The Iterative Unitary Matrix Multiply Method and Its Application to Quantum Kicked Rotator

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(Dated: February 1, 2008)

We use the iterative unitary matrix multiply method to calculate the long time behavior of the resonant quantum kicked rotator with a large denominator. The delocalization time is exponentially large. The quantum wave delocalizes through degenerate states. At last we construct a nonresonant quantum kicked rotator with delocalization.

Introduction.—The quantum kicked rotator (QKR) [1], which describes a periodically kicked rotator, is one of the most studied model of quantum chaos [2]. The classical correspondence of QKR is the standard map [3, 4]. Classically, the energy of the rotator grows without a limit. But to a quantum rotator, if the kick frequency and the rotator frequency is commensurate, QKR delocalizes in the momentum space and if incommensurate, QKR generally localizes [1]. Fishman *et al* explained the classical and quantum difference by transforming QKR into an Anderson localization problem [5].

In the paper, we try to understand how the delocalization of commensurate cases happens. This is important for several reasons. First, the commensurate (incommensurate) case is described by a rational (irrational) number. The qualitative statement that delocalization happens to the commensurate cases is correct but incomplete. We want to gain a quantitative understanding. Second, no physical quantity is rational or irrational. A physical quantity has only several significant digits, while the distinction between rational and irrational numbers depends on infinite significant digits. An infinitesimal error can change a rational (irrational) number into an irrational (rational) number. While we expect the system changes little from our experiences of studying physics as was emphasized by Hofstadter [6]. To recognize and reconcile the conflict is one aim of quantum chaos. Third, Fishman *et al*'s result [5] seems to tell us localization happens to all the incommensurate cases. Is there at least one incommensurate case for which delocalization happens? Casati *et al* has derived a quantum Lyapunov equation to describe the difference between the dynamics of commensurate and incommensurate cases [7, 8]. Based on the formula, Casati *et al* claimed there are some incommensurate cases of delocalization [7]. But their argument is problematic [8] from the perspective of the exponentially large delocalization time discovered in the paper.

In the paper, we prove by numerical calculation for the commensurate case with a large denominator, the delocalization time is exponentially large. Such a large denominator effect is explained by the degenerate perturbation theory, which is based on the observation that degenerate states are the delocalization path. Localization of incommensurate cases can be understood to be caused by the large denominator effect. The large denominator effect and the quantum Lyapunov equation [7, 8] partially reconcile the conflict between commensurate and incommensurate cases and naturally lead to an incommensurate case of delocalization. This partially solves the problem: to find an incommensurate case of delocalization, posed by Casati *et al* [7, 9] and gives a counterexample to Fishman's argument [5], although a very weak one.

Numerical methods.—For a system with a periodical Hamiltonian, the unitary operator of one period is the Floquet operator F. The unitary operator of 2^N periods is F^{2^N}.

$$
\begin{aligned}
F^{2^N} &= (F^{2^{N-1}})^2; \\
F^{2^{N-1}} &= (F^{2^{N-2}})^2; \\
&\cdots \\
F^4 &= (F^2)^2.
\end{aligned}
\tag{1}
$$

From Eq. (1), we can calculate F^{2^N} from F by iteratively multiplying the unitary matrices for N times. This method is referred as the iterative unitary matrix multiply method (IUMM). It is impossible to calculate very long time behavior of QKR using the usual fast Fourier transform method [5]. IUMM is actually the same method as direct diagonalization or the matrix vector multiply method used in the original paper [1] of QKR. See the section III and IV of [1].

Calculation results.—The Hamiltonian of QKR is

$$
H = -\frac{1}{2}\hbar^2\frac{\partial^2}{\partial\theta^2} + k\cos\theta\sum_{n=1}^{\infty}\delta(t-n\tau),
\tag{2}
$$

where $\hbar$ is the Planck constant, τ the kick period and k the kick strength. The matrix element of F is

$$
F_{nm} = \langle n|F|m\rangle = \exp(-i\hbar\tau\frac{m^2}{2})i^{m-n}J_{n-m}(\frac{k}{\hbar}),
\tag{3}
$$

where $|n\rangle = \frac{1}{\sqrt{2\pi}}e^{in\theta}$. We apply IUMM to QKR. In the calculation $\hbar = 1$, $k = 1$ and $\tau = \frac{2\pi}{q} = \frac{\pi}{10}$. The initial state is $|0\rangle$. This is the commensurate/resonant case with a large denominator $q = 20$. $F_{nm} = F_{n+20,m+20}$ for
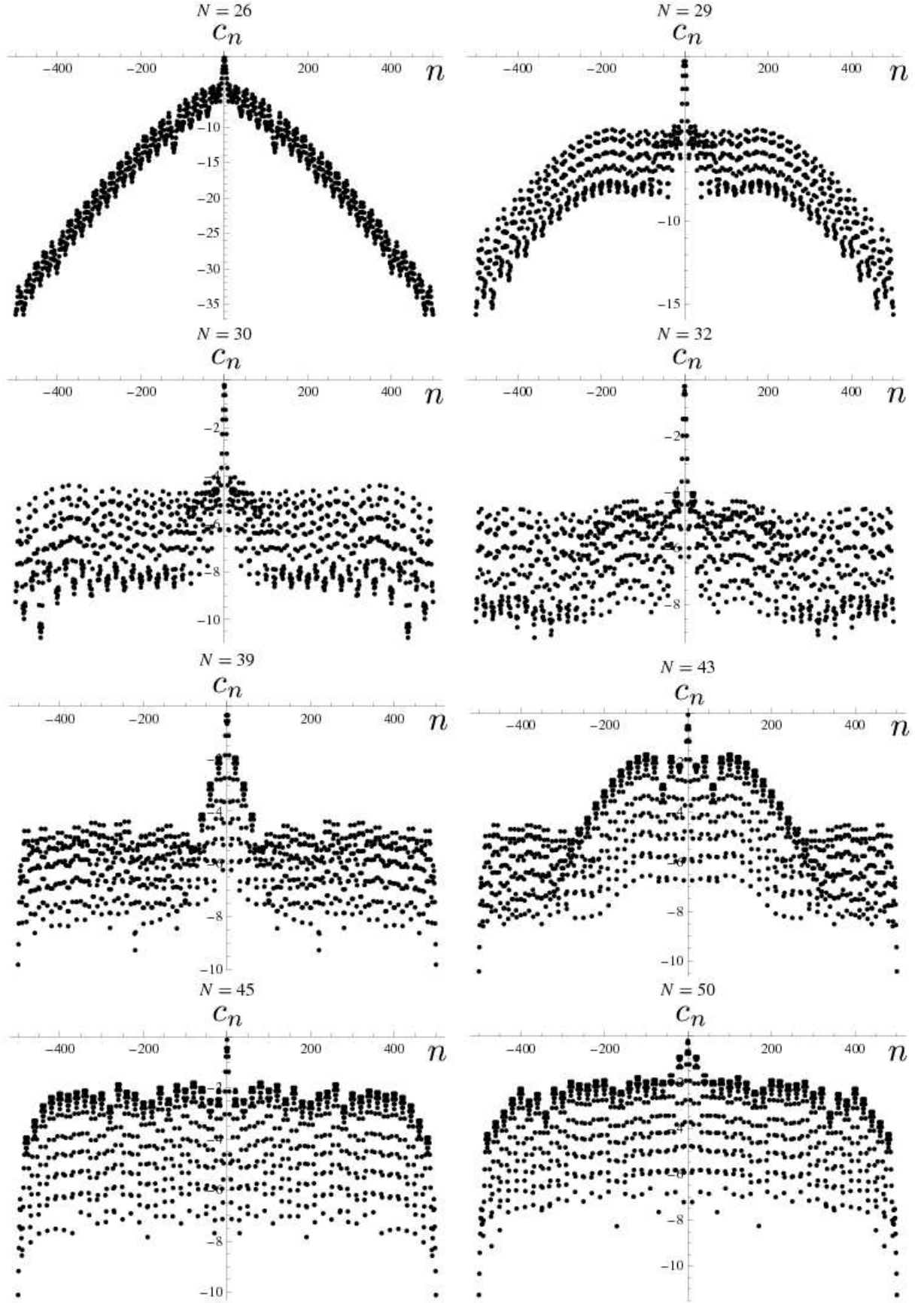
FIG. 1: QKR wave function at different time. N is at time $2^N \tau$. n is $|n\rangle$ and c_n is the base-10 logarithm of the absolute value of the wave function on $|n\rangle$. n is from -500 to 500 in our calculation.
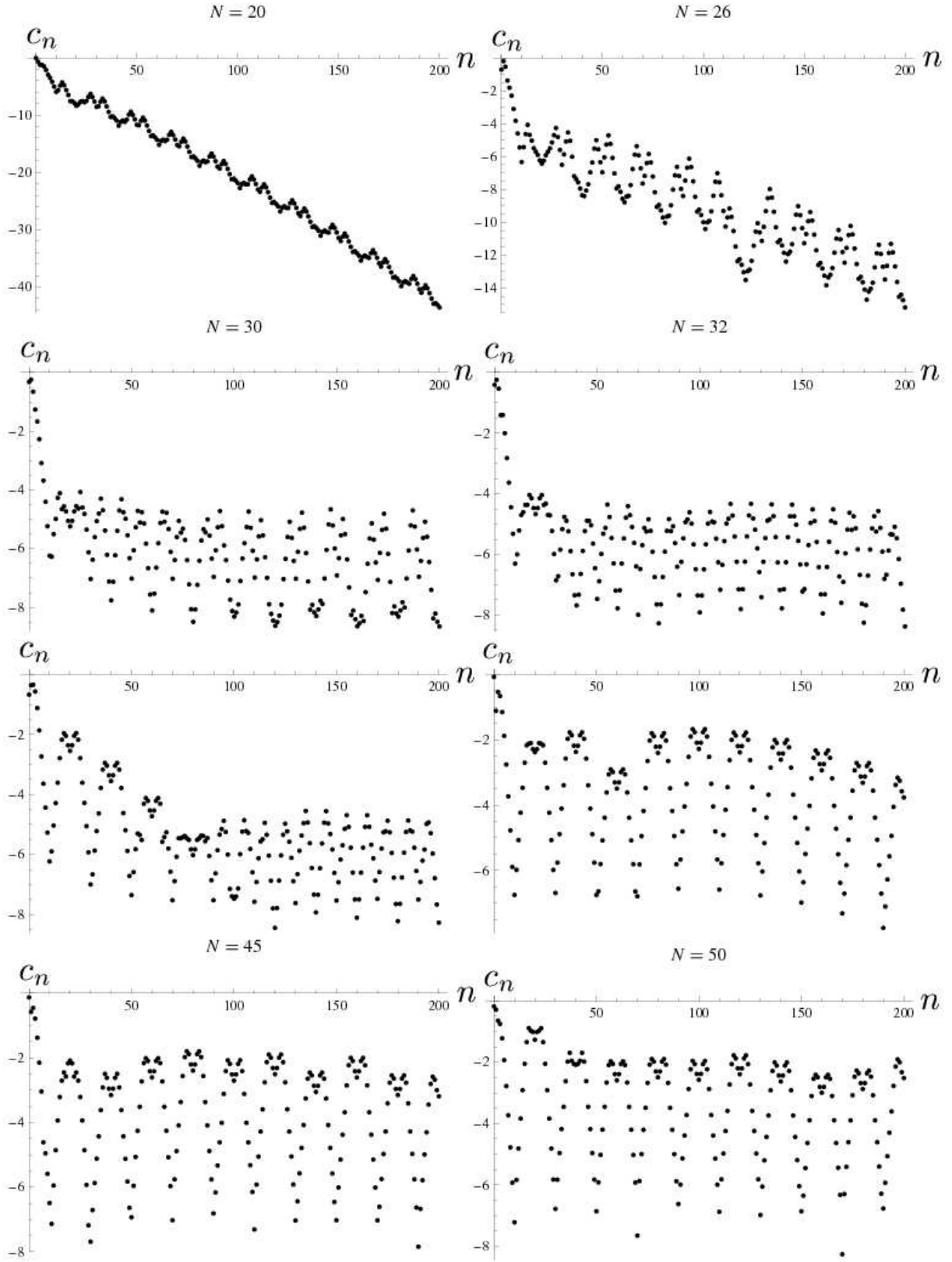
FIG. 2: Clearer figures of QKR wave function. Note the changes of peaks and valleys from $N = 26$ to $N = 43$. The peaks indicate what $|n\rangle$s actively contribute to the wave propagation.

every n and m. The rotator will delocalize in the future, nevertheless it delocalizes very slowly.

FIG. 1 and FIG. 2 show the same calculation results. The distribution of QKR wave function is clearer in FIG. 2. Before $N = 20$ (the time is $2^{20}\tau = 1.05 \times 10^6\tau$), QKR does not delocalize at all. To a resonant case, this is unexpected. At $N = 20$, peaks (local maxima) and valleys (local minima) appear. Naively one expects, from $N = 20$ to 50, peaks should be at $|n = 20 \times I\rangle$s, where I is an integer, because such states are resonant with the $|0\rangle$. Actually $|n = 20 \times I\rangle$s are always valleys. Before $N = 39$, the peaks are at $|n = 10 \times O \pm 3\rangle$s, where O is an odd integer, such as $n = 13, 27, 33, 47, 53, \cdots$. At $N = 50$, the peaks are at $|n = 20 \times I \pm 3\rangle$s, such as $n = 17, 23, 37, 43, \cdots$.

At $N = 26$, the triangle like wave function outside $n = 0$ forms and at $N = 30$ it flattens. The height of the flattened wave function is approximately 10^{-4}. The main distribution is still at $|0\rangle$. From $N = 30$ to 39, the needle like wave function around $n = 0$ becomes a triangle like one. From $N = 39$ the triangle expands and at $N = 45$ the wave function flattens again. The height of the QKR wave function is approximately 10^{-2}. At $N = 50$, a triangle like wave function forms again.

Around $n = 0$, the circulation: needle $\rightarrow$ triangle $\rightarrow$ expanded triangle $\rightarrow$ flattened, drives the whole delocalization process. Will the wave function around $n = 0$ be totally flattened by one more circulation or several more after $N = 50$? Will $|n = 20 \times I\rangle$s finally become the only peaks when N is very large and how? We do not know! Our calculation overflows around $N = 60$, which may be caused by the ununitarity of the truncated Floquet operator.

At what N should the wave function be considered as delocalized? At $N = 30$, there is still lots of distribution of the wave function at $n = 0$. $N = 45$ is more proper than $N = 30$. We define $T_{\tau/2\pi} \times \tau$ as the delocalization time of QKR with the period τ. For simplicity, we also refer $T_{\tau/2\pi}$ as the delocalization time. For $\tau = 2\pi/q$, QKR delocalizes exponentially slowly. We estimate $T_{1/q} \approx \exp(cq/k)$, where c is a factor that depends weakly on k and q. If we assume QKR is delocalized at $N = 45$, $e^{20c}\tau \approx 2^{45}\tau$ and $c \approx 1.56$.

Delocalization path and degenerate perturbation theory.—The delocalization time can be estimated from the degenerate perturbation theory. The sequence $\{-\frac{1}{40}n^2\text{Mod}(1)\}_{n=0,1,\cdots,20}$, which are the phases of F_{nn} divided by 2π, is

$$
\begin{aligned}
\{0, & \frac{39}{40}, \frac{9}{10}, \frac{31}{40}, \frac{3}{5}, \frac{3}{8}, \frac{1}{10}, \frac{31}{40}, \frac{2}{5}, \frac{39}{40}, \frac{1}{2}, \\
& \frac{39}{40}, \frac{2}{5}, \frac{31}{40}, \frac{1}{10}, \frac{3}{8}, \frac{3}{5}, \frac{31}{40}, \frac{9}{10}, \frac{39}{40}, 0\}.
\end{aligned} \tag{4}
$$

In a period, there is four $\frac{39}{40}$s, four $\frac{31}{40}$s, two $\frac{9}{10}$s, two $\frac{3}{5}$s, two $\frac{2}{5}$s, two $\frac{3}{8}$s, two $\frac{1}{10}$s, one $\frac{1}{2}$ and one 0. The quantum wave is easier to propagate between $\frac{31}{40}$s or between $\frac{39}{40}$s. So $|n = 20 \times I\rangle$s are valleys from $N = 20$ to 50. When $n = 1, 9, 11, 19, 21, 29, 31, 39$, the phases are $\frac{39}{40} \times 2\pi$.

The intervals between two degenerate states are 1 and 8. When $n = 3, 7, 13, 17, 23, 27, 33, 37$, the phases are $\frac{31}{40} \times 2\pi$. The intervals are 4 and 6. So the wave is easiest to propagate between $n = 3, 7, 13, 17, 23, 27, 33, 37$, which are peaks in FIG. 2. But we do not know why peaks are at only some of the $|n = 10 \times I \pm 3\rangle$s. Peaks even change from the $|n = 10 \times O \pm 3\rangle$s to the $|n = 20 \times I \pm 3\rangle$s as we discussed above.

From the degenerate perturbation theory, if the wave propagates through the path $|0\rangle \rightarrow |20\rangle$, F is approximated by

$$
F_{\text{appr}} = \begin{pmatrix} F_{0,0} & F_{0,20} \\ F_{20,0} & F_{20,20} \end{pmatrix} = \begin{pmatrix} J_0(k) & J_{-20}(k) \\ J_{20}(k) & J_0(k) \end{pmatrix}. \tag{5}
$$

The eigenvalues of F_{appr} is $J_0(1) \pm J_{20}(1)$. So after approximate $J_0(1)/J_{20} = 1.98 \times 10^{24}$-time kicks, the wave function will be transferred from $|0\rangle$ to $|20\rangle$. This is far larger than $2^{45} = 3.52 \times 10^{13}$ of our numerical result. A more exact estimate has to take into account other degenerate states. The quantum wave can propagate through the path $|0\rangle \rightarrow |1\rangle \rightarrow |9\rangle \rightarrow |11\rangle \rightarrow |19\rangle \rightarrow |20\rangle$. The states contributing to the wave propagation are mainly these states. So F is approximated by F_{appr}, which only considers the states in the delocalization path.

$$
F_{\text{appr}} = \begin{pmatrix} F_{0,0} & F_{0,1} & & & & \\ F_{1,0} & F_{1,1} & F_{1,9} & & & \\ & F_{9,1} & F_{9,9} & F_{9,11} & & \\ & & F_{11,9} & F_{11,11} & F_{11,19} & \\ & & & F_{19,11} & F_{19,19} & F_{19,20} \\ & & & & F_{20,19} & F_{20,20} \end{pmatrix}. \tag{6}
$$

The propagation time from $|0\rangle$ to $|1\rangle$ is $J_0(1)/J_1(1)$; from $|1\rangle$ to $|9\rangle$ is $J_0(1)/J_8(1)$; from $|9\rangle$ to $|11\rangle$ is $J_0(1)/J_2(1)$; and so on. The delocalization time from $|0\rangle$ to $|20\rangle$ is estimated to be

$$
T_{1/q} \approx \frac{J_0(1)J_0(1)J_0(1)J_0(1)J_0(1)}{J_1(1)J_8(1)J_2(1)J_8(1)J_1(1)} = 1.33 \times 10^{15}. \tag{7}
$$

This is more realistic than Eq. 5. Another path of wave propagation, $|0\rangle \rightarrow |3\rangle \rightarrow |7\rangle \rightarrow |13\rangle \rightarrow |17\rangle \rightarrow |20\rangle$, gives

$$
T_{1/q} \approx \frac{J_0(1)J_0(1)J_0(1)J_0(1)J_0(1)}{J_3(1)J_4(1)J_6(1)J_4(1)J_3(1)} = 5.34 \times 10^{12}, \tag{8}
$$

which is close to the delocalization time $2^{45} = 3.52 \times 10^{13}$. One problem of the degenerate perturbation theory is F_{appr} is not unitary.

An incommensurate case of delocalization.—The smaller q, the faster the delocalization. If $\tau/2\pi = p/q \approx p'/q'$ and $q' \ll q$, QKR with $\tau = 2\pi p/q$ delocalizes quicker because it is closer to a stronger resonance. But $\tau = 2\pi/q$ is far from any strong resonance in all the ($\tau = 2\pi p/q$)s, where $p = 1, 2, \cdots, q - 1, q$. So it has the largest delocalization time and $T_{1/q} \approx \exp(cq/k)$ is the upper limit of delocalization time in all the ($\tau = 2\pi p/q$)s.

Now we construct irrational $\tau/2\pi$ with delocalization. Imagine two QKRs with almost equal kick period τ and τ' and the equal kick strength k. $\delta\tau = |\tau - \tau'| \ll 1$. $U(M,\tau)$ is the M-period unitary operator with the kick periods τ and $U(M,\tau')$ with τ'. The difference between the matrix elements of two unitary operators [7, 8]

$$|U(M,\tau)_{nm} - U(M,\tau')_{nm}| \leq \gamma M^3 k^2 \delta\tau. \qquad (9)$$

As the particular value of γ is not important, we set $\gamma = 1$. Before $(\epsilon/(k^2\delta\tau))^{1/3}$-time kicks, $|U(M,\tau)_{nm} - U(M,\tau')_{nm}| \leq \epsilon$.

We consider $k = 1$ and construct

$$\begin{aligned}
\tau/2\pi =& 1/q + 1/\lfloor\exp(3c_1 q)\rfloor + 1/\lfloor\exp(3c_2\exp(3c_1 q))\rfloor \\
& + 1/\lfloor\exp(3c_3\exp(3c_2\exp(3c_1 q)))\rfloor \\
& + \cdots \\
& + 1/\lfloor\exp(3c_n\cdots\exp(3c_3\exp(3c_2\exp(3c_1 q))))\rfloor \\
& + \cdots.
\end{aligned} \qquad (10)$$

$\lfloor x\rfloor$ is an integer around the real number x (For the convenience of the argument below, $\lfloor x\rfloor$ is not the same as the floor function in mathematics.) and ensures every term is a rational number. q is a positive integer such as 20. $c_1 > c_{1\mathrm{d}}$ and $c_{1\mathrm{d}}$ is the factor in the delocalization time $T_{1/q} = \exp(c_{1\mathrm{d}}q)$. $c_2 > c_{2\mathrm{d}}$ and $c_{2\mathrm{d}}$ is the factor in the delocalization time $T_{1/q_1} = \exp(c_{2\mathrm{d}}q_1)$, where $q_1 = \lfloor\exp(3c_1 q)\rfloor$. And so on.

$$|\tau/2\pi - 1/q| \approx 1/\exp(3c_1 q). \qquad (11)$$

After $\sqrt[3]{q_1}$-time kicks, the dynamics of τ and $\tau' = 2\pi/q$ will not diverge from each other much due to Eqs. (9) and (11). QKR with τ propagates to a domain in the momentum space as large as l_1. We choose $c_1 \gg c_{1\mathrm{d}}$ to ensure $l_1 \gg k^2/4 = 1/4$. We choose $\lfloor\exp(3c_1 q)\rfloor$ to be an integer approximately $\exp(3c_1 q)$ and to be multiples of q. So from $\sqrt[3]{q_1}$ kicks to $\sqrt[3]{q_2}$ kicks, the delocalization speed of QKR is larger than or equal to QKR with $2\pi/q_1$. After $\sqrt[3]{q_2}$-time kicks, QKR propagates to a larger domain $l_2 > l_1$. We choose $c_2 \gg c_{2\mathrm{d}}$ to ensure $l_2 \gg l_1$. And so on. $l_\infty = \infty$. So QKR with the kick period τ will delocalize.

Even if $T_{1/q} \neq \exp(cq/k)$, we can always construct

$$\frac{\tau}{2\pi} = \frac{1}{q} + \frac{1}{\lfloor T_{1/q}^3\rfloor} + \frac{1}{\lfloor T_{1/(T_{1/q}^3)}^3\rfloor} + \cdots. \qquad (12)$$

QKR with τ in Eq. (12) delocalizes.

We note similar irrational numbers have been constructed by Avron *et al* concerning the Harper equation [10] and by Berry [11] and Prange *et al* [12] concerning the Maryland model. It cannot be a coincidence that similar numbers are constructed to three totally different problems. We think such irrational numbers *universally* have similar behavior with rational numbers in problems of quantum chaos. The way to construct irrational numbers in Eqs. (10) and (12) is very general and our argument depends on Eq. 9, which is a *universal* quantum Lyaponov equation [8].

Problems.—Some problems remain. First, how does k influence $T_{1/q}$? Second, how to estimate $T_{p/q}$? Do $T_{p/q}$s generally approximate to $T_{1/q}$? Third, is there one incommensurate case of delocalization, which is not similar to Eqs. (10) and (12)? We think localization happens to the general Liouville number $\tau/2\pi$, such as the Liouville constant.

Conclusion.—First, we have calculated the long time behavior of QKR using IUMM. It is discovered the delocalization time is exponentially large for large denominators. Second, we have constructed an irrational number of delocalization. Concerning QKR, Eqs. (10) and (12) are the first irrational number with delocalization ever known. Both results have important meaning for the theory of QKR. Third, the large delocalization time is explained by the degenerate perturbation theory, which is suggested by and consistent with the delocalization path of the numerical calculation. The phenomena that the wave propagates between degenerate or almost degenerate states may be found in many other systems.

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