Module 0 - Material Notes

Propositional Logic

Propositional logic, also known as propositional calculus or Boolean logic, is a branch of logic that studies the ways statements can interact with each other and the manipulations that can be performed on them. It primarily deals with propositions (statements that can be true or false) and logical connectives that combine them.

Here are some key terms in propositional logic:

1. **Proposition**: A proposition, or statement, is a declarative sentence that is either true or false, but not both. For example, "It is raining" is a proposition because it can be either true or false.
2. **Truth Value**: The truth value of a proposition is 'true' (often represented by 'T' or '1') if it is a true statement, and 'false' (often represented by 'F' or '0') if it is a false statement.
3. **Logical Connectives**: Logical connectives, also known as logical operators, are symbols or words used to connect two or more propositions. The most common logical connectives are:
   - **Conjunction** ('and', represented as '∧'): A conjunction of two propositions is true if and only if both propositions are true.
   - **Disjunction** ('or', represented as '∨'): A disjunction of two propositions is true if at least one of the propositions is true.
   - **Negation** ('not', represented as '¬' or '~'): The negation of a proposition is true if and only if the proposition is false.
   - **Conditional** ('if...then', represented as '→'): A conditional of two propositions is false if the first proposition is true and the second proposition is false; otherwise, it is true.
   - **Biconditional** ('if and only if', represented as '↔'): A biconditional of two propositions is true if both propositions have the same truth value.
4. **Truth Table**: A truth table is a mathematical table used to determine the truth value of a proposition or a logical expression, based on all possible truth values of its elementary propositions. Each row of the truth table represents a possible assignment of truth values to the elementary propositions, and the corresponding truth value of the proposition or logical expression.
5. **Tautology**: A tautology is a proposition or logical expression that is always true, regardless of the truth values of its elementary propositions.
6. **Contradiction**: A contradiction is a proposition or logical expression that is always false, regardless of the truth values of its elementary propositions.
7. **Contingency**: A contingency is a proposition or logical expression that can be either true or false, depending on the truth values of its elementary propositions.

Rules of Inference

Rules of inference are logical rules that are applied to propositions to derive conclusions. They are the fundamental principles of deductive reasoning, allowing us to infer new statements from given statements. Each rule of inference is justified by the principle of truth preservation: if the given statements are true, then the inferred statement is also true.

Here are the most common rules of inference:

1. **Modus Ponens (MP)**: If 'P implies Q' is true and 'P' is true, then 'Q' must be true. For example, if it is true that 'if it is raining then the ground is wet' and 'it is raining' is true, then it must be true that 'the ground is wet'.
2. **Modus Tollens (MT)**: If 'P implies Q' is true and 'not Q' is true, then 'not P' must be true. For example, if it is true that 'if it is raining then the ground is wet' and 'the ground is not wet' is true, then it must be true that 'it is not raining'.
3. **Hypothetical Syllogism (HS)**: If 'P implies Q' and 'Q implies R' are both true, then 'P implies R' must be true. For example, if it's true that 'if it rains, then the ground gets wet' and 'if the ground gets wet, then grass grows', then it must be true that 'if it rains, then grass grows'.
4. **Disjunctive Syllogism (DS)**: If 'P or Q' is true and 'not P' is true, then 'Q' must be true. For example, if 'it is raining or it is sunny' is true and 'it is not raining' is true, then 'it is sunny' must be true.
5. **Conjunction (CON)**: If 'P' is true and 'Q' is true, then 'P and Q' must be true.
6. **Simplification (SIM)**: If 'P and Q' is true, then 'P' is true, and 'Q' is true.
7. **Addition (ADD)**: If 'P' is true, then 'P or Q' is true, regardless of the truth value of 'Q'.

Predicate Logic

Predicate logic, also known as first-order logic or quantificational logic, extends propositional logic to include elements such as variables, quantifiers, and predicates, which allows for more complex statements and relationships to be expressed.

Here are some key terms in predicate logic:

1. **Predicate**: A predicate is a function that takes an object or objects in a particular domain of discourse and assigns it a truth value. For example, in the statement "x is a cat," "is a cat" is the predicate and "x" is the object.
2. **Quantifiers**: Quantifiers express the quantity of instances for which the predicate is true. There are two main types of quantifiers in predicate logic:

- **Universal Quantifier** ('for all', represented as '∀'): The universal quantifier states that a certain property holds for all members of a specific set. For instance, "For all x, x is a cat" means that every object in the domain is a cat.
- **Existential Quantifier** ('there exists', represented as '∃'): The existential quantifier states that there is at least one member of a specific set for which a certain property holds. For example, "There exists an x such that x is a cat" means that there is at least one object in the domain that is a cat.

3. **Variables**: Variables are symbols that can represent any object within a specific domain of discourse. In the statement "For all x, x is a cat," "x" is a variable that represents any object in the domain.
4. **Domain of Discourse**: The domain of discourse, also known as the universe of discourse, is the set of all possible objects that the variables in a logical statement may represent.
5. **Arguments and Parameters**: In a predicate, the objects that it is evaluating are called arguments or parameters. For instance, in the predicate P(x), x is the argument or parameter.
6. **Logical Connectives**: Just like in propositional logic, logical connectives (and, or, not, if...then, if and only if) are used in predicate logic to combine predicates and form more complex expressions.
7. **Truth Value**: The truth value of a statement in predicate logic is determined by the truth values of its predicates and the truth values assigned to its variables by a particular interpretation.

Set Theory

Set Theory is the mathematical theory of well-determined collections, called sets, of objects that are called members, or elements, of the set.

Here are some key terms and concepts in Set Theory:

1. **Set**: A set is a collection of distinct objects, considered as an object in its own right. Sets are usually denoted by uppercase letters, and the elements are listed in lowercase letters. For example, the set A = {a, b, c} has the elements a, b, and c.
2. **Element**: An element or member of a set is any one of the distinct objects that make up that set.
3. **Subset**: A set A is a subset of a set B (denoted as A ⊆ B) if every element of A is also an element of B.
4. **Universal Set**: The universal set, usually denoted by the symbol ξ, is the set that contains all objects or elements under consideration for a particular discussion or problem.
5. **Empty Set or Null Set**: The empty set, also known as the null set, is the unique set that contains no elements. It is usually denoted by the symbol ∅.
6. **Intersection**: The intersection of two sets A and B, denoted by A ∩ B, is the set of elements that are in both A and B.
7. **Union**: The union of two sets A and B, denoted by A ∪ B, is the set of elements that are in A, or in B, or in both.
8. **Difference**: The difference of two sets A and B, denoted by A - B, is the set of elements that are in A but not in B.
9. **Complement**: The complement of a set A, denoted by A', is the set of all elements in the universal set that are not in A.
10. **Venn Diagrams**: A Venn diagram is a diagrammatic representation of sets where each set is represented by a circle, usually inside a rectangle representing the universal set. Overlapping areas represent intersection between sets.
11. **Cardinality**: The cardinality of a set is a measure of the "number of elements in the set". It is often denoted with the symbol |A| to denote the cardinality of a set A.

Relations

In mathematics, a relation is a set of ordered pairs of elements. Essentially, it is a way of associating elements from one set to another.

Here are some key terms and concepts in the study of Relations:

1. **Relation**: A relation R from a set A to a set B is a set of ordered pairs (a, b), where a is a member of A, and b is a member of B. Relations are often represented as a matrix or as a graph.
2. **Domain**: The set of all first elements (a) of the ordered pairs in a relation is called the domain.
3. **Range**: The set of all second elements (b) of the ordered pairs in a relation is called the range.
4. **Inverse Relation**: If a relation R consists of ordered pairs (a, b), the inverse relation R^-1 consists of the ordered pairs (b, a).
5. **Reflexive Relation**: A relation R on a set A is called reflexive if every element in A is related to itself. That is, for all a in A, the pair (a, a) is in R.
6. **Symmetric Relation**: A relation R on a set A is called symmetric if for all (a, b) in R, the pair (b, a) is also in R.
7. **Transitive Relation**: A relation R on a set A is called transitive if whenever (a, b) and (b, c) are in R, then (a, c) is also in R.
8. **Equivalence Relation**: A relation R is an equivalence relation if it is reflexive, symmetric, and transitive.
9. **Partial Order**: A relation R is a partial order if it is reflexive, antisymmetric (if (a, b) and (b, a) are in R, then a = b), and transitive.

Functions

In mathematics, a function is a special type of relation that uniquely associates members of one set with members of another set. More specifically, a function from A to B is an object f such that every a in A is uniquely associated with an object f(a) in B.

Here are some key terms and concepts in the study of Functions:

1. **Function**: A function f from a set A (the domain) to a set B (the codomain) is a rule that assigns to each element a in A exactly one element b in B. The element b is called the image of a under f, or the value of f at a, and is denoted as f(a).
2. **Domain**: The domain of a function is the set of all possible inputs for the function, usually denoted as x.
3. **Range**: The range of a function is the set of all possible outputs, usually denoted as y.
4. **Function Notation**: Function notation involves writing the function name followed by the input value in parentheses. For example, if f is a function, then the output value for the input x is denoted by f(x).
5. **One-to-One Function (Injective)**: A function f is one-to-one (or injective) if and only if different inputs yield different outputs. That is, if a ≠ b, then f(a) ≠ f(b).
6. **Onto Function (Surjective)**: A function f is onto (or surjective) if and only if for every element in the codomain B, there is at least one element in the domain A such that f(a) = b.
7. **One-to-One Correspondence (Bijective)**: A function f is a one-to-one correspondence, or bijective, if it is both one-to-one (injective) and onto (surjective). This means that every element of the domain is associated with exactly one element of the codomain, and vice versa.
8. **Inverse Function**: The inverse function of a bijective function f, denoted by f^-1, is the function that assigns to an element b in B the unique element a in A such that f(a) = b.
9. **Composition of Functions**: If f and g are functions, the composition of f and g, denoted by f ∘ g, is the function defined by (f ∘ g)(x) = f(g(x)).