

Finite Elements: examples 3

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1. Let L be a linear functional on a Hilbert space. Prove that L is continuous if and only if L is bounded.
2. Consider the variational problem with bilinear form

$$a(u, v) = \int_0^1 (u'v' + u'v + uv) \, dx.$$

Prove that $a(\cdot, \cdot)$ is coercive on $H^1([0, 1])$.

3. For the differential equation $-u'' + ku' + u = f$, find a value for k such that $a(v, v) = 0$ but $v \neq 0$ for some $v \in H^1([0, 1])$.
4. Let $a(\cdot, \cdot)$ be the inner product for a Hilbert space V . For $F \in V'$, and U an arbitrary (closed) subspace U of V , show that the following two statements are equivalent:
 - (a) $u \in U$ satisfies $a(u, v) = F(v) \, \forall v \in U$.
 - (b) u minimises $\frac{1}{2}a(u, v) - F(v)$ over $v \in U$.

5. Let

$$a(u, v) = \int_0^1 (u'v' + u'v + uv) \, dx,$$

with

$$V = \{v \in H^1([0, 1]) : v(0) = v(1) = 0\}.$$

Prove that

$$a(u, v) = \int_0^1 ((v')^2 + v^2) \, dx, \quad \forall v \in V.$$

6. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a variational formulation of the problem

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sigma(x) \frac{\partial u}{\partial x_i} \right) = f.$$

- (b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show that a finite element discretisation of this problem based on Lagrange elements has a unique solution, and give the rate of convergence to zero with h of the H^1 norm of the error.

7. Find a variational formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$

for a function g defined on $\partial\Omega$. Derive conditions under which a finite element discretisation of this problem based on Lagrange elements has a unique solution.

8. Find a variational formulation for the Poisson equation

$$-\nabla^2 u = f, \quad u + \frac{\partial u}{\partial n} = r \text{ on } \partial\Omega,$$

for a function r defined on $\partial\Omega$. Derive conditions under which a finite element discretisation of this problem based on Lagrange elements has a unique solution.