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CONSTRUCTION OF MATRICES WITH PRESCRIBED SINGULAR VALUES AND EIGENVALUES *

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Abstract.

Two issues concerning the construction of square matrices with prescribed singular values and eigenvalues are addressed. First, a necessary and sufficient condition for the existence of an $n \times n$ complex matrix with n given nonnegative numbers as singular values and $m(\leq n)$ given complex numbers to be m of the eigenvalues is determined. This extends the classical result of Weyl and Horn treating the case when $m = n$. Second, an algorithm is given to generate a triangular matrix with prescribed singular values and eigenvalues. Unlike earlier algorithms, the eigenvalues can be arranged in any prescribed order on the diagonal. A slight modification of this algorithm allows one to construct a real matrix with specified real and complex conjugate eigenvalues and specified singular values. The construction is done by multiplication by diagonal unitary matrices, permutation matrices and rotation matrices. It is numerically stable and may be useful in developing test software for numerical linear algebra packages.

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1 Introduction.

A classical result of Weyl [11] and Horn [5] completely determines the relations between the singular values and eigenvalues of an $n \times n$ complex matrix as follows.

THEOREM 1.1. *There exists an $n \times n$ complex matrix with singular values $s_1 \geq \cdots \geq s_n \geq 0$ and eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ if and only if $|\prod_{j=1}^n \lambda_n| = \prod_{j=1}^n s_j$ and for $k = 1, \ldots, n-1$,*

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j. \tag{1.1}$$

The necessity proof can be done by first establishing the basic result:

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If $Ax = \lambda x$ for some unit vector x, then

$$|\lambda| = \|\lambda x\| = \|Ax\| \leq \max\{\|Ay\| : \|y\| = 1\} = s_1;$$

and then applying the result to the compound matrices $C_k(A)$ for $k = 1, \ldots, n$ (see [9, 9.E.1], [2, II.3.6]). Alternatively, one can use the Schur triangular form (see [6, Theorem 3.3.2]).

The sufficiency part is usually not presented in textbooks ([2, 5, 9]). A. Horn's original proof was by induction, and proved the special case where $\lambda_i \neq 0$ for all i first, and then extended it to the general case using an idea of Kaplansky [5]. Horn's algorithm always generated a matrix with diagonal entries ordered in decreasing absolute value.

In this note, two issues concerning the construction of square matrices with prescribed singular values and eigenvalues are addressed. First, in Section 2, we give a necessary and sufficient condition for the existence of an $n \times n$ complex matrix with n given nonnegative numbers as singular values and $m(\leq n)$ given complex numbers to be m of the eigenvalues. This extends the classical result of Weyl and Horn.

Second, in Section 3, we present an algorithm for generating a triangular matrix with prescribed singular values and prescribed eigenvalues. Unlike earlier algorithms the eigenvalues can be arranged in any specified order on the diagonal. In principle, once one has a triangular matrix with the desired singular values and eigenvalues (in any order on the diagonal) one can then use an eigenvalue reordering algorithm to put the eigenvalues in any desired order. However, these algorithms are rather complicated to implement accurately—see [3] for the details. Our construction is done by multiplication by unitary diagonal matrices, permutation matrices and rotation matrices. It is numerically stable. This allows one to generate matrices with prescribed singular values and eigenvalues which may be useful in testing numerical linear algebra packages.

In Section 4, we show how one can extend the ideas in Section 3 to construct a real matrix with specified real and complex conjugate eigenvalues and specified singular values, and in Section 5 we briefly compare our work with two other papers. A Matlab program implementing our algorithm is available at `http://www.math.wm.edu/~mathias/evsv.html`.

2 An extension of the result of Weyl and Horn.

THEOREM 2.1. *Let $1 \leq m \leq n$. Suppose the nonnegative numbers $s_1 \geq \cdots \geq s_n \geq 0$ and complex numbers $\lambda_1, \ldots, \lambda_m$ are given such that $|\lambda_1| \geq \cdots \geq |\lambda_m|$. The following conditions are equivalent:*

(a) *There exists an $n \times n$ complex matrix with $s_1, \ldots, s_n$ as singular values and $\lambda_1, \ldots, \lambda_m$ as m of the eigenvalues.*

(b) *There exists an $n \times n$ complex matrix with singular values $s_1, \ldots, s_n$ and* *eigenvalues* $\lambda_1, \ldots, \lambda_m, \underbrace{\gamma, \ldots, \gamma}_{n-m}$, *where*

$$\gamma = \begin{cases} 0 & \text{if } \lambda_m = 0, \\ \left|(\prod_{j=1}^{n} s_j / \prod_{j=1}^{m} \lambda_j)\right|^{1/(n-m)} & \text{otherwise.} \end{cases}$$

(c) $\prod_{j=1}^{k} s_{n-j+1} \leq \prod_{j=1}^{k} |\lambda_{m-j+1}|$ *and* $\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j$ *for all* $k = 1, \ldots, m$.

Proof. The implication $(b) \Rightarrow (a)$ is clear. The implication $(a) \Rightarrow (c)$ follows from the result of Weyl [11]. It remains to prove $(c) \Rightarrow (b)$. If $m = n$, the result reduces to Theorem 1.1. Thus, we assume that $m < n$.

Consider the vector

$$v = (\lambda_1, \ldots, \lambda_m, \underbrace{\gamma, \ldots, \gamma}_{n-m}),$$

where γ is defined as in the theorem. Clearly, the product of the entries of v is the same as $\prod_{j=1}^{n} s_j$ by construction. Using the result of Horn [5] (see also [9, 9.E.2]), we need only to show that for $k = 1, \ldots, n-1$, the magnitude of the product of any k entries of v is bounded below by $\prod_{j=1}^{k} s_{n-j+1}$ and above by $\prod_{j=1}^{k} s_j$. If $\lambda_m = 0 = \gamma$, then $s_n = 0$ and the result can be readily verified. So, we assume that λ_m, γ and s_n are all nonzero.

Let us address the upper bound. Our proof is by contradiction. Take $1 \leq k < n$. Let $\gamma^q \prod_{j=1}^{p} |\lambda_j|$ be the product of the absolute values of the k entries in v with largest magnitudes, where $k = p + q$. We claim that it is not larger than $\prod_{j=1}^{k} s_j$. If it is not true, then

$$q \ln \gamma + \sum_{j=1}^{p} \ln |\lambda_j| > \sum_{j=1}^{k} \ln s_j. \tag{2.1}$$

Since the absolute value of the product of the entries of v equals $\prod_{j=1}^{n} s_j > 0$, we also have

$$(n - m - q) \ln \gamma + \sum_{j=p+1}^{m} \ln |\lambda_j| < \sum_{j=k+1}^{n} \ln s_j. \tag{2.2}$$

By (c), we have

$$\sum_{j=1}^{m-p} \{\ln |\lambda_{n-j+1}| - \ln s_{m-j+1}\} \leq 0 \tag{2.3}$$

and

$$\sum_{j=1}^{p} \{\ln |s_j| - \ln |\lambda_j|\} \geq 0. \tag{2.4}$$

Let α be the average of the numbers $\ln s_{k+1}, \ldots, \ln s_{n-m+p}$, and let β be the average of $\ln s_{p+1}, \ldots, \ln s_k$. Then since the s_i are non-increasing, $\alpha \leq s_{k+1} \leq$

$s_k \le \beta$. However, using (2.2) and (2.1) for the strict inequalities, and (2.3) and (2.4) for the weak inequalities we obtain the contradiction $\alpha > \beta$:

$$\begin{aligned}
\alpha &= \Big(\sum_{j=k+1}^{n-m+p} \ln s_j \Big) / (n-m-q) \\
&= \Big(\sum_{j=k+1}^{n} \ln s_j - \sum_{j=n-m+p+1}^{n} \ln s_j \Big) / (n-m-q) \\
&\ge \Big(\sum_{j=k+1}^{n} \ln s_j - \sum_{j=p+1}^{m} \ln |\lambda_j| \Big) / (n-m-q) \\
&> \ln \gamma \\
&> \Big(\sum_{j=1}^{k} \ln s_j - \sum_{j=1}^{p} \ln |\lambda_j| \Big) / q \\
&\ge \Big(\sum_{j=1}^{k} \ln s_j - \sum_{j=1}^{p} \ln s_j \Big) / q \\
&= \Big(\sum_{j=p+1}^{k} \ln s_j \Big) / q \\
&= \beta.
\end{aligned}$$

Similarly, we can show that the product of the absolute values of the k entries in v with smallest magnitudes is bounded below by $\prod_{j=1}^{k} s_{n-j+1}$, as required. □

Bebiano, Li and da Providencia have given necessary and sufficient conditions for the existence of a complex $n \times n$ Hermitian (real symmetric) matrices to have n prescribed eigenvalues and k (with $k < n$) prescribed diagonal entries (see the proof of [1, Theorem 2.4]). Tam studied the necessary and sufficient conditions relating the n singular values and the first k ($\le n$) diagonal entries of an $n \times n$ general or symmetric matrix over $\mathbf{R}$ or $\mathbf{C}$. He also studied the conditions on the singular values and the $(1, k+1)$, $(2, k+2)$, ..., $(k, 2k)$ entries of a skew-symmetric matrix over $\mathbf{R}$ or $\mathbf{C}$ [10].

3 Our construction.

Let x and y be two nonnegative real vectors with n entries. We say that x is *log majorized* by y, denoted by $x \prec_{\log} y$, if the product of the k largest entries of x is not larger than that of y for $k = 1, \ldots, n$, and the products are equal when $k = n$.

Suppose (1.1) holds, and that equality holds when $k = n$, i.e.,

$$(|\lambda_1|, \ldots, |\lambda_n|) \prec_{\log} (s_1, \ldots, s_n). \tag{3.1}$$

We shall construct a triangular matrix with diagonal entries $|\lambda_i|$'s and then multiply it by a suitable diagonal matrix $\mathrm{diag}(e^{it_1}, \ldots, e^{it_n})$ to get the desired matrix.

Suppose we want to have $|\lambda_j|$ at the $(1,1)$ entry of the resulting triangular matrix. We derive a *basic procedure* to construct a lower triangular matrix A such that:

(1) $\sigma(A) = (s_1, \ldots, s_n)$;

(2) the $(1,1)$ entry of A equals $|\lambda_j|$;

(3) removing the first row and the first column of A results in a matrix in diagonal form, say, $\mathrm{diag}(y_1, \ldots, y_{n-1})$, such that

$$(|\lambda_1|, \ldots, |\lambda_{j-1}|, |\lambda_{j+1}|, \ldots, |\lambda_n|) \prec_{\log} (y_1, \ldots, y_{n-1}).$$

Once this is done, we can focus on the $(n-1) \times (n-1)$ submatrix of A in the bottom right corner and apply the basic procedure to it. Then we get a triangular matrix with the desired $(1,1)$, $(2,2)$ entries, and the prescribed singular values, whose $(n-2) \times (n-2)$ submatrix in the bottom right corner is in diagonal form that allows one to repeat the basic procedure. After $(n-1)$ steps one will get the required matrix.

The construction is rather detailed and the reader may lose sight of the goal. So we first point out the key ideas that ensure that the conditions (1)–(3) are satisfied. In every case the matrix A is obtained from

$$S = \mathrm{diag}(s_1, \ldots, s_n)$$

by multiplication by permutation matrices and (2×2) orthogonal matrices. This guarantees (1). The condition (2) is ensured by finding an explicit formula to solve the 2×2 case at the left top corner of S (or a modified S) so that $|\lambda_j|$ will lie at the $(1,1)$ position in the resulting matrix. To achieve that, we often have to permute the diagonal of S so that $|\lambda_j|$ lies between the first two diagonal elements of S. There may well be many ways to do this, but not every one of these results in a matrix that satisfies condition (3). One simple way to ensure that (3) is satisfied is to permute s_{k-1} and s_k to the top where k is chosen so that $s_{k-1} \geq |\lambda_j| \geq s_k$.

To obtain the matrix satisfying (1)–(3), we consider 2 cases, each of which is divided into two subcases.

Case 1. Assume that $\lambda_j = 0$. Then $s_n = 0$ and we may assume that $j = n$. Let P be a permutation matrix such that PSP^t has s_n and s_{n-1} at the $(1,1)$ and $(2,2)$ position. Now consider two subcases:

(a) If $\lambda_{n-1} = 0$, interchange the first two columns of PSP^t to obtain A. (This interchange is of vital importance when $s_{n-1} \neq 0$. It makes no difference whether we do it or not when $s_{n-1} = 0$.)

(b) If $\lambda_{n-1} \neq 0$, multiply the first two columns of PSP^t on the right by W to obtain A, where

$$W = \begin{pmatrix} \cos w & \sin w \\ -\sin w & \cos w \end{pmatrix}, \qquad \cos w = \frac{|\lambda_1 \cdots \lambda_{n-1}|}{s_1 \cdots s_{n-1}}.$$

One easily checks that (1)–(3) hold.

Case 2. Assume that $\lambda_j \neq 0$. Note that $s_1 \geq |\lambda_1| \geq |\lambda_j|$. If $\lambda_n = 0$, then $s_n = 0$ and $|\lambda_j| > s_n$. If $\lambda_n \neq 0$, then (3.1) implies that $s_n \leq |\lambda_n| \leq |\lambda_j|$. Consequently, either

(a) $|\lambda_j| = s_k$ for some $1 \leq k \leq n$, or

(b) there exists k with $1 \leq k < n$ such that $s_{k-1} > |\lambda_j| > s_k$.

If (a) holds, let P be a permutation so that $A = PSP^t$ has s_k in its (1,1) position. Then clearly A satisfies (1)–(2), and the $(n-1) \times (n-1)$ submatrix in the right bottom corner of A is in diagonal form, say, $\operatorname{diag}(y_1, \ldots, y_{n-1})$. To verify (3), let $a(r)$ be the product of the r largest entries of $(|\lambda_1|, \ldots, |\lambda_{j-1}|, |\lambda_{j+1}|, \ldots, |\lambda_n|)$ and let $b(r)$ be the product of the r largest entries in $(y_1, \ldots, y_{n-1})$. It is clear that $a(n-1) = b(n-1)$.

Suppose $r < j$. Then $a(r) = \prod_{i=1}^{r} |\lambda_i|$. If $r < k$, then $b(r) = \prod_{i=1}^{r} s_i \geq a(r)$ by (3.1). Suppose $k \geq r$. Since $j \geq r+1$, we have $|\lambda_{r+1}| \geq |\lambda_j|$ and hence by (3.1),

$$b(r) = \prod_{i=1}^{r+1} s_i / s_k = \prod_{i=1}^{r+1} s_i / |\lambda_j| \geq \prod_{i=1}^{r+1} s_i / |\lambda_{r+1}| \geq a(r).$$

Next, assume $r \geq j$. Then $a(r) = \prod_{i=1}^{r+1} |\lambda_i| / |\lambda_j|$. Suppose $r < k$. Since $r \geq j$, we have $|\lambda_j| \leq |\lambda_{r+1}|$ and hence by (3.1),

$$b(r) = \prod_{i=1}^{r} s_i \geq \prod_{i=1}^{r} |\lambda_i| \geq a(r).$$

If $k \leq r$, then $b(r) = \prod_{i=1}^{r+1} s_i / s_k = \prod_{i=1}^{r+1} s_i / |\lambda_j| \geq a(r)$ by (3.1).

Now suppose (b) holds. Let P be a permutation matrix so that PSP^t has s_k and s_{k-1} in the $(1,1)$ and $(2,2)$ positions. Let

$$U = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix}$$

so that

$$\cos u = \sqrt{\frac{s_{k-1}^2 - |\lambda_j|^2}{s_{k-1}^2 - s_k^2}}, \qquad \sin u = \sqrt{\frac{|\lambda_j|^2 - s_k^2}{s_{k-1}^2 - s_k^2}},$$

$$\cos v = \frac{s_k}{|\lambda_j|} \cos u = \frac{s_k}{|\lambda_j|} \sqrt{\frac{s_{k-1}^2 - |\lambda_j|^2}{s_{k-1}^2 - s_k^2}},$$

$$\sin v = \frac{s_{k-1}}{|\lambda_j|} \sin u = \frac{s_{k-1}}{|\lambda_j|} \sqrt{\frac{|\lambda_j|^2 - s_k^2}{s_{k-1}^2 - s_k^2}}.$$

Then

$$U \begin{pmatrix} s_k & 0 \\ 0 & s_{k-1} \end{pmatrix} V = \begin{pmatrix} |\lambda_j| & 0 \\ * & t \end{pmatrix}, \quad t = s_k s_{k-1} / |\lambda_j|.$$

Multiply the first and second rows of PSP^t by U, and then multiply the first and second columns by V to obtain A. The resulting matrix with have $|\lambda_j|$ and t in its $(1,1)$ and $(2,2)$ positions. Clearly, A satisfies (1)–(2) and the $(n-1)\times(n-1)$ submatrix in the right bottom corner is in diagonal form, say, $\mathrm{diag}(y_1,\ldots,y_{n-1})$. To verify (3), let $a(r)$ be the product of the r largest entries of of $(|\lambda_1|,\ldots,|\lambda_{j-1}|,|\lambda_{j+1}|,\ldots,|\lambda_n|)$ and let $b(r)$ be the product of the r largest entries in $(y_1,\ldots,y_{n-1})$. It is clear that $a(n-1)=b(n-1)$.

Suppose $r<j$. Then $a(r)=\prod_{i=1}^r|\lambda_i|$. If $r<k-1$, then $b(r)=\prod_{i=1}^r s_i\geq a(r)$ by (3.1). Suppose $k-1\geq r$. Since $j\geq r+1$, we have $|\lambda_{r+1}|\geq|\lambda_j|$ and hence by (3.1),

$$b(r)=\prod_{i=1}^{r+1}s_i/|\lambda_j|\geq\prod_{i=1}^{r+1}s_i/|\lambda_{r+1}|\geq a(r).$$

Next, assume $r\geq j$. Then $a(r)=\prod_{i=1}^{r+1}|\lambda_i|/|\lambda_j|$. Suppose $r<k-1$. Since $r\geq j$, we have $|\lambda_j|\leq|\lambda_{r+1}|$ and hence by (3.1),

$$b(r)=\prod_{i=1}^{r}s_i\geq\prod_{i=1}^{r}|\lambda_i|\geq a(r).$$

If $k-1\leq r$, then $b(r)=\prod_{i=1}^{r+1}s_j/|\lambda_j|\geq a(r)$ by (3.1). □

As mentioned in the introduction, if the data are real our construction will produce a real matrix satisfying the requirement. If $\lambda_1=\cdots=\lambda_n\geq 0$, one can actually construct a nonnegative lower triangular matrix with the desired property.

Below we describe an algorithm to construct a lower triangular matrix A with prescribed singular values $s_1,\ldots,s_n$, and diagonal entries $(x_1,\ldots,x_n)$, where $x_i=|\lambda_{j_i}|$ for a permutation $(j_1,\ldots,j_n)$ of $(1,\ldots,n)$. Notice that on any particular iteration one will perform exactly one of the steps 2a, 2b, 2c and 2d, and that these four steps correspond to the four cases in the constructive proof in the previous section. In Step 1 we denote the last $n-r+1$ diagonal entries of A_{r-1} by $y_i, i=1,\ldots,n-r+1$. This is merely for notational convenience in the rest of the algorithm.

Algorithm 3.1.

0. Set $r=1$ and $A_0=\mathrm{diag}(s_n,\ldots,s_1)$.

1. If $r=n$, stop. Otherwise, set $y_{n-i+1}=(A_r)_{ii}$ for $i=r+1,\ldots,n$ and determine the smallest integer $k\geq r$ so that

$$x_r\geq y_k.$$

2a. If $0=x_r$ and there exists $j>r$ such that $x_j=0$, interchange the r and $(r+1)$st columns of A_{r-1} to obtain A_r; set $r=r+1$ and go to Step 1.

2b. If $0 = x_r$ and no other x_j with $j > r$ equal to 0, obtain A_r from A_{r-1} by changing its $(r,r), (r,r+1), (r+1,r), (r+1,r+1)$ entries to

$$0,\ 0,\ \sqrt{y_{n-r}^2 - t^2},\ t,$$

where $t = (x_{r+1} \cdots x_n)/(y_{n-r-1} \cdots y_1)$; set $r = r+1$ and go to Step 1.

2c. If $0 < x_r = y_k$ apply a permutation to the rows and to the columns, accordingly, of A_{r-1} to obtain A_r so that the last $n-r-1$ diagonal entries of A_r equal

$$y_k, y_n, \ldots, y_{k+1}, y_{k-1}, \ldots, y_r;$$

set $r = r+1$ and go to Step 1.

2d. Otherwise, apply a permutation to the rows and to the columns, accordingly, of A_{r-1} to obtain $\tilde{A}_r$ with y_k and y_{k-1} as its (r,r) and $(r+1,r+1)$ entries. If $r > 1$, multiply the $2 \times (r-1)$ submatrix of $\tilde{A}_r$ lying in the r and $(r+1)$st rows and the first $(r-1)$ columns by

$$U = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix},$$

where

$$\cos u = \sqrt{\frac{y_{k-1}^2 - x_r^2}{y_{k-1}^2 - y_k^2}}, \qquad \sin u = \sqrt{\frac{x_r^2 - y_k^2}{y_{k-1}^2 - y_k^2}}.$$

Set the $(r,r), (r,r+1), (r+1,r), (r+1,r+1)$ entries of the resulting matrix to

$$x_r,\ 0,\ \sqrt{(y_{k-1}^2 - x_r^2)(x_r^2 - y_k^2)}/x_r, \qquad y_{k-1}y_k/x_r,$$

respectively. Then obtain A_r from the resulting matrix by permuting the last $n-r$ rows and columns, accordingly, so that the last $n-r$ diagonal entries are in ascending order. Set $r = r+1$ and go to Step 1.

4 Real matrices with complex eigenvalues.

Chu mentions that the construction of a real matrix with prescribed singular values and complex eigenvalues occuring in conjugate pairs is an open problem [4]. This is a very natural question when constructing matrices to test numerical software. We provide a construction using a slight modification of the ideas in the previous section. Note that we are able to maintain a block triangular structure, and control the order of the eigenvalues appearing as the eigenvalues of the 1×1 and 2×2 diagonal blocks of the resulting matrix.

THEOREM 4.1. *Let $s_1 \geq \cdots \geq s_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ be given such that the non-real complex numbers in $\lambda_1, \ldots, \lambda_n$ occur in conjugate pairs. Suppose $(|\lambda_1|, \ldots, |\lambda_n|) \prec_{\log} (s_1, \ldots, s_n)$, and $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$ such that complex conjugate pairs occur in consecutive positions in $\lambda_{i_1}, \ldots, \lambda_{i_n}$. Then there exists a real lower block matrix A with singular values $s_1, \ldots, s_n$, and*

$\lambda_{i_1},\dots,\lambda_{i_n}$ *as the eigenvalues of the* 1×1 *and* 2×2 *diagonal blocks of* A *in the specified order.*

PROOF. We prove the result by induction on n. The result is trivial when $n=1$. Suppose $n\geq 2$ and that the result is true for dimensions lower than n.

If λ_{i_1} is real, then using the method in Section 3, we can construct a real matrix

$$B=\begin{pmatrix}|\lambda_{i_1}| & 0\\ * & D\end{pmatrix},$$

where D is a diagonal matrix with diagonal entries $y_1\geq\cdots\geq y_{n-1}\geq 0$ so that

$$(|\lambda_{i_2}|,\dots,|\lambda_{i_n}|)\prec_{\log}(y_1,\dots,y_{n-1}).$$

By the induction assumption, there exist real orthogonal matrices U_1,V_1 such that U_1DV_1 is in lower block triangular form with $\lambda_{i_2},\dots,\lambda_{i_n}$ as the eigenvalues of the 1×1 and 2×2 diagonal blocks of U_1DV_1 in the specified order. Let $U=[1]\oplus U_1$ and $V=[\mu]\oplus V_1$ so that $\mu=\pm 1$ satisfying $\mu|\lambda_{i_1}|=\lambda_{i_1}$. Then $A=UBV$ is the desired matrix.

Suppose $\lambda_{i_1}=\bar{\lambda}_{i_2}=a+ib$, where $a,b\in\mathbf{R}$ with $b\neq 0$. If $n=2$, then $s_1s_2=|\lambda_1\lambda_2|$. Since $s_2\leq|\lambda_1|=|\lambda_2|\leq s_1$, we have $s_1^2+s_2^2\geq|\lambda_1|^2+|\lambda_2|^2$. Consider the family of matrices

$$A(r)=\begin{pmatrix}a & b/r\\ -br & a\end{pmatrix},\qquad r>0.$$

Evidently, each $A(r)$ has eigenvalues $a+ib$ and $a-ib$. If $s_1(r)\geq s_2(r)$ are the singular values of $A(r)$, then

$$s_1(r)^2+s_2(r)^2=\operatorname{tr}A(r)A(r)^t=2a^2+(rb)^2+(b/r)^2.$$

Since $s_1^2+s_2^2\geq 2(a^2+b^2)=s_1(0)^2+s_2(0)^2$, and $\lim_{r\to\infty}s_1(r)^2+s_2(r)^2=\infty$, there exists r_0 such that $s_1(r_0)^2+s_2(r_0)^2=s_1^2+s_2^2$. One readily checks that $A(r_0)$ has singular values s_1 and s_2. Consequently, there are real orthogonal matrices U and V such that

$$U\begin{pmatrix}s_1 & 0\\ 0 & s_2\end{pmatrix}V^T=A(r_0).$$

One can obtain explicit formulae for r_0 and U and V. We do not do this in the interest of brevity.

Now suppose $n\geq 3$. Since $s_1s_2\geq|\lambda_{i_1}\lambda_{i_2}|\geq s_{n-1}s_n$, one can find the largest integer k with $1\leq k<n$ such that $s_ks_{k+1}\geq|\lambda_{i_1}\lambda_{i_2}|$. We consider two cases—$k=n-1$ and $k<n-1$.

If $k=n-1$, we have $|\lambda_{i_1}\lambda_{i_2}|=s_{n-1}s_n$. Using the construction in the case $n=2$ we can construct a real 2×2 matrix A_1 with singular values s_{n-1},s_n and eigenvalues $\lambda_{i_1},\lambda_{i_2}$. It is almost immediate that

$$(|\lambda_{i_3}|,\dots,|\lambda_{i_n}|)\prec_{\log}(s_1,\dots,s_{n-2}).$$

So by the induction assumption, there exists a lower block triangular matrix A_2 with singular values $s_1, \ldots, s_{n-2}$ and $\lambda_{i_3}, \ldots, \lambda_{i_n}$ as eigenvalues of the 1×1 and 2×2 diagonal blocks of A_2 in the specified order. The matrix $A = A_1 \oplus A_2$ has the desired eigenvalues and singular values.

If $k < n-1$, let $y = s_k s_{k+1} s_{k+2}/|\lambda_{i_1}\lambda_{i_2}|$, and let $z = s_k s_{k+2}/y$. By the case $n = 2$ in Section 3, there are real orthogonal matrices U and V such that

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} s_{k+1} & 0 & 0 \\ 0 & s_k & 0 \\ 0 & 0 & s_{k+2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V^T \end{pmatrix} = \begin{pmatrix} s_{k+1} & 0 & 0 \\ 0 & z & 0 \\ 0 & * & y \end{pmatrix}.$$

Now notice that $s_{k+1} z = |\lambda_{i_1}\lambda_{i_2}|$. So, by the case $n = 2$ just proved, there are real orthogonal matrices U_1 and V_1 such that

$$\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_{k+1} & 0 & 0 \\ 0 & z & 0 \\ 0 & * & y \end{pmatrix} \begin{pmatrix} V_1^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/r_0 & 0 \\ -br_0 & a & 0 \\ * & * & y \end{pmatrix} \equiv B.$$

This matrix has λ_{i_1} and λ_{i_2} as the eigenvalues of its leading 2×2 block, and has singular values s_k, s_{k+1}, s_{k+2}.

Now let $R = B \oplus \mathrm{diag}(s_1, \ldots, s_{k-1}, s_{k+3}, \ldots, s_n)$. The trailing $(n-2)\times(n-2)$ principal submatrix of R is

$$\tilde{D} = \mathrm{diag}(y, s_1, \ldots, s_{k-1}, s_{k+3}, \ldots, s_n).$$

We need to show that the singular values of $\tilde{D}$ majorize $|\lambda_{i_3}|, \ldots, |\lambda_{i_n}|$. Clearly, the product of the singular values of $\tilde{D}$ equals $\prod_{j=3}^n |\lambda_{i_j}|$. Suppose $1 \le m < n-2$, and let $\mu_1, \ldots, \mu_m$ be any m numbers chosen from $|\lambda_{i_3}|, \ldots, |\lambda_{i_n}|$. We must show that the the product of the largest m numbers from $s_1, \ldots, s_{k-1}, y, s_{k+3}, \ldots, s_n$ is as least as large as $\prod_{j=1}^m \mu_j$. If $m \le k-1$, then clearly $\prod_{j=1}^m \mu_j \le \prod_{j=1}^m s_j$. If $k-1 < m$, then

$$|\lambda_{i_1}\lambda_{i_2}| \prod_{j=1}^m \mu_j \le \prod_{j=1}^{m+2} s_j$$

and hence

$$\begin{aligned} \prod_{j=1}^m \mu_j \le \frac{1}{|\lambda_{i_1}\lambda_{i_2}|} \prod_{j=1}^{m+2} s_j &= \Big(\prod_{j=1}^{k-1} s_j\Big) \frac{s_k s_{k+1} s_{k+2}}{|\lambda_{i_1}\lambda_{i_2}|} \Big(\prod_{j=k+3}^{m} s_j\Big) \\ &= \Big(\prod_{j=1}^{k-1} s_j\Big) y \Big(\prod_{j=k+3}^{m} s_j\Big). \end{aligned}$$

This establishes the required majorization. By the induction assumption, there exist unitary matrices U_2, V_2 such that $U_2 \tilde{D} V_2^T$ is a lower block triangular matrix with eigenvalues $\lambda_{i_3}, \ldots, \lambda_{i_n}$ arising from the 1×1 and 2×2 diagonal blocks of A in the specified order. Now, $A = (I_2 \oplus U_2) R (I_2 \oplus V_2^T)$ is a block lower triangular matrix with the desired eigenvalues and singular values. $\square$

One may wonder about the existence of a real matrix with all singular values and some eigenvalues prescribed. We have the following result:

COROLLARY 4.2. *Suppose $1 \leq m \leq n$. Let $s_1 \geq \cdots \geq s_n \geq 0$ and the complex numbers $\lambda_1, \ldots, \lambda_m$ are given. Let $\mu_1, \ldots, \mu_r$ be the smallest collection of complex numbers such that the non-real complex numbers in $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_r$ occur in conjugate pairs. The following conditions are equivalent:*

(a) *There exists a real matrix A with singular values $s_1, \ldots, s_n$ and $\lambda_1, \ldots, \lambda_m$ as m of the eigenvalues.*

(b) *$m + r \leq n$ and there exists a real matrix with singular values $s_1, \ldots, s_n$, and eigenvalues $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_r, \underbrace{\gamma, \ldots, \gamma}_{n-m-r}$, where*

$$\gamma = \begin{cases} 0 & \text{if } \prod_{i=1}^m \lambda_i = 0, \\ \left\{\prod_{j=1}^n s_j / \left|(\prod_{i=1}^m \lambda_i)(\prod_{j=1}^r \mu_j)\right|\right\}^{1/(n-m-r)} & \text{otherwise }. \end{cases}$$

(c) *$m + r \leq n$, and for any $1 \leq k \leq m + r$ the product of any k numbers from*

$$|\lambda_1|, \ldots, |\lambda_m|, |\mu_1|, \ldots, |\mu_r|$$

lies in the interval $\left[\prod_{j=1}^k s_{n-j+1}, \prod_{j=1}^k s_j\right]$.

5 Relation to other research.

Smoktunowicz and Kosowski [7, 8] showed how to construct an upper triangular matrix with singular values whose product is 1, and whose eigenvalues are all equal to 1. Chu [4], independently of our work, showed how to construct a matrix with specified singular values and eigenvalues that satisfy (1.1). His algorithm produces a matrix that is permutation similar to a triangular matrix; however, his algorithm does not allow one to specify the order of the eigenvalues on the diagonal. The first version of our paper was independent of Chu's work. We were informed of [4], and so added Section 4 where we show how to construct a real matrix with specified real and complex conjugate eigenvalues and specified singular values, to answer an open problem mentioned by Chu. A Matlab program implementing our algorithm is available from the authors.

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