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Estimates for the Eigenvalues of Matrices

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Abstract—For matrices whose eigenvalues are real (such as Hermitian or real symmetric matrices), we derive simple explicit estimates for the maximal ($\lambda_{\max}$) and the minimal ($\lambda_{\min}$) eigenvalues in terms of determinants of order less than 3. For 3×3 matrices, we derive sharper estimates, which use $\det A$ but do not require to solve cubic equations.

KEY WORDS: *eigenvalue problem for 3×3 matrices, estimates of minimal and maximal eigenvalues, characteristic polynomial.*

1. BASIC RELATIONS AND INVARIANTS

The eigenvalues λ_i of the matrix $A = (a_{ij})_{i,j=1,\dots,n}$ are the roots of its characteristic polynomial

$$P(\lambda) \equiv \det(\lambda E - A) = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} - \dots + (-1)^n\sigma_n, \qquad \sigma_n = \det A, \tag{1}$$

where E is the unit matrix,

$$\sigma_1 = \sum_{i=1}^{n} a_{ii}, \qquad \sigma_2 = \sum_{1\le i<j\le n} M_{ij}, \quad M_{ij} = a_{ii}a_{jj} - a_{ij}a_{ji}. \tag{2}$$

The Vièta theorem and Eq. (1) imply that

$$\sigma_1 = \lambda_1 + \dots + \lambda_n, \qquad \sigma_2 = \lambda_1(\lambda_2 + \dots + \lambda_n) + \lambda_2(\lambda_3 + \dots + \lambda_n) + \dots + \lambda_{n-1}\lambda_n. \tag{3}$$

The numbers a_{ij} and λ_k in Eqs. (1)–(3) may be real or complex. Further, we will suppose that the coefficients a_{ij} are real.

The eigenvalues λ_k and functions of $\{\lambda_k\}$ are invariants of orthogonal transformations, i.e., they are the same for matrices A and C^TAC, where C any orthogonal matrix, and C^T is the transposed matrix. In particular, the "quadratic norm"

$$\|A\| = \biggl(\sum_{i,j=1}^{n} a_{ij}^2\biggr)^{1/2} \equiv (\sigma_1^2 - 2\sigma_2)^{1/2} \equiv (\lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2)^{1/2} \tag{4}$$

and the "dispersion" of the eigenvalues

$$R = \sum_{1\le j<k\le n} (\lambda_j - \lambda_i)^2 \equiv (n-1)\sigma_1^2 - 2n\sigma_2 \equiv n\|A\|^2 - \sigma_1^2. \tag{5}$$

are such invariants. This follows from the calculation rule for the coefficients at $\lambda_j\lambda_k$ and from Eq. (4).

If all diagonal elements a_{ii} of the matrix A are given an increment δ, the eigenvalues λ_i acquire the same increment (this readily follows from Eq. (1)). For the matrix $B = A + \delta E$, the numbers

$$\tau_1 = \sigma_1 + n\delta, \qquad \tau_2 = \sigma_2 + (n-1)\sigma_1\delta + n(n-1)\frac{\delta^2}{2} \tag{6}$$

play the same role as σ_1 and σ_2 do for A.

The numbers R and $R = nS_d - \sigma_1^2$ (S_d is the sum of the squares of the diagonal elements of the matrix) are the same for the matrices $A + \delta E$ and A, and the numbers S_d and $\|A\|$ are replaced by the numbers

$$S_d(\delta) = S_d + 2\sigma_1\delta + n\delta^2, \qquad \|A + \delta E\|^2 = \|A\|^2 + 2\sigma_1\delta^2.$$

The values of $S_d(\delta)$ and $\|A+\delta E\|^2$ on $-\infty < \delta < \infty$ attain their minima at the point $\delta = -\sigma_1/n$. For this δ, we have

$$S_d\left(-\frac{\sigma_1}{n}\right) = S_d - \frac{\sigma_1^2}{n} \equiv \frac{R_d}{n}, \qquad \left\|A - E\frac{\sigma_1}{n}\right\|^2 = \|A\|^2 - \frac{\sigma_1^2}{n} = \frac{R}{n}, \tag{7}$$

$$\tau_1 = 0, \qquad \tau_2 = \sigma_2 - (n-1)\frac{\sigma_1^2}{2n} \equiv -\frac{R}{2n}. \tag{8}$$

If all the numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ are known for the matrix A, then the value of $\det(A + \delta E)$ can be obtained for any δ by setting $\lambda = -\delta$ in Eq. (1). In particular, for $n = 3$,

$$\det(A + \delta E) = \det A + \sigma_2\delta + \sigma_1\delta^2 + \delta^3. \tag{9}$$

2. ESTIMATES FOR EIGENVALUES

For a real matrix of nth order with real eigenvalues λ_i, Lemmas 1 and 2 will provide an estimate of the interval $\lambda_{\min} \le \lambda_i \le \lambda_{\max}$ containing all the eigenvalues of the given matrix. More accurate estimates are given in Sec. 3.

It is well known [1, Sec. 13] that, for all λ_i, the number $|\lambda_i|$ is less than or equal to any norm of the matrix. In particular,

$$|\lambda_i| \le \|A\| = \left(\sum_{k,j=1}^{n} a_{kj}^2\right)^{1/2}, \qquad |\lambda_i| \le \|A\|_1 = \max_k \sum_{j=1}^{n} |a_{kj}|.$$

Sharper estimates are based on the fact that the numbers $\lambda_1, \dots, \lambda_n$ are the roots of the characteristic equation $P(\lambda) = 0$ (see (1)) whose coefficients satisfy relations (3) and can be expressed in terms of the matrix elements a_{ij} (see (2)).

Lemma 1. *If the eigenvalues $\lambda_1, \dots, \lambda_n$ are real and the numbers σ_1 and σ_2 (see Eq. (3)) are given, then all λ_i belong to the segment $[\lambda_*, \lambda^*]$, where*

$$\lambda_* = \frac{\sigma_1}{n} - (n-1)\rho, \qquad \lambda^* = \frac{\sigma_1}{n} + (n-1)\rho, \qquad \rho = \frac{1}{n}\sqrt{\frac{R}{n-1}} = \frac{1}{n}\sqrt{\sigma_1^2 - \frac{2n}{n-1}\sigma_2}. \tag{10}$$

Proof. Suppose that λ_1 is any eigenvalue from the collection $\{\lambda_i\}$ and $\Lambda = \sigma_1 - \lambda_1$. Then

$$\Lambda^2 = (\lambda_2 + \dots + \lambda_n)^2 = \lambda_2^2 + \dots + \lambda_n^2 + 2 \sum_{2 \le i < j \le n} \lambda_i \lambda_j.$$

Since $2\lambda_i\lambda_j \le \lambda_i^2 + \lambda_j^2$ and each λ_i appears $n-2$ times in the sum $2\sum \lambda_i\lambda_j$, it follows that this sum is less than or equal to $(n-1)(\lambda_2^2 + \dots + \lambda_n^2)$, and hence

$$(\sigma_1 - \lambda_1)^2 = \Lambda^2 \le (n-1)(\lambda_2^2 + \dots + \lambda_n^2) = (n-1)(Q - \lambda_1^2),$$

where $Q = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2 \equiv \sigma_1^2 - 2\sigma_2$. Therefore,

$$\sigma_1^2 - 2\sigma_1\lambda_1 + \lambda_1^2 \le (n-1)(\sigma_1^2 - 2\sigma_2) - (n-1)\lambda_1^2. \tag{11}$$

As is well known, the inequality $\alpha\lambda_1^2 + \beta\lambda_1 + \gamma \le 0$, $\alpha > 0$, holds for all $\lambda_1 \in [\lambda_*, \lambda^*]$, where λ_*, λ^* are the roots of the corresponding quadratic equation. For inequality (11), these λ_* and λ^* are of the form (10), and this proves the lemma. □

Lemma 2. *The segment of length d contains n points λ_i, $i = 1, \dots, n$, $\lambda_1 - \lambda_n = d$, and*

$$\frac{nd^2}{2} \le \sum_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 \le \frac{n^2 - \theta}{4} d^2, \qquad \theta = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k+1. \end{cases} \tag{12}$$

Proof. Denote by R the sum in (12). If $n = 2$, then $R = (\lambda_1 - \lambda_2)^2$ and estimate (12) holds. If $n \ge 3$, we set $\lambda_i = c + y$, $c = (\lambda_1 + \lambda_n)/2$, for $1 < i < n$. Therefore,

$$s_i = (\lambda_1 - \lambda_i)^2 + (\lambda_i - \lambda_n)^2 = \left(\frac{d}{2} - y\right)^2 + \left(\frac{d}{2} + y\right)^2 = \frac{d^2}{2} + 2y^2.$$

Hence $s_i \ge d^2/2$. Since $R \ge s_2 + \dots + s_{n-1} + (\lambda_1 - \lambda_n)^2$, we have

$$R \ge (n-2)\frac{d^2}{2} + d^2 = n\frac{d^2}{2}.$$

This proves the lower bound in Eq. (12).

Let us prove the upper bound. The continuous function $R(\lambda_2, \dots, \lambda_{n-1})$ attains its maximum at a point of the compact set K: $\lambda_n \le x_i \le \lambda_1$, $i = 2, \dots, n-1$. Since $\partial^2 R/\partial \lambda_i > 0$, $i = 2, \dots, n-1$, this point is the end-point of K (otherwise, K contains greater values of R). Hence, at the maximum point, each λ_i coincides either with λ_1 or with λ_n.

Suppose that exactly p numbers from the set $\{\lambda_i\}$ coincide with λ_1 and the remaining $q = n - p$ numbers coincide with λ_n. Then $R = pqd^2$. For integer p and q such that $p + q = n$, $\max pq$ is equal to k^2 if $n = 2k$, or $k(k+1)$ if $n = 2k+1$. This completes the proof of the lemma. □

Corollary. *If all eigenvalues of the real matrix A are real, then, for $d = \lambda_{\max} - \lambda_{\min}$, we have*

$$\frac{2}{n}\sqrt{R} \le d \le \sqrt{\frac{2R}{n}} \tag{13}$$

(For odd n, the fraction $2/n$ in the left-hand side of (13) *should be replaced by $2/\sqrt{n^2 - 1/4} \cdot R$, see Eq.* (5)*).*

By resolving inequality (12) with respect to d, we obtain inequality (13). Although we have $d < \lambda^* - \lambda_*$, the corollary of Lemma 2 (together with the two obvious inequalities $\lambda_{\max} \ge \sigma_1/n \ge \lambda_{\min}$) yields an improvement of the estimate of Lemma 1 only if additional information about the matrix A, besides the numbers σ_1 and σ_2, is available.

One can characterize the relative errors λ^* and λ_* for the estimates of $\lambda_{\max}$ and $\lambda_{\min}$ by using the numbers

$$\chi_1 = \frac{\lambda^* - \lambda_{\max}}{d_0}, \quad \chi_n = \frac{\lambda_{\min} - \lambda_*}{d_0}, \qquad d_0 = \lambda_{\max} - \lambda_{\min}.$$

Motivation. The eigenvalues λ_i, σ_1, σ_2, λ^*, λ_* are invariant under diagonalization of the matrix, and the numbers χ_1 and χ_n do not also vary under the transformations $A \to A + \delta E$ or $A \to \gamma A$, where $\gamma > 0$ is any number. On the other hand, such transformations do not significantly change the process of calculating the estimates λ^* and λ_* and the accuracy of the result (after restoring the matrix by using the inverse transformation). $\square$

Example 1. The matrix

$$W = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix}$$

from [1, Sec. 15] has the following eigenvalues (see [1, Sec. 42]):

$$\lambda_1 = 30.289, \qquad \lambda_2 = 3.858, \qquad \lambda_3 = 0.843, \qquad \lambda_4 = 0.010, \qquad d_0 = \lambda_1 - \lambda_4 = 30.279.$$

We have $\chi_1 = 0.0047$, $\chi_4 = 0.428$. The large variation of accuracy here is due to the fact that the matrix has the eigenvalue, λ_4, which is rather close to $\lambda_{\min}$ (the distance from $\lambda_{\min}$ to λ_4 is several times smaller than d_0) and no nearby eigenvalues exist close to $\lambda_{\max}$.

Moreover, according to (5), $R = (n-1)\sigma_1^2 - 2n\sigma_2 = 2507$, and by the corollary of Lemma 2, we have

$$25.035 \le \lambda_{\max} - \lambda_{\min} \le 34.405. \tag{14}$$

This does not allow one to improve the above estimates λ^* and λ_* obtained by using only the values σ_1 and σ_2. But if one takes into account some additional information about the matrix W, for example, its positive definiteness, then it is obvious that $\lambda_i > 0$ for all i. One can take $\lambda_* = 0$ and then Eq. (14) implies the lower bound $\lambda_{\max} \ge 25.035$.

Let us prove that it is impossible to improve the above estimate by using the numbers σ_1 and σ_2 solely.

Example 2. Set

$$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

The numbers $\sigma_1 = 6$ and $\sigma_2 = 9$ are the same for both matrices. Lemma 1 implies the estimates $\lambda^* = 4$, $\lambda_* = 0$, which are the same for both matrices A_1 and A_2. On the other hand, for A_1, we have $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 1$ and, for A_2, we have $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 0$. Thus, for both matrices, all eigenvalues λ_i belong to the previously obtained segment $[\lambda^*, \lambda_*] = [0, 4]$. Any estimate $\varphi(\sigma_1, \sigma_2)$ of the number $\lambda_{\max}$ from above which depends only on σ_1, σ_2, gives $\varphi(\sigma_1, \sigma_2) \ge 4$ for the matrix A_1. Hence any estimate $\lambda_{\max} \le \varphi(\sigma_1, \sigma_2)$ (which holds for all matrices with real λ_1, λ_2, and λ_3) applied to the matrix A_2 does not imply a better estimate of $\lambda_{\max}$ than λ^* from Lemma 1. The same remains true for λ_*.

Let us prove that, for matrices of order 3, the relative errors χ_1 and χ_3 of the estimates λ^* and λ_* in Lemma 1 depend only on the fraction $\mu = (\lambda_2 - \lambda_3)/(\lambda_1 - \lambda_3)$. Let us consider a convenient estimate of χ_1.

Lemma 3. *For all $\mu \in [0, 1]$, we have*

$$\chi_1 = \chi_1(\mu) = \frac{2}{3}\left(\sqrt{1 - \mu + \mu^2} - 1 + \frac{\mu}{2}\right) \le \frac{\mu^2}{3}.$$

Proof. Suppose that the matrix A of order 3 has the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\lambda_1 > \lambda_3$ (if $\lambda_1 = \lambda_2 = \lambda_3$, then $\lambda^* = \lambda_* = \lambda_1$). Then, for $\gamma = 1/(\lambda_1 - \lambda_3)$, the matrix $B = \gamma(A - \lambda_3 E)$ has the eigenvalues 1, μ, 0 (where $0 \leq \mu \leq 1$), and the numbers χ_1, χ_3 coincide with those for A. Since similarly to Eq. (3), we have $\sigma_1 = 1 + \mu$ and $\sigma_3 = \mu$ for B, Lemma 1 implies that

$$\lambda^*(B) = \frac{1}{3}(1 + \mu + 2\sqrt{1 - \mu + \mu^2}), \tag{15}$$

$$\chi_1(B) = \lambda^*(B) - 1 = \frac{2}{3}\left(\sqrt{1 - \mu + \mu^2} - 1 + \frac{\mu}{2}\right). \tag{16}$$

Since $(\sqrt{1-x}\,)^n < 0$, we have $\sqrt{1-x} \leq 1 - x/2$, $x \leq 1$ by Taylor's formula and, for $0 \leq \mu \leq 1$, $x = \mu - \mu^2$, Eq. (16) implies that $\chi_1(B) \leq \mu^2/3$. This proves formula (15). $\square$

The eigenvalues of the matrix $E - B$ are equal to 0, $1 - \mu$, 1 and $\chi(B) = \chi(E - B) = (1 - \mu)^2$.

Remark 1. For complex a_{ij} and λ_k, various estimates of eigenvalues are considered in [2] under certain conditions which ensure that the off-diagonal elements of the matrix A are small with respect to the diagonal elements. Let us derive a simple estimate of eigenvalues for which this condition is not required.

Set $|z| = (x^2 + y^2)^{1/2}$ for $z = x + iy$ and

$$|z| = (|z_1|^2 + \dots + |z_n|^2)^{1/2}, \qquad \|A\| = \biggl(\sum_{jk} |a_{jk}|^2\biggr)^{1/2}$$

for the vector $z = (z_1, \dots, z_n)$.

Then for any complex δ, all eigenvalues λ_k of the matrix A belong to the disk of radius $r = \|A + \delta E\|$ centered at the point $z = -\delta$ in the complex plane.

Proof. The matrix $B = A + \delta E$ has the eigenvalues $\mu_k = \lambda_k + \delta$, $k = 1, \dots, n$. Then

$$|\mu_k| \leq \|B\| = \|A + \delta E\|, \qquad |\lambda_k + \delta| = |\mu_k| \leq \|A + \delta E\|, \quad k = 1, \dots, n,$$

i.e., all λ_k belongs to this disk. $\square$

Sometimes, a sharper estimate can be obtained if one considers several disks of different radii $r_s = \|A + \delta_s E\|$ centered at the different point $z_s = -\delta_s$, $s = 1, \dots, m$. Then all the eigenvalues λ_k belong to the intersection of these disks.

3. SHARPER ESTIMATES

Sharper estimates require more information than the numbers σ_1 and σ_2 from Sec. 2 provide. The eigenvalues of the $n \times n$ matrix A are the roots of the characteristic polynomial (1); however, for large n, the calculation of the coefficients of this polynomial is a cumbersome task (see [1], Introduction to Chap. 4). Hence we consider the case $n = 3$; as before, we suppose that all coefficients of the characteristic equation are real and its roots are also real. It is well known that, in the general case, it is impossible to calculate the roots λ_i in terms of the coefficients of the characteristic equation by using a finite number of arithmetic operations and extractions of (cube) roots of real numbers: the calculation of the eigenvalues λ_i requires either iterations or more complicated procedures, such as the calculation of trigonometric functions.

Hence, in order to solve the characteristic equation

$$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0, \qquad \sigma_3 = \det A, \tag{17}$$

with the same σ_1 and σ_2 as in Eq. (2), we suppose that the cubic polynomial can be replaced by a simpler function in the considered interval (or in its parts) and the resulting error can be estimated. This allows us to derive a simpler formula for λ_i, which ensures that the relative error is less than or equal to 1.2% of $\lambda_{\max} - \lambda_{\min}$.

According to Sec. 2, the replacement $\lambda = \mu + \sigma_1/3$ reduces Eq. (17) to the following form:

$$\mu^3 + \tau_2\mu = d, \qquad d = \det B, \qquad B = A - \frac{\sigma_1}{3}E, \tag{18}$$

$\tau_2 = \sigma_2 - \sigma_1/3$ (see (8)). As in the case $\tau_2 \geq 0$, the function $\mu^3 + \tau_2\mu$ is monotone; hence, for real roots, only $\tau_2 < 0$ may occur.

The replacements $\mu = \rho\nu$, $d = \rho^3 h$, where $\rho = \sqrt{-\tau_2/3}$, reduce Eq. (18) to the form

$$\nu^3 - 3\nu = h. \tag{19}$$

For $-1 \leq \nu < 1$, the function $\nu^3 - 3\nu$ decreases from 2 to -2 and in the intervals $(-\infty, -1)$ and $(1, \infty)$ it increases. In the case $|h| \leq 2$, there exist three real roots; the maximal root $\nu_{\max}$ belongs to the segment $1 \leq \nu \leq 2$. In order to find an estimate of this root, let us replace the function $\varphi(\nu) = \nu^3 - 3\nu$ by first- or second-order polynomials on the segment $[1, 2]$ (or on its parts) by using Taylor's formula

$$\begin{aligned} &\varphi(\nu) \equiv -2 + 3(\nu-1)^2 + (\nu-1)^3 > -2 + 3(\nu-1)^2 = \varphi_1(\nu), \\ &\varphi(\nu) \equiv 6(\nu\sqrt{3}) + 3\sqrt{3}(\nu-\sqrt{3})^2 + (\nu-\sqrt{3})^3 > 6(\nu-\sqrt{3}) = \varphi_2(\nu), \\ &\varphi(\nu) \equiv 2 + 9(\nu-2) + 6(\nu-2)^2 + (\nu-2)^3 > 2 + 9(\nu-2) = \varphi_3(\nu). \end{aligned} \tag{20}$$

These inequalities hold for $\nu \geq 1$. Since these functions increase whenever we replace Eq. (19) by any equation $\varphi_i(\nu_i) = h$, $i = 1, 2, 3$, we obtain an estimate ν_i of the root ν from above, i.e., $\nu_i \geq \nu$.

Let us split the segment $1 \leq \nu \leq 2$ into three parts so that one of the functions φ_i ensures a smaller error on each part than the remaining two functions. It is obvious that at the end-points of these subintervals the values of a pair of functions φ_i, φ_j coincide:

$$\varphi_1(\nu) = \varphi_2(\nu) \implies \nu_a = 1.5499, \qquad h = \varphi_1(\nu) = \varphi_2(\nu) = -1.0927, \tag{21a}$$

$$\varphi_2(\nu) = \varphi_3(\nu) \implies \nu_b = 1.8692, \qquad h = \varphi_2(\nu) = \varphi_3(\nu) = 0.8231. \tag{21b}$$

For such h, the Newton method gives the following solutions of Eq. (19):

$$\nu = 1.50852, \qquad \nu = 1.85568. \tag{22}$$

The errors $\nu_a - \nu$ and $\nu_b - \nu$ are equal to 0.0413 and 0.0135 respectively, i.e., these values are

(1) less than 1.2% of $\nu_{\max} - \nu_{\min}$: in this case $\nu_{\max} - \nu_{\min} = 3.403$;

(2) less than 0.4% of $\nu_{\max} - \nu_{\min}$: in this case $\nu_{\max} - \nu_{\min} = 3.430$.

Since the first part T_1 (i.e., $1 \leq \nu \leq \nu_a$) contains the point $\nu = 1$, where $\varphi_1 = \varphi$, the second part T_2 (i.e., $\nu_a \leq \nu \leq \nu_b$) contains the point $\nu = \sqrt{3}$, where $\varphi_2 = \varphi$, and the third part T_3 (i.e., $\nu_b \leq \nu \leq 2$) contains the point $\nu = 2$, where $\varphi_3 = \varphi$, it turns out that the function φ_i ensures a higher accuracy on each part of T_i than the remaining two functions.

Let us prove that at any other point ν, with the exception of ν_a and ν_b, the error is less than at these two points. The function $\varphi_2(\nu)$ is linear and its graph is the linear function tangent to the graph of the function $\varphi(\nu) = \nu^3 - 3\nu$. The curve $y = \varphi(\nu)$ is convex on the segment $1 \leq \nu \leq 2$ ($\varphi''(\nu) = 6\nu > 0$); hence it lies above the tangent line. The error $\nu_2 - \nu_0$ of the root of the equation $\varphi_2(\nu_2) = h$, where $\varphi(\nu_0) = h$, is the horizontal distance between the intersection points of the line $y = h$ and the curves $y = \varphi(\nu)$ and $y = \varphi_2(\nu)$. Since the curve $y = \varphi(\nu)$ is convex, this distance increases as the line $y = h$ goes off the point of tangency. Hence, at the end-points of the segment T_2, the error is larger than in the interior. The arguments for $\varphi_3(\nu)$ are similar.

For $\varphi_1(\nu)$, the proof is more complicated. We have $\varphi(\nu) = h$, $\varphi_1(\nu_1) = h$. Let us express ν_1 in terms of ν and prove that $\nu_1 - \nu$ increases starting from the value $\nu_1 - \nu = 0$ for $\nu = 1$. Equation (20) and the equality $\varphi_1(\nu_1) = h$ imply that $\nu_1 = 1 + \sqrt{(h+2)/3}$. Since $h = \varphi(\nu) = \nu^3 - 3\nu$, it follows that

$$\frac{d(\nu_1 - \nu)}{d\nu} = \frac{\varphi'(\nu)}{6\sqrt{(h+2)/3}} - 1. \tag{23}$$

We must prove that the expression (23) is positive, i.e., that $\varphi(\nu) > \sqrt{(\varphi(\nu)+2)/3}$. By raising it to the second power, we obtain $(\varphi(\nu))^2 > 12(\varphi(\nu)+2)$, i.e.,

$$9(\nu^2 - 1)^2 > 12(\nu^3 - 3\nu + 2).$$

Dividing this inequality by $3(\nu - 1)^2$, we obtain the inequality $3(\nu+1)^2 > 4(\nu+2)$ which holds for $\nu > 1$. This proves the required inequality. Hence, by identity (23), $\nu_1 - \nu$ increases for $\nu > 1$, and the error on the segment T_1 $(1 \le \nu \le \nu_n)$ is less than or equal to that for $\nu = \nu_a$.

The above considerations justify the following statement.

Theorem. *Suppose that the 3×3 matrix A has all a_{ij} and λ_k real, σ_1 and σ_2 are the same as in Eq.* (2), *and*

$$\tau_2 = \sigma_2 - \frac{\sigma_1}{3}, \qquad \rho = \sqrt{-\tau_2/3}, \qquad d = \det\left(A - \frac{\sigma_1}{3}E\right), \qquad h = \frac{d}{\rho^3}.$$

Then $\lambda_i = \mu_i + \sigma_i/3$, $i = 1, 2, 3$, where $\mu_i = \mu^3 - c\mu = d$ and ν_i are the solutions of the equation $\nu^3 - 3\nu = h$, and for the matrix A the following estimates hold:

(1) *if $-2 \le h \le -1.0927$, then $\nu^* - 0.0413 \le \max \nu_i \le \nu^* = (1 + \sqrt{(2+h)/3})$;*

(2) *if $-1.0927 \le h \le 0.8231$, then $\nu^* - 0.0413 \le \max \nu_i \le \nu^* = \sqrt{3} + h/6$; $\mu \le \sqrt{c} + d/2c$ (for $h > 0$, where 0.0413 can be replaced by 0.0135);*

(3) *if $0.8231 \le h \le 2$, then $\nu^* - 0.01356 \le \max \nu_i \le \nu^*$, where $\nu^* = 2 - (2-h)/9$; $\mu_1 \le 16\rho/9 + d/3c$, $c = \sqrt{-\tau_2}$;*

(4) *if $|h| > 2$, then there exists a unique real root and its upper estimate follows from Lemma* 1.

On each segment T_i, the above inequalities for ν^* readily follow from the solutions of the equations $\varphi_i(\nu^*) = h$ with respect to ν^*, $i = 1, 2, 3$, and from the estimates of the error given after Eq. (22).

Remark 2. In order to obtain a lower bound for the eigenvalues of the matrix A, it suffices to find an upper estimate for the matrix $-A$ and take it with opposite sign, because $\lambda(-A) = -\lambda(A)$.

Remark 3. If sharper estimates of eigenvalues than the theorem ensures are required, then the well-known Newton algorithm can be used:

$$\nu^{(k)} = \nu^{(k-1)} - \frac{\psi(\nu^{(k-1)})}{\psi'(\nu^{(k-1)})}, \quad k = 1, 2, \dots, \qquad \psi(\nu) = \nu^3 - 3\nu - h,$$

where $\nu^{(0)}$ is equal to the value ν^* defined in the theorem. Then the sequence $\nu^{(k)}$, $k = 1, 2, \dots$, converges rapidly to the exact solution of the equation $\nu^3 - 3\nu = h$.

State-of-the-art algorithms for solving eigenvalue problems are described in [3, Chap. 6, Sec. 12].

Example 3. Let us estimate the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1+i \\ 1 & 3 & 1 \\ 1-i & 1 & 4 \end{bmatrix}.$$

Equation (17) implies that $\lambda^3 - 9\lambda^2 + 22\lambda - 14 = 0$. The exact values of the roots are $\lambda_1 = 1$, $\lambda_2 = 4 - \sqrt{2}$, and $\lambda_3 = 4 + \sqrt{2}$.

In order to illustrate the statement of the theorem and the accuracy of previously proved estimates, let us calculate the eigenvalues approximately according to Theorem.

Under the replacement $\lambda = \mu + 3$, we have

$$\mu^3 - 5\mu - 2 = 0, \qquad d = 2, \qquad \rho = \sqrt{5/3} = 1.291.$$

Further, $\mu = \rho\nu$, $h = 2/\rho^3 = 0.9295$. Hence case (3) of the theorem occurs and for the equation

$$\nu^3 - 3\nu - 0.92955 = 0 \tag{24}$$

we get the estimate $\nu^* - 0.0135 \le \max \nu_i \le \nu^* = 1.881$, $\lambda^* = \rho\nu^* + \sigma_1/3 = 5.4284$, i.e., $1.8674 \le \max \nu_i \le 1.881$; in other words, $\mu^* = 16\rho/9 + d/(3c) = 2.4284$. According to the theorem, we can write $\mu^* - 0.0135\rho < \max \mu \le \mu^*$, i.e., $2.4108 \le \max \mu \le 2.4284$.

In order to estimate λ from below, we note that, according to Remark 1, $\lambda_{\min}(A) = -\lambda_{\max}(-A)$.

For the matrix $-A$, the parameters $\mu^3 - 5\mu = -d$, $\nu^3 - 5\nu = -h$; d, and h are the same as for A. Since h and d in Eq. (24) are replaced by $h^\sim = -0.9295$ and $d^\sim = -2$, case (2) of the theorem holds. Hence, for the matrix $-A$, we have

$$\nu_1 \le \nu^* = \sqrt{3} + \frac{h^\sim}{6} = 1.5772 \qquad \text{or} \qquad \mu_1 \le \sqrt{c} + \frac{d^\sim}{2c} = 2.0361.$$

Returning from ν to λ, we obtain the estimate for A

$$0.9639 \le \lambda_{\min} \le 1.0172 \qquad \text{and} \qquad 5.4108 \le \lambda_{\max} \le 5.4284.$$

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