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Deriving the Euler and Milstein Schemes for the Heston Model

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Problem Statement

The Heston model is given by the following system of stochastic differential equations (SDEs) for the stock price S_t and its variance v_t:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1$$
$$dv_t = \kappa(\bar{v} - v_t)dt + \sigma\sqrt{v_t} dW_t^2$$

where W_t^1 and W_t^2 are Wiener processes with a correlation $\mathbb{E}[dW_t^1 dW_t^2] = \rho dt$.

Step 1: Decouple the System Using Cholesky Decomposition

Since W_t^1 and W_t^2 are correlated, we first decouple them using the Cholesky decomposition. We can express dW_t^2 as:

$$dW_t^2 = \rho dW_t^1 + \sqrt{1 - \rho^2} dZ_t$$

where dZ_t is a Wiener process independent of dW_t^1. This transformation simplifies the SDEs to:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1$$
$$dv_t = \kappa(\bar{v} - v_t)dt + \sigma\sqrt{v_t}(\rho dW_t^1 + \sqrt{1 - \rho^2} dZ_t)$$

Step 2: Euler Scheme

The Euler scheme is the simplest method for numerically solving SDEs. For the Heston model, the Euler discretization is as follows:

Discretization for S_t:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t} S_t \Delta W_t^1$$

where ΔW_t^1 is a Wiener increment over Δt, typically $\Delta W_t^1 \sim \mathcal{N}(0, \Delta t)$.

Discretization for v_t:

$$v_{t+\Delta t} = v_t + \kappa(\bar{v} - v_t)\Delta t + \sigma\sqrt{v_t}(\rho\Delta W_t^1 + \sqrt{1 - \rho^2}\Delta Z_t)$$

where ΔZ_t is another independent Wiener increment.

Step 3: Milstein Scheme

The Milstein scheme extends the Euler method by including a correction term that accounts for the derivative of the diffusion term.

Milstein Discretization for S_t:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t} S_t \Delta W_t^1 + \frac{1}{2} \cdot v_t S_t((\Delta W_t^1)^2 - \Delta t)$$

The Milstein correction term $\frac{1}{2} \cdot v_t S_t((\Delta W_t^1)^2 - \Delta t)$ accounts for the curvature of the diffusion term.

Milstein Discretization for v_t:

$$v_{t+\Delta t} = v_t + \kappa(\bar{v} - v_t)\Delta t + \sigma\sqrt{v_t}(\rho\Delta W_t^1 + \sqrt{1-\rho^2}\Delta Z_t) + \frac{1}{2}\cdot\sigma^2 v_t((\Delta W_t^2)^2 - \Delta t)$$

The Milstein correction here uses ΔW_t^2 in the same manner as for S_t, but with the appropriate derivative terms for the variance process.

Summary

- **Euler Scheme:** The simplest numerical method, providing first-order weak convergence by approximating the SDEs using the current value of the process and the Wiener increments.
- **Milstein Scheme:** Extends the Euler method by including a correction term for the diffusion component, leading to first-order strong convergence.