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FACTORING MATRICES INTO THE PRODUCT OF TWO MATRICES*

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Abstract.

Linear algebra of factoring a matrix into the product of two matrices with special properties is developed. This is accomplished in terms of the so-called inverse of a matrix subspace which yields an extended notion for the invertibility of a matrix. The product of two matrix subspaces gives rise to a natural generalization of the concept of matrix subspace. Extensions of these ideas are outlined. Several examples on factoring are presented.

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1 Introduction.

Almost without exception, to any interesting splitting of a matrix $M \in \mathbb{C}^{n\times n}$ corresponds an interesting product of the parts. For instance, for the ubiquitous splitting

$$M = H + iK, \quad \text{with } H = \frac{1}{2}(M + M^*) \text{ and } K = \frac{1}{2i}(M - M^*), \tag{1.1}$$

the product HK obviously has a real determinant. It is the converse that is intriguing, i.e., which matrices can be factored as the product of two Hermitian matrices [12]. For factoring, see the thorough review [15] and references therein. See also [5]. In this paper we consider the task of factoring matrices into the product of two matrices of particular type by devising a general, numerically reliable finite step algorithm to this end. This is accomplished in terms of the so-called inverse of a matrix subspace, allowing us to convert the problem into an almost equivalent one of computing the nullspace of an ap-

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propriate linear operator. In case the matrix to be factored is invertible, we provide all the factorizations parametrised as a manifold depending on the matrix.

The problem has several motivations. Certainly, factorizations play a central role in classical numerical linear algebra. In preconditioning linear systems, the splitted parts of a matrix appear as the product, once one of them has been inverted. (For the splitting (1.1), see [1].) Our most recent interest in factorization problems originates from the algorithmic construction of diffractive optical systems, where matrices of particular type have a direct physical interpretation [10, 9]. Then, somewhat unconventionally in numerical linear algebra, one may need to have all the possible factorizations available for singling out one with most desirable properties.

The paper is organised as follows. In Section 2 we introduce the notion of the inverse of a matrix subspace. Then we look at products of two matrix subspaces. With these tools the problem of factoring a matrix into the product of two matrices is converted into finding a nullspace, assuming one of the matrix subspaces involved is invertible. Extensions of these ideas are discussed. In Section 3 several examples are presented to illustrate how factorization problems can be systematically and completely solved with the approach proposed. We also demonstrate what type of conditions, both necessary and sufficient, arise in factoring matrices.

2 Linear algebra of factoring into the product of two matrices.

Necessary linear algebraic notions are defined and then applied to factoring matrices.

2.1 Matrix subspaces and inversion.

Let us set the following linear algebraic concept, where $\mathrm{GL}(n,\mathbb{C})$ denotes the general linear group of invertible n-by-n complex matrices.

DEFINITION 2.1.[1] *Let $\mathcal{V}$ and $\mathcal{W}$ be two subspaces of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ (or $\mathbb{R}$). Then $\mathcal{W}$ is the inverse of $\mathcal{V}$ if*

$$\{V^{-1} :\ V \in \mathrm{GL}(n,\mathbb{C}) \cap \mathcal{V}\} = \mathrm{GL}(n,\mathbb{C}) \cap \mathcal{W} \neq \emptyset.^2$$

In other words, $\mathcal{V}$ has invertible elements such that the inverses of the invertible elements are the invertible elements of $\mathcal{W}$. When this holds, we say that the subspace $\mathcal{V}$ is invertible. The condition is clearly reflective, i.e., $\mathcal{W}$ is the inverse of $\mathcal{V}$ if and only if $\mathcal{V}$ is the inverse of $\mathcal{W}$.

In the special case of $\mathcal{V} = \mathrm{span}\{A\}$ we have an invertible subspace (with the inverse $\mathcal{W} = \mathrm{span}\{A^{-1}\}$) if and only if $A \in \mathbb{C}^{n\times n}$ is invertible. Hence we have a natural generalized notion for the invertibility of a matrix.

[1] The definition can also be given in a more abstract setting.
[2] We exclude the noninteresting case of having no invertible elements in $\mathcal{V}$.

Seemingly the most interesting case is $\mathcal{W} = \mathcal{V}$. Then we say that $\mathcal{V}$ is closed under inversion. Of course, if $\mathcal{V}$ is a subalgebra of $\mathbb{C}^{n\times n}$, then this holds automatically (as long as there are invertible elements in $\mathcal{V}$). Due to the LU factorization, lower and upper triangular matrices are probably the most important subalgebras from the practical point of view.

Let us give less straightforward examples.

EXAMPLE 2.1. The set of symmetric (similarly, the set of skew-symmetric matrices) is a subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ that is closed under inversion. Similarly, the set of persymmetric matrices is a subspace of $\mathbb{C}^{n\times n}$ that is closed under inversion.

EXAMPLE 2.2. Let $\mathcal{V}$ be the set of symmetric matrices and $V \in \mathcal{V}$. Then

$$p(V) \in \mathcal{V} \quad \text{for any polynomial } p. \tag{2.1}$$

From this it follows that the inverse is symmetric as well. More generally, any subspace $\mathcal{V}$ of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ whose elements satisfy the property (2.1) is closed under inversion (assuming there are invertible elements in $\mathcal{V}$).

EXAMPLE 2.3. The set of Hermitian matrices (similarly, the set of skew-Hermitian matrices) is a subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{R}$ that is closed under inversion.

EXAMPLE 2.4. Fix a matrix $U \in \mathbb{C}^{n\times k}$. Then $\mathbb{C}I + U\mathbb{C}^{k\times n}$ is a subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ that is closed under inversion. For this, use the Sherman–Morrison–Woodbury formula. (Matrix subspaces of this type arise in control theory, where the freedom of choosing a "feedback" means that one deals with $U\mathbb{C}^{k\times n}$.)

Suppose $\mathcal{W}$ is the inverse of $\mathcal{V}$ and $X, Y \in \mathrm{GL}(n, \mathbb{C})$. Then

$$Y\mathcal{W}X^{-1} = \{\, YWX^{-1} :\ W \in \mathcal{W}\}$$

is the inverse of $X\mathcal{V}Y^{-1}$. Let

$$\mathcal{V}^T = \{V^T \in \mathbb{C}^{n\times n} :\ V \in \mathcal{V}\}.$$

If $\mathcal{W}$ is the inverse of $\mathcal{V}$, then $\mathcal{W}^T$ is the inverse of $\mathcal{V}^T$. The same holds for the Hermitian transposition.

Invertibility is preserved in taking direct sums of matrix subspaces that are invertible. Also, if $\mathcal{V}$ is invertible, then so is $\mathcal{V} \otimes \mathbb{C}I$, where I is the identity matrix.

The following theorem is of importance in practice.

THEOREM 2.1. *Assume a subspace $\mathcal{V}$ of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ (or $\mathbb{R}$) contains an invertible element. Then the set of invertible elements is open and dense in $\mathcal{V}$.*

PROOF. Take any element $V \in \mathcal{V}$ and an invertible element $W \in \mathcal{V}$. With these, look at the map

$$z \longmapsto \det(zV + (1-z)W) \tag{2.2}$$

on $\mathbb{C}$. This is a nonzero polynomial in the complex variable z by the fact that at $z = 0$ its value is nonzero. Since it has a finite number of zeros, we can conclude there are invertible matrices arbitrarily close to V, i.e., invertible elements are dense in $\mathcal{V}$. To prove that the set of invertible elements in an open subset of $\mathcal{V}$, we have $W + V = W(I + W^{-1}V)$ which is invertible as soon as $\|W^{-1}V\| < 1$. The proof is complete for a subspace over $\mathbb{C}$.

For a subspace over $\mathbb{R}$, restrict z to be real to draw the same conclusion. $\square$

Consider $\mathcal{V}_0 = \{V \in \mathcal{V} : \det(V) = 0\}$. To show that this set is an embedded submanifold[3] of $\mathcal{V}$ of dimension $\dim(\mathcal{V}) - 1$, suppose $V \in \mathcal{V}_0$. We should have

$$\frac{\det(V - zW) - \det(V)}{z} = \frac{\det(VW^{-1} - zI)\det(W)}{z} \neq 0$$

for some invertible $W \in \mathcal{V}$. This means that the algebraic multiplicity of the eigenvalue 0 of VW^{-1} should be 1. This cannot be imposed in general to have a stronger claim in Theorem 2.1.

Since we assume there are invertible elements in an invertible subspace, we have the following corollaries.

COROLLARY 2.2. *Let a subspace $\mathcal{V} \subset \mathbb{C}^{n\times n}$ over $\mathbb{C}$ (or $\mathbb{R}$) be invertible. Then the set of invertible elements is open and dense in $\mathcal{V}$.*

COROLLARY 2.3. *Let a subspace $\mathcal{V} \subset \mathbb{C}^{n\times n}$ over $\mathbb{C}$ (or $\mathbb{R}$) be invertible with the inverse $\mathcal{W}$. Then* $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

PROOF. Use the étale mapping theorem. $\square$

Although a subspace $\mathcal{V}$ may not be invertible in the sense of Definition 2.1, the inverses of its invertible elements may still be readily characterisable. Such a property is of equal interest. Let us illustrate this with the following example.

EXAMPLE 2.5. Consider the subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{R}$ consisting of matrices of the form $isI + H$ with $s \in \mathbb{R}$ and $H \in \mathbb{C}^{n\times n}$ Hermitian. We have for $s \neq 0$

$$(isI + H)^{-1} = \frac{i}{2s}(Q - I),$$

where Q is the Cayley transform $(iH/s + I)(iH/s - I)^{-1}$ of iH/s, i.e., Q is unitary. (Of course, any unitary Q with $Q - I$ invertible is the Cayley transform of a skew-Hermitian matrix.) For $s = 0$ we have the inverse of H, whenever invertible.

This example is continued in Section 3.4.

Finally, we have been concerned with the inversion on matrix subspaces with a special structure. Similarly, other functions give rise to corresponding structures. For instance, with the exponential function one is interested in matrix subspaces that are also Lie algebras. The reason for this is that the exponential maps a Lie algebra to the respective matrix Lie group.

[3] For embedded submanifold, see, e.g., [11, p. 238 and p. 343].

2.2 Products of matrix subspaces, factoring and computing nullspaces.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two matrix subspaces of $\mathbb{C}^{n\times n}$ with $\mathcal{V}_2$ being invertible with the inverse $\mathcal{W}$. In what follows we are concerned with the set

$$\mathcal{V}_1\mathcal{V}_2 = \left\{V_1V_2 \in \mathbb{C}^{n\times n} :\ V_1 \in \mathcal{V}_1 \text{ and } V_2 \in \mathcal{V}_2\right\}$$

consisting of the products of the elements of $\mathcal{V}_1$ and $\mathcal{V}_2$ (in this order). If the dimension of both of the subspaces exceeds 1, then $\mathcal{V}_1\mathcal{V}_2$ typically is not a subspace of $\mathbb{C}^{n\times n}$ unless $\mathcal{V}_1 = \mathcal{V}_2$ is an algebra. The case of $\mathcal{V}_1 = \mathcal{V}_2$ being an algebra is not much of interest to us since then $\mathcal{V}_1\mathcal{V}_2 = \mathcal{V}_1$ is a subspace. (For a subspace $\mathcal{V}_1\mathcal{V}_2$ all the questions concerning factoring a given matrix $A \in \mathcal{V}_1\mathcal{V}_2$ are readily solved.)

Clearly, the map

$$(V_1, V_2) \longmapsto V_1V_2$$

is smooth from $\mathcal{V}_1 \times \mathcal{V}_2$ to $\mathbb{C}^{n\times n}$ and hence its rank is well-defined at every point (V_1, V_2). This, however, is not the main object of interest to us. Rather, fix a matrix $A \in \mathbb{C}^{n\times n}$ and suppose having $A \in \mathcal{V}_1\mathcal{V}_2$, i.e.,

$$A = V_1V_2 \tag{2.3}$$

for some $V_1 \in \mathcal{V}_1$ and $V_2 \in \mathcal{V}_2$. If the latter factor is invertible, then the task of recovering this factorization can be converted into computing the nullspace of a linear operator. To this end, denote by P_1 the orthogonal projection on $\mathbb{C}^{n\times n}$ onto $\mathcal{V}_1$, for instance, with respect to the standard inner product

$$(M, N) = \operatorname{trace}(N^*M) \quad \text{for } M, N \in \mathbb{C}^{n\times n}. \tag{2.4}$$

Define a linear operator $L : \mathcal{W} \to \mathbb{C}^{n\times n}$ as[4]

$$W \longmapsto (I - P_1)AW. \tag{2.5}$$

With this formulation, if W is in the nullspace of L, we necessarily have $AW = V_1 \in \mathcal{V}_1$. Hence, for any invertible element W in the nullspace of this linear operator we can set $A = V_1V_2$ with $V_2 = W^{-1}$ to have a desired factorization (2.3). The success of this approach relies on the fact that there are several numerically stable algorithms for computing the whole nullspace. As is well-known, some of these algorithms are even finite step methods.

An inexpensive alternative to provide a sufficient condition on (2.3) to hold consists of computing a random element from the nullspace of the linear operator (2.5) and checking its invertibility. In the positive case the nullspace has a dense open subset of invertible elements by Theorem 2.1, so that with probability one we get the correct answer. This probabilistic approach is clearly of great practical importance.

[4] The identity matrix I is regarded as acting on $\mathbb{C}^{n\times n}$.

If the matrix is invertible, then we have a complete solution to the factorization problem since then only invertible factors can appear in (2.3). Let us state this as follows.

THEOREM 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be invertible and $\mathcal{V}_1$ and $\mathcal{V}_2$ be subspaces of $\mathbb{C}^{n\times n}$ with $\mathcal{V}_2$ being invertible. Then* (2.3) *holds if and only if there exists an invertible element in the nullspace of* (2.5).

To characterise the number of factorizations in terms of a dimension, use Theorem 2.1 with the nullspace.

Clearly, the order of $\mathcal{V}_1$ and $\mathcal{V}_2$ makes a difference. If the matrix subspace $\mathcal{V}_1$ is invertible instead, then one option is to start from the identity $A^T = V_2^T V_1^T$ and proceed analogously with $\hat{\mathcal{V}}_1 = \mathcal{V}_2^T$ and $\hat{\mathcal{V}}_2 = \mathcal{V}_1^T$.

We have been concerned with square matrices. However, in this factoring approach only $\mathcal{V}_2$ needs to be a subspace of square matrices.

Products of two matrix subspaces have also other applications once standard operations with subspaces are extended accordingly. For instance, the sum operation has turned out to be useful, as the following examples illustrate.

EXAMPLE 2.6. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the subspaces of circulant and diagonal matrices, respectively. Motivated by diffractive optics, in [9] we considered the problem of approximating in the Frobenius norm a given matrix $A \in \mathbb{C}^{n\times n}$ with elements from

$$\underbrace{\mathcal{V}_1\mathcal{V}_2 + \mathcal{V}_1\mathcal{V}_2 + \cdots + \mathcal{V}_1\mathcal{V}_2}_{k \text{ sums}}$$

with $1 \le k \le n$. Clearly, with $k = 1$ we are concerned with approximate factoring of A as the product of a circulant and a diagonal matrix.

EXAMPLE 2.7. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the subspaces of circulant and skew-circulant matrices, respectively. In the inversion of Toeplitz matrices

$$\mathcal{V}_1\mathcal{V}_2 + \mathcal{V}_1\mathcal{V}_2$$

plays a central role since the inverses of invertible Toeplitz matrices belong to this sum [3, Theorem 3.1].

2.3 Remarks on approximate factoring.

In practice, approximate factorizations are occasionally of importance when A cannot be represented exactly as the product (2.3), i.e., when the nullspace of the linear operator (2.5) is zero dimensional, or does not contain invertible matrices. (Also, due to finite precision, computing the nullspace can be a delicate task [4, Chapter 5.5].) As is well-known, in preconditioning iterative methods, approximate factoring is of central importance and may be the only realistic alternative even if an exact factorization exists. Then one option consists of considering the "residual error"

$$\|(I - P_1)AW\|$$

when W, bounded away from the zero matrix, varies in $\mathcal{W}$. Denoting by $\|\cdot\|_2$ and $\|\cdot\|_F$ the operator and Frobenius norms, we have for any invertible W

$$\|(I-P_1)AW\|_F = \big\|\big(A - V_1W^{-1}\big)W\big\|_F \geq \big\|\big(A - V_1W^{-1}\big)W\big\|_2 \geq \frac{\|A - V_1V_2\|_2}{\|W^{-1}\|_2} \tag{2.6}$$

on the error corresponding to the approximating factors $V_1 = P_1AW \in \mathcal{V}_1$ and $V_2 = W^{-1} \in \mathcal{V}_2$. Hence the error is small if the residual error in the Frobenius norm can be made small with some invertible $W \in \mathcal{W}$ whose inverse has a moderate operator norm.

Clearly, for approximate factoring there is always the option of executing an alternating iteration. This is due to the fact that $V_1\mathcal{V}_2$ and $\mathcal{V}_1V_2$ are matrix subspaces for any fixed matrices V_1 and V_2. Therefore solving the matrix nearness problems

$$\min_{V_2 \in \mathcal{V}_2} \|A - V_1V_2\|_F \quad \text{and} \quad \min_{V_1 \in \mathcal{V}_1} \|A - V_1V_2\|_F$$

consecutively is straightforward, albeit very expensive. On the positive side, then $\mathcal{V}_2$ (or $\mathcal{V}_1$) does not need to be an invertible subspace.

3 Examples on factoring.

We consider a number of more or less classical examples to illustrate that we have a systematic approach to solve factorization problems with the nullspace of the linear operator (2.5). Even though the approach is unified, each case has its special character that gets expressed in terms of the respective necessary and sufficient conditions on the matrix A involved. We provide references, to the best of our knowledge, to the publications where these factorizations have been considered for the first time.

Undoubtedly, the most encountered case in practice consists of taking $\mathcal{V}_1$ and $\mathcal{V}_2$ to be the set of lower and upper triangular matrices. In case a solution exists, the result is an LU factorization of the matrix. We do not pay any particular attention to this problem since it is so extensively covered in the literature; see [4] and references therein.

3.1 Factoring into the product of two symmetric matrices.

As emphasized, e.g., in [14, 7], the role of symmetric matrices is central in matrix analysis. In fact, any square matrix $A \in \mathbb{C}^{n\times n}$ is the product of two symmetric matrices[5] of which at least one can be chosen to be invertible [2]. This is a classical result; see also [15, p. 37] for other references, as well as [5]. The demonstration is typically almost like an existence proof and only formally constructive based on a clever trick, yielding a single factorization in terms of the rational form of A.

[5] Equivalently, any $A \in \mathbb{C}^{n\times n}$ is the product of two self-adjoint antilinear operators.

For a demonstration that also leads to a numerically reliable and complete construction of the solutions, let $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{W}$ be the set of symmetric matrices. Then in (2.5) the operator $I - P_1$ consists of computing the skew-symmetric part of a matrix. Hence a symmetric $W \in \mathbb{C}^{n\times n}$ is mapped as

$$W \longmapsto (I - P_1)AW = \frac{1}{2}(AW - WA^T). \tag{3.1}$$

To solve for the nullspace, there are several ways. For one that only mildly changes the formulation and is numerically stable, compute the Schur decomposition $A = UTU^*$ of A, i.e., U is unitary and T upper triangular. Then the equation

$$AW - WA^T = 0 \tag{3.2}$$

is equivalent to

$$T\hat{W} - \hat{W}T^T = 0 \tag{3.3}$$

with symmetric $\hat{W} = U^* W\overline{U}$. Since T is upper triangular, it is straightforward to construct the solutions $\hat{W}$ and then put $W = U\hat{W}U^T$ to have the elements of the nullspace. For the number of solutions, suppose A has p distinct eigenvalues with the Jordan blocks of size $n_1 \geq \cdots \geq n_{l_1}$ for the eigenvalue λ_1 and $n_{l_1+1} \geq \cdots \geq n_{l_2}$ for the eigenvalue λ_2 etc. (We set $l_0 = 0$ below.)

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$. Then the nullspace of the map* (3.1) *is of dimension $n + \sum_{j=0}^{p-1} \sum_{k=2}^{l_{j+1}-l_j} (k-1)n_{l_j+k}$ with invertible elements open and dense.*

PROOF. First we prove by induction on n that there exists an invertible element in the nullspace. For openness and denseness this suffices, by Theorem 2.1. For $n = 1$ the claim is obviously true. Consider (3.3) with n. Striking the last column and row of T, we have an invertible element $\hat{W}_{n-1} \in \mathbb{C}^{(n-1)\times(n-1)}$ in the nullspace of (3.3), by the induction assumption. Augment $\hat{W}_{n-1}$ with a row and column of zeros to have an n-by-n symmetric matrix W such that $\hat{W}_{n-1}$ is its top left block. Consider $W + ww^T$ with an arbitrary $w \in \mathbb{C}^n$ and insert it into the equation (3.3). This gives us $Tww^T - ww^T T^T = 0$, which is satisfied by any eigenvector w of T. Choose an eigenvector w whose nth entry is nonzero. (This obviously can be done, for instance, by considering the eigenvalue t_{nn} of T.) Then $\hat{W}_n = W + ww^T$ is necessarily invertible, since the only solution to $\hat{W}_n x = 0$ is $x = 0$ by the fact that $\hat{W}_{n-1}$ is invertible.

To have the dimension of the nullspace, let $A = XJX^{-1}$ be the Jordan canonical form of A with $J = \operatorname{diag}(J_{n_1}(\lambda_1), \ldots, J_{n_{l_p}}(\lambda_p))$. Then

$$AW - WA^T = 0 \iff J\tilde{W} - \tilde{W}J^T = 0 \tag{3.4}$$

with a symmetric $\tilde{W} = X^{-1}WX^{-T}$. Because $\tilde{W}$ is symmetric, it suffices to solve on the diagonal

$$J_{n_i}(0)S - SJ_{n_i}(0)^T = 0 \quad \text{with a symmetric } S \in \mathbb{C}^{n_i\times n_i}$$

and for coinciding eigenvalues for the respective blocks

$$J_{n_i}(0)M - MJ_{m_j}(0)^T = 0 \quad \text{with } M \in \mathbb{C}^{n_i \times m_j}$$

in the strictly upper triangular part of $\tilde{W}$. The dimension of the latter is $\min\{n_i, m_j\}$ [8, Theorem 4.4.14], summing up to $\sum_{j=1}^{p} \sum_{k=2}^{l_{j+1}-l_j} (k-1)n_{l_j+k}$. For the former, there are $(n_i+1)n_i/2$ free parameters and $n_i(n_i-1)/2$ independent equations in the upper triangular part (start the inspection of the equations from the last column), so that the dimension is n_i. Hence the diagonal blocks sum up to n all together. □

The dimension of the nullspace is hence at least n. Generically, a matrix $A \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues and therefore the dimension is also generically n.

EXAMPLE 3.1. Suppose $A \in \mathbb{C}^{10\times 10}$ has a single eigenvalue λ_1 with the Jordan blocks of size 3, 2, 2, 1, 1 and 1. Then the dimension of the nullspace of (3.1) is $10 + 1*2 + 2*2 + 3*1 + 4*1 + 5*1 = 28$.

In [5, p. 2] there is a comment on factoring a matrix into the product of two symmetric matrices stating that "factoring is far from unique ... ". The approach presented makes this completely quantitative and is simultaneously fully constructive. For an illustration of a particularly simple parametrization of all the factorizations, consider normal matrices with distinct eigenvalues.

EXAMPLE 3.2. Suppose $A \in \mathbb{C}^{n\times n}$ is normal with $A = U\Lambda U^*$, where U is unitary and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. If $\lambda_j \neq \lambda_k$ for $j \neq k$, then the invertible solutions to (3.3) can be computed to give a factorization

$$A = V_1 V_2$$

of A with $V_1 = U\Lambda\hat{\Lambda}U^T$ and $V_2 = \overline{U}\hat{\Lambda}^{-1}U^*$, where $\hat{\Lambda}$ is an arbitrary invertible diagonal matrix. Clearly, the nullspace is exactly n dimensional.

Theorem 3.1 yields also the following corollary.

COROLLARY 3.2. *If $A \in \mathbb{R}^{n\times n}$, then A is the product of two real symmetric matrices.*

PROOF. Let $W = W_1 + iW_2$ be an invertible symmetric solution of the equation (3.2) with $W_1 = \operatorname{Re} W$ and $W_2 = \operatorname{Im} W$. Since A is real, also W_1 and W_2 are symmetric solutions of the equation (3.2). (That is, now AW is symmetric if and only if AW_1 and iAW_2 are symmetric.) Hence so are their linear combinations. Consider the map

$$z \longmapsto \det(W_1 + zW_2)$$

on $\mathbb{C}$. This is a nonzero polynomial in the complex variable z since it is invertible at $z = i$. Hence it has a finite number of zeros and therefore also for real z there must exist invertible solutions of the equation (3.2). □

From the point of view of iterative methods, the condition numbers of an ideal factorization $A = V_1V_2$ of an invertible $A \in \mathbb{R}^{n\times n}$ satisfy $\kappa(V_1) \approx \kappa(V_2) \approx \sqrt{\kappa(A)}$. We do not know if this can be achieved in general with the degrees of freedom provided by Corollary 3.2.

3.2 Factoring into the product of a skew-symmetric and a symmetric matrix.

Any matrix $M \in \mathbb{C}^{n\times n}$ can be split as

$$M = S + T, \quad \text{with } S = \frac{1}{2}(M + M^T) \text{ and } T = \frac{1}{2}(M - M^T). \tag{3.5}$$

Related to this, look at the product $\mathcal{V}_1\mathcal{V}_2$ of the subspaces $\mathcal{V}_1$ of skew-symmetric and $\mathcal{V}_2 = \mathcal{W}$ of symmetric matrices. With these, consider the factorization problem (2.3) studied also in [13]. Observe that if the dimension n is odd, then skew-symmetric matrices are not invertible. For this reason we are primarily interested in even dimensions.

Now in (2.5) the operator $I - P_1$ consists of computing the symmetric part of a matrix. Therefore a symmetric $W \in \mathbb{C}^{n\times n}$ is mapped as

$$W \longmapsto (I - P_1)AW = \frac{1}{2}(AW + WA^T). \tag{3.6}$$

For numerical stability, take again the Schur decomposition $A = UTU^*$ of A. Then the nullspace is obtained with the symmetric solutions $\hat{W}$ to

$$T\hat{W} + \hat{W}T^T = 0, \tag{3.7}$$

after setting $W = U\hat{W}U^T$. Considering the upper triangular part, we have $(n+1)n/2$ linear equations and $(n+1)n/2$ free parameters. Hence the problem is more intricate now since the difference is zero. To have nonzero solutions in the first place, necessary pairing conditions on the eigenvalues of A arise. Namely, ordering the entries of the upper triangular part of $\hat{W}$ row-wise, starting from the top row, gives an upper triangular linear system for the entries of $\hat{W}$ solving (3.7). The diagonal entries in this $(n+1)n/2$-by-$(n+1)n/2$ upper triangular matrix are $t_{jj} + t_{kk}$, with $1 \le j \le k \le n$, in the order

$$2t_{11}, t_{11} + t_{22}, t_{11} + t_{33}, \dots, t_{11} + t_{nn}, 2t_{22}, t_{22} + t_{33}, t_{22} + t_{44}, \dots, t_{22} + t_{nn}, \dots, \tag{3.8}$$

i.e., for nonzero solutions $\hat{W}$, necessarily $t_{jj} + t_{kk} = 0$ for some $1 \le j \le k \le n$.

Whether there are invertible solutions $\hat{W}$ at all, a more careful inspection of the eigenvalues is needed. In fact, suppose the matrices V_1 and V_2 are skew-symmetric and symmetric. The products V_1V_2 and V_2V_1 have the same eigenvalues. They are, moreover, similar if V_1V_2 is invertible. Since $(V_1V_2)^T = -V_2V_1$, we can conclude that an invertible A must be similar to $-A$ for $A = V_1V_2$ to hold. Assuming this, suppose an invertible A has p distinct eigenvalues ordered in pairs $\lambda_2 = -\lambda_1$, $\lambda_4 = -\lambda_3$ etc., with the Jordan blocks of size $n_1 \geq \cdots \geq n_{l_1}$ for the eigenvalue λ_1 and (of the same size) $n_{l_1+1} \geq \cdots \geq n_{l_2}$ for the eigenvalue $\lambda_2 = -\lambda_1$ etc. (We set $l_0 = 0$ below.)

THEOREM 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be invertible. Then there are invertible elements in the nullspace of the map* (3.6) *if and only if A is similar to $-A$. In the positive case, the nullspace is of dimension $\frac{1}{2}\sum_{j=0}^{p-1}\sum_{k=1}^{l_{j+1}-l_j}(2k-1)n_{l_j+k}$ with invertible elements open and dense.*

PROOF. Necessity has been shown. For sufficiency, suppose A is invertible and similar to $-A$. Let $A = XJX^{-1}$ be the Jordan canonical form of A. Then the nullspace of the map (3.6) can also be recovered with the symmetric solutions $\tilde{W}$ to

$$J\tilde{W} + \tilde{W}J^T = 0, \tag{3.9}$$

after setting $W = X\tilde{W}X^T$.

It suffices to consider a double block

$$\begin{bmatrix} J_{n_i}(\lambda_j) & 0 \\ 0 & J_{m_j}(-\lambda_j) \end{bmatrix}. \tag{3.10}$$

With this, since $\lambda_j \neq 0$, for the diagonal blocks we do not have any solutions by [8, Theorem 4.4.14]. For the $(1,2)$-block we have to solve

$$J_{n_i}(0)M + MJ_{m_j}(0)^T = 0 \quad \text{ with } M \in \mathbb{C}^{n_i\times m_j}.$$

The dimension of the solution space of this equation is $\min\{n_i, m_j\}$ by [8, Theorem 4.4.14]. This gives us the dimension $\sum_{j=1}^{l_{j+1}-l_j}(2k-1)n_{l_j+k}$ for the eigenvalue pair λ_j and $-\lambda_j$. In all these sum up as claimed.

For the existence of a symmetric invertible solution, look at (3.10) with $n_i = m_j$. It suffices to find an invertible solution $M \in \mathbb{C}^{n_i\times n_i}$ to

$$J_{n_i}(0)M + MJ_{n_i}(0)^T = 0$$

and put $\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}$. For this, take M having the entries $(-1)^k$ at the antidiagonal positions $(n_i - k + 1, k)$ for $k = 1, \ldots, n_i$. Repeating this for each pair of blocks of the same size, with every pair λ_j and $-\lambda_j$, gives an invertible solution to (3.9). For openness and denseness this suffices, by Theorem 2.1. ∎

To give an example, if $A \in \mathbb{C}^{n\times n}$ satisfying the assumptions of this theorem has n distinct eigenvalues, then the dimension is $n/2$.

A practical consequence of this theorem is that the symmetric-skew-symmetric splitting (3.5) is not very attractive for preconditioning. As is well known, iterative methods can be expected to converge poorly for matrices with eigenvalues surrounding the origin. This is the case if A is similar to $-A$.

With the techniques of the proof of Corollary 3.2, we have the following analogy.

COROLLARY 3.4. *If $A \in \mathbb{R}^{n\times n}$ is invertible and similar to $-A$, then A is the product of a real skew-symmetric and a real symmetric matrix.*

3.3 Factoring into the product of two Hermitian matrices.

In a completely similar vain, let us briefly go over the factorization problem (2.3) with the subspaces $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{W}$ being the set of Hermitian matrices corresponding to the splitting (1.1). For this factorization, see [12]. In this case the operator $I - P_1$ in (2.5) consists of computing the skew-Hermitian part of a matrix, so that a Hermitian $W \in \mathbb{C}^{n\times n}$ is mapped as

$$W \longmapsto (I - P_1)AW = \frac{1}{2}(AW - WA^*). \tag{3.11}$$

With the Schur decomposition $A = UTU^*$ of A, the nullspace can be recovered numerically reliably by finding the Hermitian solutions $\hat{W}$ to

$$T\hat{W} - \hat{W}T^* = 0 \tag{3.12}$$

and setting $W = U\hat{W}U^*$. The difference now, compared with the matrix equations of the preceding subsections, is that the map (3.11) is $\mathbb{R}$-linear (by the fact that the set of Hermitian matrices is a subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{R}$).

Taking the real and imaginary parts, we have n^2 equations and n^2 free real parameters, so that the difference is again zero. To have nonzero solutions, necessary pairing conditions on the eigenvalues of A arise. Ordering the entries of the upper triangular part of a Hermitian $\hat{W} \in \mathbb{C}^{n\times n}$ row-wise, starting from the top row, gives us an upper triangular $\mathbb{R}$-linear system for the entries of $\hat{W}$ solving (3.12). For nonzero solutions, necessarily

$$t_{jj} - \overline{t_{kk}} = 0 \tag{3.13}$$

for some $1 \le j \le k \le n$. To the case $j = k$ corresponds a free real parameter (diagonal entries of $\hat{W}$) and to the case $j \neq k$ a free complex parameter (strictly upper-triangular entries of $\hat{W}$).

Again, whether there are invertible solutions $\hat{W}$ at all, a more careful inspection of the eigenvalues is needed. For this, suppose the matrices V_1 and V_2 are Hermitian. The products V_1V_2 and V_2V_1 have the same eigenvalues. They are, moreover, similar if V_1V_2 is invertible. Since $(V_1V_2)^* = V_2V_1$, we can conclude that an invertible A must be similar to A^* for $A = V_1V_2$ to hold. Assuming this, suppose A has p distinct eigenvalues of which the first s are real and the remaining complex eigenvalues are ordered such that $\lambda_{s+2} = \overline{\lambda_{s+1}}$, $\lambda_{s+4} = \overline{\lambda_{s+3}}$ etc. Let the Jordan blocks of A be of size $n_1 \ge \cdots \ge n_{l_1}$ for the eigenvalue λ_1 and $n_{l_1+1} \ge \cdots \ge n_{l_2}$ for the eigenvalue λ_2 etc. (We set $l_0 = 0$ below.)

THEOREM 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be invertible. Then there are invertible elements in the nullspace of the map* (3.11) *if and only if A is similar to A^*. In the positive case, the nullspace is of real dimension*

$$\sum_{j=0}^{s-1}\sum_{k=1}^{l_{j+1}-l_j} n_{l_j+k} + 2\sum_{j=0}^{s-1}\sum_{k=2}^{l_{j+1}-l_j}(k-1)n_{l_j+k} + \sum_{j=s}^{p-1}\sum_{k=1}^{l_{j+1}-l_j}(2k-1)n_{l_j+k}$$

with invertible elements open and dense.

PROOF. Necessity has been shown. For sufficiency, suppose A is invertible and similar to A^*. Let $A = XJX^{-1}$ be the Jordan canonical form of A. Then the nullspace of the map (3.11) can also be recovered with the Hermitian solutions $\tilde{W}$ to

$$J\tilde{W} - \tilde{W}J^* = 0, \tag{3.14}$$

after setting $W = X\tilde{W}X^*$.

For the complex eigenvalues of A it suffices to consider a double block

$$\begin{bmatrix} J_{n_i}(\lambda_j) & 0 \\ 0 & J_{m_j}(\overline{\lambda_j}) \end{bmatrix}. \tag{3.15}$$

For the diagonal blocks we do not have any solutions by [8, Theorem 4.4.14] while for the $(1,2)$-block we have to solve

$$J_{n_i}(0)M - MJ_{m_j}(0)^* = 0 \quad \text{with } M \in \mathbb{C}^{n_i\times m_j}.$$

The complex dimension of the solution space for this equation is $\min\{n_i, m_j\}$ by [8, Theorem 4.4.14]. Consequently, this gives us the complex dimension $\sum_{k=1}^{l_{j+1}-l_j}(2k-1)n_{l_j+k}$ for the eigenvalue pair λ_j and $\overline{\lambda_j}$.

For real eigenvalues, suppose $\lambda_j \in \mathbb{R}$ in the block (3.15). Then for the diagonal blocks, Hermitian solutions are acceptable. Since λ_j is real, Hermitian solutions are the Hermitian parts of any solutions. These have been constructed in the proof of Theorem 3.1. Hence the real dimension of the nullspace for the first diagonal block is n_j and for the second m_j. Corresponding to the $(1,2)$-block, the complex dimension of the solution space is $\min\{n_j, m_i\}$ by [8, Theorem 4.4.14]. Summing these give the real dimension of the nullspace as claimed.

For the existence of a Hermitian invertible solution, look at first complex eigenvalues in (3.15) with $n_i = m_j$. It suffices to find an invertible solution $M \in \mathbb{C}^{n_i\times n_i}$ to

$$J_{n_i}(0)M - MJ_{n_i}(0)^* = 0$$

and put $\begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix}$. For this, take M having ones at the antidiagonal positions. Then to deal with real eigenvalues of A, consider a Jordan block on the diagonal. Use Corollary 3.2 to have an invertible real symmetric, i.e., Hermitian solution for the respective equation.

Collecting these invertible blocks gives an invertible Hermitian solution to the equation (3.14). For openness and denseness this suffices, by Theorem 2.1. □

EXAMPLE 3.3. As a mixture of Sections 3.1 and 3.3 one can take $\mathcal{V}_1$ to be the set of Hermitian and $\mathcal{V}_2 = \mathcal{W}$ the set of symmetric matrices. For the conditions on $A \in \mathbb{C}^{n\times n}$, proceed analogously by using its concanonical form. (For the concanonical form, see [6].)

3.4 Other factorizations.

Suppose having a matrix subspace that is not invertible. Occasionally the inverses of invertible elements can still be readily characterised, giving rise to a natural family of matrices. When this is the case, the computational approach proposed can be executed for factoring matrices accordingly, with elements from this family of matrices. Let us illustrate this by pursuing Example 2.5 further.

EXAMPLE 3.4. Let $\mathcal{W}$ be the subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{R}$ consisting of matrices of the form $isI+H$ with $s \in \mathbb{R}$ and $H \in \mathbb{C}^{n\times n}$ Hermitian. Let $\mathcal{V}_1$ be any subspace of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$. Then the problem of factoring a given invertible $A \in \mathbb{C}^{n\times n}$ as

$$A = V_1(Q - I) \tag{3.16}$$

with $V_1 \in \mathcal{V}_1$ and Q unitary can be converted into an equivalent problem of finding the nullspace of the map from $\mathcal{W}$ to $\mathbb{C}^{n\times n}$ defined by

$$W \longmapsto (I - P_1)AW, \tag{3.17}$$

where P_1 is the orthogonal projector onto $\mathcal{V}_1$. If there exists an invertible element $W = isI + H$ in the nullspace, with $s \neq 0$, then put $V_1 = \frac{i}{2s}AW$ and $(Q - I) = \frac{2s}{i}W^{-1}$. Again, the number of such factorizations is characterised by the dimension of the nullspace of the operator (3.17).

3.5 Approximate factoring and preconditioning.

Next we outline some preliminary ideas how approximate factoring of Section 2.3 can be used in preconditioning a linear system

$$Ax = b \tag{3.18}$$

with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Sometimes one takes the symmetric (Hermitian since A is real) part $\frac{1}{2}(A + A^T)$ of A to this end, assuming that an efficient solver for the respective symmetric problem is available [1]. In the examples that follow, we improve this approximation in terms of products of two real symmetric matrices.

EXAMPLE 3.5. Take $\mathcal{V}_1$ to be the set of symmetric matrices and $\mathcal{V}_2 = \mathcal{W}$ the set of diagonal matrices. Let the diagonal entries of $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$

be the variables. Then look at

$$\begin{aligned}
&\|(I-P_1)AW\|_F^2 + \left(\|W\|_F^2 - n\right)^2 \\
&\quad = \frac{1}{4}\|AW - WA^T\|_F^2 + \left(\|W\|_F^2 - n\right)^2 \\
&\quad = \frac{1}{4}\sum_{k=1}^{n}\sum_{j=1}^{n}(w_j a_{kj} - w_k a_{jk})^2 + \Big(\sum_{j=1}^{n}\left(w_j^2 - 1\right)\Big)^2.
\end{aligned}$$

The jth entry of its gradient is

$$\frac{1}{2}w_j \sum_{k=1,\, k\neq j}^{n} a_{kj}^2 - \frac{1}{2}\sum_{k=1,\, k\neq j}^{n} w_k a_{jk} a_{kj} + 4\sum_{k=1}^{n}\left(w_k^2 - 1\right)w_j$$

for $j = 1, 2, \ldots, n$. For sparse A this is not expensive to evaluate. Descending with this, using $W_0 = I$ as a starting point, improves the preconditioner $\frac{1}{2}(A + A^T)$. Namely, after taking k steps, suppose W_k is invertible. Then take the product V_1V_2, with $V_1 = \frac{1}{2}(AW_k + W_kA^T)$ and $V_2 = W_k^{-1}$, to approximate A.

EXAMPLE 3.6. For another approach, let $\mathcal{V}_1$ be again the set of symmetric matrices and $\mathcal{V}_2 = \mathcal{W}$ the set of polynomials in a symmetric matrix $S \in \mathbb{R}^{n\times n}$. Then consider

$$\min \|Ap(S) - p(S)A^T\| \tag{3.19}$$

while p varies in the set of monic polynomials of degree $k \ll n$. If q solves this, then take the product $\frac{1}{2}(Aq(S) + q(S)A^T)q(S)^{-1}$ to approximate A, assuming $q(S)$ to be invertible. Instead of minimizing (3.19), a less expensive alternative to have a monic polynomial is to solve

$$\min \|Ap(S)b - p(S)A^Tb\|$$

with Krylov subspace methods.

4 Conclusions.

A systematic approach to solve factorization problems has been proposed together with motivations. The notion of the inverse of a matrix subspace allows a general treatment of the problem of factoring into the product of two matrices. Numerically stable factorization algorithms result. Extensions of these ideas were outlined.

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