

Finite Elements: examples 2

Colin Cotter

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1. Show that affine equivalence between finite elements is an equivalence relation.
2. For given degree k , show that there exist nodal placements such that all Lagrange elements are affine equivalent.
3. Suppose that Ω is any bounded domain, $k, m > 0$ integers with $k \leq m$, and $1 \leq p < \infty$. Show that $W_p^m(\Omega) \subset W_p^k(\Omega)$.
4. Suppose that Ω is any bounded domain, $k > 0$ an integer, and $1 \leq p_0 \leq p_1 \leq \infty$ are integers. Show that $W_{p_1}^k(\Omega) \subset W_{p_0}^k(\Omega)$.
5. Let α be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D_w^\alpha \psi = D^\alpha \psi$.
6. Let Δ be the triangle with vertices (x_i, y_j) , (x_{i+1}, y_j) , (x_i, y_{j+1}) , with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to K , and show that

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_K u) \right|^2 dx dy = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0, 0) + \bar{u}(1, 0) \right|^2 d\xi d\eta,$$

where $\bar{u} = u \circ g$, and ξ and η are the coordinates on K .

7. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_K u) \right|^2 dx dy \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

Hence show that

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_K u) \right|^2 dx dy \leq Ch^2 \int_{\Delta} \left| \frac{\partial^2 u}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

8. Consider a triangulation \mathcal{T} of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_{\mathcal{T}}\|_E \leq ch \|u\|_{H^2(\Omega)},$$

where

$$\|f\|_E^2 = \int_{\Omega} \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 dx dy, \quad \|u\|_{H^2(\Omega)}^2 = \int_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial xy} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 dx dy.$$

9. Let \mathcal{T} be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let f be a P_k Lagrange finite element function on \mathcal{T} . Show that the weak first derivatives of f exist.