# Propositional Logic

CS 5012

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# What is Logic?

- *A science that deals with the principles and criteria of the validity of inference. Any formal system can be considered a logic if it has*:
  - a well-defined **syntax**
  - well-defined **semantics** and
  - a well-defined **proof-theory**

# Syntax

- Syntax is concerned with the **symbols** and **rules** used in Propositional logic
  - without regard to meanings (that is *semantics*)
- **Logical constants**:
  - True, False *(describe the truth value of a proposition)*
- **Propositional symbols**:
  - P, Q, R, S, … *(atomic sentences)*
- Wrapping **parenthesis**: (...)
- And the **symbolic connectives** (also called *operators*):
  - $\land$, $\lor$, $\neg$, $\rightarrow$, $\equiv$ / $\leftrightarrow$
  - *and, or, not, implication/conditional, biconditional ($\equiv$ or $\leftrightarrow$)*

# Operator Order of Precedence

- Order of precedence: $\neg$ $\wedge$ / $\vee$ $\rightarrow$ $\leftrightarrow$

- In the same way that there is an order of precedence for mathematical operations: parenthesis, exponents, multiple or divide, then add or subtract
  there is an order of precedence for logical operators (shown above)
- Like mathematics, parentheses can be used to change the order of precedence
- There are many other symbols used in propositional logic

# Semantics

- Semantics of logic are concerned with **meanings** and **interpretations**

- How we interpret various rules of propositional logic come under this category
- What do we understand from the placement of various connectives between sentences (atomic propositions) is part of semantics

# A Sample of Semantics (1)

- An interpretation of a set of propositions is the assignment of a truth value, either **T (true)** or **F (false)** to each propositional symbol
- Negation:
  - The truth assignment of negation, ¬P, where P is any propositional symbol, is F if the assignment to P is T, and it is T if the assignment to P is F
- Conjunction:
  - The truth assignment of conjunction, ∧, is T only when both conjuncts have truth value T; otherwise it is F

# A Sample of Semantics (2)

- Disjunction:
  - The truth assignment of disjunction, $\vee$, is F only when both disjuncts have truth value F; otherwise it is T
- Implication:
  - The truth assignment of implication, $\rightarrow$, is F only when the premise or symbol before the implication is T and the truth value of the consequent (or symbol after the implication) is F; otherwise it is T

# A Sample of Semantics (3)

- Equivalence:
  - The truth assignment of equivalence, ≡, is T only when both expressions have the same truth assignment for all possible interpretations; otherwise it is F

# Proof-Theory

- The **rules, regulations and procedures** that help in determining the validity of propositions form part of the proof theory

- This course will examine a couple rules to help you determine whether or not a propositional statement is true or false. This helps in determining the validity of arguments

# Why Study Logic? (1)

- Advanced Communication identifies Homo Sapiens
- Communication: Sounds, symbols or images **logically connected**
- We start developing logical concepts from childhood, yet ordinary communication is not always **precise** or **logically valid**

# Why Study Logic? (2)

- Logic improves the **quality** of our arguments
- Helps us evaluate others' reasoning
- Helps us analyze others' beliefs
- Improves our ability to spot **fallacies**
- May help in our relationships with better communication and conflict resolution
- Greatly enhances our skill for thinking clearly

# Why Study Logic? (3)

- The concepts of the **general purpose computer** and the **Turing Machine** started during research in logic

- Computer languages contain **logical symbolism**

- Logic helps in **analyzing computer programs**

- Artificial intelligence, robotics, circuit design heavily depend on logic

# Why Study Logic? (4)

- Many data query languages such as **SQL** heavily depend of logic

- If we want to determine that a **program** will *do what it is supposed to do*, we resort to logic
  - *Basis for testing!*

- Computers process information, so does logic
  A smart computer needs smart logic

# Propositional Logic

- Since **logic** is a big subject, many sub-disciplines have developed over the years
- Each branch of logic has its own symbols and its own rules. Many symbols and rules are common for various areas
- Today, we will start the study of **Propositional Logic**

# Some useful Terms (1)

- A *valid sentence* or **tautology** is a sentence that is **True under all interpretations**, no matter what the world is actually like or how the semantics are defined. Example: **“It’s raining or it’s not raining”**

- An *inconsistent sentence* or **contradiction** is a sentence that is **False under all interpretations**. The world is never like what it describes, as in **“It’s raining and it’s not raining”**

# Some useful Terms (2)

- Two statements ate **logically equivalent** when they **have exactly the same logical content**. A statement (say P) and its *double negation* ¬ (¬P) is a simple example of logical equivalence
- **Valid and Meaningful**: An argument does not have to be meaningful to be valid

“**Dolphins live underground** OR **Dolphins live in the trees.** Since Dolphins don’t live underground, **Dolphins live in the trees**”

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A perfectly valid statement – not very *meaningful*

# Some useful Terms (3)

- **Logical Equivalence**: Two compound propositions P and Q are logically equivalent if the columns in a **truth table** giving their truth values agree
  - This is written as $P \equiv Q$

- **Implication Law**: $P \rightarrow Q$ and $\neg P \vee Q$ are logically equivalent
  - (using **truth tables**, we can prove the implication law)

# What is a Proposition?

- A **proposition** (*statement*) is a sentence which is **either true or false**
  - The moon is made of milk chocolate
  - Grass is green
  - CS5012 class has 48 students
  - 2+1 = 5
- If a proposition is true, we say its **truth value** is **T** (*true*). If it is false we say its truth value is **F** (*false*)

- A sentence which cannot be classified as true or false is **not a proposition**
- Typical examples are statements that are questions, instructions, exclamations, remarks, greetings or statements with variables
  - Is it windy?
  - Don't leave the room
  - Great!
  - Our team played well
  - Hello!
  - A+B = 6
  - The car is beautiful

# Propositional Symbols/Variables

- For showing key ideas and definitions, and for *simplification*
- User defines a set of propositional symbols/variables, like P and Q
- User defines the ***semantics*** of each propositional symbol:
  - S means "He is a student"
  - T means "He is a teacher"
  - X means "It's cold in Richmond today"
- A sentence (***well formed formula***) is defined as follows:
  - A symbol is a sentence
  - If S is a sentence, then ¬S is a sentence
  - If S is a sentence, then (S) is a sentence
  - If S and T are sentences, then (S ∨ T), (S ∧ T), (S → T), and (S ↔ T) are sentences

# Argument

- An argument consists of a sequence of statements called **premises** and a statement called a **conclusion**
- Examples
  - He is sick OR he is lazy. He is NOT sick so he is lazy
  - My computer has a grasshopper inside OR my program has a bug. My computer does NOT have a grasshopper inside. Therefore my program has a bug 😊

Let P be: He is sick
Let Q be: He is lazy
- P OR Q
- NOT P
- Therefore Q

# Some English words for Logic

- Typically any of the following words is a **premise indicator**:
    - » Since, because, for, as

- Any of the following words signifies a **conclusion**:
    - » Therefore, hence, thus, so , consequently

- NOTE: The word "**but**" is sometimes used as a substitute for "**and**"
- **"He is smart but he is not arrogant"** is the same thing as saying **"He is smart and he is not arrogant"**

# Inclusive OR vs excusive OR (1)

- The word “or” is used loosely in everyday English language conversation

1. **John will enroll in CS 5012 or he will enroll in CS 2110** (maybe he will enroll in both courses)
2. **For dinner, Steve will eat a 16 Oz steak or a double Whopper** (hopefully not both!)
3. **John will fly from Richmond to London in the early morning or late afternoon** (definitely not at both times)

- 1 is called **inclusive** OR,
  2 and 3 are examples of **exclusive** OR

# Inclusive OR vs excusive OR (2)

- As it can be seen from the previous slide, the connective "OR" can create different meanings and interpretations in the English language. This can be **ambiguous** and confusing
- We accept it: logic does not and computers don't. Science, logic and technology need precision.
- In logic, "Inclusive Or" and "Exclusive Or" are two separate connectives with symbols being $\vee$ and $\veebar$ (also $\oplus$) respectively

# Atomic and Complex statements

- An **atomic proposition** is a statement which cannot be broken down into smaller statements, also simply called an "atom" (**rose is red**)

- A **compound proposition** is constructed by joining atomic statements (**rose is red AND grass is green**)

- *Logical connectives (logical operators) can be used to make compound propositions from atomic propositions*

# Truth Values and Truth Tables

- Before we start discussing compound statements, it is important to know about **truth tables**
- Every proposition is true or false. When the statement is true, we say its truth value is ‘T’ (True)

- A truth table gives possible values (being true or false) for atomic statements and the consequent truth values of compound statements

# Negation

- Truth Table of a simple statement P is:

| P |
|---|
| T |
| F |

- The next simplest truth table is for the negation of P (¬P)

| P | ¬P |
|---|---|
| T | F |
| F | T |

# Conjunction

- P = Sara is rich
- Q = Sara is smart
- The conjunction of propositions P and Q is denoted by $P \wedge Q$ (P AND Q) and has this truth table:

| P | Q | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Disjunction

- P = Sara is rich
- Q = Sara is smart
- The disjunction of propositions P and Q is denoted by P ∨ Q (P OR Q) and has this truth table:

| P | Q | P ∨ Q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Implication

- P = It's wet outside, Q = It's raining
- If P and Q are propositions then P → Q is a conditional statement (implication) which means if P then Q and has the following truth table:

| P | Q | P → Q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

- "if it's wet outside then it's raining"

The compound proposition P → Q (If it's wet then it's raining) will be **false** only if we know *that it's wet outside but it isn't raining*

# Implication Reminder

- There doesn't need to be any connection between P and Q
- Actual meanings of the statements are not always relevant in this situation
  - We just look at the truth values to establish if the statement is true or false
- Implication/conditional statements DO NOT imply **causation** (i.e. that P causes Q)!
  - It *may* happen to be true under certain situations
  - However, it is not a requirement for it to be true
- Think of it as a *promise*
  - *"If I become president, then I will reduce taxes"*

# Implication: *Further clarifications*

*“If I become president, then I will lower taxes”*

| P | Q | P → Q |
|---|---|---|
| T | T | T |
| T | F | F |
| **F** | T | T |
| **F** | F | T |

If I become president (T), Lower taxes (T) □ T

If I become president (T), Don’t lower taxes (F) □ F (“broke the promise”)

For the last two rows, since proposition P isn’t true (“become president”) then it doesn’t matter whether or not taxes were lowered, so both of them □ T

Remember: “**IF** P, then Q”

*(If P is **not true** then no matter what Q is, the implication P→Q will always be **true**!)*

***Implication / conditional statements DO NOT imply causation,*** *because if it did, then proposition Q would have no choice but to be true (T) if proposition P was true (T) – but this is NOT the case!*

# Biconditional

- The biconditional of statements P and Q ( $P \leftrightarrow Q$) read as "**P if and only if Q**" is true only if both statements have that same truth values (P and Q are both true or both false). If one statement is false and the other is true then the compound statement is false
- The two situations $P \leftrightarrow Q$ will be true is either "Sara is smart & rich" or "Sara is not smart & not rich"
- If Sara is smart but not rich then $P \leftrightarrow Q$ is false
- If Sara is not smart but she is rich then $P \leftrightarrow Q$ is also false

# Truth Table for Bidirectional

| P | Q | $P \leftrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Sound Rules of Inference

- Here are some examples of sound rules of inference
  - *A rule is sound if its conclusion is true whenever the premise is true*
- Each can be shown to be sound using a truth table

| RULE | PREMISE | CONCLUSION |
| --- | --- | --- |
| Modus Ponens | $A, A \rightarrow B$ | $B$ |
| And Introduction | $A, B$ | $A \wedge B$ |
| OR Elimination | $A \vee B \rightarrow C$ | $A \rightarrow C , B \rightarrow C$ |
| Double Negation | $\neg\neg A$ | $A$ |
| Unit Resolution | $A \vee B, \neg B$ | $A$ |
| Resolution | $A \vee B, \neg B \vee C$ | $A \vee C$ |

# De Morgan’s Laws (1)

- **Augustus De Morgan** (27 June 1806 – 18 March 1871) was a British mathematician and Logician
- In Propositional logic **De Morgan's laws** are transformation rules that are both valid rules of inference
- These rules show that two sets of equivalent propositions can be created by
  - exchanging the connectives of conjunction and disjunction and
  - relocating parentheses and negation signs

# De Morgan’s Laws (2)

- **Rule # 1** (Changed from conjunction to disjunction)

¬(P Λ Q) is logically equivalent to ¬P ∨ ¬Q

- **Rule # 2** (Changed from disjunction to conjunction) ¬(P ∨ Q) is logically equivalent to ¬P Λ ¬Q

# Converse, Contrapositive, and Inverse

- From $P \rightarrow Q$ we can form new conditional statements

- **converse** of $P \rightarrow Q$ is $Q \rightarrow P$
- **contrapositive** of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$
- **inverse** of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$