Finite Elements: examples 3

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1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation $\mathcal{T}$ of a domain Ω. Show that functions in V do not have weak derivatives in general.
Solution: *Choose a triangle $K_0 \in \mathcal{T}$, and define $u \in V$ as*

$$u(x) = \begin{cases} 1 & \text{if } x \in K_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $D^x_w u$ exists,

$$\begin{aligned}\int_\Omega D^x_w u\phi \,\mathrm{d}\,x &= -\int_\Omega \phi_x u \,\mathrm{d}\,x, \\ &= -\int_{K_0} \phi_x \,\mathrm{d}\,x, \\ &= -\int_{\partial K_0} \phi n_1 \,\mathrm{d}\,S,\end{aligned}$$

where n_1 is the x-component of the outward pointing normal n to ∂K_0.
Now we choose a sequence $\phi_i \in C^\infty_0(\Omega)$ such that

$$\phi_i|_{\partial K_0} \to 1 \text{ in } L^2(\partial K_0), \quad \phi_i|_\Omega \to 0 \text{ in } L^2(\Omega).$$

Then,

$$\int_\Omega \phi_i D^x_w \,\mathrm{d}\,x \to 0,$$

but we have just shown that

$$\int_\Omega D^x_w \phi_i \,\mathrm{d}\,x \to \int_{\partial K_0} n_1 \,\mathrm{d}\,S \neq 0,$$

so the weak derivative does not exist.

2. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y = \int_K \left|\frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta,$$

where $\bar{u} = u \circ g$, ξ and η are the coordinates on K, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ.
Solution: *The mapping is defined by*

$$x = x_i + \xi h, \quad y = y_i + \eta h.$$

Defining $\bar{u}(\xi,\eta) = u(x,y)$, we have

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{h}\frac{\partial u}{\partial x}, \quad \frac{\partial \bar{u}}{\partial \eta} = \frac{1}{h}\frac{\partial u}{\partial y},$$

and the Jacobian of the mapping is

$$|J| = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = h^2.$$

We have

$$\mathcal{I}_\Delta u \circ g = (1-\xi-\eta)\bar{u}(0,0) + \xi\bar{u}(1,0) + \eta\bar{u}(0,1).$$

Hence

$$\left(\frac{\partial}{\partial x}\mathcal{I}_\Delta u\right)\circ g = \frac{-\bar{u}(0,0)+\bar{u}(1,0)}{h}.$$

Substitution gives the result.

3. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left|\frac{\partial}{\partial x}(u-\mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \leq C\int_K \left|\frac{\partial^2\bar{u}}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Solution:

$$\begin{aligned}
\int_K \left|\frac{\partial\bar{u}}{\partial\xi} - \bar{u}(0,0)+\bar{u}(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \quad &\leq \quad \int_K \left|\frac{\partial\bar{u}}{\partial\xi}(\xi,\eta) - \int_0^1 \frac{\partial\bar{u}}{\partial\xi}(\sigma,0)\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \\
&= \quad \int_{\xi=0}^1\int_{\eta=0}^\xi \left|\int_0^1 \left(\frac{\partial\bar{u}}{\partial\xi}(\xi,\eta)-\frac{\partial\bar{u}}{\partial\xi}(\sigma,\eta)\right)\mathrm{d}\,\sigma\right. \\
&\qquad\qquad \left. + \int_0^1 \left(\frac{\partial\bar{u}}{\partial\xi}(\sigma,\eta)-\frac{\partial\bar{u}}{\partial\xi}(\sigma,0)\right)\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&= \quad \int_{\xi=0}^1\int_{\eta=0}^\xi \left|\int_0^1\int_\sigma^\xi \frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\,\mathrm{d}\,\sigma\right. \\
&\qquad\qquad \left. + \int_0^1\int_0^\eta \frac{\partial^2\bar{u}}{\partial\xi\eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\leq \quad \int_{\xi=0}^1\int_{\eta=0}^\xi \left(\int_0^1 \left|\int_\sigma^\xi \frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\right|^2 \mathrm{d}\,\sigma\right. \\
&\qquad\qquad \left. + \int_0^1 \left|\int_0^\eta \frac{\partial^2\bar{u}}{\partial\xi\partial\eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\right|^2 \mathrm{d}\,\sigma\right)\mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\leq \quad \int_{\xi=0}^1\int_{\eta=0}^\xi \left(\int_0^1 |\xi-\sigma|\int_\sigma^\xi \left|\frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\right|^2 \mathrm{d}\,\gamma\,\mathrm{d}\,\sigma\right. \\
&\qquad\qquad \left. + \int_0^1 |\eta|\int_0^\eta \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}(\sigma,\alpha)\right|^2 \mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right)\mathrm{d}\,\eta\,\mathrm{d}\,\xi
\end{aligned}$$

$$\leq \quad C\int_K \left|\frac{\partial^2\bar{u}}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Hence show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u-\mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \leq Ch^2\int_\Delta \left|\frac{\partial^2 u}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2 u}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Solution: *We take the previous result and change variables back, so that e.g.* $\frac{\partial^2\bar{u}}{\partial\xi^2}$ *becomes* $\frac{\partial^2 u}{\partial x^2}$. *Hence, the second derivatives produce factors of* h^2 *that get squared, and we divide by* h^2 *from the Jacobian factor, leaving a factor of* h^2.

4. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u-\mathcal{I}_\mathcal{T}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.$$

Solution: *All right-angled triangles can be transformed to the reference element by the transformation given above, plus a rotation. Hence, the estimate of the previous section applies to any triangle in the mesh. Summing over elements and taking square roots gives the result with* $c = \sqrt{C}$.

5. Show that

$$D^\beta Q^k_B f = Q_B^{k-|\beta|} D^\beta f,$$

where Q^l_B is the degree l averaged Taylor polynomial of f, and D^β is the β-th derivative where β is a multi-index.

Solution: *We assume that* $f \in C^\infty(B)$ *and then pass to the limit.*

$$\begin{aligned}
D_x^\beta (T_y^k f)(x) &= \sum_{|\alpha|\le k} D_y^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}, && (1)\\
&= \sum_{|\alpha|\le k} D_y^\alpha f(y) \frac{\alpha!}{(\alpha-\beta)!}\frac{(x-y)^{\alpha-\beta}}{\alpha!}, && (2)\\
&= \sum_{|\alpha'|\le k-|\beta|} D_y^{\alpha'} D_y^\beta f(y) \frac{(x-y)^{\alpha'}}{\alpha'!}, && (3)\\
&= (T_y^{k-|\beta|} D^\beta f)(x), && (4)
\end{aligned}$$

after setting $\alpha' = \alpha - \beta$.