

Finite Elements: examples 3

Colin Cotter

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1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation \mathcal{T} of a domain Ω . Show that functions in V do not have weak derivatives in general.

Solution: Choose a triangle $K_0 \in \mathcal{T}$, and define $u \in V$ as

$$u(x) = \begin{cases} 1 & \text{if } x \in K_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $D_w^x u$ exists,

$$\begin{aligned} \int_{\Omega} D_w^x u \phi \, dx &= - \int_{\Omega} \phi_x u \, dx, \\ &= - \int_{K_0} \phi_x \, dx, \\ &= - \int_{\partial K_0} \phi n_1 \, dS, \end{aligned}$$

where n_1 is the x -component of the outward pointing normal n to ∂K_0 .

Now we choose a sequence $\phi_i \in C_0^\infty(\Omega)$ such that

$$\phi_i|_{\partial K_0} \rightarrow 1 \text{ in } L^2(\partial K_0), \quad \phi_i|_{\Omega} \rightarrow 0 \text{ in } L^2(\Omega).$$

Then,

$$\int_{\Omega} \phi_i D_w^x u \, dx \rightarrow 0,$$

but we have just shown that

$$\int_{\Omega} D_w^x \phi_i \, dx \rightarrow \int_{\partial K_0} n_1 \, dS \neq 0,$$

so the weak derivative does not exist.

2. Let Δ be the triangle with vertices (x_i, y_j) , (x_{i+1}, y_j) , (x_i, y_{j+1}) , with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to K , and show that

$$\int_{\Delta} \left| \frac{\partial}{\partial x} (u - \mathcal{I}_{\Delta} u) \right|^2 \, dx \, dy = \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0, 0) + \bar{u}(1, 0) \right|^2 \, d\xi \, d\eta,$$

where $\bar{u} = u \circ g$, ξ and η are the coordinates on K , and \mathcal{I}_{Δ} is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ .

Solution: The mapping is defined by

$$x = x_i + \xi h, \quad y = y_i + \eta h.$$

Defining $\bar{u}(\xi, \eta) = u(x, y)$, we have

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{h} \frac{\partial u}{\partial x}, \quad \frac{\partial \bar{u}}{\partial \eta} = \frac{1}{h} \frac{\partial u}{\partial y},$$

and the Jacobian of the mapping is

$$|J| = \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| = h^2.$$

We have

$$\mathcal{I}_\Delta u \circ g = (1 - \xi - \eta)\bar{u}(0, 0) + \xi\bar{u}(1, 0) + \eta\bar{u}(0, 1).$$

Hence

$$\left(\frac{\partial}{\partial x} \mathcal{I}_\Delta u \right) \circ g = \frac{-\bar{u}(0, 0) + \bar{u}(1, 0)}{h}.$$

Substitution gives the result.

3. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left| \frac{\partial}{\partial x} (u - \mathcal{I}_\Delta u) \right|^2 dx dy \leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

Solution:

$$\begin{aligned} \int_K \left| \frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0, 0) + \bar{u}(1, 0) \right|^2 d\xi d\eta &\leq \int_K \left| \frac{\partial \bar{u}}{\partial \xi}(\xi, \eta) - \int_0^1 \frac{\partial \bar{u}}{\partial \xi}(\sigma, 0) d\sigma \right|^2 d\xi d\eta \\ &= \int_{\xi=0}^1 \int_{\eta=0}^\xi \left| \int_0^1 \left(\frac{\partial \bar{u}}{\partial \xi}(\xi, \eta) - \frac{\partial \bar{u}}{\partial \xi}(\sigma, \eta) \right) d\sigma \right. \\ &\quad \left. + \int_0^1 \left(\frac{\partial \bar{u}}{\partial \xi}(\sigma, \eta) - \frac{\partial \bar{u}}{\partial \xi}(\sigma, 0) \right) d\sigma \right|^2 d\eta d\xi \\ &= \int_{\xi=0}^1 \int_{\eta=0}^\xi \left| \int_0^1 \int_\sigma^\xi \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) d\gamma d\sigma \right. \\ &\quad \left. + \int_0^1 \int_0^\eta \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}(\sigma, \alpha) d\alpha d\sigma \right|^2 d\eta d\xi \\ &\leq \int_{\xi=0}^1 \int_{\eta=0}^\xi \left(\int_0^1 \left| \int_\sigma^\xi \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) d\gamma \right|^2 d\sigma \right. \\ &\quad \left. + \int_0^1 \left| \int_0^\eta \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}(\sigma, \alpha) d\alpha \right|^2 d\sigma \right) d\eta d\xi \\ &\leq \int_{\xi=0}^1 \int_{\eta=0}^\xi \left(\int_0^1 |\xi - \sigma| \int_\sigma^\xi \left| \frac{\partial^2 \bar{u}}{\partial \xi^2}(\gamma, \eta) \right|^2 d\gamma d\sigma \right. \\ &\quad \left. + \int_0^1 |\eta| \int_0^\eta \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}(\sigma, \alpha) \right|^2 d\alpha d\sigma \right) d\eta d\xi \\ &\leq C \int_K \left| \frac{\partial^2 \bar{u}}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} \right|^2 d\xi d\eta. \end{aligned}$$

Hence show that

$$\int_\Delta \left| \frac{\partial}{\partial x} (u - \mathcal{I}_\Delta u) \right|^2 dx dy \leq Ch^2 \int_\Delta \left| \frac{\partial^2 u}{\partial \xi^2} \right|^2 + \left| \frac{\partial^2 u}{\partial \xi \partial \eta} \right|^2 d\xi d\eta.$$

Solution: We take the previous result and change variables back, so that e.g. $\frac{\partial^2 \bar{u}}{\partial \xi^2}$ becomes $\frac{\partial^2 u}{\partial x^2}$. Hence, the second derivatives produce factors of h^2 that get squared, and we divide by h^2 from the Jacobian factor, leaving a factor of h^2 .

4. Consider a triangulation \mathcal{T} of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_\mathcal{T}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_{\Omega} \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 dx dy, \quad |u|_{H^2(\Omega)}^2 = \int_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial xy} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 dx dy.$$

Solution: All right-angled triangles can be transformed to the reference element by the transformation given above, plus a rotation. Hence, the estimate of the previous section applies to any triangle in the mesh. Summing over elements and taking square roots gives the result with $c = \sqrt{C}$.

5. Show that

$$D^\beta Q_B^k f = Q_B^{k-|\beta|} D^\beta f,$$

where Q_B^l is the degree l averaged Taylor polynomial of f , and D^β is the β -th derivative where β is a multi-index.

Solution: We assume that $f \in C^\infty(B)$ and then pass to the limit.

$$D_x^\beta (T_y^k f)(x) = \sum_{|\alpha| \leq k} D_y^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}, \quad (1)$$

$$= \sum_{|\alpha| \leq k} D_y^\alpha f(y) \frac{\alpha!}{(\alpha-\beta)!} \frac{(x-y)^{\alpha-\beta}}{\alpha!}, \quad (2)$$

$$= \sum_{|\alpha'| \leq k-|\beta|} D_y^{\alpha'} D_y^\beta f(y) \frac{(x-y)^{\alpha'}}{\alpha'!}, \quad (3)$$

$$= (T_y^{k-|\beta|} D^\beta f)(x), \quad (4)$$

after setting $\alpha' = \alpha - \beta$.