Finite Elements: examples 2

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1. Show that affine equivalence between finite elements is an equivalence relation.

2. For given degree k, show that there exist nodal placements such that all Lagrange elements are affine equivalent.

3. Suppose that Ω is any bounded domain, $k, m > 0$ integers with $k \leq m$, and $1 \leq p < \infty$. Show that $W_p^m(\Omega) \subset W_p^k(\Omega)$.

4. Suppose that Ω is any bounded domain, $k > 0$ an integer, and $1 \leq p_0 \leq p_1 \leq \infty$ are integers. Show that $W_{p_1}^k(\Omega) \subset W_{p_0}^k(\Omega)$.

5. Let α be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D_w^\alpha \psi = D^\alpha \psi$.

6. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_K u)\right|^2 \mathrm{d}\,x \,\mathrm{d}\,y = \int_K \left|\frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0)\right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta,$$

   where $\bar{u} = u \circ g$, and ξ and η are the coordinates on K.

7. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_K u)\right|^2 \mathrm{d}\,x \,\mathrm{d}\,y \leq C \int_K \left|\frac{\partial^2 \bar{u}}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi \,\mathrm{d}\,\eta.$$

   Hence show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_K u)\right|^2 \mathrm{d}\,x \,\mathrm{d}\,y \leq C h^2 \int_\Delta \left|\frac{\partial^2 u}{\partial x^2}\right|^2 + \left|\frac{\partial^2 u}{\partial xy}\right|^2 \mathrm{d}\,x \,\mathrm{d}\,y.$$

8. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_\mathcal{T}\|_E \leq ch\|u\|_{H^2(\Omega)},$$

   where

$$\|f\|_E^2 = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x \,\mathrm{d}\,y, \quad \|u\|_{H^2(\Omega)}^2 = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x \,\mathrm{d}\,y.$$

9. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let f be a P_k Lagrange finite element function on $\mathcal{T}$. Show that the weak first derivatives of f exist.