

Finite Elements: examples 3

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1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation \mathcal{T} of a domain Ω . Show that functions in V do not have weak derivatives in general.
2. Let Δ be the triangle with vertices (x_i, y_j) , (x_{i+1}, y_j) , (x_i, y_{j+1}) , with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to K , and show that

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_{\Delta} u) \right|^2 dx dy = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0, 0) + \bar{u}(1, 0) \right|^2 d\xi d\eta,$$

where $\bar{u} = u \circ g$, ξ and η are the coordinates on K , and \mathcal{I}_{Δ} is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ .

3. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_{\Delta} u) \right|^2 dx dy \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

Hence show that

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_{\Delta} u) \right|^2 dx dy \leq Ch^2 \int_{\Delta} \left| \frac{\partial^2 u}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

4. Consider a triangulation \mathcal{T} of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_{\mathcal{T}}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_{\Omega} \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 dx dy, \quad |u|_{H^2(\Omega)}^2 = \int_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial xy} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 dx dy.$$

5. Show that

$$D^{\beta} Q_B^k f = Q_B^{k-|\beta|} D^{\beta} f,$$

where Q_B^l is the degree l averaged Taylor polynomial of f , and D^{β} is the β -th derivative where β is a multi-index.