Finite Elements: examples 1

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1. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise linear finite element space known as P1. Compute the entries of the matrix
$$K_{ij} = a(\phi_i, \phi_j),$$
and the right-hand side vector
$$F_i = (\phi_i, f_I),$$
where $f_I \in S$ is the interpolant of f.
For an equispaced mesh with $h_i = x_i - x_{i-1} := h$, write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

2. Give a weak formulation for the following ODEs,
   (a)
$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$
   (b)
$$-u'' + u = f, \quad u'(0) = 0, \, u'(1) = \alpha.$$
   (c)
$$-u'' = f, \quad u'(0) = u'(1) = 0.$$
   What is wrong with the formulation of the last ODE?

3. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise quadratic finite element space known as P2, defined by the following:
   (a) $S \subset C^0([0, 1])$.
   (b) For $v \in S$, $v|_{[x_{j-1}, x_j]}$ is a quadratic function of x.
   (c) $v(0) = 0$.

   Find a nodal basis for S, using the nodes for P1 plus nodes at element midpoints $(x_j + x_{j+1})/2$. Evaluate the matrix K for this basis.

4. Under the same assumptions as Theorem 1.8, prove that
$$\|u - u_I\| \leq Ch^2 \|u''\|.$$
(hint: make use of the fact that $u(0) = 0$ to write u in terms of u'.)

5. Under the same assumptions as Theorem 1.8, prove the following version of *Sobolev's inequality*:
$$\|v\|^2_{\max} \leq Ca(v, v), \quad \forall v \in V \cap C^1([0, 1]).$$
Give a value for C.

6. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.

7. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.

8. Let K be a rectangle, Q_2 be the space of biquadratic polynomials, and let $\mathcal{N}$ be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that $\mathcal{N}$ determines the finite element.

9. For K being the unit square, determine the nodal basis for the element in the previous question.