# Finite Elements: examples 2

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March 7, 2016

1. Let $(T_1, P_k, \mathcal{N}_{k,1})$ and $(T_2, P_k, \mathcal{N}_{k,2})$ be degree $k$ Lagrange elements defined on two different triangles $T_1$ and $T_2$. Show that there exists an affine (translation plus linear transformation) map $g : T_1 \to T_2$ such that $g^* N_{i,2} = N_{i,1}$ where $N_{i,1} \in \mathcal{N}_{k,1}$ and $N_{i,2} \in \mathcal{N}_{k,2}$ are corresponding pairs of dual basis functions and

$$(g^* N_{i,2})[u(x)] = N_{i,2}[u(g(x))].$$

**Solution:** *For any triangle $K$, define barycentric coordinates $(b_1, b_2, b_3)$ defined by $b_i(x, y) = \phi_i(x, y)$ where $\phi_i$ is the ith linear Lagrange basis function (equal to perpendicular distance from the edge opposite vertex $i$). We have $b_1 + b_2 + b_3 = 1$ for any point in the triangle. The mapping $(x, y) \to (b_1, b_2, b_3)$ is an affine map from the triangle to*

$$\{b \in [0, 1]^3 : b_1 + b_2 + b_3 = 1\}.$$

*Hence, we can define an affine mapping between two triangles by tranforming from $T_1$ to barycentric coordinates, then from barycentric coordinates to $T_2$.*

*For degree $k$ Lagrange elements, pick nodes with barycentric coordinates*

$$\left(\frac{i}{k}, \frac{j}{k}, \frac{l}{k}\right), \quad 0 \leq i, j, l \leq k, \quad i + j + l = k.$$

*The dual basis functions represent function evaluate at these nodes after transformation to barycentric coordinates.*

2. Suppose that $\Omega$ is any bounded domain, $k, m > 0$ integers with $k \leq m$, and $1 \leq p < \infty$. Show that $W_p^m(\Omega) \subset W_p^k(\Omega)$.
**Solution:** *If $f \in W_p^m(\Omega)$ then*

$$\|f\|^p_{W_p^m(\Omega)} = \sum_{|\alpha| \leq m} \|D_w^\alpha f\|^p_{L^p} \leq \infty.$$

*Since, $m \geq k$, we have*

$$\|f\|^p_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k} \|D_w^\alpha f\|^p_{L^p} \leq \sum_{|\alpha| \leq m} \|D_w^\alpha f\|^p_{L^p} \leq \infty,$$

*so $f \in W_p^k(\Omega)$.*

3. Suppose that $\Omega$ is any bounded domain, $k > 0$ an integer, and $1 \leq p_0 \leq p_1 \leq \infty$ are integers. Show that $W_{p_1}^k(\Omega) \subset W_{p_0}^k(\Omega)$.
**Solution:** *If $f \in W_{p_1}^k$ then $D_w^\alpha f \in L^{p_1}$ for $\alpha \leq k$. Using Hölder's inequality with $p = p_1/p_0 > 1$, $q = p_1/(p_1 - p_0) > 1$, we get*

$$\|D_w^\alpha f\|_{L^{p_0}(\Omega)} = \|1 \times |D_w^\alpha f|^{p_0}\|_{L^1(\Omega)} \leq \||D_w^\alpha f|^{p_0}\|_{L^{p_1/p_0}(\Omega)} \|1\|_{L^{\frac{p_1}{p_1 - p_0}}(\Omega)} \leq \|D_w^\alpha f\|_{L^{p_1}(\Omega)} \leq \infty,$$

*for all $|\alpha| \leq k$, therefore $f \in W_{p_0}^k$.*

4. Let $\alpha$ be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D_w^\alpha \psi = D^\alpha \psi$.
**Solution:** *Taking $\phi \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega \phi D^\alpha \psi \, \mathrm{d}\, x &= (-1)^{|\alpha|} \textstyle\int_\Omega D^\alpha \phi \psi \, \mathrm{d}\, x, \\ &= (-1)^{|\alpha|} \textstyle\int_\Omega D_w^\alpha \phi \psi \, \mathrm{d}\, x, \end{aligned}$$

*for all test functions $\phi$. Further, since $D_w^\alpha \phi$ is continuous, it is bounded in all closed domains $K$ contained in the interior of $\Omega$, i.e. $D_w^\alpha \phi \in L^1_{loc}(\Omega)$ as required.*

5. Let $\Delta$ be the triangle with vertices $(x_i, y_j)$, $(x_{i+1}, y_j)$, $(x_i, y_{j+1})$, with $x_i = hi$, $y_j = hj$. Define a transformation $g$ from the reference element $K$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to $K$, and show that

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\, x \, \mathrm{d}\, y = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0) \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta,$$

where $\bar{u} = u \circ g$, $\xi$ and $\eta$ are the coordinates on $K$, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on $\Delta$.
**Solution:** *The mapping is defined by*

$$x = x_i + \xi h, \quad y = y_i + \eta h.$$

*Defining $\bar{u}(\xi, \eta) = u(x, y)$, we have*

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{h}\frac{\partial u}{\partial x}, \quad \frac{\partial \bar{u}}{\partial \eta} = \frac{1}{h}\frac{\partial u}{\partial y},$$

*and the Jacobian of the mapping is*

$$|J| = \left| \frac{\partial(x,y)}{\partial(\xi,\eta)} \right| = h^2.$$

*We have*

$$\mathcal{I}_\Delta u \circ g = (1 - \xi - \eta)\bar{u}(0,0) + \xi \bar{u}(1,0) + \eta \bar{u}(0,1).$$

*Hence*

$$\left( \frac{\partial}{\partial x} \mathcal{I}_\Delta u \right) \circ g = \frac{-\bar{u}(0,0) + \bar{u}(1,0)}{h}.$$

*Substitution gives the result.*

6. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\, x \, \mathrm{d}\, y \le C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta.$$

**Solution:**

$$\begin{aligned}
\int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0) \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta \quad &\le \quad \int_K \left| \frac{\partial \bar{u}}{\partial \xi}(\xi, \eta) - \int_0^1 \frac{\partial \bar{u}}{\partial \xi}(\sigma, 0) \, \mathrm{d}\, \sigma \right|^2 \mathrm{d}\, \xi \, \mathrm{d}\, \eta \\
&= \quad \int_{\xi=0}^1 \int_{\eta=0}^\xi \left| \int_0^1 \left( \frac{\partial \bar{u}}{\partial \xi}(\xi, \eta) - \frac{\partial \bar{u}}{\partial \xi}(\sigma, \eta) \right) \mathrm{d}\, \sigma \right. \\
&\qquad \left. + \int_0^1 \left( \frac{\partial \bar{u}}{\partial \xi}(\sigma, \eta) - \frac{\partial \bar{u}}{\partial \xi}(\sigma, 0) \right) \mathrm{d}\, \sigma \right|^2 \mathrm{d}\, \eta \, \mathrm{d}\, \xi \\
&= \quad \int_{\xi=0}^1 \int_{\eta=0}^\xi \left| \int_0^1 \int_\sigma^\xi \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) \, \mathrm{d}\, \gamma \, \mathrm{d}\, \sigma \right. \\
&\qquad \left. + \int_0^1 \int_0^\eta \frac{\partial^2 \bar{u}}{\partial \xi \eta}(\sigma, \alpha) \, \mathrm{d}\, \alpha \, \mathrm{d}\, \sigma \right|^2 \mathrm{d}\, \eta \, \mathrm{d}\, \xi \\
&\le \quad \int_{\xi=0}^1 \int_{\eta=0}^\xi \left( \int_0^1 \left| \int_\sigma^\xi \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) \, \mathrm{d}\, \gamma \right|^2 \mathrm{d}\, \sigma \right. \\
&\qquad \left. + \int_0^1 \left| \int_0^\eta \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}(\sigma, \alpha) \, \mathrm{d}\, \alpha \right|^2 \mathrm{d}\, \sigma \right) \mathrm{d}\, \eta \, \mathrm{d}\, \xi \\
&\le \quad \int_{\xi=0}^1 \int_{\eta=0}^\xi \left( \int_0^1 |\xi - \sigma| \int_\sigma^\xi \left| \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) \right|^2 \mathrm{d}\, \gamma \, \mathrm{d}\, \sigma \right. \\
&\qquad \left. + \int_0^1 |\eta| \int_0^\eta \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}(\sigma, \alpha) \right|^2 \mathrm{d}\, \alpha \, \mathrm{d}\, \sigma \right) \mathrm{d}\, \eta \, \mathrm{d}\, \xi
\end{aligned}$$

$$\leq \quad C \int_K \left|\frac{\partial^2 \bar{u}}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Hence show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \leq Ch^2 \int_\Delta \left|\frac{\partial^2 u}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 u}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

**Solution:** *We take the previous result and change variables back, so that e.g.* $\frac{\partial^2 \bar{u}}{\partial \xi^2}$ *becomes* $\frac{\partial^2 u}{\partial x^2}$*. Hence, the second derivatives produce factors of* $h^2$ *that get squared, and we divide by* $h^2$ *from the Jacobian factor, leaving a factor of* $h^2$*.*

7. Consider a triangulation $\mathcal{T}$ of points $x_i$ and $y_j$ arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_\mathcal{T}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.$$

**Solution:** *All right-angled triangles can be transformed to the reference element by the transformation given above, plus a rotation. Hence, the estimate of the previous section applies to any triangle in the mesh. Summing over elements and taking square roots gives the result with* $c = \sqrt{C}$*.*

8. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let $f$ be a $P_k$ Lagrange finite element function on $\mathcal{T}$. Show that the weak first derivatives of $f$ exist.
**Solution:** *We claim that the weak first derivative of* $f$ *is given by* $g \in L^1_{loc}(\Omega)$ *with*

$$g|_e(x) = D^\alpha f|_e(x),$$

*for each element* $e$*, where* $\alpha = (0,1)$ *or* $(1,0)$*, and* $|_e$ *indicates the restriction of functions to* $e$*. To check this, we take* $\phi \in C_0^\infty(\Omega)$*, and calculate,*

$$\begin{aligned}
\int_\Omega \phi g \,\mathrm{d}\,x = & \quad \textstyle\sum_e \int_e \phi D^\alpha f|_e(x), \\
= & \quad \textstyle\sum_e \left(-\int_e (D^\alpha \phi)\, f \,\mathrm{d}\,x + \int_{\partial e} \phi (n \cdot \alpha) f \,\mathrm{d}\,S\right), \\
= & \quad -\textstyle\int_\Omega (D^\alpha \phi)\, f \,\mathrm{d}\,x,
\end{aligned}$$

*where* $n$ *is the unit outward normal to* $\partial e$*. The surface integrals cancel since* $(n \cdot \alpha)$ *takes the same value with opposite sign on each side of* $\partial e$*, whilst* $\phi f$ *is continuous.*