Finite Elements: examples 1

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1. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise linear finite element space known as P1. A finite element discretisation of the 1D Poisson equation

$$-\frac{\partial^2 u}{\partial x^2} = f, \quad u(0) = u(1) = 0,$$

defines the numerical solution $u \in V$ such that

$$a(u, v) = F[v], \quad \forall v \in V,$$

where V is the supspace of P1 satisfying the boundary conditions.

$$a(u, v) = \int_0^1 u'v' \,\mathrm{d}\, x, \quad F[v] = \int_0^1 fv \,\mathrm{d}\, x.$$

Where V Compute the entries of the matrix

$$K_{ij} = a(\phi_i, \phi_j),$$

and the right-hand side vector

$$F_i = (\phi_i, f_I),$$

where $f_I \in S$ is the interpolant of f.

Solution: *The local shape functions on the reference element interval* $[0, 1]$ *are*

$$\psi_0(x) = 1 - x, \quad \psi_1(x) = x,$$

and so

$$\psi_0'(x) = -1, \quad \psi_1'(x) = 1.$$

For $i = j$ *we have*

$$\begin{aligned}
K_{ii} &= \int_0^1 \phi_i'(x)\phi_i'(x) \,\mathrm{d}\, x, && (1)\\
&= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \phi_i'(x)\phi_i'(x) \,\mathrm{d}\, x, && (2)\\
&= \frac{1}{x_i - x_{i-1}} \int_0^1 (\psi_1'(x'))^2 \,\mathrm{d}\, x' + \frac{1}{x_{i+1} - x_i} \int_0^1 (\psi_0'(x'))^2 \,\mathrm{d}\, x', && (3)\\
&= \frac{1}{h_{i+1}} + \frac{1}{h_i}, && (4)
\end{aligned}$$

where we have used the change of variables

$$x = x_{i-1} + x'(x_i - x_{i-1}).$$

The first term is removed for $i = n$. For $i = j - 1$ we have

$$K_{i,i+1} = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \phi_i'(x)\phi_{i+1}'(x) \,\mathrm{d}\, x, \tag{5}$$

$$= \frac{1}{x_{i+1} - x_i} \int_0^1 \psi_0'(x')\psi_1'(x) \,\mathrm{d}\, x', \tag{6}$$

$$= -\frac{1}{h_{i+1}}. \tag{7}$$

Similarly, for $i = j + 1$, we have

$$K_{i,i-1} = -\frac{1}{h_{i-1}}.$$

Otherwise, $K_{ij} = 0$. Similarly,

$$F_i = \int_0^1 \phi_i(x) f_I(x) \,\mathrm{d}\, x, \tag{8}$$

$$= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \phi_i(x) f_I(x) \,\mathrm{d}\, x, \tag{9}$$

$$= h_i \int_0^1 \psi_1(x')(f_{i-1} + x'(f_i - f_{i-1})) \,\mathrm{d}\, x' + h_{i+1} \int_0^1 \psi_0(x')(f_i + x'(f_{i+1} - f_i)) \,\mathrm{d}\, x' \tag{10}$$

$$= h_i \left(\frac{f_{i-1}}{6} + \frac{f_i}{3} \right) + h_{i+1} \left(\frac{f_i}{3} + \frac{f_{i+1}}{6} \right). \tag{11}$$

For an equispaced mesh with $h_i = x_i - x_{i-1} := h$, write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?
Solution: *For $h_i := h$, we have*

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f_{i-1}}{6} + \frac{2f_i}{3} + \frac{f_{i+1}}{6},$$

for $i < n$. This is the centred difference formula for the second derivative with an averaged value of f on the right-hand side. The classical centred difference formula is

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i.$$

2. Using the methodology of the introductory lecture, develop an integral formulation that can be used to build a finite element discretisation for the following ODEs,

   (a)

   $$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$

   Solution: *Let $b(u, v)$ be*

   $$b(u, v) = \int_0^1 u'v' + uv \,\mathrm{d}\, x.$$

   Define V by

   $$V = \{u \in L^2([0, 1]) : b(u, u) < \infty\}.$$

   Then, find $u \in V$ such that

   $$b(u, v) = (f, v), \quad \forall v \in V.$$

   (b)

   $$-u'' + u = f, \quad u'(0) = 0, \ u'(1) = \alpha.$$

   Solution: *Under same definitions, find $u \in V$ such that*

   $$b(u, v) = (f, v) - v(1)\alpha, \quad \forall v \in V.$$

(c)
$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

Solution: *Let $a(u,v)$ be*
$$a(u,v) = \int_0^1 u'v' \,\mathrm{d}\, x.$$
Define V by
$$V = \{u \in L^2([0,1]) : a(u,u) < \infty\}.$$
Then, find $u \in V$ such that
$$a(u,v) = (f,v), \quad \forall v \in V.$$

3. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0,1]$, let S be the piecewise quadratic finite element space known as P2, defined by the following:

   (a) $S \subset C^0([0,1])$.

   (b) For $v \in S$, $v|_{[x_{j-1},x_j]}$ is a quadratic function of x.

   (c) $v(0) = 0$.

   Find a nodal basis for S, using the nodes for P1 plus nodes at element midpoints $(x_j + x_{j+1})/2$.
   Solution: *In the reference element $[0,1]$, the nodal points are $(z_0, z_1, z_2) = (0, 0.5, 1)$. The nodal basis in the reference element satisfies*
$$\psi_i(z_j) = \delta_{ij}.$$
   We get
$$\psi_i(x') = \frac{\prod_{j\neq i}(x' - x_j)}{\prod_{j\neq i}(x_i - x_j)},$$
   hence,
$$\psi_0(x') = \frac{(x'-0.5)(x'-1)}{0.5 \times 1} = 2{x'}^2 - 3x' + 1, \tag{12}$$
$$\psi_1(x') = \frac{x'(x'-1)}{-0.5 \times 0.5} = -4{x'}^2 + 4x', \tag{13}$$
$$\psi_2(x') = \frac{x'(x'-0.5)}{0.5} = 2{x'}^2 - x'. \tag{14}$$
   Derivatives are
$$\psi_0'(x') = 4x' - 3, \tag{15}$$
$$\psi_1'(x') = -8x' + 4, \tag{16}$$
$$\psi_2'(x') = 4x' - 1. \tag{17}$$

   Evaluate the matrix K for this basis.
   Solution: *The local matrix is*
$$\hat{K}_{ij} = \frac{1}{h}\int_0^1 \psi_i'(x)\psi_j'(x) \,\mathrm{d}\, x,$$

so we have

$$\hat{K}_{00} = \frac{1}{h}\int_0^1 (4x'-3)^2 \,\mathrm{d}\,x = \frac{7}{3h},$$
$$\hat{K}_{11} = \frac{1}{h}\int_0^1 (-8x'+4)^2 \,\mathrm{d}\,x = \frac{16}{3h},$$
$$\hat{K}_{22} = \frac{1}{h}\int_0^1 (4x'-1)^2 \,\mathrm{d}\,x = \frac{7}{3h},$$
$$\hat{K}_{01} = \hat{K}_{10} = \frac{1}{h}\int_0^1 (4x'^2-3)(-8x'^2+4) \,\mathrm{d}\,x = -\frac{8}{3h},$$
$$\hat{K}_{02} = \hat{K}_{20} = \frac{1}{h}\int_0^1 (4x'^2-3)(4x'^2-1) \,\mathrm{d}\,x = \frac{1}{3h},$$
$$\hat{K}_{12} = \hat{K}_{21} = \frac{1}{h}\int_0^1 (-8x'^2+4)(4x'^2-1) \,\mathrm{d}\,x = \frac{-8}{3h}.$$

The local matrix is

$$\hat{K} = \begin{pmatrix} \frac{7}{3h} & -\frac{8}{3h} & \frac{1}{3h} \\ -\frac{8}{3h} & \frac{16}{3h} & \frac{-8}{3h} \\ \frac{1}{3h} & \frac{-8}{3h} & \frac{7}{3h} \end{pmatrix}.$$

Hence, if we adopt a global node numbering such that odd numbered nodes are located at subinterval midpoints, and even numbered nodes are located at subinterval boundaries, then for odd i,

$$K_{ii} = \frac{16}{3h_{i/2}}, \quad K_{i,i+1} = K_{i+1,i} = K_{i,i-1} = K_{i-1,i} = \frac{-8}{3h_{i/2}}.$$

For even i,

$$K_{ii} = \frac{7}{3}\left(\frac{1}{h_{i/2}} + \frac{1}{h_{i/2+1}}\right),$$
$$K_{i,i-1} = -\frac{8}{3}\frac{1}{h_{i/2}},$$
$$K_{i,i-2} = \frac{1}{3}\frac{1}{h_{i/2}},$$
$$K_{i,i+1} = -\frac{8}{3}\frac{1}{h_{i/2+1}},$$
$$K_{i,i+2} = \frac{1}{3}\frac{1}{h_{i/2+1}}.$$

4. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.
Solution: *Let P be a cubic polynomial on the triangle K such that $N_i(P) = 0$, $i = 1,2,3,4,5,6$. Let L_1, L_2 and L_3 be the three edges of the triangle opposite vertices z_1, z_2 and z_3 respectively. Restricting P to L_1, we find that $P(z_0) = P'(z_0) = P(z_1) = P'(z_1)$ where $'$ indicates differentiation in the direction of L_1. Therefore $P|_{L_1}$ has double roots at z_0 and z_1, and so $P|_{L_1} = 0$. Similar argument for the other three edges means that $P = cL_1L_2L_3$ for constant c. But, P also vanishes at the midpoint z_4, so $c = 0$.*

5. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.
Solution:

$$\begin{aligned}\mathcal{I}_K(f+\alpha g) &= \sum_{i=1}^n \phi_i(x)(f+\alpha g)(x_i), \\ &= \sum_{i=1}^n \phi_i(x) f(x_i) + \alpha \sum_{i=1}^n \phi_i(x) g(x_i), \\ &= \mathcal{I}_K f + \alpha \mathcal{I}_K g.\end{aligned}$$

6. Let K be a rectangle, Q_2 be the space of biquadratic polynomials, and let $\mathcal{N}$ be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that $\mathcal{N}$ determines the finite element.
Solution: *Let z_i, $i = 1, 2, 3, 4$ be the rectangle vertices, z_i, $i = 5, 6, 7, 8$ be the edge midpoints, and z_9 be the rectangle centre. Let L_1, L_2, L_3 and L_4 be the linear functions determining the top, bottom, left and right rectangle edges respectively. Then L_1 and L_2 are functions of x only, whilst L_3 and L_4 are functions of y only. If P is biquadratic, then P restricted to L_1 is quadratic, and vanishes at three points on L_1. Therefore, $P|_{L_1} = 0$. The same reasoning for the other three edges means that we have*

$$P = cL_1L_2L_3L_4,$$

where c must be constant since $L_1L_2L_3L_4$ is biquadratic. Since P also vanishes at the rectangle centre, we conclude that $P = 0$ as required.

7. For K being the unit square, determine the nodal basis for the element in the previous question.
Solution: *The local basis functions are determined as products*

$$\Phi_{3i+j}(x, y) = \phi_i(x)\phi_j(y), \quad 0 \leq i, j \leq 2,$$

where $\phi_i(x)$ is the 1D quadratic Lagrange basis function.