Finite Elements: examples 2

Colin Cotter

March 2, 2015

1. Show that affine equivalence between finite elements is an equivalence relation.
Solution: *Reflexivity: a finite element* $(K, \mathcal{P}, \mathcal{N})$ *is equivalent to itself with* $F = I$*. Symmetry: If* F *maps from* K *to* $\hat{K}$*, then* F^{-1} *maps from* $\hat{K}$ *to* K*. We have* $F^*\hat{p} = \hat{p} \circ F = p$*, so* $(F^{-1})^* p = p \circ F^{-1} = \hat{p}$*. If* $F_*\mathcal{N} = \hat{\mathcal{N}}$*, we have a one-to-one correspondence between* $\hat{N} \in \hat{\mathcal{N}}$ *and* $N \in \mathcal{N}$ *such that* $N(f \circ F) = \hat{N}(f)$ *for all* $f \in \mathcal{P}$*. Therefore,* $\hat{N}(\hat{f} \circ F^{-1}) = N(\hat{f})$ *for all* $\hat{f} \in \hat{\mathcal{P}}$*, as required. Transivity: If* $(K, \mathcal{P}, \mathcal{N})$ *is equivalent to* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *via* F*, and* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *is equivalent to* $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ *via* $\hat{F}$*, then we can perform similar calculations to the above with* $\hat{F} \circ F$*.*

2. For given degree k, show that there exist nodal placements such that all Lagrange elements are affine equivalent.
Solution: *For any triangle* K*, define barycentric coordinates* (b_1, b_2, b_3) *defined by* $b_i(x, y) = \phi_i(x, y)$ *where* ϕ_i *is the* i*th linear Lagrange basis function (equal to perpendicular distance from the edge opposite vertex* i*). We have* $b_1 + b_2 + b_3 = 1$ *for any point in the triangle. The mapping* $(x, y) \to (b_1, b_2, b_3)$ *is an affine map from the triangle to*
$$\{b \in [0, 1]^3 : b_1 + b_2 + b_3 = 1\}.$$
Hence, we can define an affine mapping between two triangles by tranforming from T_1 *to barycentric coordinates, then from barycentric coordinates to* T_2*.*

For degree k *Lagrange elements, pick nodes with barycentric coordinates*
$$\left(\frac{i}{k}, \frac{j}{k}, \frac{l}{k}\right), \quad 0 \leq i, j, l \leq k, \quad i + j + l = k.$$

3. Suppose that Ω is any bounded domain, $k, m > 0$ integers with $k \leq m$, and $1 \leq p < \infty$. Show that $W_p^m(\Omega) \subset W_p^k(\Omega)$.
Solution: *If* $f \in W_p^m(\Omega)$ *then*
$$\|f\|^p_{W_p^m(\Omega)} = \sum_{|\alpha| \leq m} \|D_w^\alpha f\|^p_{L^p} \leq \infty.$$
Since, $m \geq k$*, we have*
$$\|f\|^p_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k} \|D_w^\alpha f\|^p_{L^p} \leq \sum_{|\alpha| \leq m} \|D_w^\alpha f\|^p_{L^p} \leq \infty,$$
so $f \in W_p^k(\Omega)$*.*

4. Suppose that Ω is any bounded domain, $k > 0$ an integer, and $1 \leq p_0 \leq p_1 \leq \infty$ are integers. Show that $W_{p_1}^k(\Omega) \subset W_{p_0}^k(\Omega)$.
Solution: *If* $f \in W_{p_1}^k$ *then* $D_w^\alpha f \in L^{p_1}$ *for* $\alpha \leq k$*. Using Hölder's inequality with* $p = p_1/p_0 > 1$*,* $q = p_1/(p_1 - p_0) > 1$*, we get*
$$\|D_w^\alpha f\|_{L^{p_0}(\Omega)} = \|1 \times |D_w^\alpha f|^{p_0}\|_{L^1(\Omega)} \leq \||D_w^\alpha f|^{p_0}\|_{L^{p_1/p_0}(\Omega)} \|1\|_{L^{\frac{p_1}{p_1 - p_0}}(\Omega)} \leq \|D_w^\alpha f\|_{L^{p_1}(\Omega)} \leq \infty,$$
for all $|\alpha| \leq k$*, therefore* $f \in W_{p_0}^k$*.*

5. Let α be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D_w^\alpha \psi = D^\alpha \psi$.
Solution: *Taking $\phi \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega \phi D^\alpha \psi \,\mathrm{d}\,x &= (-1)^{|\alpha|} \textstyle\int_\Omega D^\alpha \phi \psi \,\mathrm{d}\,x, \\ &= (-1)^{|\alpha|} \textstyle\int_\Omega D_w^\alpha \phi \psi \,\mathrm{d}\,x, \end{aligned}$$

for all test functions ϕ. Further, since $D_w^\alpha \phi$ is continuous, it is bounded in all closed domains K contained in the interior of Ω, i.e. $D_w^\alpha \phi \in L^1_{loc}(\Omega)$ as required.

6. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_K u) \right|^2 \mathrm{d}\,x \,\mathrm{d}\,y = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0) \right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta,$$

where $\bar{u} = u \circ g$, and ξ and η are the coordinates on K.
Solution: *The mapping is defined by*

$$x = x_i + \xi h, \quad y = y_i + \eta h.$$

Defining $\bar{u}(\xi, \eta) = u(x, y)$, we have

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{h} \frac{\partial u}{\partial x}, \quad \frac{\partial \bar{u}}{\partial \eta} = \frac{1}{h} \frac{\partial u}{\partial y},$$

and the Jacobian of the mapping is

$$|J| = \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| = h^2.$$

We have

$$\mathcal{I}_K u \circ g = (1 - \xi - \eta)\bar{u}(0,0) + \xi \bar{u}(1,0) + \eta \bar{u}(0,1).$$

Hence

$$\left(\frac{\partial}{\partial x} \mathcal{I}_K u \right) \circ g = \frac{-\bar{u}(0,0) + \bar{u}(1,0)}{h}.$$

Substitution gives the result.

7. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_K u) \right|^2 \mathrm{d}\,x \,\mathrm{d}\,y \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta.$$

Solution:

$$
\begin{aligned}
\int_K \left|\frac{\partial \bar u}{\partial \xi} - \bar u(0,0) + \bar u(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \quad &\le \quad \int_K \left|\frac{\partial \bar u}{\partial \xi}(\xi,\eta) - \int_0^1 \frac{\partial \bar u}{\partial \xi}(\sigma,0)\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \\
&= \int_{\xi=0}^1 \int_{\eta=0}^{\xi} \left|\int_0^1 \left(\frac{\partial \bar u}{\partial \xi}(\xi,\eta) - \frac{\partial \bar u}{\partial \xi}(\sigma,\eta)\right)\mathrm{d}\,\sigma\right. \\
&\qquad \left. + \int_0^1 \left(\frac{\partial \bar u}{\partial \xi}(\sigma,\eta) - \frac{\partial \bar u}{\partial \xi}(\sigma,0)\right)\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&= \int_{\xi=0}^1 \int_{\eta=0}^{\xi} \left|\int_0^1 \int_\sigma^\xi \frac{\partial^2 \bar u}{\partial \xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\,\mathrm{d}\,\sigma\right. \\
&\qquad \left. + \int_0^1 \int_0^\eta \frac{\partial^2 \bar u}{\partial \xi \eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\le \int_{\xi=0}^1 \int_{\eta=0}^{\xi} \left(\int_0^1 \left|\int_\sigma^\xi \frac{\partial^2 \bar u}{\partial \xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\right|^2 \mathrm{d}\,\sigma\right. \\
&\qquad \left. + \int_0^1 \left|\int_0^\eta \frac{\partial^2 \bar u}{\partial \xi \partial \eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\right|^2 \mathrm{d}\,\sigma\right) \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\le \int_{\xi=0}^1 \int_{\eta=0}^{\xi} \left(\int_0^1 |\xi-\sigma| \int_\sigma^\xi \left|\frac{\partial^2 \bar u}{\partial \xi^2}(\gamma,\eta)\right|^2 \mathrm{d}\,\gamma\,\mathrm{d}\,\sigma\right. \\
&\qquad \left. + \int_0^1 |\eta| \int_0^\eta \left|\frac{\partial^2 \bar u}{\partial \xi \partial \eta}(\sigma,\alpha)\right|^2 \mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right) \mathrm{d}\,\eta\,\mathrm{d}\,\xi
\end{aligned}
$$

$$
\le \quad C \int_K \left|\frac{\partial^2 \bar u}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 \bar u}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.
$$

Hence show that

$$
\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_K u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \le Ch^2 \int_\Delta \left|\frac{\partial^2 u}{\partial x^2}\right|^2 + \left|\frac{\partial^2 u}{\partial xy}\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y.
$$

Solution: *We take the previous result and change variables back, so that e.g.* $\frac{\partial^2 \bar u}{\partial \xi^2}$ *becomes* $\frac{\partial^2 u}{\partial x^2}$. *Hence, the second derivatives produce factors of* h^2 *that get squared, and we divide by* h^2 *from the Jacobian factor, leaving a factor of* h^2.

8. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$
\|u - \mathcal{I}_{\mathcal{T}}\|_E \le ch\|u\|_{H^2(\Omega)},
$$

where

$$
\|f\|_E^2 = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y, \quad \|u\|_{H^2(\Omega)}^2 = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.
$$

Solution: *All right-angled triangles can be transformed to the reference element by the transformation given above, plus a rotation. Hence, the estimate of the previous section applies to any triangle in the mesh. Summing over elements and taking square roots gives the result with* $c = \sqrt{C}$.

9. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let f be a P_k Lagrange finite element function on $\mathcal{T}$. Show that the weak first derivatives of f exist.
Solution: *We claim that the weak first derivative of* f *is given by* $g \in L^1_{loc}(\Omega)$ *with*

$$
gf|_e(x) = D^\alpha f|_e(x),
$$

for each element e, where $\alpha = (0,1)$ or $(1,0)$, and $|_e$ indicates the restriction of functions to e. To check this, we take $\phi \in C_0^\infty(\Omega)$, and calculate,

$$\begin{aligned}\int_\Omega \phi g \,\mathrm{d}\, x &= \textstyle\sum_e \int_e \phi D^\alpha f|_e(x), \\ &= \textstyle\sum_e \left(-\int_e (D^\alpha \phi)\, f \,\mathrm{d}\, x + \int_{\partial e} \phi(n\cdot\alpha) f \,\mathrm{d}\, S\right), \\ &= -\textstyle\int_\Omega (D^\alpha \phi)\, f \,\mathrm{d}\, x,\end{aligned}$$

where n is the unit outward normal to ∂e. The surface integrals cancel since $(n \cdot \alpha)$ takes the same value with opposite sign on each side of ∂e, whilst ϕf is continuous.