Multi scale unfolding of real networks by geometric renormalization

Geometric renormalization of the $\mathcal{S}^1$ model

Introduction

Many of the usual ways of renormalizing networks make use of its topology (using the adjacency, Laplacian or correlation matrices). This way, the "length" to be used in renormalization is related to minimum distances between nodes, and this can be a problem in complex networks, in which the small-world effect correlates different scales, and makes the layers dependent of each other.

The article [1] thus proposes to make a "geometric renormalization", by finding a hidden metric space that well describes properties of the network, and proceeding with renormalization in that space.

Turns out that the hyperbolic disk is a good way to represent complex networks, because it can embed two degrees of freedom (popularity and similarity of nodes) in a way that creates few nodes that have very high popularity (thus depicting the degree power law present in many complex networks). This space can be represented either in $\mathcal{S}^1$ (1-dimensional sphere) or in $\mathcal{H}^2$ (hyperbolic disk).

In the first case, the variables used are the angular distance between two nodes, representing similarity, and a hidden variable κ, representing popularity. These two variables are used in the second model almost in the same fashion, but changing the hidden degree to the radial coordinates. Nodes closer to the center of the disk are more popular. Here we will use the $\mathcal{S}^1$ model, as it is easier to deal with. It is important to notice that even though the networks themselves can be in D-dimensional space, the strong coupling between scales makes the one-dimensional sphere a faithful model.

Derivations

We proceed by defining non-overlapping blocks of consecutive nodes of size r in the circle, and then coarse graining those blocks into super-nodes, and we want to find how the hidden degrees and angular distances should scale to make the probability distribution of linking two nodes link to remain invariant with respect to the change in scale. As long as the real network remains well-embedded in the metric space, this scaling law will also renormalize the network.

In this derivation, primed terms refer to layer l and unprimed ones to layer $l-1$. We assume that all supernodes contain the **same number of nodes** r. This is not necessary to the final result, though.

Let p'_{ij} be the probability that 2 supernodes i e j be connected in layer l. This is equivalent to the probability that there is at least 1 link between some internal node from i and one from j (or 1 minus the probability of no connections between any node of a cluster and another node of the other cluster):

$$p'_{ij} = 1 - \prod_{e=1}^{r^2}(1 - p_e) \tag{1}$$

The condensed index e sweeps all pairs of nodes (m, n) between both groups, and p_e is the probability that node m, inside supernode i, and node n, inside j be connected on the previous layer $(l-1)$:

$$p_e = \frac{1}{1 + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{\beta}} \tag{2}$$

This term is a choice of probability density for node connection that comes from the derivation presented in [2].

This way:

$$\begin{aligned}
p'_{ij} &= 1 - \prod_{e=1}^{r^2}\left[1 - \frac{1}{1 + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{\beta}}\right] = 1 - \prod_{e=1}^{r^2}\left[\frac{\cancel{1} + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{\beta} - \cancel{1}}{1 + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{\beta}}\right] \\
&= 1 - \prod_{e=1}^{r^2}\left[\frac{1}{1 + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{-\beta}}\right] = 1 - \frac{1}{\prod_{e=1}^{r^2}\left[1 + \left(\frac{R\Delta\theta_e}{\mu(\kappa_m\kappa_n)_e}\right)^{-\beta}\right]} \\
&= 1 - \frac{1}{\prod_{e=1}^{r^2}\left[1 + \left(\frac{\mu(\kappa_m\kappa_n)_e}{R\Delta\theta_e}\right)^{\beta}\right]}
\end{aligned}$$

Where, in the last term, we just inverted the fraction and changed the sign of the exponent β.

The denominator of the last term can be expanded into r^2 factors, namely:

$$\prod_{e=1}^{r^2}\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_e}{R\Delta\theta_e}\right)^{\beta}\right]=\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_1}{R\Delta\theta_1}\right)^{\beta}\right]\cdot\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_2}{R\Delta\theta_2}\right)^{\beta}\right]\cdot\ldots$$
$$\ldots\cdot\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_{r^2}}{R\Delta\theta_{r^2}}\right)^{\beta}\right]$$

This product can be ordered so it generates terms of growing order, since it is a finite product. We can first pick the unitary terms from all parentheses, and then all unitary but one, and so on, until we have reach the term of largest order, that corresponds to the product of all variable terms, that is:

$$\begin{aligned}D&=\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_1}{R\Delta\theta_1}\right)^{\beta}\right]\cdot\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_2}{R\Delta\theta_2}\right)^{\beta}\right]\cdot\ldots\cdot\left[1+\left(\frac{\mu(\kappa_m\kappa_n)_{r^2}}{R\Delta\theta_{r^2}}\right)^{\beta}\right]\\&=1+\left[\left(\frac{\mu(\kappa_m\kappa_n)_1}{R\Delta\theta_1}\right)^{\beta}+\left(\frac{\mu(\kappa_m\kappa_n)_2}{R\Delta\theta_2}\right)^{\beta}+\left(\frac{\mu(\kappa_m\kappa_n)_{r^2}}{R\Delta\theta_{r^2}}\right)^{\beta}\right]+\\&+\sum_{e=1}^{r^2-1}\sum_{f=e+1}^{r^2}\left(\frac{\mu(\kappa_m\kappa_n)_e}{R\Delta\theta_e}\right)^{\beta}\left(\frac{\mu(\kappa_m\kappa_n)_f}{R\Delta\theta_f}\right)^{\beta}+\ldots\end{aligned}$$

If we can approximate the angular distances between nodes within each layer to be much smaller than those between nodes in different layers, those distances between inner nodes can be taken to be the same, for any pair $e=(m,n)$, and we can write: $\Delta\theta_e\approx\Delta\theta$, $\forall e$, so the denominator expanded previously can be simplified:

$$D\approx 1+\left(\frac{\mu}{R\Delta\theta}\right)^{\beta}\sum_{e=1}^{r^2}(\kappa_m\kappa_n)_e^{\beta}+\left(\frac{\mu}{R\Delta\theta}\right)^{2\beta}\sum_{e=1}^{(r^2-1)}\sum_{f=e+1}^{r^2}(\kappa_m\kappa_n)_e^{\beta}(\kappa_m\kappa_n)_f^{\beta}+\ldots$$

The model of the $\mathcal{S}^1$ sphere for embedding networks assumes that the density of nodes is constant and equal to 1, so that the radius of the sphere is adjusted to get this uniform unitary density, depending on N as $R=\frac{N}{2\pi}$, while μ is independent of the number of nodes N. Following the approximations we were doing, we are usually looking into very large networks, so that, with μ being a constant factor that regulates the average degree of each layer (it connects k

to κ [2]), $\frac{\mu}{R} << 1$, and we can discard the higher order terms in the previous equation, sticking to the form:

$$D \approx 1 + \left(\frac{\mu}{R\Delta\theta}\right)^{\beta} \sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}$$

And this leads us to an approximate formula for the probability of the two supernodes, i and j, to be connected:

$$p'_{ij} \approx 1 - \frac{1}{1 + \left(\frac{\mu}{R\Delta\theta}\right)^{\beta} \sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}} = \frac{\left(\frac{\mu}{R\Delta\theta}\right)^{\beta} \sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}}{1 + \left(\frac{\mu}{R\Delta\theta}\right)^{\beta} \sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}} \implies$$

$$\boxed{p'_{ij} \approx \frac{1}{1 + \left(\frac{R\Delta\theta}{\mu}\right)^{\beta} \frac{1}{\sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}}}}$$

We require that this probability on a coarsened scale match the form we had started with. To achieve that, we need to find some parameters such that the blue part becomes:

$$\left(\frac{R\Delta\theta}{\mu}\right)^{\beta} \frac{1}{\sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta}} = \left(\frac{R'\Delta\theta'_{ij}}{\mu' \kappa'_i \kappa'_j}\right)^{\beta'}$$

Assuming further that the angular separation between two renormalized nodes is a generalized center of mass of the nodes that composes it. So the angular distance between supernodes i and j is approximately the angular separation between the centers of mass of the supernodes: $\Delta\theta'_{ij} \approx \Delta\theta \implies \beta' = \beta$, given that no other variable can depend on $\Delta\theta$.

Also, given that the density of nodes must be unitary in all scales (as required before), the radius has to scale with the size of the clusters r: the larger the number of nodes r composing a cluster, the smaller should be the radius, since the number of supernodes in a layer is r times smaller than in the previous one. Therefore, $R' = R/r$. This is the rescaling step, and the constant r can be absorbed by μ as: $\mu' = \mu/r$. All these requirements together make up for the following transformation:

$$(\kappa'_i \kappa'_j)^{\beta} = \sum_{e=1}^{r^2} (\kappa_m \kappa_n)_e^{\beta} \implies \boxed{\kappa'_i = \left(\sum_{j=1}^{r} \kappa_j^{\beta}\right)^{1/\beta}}$$

The right hand side comes from the fact that the left side is the distribution of the product:

$$\kappa_i'\kappa_j' = \left(\sum_{m=1}^{r} \kappa_m^\beta\right)^{1/\beta} \cdot \left(\sum_{n=1}^{r} \kappa_n^\beta\right)^{1/\beta} = \left[\kappa_{i_1}^\beta + \kappa_{i_2}^\beta + \ldots + \kappa_r^\beta\right]^{1/\beta} \cdot \left[\kappa_{j_1}^\beta + \kappa_{j_2}^\beta + \ldots + \kappa_r^\beta\right]^{1/\beta}$$
$$= \left(\sum_{m,n=1}^{r} \kappa_m^\beta \kappa_n^\beta\right)^{1/\beta}$$

This transformation preserves the semigroup structure, since applying the transformation once, obtaining a cluster of size r, and then applying it again is the same as applying it once to a cluster of size r^2.

In order for the angle transformation to represent a generalized center of mass, as stated before, and also respect the semigroup (associative) property, we can choose the transformation:

$$\boxed{\theta_i' \;=\; \left(\frac{\sum_{j=1}^{r} (\theta_j \kappa_j)^\beta}{\sum_{j=1}^{r} \kappa_j^\beta}\right)^{1/\beta}}$$

1 RG flow of the average degree

By the $\mathcal{S}^1$ model defined in [2], the density of nodes in this simple metric space is a power law in the expected degree of each node, κ: $\rho(\kappa) \sim \kappa^{-\gamma}$. This is a good model for scale-free (power-law distributed) networks, in the sense that it represents important features of the network with fidelity.

More precisely, this power law also considers that the network has a minimum κ_0 and a maximum κ_c value for the hidden degree:

$$\rho(\kappa) = \frac{1-\gamma}{\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma}} \kappa^{-\gamma} \quad , \quad \kappa \in [\kappa_0, \kappa_c]$$

After some derivations [2], this leads to a relation between the average degree of the renormalized network, $\langle k' \rangle$ and the average expected degree, $\langle \kappa' \rangle$:

$$\langle k' \rangle = C_0 \mu' \langle \kappa' \rangle^2$$

To get from the probability law to the average value of κ', the article follows these steps:

- Define another variable $z = \kappa^\beta$ and find its degree distribution
- Find the Laplace transform of the distribution of z, and multiply it by itself r times to get the transform of the convolution: $z' = \sum_r z$
- Compute $\langle \kappa' \rangle$ as the $1/\beta$-th moment of the distribution of z

Proceeding as described, we make:

Step 1:

$$\rho(\kappa)d\kappa = \frac{1-\gamma}{\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma}} \kappa^{-\gamma} d\kappa = \frac{1-\gamma}{\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma}} z^{-\gamma/\beta} d(z^{1/\beta}) = \frac{1-\gamma}{\beta(\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma})} z^{\frac{1-\gamma}{\beta}-1} dz$$

By defining $\eta = \frac{\gamma-1}{\beta} + 1$, we can write:

$$\rho_z(z)dz = \frac{1-\gamma}{\beta(\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma})} z^{-\eta} dz$$

Step 2:

If we have $\gamma < 2\beta + 1 \implies \eta < 3$. Applying the Laplace transform to the previous equation (remembering that κ is bounded by the constants κ_0 and κ_c):

$$\hat{\rho}_z(s) = \int_0^\infty \rho_z(z) e^{-sz} dz = \int_{\kappa_0^\beta}^{\kappa_c^\beta} \rho_z(z) e^{-sz} dz$$

By defining the upper (incomplete) Gamma function:

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$$

We can solve the Laplace transform in terms of it:

$$\hat{\rho}_z(s) = \frac{(1-\gamma)\left[\Gamma(1-\eta, s\kappa_0^\beta) - \Gamma(1-\eta, s\kappa_c^\beta)\right]}{\beta\left(\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma}\right)} s^{\eta-1} \tag{3}$$

And finally, defining the transformed variable $z' = \sum_r z = \kappa'^\beta$, we can see that $\hat{\rho}_z(s)^r$ is the Laplace transform of z':

$$\hat{\rho}_{z'}(s) = \left[\frac{(1-\gamma)\left[\Gamma(1-\eta, s\kappa_0^\beta) - \Gamma(1-\eta, s\kappa_c^\beta)\right]}{\beta\left(\kappa_c^{1-\gamma} - \kappa_0^{1-\gamma}\right)} s^{\eta-1}\right]^r \tag{4}$$

Step 3:

To get the average value of κ' in terms of the distribution of $z' = \kappa'^\beta$, we need to get its $1/\beta$-th moment. Remember that z' is a function of κ, which begins at zero, the integral also starts at zero:

$$\langle \kappa' \rangle = \langle z'^{1/\beta} \rangle = \int_0^\infty z'^{1/\beta} \rho_{z'}(z') dz'$$

After some substitutions, we can write this as:

$$\langle \kappa' \rangle = -\frac{1}{\Gamma\left(1 - \frac{1}{\beta}\right)} \int_0^\infty s^{-1/\beta} \hat{\rho}'_{z'}(s) ds$$

This integral can be very hard to compute exactly, so we can make an assumption on the form of the distribution for z': if we assume that $\rho_{\kappa'}(\kappa')$ is also a power law, the computation of the average hidden degree $\langle \kappa' \rangle$ is simpler.

With (a lot of) calculations, we reach two different distributions, depending on the value of η. We can also take the limit as r goes to infinity (that is, the cluster size grows to the full size of a large network):

- $1 < \eta < 2$:

$$\langle k' \rangle = C_0 \frac{\mu}{r} r^{\frac{2}{\gamma-1}} \langle \kappa \rangle = r^{\frac{2}{\gamma-1}-1} \langle k \rangle \longrightarrow \begin{cases} \infty, \text{ if } \gamma < 3 \\ \text{cte, if } \gamma = 3 \\ 0, \text{ if } \gamma > 3 \end{cases}$$

- $2 < \eta < 3$:

$$\langle k' \rangle = C_0 \frac{\mu}{r} r^{\frac{2}{\beta}} \langle \kappa \rangle^2 = r^{\frac{2}{\beta}-1} \langle k \rangle \longrightarrow \begin{cases} \infty, \text{ if } \beta < 2 \\ \text{cte, if } \beta = 2 \\ 0, \text{ if } \beta > 2 \end{cases}$$

Therefore, we can conclude that the network flows towards a fully connected graph if $\gamma < 3$ or $\beta < 2$. The lines $\gamma = 3$, $\beta > 2$ and $\beta = 2$, $\gamma > 3$ are unstable fixed points (transition between small-world and non small-world phases) and the average degree decays to zero for $\gamma > 3$ and $\beta > 2$ - fixed point is a 1-dimensional ring structure.

This is only valid for power law distributions of the renormalized variable, but this approximation gets better for $r \to \infty$ (asymptotic limit).

References

[1] García-Pérez, G., Boguñá, M., and Ángeles Serrano, M. Multiscale unfolding of real networks by geometric renormalization, Nature Physics **14**, 583 (2018).

[2] Krioukov D, Papadopoulos F, Kitsak M, Vahdat A, Boguñá M. *Hyperbolic geometry of complex networks.* Phys Rev E Stat Nonlin Soft Matter Phys. (2010).