y, \\
\leq& 2^{n-1}|B| \int_\Omega |D^\alpha f(z)|^2 \, \mathrm{d}z.
\end{aligned}$$

Proof (Cont.)

$$
\begin{aligned}
II &= \int_B \int_{1/2}^1 \int_{\mathbb{R}^n} |g_\alpha(ty + (1-t)x)|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y, \\
&= \int_B \int_{1/2}^1 \int_{\mathbb{R}^n} |g_\alpha(ty)|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y, \\
&= \int_B \int_{1/2}^1 \int_{\mathbb{R}^n} |g_\alpha(z)|^2 t^{-n} \,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}y, \\
&\leq 2^{n-1}|B| \int_\Omega |D^\alpha f(z)|^2 \,\mathrm{d}z.
\end{aligned}
$$

Proof (Cont.)

We obtain the required bounds for $|\beta| = 0$. For higher derivatives we use the fact that

$$D^\beta Q_{k,B} f(x) = Q_{k-|\beta|,B} D^\beta f(x),$$

which immediately leads to the estimate for $|\beta| > 0$. $\square$

Proposition 21

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element such that K has diameter d, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$, and $\mathcal{P}$ contain all polynomials of degree $m-1$ and below, with $l < m$. Let $u \in H^m$. Then for $m > k$, the local interpolation operator satisfies

$$\|\mathcal{I}_K u - u\|_{H^k(K)} \leq C d^{m-k} |u|_{H^m(K)}, \tag{3}$$

where C depends only on the shape of K (but not the diameter).

Now we will use the Taylor polynomial estimates to derive error estimates for the local interpolation operator. First we need to obtain the following bound.

Proposition 22

Let $(K_1, \mathcal{P}, \mathcal{N})$ be a finite element such that K_1 be a triangle with diameter 1, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$. Let $u \in H^k(K_1)$ with $l < k$. Then

$$\|\mathcal{I}_{K_1} u\|_{H^k(K_1)} \leq C\|u\|_{H^k(K_1)}. \tag{4}$$

Proof.

We will sketch for the case $l = 0$. Assuming $u \in C^\infty$, we have $\int_{K_1} (\mathcal{I}_{K_1} u)^2 \,\mathrm{d}x = \sum_{ij} \int_K \phi_i \phi_j \,\mathrm{d}x N_i(u) N_j(u)$.
Then writing
$u(x_i) = u(x) + \int_0^1 (x - x_i) \cdot \nabla u(x_i + s(x - x_i)) \,\mathrm{d}s$,
we get

$$|K_1| u(x_i)^2 = \int_{K_1} \left(u(x) + \int_0^1 (x - x_i) \cdot \nabla u(x_i + s(x - x_i)) \,\mathrm{d}s \right)^2 \mathrm{d}x,$$

$$\leq 2 \int_{K_1} u(x)^2 + \left(\int_0^1 (x - x_i) \cdot \nabla u(x_i + s(x - x_i)) \,\mathrm{d}s \right)^2 \mathrm{d}x,$$

and we eventually (following similar methods to above) get
$u(x_i)^2 \leq C_1 \int_{K_1} u(x)^2 + |\nabla u|^2 \,\mathrm{d}x$,
and hence the result after passing to the H^k limit. $\square$

Corollary 23

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$. Let $u \in C^k$ with $l < k$. Then

$$\|\mathcal{I}_K u\|_{H^k(K)} \leq C\|u\|_{H^k(K)}, \tag{5}$$

where C is a constant that depends only on the shape of K and not the diameter.

Proof.

Use a homogeneity argument (change variables to a unit reference element). □

Proposition 24

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element such that K has diameter d, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$, and $\mathcal{P}$ contain all polynomials of degree $m-1$ and below, with $l < m$. Let $u \in H^m$. Then for $m > k$, the local interpolation operator satisfies

$$\|\mathcal{I}_K u - u\|_{H^k(K)} \leq Cd^{m-k}|u|_{H^m(K)}, \tag{6}$$

where C depends only on the shape of K (but not the diameter).

Proof.

Picking y in K, we have

$$
\begin{aligned}
\|\mathcal{I}_K u - u\|^2_{H^k(K)} &= \|\mathcal{I}_K u - T^m_y u + T^m_y u - u\|^2_{H^k(K)} \\
&\leq \|T^m_y u - u\|^2_{H^k(K)} + \|\mathcal{I}(u - T^m_y u)\|^2_{H^k(K)}, \\
&\leq (1 + C)\|T^m_y u - u\|^2_{H^k(K)} \\
&\leq (1 + C)d^{2(m-k)}|u|^2_{H^m(K)},
\end{aligned}
$$

as required. □

We now consider norms for the error over the entire mesh.

Proposition 25

Let $\mathcal{T}$ be a triangulation with finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$, such that the minimum aspect ratio r of the triangles K_i satisfies $r > 0$, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$, and $\mathcal{P}$ contain all polynomials of degree $m-1$ and below, with $l < m$. Let $u \in H^m(\Omega)$. Let h be the maximum over all of the triangle diameters. Let V be the corresponding C^n finite element space. Then for $m > k$ and $k \leq n+1$, the global interpolation operator satisfies

$$\|\mathcal{I}_h u - u\|_{H^k(\Omega)} \leq Ch^{m-k}|u|_{H^m(\Omega)}. \tag{7}$$

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Proof.

If V is a C^n finite element space, then the $n+1$-th derivatives are defined in the finite element sense. Then we may write

$$\begin{aligned}
\|\mathcal{I}_K u - u\|^2_{H^k(\Omega)} &= \sum_{K\in\mathcal{T}} \|\mathcal{I}_K u - u\|^2_{H^k(K)}, && (8)\\
&\leq \sum_{K\in\mathcal{T}} C_K d_K^{2(m-k)} |u|^2_{H^m(K)}, && (9)\\
&\leq C_{\max} h^{2(m-k)} \sum_{K\in\mathcal{T}} |u|^2_{H^m(K)}, && (10)\\
&= C_{\max} h^{2(m-k)} |u|^2_{H^m(\Omega)}, && (11)
\end{aligned}$$

where the existence of the $C_{\max} = \max_K C_K < \infty$ is due to the lower bound in the aspect ratio. □

Corollary 26

The solution u of the variational Helmholtz problem satisfies

$$\|u_h - u\|_{H^1(\Omega)} \leq Cd^{m-1}\|u\|_{H^2(\Omega)}.$$

Proof.

We combine Céa's Theorem with the previous estimate, since

$$\min_{v\in V_h} \|u - v\|_{H^1(\Omega)} \leq \|u - I_h u\|_{H^1(\Omega)}.$$

$\square$

We would also like estimates of the error in the L^2 norm. To do this we quote the following without proof.

Theorem 27 (Elliptic regularity)

Let w solve the equation

$$w - \nabla^2 w = f, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega,$$

on a convex domain Ω, with $f \in L^2$. Then there exists constant $C > 0$ such that

$$|w|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Theorem 28

The solution u of the variational Helmholtz problem satisfies

$$\|u_h - u\|_{L^2(\Omega)} \leq Cd^m \|u\|_{H^2(\Omega)}.$$

Proof

We use the Aubin-Nitsche duality argument. Let w be the solution of

$$w - \nabla^2 w = u - u_h.$$

Proof (Cont.)

Then equivalently,

$$b(w, v) = (u - u_h, v)_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega).$$

$$\begin{aligned} \|u - u_h\|^2_{L^2(\Omega)} &= (u - u_h, u - u_h) = b(w, u - u_h), \\ &= b(w - \mathcal{I}_h w, u - u_h) \text{ (orthogonality) }, \\ &\leq C\|u - u_h\|_{H^1(\Omega)}\|w - \mathcal{I}_h w\|_{H^1(\Omega)}, \\ &\leq Ch\|u - u_h\|_{H^1(\Omega)}|w|_{H^2(\Omega)} \\ &\leq C_1 h^{m+1}\|u - u_h\|_{L^2(\Omega)}|u|_{H^2(\Omega)} \end{aligned}$$

and dividing both sides by $\|u - u_h\|_{L^2(\Omega)}$ gives the result. $\square$