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Finite Elements

Colin Cotter

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The finite element

Given a triangulation $\mathcal{T}$ of a domain Ω, finite element spaces are defined according to

1. the form the functions take (usually polynomial) when restricted to each cell,
2. the continuity of the functions between cells.

We also need a mechanism to explicitly build a basis for the finite element space.

Definition 1 (Ciarlet's finite element)

Let

1. the element domain $K \subset \mathbb{R}^n$ be some bounded closed set with piecewise smooth boundary,
2. the space of shape functions $\mathcal{P}$ be a finite dimensional space of functions on K, and
3. the set of nodal variables $\mathcal{N} = (N_0, \ldots, N_k)$ be a basis for the dual space P'.

Then $(K, \mathcal{P}, \mathcal{N})$ is called a finite element.

P' is the dual space to P, defined as the space of linear functions from P to $\mathbb{R}$.

Examples of dual functions to P include:

1. The evaluation of $p \in P$ at a point $x \in K$.
2. The integral of $p \in P$ over a line $l \in K$.
3. The integral of $p \in P$ over K.
4. The evaluation of a component of the derivative of $p \in P$ at a point $x \in K$.

Exercise 2

Show that the four examples above are all linear functions from P to $\mathbb{R}$.

Ciarlet's finite element provides us with a standard way to define a basis for the P.

Definition 3 (Nodal basis)

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The nodal basis is the basis $\{\phi_0, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ that is dual to $\mathcal{N}$, i.e.

$$N_i(\phi_j) = \delta_{ij}, \quad 0 \leq i, j \leq k. \tag{1}$$

Example 4 (The 1-dimensional Lagrange element)

The 1-dimensional Lagrange element $(K, \mathcal{P}, \mathcal{N})$ of degree k is defined by

1. K is the interval $[a, b]$ for $-\infty < a < b < \infty$.
2. $\mathcal{P}$ is the $(k+1)$-dimensional space of degree k polynomials on K,
3. $\mathcal{N} = \{N_0, \ldots, N_k\}$ with

$$N_i(v) = v(x_i),\ x_i = a + (b-a)i/k, \quad \forall v \in \mathcal{P},\ i = 0, \ldots, k. \tag{2}$$

Exercise 5

Show that the nodal basis for $\mathcal{P}$ is given by

$$N_i(x) = \frac{\prod_{j=0,j\neq i}^{k}(x - x_i)}{\prod_{j=0,j\neq i}^{k}(x_j - x_i)}, \quad i = 0, \ldots, k. \tag{3}$$

It is useful computationally to write the nodal basis in terms of another arbitrary basis $\{\psi_i\}_{i=0}^{k}$.

Definition 6 (Vandermonde matrix)

Given a dual basis $\mathcal{N}$ and a basis $\{\psi_j\}_{i=0}^{k}$, the Vandermonde matrix is the matrix V with coefficients

$$V_{ij} = N_j(\psi_i). \tag{4}$$

Lemma 7

The expansion of the nodal basis $\{\phi_i\}_{i=0}^k$ *in terms of another basis* $\{\psi_i\}_{i=0}^k$ *for* $\mathcal{P}$*,*

$$\phi_i(x) = \sum_{j=0}^{k} \mu_{ij}\psi_j(x), \tag{5}$$

has coefficients μ_{ij}, $0 \leq i, j \leq k$ *given by*

$$\mu = V^{-1}, \tag{6}$$

where μ *is the corresponding matrix.*

Proof.

The nodal basis definition becomes

$$\delta_{ki} = N_k(\phi_i) = \sum_{j=0}^{k} \mu_{ij} N_k(\psi_j) = (\mu V)_{ki}, \tag{7}$$

where μ is the matrix with coefficients μ_{ij}, and V is the matrix with coefficients $N_k(\psi_j)$. $\square$

Lemma 8

Let $K, \mathcal{P}$ be as defined above, and let $\{N_0, N_1, \ldots, N_k\} \in \mathcal{P}'$. Let $\{\psi_0, \psi_1, \ldots, \psi_k\}$ be a basis for $\mathcal{P}$.
Then the following three statements are equivalent.

1. *$\{N_0, N_1, \ldots, N_k\}$ is a basis for $\mathcal{P}'$.*
2. *The Vandermonde matrix with coefficients*
$$V_{ij} = N_j(\psi_i),\ 0 \leq i, j \leq k, \tag{8}$$
is invertible.
3. *If $v \in \mathcal{P}$ satisfies $N_i(v) = 0$ for $i = 0, \ldots, k$, then $v \equiv 0$.*

Proof

Let $\{N_0, N_1, \ldots, N_k\}$ be a basis for $\mathcal{P}'$. This is equivalent to saying that given element E of $\mathcal{P}'$, we can find basis coefficients $\{e_i\}_{i=0}^k \in \mathbb{R}$ such that

$$E = \sum_{i=0}^{k} e_i N_i. \tag{9}$$

Proof (Cont.)

This in turn is equivalent to being able to find a vector $\boldsymbol{e} = (e_0, e_1, \ldots, e_k)^T$ such that

$$b_i = E(\psi_i) = \sum_{j=0}^{k} e_i N_j(\psi_i) = \sum_{j=0}^{k} e_i V_{ij}, \qquad (10)$$

i.e. the equation $V\boldsymbol{e} = \boldsymbol{b}$ is solvable. This means that (1) is equivalent to (2).

Proof (Cont.)

On the other hand, we may expand any $v \in \mathcal{P}$ according to

$$v(x) = \sum_{i=0}^{k} f_i \psi_i(x). \tag{11}$$

Then

$$N_i(v) = 0 \iff \sum_{j=0}^{k} f_j N_i(\psi_j) = 0, \quad i = 0, 1, \ldots, k, \tag{12}$$

by linearity of N_i.

Proof (Cont.)

So (2) is equivalent to

$$\sum_{j=0}^{k} f_j N_i(\psi_j) = 0, \quad i = 0, 1, \ldots, k \implies f_j = 0, j = 0, 1, \ldots, k, \tag{13}$$

which is equivalent to V^T being invertible, which is equivalent to V being invertible, and so (3) is equivalent to (2). $\square$

Definition 9

We say that $\mathcal{N}$ **determines** $\mathcal{P}$ if it satisfies condition 3 of Lemma 8. If this is the case, we say that $(K, \mathcal{P}, \mathcal{N})$ is **unisolvent**.

Corollary 10

The 1D degree k Lagrange element is a finite element.

Proof.

Let $(K, \mathcal{P}, \mathcal{N})$ be the degree k Lagrange element. We need to check that $\mathcal{N}$ determines $\mathcal{P}$. Let $v \in \mathcal{P}$ with $N_i(v) = 0$ for all $N_i \in \mathcal{N}$. This means that

$$v(a + (b-a)i/k) = 0, \; i =, 0, 1, \ldots, k, \tag{14}$$

which means that v vanishes at $k+1$ points in K. Since v is a degree k polynomial, it must be zero by the fundamental theorem of algebra. $\square$

We would like to construct some finite elements with 2D and 3D domains K. The following lemma is useful when checking that $\mathcal{N}$ determines $\mathcal{P}$ in those cases.

Lemma 11

Let $p(x) : \mathbb{R}^d \to \mathbb{R}$ be a polynomial of degree $k \geq 1$ that vanishes on a hyperplane Π_L defined by

$$\Pi_L = \{x : L(x) = 0\}, \tag{15}$$

for a non-degenerate affine function $L(x) : \mathbb{R}^d \to \mathbb{R}$. Then $p(x) = L(x)q(x)$ where $q(x)$ is a polynomial of degree $k-1$.

Proof

Choose coordinates (by shifting the origin and applying a linear transformation) such that $x = (x_0, x_1, \ldots, x_k)$ with $L(x) = x_k$, so Π_L is defined by $x_k = 0$. Then the general form for a polynomial is

$$P(x_0, x_1, \ldots, x_k) = \sum_{i_k=0}^{k} \left(\sum_{|i_0+i_1+\ldots+i_{k-1}|\leq k-i_k} c_{i_0,i_1,\ldots,i_{k-1},i_k} x_k^{i_k} \prod_{l=0}^{k-1} x_l^{i_l} \right), \tag{16}$$

Proof (Cont.)

Then, $p(x_0, x_1, \ldots, x_{k-1}, 0) = 0$ for all $(x_0, x_1, \ldots, x_{k-1})$, so

$$0 = \left(\sum_{|i_0+i_1+\ldots+i_{k-1}|\leq k} c_{i_0,i_1,\ldots,i_{k-1},0} \prod_{l=0}^{k-1} x_l^{i_l} \right) \tag{17}$$

which means that

$$c_{i_0,i_1,\ldots,i_{k-1},0} = 0, \quad \forall |i_0 + i_1 + \ldots + i_{k-1}| \leq k. \tag{18}$$

Proof (Cont.)

This means we may rewrite

$$P(x) = \underbrace{x_k}_{L(x)} \underbrace{\left(\sum_{|i_0+i_1+\ldots+i_{k-1}|\leq k} c_{i_0,i_1,\ldots,i_{k-1},i_k} \prod_{l=0}^{k-1} x_l^{i_l} \right)}_{Q(x)}, \qquad (19)$$

with $\deg(Q) = k - 1$. $\square$

Equipped with this tool we can consider some finite elements in two dimensions.

Definition 12 (Lagrange elements on triangles)

The triangular Lagrange element of degree k $(K, \mathcal{P}, \mathcal{N})$, denoted Pk, is defined as follows.

1. K is a (non-degenerate) triangle with vertices z_1, z_2, z_3.
2. $\mathcal{P}$ is the space of degree k polynomials on K.
3. $\mathcal{N} = \{N_{i,j} : 0 \leq i \leq k,\ 0 \leq j \leq i\}$ defined by $N_{i,j}(v) = v(x_{i,j})$ where

$$x_{i,j} = z_1 + (z_2 - z_1)i/k + (z_3 - z_1)j/k. \tag{20}$$

Example 13 (P1 elements on triangles)

The nodal basis for P1 elements is point evaluation at the three vertices.

Example 14 (P2 elements on triangles)

The nodal basis for P2 elements is point evaluation at the three vertices, plus point evaluation at the three edge centres.

We now need to check that that the degree k Lagrange element is a finite element, i.e. that $\mathcal{N}$ determines $\mathcal{P}$. We will first do this for cases $P1$ and $P2$.

Lemma 15

The degree 1 Lagrange element is a finite element.

Proof.

Let Π_1, Π_2, Π_3 be the three lines containing the vertices z_2 and z_3, z_1 and z_3, and z_1 and z_3 respectively, and defined by $L_1 = 0$, $L_2 = 0$, and $L_3 = 0$ respectively. Consider a linear polynomial p vanishing at z_1, z_2, and z_3. The restriction $p|_{\Pi_1}$ of p to Π_1 is a linear function vanishing at two points, and therefore $p = 0$ on Π_1, and so $p = L_1(x)Q(x)$, where $Q(x)$ is a degree 0 polynomial, i.e. a constant c. We also have

$$0 = p(z_1) = cL_1(z_1) \implies c = 0, \tag{21}$$

since $L_1(z_1) \neq 0$, and hence $p(x) \equiv 0$. This means that $\mathcal{N}$ determines $\mathcal{P}$. $\square$

Lemma 16

The degree 2 Lagrange element is a finite element.

Proof.

Let p be a degree 2 polynomial with $N_i(p)$ for all of the degree 2 dual basis elements. Let Π_1, Π_2, Π_3, L_1, L_2 and L_3 be defined as for the proof of Lemma 15. $p|_{\Pi_1}$ is a degree 2 scalar polynomial vanishing at 3 points, and therefore $p = 0$ on Π_1, and so $p(x) = L_1(x)Q_1(x)$ with $\deg(Q_1) = 1$. We also have $0 = p|_{\Pi_2} = L_1 Q_1|_{\Pi_2}$, so $Q_1|_{\Pi_2} = 0$ and we conclude that $p(x) = cL_1(x)L_2(x)$. Finally, p also vanishes at the midpoint of L_3, so we conclude that $c = 0$ as required. $\square$

Exercise 17

Show that the degree 3 Lagrange element is a finite element.

Lemma 18

The degree k Lagrange element is a finite element for $k > 0$.

Proof

We prove by induction. Assume that the degree $k-3$ Lagrange element is a finite element. Let p be a degree k polynomial with $N_i(p)$ for all of the degree k dual basis elements. Let Π_1, Π_2, Π_3, L_1, L_2 and L_3 be defined as for the proof of Lemma 15. The restriction $p|_{\Pi_1}$ is a degree k polynomial in one variable that vanishes at $k+1$ points, and therefore $p(x) = L_1(x)Q_1(x)$, with $\deg(Q_1) = k-1$. p and therefore Q also vanishes on Π_2, so $Q_1(x) = L_2(x)Q_2(x)$.

Proof(Cont.)

Repeating the argument again means that $p(x) = L_1(x)L_2(x)L_3(x)Q_3(x)$, with $\deg(Q_3) = k - 3$. Q_3 must vanish on the remaining points in the interior of K, which are arranged in a smaller triangle K' and correspond to the evaluation points for a degree $k-3$ Lagrange finite element on K'. From the inductive hypothesis, and using the results for $k = 1, 2, 3$, we conclude that $Q_3 \equiv= 0$, and therefore $p \equiv 0$ as required. $\square$

We now consider some finite elements that involve derivative evaluation.

Example 19 (Cubic Hermite element)

The cubic Hermite element is defined as follows:

1. K is a (nondegenerate) triangle,
2. $\mathcal{P}$ is the space of cubic polynomials on K,
3. $\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$ defined as follows:
   3.1 $(N_1, \ldots, N_3)$: evaluation of p at vertices,
   3.2 $(N_4, \ldots, N_9)$: evaluation of the gradient of p at the 3 triangle vertices.
   3.3 N_{10}: evaluation of p at the centre of the triangle.

Example 20 (Quintic Argyris element)

The quintic Hermite element is defined as follows:

1. K is a (nondegenerate) triangle,
2. $\mathcal{P}$ is the space of quintic polynomials on K,
3. $\mathcal{N}$ defined as follows:
   3.1 evaluation of p at 3 vertices,
   3.2 evaluation of gradient of p at 3 vertices,
   3.3 evaluation of Hessian of p at 3 vertices,
   3.4 evaluation of the gradient normal to 3 triangle edges.

Next we need to know how to glue finite elements together to form spaces defined over a mesh. To do this we need to develop a language for specifying connections between finite element functions between element domains.

Definition 21 (Finite element space)

Let $\mathcal{T}$ be a triangulation made of triangles K_i, with finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$. A space V of functions on $\mathcal{T}$ is called a finite element space if for each $u \in V$, and for each $K_i \in \mathcal{T}$, $u|_{K_i} \in \mathcal{P}_i$.

Note that the set of finite elements do not uniquely determine a finite element space, since we also need to specify continuity requirements between triangles, which we will do in this chapter.

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Definition 22 (C^m finite element space)

A finite element space V is a C^m finite element space if $u \in C^m$ for all $u \in V$.

Lemma 23

Let $\mathcal{T}$ be a triangulation on Ω, and let V be a finite element space defined on $\mathcal{T}$. The following two statements are equivalent.

1. *V is a C^m finite element space.*
2. *The following two conditions hold.*
   - 2.1 *For each vertex z in $\mathcal{T}$, let $\{K_i\}_{i=1}^m$ be the set of triangles that contain z. Then $u|_{K_1}(z) = u|_{K_2}(z) = \ldots = u|_{K_m}(z)$, for all functions $u \in V$, and similarly for all of the partial derivatives of degrees up to m.*
   - 2.2 *For each edge e in $\mathcal{T}$, let K_1, K_2 be the two triangles containing e. Then $u|_{K_1}(z) = u|_{K_2}(z)$, for all points z on the interior of e, and similarly for all of the partial derivatives of degrees up to m.*

Proof.

V is polynomial on each triangle K, so continuity at points on the interior of each triangle K is immediate. We just need to check continuity at points on vertices, and points on the interior of edges, which is equivalent to the two parts of the second condition. $\square$

This means that we just need to guarantee that the polynomial functions and their derivatives agree at vertices and edges (similar ideas extend to higher dimensions).

Definition 24 (Mesh entities)

Let K be a triangle. The (local) mesh entities of K are the vertices, the edges, and K itself. The global mesh entities of a triangulation $\mathcal{T}$ are the vertices, edges and triangles comprising $\mathcal{T}$.

Definition 25 (Geometric decomposition)

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. We say that the finite element has a (local) geometric decomposition if each dual basis function N_i can be associated with a single mesh entity $w \in W$ such that for any $f \in \mathcal{P}$, $N_i(f)$ can be calculated from f and derivatives of f restricted to w.

Definition 26 (Closure of a local mesh entity)

Let w be a local mesh entity for a triangle. The closure of w is the set of local mesh entities contained in w (including w itself).

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Definition 27 (C^m geometric decomposition)

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element with geometric decomposition W. We say that W is a C^m geometric decomposition, if for each local mesh entity w, for any $f \in \mathcal{P}$, the restriction $f|_w$ of f (and the restriction $D^k f|_w$ of the k-th derivative of f to w for $k \leq m$) can be obtained from the set of dual basis functions associated with entities in the closure of w, applied to f.

Exercise 28

Show that the Lagrange elements of degree k have C^0 geometric decompositions.

Exercise 29

Show that the Argyris element has a C^1 geometric decomposition.

Definition 30 (Discontinuous finite element space)

Let $\mathcal{T}$ be a triangulation, with finite elements $(K_i, P_i, \mathcal{N}_i)$ for each triangle K_i. The corresponding discontinuous finite element space V, is defined as

$$V = \left\{u : u|_{K_i} \in P_i, \, \forall K_i \in \mathcal{T}\right\}. \tag{22}$$

This defines families of discontinuous finite element spaces.

Example 31 (Discontinuous Lagrange finite element space)

Let $\mathcal{T}$ be a triangulation, with Lagrange elements of degree k, $(K_i, P_i, \mathcal{N}_i)$, for each triangle $K_i \in \mathcal{T}$. The corresponding discontinuous finite element space, denoted PkDG, is called the discontinuous Lagrange finite element space of degree k.

Definition 32 (Global C^m geometric decomposition)

Let $\mathcal{T}$ be a triangulation with finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$, each with a C^m geometric decomposition. Assume that for each global mesh entity w, the n_w triangles containing w have finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$ each with M_w dual basis functions associated with w. Further, each of these basis functions can be enumerated $N^w_{i,j} \in \mathcal{N}_i$, $j = 1, \ldots, M_w$, such that $N^w_{1,j}(u|_{K_1}) = N^w_{2,j}(u|_{K_2}) = \ldots = N^w_{n_w,j}(u|_{K_n}), \quad , j = 1, \ldots, M_w$, for all functions $u \in C^m(\Omega)$.
This combination of finite elements on $\mathcal{T}$ together with the above enumeration of dual basis functions on global mesh entities is called a global C^m geometric decomposition.

Definition 33 (Finite element space from a global C^m geometric decomposition)

Let $\mathcal{T}$ be a triangulation with finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$, each with a C^m geometric decomposition, and let $\hat{V}$ be the corresponding discontinuous finite element space. Then the global C^m geometric decomposition defines a subspace V of $\hat{V}$ consisting of all functions that u satisfy $N^w_{1,j}(u|_{K_1}) = N^w_{2,j}(u|_{K_2}) = \ldots = N^w_{n_w,j}(u|_{K_{n_w}}), \quad , j = 1, \ldots, M_w$ for all mesh entities $w \in \mathcal{T}$.

Proposition 34

Let V be a finite element space defined from a global C^m geometric decomposition. Then V is a C^m finite element space.

Proof.

From the local C^m decomposition, functions and derivatives up to degree m on vertices and edges are uniquely determined from dual basis elements associated with those vertices and edges, and from the global C^m decomposition, the agreement of dual basis elements means that functions and derivatives up to degree m agree on vertices and edges, and hence the functions are in C^m from Lemma 23. $\square$

Example 35

The finite element space built from the C^0 global decomposition built from degree k Lagrange element is called the degree k continuous Lagrange finite element space, denoted $\mathrm{P}k$.

Example 36

The finite element space built from the C^1 global decomposition built from the quintic Argyris element is called the Argyris finite element space.

Next we investigate how continuous functions can be approximated by finite element functions.

Definition 37 (Local interpolant)

Given a finite element $(K, \mathcal{P}, \mathcal{N})$, with corresponding nodal basis $\{\phi_i\}_{i=0}^k$. Let v be a function such that $N_i(v)$ is well-defined for all i. Then the local interpolant $\mathcal{I}_K$ is an operator mapping v to $\mathcal{P}$ such that

$$(I_K v)(x) = \sum_{i=0}^{k} N_i(v)\phi_i(x). \tag{23}$$

Lemma 38

The operator I_K is linear.

Lemma 39

$$N_i(I_K(v)) = N_i(v),\ \forall 0 \leq i \leq k. \quad (24)$$

Lemma 40

I_K is the identity when restricted to $\mathcal{P}$.

Exercise 41

Prove lemmas 38-40.

We would like to estimate the error $I_K u - u$. For a C^m function, this is achieved by the Taylor remainder theorem.

Definition 42 (Taylor polynomial)

Let $u \in C^m(K)$. Then the degree m Taylor polynomial for u at a point $y \in K$ is

$$T_y^m u(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha, \tag{25}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $D^\alpha = \prod_{i=1}^d \frac{\partial}{\partial x_i^{\alpha_i}}$, $x^\alpha = \prod_{i=1}^d x^{\alpha_i}$.

Theorem 43 (Taylor remainder theorem)

Let $u \in C^m(K)$. Then, for $y \in K$,

$$u(y) = T_y^m u(x) + m \sum_{|\alpha|=m} (x-y)^\alpha \frac{1}{\alpha!} \int_0^1 s^{m-1} D^\alpha u(y+s(x-y))\,\mathrm{d}s.$$

Sketch of proof.

Start from the 1D Taylor remainder theorem for $f \in C^m([0,1])$,

$$f(1) = \sum_{k=0}^{m-1} \frac{1}{k!} f(k)(0) + m \int_0^1 \frac{1}{m!} s^{m-1} f^{(m)}(1-s)\,\mathrm{d}s,$$

and substitute $f(s) = u(y + s(x-y))$. $\square$

Proposition 44 (L_2 estimate for Taylor polynomial for unit K)

Let K_1 be a triangle with diameter 1 and centre at the origin, and let $u \in C^m(K_1)$. Then for $y \in K_1$ we have

$$\|u - T_y^m u\|^2_{L^2(K_1)} \leq C_{K_1} \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(K_1)}. \tag{26}$$

Proof

$$
\begin{aligned}
&\|u - T_y^m u\|^2_{L^2(K_1)} \\
&= \int_{K_1} \left(m \sum_{|\alpha|=m} (x-y)^\alpha \frac{1}{\alpha!} \int_0^1 s^{m-1} D^\alpha u(y + s(x-y))\, \mathrm{d}s \right)^2 \mathrm{d}x, \\
&\leq C_0 \sum_{|\alpha|=m} \int_{K_1} \int_0^1 (D^\alpha u(y + s(x-y)))^2\, \mathrm{d}s\, \mathrm{d}x,
\end{aligned}
$$

having used the Cauchy-Schwartz inequality $\int_0^1 f(s)g(s)\,\mathrm{d}s \leq \left(\int_0^1 (f(s))^2\,\mathrm{d}s\right)^{1/2} \left(\int_0^1 (g(s))^2\,\mathrm{d}s\right)^{1/2}$.

Proof (Cont.)

Then, after a change of variables $z = y + s(x-y)$,

$$
\begin{aligned}
&\int_0^1 \int_{K_1} (D^\alpha u(y+s(x-y)))^2 \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^1 s^d \int_{K_1^s} (D^\alpha u(z))^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_0^1 s^d \int_{K_1} (D^\alpha u(z))^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leq C_1 \|D^\alpha u\|^2_{L^2(K_1)},
\end{aligned}
$$

where K_1^s is the image of K_1 under the change of variables and we note that $K_1^s \subset K_1$. $\square$

Proposition 45 (General L^2 estimate for Taylor polynomial)

Let K be a triangle with diameter d, and let $u \in C^m(K)$. Then for $y \in K$ we have

$$\|u - T_y^m u\|^2_{L^2(K)} \leq C_{K_1} d^{2m} \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(K)}, \tag{27}$$

where C_{K_1} is a constant that depends only on the shape of K but not the diameter.

Proof (homogeneity argument).

Take the coordinate transformation $x = x_0 + dz$ where x_0 is the centre of K, so that K transforms to K_1, and define $v(z) = u(x_0 + dz)$. Then,

$$
\begin{aligned}
\|u - T_y^m u\|_{L^2(K)}^2 &= d\|v - T_y^m v\|_{L^2(K_1)}^2, \\
&\leq C_{K_1} d \sum_{|\alpha|=m} \int_{K_1} (D^\alpha v(z))^2 \, \mathrm{d}z, \\
&\leq C_{K_1} d^{2m} \sum_{|\alpha|=m} \int_{K} (D^\alpha u(x))^2 \, \mathrm{d}x,
\end{aligned}
$$

as required. □

Proposition 46 (General L^2 estimate for derivative of Taylor polynomial)

Let K be a triangle with diameter d, and let $u \in C^m(K)$. Let β be a multi-index with $0 \leq |\beta| \leq m$. Then Then for $y \in K$ we have

$$\|D^\beta(u - T_y^m u)\|_{L^2(K)}^2 \leq C_{K_1} d^{2(m-|\beta|)} \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(K)}^2, \qquad (28)$$

where C_{K_1} is a constant that depends only on the shape of K but not the diameter.

Proof.

Use $D^\beta T_y^m u = T_y^{m-|\beta|} D^\beta u$, and apply the previous error estimate to $D^\beta u$. □

This gives us a toolkit to estimate errors in the L_2 norm, as well as norms that take account of higher derivatives.

Definition 47 (H^k norms and seminorms)

For C^k functions, the H^k norm is defined as

$$\|u\|^2_{H^k(K)} = \sum_{|\alpha|\leq k} \int_K (D^\alpha u)^2 \, \mathrm{d}x. \tag{29}$$

The H^k semi-norm is defined as

$$|u|^2_{H^k(K)} = \sum_{|\alpha|=k} \int_K (D^\alpha u)^2 \, \mathrm{d}x. \tag{30}$$

Now we will use the Taylor polynomial estimates to derive error estimates for the local interpolation operator. First we need to obtain the following bound.

Proposition 48

Let $(K_1, \mathcal{P}, \mathcal{N})$ be a finite element such that K_1 be a triangle with diameter 1, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$. Let $u \in C^k$ with $l < k$. Then

$$\|\mathcal{I}_{K_1} u\|_{H^k(K_1)} \leq C\|u\|_{H^k(K_1)}. \tag{31}$$

Proof.

We will sketch for the case $l = 0$. We have
$\int_{K_1}(\mathcal{I}_{K_1}u)^2\,\mathrm{d}x = \sum_{ij}\phi_i\phi_j\,\mathrm{d}xN_i(u)N)_j(u)$.
Then writing
$u(x_i) = u(x) + \int_0^1 (x-x_i)\cdot\nabla u(x_i + s(x-x_i))\,\mathrm{d}s$,
we get

$$
\begin{aligned}
|K_1|u(x_i)^2 &= \int_{K_1}\left(u(x) + \int_0^1 (x-x_i)\cdot\nabla u(x_i+s(x-x_i))\,\mathrm{d}s\right)^2 \mathrm{d}x, \\
&\leq 2\int_{K_1} u(x)^2 + \left(\int_0^1 (x-x_i)\cdot\nabla u(x_i+s(x-x_i))\,\mathrm{d}s\right)^2 \mathrm{d}x,
\end{aligned}
$$

and we eventually (following similar methods to above) get
$u(x_i) \leq C_1\int_{K_1} u(x))^2 + |\nabla u|^2\,\mathrm{d}x$,
and hence the result. $\square$

Corollary 49

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$. Let $u \in C^k$ with $l < k$. Then

$$\|\mathcal{I}_K u\|_{H^k(K)} \leq C\|u\|_{H^k(K)}, \tag{32}$$

where C is a constant that depends only on the shape of K and not the diameter.

Proof.

Use a homogeneity argument. □

Proposition 50

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element such that K has diameter d, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$, and $\mathcal{P}$ contain all polynomials of degree $m-1$ and below, with $l < m$. Let $u \in C^m$. Then for $m > k$, the local interpolation operator satisfies

$$\|\mathcal{I}_K u - u\|_{H^k(K)} \leq C d^{m-k} |u|_{H^m(K)}, \tag{33}$$

where C depends only on the shape of K (but not the diameter).

Proof.

Picking y in K, we have

$$\begin{aligned}
\|\mathcal{I}_K u - u\|^2_{H^k(K)} &= \|\mathcal{I}_K u - T^m_y u + T^m_y u + u\|^2_{H^k(K)} \\
&\leq \|T^m_y u - u\|^2_{H^k(K)} + \|\mathcal{I}(u - T^m_y u)\|^2_{H^k(K)}, \\
&\leq (1 + C)\|T^m_y u - u\|^2_{H^k(K)} \\
&\leq (1 + C) d^{2(m-k)} |u|^2_{H^m(K)},
\end{aligned}$$

as required. □

We now consider norms for the error over the entire mesh.

Proposition 51

Let $\mathcal{T}$ be a triangulation with finite elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)$, such that the minimum aspect ratio r of the triangles K_i satisfies $r > 0$, and such that the nodal variables in $\mathcal{N}$ involve only evaluations of functions or evaluations of derivatives of degree $\leq l$, and $\mathcal{P}$ contain all polynomials of degree $m-1$ and below, with $l < m$. Let $u \in C^m(\Omega)$. Let h be the maximum over all of the triangle diameters. Let V be the corresponding C^n finite element space. Then for $m > k$ and $k \leq n+1$, the global interpolation operator satisfies

$$\|\mathcal{I}_K u - u\|_{H^k(\Omega)} \leq Ch^{m-k}|u|_{H^m(\Omega)}. \tag{34}$$

Proof.

If V is a C^n finite element space, then the $n+1$-th derivatives are defined in the finite element sense. Then we may write

$$\begin{aligned}
\|\mathcal{I}_K u - u\|^2_{H^k(\Omega)} &= \sum_{K\in\mathcal{T}} \|\mathcal{I}_K u - u\|^2_{H^k(K)}, && (35)\\
&\leq \sum_{K\in\mathcal{T}} C_K d_K^{2(m-k)} |u|^2_{H^m(K)}, && (36)\\
&\leq C_{\max} h^{2(m-k)} \sum_{K\in\mathcal{T}} |u_H^m(K)^2, && (37)\\
&= C_{\max} h^{2(m-k)} |u|^2_{H^m(\Omega)}, && (38)
\end{aligned}$$

where the existence of the $C_{\max} = \max_K C_K$ is due to the lower bound in the aspect ratio. □