

# Finite Elements: examples 1

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1. For a general partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the interval  $[0, 1]$ , let  $S$  be the piecewise linear finite element space known as P1. Compute the entries of the matrix

$$K_{ij} = a(\phi_i, \phi_j),$$

and the right-hand side vector

$$F_i = (\phi_i, f_I),$$

where  $f_I \in S$  is the interpolant of  $f$ .

For an equispaced mesh with  $h_i = x_i - x_{i-1} := h$ , write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

2. Give a weak formulation for the following ODEs,

(a)

$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$

(b)

$$-u'' + u = f, \quad u'(0) = 0, \quad u'(1) = \alpha.$$

(c)

$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

What is wrong with the formulation of the last ODE?

3. For a general partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the interval  $[0, 1]$ , let  $S$  be the piecewise quadratic finite element space known as P2, defined by the following:

(a)  $S \subset C^0([0, 1])$ .

(b) For  $v \in S$ ,  $v|_{[x_{j-1}, x_j]}$  is a quadratic function of  $x$ .

(c)  $v(0) = 0$ .

Find a nodal basis for  $S$ , using the nodes for P1 plus nodes at element midpoints  $(x_j + x_{j+1})/2$ . Evaluate the matrix  $K$  for this basis.

4. Under the same assumptions as Theorem 1.8, prove that

$$\|u - u_I\| \leq Ch^2 \|u''\|.$$

(hint: make use of the fact that  $u(0) = 0$  to write  $u$  in terms of  $u'$ .)

5. Under the same assumptions as Theorem 1.8, prove the following version of *Sobolev's inequality*:

$$\|v\|_{\max}^2 \leq Ca(v, v), \quad \forall v \in V \cap C'(0, 1).$$

Give a value for  $C$ .

6. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.
7. Let  $\mathcal{I}_K f$  be the interpolant for a finite element  $K$ . Show that  $\mathcal{I}_K$  is a linear operator.
8. Let  $K$  be a rectangle,  $Q_2$  be the space of biquadratic polynomials, and let  $\mathcal{N}$  be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that  $\mathcal{N}$  determines the finite element.
9. For  $K$  being the unit square, determine the nodal basis for the element in the previous question.