LOW-RANK MATRIX RECOVERY VIA RANK ONE TIGHT FRAME MEASUREMENTS

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ABSTRACT. The task of reconstructing a low rank matrix from incomplete linear measurements arises in areas such as machine learning, quantum state tomography and in the phase retrieval problem. In this note, we study the particular setup that the measurements are taken with respect to rank one matrices constructed from the elements of a random tight frame. We consider a convex optimization approach and show both robustness of the reconstruction with respect to noise on the measurements as well as stability with respect to passing to approximately low rank matrices. This is achieved by establishing a version of the null space property of the corresponding measurement map.

1. INTRODUCTION

Compressed sensing [3] predicts that sparse vectors can be reconstructed stably from an incomplete and possibly noisy set of linear (random) measurements via efficient algorithms including ℓ_1-minimization. This theory has been extended to the reconstruction of low rank matrices from incomplete measurements. Initial contributions [2] analyzed Gaussian random measurement maps having no structure at all. However, in quantum state tomography, for instance, one does require structure. A particular setup considers measurements with respect to rank-one matrices which makes the analysis more difficult because of a reduced amount of (stochastic) independence. In [4] and [5], such rank one measurements consisting of projections on random vectors drawn from a Gaussian distribution or from a complex projective 4-design are considered. In this note, we extend this to the case that the measurements project onto the elements from a random tight frame. In order to make this more precise we recall the setup of [5].

We consider rank one measurements of an (approximately) low-rank Hermitian matrix $X \in \mathcal{H}_n$ of the form $\mathcal{A}(X)$, where the linear measurement map $\mathcal{A}$ is given as

$$\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m, \quad Z \mapsto \sum_{j=1}^{m} \text{tr}(Z a_j a_j^*) e_j. \tag{1}$$

Here, $\mathcal{H}_n$ denotes the space of $n \times n$ complex Hermitian matrices, $e_1, \ldots, e_m$ denote the standard basis vectors in $\mathbb{R}^m$ and $a_1, \ldots, a_m \in \mathbb{R}^n$ are measurement vectors. Taking into consideration the presence of noise we write

$$b = \mathcal{A}(X) + w, \tag{2}$$

where $w \in \mathbb{R}^m$ is a vector of perturbations. We consider the following noise constrained nuclear norm minimization problem

$$\min_{Z \in \mathcal{H}_n} \|Z\|_1 \quad \text{subject to} \quad \|\mathcal{A}(Z) - b\|_{\ell_q} \leq \eta, \tag{3}$$

where η denotes a known estimate of the noise level, i.e., $\|w\|_{\ell_q} \leq \eta$ for some $q \geq 1$ (including the case $q = \infty$). Here and in the sequel, we denote by $\|Z\|_1$ the nuclear norm of Z (the sum of its singular values). Similarly, $\|Z\|_2$ denotes the Frobenius norm of Z (the ℓ_2-norm of the vector of singular values of Z), and $\|Z\|_\infty$ denotes the maximal singular value of Z.

We are interested in choosing a minimal number m of measurements (ideally smaller than n^2) that still allows reconstruction of X of (approximately) rank r from $b = \mathcal{A}(X) + w$. In this note, we choose the measurement vectors $a_1, \ldots, a_m$ from a tight frame, that is, the matrix $M \in \mathbb{R}^{m \times n}$ whose rows are the

vectors a_j satisfies $M^*M = \mathrm{id}$. In order to analyze this setup rigorously, we introduce randomness. To this end we adopt the notation of [6], and denote by $\mathbb{V}_n^m$ the Stiefel manifold consisting of the real valued $m \times n$ matrices M with the property $M^*M = \mathrm{id}$.

Definition 1. Let $m \geq n$. A $m \times n$ random tight frame is the set of rows of a $m \times n$ matrix with orthonormal columns which is drawn uniformly from the Stiefel manifold $\mathbb{V}_n^m$.

We also use the following notation. Given a complex valued $n_1 \times n_2$ matrix M and a non negative integer $r \leq n_1, n_2$, we denote by M_r the diagonal matrix whose diagonal entries are the r largest singular values of M, and by $M_{r,c}$ the diagonal matrix whose diagonal entries are the remaining singular values of M.

The following theorem concerning reconstruction with respect to rank one measurements corresponding to the elements of a random tight frame is the main result of this note.

Theorem 2. *Consider the above measurement process with m measurement vectors a_i which are the (transposed) elements of a random tight frame multiplied by $\sqrt{m}$. Let $r \leq n$ and suppose that*

$$m \geq C_1 n r. \tag{4}$$

Then with probability at least $1 - 3\mathrm{e}^{-C_2 m}$ it holds that for any $X \in \mathcal{H}_n$, any solution $X^\sharp$ to the above convex optimization problem with noisy measurements $b = \mathcal{A}(X) + w$, where $\|w\|_{\ell_q} \leq \eta$, obeys

$$\|X - X^\sharp\|_2 \leq \frac{D_1}{\sqrt{r}}\|X_{r,c}\|_1 + D_2 \cdot \frac{\eta}{m^{1/q}}. \tag{5}$$

Here C_1, C_2, D_1, D_2 denote positive universal constants. (In particular, for $\eta = 0$ and X of rank at most r one has exact reconstruction.)

This result is of similar nature as the main results of [5], where however different types of measurements were analyzed. The bound (4) on the number of measurements is optimal. In the special case $r = 1$ it implies that a vector $x \in \mathbb{R}^n$ (or in $\mathbb{C}^n$) can be reconstructed robustly from noisy phaseless measurements $y_j = |\langle x, a_j \rangle|^2 + w_j$ via the PhaseLift approach, see [1, 4, 5] for details.

The above result is shown via establishing a version of the null space property of the measurement map $\mathcal{A}$, namely the Frobenius-robust rank null space property with respect to ℓ_q. Let us recall from [5] this notion (in the Hermitian case) which serves as a useful recovery criterion for the above measurement process (1) via the minimization problem (3).

Definition 3. For $q \geq 1$, we say that $\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m$ satisfies the Frobenius-robust rank null space property with respect to ℓ_q of order r with constants $0 < \rho < 1$ and $\tau > 0$ if for all $M \in \mathcal{H}_n$, the singular values of M satisfy

$$\|M_r\|_2 \leq \frac{\rho}{\sqrt{r}}\|M_{r,c}\|_1 + \tau\|\mathcal{A}(M)\|_{\ell_q}.$$

In [5, Theorem 3.1 and Remark 3.2], it is shown for the above measurement process (1) that if $\mathcal{A}$ satisfies the Frobenius-robust rank null space property with respect to ℓ_q of order r (with constants $0 < \rho < 1$ and $\tau > 0$) then any solution $X^\sharp$ of (3) approximates X with error

$$\|X - X^\sharp\|_2 \leq \frac{C_1(\rho)}{\sqrt{r}}\|X_{r,c}\|_1 + C_2(\rho)\tau\eta.$$

Here $C_1(\rho)$ and $C_2(\rho)$ are explicit positive constants depending only on ρ.

Proof of Theorem 2. Let Q be the matrix whose rows q_i^* are the elements of our random tight frame multiplied by $\sqrt{m}$. Recall that an $n \times n$ Wishart matrix with m degrees of freedom is a random matrix of the form AA^*, where A is an $n \times m$ Gaussian matrix. Let W be $1/\sqrt{m}$ times the square root of an $n \times n$ Wishart matrix with m degrees of freedom (i.e. $W = \frac{1}{\sqrt{m}}\sqrt{AA^*}$, where A is as above), independent of Q. Then $G := QW$ is an $m \times n$ Gaussian matrix, cf. [6, Proposition 9]. Let $g_1^*, \ldots, g_m^*$ be the rows of G. Then $g_i = W^*q_i$. We denote by $\mathcal{A}^{(G)}$ the above measurement map $\mathcal{A}$ with measurement vectors $a_i = g_i$. Similarly, we denote by $\mathcal{A}^{(Q)}$ be the above map $\mathcal{A}$ with measurement vectors $a_i = q_i$. It follows that for any Hermitian $n \times n$ matrix X

$$\mathcal{A}^{(G)}(X)_i = \mathrm{tr}(Xg_ig_i^*) = \mathrm{tr}(XW^*q_iq_i^*W) = \mathrm{tr}(WXWq_iq_i^*) = \mathcal{A}^{(Q)}(WXW)_i.$$

Hence

$$\mathcal{A}^{(Q)}(X) = \mathcal{A}^{(G)}(W^{-1}XW^{-1})$$

for all X.

Now [5, Lemma 8.1] tells us that if $\mathcal{A}^{(G)}$ satifies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants ρ, τ, and if $\kappa(W)^2\rho < 1$, then the map $X \mapsto \mathcal{A}^{(G)}(W^{-1}XW^{-1})$ satisfies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants $\kappa(W)^2\rho, \|W\|_\infty^2\tau$. (Here $\kappa(W) = \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$ is the quotient of the maximal and the minimal eigenvalue of W.) Assuming that $\mathcal{A}^{(G)}$ satifies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants ρ, τ such that $\kappa(W)^2\rho < 1$, it follows that $\mathcal{A}^{(Q)}$ satisfies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants $\kappa(W)^2\rho, \|W\|_\infty^2\tau$.

It remains to show that with high probability $\mathcal{A}^{(G)}$ satifies the Frobenius robust rank null space property with respect to ℓ_q with suitable constants $0 < \rho < 1$ and $\tau > 0$ such that $\kappa(W)^2\rho < 1$.

It is shown in the proof of [5, Proposition 8.1] that for any $c_1 > 1$ there are positive constants c_2, c_3 such that for $m \geq c_2 n$ with probability at least $1 - 2e^{-c_3 m}$ we have

$$\max(\kappa(W), \|W\|_\infty) \leq c_1.$$

Fix now $\rho = 1/2$. Then, as shown in the proof of [5, Theorem 1.2], there are positive constants c_4, c_5, c_6 such that, for $m \geq c_4 rn$, with probabilty at least $1 - e^{-c_5 m}$, the map $\mathcal{A}^{(G)}$ satisfies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants $\rho = 1/2, \tau = c_6/m^{1/q}$. (This result makes heavy use of Mendelson's small ball method, [7, 8, 9].) Choose now $1 < c_1 < \sqrt{2}$. Then with probability at least $1 - 2e^{-c_3 m}$ we have $\kappa(W)^2\rho \leq c_1^2/2 < 1$ and $\|W\|_\infty^2 \leq c_1^2 < 2$. It follows that for m large enough (as above) with probability at least $1 - 3e^{-C_2 m}$, the map $\mathcal{A}^{(Q)}$ satisfies the Frobenius robust rank null space property with respect to ℓ_q of order r with constants $\kappa(W)^2/2 < 1$ and $2c_6/m^{1/q}$. Hence by the above mentioned recovery result [5, Theorem 3.1], for $m \geq C_1 nr$, with probability at least $1 - 3e^{-C_2 m}$ the recovery result of the Theorem holds (where $C_1 = c_4$ and $C_2 = \min\{c_3, c_5\}$ are universal positive constants). $\qquad\square$

Remark 4. The multiplication of the q_i by $\sqrt{m}$ yields the correct normalization compared to Gaussian measurements since the sum of the squared ℓ_2-norms of all m elements in the random tight frame is n, hence on average each row vector has squared ℓ_2-norm equal to n/m, whereas for Gaussian measurement vectors the squared ℓ_2-norm has expectation n.

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