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Deriving the Explicit Solution to an SDE with a Law but not necessarily a Density

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Stochastic Differential Equations

Exercise

X_t is a process which for each fixed t, X_t is a random variable and hence it has a law but not necessarily a density. Hence, it is not even clear whether the solution to an SDE has a density! A sufficient condition for X_t to admit a density is the following Novikov's condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T a(X_t)^2\, dt\right)\right] < \infty$$

Solution

To derive the explicit solution for a general SDE, let's consider the following form of an SDE:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t,$$

where:

- X_t is the stochastic process.
- $\mu(X_t, t)$ is the drift term.
- $\sigma(X_t, t)$ is the diffusion term.
- dW_t is the differential of a Wiener process (standard Brownian motion).

Step 1: Apply Ito's Lemma

To solve the SDE explicitly, we apply **Ito's Lemma** to transform the equation into an integrable form.

For a function $f(X_t, t)$, Ito's Lemma states:

$$df(X_t, t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X_t}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma(X_t, t)^2\, dt.$$

Step 2: Choose a Suitable Function for Transformation

We will use the logarithm function $f(X_t) = \ln(X_t)$. Taking $f(X_t) = X_t$, we substitute it into Ito's Lemma to get:

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \mu(X_t, t)\frac{\partial X_t}{\partial X_t} + \frac{1}{2}\sigma(X_t, t)^2\frac{\partial^2 X_t}{\partial X_t^2}\right)dt + \sigma(X_t, t)\frac{\partial X_t}{\partial X_t}dW_t.$$

Simplify:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t.$$

Step 3: Integrate the SDE

Integrate both sides from 0 to t:

$$X_t = X_0 + \int_0^t \mu(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dW_s.$$

This is the explicit solution for the SDE in integral form, where the first integral represents the deterministic component (drift), and the second integral represents the stochastic component (diffusion).

Step 4: Apply Novikov's Condition

To ensure that the solution admits a density, we apply Novikov's condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T a(X_t)^2\, dt\right)\right] < \infty.$$

If this condition holds, it guarantees that the stochastic exponential martingale remains integrable and that the process X_t has a density.

The explicit solution in integral form involves the integration of both the drift and diffusion components over time. Novikov's condition ensures the solution's density exists. To get a specific explicit form, we need the exact forms of $\mu(X_t, t)$ and $\sigma(X_t, t)$. To apply Novikov's condition to the solution of a stochastic differential equation (SDE), we ensure that the solution admits a density by proving that the stochastic exponential martingale remains integrable.

Given SDE and Novikov's Condition

For the SDE:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t,$$

we aim to check if the solution admits a density using Novikov's condition. Novikov's condition is:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T a(X_t)^2\, dt\right)\right] < \infty,$$

where $a(X_t) = \frac{\sigma(X_t,t)}{\sigma(X_0,0)}$

Applying Novikov's Condition

Step 1: Identify $a(X_t)$

For our SDE, let's assume that the diffusion term $\sigma(X_t, t)$ is deterministic and that Novikov's condition applies to this term:

$$a(X_t) = \sigma(X_t, t).$$

Novikov's condition becomes:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sigma(X_t, t)^2\, dt\right)\right] < \infty.$$

Step 2: Check the Condition for Integrability

To apply Novikov's condition, consider the exponential term:

$$\exp\left(\frac{1}{2}\int_0^T \sigma(X_t,t)^2\,dt\right).$$

For the condition to hold, the expected value must be finite. This means we need to show that:

$$\int_0^T \sigma(X_t,t)^2\,dt \quad \text{is integrable, such that:}$$

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sigma(X_t,t)^2\,dt\right)\right] < \infty.$$

Step 3: Ensure the Condition Holds

If $\sigma(X_t,t)$ is bounded, say $\sigma(X_t,t) \leq C$ for all $t \in [0,T]$ and some constant $C > 0$, then:

$$\int_0^T \sigma(X_t,t)^2\,dt \leq C^2 T.$$

Thus:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}C^2 T\right)\right] = \exp\left(\frac{1}{2}C^2 T\right) < \infty.$$

This confirms that Novikov's condition is satisfied if $\sigma(X_t,t)$ is bounded.

Conclusion

To ensure that the solution X_t of the SDE has a density, Novikov's condition must hold:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sigma(X_t,t)^2\,dt\right)\right] < \infty.$$

If $\sigma(X_t,t)$ is bounded, this condition is satisfied, ensuring the solution admits a density. Therefore, the sufficient condition for the density of X_t is that the diffusion term $\sigma(X_t,t)$ does not grow too rapidly.