Finite Elements: examples 2

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1. Let $(T_1, P_k, \mathcal{N}_{k,1})$ and $(T_2, P_k, \mathcal{N}_{k,2})$ be degree k Lagrange elements defined on two different triangles T_1 and T_2. Show that there exists an affine (translation plus linear transformation) map $g : T_1 \to T_2$ such that $g^* N_{i,2} = N_{i,1}$ where $N_{i,1} \in \mathcal{N}_{k,1}$ and $N_{i,2} \in \mathcal{N}_{k,2}$ are corresponding pairs of dual basis functions and
$$(g^* N_{i,2})[u(x)] = N_{i,2}[u(g(x))].$$

2. Suppose that Ω is any bounded domain, $k, m > 0$ integers with $k \leq m$, and $1 \leq p < \infty$. Show that $W_p^m(\Omega) \subset W_p^k(\Omega)$.

3. Suppose that Ω is any bounded domain, $k > 0$ an integer, and $1 \leq p_0 \leq p_1 \leq \infty$ are integers. Show that $W_{p_1}^k(\Omega) \subset W_{p_0}^k(\Omega)$.

4. Let α be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D_w^\alpha \psi = D^\alpha \psi$.

5. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that
$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\,x \,\mathrm{d}\,y = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0) \right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta,$$
where $\bar{u} = u \circ g$, ξ and η are the coordinates on K, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ.

6. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain
$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\,x \,\mathrm{d}\,y \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta.$$
Hence show that
$$\int_\Delta \left| \frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u) \right|^2 \mathrm{d}\,x \,\mathrm{d}\,y \leq Ch^2 \int_\Delta \left| \frac{\partial^2 u}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta.$$

7. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain
$$\|u - \mathcal{I}_\mathcal{T}\|_E \leq ch|u|_{H^2(\Omega)},$$
where
$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x \,\mathrm{d}\,y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x \,\mathrm{d}\,y.$$

8. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let f be a P_k Lagrange finite element function on $\mathcal{T}$. Show that the weak first derivatives of f exist.