Stat 6021: Review of Matrices

1 Matrix Notation

Consider two $m \times n$ (where m denotes the number of rows and n denotes number of columns) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Sometimes a matrix could be written as $\boldsymbol{A}_{m\times n}$ with the dimensions in the subscript. The entries in matrix are denoted by $a_{i,j}$ for the (i,j)th entry in matrix $\boldsymbol{A}$.

Example: Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 7 & 3 \end{bmatrix}.$$

$\boldsymbol{A}$ is a 2×3 matrix. The value of the $(2,1)$th entry is $a_{2,1} = 6$.

2 Matrix Addition

Two matrices can only be added together if and only if they have the same dimension. The (i,j)th entries are added together.

Example: Consider the following matrices

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 7 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 5 & 11 & 8 \\ 9 & 4 & 2 \end{bmatrix}.$$

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be added since they have the same dimension, and their addition becomes

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 4+5 & 2+11 & 1+8 \\ 6+9 & 7+4 & 3+2 \end{bmatrix} = \begin{bmatrix} 9 & 13 & 9 \\ 15 & 11 & 5 \end{bmatrix}.$$

Some properties with matrix addition:

- Matrix addition is commutative, i.e., $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.
- Matrix addition is associative, i.e., $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$.

3 Matrix Multiplication

Consider the matrices $\boldsymbol{A}_{m\times n}$ and $\boldsymbol{E}_{n\times q}$. The product of two matrices exists only if the number of columns of the first matrix is equal to the number of rows of the second matrix. The product is a $m \times q$ matrix whose (i,j)th entry is the inner product between the ith row of $\boldsymbol{A}$ and the jth column of $\boldsymbol{E}$.

Example: Consider the following matrices

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 7 & 3 \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

The product $\boldsymbol{AE}$ exists since $\boldsymbol{A}$ has 3 columns and $\boldsymbol{E}$ has 3 rows, and $\boldsymbol{AE}$ will be a 2×1 matrix . The product $\boldsymbol{EA}$ does NOT exist since $\boldsymbol{E}$ has 1 column and $\boldsymbol{A}$ has 2 rows. The product $\boldsymbol{AE}$ is written as

$$\boldsymbol{AE} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 33 \end{bmatrix}$$

- The $(1,1)$th entry in $\boldsymbol{AE}$ is found by taking the inner product between row 1 of $\boldsymbol{A}$ and column 1 of $\boldsymbol{E}$, i.e. $4 \times 5 + 2 \times 0 + 1 \times 1 = 21$.
- The $(2,1)$th entry in $\boldsymbol{AE}$ is found by taking the inner product between row 2 of $\boldsymbol{A}$ and column 1 of $\boldsymbol{E}$, i.e. $6 \times 5 + 7 \times 0 + 3 \times 1 = 33$.

Practice question: Consider the following matrices

$$\boldsymbol{B} = \begin{bmatrix} -2 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}.$$

1. Does $\boldsymbol{BC}$ exist? If so, what is $\boldsymbol{BC}$?
2. Does $\boldsymbol{CB}$ exist? If so, what is $\boldsymbol{CB}$?

Some properties with matrix multiplication:

- Matrix multiplication is associative, i.e., $\boldsymbol{F}(\boldsymbol{GH}) = (\boldsymbol{FG})\boldsymbol{H}$.
- Matrix multiplication is distributive, i.e., $(\boldsymbol{F} + \boldsymbol{G})\boldsymbol{H} = \boldsymbol{FH} + \boldsymbol{FH}$.
- Matrix multiplication is NOT commutative, i.e., $\boldsymbol{FG} \neq \boldsymbol{GH}$. The order matters.

4 Transpose

The transpose of a matrix $\boldsymbol{A}_{m\times n}$ is a matrix whose rows are the columns of $\boldsymbol{A}$. The transpose is written as $\boldsymbol{A'}$. The (j,i)th entry in $\boldsymbol{A'}$ is the (i,j)th entry in $\boldsymbol{A}$. Using the same matrix $\boldsymbol{A}$ defined earlier,

$$\boldsymbol{A'} = \begin{bmatrix} 4 & 6 \\ 2 & 7 \\ 1 & 3 \end{bmatrix}$$

A few properties with the transpose:

- $(\boldsymbol{A'})' = \boldsymbol{A}$.
- $(\boldsymbol{A}+\boldsymbol{B})' = \boldsymbol{A'}+\boldsymbol{B'}$.
- $(\boldsymbol{AE})' = \boldsymbol{E'A'}$.
- $(\boldsymbol{FGH})' = \boldsymbol{H'G'F'}$.

5 Inverse

Some additional terminology:

- A matrix $\boldsymbol{K}$ is square if the dimensions of its column and row are the same.
- The identity matrix $\boldsymbol{I}$ is a square matrix where the diagonal entries are equal to 1, and the other entries are all equal to 0.

The inverse of $\boldsymbol{K}$, if it exists, is the matrix $\boldsymbol{K^{-1}}$ such that $\boldsymbol{KK^{-1}} = I$. We say that $\boldsymbol{K}$ is invertible if its inverse exists.

Example: Consider the following matrix

$$\boldsymbol{D} = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix},$$

Its inverse, $\boldsymbol{D^{-1}}$, is

$$\boldsymbol{D} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix},$$

Verify that $\boldsymbol{DD^{-1}}$ is an identity matrix.
A few properties with the inverse:

- $(\boldsymbol{AB})^{-1} = \boldsymbol{B^{-1}A^{-1}}$.
- $(\boldsymbol{A^{-1}})^{-1} = \boldsymbol{A}$.
- $(\boldsymbol{A'})^{-1} = (\boldsymbol{A^{-1}})'$.

6 Common Matrices in MLR

The vector of response variables for the observations is denoted as

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The **design matrix** is denoted as

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & & & \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix}$$

The vector of coefficients is denoted as

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

The vector of error terms is denoted as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

In matrix form, the MLR model is written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & & & \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$