CHAPTER NINE

Spatial Cartesian Kinematics

The same approach developed for planar kinematic analysis in Chapter 3 is employed in this chapter for the kinematic analysis of spatial systems. Some extension to concepts of vector analysis in three-dimensional space is required, but the same basic algebraic vector approach and notation remain valid. The principal extension from planar to spatial kinematics is the complexity of defining the orientation of a body in space. Coordinate transformations are defined, as in the plane, and concepts of angular velocity and virtual rotation are introduced as vector quantities that are required for kinematic and dynamic analysis. A fundamental difference between planar and spatial kinematic analysis is the use of angular velocity and acceleration variables in spatial analysis that are not time derivatives of generalized coordinates. Euler parameter orientation generalized coordinates are defined, and their properties are developed for use in kinematic and dynamic analysis. A library of kinematic and driving constraints is derived, and their variations, in terms of both virtual rotation and Euler parameter variations, are calculated. Finally, velocity and acceleration equations are derived.

The reader is cautioned that the analytical sophistication of spatial kinematics, particularly in defining orientation of bodies in space, is more demanding than corresponding concepts in planar system kinematics. It is, nevertheless, critically important that the geometric aspects of the spatial kinematic system definition be clearly understood, in order for the engineer to effectively model and analyze the spatial kinematics of mechanical systems. This chapter has been written presuming that the reader has a clear physical understanding of the concepts of planar kinematics of mechanical systems that are developed in Part One. Even though substantial algebraic and analytical intricacy arises in the kinematics of spatial systems, geometric concepts of kinematic constraints, mechanical system modeling, and kinematic analysis are very similar to corresponding concepts in planar kinematics. The reader is, therefore, encouraged to draw parallels and analogies between spatial kinematics and kinematics of planar systems in order to exploit an intuitive mastery of planar kinematics. Adherence to physically based concepts will assist the reader in maintaining a degree of sanity while making his or her way through the following sections, which involve seemingly endless analytical and algebraic manipulations.

9.1 VECTORS IN SPACE

Vectors in space have many of the same properties as in the plane, but they have some additional properties that were not encountered in Chapter 2. The principal focus of this section is on the properties of vectors in space, building on the basic properties of vectors in the plane that were developed in Chapter 2.

Geometric Vectors The concept of a vector in space may be introduced in a geometric setting, with no requirement for identification of a reference frame. In this setting, a *geometric vector,* or simply a vector, is defined as the directed line segment from one point in space to another point in space. Vector $\vec{a}$ in Fig. 9.1.1, beginning at point A and ending at point B, is denoted by the $\vec{\ }$ notation in its geometrical sense. The *magnitude of a vector* $\vec{a}$ is its length (the distance between A and B) and is denoted by a or $|\vec{a}|$. Note that the length of a vector is positive if points A and B do not coincide and is zero only when they coincide. A vector with zero length is denoted as $\vec{0}$ and is called the *zero vector.*

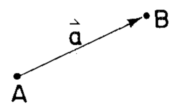

Figure 9.1.1 Vector from point A to point B.

Multiplication of a vector $\vec{a}$ by a nonnegative scalar $\alpha \geqslant 0$ is defined as a vector $\alpha\vec{a}$ in the same direction as $\vec{a}$, but having magnitude αa. *A unit vector,* having a length of 1 unit, in the direction $\vec{a} \neq \vec{0}$ is $(1/a)\vec{a}$. Multiplication of a vector $\vec{a}$ by a negative scalar $\beta < 0$ is defined as the vector with magnitude $|\beta|\, a$ and direction opposite to that of $\vec{a}$. The *negative of a vector* is obtained by multiplying the vector by -1. It is the vector with the same magnitude but opposite direction. These definitions are identical to those for the planar case.

Example 9.1.1: Let points A and B in Fig. 9.1.1 be located in an orthogonal x-y-z reference frame, as shown in Fig. 9.1.2. The distance between

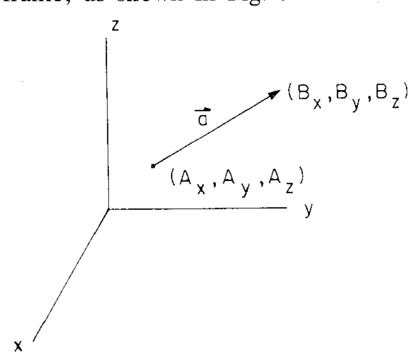

Figure 9.1.2 Vector located in orthogonal reference frame.

points A and B, with coordinates (A_x, A_y, A_z) and (B_x, B_y, B_z), respectively, is the length of $\vec{a}$; that is,

$$|\vec{a}| = [(B_x - A_x)^2 + (B_y - A_y)^2 + (B_z - A_z)^2]^{1/2}$$

Two vectors $\vec{a}$ and $\vec{b}$ are added according to the *parallelogram rule,* as shown in Fig. 9.1.3. The parallelogram used in this construction is formed in the plane that contains the intersecting vectors $\vec{a}$ and $\vec{b}$. The *vector sum* is written as

$$\vec{c} = \vec{a} + \vec{b} \tag{9.1.1}$$

Addition of vectors and multiplication of vectors by scalars obey the following rules [21]:

$$\begin{aligned} \vec{a} + \vec{b} &= \vec{b} + \vec{a} \\ (\alpha + \beta)\vec{a} &= \alpha\vec{a} + \beta\vec{a} \end{aligned} \tag{9.1.2}$$

where α and β are scalars.

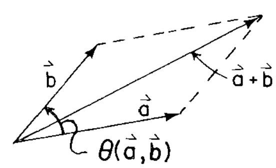

Figure 9.1.3 Addition of vectors.

Example 9.1.2: Using the definition of vector addition, the plane formed by intersecting vectors $\vec{a}$ and $\vec{b}$ in Fig. 9.1.3 may be called the x'-y' plane, with the x' axis along vector $\vec{a}$, as shown in Fig. 9.1.4. Plane trigonometry can now be used to calculate the length of vector $\vec{a} + \vec{b}$ and the angle $\theta(\vec{a}, \vec{a} + \vec{b})$ between $\vec{a}$ and $\vec{a} + \vec{b}$:

$$|\vec{a} + \vec{b}| = [(a + b \cos \theta(\vec{a}, \vec{b}))^2 + (b \sin \theta(\vec{a}, \vec{b}))^2]^{1/2}$$

$$\theta(\vec{a}, \vec{a} + \vec{b}) = \text{Arctan}\left[\frac{b \sin \theta(\vec{a}, \vec{b})}{a + b \cos \theta(\vec{a}, \vec{b})}\right]$$

These calculations characterize the vector $\vec{a} + \vec{b}$, but they are not convenient.

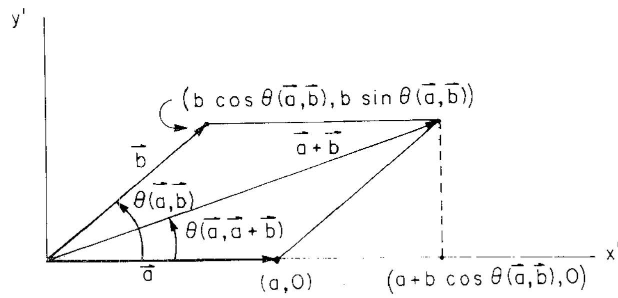

Figure 9.1.4 Sum of vectors in plane of intersection.

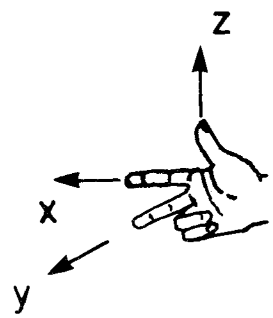

Figure 9.1.5 Right-hand orthogonal reference frame.

Orthogonal reference frames are used extensively in representing vectors. Use in this text is limited to *right-handed* x-y-z orthogonal reference frames, that is, with mutually orthogonal x, y, and z axes that are ordered by the finger structure of the right hand, as shown in Fig. 9.1.5. Such a frame is called a *Cartesian reference frame.*

A vector $\vec{a}$ can be resolved into components a_x, a_y, and a_z along the x, y, and z axes of a Cartesian reference frame, as shown in Fig. 9.1.6. These components are called the *Cartesian components of the vector.* The *unit coordinate vectors* $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors that are directed along the x, y, and z axes, respectively, as shown in Fig. 9.1.6. In vector notation,

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \tag{9.1.3}$$

Denote the angle from vector $\vec{a}$ to vector $\vec{b}$ in the plane that contains them by $\theta(\vec{a}, \vec{b})$, with counterclockwise as positive about the normal to the plane of the vectors that points toward the viewer, as shown in Fig. 9.1.3. In terms of angles between the vector $\vec{a}$ and the positive x, y, and z axes, the *components of a vector* $\vec{a}$ are

$$\begin{aligned} a_x &= a \cos \theta(\vec{i}, \vec{a}) \\ a_y &= a \cos \theta(\vec{j}, \vec{a}) \\ a_z &= a \cos \theta(\vec{k}, \vec{a}) \end{aligned} \tag{9.1.4}$$

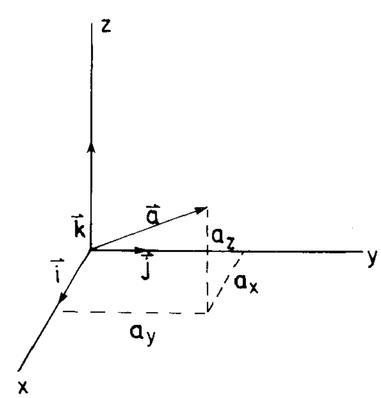

Figure 9.1.6 Components of a vector.

The quantities $\cos\theta(\vec{i}, \vec{a})$, $\cos\theta(\vec{j}, \vec{a})$, and $\cos\theta(\vec{k}, \vec{a})$ are called the *direction cosines of vector* $\vec{a}$. Note that if the viewer had been on the back side of the plane of Fig. 9.1.3 the counterclockwise angle from $\vec{a}$ and $\vec{b}$ would be $2\pi - \theta(\vec{a}, \vec{b})$. However, $\cos(2\pi - \theta) = \cos\theta$, so the viewpoint does not influence the direction cosines.

Addition of vectors $\vec{a}$ and $\vec{b}$ may be expressed in terms of their components, using Eq. 9.1.2, as

$$\begin{aligned}\vec{c} = \vec{a} + \vec{b} &= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} + (a_z + b_z)\vec{k} \\ &\equiv c_x\vec{i} + c_y\vec{j} + c_z\vec{k}\end{aligned} \qquad \textbf{(9.1.5)}$$

where c_x, c_y, and c_z are the Cartesian components of vector $\vec{c}$. Thus, addition of vectors occurs component by component. Using this idea, three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ may be added to show (Prob. 9.1.1) that

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \textbf{(9.1.6)}$$

The *scalar product* (sometimes called the *dot product*) of two vectors $\vec{a}$ and $\vec{b}$ is defined as the product of their magnitudes and the cosine of the angle between them; that is,

$$\vec{a} \cdot \vec{b} = ab\cos\theta(\vec{a}, \vec{b}) \qquad \textbf{(9.1.7)}$$

This definition is purely geometric, so it is independent of the reference frame in which the vectors are represented.

Note that if two vectors $\vec{a}$ and $\vec{b}$ are nonzero (i.e., $a \neq 0$ and $b \neq 0$) then their scalar product is zero if and only if $\cos\theta(\vec{a}, \vec{b}) = 0$. Two nonzero vectors are said to be *orthogonal vectors* if their scalar product is zero.

Since $\theta(\vec{b}, \vec{a}) = 2\pi - \theta(\vec{a}, \vec{b})$ and $\cos(2\pi - \theta) = \cos\theta$, the order of terms appearing on the right side of Eq. 9.1.7 is immaterial. Thus,

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \textbf{(9.1.8)}$$

The scalar product of two vectors may be interpreted as the product of the magnitude of one of the vectors times the projection of the other vector onto that vector. To see this, refer to Fig. 9.1.4, where the projection of vector $\vec{b}$ onto $\vec{a}$ has length $b\cos(\vec{a}, \vec{b})$.

Based on the definition of the scalar product, the following identities hold for the unit coordinate vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$:

$$\begin{aligned}\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \\ \vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1\end{aligned} \qquad \textbf{(9.1.9)}$$

For any vector $\vec{a}$,

$$\vec{a} \cdot \vec{a} = aa\cos 0 = a^2$$

While not obvious on geometrical grounds, the scalar product satisfies the relation [21]

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} \qquad \textbf{(9.1.10)}$$

Using Eq. 9.1.10 and the identities of Eq. 9.1.9, a direct calculation (Prob. 9.1.2) yields

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \tag{9.1.11}$$

Note that the concepts and properties of vectors in space discussed thus far are elementary extensions of ideas from vectors in the plane. A new concept for spatial vectors is the *vector product* (sometimes called the *cross product*) of two vectors $\vec{a}$ and $\vec{b}$, defined as the vector

$$\vec{a} \times \vec{b} = ab \sin \theta(\vec{a}, \vec{b})\vec{u} \tag{9.1.12}$$

where $\vec{u}$ is the unit vector that is orthogonal (perpendicular) to the plane of intersection of vectors $\vec{a}$ and $\vec{b}$, taken in the positive right-hand coordinate direction, as shown in Fig. 9.1.7. If the viewer were behind the plane of Fig. 9.1.7, the unit normal to the plane would be $-\vec{u}$ and the counterclockwise angle from $\vec{a}$ to $\vec{b}$ would be $2\pi - \theta(\vec{a}, \vec{b})$. Then the vector product would be

$$\begin{aligned} \vec{a} \times \vec{b} &= ab \sin(2\pi - \theta(\vec{a}, \vec{b}))(-\vec{u}) \\ &= ab \sin \theta(\vec{a}, \vec{b})\vec{u} \end{aligned}$$

since $\sin(2\pi - \theta) = -\sin \theta$. This is the same result as in Eq. 9.1.12, so the viewpoint does not influence evaluation of the vector product. Since the definition of vector product is purely geometrical, the result is independent of the reference frame in which the vectors are represented.

Since reversal of the order of vectors $\vec{a}$ and $\vec{b}$ in Eq. 9.1.12 yields the opposite direction for the unit vector $\vec{u}$,

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} \tag{9.1.13}$$

Analogous to Eq. 9.1.10, the vector product satisfies [21]

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \tag{9.1.14}$$

Since $\theta(\vec{a}, \vec{a}) = 0$, for any vector $\vec{a}$,

$$\vec{a} \times \vec{a} = a^2 \sin 0\vec{u} = \vec{0}$$

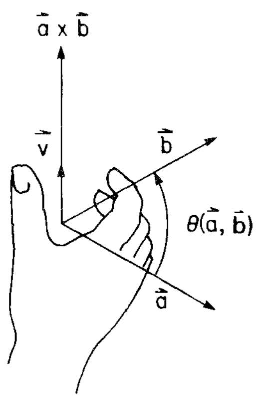

Figure 9.1.7 Vector product.

From the definition of unit coordinate vectors and vector product, the following identities are valid:

$$
\begin{aligned}
&\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}\\
&\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}\\
&\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}\\
&\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}
\end{aligned}
\tag{9.1.15}
$$

Using the identities of Eq. 9.1.15 and the property of vector product of Eq. 9.1.14, the vector product of two vectors may be expanded and written in terms of their components as (Prob. 9.1.3)

$$
\begin{aligned}
\vec{c}=\vec{a}\times\vec{b}&=(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}\\
&\equiv c_x\vec{i}+c_y\vec{j}+c_z\vec{k}
\end{aligned}
\tag{9.1.16}
$$

Algebraic Vectors Recall from Eq. 9.1.3 that a geometric vector $\vec{a}$ can be written in component form in a Cartesian x-y-z frame as

$$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$$

The geometric vector $\vec{a}$ is thus uniquely defined by its Cartesian components, which may be written in matrix notation as

$$\mathbf{a}=\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix}=[a_x,\ a_y,\ a_z]^T \tag{9.1.17}$$

This is the *algebraic representation of a geometric vector.*

Note that the algebraic representation of vectors is dependent on the Cartesian reference frame selected, that is, vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. Some of the purely geometric properties of vectors are thus lost, and the properties of the reference frame that is used in defining components of vectors come into play. This apparent problem will in fact become a valuable tool in spatial kinematics and dynamics.

An *algebraic vector* is defined as a column matrix. When an algebraic vector represents a geometric vector in three-dimensional space, it has three components. Algebraic vectors with more than three components will also be employed in the kinematics and dynamics of multibody systems. In the case where $\mathbf{a}=[a_1,\ldots,a_n]^T$, the algebraic vector $\mathbf{a}$ is called an *n vector* and is said to belong to *n-dimensional real space,* denoted R^n.

If two vectors $\vec{a}$ and $\vec{b}$ are represented in algebraic form as

$$
\begin{aligned}
\mathbf{a}&=[a_x,\ a_y,\ a_z]^T\\
\mathbf{b}&=[b_x,\ b_y,\ b_z]^T
\end{aligned}
\tag{9.1.18}
$$

then their vector sum $\vec{c}=\vec{a}+\vec{b}$ of Eq. 9.1.5 is represented in algebraic form by

(Prob. 9.1.4)

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{9.1.19}$$

Example 9.1.3: The algebraic representation of vectors

$$\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$$
$$\vec{b} = -\vec{i} + \vec{j} - \vec{k}$$

is

$$\mathbf{a} = [1, 2, 3]^T$$
$$\mathbf{b} = [-1, 1, -1]^T$$

The algebraic representation of the sum $\vec{c} = \vec{a} + \vec{b}$ is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = [0, 3, 2]^T$$

Two geometric vectors are equal if and only if the Cartesian components of the vectors are equal; that is, $\mathbf{a} = \mathbf{b}$. Multiplication of a vector $\vec{a}$ by a scalar α occurs component by component, so the geometric vector $\alpha\vec{a}$ is represented by the algebraic vector $\alpha\mathbf{a}$. Since there is a one-to-one correspondence between geometric vectors and 3×1 algebraic vectors that are formed from their Cartesian components in a specified Cartesian reference frame, no distinction other than notation will be made between them in the remainder of this text.

The scalar product of two geometric vectors may be expressed in algebraic form, using the result of Eq. 9.1.11, as

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \mathbf{a}^T\mathbf{b} \tag{9.1.20}$$

Example 9.1.4: The scalar product of vectors $\vec{a}$ and $\vec{b}$ (or $\mathbf{a}$ and $\mathbf{b}$) in Example 9.1.3 is

$$\vec{a} \cdot \vec{b} = \mathbf{a}^T\mathbf{b} = [1, 2, 3]\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -2$$

From the definition of scalar product,

$$\mathbf{a}^T\mathbf{b} = ab \cos \theta(\mathbf{a}, \mathbf{b})$$

and, with $a = (\mathbf{a}^T\mathbf{a})^{1/2} = \sqrt{14}$ and $b = (\mathbf{b}^T\mathbf{b})^{1/2} = \sqrt{3}$, $\cos \theta(\mathbf{a}, \mathbf{b}) = -2/\sqrt{42}$. Thus, $\theta(\mathbf{a}, \mathbf{b}) = 1.88$ or 4.40 rad.

A skew-symmetric matrix $\tilde{\mathbf{a}}$ associated with an algebraic vector $\mathbf{a} = [a_x, a_y, a_z]^T$ is defined as

$$\tilde{\mathbf{a}} \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{9.1.21}$$

Note that an overhead ~ (pronounced tilde) indicates that the components of the vector are used to generate a skew-symmetric 3×3 matrix. Conversely, any 3×3 skew-symmetric matrix of the form

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix}$$

can be written as $\mathbf{B} = \tilde{\mathbf{b}}$, where $\mathbf{b} = [b_1, b_2, b_3]^T$, with $b_1 = -b_{23}$, $b_2 = b_{13}$, and $b_3 = -b_{12}$; that is,

$$\mathbf{B} = \tilde{\mathbf{b}} = \widetilde{\begin{bmatrix} -b_{23} \\ b_{13} \\ -b_{12} \end{bmatrix}}$$

The vector product $\vec{c} = \vec{a} \times \vec{b}$, which is expanded in component form in Eq. 9.1.6, can thus be written in algebraic vector form as

$$\mathbf{c} = \tilde{\mathbf{a}}\mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \tag{9.1.22}$$

This result is the reason the ~ operator is introduced. It gives a convenient and computationally practical means of evaluating the vector product of two vectors that are represented in algebraic form.

Example 9.1.5: The algebraic representation of the vector product $\vec{c} = \vec{a} \times \vec{b}$ of $\vec{a}$ and $\vec{b}$ in Example 9.1.3 is

$$\mathbf{c} = \tilde{\mathbf{a}}\mathbf{b} = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix}$$

For later use, it is helpful to develop some standard properties of the ~ operation. First, note that

$$\tilde{\mathbf{a}}^T = \begin{bmatrix} 0 & a_z & -a_y \\ -a_z & 0 & a_x \\ a_y & -a_x & 0 \end{bmatrix} = -\tilde{\mathbf{a}} \tag{9.1.23}$$

Also, for a scalar α,

$$\alpha\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -\alpha a_z & \alpha a_y \\ \alpha a_z & 0 & -\alpha a_x \\ -\alpha a_y & \alpha a_x & 0 \end{bmatrix} = \widetilde{(\alpha \mathbf{a})} \tag{9.1.24}$$

For any vectors **a** and **b**, a direct calculation (Prob. 9.1.5) shows that

$$\tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a} \tag{9.1.25}$$

which agrees with the vector product relation $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ of Eq. 9.1.13.

Direct calculation shows that

$$\tilde{\mathbf{a}}\mathbf{a} = \mathbf{0} \tag{9.1.26}$$

Hence, by Eq. 9.1.23,

$$(\tilde{\mathbf{a}}\mathbf{a})^T = \mathbf{a}^T\tilde{\mathbf{a}}^T = -\mathbf{a}^T\tilde{\mathbf{a}} = \mathbf{0} \tag{9.1.27}$$

It can be verified by direct calculation (Prob. 9.1.6) that

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I} \tag{9.1.28}$$

where **I** is the 3×3 identity matrix. Direct calculation also shows that (Prob. 9.1.7)

$$\widetilde{(\tilde{\mathbf{a}}\mathbf{b})} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T \tag{9.1.29}$$

$$= \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}} \tag{9.1.30}$$

From Eqs. 9.1.29 and 9.1.30,

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} + \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{b}}\tilde{\mathbf{a}} + \mathbf{b}\mathbf{a}^T \tag{9.1.31}$$

Finally, it can be verified by direct calculation that

$$\widetilde{(\mathbf{a}+\mathbf{b})} = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \tag{9.1.32}$$

Matrix implementation of vector operations permits the systematic organization of calculations, which is especially valuable for computer applications.

Example 9.1.6: To define the position and orientation of a Cartesian x'-y'-z' reference frame with its origin at point P, in the x-y-z reference frame, it is first essential to locate point P with vector $\mathbf{r}^P$, as shown in Fig. 9.1.8. If a second point Q is located on the x' axis by vector $\mathbf{r}^Q$, the unit vector $\mathbf{f}$ along the x' axis is

$$\mathbf{f} = \frac{1}{|\mathbf{r}^Q - \mathbf{r}^P|}(\mathbf{r}^Q - \mathbf{r}^P)$$

If a third point R that is not on the x' axis is located in the x'-y' plane by vector $\mathbf{r}^R$, then the vector product of $\mathbf{f}$ and $\mathbf{r}^R - \mathbf{r}^P$ is a vector in the direction of the z' axis; that is,

$$\mathbf{h} = \frac{1}{|\tilde{\mathbf{f}}(\mathbf{r}^R - \mathbf{r}^P)|}\tilde{\mathbf{f}}(\mathbf{r}^R - \mathbf{r}^P)$$

Finally, the unit vector $\mathbf{g}$ along the y' axis is

$$\mathbf{g} = \tilde{\mathbf{h}}\mathbf{f}$$

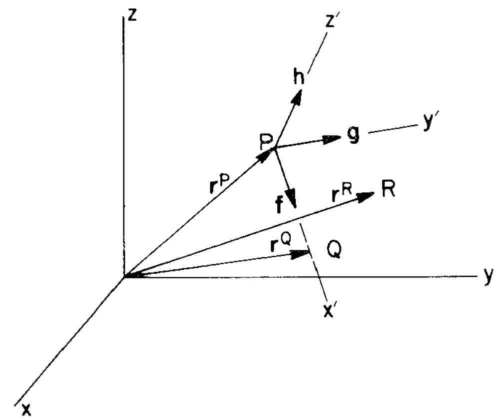

Figure 9.1.8 Points defining a Cartesian reference frame.

Thus, three distinct noncollinear points P, Q, and R in the x'-y' plane define unit vectors **f**, **g**, and **h** along the x', y', and z' axes, hence defining the position and orientation of the x'-y'-z' Cartesian reference frame.

Differentiation of Vectors In analyzing velocities and accelerations, the time derivatives of vectors that locate points must be calculated. Consider a time-dependent vector $\vec{a}(t)$ with time-dependent components $\mathbf{a} \equiv \mathbf{a}(t) = [a_x(t), a_y(t), a_z(t)]^T$ in an x-y-z *stationary Cartesian reference frame* that does not depend on time; that is, $\vec{i}$, $\vec{j}$, and $\vec{k}$ are constant. The *time derivative of a vector* $\vec{a}$ is the *velocity* of the point that is located by the vector, denoted by

$$\dot{\vec{a}} \equiv \frac{d}{dt}\vec{a}(t) = \frac{d}{dt}[a_x(t)\vec{i} + a_y(t)\vec{j} + a_z(t)\vec{k}]$$
$$= \left[\frac{d}{dt}a_x(t)\right]\vec{i} + \left[\frac{d}{dt}a_y(t)\right]\vec{j} + \left[\frac{d}{dt}a_z(t)\right]\vec{k}$$

Note that this is only valid if $\vec{i}$, $\vec{j}$, and $\vec{k}$ are not time dependent. In algebraic vector notation, this is

$$\dot{\mathbf{a}} \equiv \frac{d}{dt}\mathbf{a}(t) = \left[\frac{d}{dt}a_x(t), \frac{d}{dt}a_y(t), \frac{d}{dt}a_z(t)\right]^T \qquad \mathbf{(9.1.33)}$$

Thus, for vectors that are written in terms of their components in a stationary Cartesian reference frame, the derivative of a vector is obtained by differentiating its components.

Example 9.1.7: A particle P moves with constant speed along a circle of radius R in the x-y plane of a stationary Cartesian reference frame, as shown in

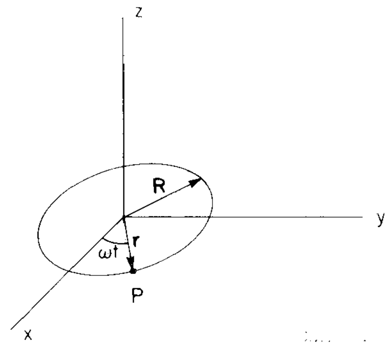

Figure 9.1.9 Particle moving on circular path with constant speed.

Fig. 9.1.9. Its position vector is

$$\mathbf{r} = \begin{bmatrix} R\cos\omega t \\ R\sin\omega t \\ 0 \end{bmatrix}$$

where R and ω are constant. Its velocity is thus

$$\dot{\mathbf{r}} = \begin{bmatrix} -R\omega\sin\omega t \\ R\omega\cos\omega t \\ 0 \end{bmatrix}$$

The derivative of the sum of two vectors is (Prob. 9.1.9)

$$\frac{d}{dt}(\mathbf{a}(t) + \mathbf{b}(t)) = \dot{\mathbf{a}} + \dot{\mathbf{b}} \qquad \textbf{(9.1.34)}$$

which is analogous to the differentiation rule of calculus that the derivative of a sum is the sum of the derivatives.

The following vector forms of the *product rule of differentiation* are also valid (Prob. 9.1.10):

$$\frac{d}{dt}(\alpha\mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}} \qquad \textbf{(9.1.35)}$$

$$\frac{d}{dt}(\mathbf{a}^T\mathbf{b}) = \dot{\mathbf{a}}^T\mathbf{b} + \mathbf{a}^T\dot{\mathbf{b}} \qquad \textbf{(9.1.36)}$$

$$\frac{d}{dt}(\tilde{\mathbf{a}}\mathbf{b}) = \dot{\tilde{\mathbf{a}}}\mathbf{b} + \tilde{\mathbf{a}}\dot{\mathbf{b}} \qquad \textbf{(9.1.37)}$$

where $\alpha(t)$ is a scalar function of time. Note also that

$$\dot{\tilde{\mathbf{a}}} = \tilde{\dot{\mathbf{a}}} \qquad \textbf{(9.1.38)}$$

Example 9.1.8: If the length of a vector $\mathbf{a}(t)$ is fixed, that is, $\mathbf{a}(t)^T\mathbf{a}(t) = c$, where c is a constant, then Eq. 9.1.36 yields

$$\dot{\mathbf{a}}^T\mathbf{a} = 0 \tag{9.1.39}$$

If $\mathbf{a}$ is a position vector that locates a point, then $\dot{\mathbf{a}}$ is the velocity of that point. Hence, Eq. 9.1.39 indicates that the velocity of a point whose distance from the origin is constant is orthogonal to the position vector of the point. Note that Eq. 9.1.39 is satisfied by the velocity vector of Example 9.1.7, which is to be expected since $\mathbf{a}^T\mathbf{a} = R^2$.

The second time derivative of $\mathbf{a}(t)$ is the *acceleration* of the point that is located by the vector, denoted as

$$\ddot{\mathbf{a}} = \frac{d}{dt}(\dot{\mathbf{a}}(t)) = \left[\frac{d^2}{dt^2}a_x(t), \frac{d^2}{dt^2}a_y(t), \frac{d^2}{dt^2}a_z(t)\right]^T \tag{9.1.40}$$

Thus, for vectors that are written in terms of their components in a stationary Cartesian reference frame, acceleration may be calculated in terms of the second time derivatives of the components of the vector.

Example 9.1.9: The acceleration of the particle that is located by the vector $\mathbf{r}(t)$ in Example 9.1.7 is

$$\ddot{\mathbf{r}} = \begin{bmatrix} -R\omega^2 \cos \omega t \\ -R\omega^2 \sin \omega t \\ 0 \end{bmatrix} = -\omega^2\mathbf{r}$$

This is the classical *centripetal acceleration* of a point that moves on a circular path, since the direction of $\ddot{\mathbf{r}}$ is opposite to the direction of $\mathbf{r}$.

9.2 KINEMATICS OF A RIGID BODY IN SPACE

A *rigid body* is defined as being made up of a continuum of particles that are constrained not to move relative to one another. While actual bodies are never perfectly rigid, deformation effects are often negligible when considering the motion of a machine that is made up multiple bodies. For this reason, most attention in the study of the dynamics of machines is devoted to modeling individual bodies in a system as being rigid. This section develops the relations needed for the kinematic and dynamic analysis of rigid bodies in space.

Substantial technical differences arise in the kinematic analysis of rigid bodies in space, as compared to rigid bodies in the plane. Most of the attention in this section is on methods of defining the orientation of rigid bodies in space.

Reference Frames for the Location and Orientation of a Body in Space Consider, for purposes of introduction, the rectangular body shown in Fig. 9.2.1. The geometry of the body, usually defined by key points on the body, may be defined in an x'-y'-z' reference frame that is fixed to the body, called a *body-fixed reference frame*. The x'-y'-z' frame is coincident with the x-y-z global reference frame in the position shown in Fig. 9.2.1. This body-fixed reference frame might be thought of as a *drafting board reference frame* in engineering practice. That is, the engineer defines points, lines, and surfaces on a machine or structural component in terms of front view, side view, and top view to create a three-dimensional definition of the component under consideration. Thus, data that define the geometry of the body are specified in the x'-y'-z' frame.

Consider next the translated and rotated position of the body shown in Fig. 9.2.2. Translation of the body may easily be defined by the vector **r** from the origin of the x-y-z reference frame to the origin of the x'-y'-z' frame. Specification of orientation is, however, more complex. Nevertheless, once the x'-y'-z' frame is oriented relative to the x-y-z reference frame, all points of interest on the body can be defined in the x-y-z reference frame. Most of the attention in this section focuses on transformations from generally oriented body-fixed frames to the x-y-z reference frame.

To see that the orientation of a body in space raises delicate technical questions, consider two rotation sequences for a body from the orientation shown in Fig. 9.2.1. Rotations of the body by angles of $\pi/2$ are to be made about its body-fixed x' and y' axes. Consider first the rotation sequence shown in Fig. 9.2.3, where the first rotation is about the body-fixed x' axis, which initially coincides with the reference frame x axis. The second rotation is about the

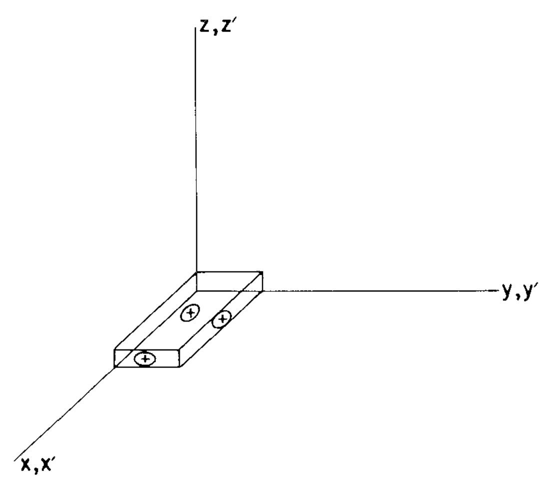

Figure 9.2.1 Body in reference position.

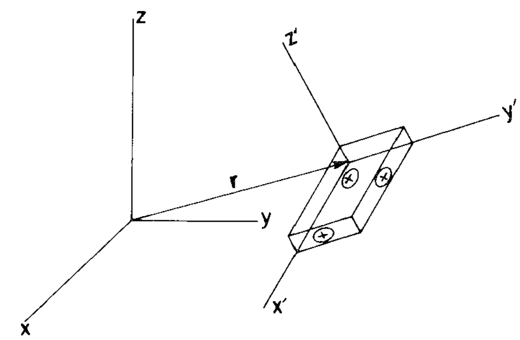

Figure 9.2.2 Body translated and rotated.

body-fixed y' axis, when it coincides with the reference frame z axis, as shown in Fig. 9.2.3a. The resulting orientation is shown in Fig. 9.2.3b.

Consider next the same rotations about body-fixed x' and y' axes, but in reverse order. First, the body is rotated $\pi/2$ about its body-fixed y' axis, which initially coincides with the reference frame y axis, as shown in Fig. 9.2.4a. The body is then rotated $\pi/2$ about its body-fixed x' axis, when it coincides with the negative reference frame z axis, as shown in Fig. 9.2.4a. The resulting orientation is shown in Fig. 9.2.4b.

Since the orientations shown in Figs. 9.2.3b and 9.2.4b are distinctly different, it is clear that the order of rotation is important in defining the orientation of a body in space. Much as in the case of matrix multiplication, in which the order of terms in the product of two matrices is critically important, the order of rotations is likewise important. This shows that large-amplitude rotation is not a vector quantity, since the order of rotations is not commutative; that is, orders of rotation cannot be inverted.

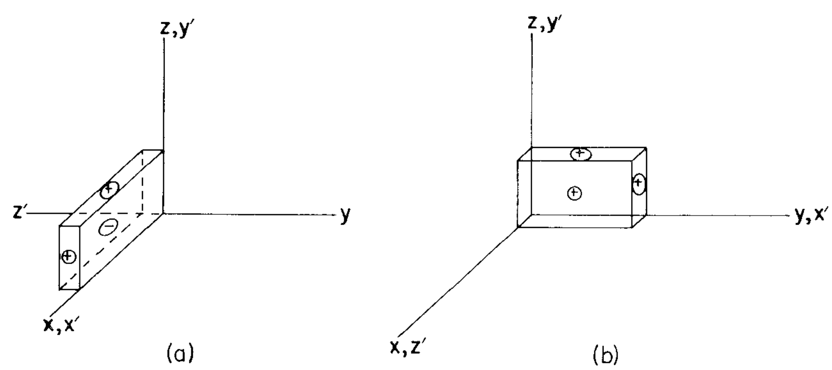

Figure 9.2.3 Rotation of body by $\pi/2$ about x' and then y' axes.

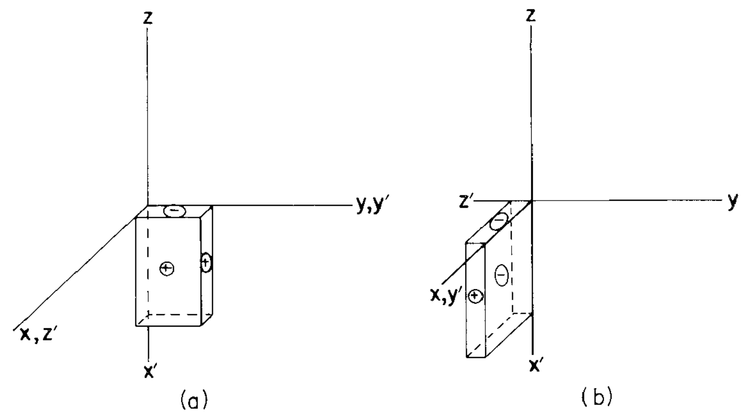

Figure 9.2.4 Rotations of body by $\pi/2$ about y' and then x' axes.

Transformation of Coordinates. It is shown in Section 9.1 that a geometric vector is uniquely represented by an algebraic vector that contains components of the geometric vector in a Cartesian reference frame. The components of a vector, however, are defined in a specific Cartesian reference frame. Consider a second Cartesian x'-y'-z' frame with the same origin as the x-y-z frame, as shown in Fig. 9.2.5. Unit x', y', and z' coordinate vectors are denoted by $\vec{f}$, $\vec{g}$, and $\vec{h}$, respectively, and unit x, y, and z coordinate vectors are denoted by $\vec{i}$, $\vec{j}$, and $\vec{k}$.

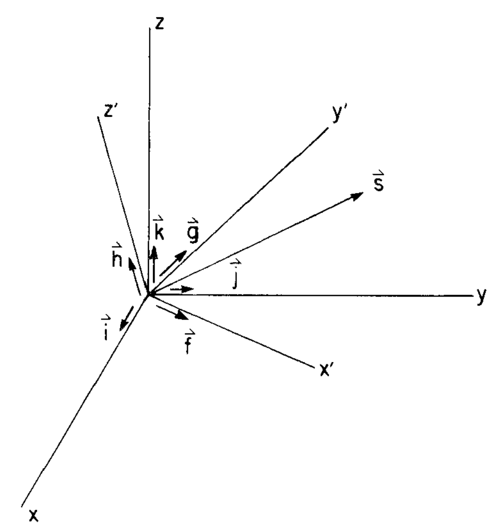

Figure 9.2.5 Two Cartesian reference frames.

A vector $\vec{s}$ in space can be represented in either of the frames as

$$\vec{s} = s_x\vec{i} + s_y\vec{j} + s_z\vec{k} \tag{9.2.1}$$

or

$$\vec{s} = s_{x'}\vec{f} + s_{y'}\vec{g} + s_{z'}\vec{h} \tag{9.2.2}$$

where

$$s_x = \vec{s}\cdot\vec{i}, \qquad s_y = \vec{s}\cdot\vec{j}, \qquad s_z = \vec{s}\cdot\vec{k} \tag{9.2.3}$$

and

$$s_{x'} = \vec{s}\cdot\vec{f}, \qquad s_{y'} = \vec{s}\cdot\vec{g}, \qquad s_{z'} = \vec{s}\cdot\vec{h} \tag{9.2.4}$$

The algebraic vectors that define $\vec{s}$ in the two frames are

$$\mathbf{s} = [s_x, s_y, s_z]^T \tag{9.2.5}$$

in the x-y-z frame and

$$\mathbf{s}' = [s_{x'}, s_{y'}, s_{z'}]^T \tag{9.2.6}$$

in the x'-y'-z' frame.

It is clear that there is a relation between $\mathbf{s}$ and $\mathbf{s}'$, since they are defined by the same geometric vector $\vec{s}$. To establish this relation, expand the $\vec{f}$, $\vec{g}$, and $\vec{h}$ unit vectors in terms of the $\vec{i}$, $\vec{j}$, and $\vec{k}$ unit vectors as

$$\begin{aligned}
\vec{f} &= a_{11}\vec{i} + a_{21}\vec{j} + a_{31}\vec{k}\\
\vec{g} &= a_{12}\vec{i} + a_{22}\vec{j} + a_{32}\vec{k}\\
\vec{h} &= a_{13}\vec{i} + a_{23}\vec{j} + a_{33}\vec{k}
\end{aligned} \tag{9.2.7}$$

where a_{ij} are the following direction cosines:

$$\begin{aligned}
a_{11} &= \vec{i}\cdot\vec{f} = \cos\theta(\vec{i}, \vec{f})\\
a_{12} &= \vec{i}\cdot\vec{g} = \cos\theta(\vec{i}, \vec{g})\\
a_{13} &= \vec{i}\cdot\vec{h} = \cos\theta(\vec{i}, \vec{h})\\
a_{21} &= \vec{j}\cdot\vec{f} = \cos\theta(\vec{j}, \vec{f})\\
a_{22} &= \vec{j}\cdot\vec{g} = \cos\theta(\vec{j}, \vec{g})\\
a_{23} &= \vec{j}\cdot\vec{h} = \cos\theta(\vec{j}, \vec{h})\\
a_{31} &= \vec{k}\cdot\vec{f} = \cos\theta(\vec{k}, \vec{f})\\
a_{32} &= \vec{k}\cdot\vec{g} = \cos\theta(\vec{k}, \vec{g})\\
a_{33} &= \vec{k}\cdot\vec{h} = \cos\theta(\vec{k}, \vec{h})
\end{aligned} \tag{9.2.8}$$

Substituting from Eq. 9.2.7 into Eq. 9.2.2 yields

$$\begin{aligned}
\vec{s} = {} & (a_{11}s_{x'} + a_{12}s_{y'} + a_{13}s_{z'})\vec{i}\\
& + (a_{21}s_{x'} + a_{22}s_{y'} + a_{23}s_{z'})\vec{j}\\
& + (a_{31}s_{x'} + a_{32}s_{y'} + a_{33}s_{z'})\vec{k}
\end{aligned}$$

Equating corresponding right sides of this representation of $\vec{s}$ and Eq. 9.2.1,

$$\begin{aligned} s_x &= a_{11}s_{x'} + a_{12}s_{y'} + a_{13}s_{z'} \\ s_y &= a_{21}s_{x'} + a_{22}s_{y'} + a_{23}s_{z'} \\ s_z &= a_{31}s_{x'} + a_{32}s_{y'} + a_{33}s_{z'} \end{aligned} \tag{9.2.9}$$

In matrix form, this is

$$\mathbf{s} = \mathbf{A}\mathbf{s}' \tag{9.2.10}$$

where $\mathbf{A}$ is called the *direction cosine matrix* or *rotation transformation matrix:*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{9.2.11}$$

The direction cosine matrix $\mathbf{A}$ has a very special property. If the x-y-z component forms of unit vectors $\vec{f}$, $\vec{g}$, and $\vec{h}$ are denoted by $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ and the x-y-z components of vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, then

$$\begin{aligned} \mathbf{i} &= [1, 0, 0]^T \\ \mathbf{j} &= [0, 1, 0]^T \\ \mathbf{k} &= [0, 0, 1]^T \end{aligned} \tag{9.2.12}$$

and Eq. 9.2.8 shows that

$$\begin{aligned} \mathbf{f} &= [a_{11}, a_{21}, a_{31}]^T \\ \mathbf{g} &= [a_{12}, a_{22}, a_{32}]^T \\ \mathbf{h} &= [a_{13}, a_{23}, a_{33}]^T \end{aligned}$$

Therefore, the matrix $\mathbf{A}$ of Eq. 9.2.11 can be written as

$$\mathbf{A} = [\mathbf{f}, \mathbf{g}, \mathbf{h}] \tag{9.2.13}$$

Since the unit vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are orthogonal, expansions of the matrix product shows that (Prob. 9.2.3)

$$\mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{9.2.14}$$

Thus, $\mathbf{A}^T = \mathbf{A}^{-1}$ and the direction cosine matrix $\mathbf{A}$ is an *orthogonal matrix.* This special property permits an easy inversion of Eq. 9.2.10 to obtain

$$\mathbf{s}' = \mathbf{A}^T\mathbf{s} \tag{9.2.15}$$

Thus, transforming between algebraic vectors that represent the same physical vector in the x-y-z and x'-y'-z' Cartesian reference frames is a trivial matter.

Example 9.2.1: The z' axis of the x'-y'-z' Cartesian frame is fixed in the x-y plane, and the x' axis is rotated an angle ϕ above the x-y plane, as shown in

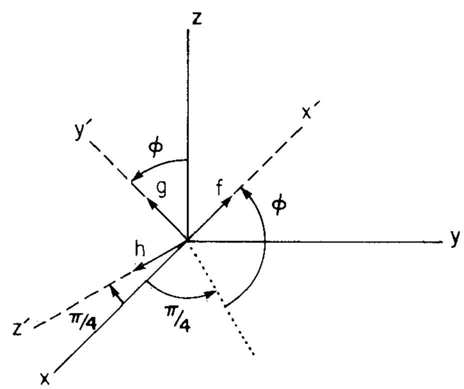

Figure 9.2.6 Orientation of a reference frame.

Fig. 9.2.6. The **f**, **g**, and **h** unit vectors are thus

$$\mathbf{f} = \left[\frac{\sqrt{2}}{2}\cos\phi, \frac{\sqrt{2}}{2}\cos\phi, \sin\phi\right]^T$$

$$\mathbf{g} = \left[-\frac{\sqrt{2}}{2}\sin\phi, -\frac{\sqrt{2}}{2}\sin\phi, \cos\phi\right]^T$$

$$\mathbf{h} = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right]^T$$

The transformation matrix from the x'-y'-z' frame to the x-y-z frame may be evaluated using Eq. 9.2.13.

When the origins of the x-y-z and x'-y'-z' frames do not coincide, the foregoing analysis is applied between the x'-y'-z' and a translated x-y-z frame, as shown in Fig. 9.2.7. If the algebraic vector $\mathbf{s}'^P$ locates point P in the x'-y'-z' frame, then in the translated x-y-z frame this vector is $\mathbf{A}\mathbf{s}'^P$. Thus,

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \tag{9.2.16}$$

where $\mathbf{r}$ is the vector from the origin of the x-y-z reference frame to the origin of the x'-y'-z' frame, as shown in Fig. 9.2.7.

The nine direction cosines in matrix $\mathbf{A}$ define the orientation of the x'-y'-z' frame relative to the x-y-z reference frame, but they are not independent. Equation 9.2.14 provides six equations among the nine direction cosines, since in component form Eq. 9.2.14 is

$$\sum_{k=1}^{3} a_{ki}a_{kj} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}, \qquad i, j = 1, 2, 3$$

and interchanging $i \neq j$ yields the same equation, because $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ is symmetric.

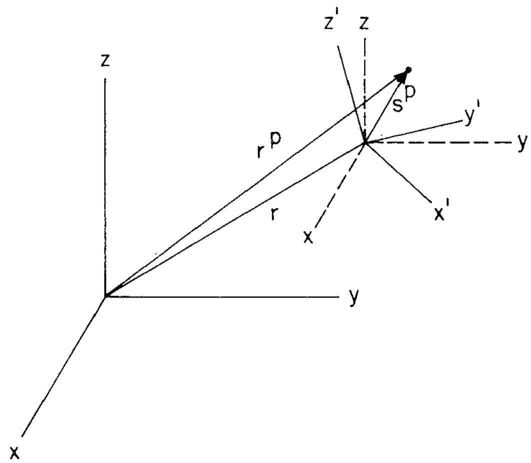

Figure 9.2.7 Translation and rotation of a reference frame.

To show that there are three rotational degrees of freedom, it must be shown that these equations impose six independent constraints on the nine components of $\mathbf{A}$.

First note that by Eq. 9.2.13, $\mathbf{A} = [\mathbf{f}, \mathbf{g}, \mathbf{h}]$, where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are unit vectors along the x', y', and z' axes, respectively. Denote the vector of nine direction cosines in $\mathbf{A}$ as

$$\mathbf{q} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{bmatrix}$$

The constraints imposed on these variables by the diagonal and upper triangular parts of Eq. 9.2.14 are

$$\mathbf{\Phi}(\mathbf{q}) \equiv \begin{bmatrix} \frac{1}{2}\mathbf{f}^T\mathbf{f} - \frac{1}{2} \\ \frac{1}{2}\mathbf{g}^T\mathbf{g} - \frac{1}{2} \\ \frac{1}{2}\mathbf{h}^T\mathbf{h} - \frac{1}{2} \\ \mathbf{f}^T\mathbf{g} \\ \mathbf{f}^T\mathbf{h} \\ \mathbf{g}^T\mathbf{h} \end{bmatrix} = \mathbf{0}$$

Let $\mathbf{q}_0$ satisfy $\mathbf{\Phi}(\mathbf{q}_0) = \mathbf{0}$. The Jacobian of the constraint function $\mathbf{\Phi}(\mathbf{q})$ is

$$\mathbf{\Phi}_{\mathbf{q}} = \begin{bmatrix} \mathbf{f}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{g}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{h}^T \\ \mathbf{g}^T & \mathbf{f}^T & \mathbf{0} \\ \mathbf{h}^T & \mathbf{0} & \mathbf{f}^T \\ \mathbf{0} & \mathbf{h}^T & \mathbf{g}^T \end{bmatrix}$$

The rows of $\mathbf{\Phi_q}$ are linearly independent if and only if the columns of $\mathbf{\Phi_q^T}$ are linearly independent, that is, if and only if

$$\mathbf{\Phi_q^T u} = \mathbf{0}$$

implies $\mathbf{u} = \mathbf{0}$. Expanding

$$\mathbf{\Phi_q^T u} = \begin{bmatrix} \mathbf{f} & \mathbf{0} & \mathbf{0} & \mathbf{g} & \mathbf{h} & \mathbf{0} \\ \mathbf{0} & \mathbf{g} & \mathbf{0} & \mathbf{f} & \mathbf{0} & \mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{h} & \mathbf{0} & \mathbf{f} & \mathbf{g} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{0}$$

and reordering variables yields

$$[\mathbf{f}, \mathbf{g}, \mathbf{h}] \begin{bmatrix} u_1 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_1 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

$$[\mathbf{f}, \mathbf{g}, \mathbf{h}] \begin{bmatrix} u_4 \\ u_2 \\ u_6 \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_4 \\ u_2 \\ u_6 \end{bmatrix} = \mathbf{0}$$

$$[\mathbf{f}, \mathbf{g}, \mathbf{h}] \begin{bmatrix} u_5 \\ u_6 \\ u_3 \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_5 \\ u_6 \\ u_3 \end{bmatrix} = \mathbf{0}$$

Since $|\mathbf{A}^T\mathbf{A}| = |\mathbf{I}| = 1 = |\mathbf{A}^T|\,|\mathbf{A}| = |\mathbf{A}|\,|\mathbf{A}| = |\mathbf{A}|^2$, $|\mathbf{A}| = 1$ and $\mathbf{A}$ is nonsingular. Thus, the previous three equations imply that $\mathbf{u} = \mathbf{0}$. Hence, $\mathbf{\Phi_q}$ has full row rank and there is a 6×6 nonsingular submatrix of $\mathbf{\Phi_q}$ with nonzero determinant. Defining the variables in $\mathbf{q}$ that correspond to columns of the nonsingular submatrix as dependent, these six variables can be written explicitly as functions of the remaining three variables uniquely in some neighborhood of $\mathbf{q}_0$ (by the implicit function theorem). Thus, there are three degrees of rotational freedom of a rigid body in space.

While the nine direction cosines, subject to the constraints of Eq. 9.2.14, could be adopted as generalized coordinates that define the orientation of the x'-y'-z' frame, this is neither practical nor convenient. Thus, other orientation generalized coordinates are sought.

Example 9.2.2: As a special case, consider the x-y-z and x'-y'-z' frames shown in Fig. 9.2.8, with z and z' axes coincident. The angle of rotation of the x' axis relative to the x axis, with counterclockwise as positive, is denoted as ϕ.

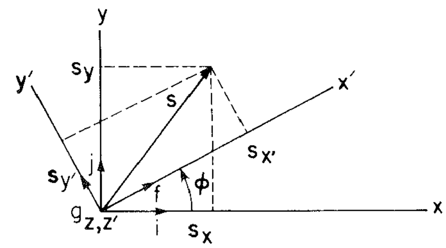

Figure 9.2.8 Reference frames with coincident z and z' axes.

From Eq. 9.2.8,

$$a_{11} = a_{22} = \cos\phi$$

$$a_{21} = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$$

$$a_{12} = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$a_{33} = 1$$

$$a_{31} = a_{32} = a_{13} = a_{23} = 0$$

Thus the direction cosine matrix of Eq. 9.2.11 yields the transformation

$$\mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x'} \\ s_{y'} \\ s_{z'} \end{bmatrix} = \mathbf{A}\mathbf{s}' \tag{9.2.17}$$

This contains the planar rotation relations from Chapter 2, that is,

$$\begin{bmatrix} s_x \\ s_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s_{x'} \\ s_{y'} \end{bmatrix} \tag{9.2.18}$$

$$\begin{bmatrix} s_{x'} \\ s_{y'} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s_x \\ s_y \end{bmatrix} \tag{9.2.19}$$

Useful relations may be obtained by noting that, for any vectors $\mathbf{s}'$ and $\mathbf{v}'$ in the x'-y'-z' frame, the vector product of $\mathbf{s}$ and $\mathbf{v}$ in the x-y-z frame may be obtained by forming the vector product of $\mathbf{s}'$ and $\mathbf{v}'$ and transforming the result to the x-y-z frame; that is,

$$\tilde{\mathbf{s}}\mathbf{v} = \mathbf{A}(\tilde{\mathbf{s}}'\mathbf{v}')$$

Using $\mathbf{v} = \mathbf{A}\mathbf{v}'$,

$$(\tilde{\mathbf{s}}\mathbf{A})\mathbf{v}' = (\mathbf{A}\tilde{\mathbf{s}}')\mathbf{v}'$$

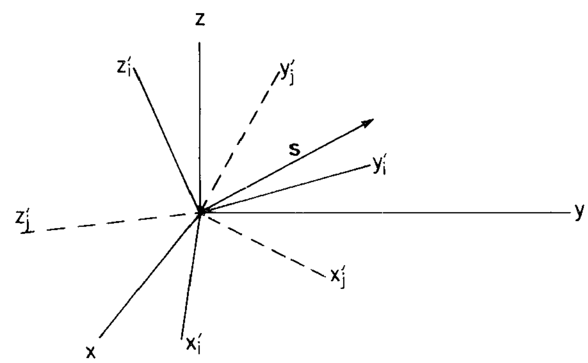

Figure 9.2.9 Reference frames with coincident origins.

Since this must hold for arbitrary $\mathbf{v}'$ (Prob. 9.2.4),

$$\tilde{\mathbf{s}}\mathbf{A} = \mathbf{A}\tilde{\mathbf{s}}' \tag{9.2.20}$$

Postmultiplying by $\mathbf{A}^T$ and using Eq. 9.2.14 yields the desired result:

$$\widetilde{(\mathbf{A}\mathbf{s}')} = \tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T \tag{9.2.21}$$

Similarly,

$$\widetilde{(\mathbf{A}^T\mathbf{s})} = \tilde{\mathbf{s}}' = \mathbf{A}^T\tilde{\mathbf{s}}\mathbf{A} \tag{9.2.22}$$

Consider the pair of Cartesian x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames shown in Fig. 9.2.9. An arbitrary vector $\mathbf{s}$ in the x-y-z frame has representations $\mathbf{s}_i'$ and $\mathbf{s}_j'$ in the x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames, respectively; that is,

$$\mathbf{s} = \mathbf{A}_i\mathbf{s}_i' = \mathbf{A}_j\mathbf{s}_j' \tag{9.2.23}$$

where $\mathbf{A}_i$ and $\mathbf{A}_j$ are transformation matrices from the x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames to the x-y-z frame, respectively.

Since $\mathbf{A}_i$ and $\mathbf{A}_j$ are orthogonal matrices, Eq. 9.2.23 yields

$$\mathbf{s}_i' = \mathbf{A}_i^T\mathbf{A}_j\mathbf{s}_j' \equiv \mathbf{A}_{ij}\mathbf{s}_j' \tag{9.2.24}$$

Since $\mathbf{s}_j'$ is an arbitrary vector,

$$\mathbf{A}_{ij} = \mathbf{A}_i^T\mathbf{A}_j \tag{9.2.25}$$

is the transformation matrix from the x_j'-y_j'-z_j' frame to the x_i'-y_i'-z_i' frame. A direct calculation shows that $\mathbf{A}_{ij}$ is an orthogonal matrix (Prob. 9.2.5).

Example 9.2.3: In numerous applications, kinematic constraints dictate that certain axes of the x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames remain parallel; for example, the vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ shown in Fig. 9.2.10 are to be parallel. The angle θ of rotation, measured positive as counterclockwise from $\mathbf{f}_i$ to $\mathbf{f}_j$, is a key quantity that is to be

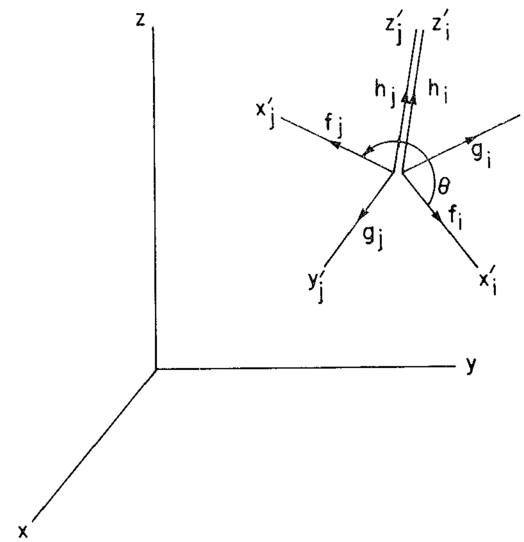

Figure 9.2.10 Triads with z' axes parallel.

calculated. From the definition of scalar product and the fact that the coordinate vectors shown in Fig. 9.2.10 are unit vectors,

$$\mathbf{f}_i^T\mathbf{f}_j = \cos\theta \tag{9.2.26}$$

Similarly, from the definition of vector product,

$$\tilde{\mathbf{f}}_i\mathbf{f}_j = \mathbf{h}_i \sin\theta$$

Taking the scalar product of both sides of this equation with $\mathbf{h}_i$ and using the fact that $\tilde{\mathbf{f}}_i\mathbf{h}_i = -\mathbf{g}_i$,

$$\sin\theta = \mathbf{h}_i^T\tilde{\mathbf{f}}_i\mathbf{f}_j = \mathbf{g}_i^T\mathbf{f}_j \tag{9.2.27}$$

Writing the unit vectors in terms of the respective reference frames in which they are fixed, and using the transformation matrices from these frames to the x-y-z frame, Eq. 9.2.26 becomes

$$\cos\theta = \mathbf{f}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{f}_j' \tag{9.2.28}$$

Similarly, Eq . 9.2.27 becomes

$$\sin\theta = \mathbf{g}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{f}_j' \tag{9.2.29}$$

If $\sin\theta$ and $\cos\theta$ are known, the value of θ, $0 \leq \theta < 2\pi$, may be uniquely determined by taking the Arcsin of both sides of Eq. 9.2.29 and using the algebraic sign of $\cos\theta$ from Eq. 9.2.28 to uniquely evaluate θ. To simplify notation, denote $s = \sin\theta$ from Eq. 9.2.29 and $c = \cos\theta$ from Eq. 9.2.28. Then (Prob. 9.2.6), with

$$\frac{-\pi}{2} \leq \text{Arcsin}\, s \leq \frac{\pi}{2} \tag{9.2.30}$$

θ is

$$\theta = \begin{cases} \text{Arcsin}\, s, & \text{if } s \geq 0 \text{ and } c \geq 0 \\ \pi - \text{Arcsin}\, s, & \text{if } s \geq 0 \text{ and } c < 0 \\ \pi - \text{Arcsin}\, s, & \text{if } s < 0 \text{ and } c < 0 \\ 2\pi + \text{Arcsin}\, s, & \text{if } s < 0 \text{ and } c \geq 0 \end{cases} \tag{9.2.31}$$

The angle calculated in Eq. 9.2.31 is in the range $0 \leq \theta < 2\pi$. If the number n of revolutions that have occurred is important, logic must be supplied to determine n. The angle $2n\pi$ is then added to the value of θ calculated in Eq. 9.2.31.

Velocity, Acceleration, and Angular Velocity An x'-y'-z' frame is fixed in a moving body, to define the body's position and orientation, relative to an x-y-z reference frame. Consider a point P that is fixed in the x'-y'-z' frame, as shown in Fig. 9.2.11. The vector that locates P in the x-y-z reference frame is given by Eq. 9.2.16 as

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \tag{9.2.32}$$

where $\mathbf{s}'^P$ is the constant vector of coordinates of P in the x'-y'-z' frame and $\mathbf{A}$ is the direction cosine matrix of the x'-y'-z' frame relative to the stationary x-y-z frame.

Since the x'-y'-z' frame is moving and changing its orientation with time, the vector $\mathbf{r}$ and transformation matrix $\mathbf{A}$ are functions of time. The differentiation rules of Section 2.5 can be used to obtain the time derivative of $\mathbf{r}^P$ as

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P \tag{9.2.33}$$

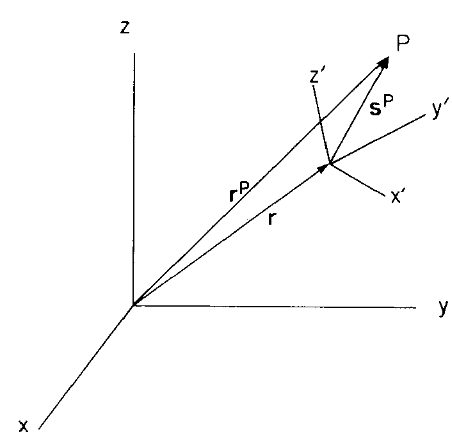

Figure 9.2.11 Point P fixed in an x'-y'-z' reference frame.

Using Eq. 9.2.15, this result may be rewritten as

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{A}^T\mathbf{s}^P \tag{9.2.34}$$

To interpret terms in Eq. 9.2.34, it is helpful to derive an identity that involves the transformation matrix $\mathbf{A}$. Differentiating both sides of $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ with respect to time,

$$\dot{\mathbf{A}}\mathbf{A}^T + \mathbf{A}\dot{\mathbf{A}}^T = \mathbf{0}$$

Thus,

$$(\dot{\mathbf{A}}\mathbf{A}^T)^T = \mathbf{A}\dot{\mathbf{A}}^T = -\dot{\mathbf{A}}\mathbf{A}^T \tag{9.2.35}$$

and $\dot{\mathbf{A}}\mathbf{A}^T$ is skew-symmetric. As noted following the definition of Eq. 9.1.21 in Section 9.1, there exists a vector $\boldsymbol{\omega}$, called the *angular velocity* of the x'-y'-z' frame, such that

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{A}}\mathbf{A}^T \tag{9.2.36}$$

This is the definition of $\boldsymbol{\omega}$ in terms of the transformation matrix $\mathbf{A}$ and its time derivatives. Using Eq. 9.2.36 in Eq. 9.2.34 yields the *velocity equation*

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{s}^P \tag{9.2.37}$$

If $\mathbf{r} = \dot{\mathbf{r}} = \mathbf{0}$, this equation defines the geometry shown in Fig. 9.2.12. In geometric vector notation, this is the familiar relationship

$$\dot{\vec{r}}^P = \dot{\vec{r}} + \vec{\omega} \times \vec{s}^P$$

where $\vec{\omega}$ is the angular velocity of the x'-y'-z' frame, relative to the x-y-z reference frame. Taking the vector product in the x'-y'-z' frame and transforming to the x-y-z frame yields (Prob. 9.2.7)

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P \tag{9.2.38}$$

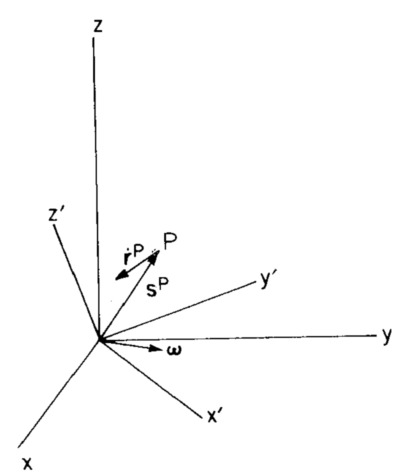

Figure 9.2.12 Angular velocity of x'-y'-z' reference frame.

Example 9.2.4: The transformation matrix $\mathbf{A}$ of Example 9.2.2 is defined in Eq. 9.2.17 as a function of the angle ϕ. For this matrix, Eq. 9.2.36 yields

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{A}}\mathbf{A}^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi}$$

Thus,

$$\boldsymbol{\omega} = [0, 0, \dot{\phi}]^T$$

which agrees with the physical concept of angular velocity about a fixed axis. Note also that $\boldsymbol{\omega}' = \mathbf{A}^T\boldsymbol{\omega} = \boldsymbol{\omega}$ in this case, which is expected since the z and z' axes coincide.

Multiplying both sides of Eq. 9.2.36 on the right by $\mathbf{A}$ yields the useful relationship

$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} \tag{9.2.39}$$

Furthermore, using Eq. 9.2.21 with $\mathbf{s} = \boldsymbol{\omega}$, Eq. 9.2.39 yields

$$\left.\begin{aligned} \dot{\mathbf{A}} &= \mathbf{A}\tilde{\boldsymbol{\omega}}' \\ \tilde{\boldsymbol{\omega}}' &= \mathbf{A}^T\dot{\mathbf{A}} \end{aligned}\right\} \tag{9.2.40}$$

Equations 9.2.39 and 9.2.40 provide frequently used relationships between the time derivative of the transformation matrix $\mathbf{A}$ and the angular velocity $\boldsymbol{\omega}$ of the x'-y'-z' frame.

Equation 9.2.33 may be differentiated with respect to time to obtain the *acceleration equation*

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{A}}\mathbf{s}'^P \tag{9.2.41}$$

Differentiating Eq. 9.2.39, the following relationships for the second time derivative of the transformation matrix $\mathbf{A}$ are obtained (Prob. 9.2.8):

$$\begin{aligned} \ddot{\mathbf{A}} &= \dot{\tilde{\boldsymbol{\omega}}}\mathbf{A} + \tilde{\boldsymbol{\omega}}\dot{\mathbf{A}} \\ &= \dot{\tilde{\boldsymbol{\omega}}}\mathbf{A} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{A} \end{aligned} \tag{9.2.42}$$

Equation 9.2.21 may be applied to $\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega}'$ and $\dot{\boldsymbol{\omega}} = \mathbf{A}\dot{\boldsymbol{\omega}}'$ to rewrite Eq. 9.2.42 in terms of $\boldsymbol{\omega}'$, yielding

$$\ddot{\mathbf{A}} = \mathbf{A}\dot{\tilde{\boldsymbol{\omega}}}' + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}' \tag{9.2.43}$$

Example 9.2.5: Using the results of Example 9.2.4 and Eq. 9.2.38, the velocity of a point P that is defined by $\mathbf{s}'^P = [1, 1, 1]$ in the x'-y'-z' frame is

$$\dot{\mathbf{r}}^P = \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P = [-\cos\phi - \sin\phi,\ \cos\phi - \sin\phi,\ 0]^T\dot{\phi}$$

where $\dot{\mathbf{r}} = \mathbf{0}$. The acceleration of point P, from Eqs. 9.2.41 and 9.2.42, is

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{A}}\mathbf{s}'^P = \begin{bmatrix} -\cos\phi - \sin\phi \\ \cos\phi - \sin\phi \\ 0 \end{bmatrix} \ddot{\phi} + \begin{bmatrix} \sin\phi - \cos\phi \\ -\cos\phi - \sin\phi \\ 0 \end{bmatrix} \dot{\phi}^2$$

The derivative relations derived here serve as the foundation for the kinematic analysis of rigid bodies in space. The reader is encouraged to become proficient in their use, in preparation for kinematic analysis in this chapter and dynamic analysis in Chapter 11 (Probs. 9.2.9 and 9.2.10). As an aid in the use of the relations derived, a summary of key formulas is provided at the end of the chapter.

Virtual Displacements and Rotations A *virtual displacement* $\delta\mathbf{r}^P$ of point P is defined as an *infinitesimal displacement* (see Section 6.1), or small displacement with time held fixed. If point P is attached to a rigid body, or equivalently if it is fixed in an x'-y'-z' frame, then a virtual displacement of point P may be represented by a virtual displacement of the origin of the x'-y'-z' frame and a variation in its direction cosines. The vector $\mathbf{r}^P$ in Eq. 9.2.32 depends on $\mathbf{r}$ and $\mathbf{A}$, so if the frame is perturbed slightly, $\mathbf{r}$ changes to $\mathbf{r} + \delta\mathbf{r}$ and $\mathbf{A}$ changes to $\mathbf{A} + \delta\mathbf{A} + \mathbf{0}(\delta\mathbf{A}^2)$.† Since the direction cosine matrix must be orthogonal, it is required that

$$(\mathbf{A} + \delta\mathbf{A} + \mathbf{0}(\delta\mathbf{A}^2))(\mathbf{A} + \delta\mathbf{A} + \mathbf{0}(\delta\mathbf{A}^2))^T$$
$$= \mathbf{A}\mathbf{A}^T + \mathbf{A}\,\delta\mathbf{A}^T + \delta\mathbf{A}\mathbf{A}^T + \delta\mathbf{A}\,\delta\mathbf{A}^T + \mathbf{0}(\delta\mathbf{A}^2) = \mathbf{I}$$

Since $\mathbf{A}$ is orthogonal, this is

$$\mathbf{A}\,\delta\mathbf{A}^T + \delta\mathbf{A}\mathbf{A}^T = \mathbf{0}(\delta\mathbf{A}^2)$$

Equating linear terms in $\delta\mathbf{A}$ on both sides (called *infinitesimals*), this is

$$\mathbf{A}\,\delta\mathbf{A}^T + \delta\mathbf{A}\mathbf{A}^T = \mathbf{0} \tag{9.2.44}$$

This result may also be obtained by taking the differential of both sides of the identity $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ (Prob. 9.2.11).

Using Eq. 9.2.44, observe that

$$(\delta\mathbf{A}\mathbf{A}^T)^T = \mathbf{A}\,\delta\mathbf{A}^T = -\delta\mathbf{A}\mathbf{A}^T \tag{9.2.45}$$

so the matrix $\delta\mathbf{A}\mathbf{A}^T$ is skew-symmetric and can be represented by

$$\delta\mathbf{A}\mathbf{A}^T = \delta\tilde{\boldsymbol{\pi}} \tag{9.2.46}$$

---

† Quantities of magnitude $0(\alpha^n)$ approach zero as α^n; that is, there is a constant c such that $|0(\alpha^n)|/|\alpha^n| \leq c$. Here, they are higher-order terms in a Taylor expansion.

where the vector $\delta\boldsymbol{\pi}$ depends on both the components of $\mathbf{A}$ and $\delta\mathbf{A}$ (Prob. 9.2.12). Multiplying Eq. 9.2.46 on the right by $\mathbf{A}$,

$$\delta\mathbf{A} = \delta\tilde{\boldsymbol{\pi}}\mathbf{A} \tag{9.2.47}$$

To use the foregoing results, take the variation of Eq. 9.2.32 to obtain (Prob. 9.2.13)

$$\delta\mathbf{r}^P = \delta\mathbf{r} + \delta\mathbf{A}\mathbf{s}'^P \tag{9.2.48}$$

Using Eq. 9.2.47, this is

$$\begin{aligned}\delta\mathbf{r}^P &= \delta\mathbf{r} + \delta\tilde{\boldsymbol{\pi}}\mathbf{A}\mathbf{s}'^P \\ &= \delta\mathbf{r} + \delta\tilde{\boldsymbol{\pi}}\mathbf{s}^P\end{aligned} \tag{9.2.49}$$

Thus, $\delta\boldsymbol{\pi}$ plays the role of a rotation, called a *virtual rotation* of the x'-y'-z' frame relative to the x-y-z reference frame, with components in the latter frame. Using $\delta\boldsymbol{\pi} = \mathbf{A}\,\delta\boldsymbol{\pi}'$ and Eq. 9.2.21 in Eq. 9.2.47,

$$\begin{aligned}\delta\mathbf{A} &= \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}' \\ \delta\tilde{\boldsymbol{\pi}}' &= \mathbf{A}^T\,\delta\mathbf{A}\end{aligned} \tag{9.2.50}$$

Using these relations in Eq. 9.2.48,

$$\delta\mathbf{r}^P = \delta\mathbf{r} + \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}'\mathbf{s}'^P \tag{9.2.51}$$

which verifies that virtual rotation is a vector quantity. This fact is very important. Even though large rotation cannot be represented as a vector quantity (see Section 9.1), virtual rotation and angular velocity are vector quantities.

For a vector $\mathbf{s}'$ that is fixed in the x'-y'-z' frame, $\mathbf{s}'$ is constant and

$$\begin{aligned}\delta\mathbf{s} &= \delta\mathbf{A}\mathbf{s}' \\ &= \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}'\mathbf{s}'\end{aligned} \tag{9.2.52}$$

Using Eq. 9.1.25, this may be written as

$$\delta\mathbf{s} = -\mathbf{A}\tilde{\mathbf{s}}'\,\delta\boldsymbol{\pi}' \tag{9.2.53}$$

The δ notation associated with virtual displacements and virtual rotations may be interpreted as a *partial differential operator* of calculus. This is the case since vectors in the x'-y'-z' frame are functions of parameters that are used to define the position and orientation of the frame.

If a vector $\mathbf{h}(\mathbf{q})$ depends on a variable $\mathbf{q}$ in a differentiable way, $\delta\mathbf{h} = \mathbf{h}_\mathbf{q}\,\delta\mathbf{q}$. Even before the variable $\mathbf{q}$ is defined, the differential $\delta\mathbf{h}$ can be defined and used in analysis, since the linear differential operator obeys the same rules of calculus as the partial derivative operator, with time held fixed. The differential of the sum of two vectors is the sum of their differentials:

$$\delta(\mathbf{g} + \mathbf{h}) = \delta\mathbf{g} + \delta\mathbf{h} \tag{9.2.54}$$

where the differential δ operates on the quantity to its immediate right. The product rule for differentials is analogous to Eq. 2.5.13; that is,

$$\delta(\mathbf{g}^T\mathbf{h}) = \mathbf{h}^T\,\delta\mathbf{g} + \mathbf{g}^T\,\delta\mathbf{h} \tag{9.2.55}$$

Since the vector product $\tilde{}$ operator is linear (see Eq. 9.1.32), the δ and $\tilde{}$ operators may be interchanged in order of application. Thus,

$$\begin{aligned}\delta(\tilde{\mathbf{g}}\mathbf{h}) &= \delta\tilde{\mathbf{g}}\mathbf{h} + \tilde{\mathbf{g}}\,\delta\mathbf{h} \\ &= -\tilde{\mathbf{h}}\,\delta\mathbf{g} + \tilde{\mathbf{g}}\,\delta\mathbf{h}\end{aligned} \tag{9.2.56}$$

Finally, the differential of a triple product may be expanded, using the product rule differentials, to obtain

$$\begin{aligned}\delta(\mathbf{g}^T\mathbf{B}\mathbf{h}) &= \delta\mathbf{g}^T\mathbf{B}\mathbf{h} + \mathbf{g}^T\,\delta\mathbf{B}\mathbf{h} + \mathbf{g}^T\mathbf{B}\,\delta\mathbf{h} \\ &= \mathbf{h}^T\mathbf{B}^T\,\delta\mathbf{g} + \mathbf{g}^T\,\delta\mathbf{B}\mathbf{h} + \mathbf{g}^T\mathbf{B}\,\delta\mathbf{h}\end{aligned} \tag{9.2.57}$$

where $\delta\mathbf{g}^T\mathbf{B}\mathbf{h} = \mathbf{h}^T\mathbf{B}^T\,\delta\mathbf{g}$, since the transpose of a scalar is the same scalar.

These and many additional identities that involve virtual displacements and rotations, or variations in position and orientation of reference frames, can be used to advantage in the analysis of the dynamics of mechanical systems. Of particular importance in analytical kinematics and dynamics is a powerful coupling between differential calculus and vector analysis. If the abundant manipulation rules of calculus and vector analysis are used to advantage, geometrically clear results can be obtained and understood.

Example 9.2.6: To illustrate the power of combining the vector properties of virtual displacements and rotations and differential calculus, consider a commonly used constraint between two vectors $\mathbf{s}'_i$ and $\mathbf{s}'_j$ that are fixed in their respective x'_i-y'_i-z'_i and x'_j-y'_j-z'_j frames; that is, they are orthogonal:

$$\Phi = \mathbf{s}_i^T\mathbf{s}_j = 0 \tag{9.2.58}$$

Taking the differential of both sides of this equation,

$$\delta\Phi = \mathbf{s}_j^T\,\delta\mathbf{s}_i + \mathbf{s}_i^T\,\delta\mathbf{s}_j = 0 \tag{9.2.59}$$

Substituting for the differentials of the vectors from Eq. 9.2.53, the differential of Eq. 9.2.59 is obtained in terms of virtual rotations as

$$\delta\Phi = -(\mathbf{s}_j'^T\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\,\delta\boldsymbol{\pi}_i' + \mathbf{s}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\,\delta\boldsymbol{\pi}_j') = 0 \tag{9.2.60}$$

This and related easily derived differential constraint expressions, or linearized constraints, are of great value in formulating the equations of motion of constrained dynamic systems.

Example 9.2.7: As a second illustration of the power of vector properties of virtual rotations, consider the reference frames shown in Fig. 9.2.10 that are constrained to permit only relative rotation about their common $\mathbf{h}_i$ and $\mathbf{h}_j$ unit vectors. Since virtual rotations $\delta\boldsymbol{\pi}_i$ and $\delta\boldsymbol{\pi}_j$ are vector quantities, the infinitesimal relative rotation is

$$\delta\theta\mathbf{h}_i = \delta\boldsymbol{\pi}_j - \delta\boldsymbol{\pi}_i$$

Taking the scalar product of both sides with $\mathbf{h}_i$ and representing vectors in their respective reference frame components,

$$\begin{aligned}\delta\theta &= \mathbf{h}_i^T(\delta\boldsymbol{\pi}_j - \delta\boldsymbol{\pi}_i) \\ &= \mathbf{h}_i'^T(\mathbf{A}_i^T\mathbf{A}_j\,\delta\boldsymbol{\pi}_j' - \delta\boldsymbol{\pi}_i')\end{aligned} \quad \textbf{(9.2.61)}$$

Similarly, the relative angular velocity is

$$\dot{\theta} = \mathbf{h}_i'^T(\mathbf{A}_i^T\mathbf{A}_j\boldsymbol{\omega}_j' - \boldsymbol{\omega}_i') \quad \textbf{(9.2.62)}$$

Example 9.2.8: To further illustrate the power of coupling differential and vector calculus, consider a vector of generalized coordinates, $\mathbf{q}$, that defines the orientation of bodies that make up a mechanical system. Such coordinates are defined in Section 9.3, where relationships between variations in generalized coordinates and virtual rotations are established in the form

$$\begin{aligned}\delta\boldsymbol{\pi}_i' &= \mathbf{B}_i\,\delta\mathbf{q} \\ \delta\boldsymbol{\pi}_j' &= \mathbf{B}_j\,\delta\mathbf{q}\end{aligned} \quad \textbf{(9.2.63)}$$

where $\delta\mathbf{q}$ is a variation in the generalized coordinates, and $\mathbf{B}_i$ and $\mathbf{B}_j$ are matrices that depend on the generalized coordinate vector $\mathbf{q}$. Once the relationship of Eq. 9.2.63 is established, it may be substituted into Eq. 9.2.60, which was derived using vector relationships, to obtain the differential of the function Φ of Eq. 9.2.58 as

$$\delta\Phi = -(\mathbf{s}_j'^T\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\mathbf{B}_i + \mathbf{s}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\mathbf{B}_j)\,\delta\mathbf{q} \quad \textbf{(9.2.64)}$$

Since the coefficient of $\delta\mathbf{q}$ in the total differential is $\partial\Phi/\partial\mathbf{q}$,

$$\frac{\partial\Phi}{\partial\mathbf{q}} \equiv -(\mathbf{s}_j'^T\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\mathbf{B}_i + \mathbf{s}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\mathbf{B}_j) \quad \textbf{(9.2.65)}$$

9.3 EULER PARAMETER ORIENTATION GENERALIZED COORDINATES

As shown in Section 9.1, the position and orientation of a rigid body in space can be defined by locating the origin of a body-fixed x'-y'-z' frame and specifying an orthogonal direction cosine matrix that defines orientation of the x'-y'-z' frame. To characterize the orientation of the body analytically, a set of orientation generalized coordinates must be defined.

Euler's Theorem and Rotation Relations An important result that is used in defining orientation generalized coordinates is the following:

> ***Euler's Theorem:*** If the origins of two right-hand Cartesian reference frames coincide, then they may be brought into coincidence by a single rotation about some axis.

This Theorem is proved in References 4 and 9, using the orthogonality of the transformation matrix **A** between the reference frames and the eigenvalue theory of matrices.

A physical interpretation of Euler's theorem is given in Fig. 9.3.1, where the orthogonal projections of **i**-**f**, **j**-**g**, and **k**-**h** pairs onto the axis of rotation are equal. The axis about which rotation is to take place is called the *orientation axis* and is defined by a unit vector **u**. The angle χ of rotation, measured in the plane perpendicular to **u**, is positive counterclockwise (i.e., using the right-hand sign convention). Note that if the vector **u** had been taken in the opposite sense along the orientation axis then the angle χ shown in Fig. 9.3.1 would be replaced by $2\pi - \chi$.

Euler's theorem states that any orientation of a body can be achieved by a single rotation from a reference orientation about some axis. It is natural, therefore, to seek a representation of the direction cosine transformation matrix **A** in terms of parameters of this rotation, that is, the angle of rotation and the components of the unit vector **u** along the orientation axis.

Figure 9.3.2 shows the orientation axis, the x'-y'-z' frame, the x-y-z reference frame, an arbitrary vector **s**, and projections (or components) of **s** in

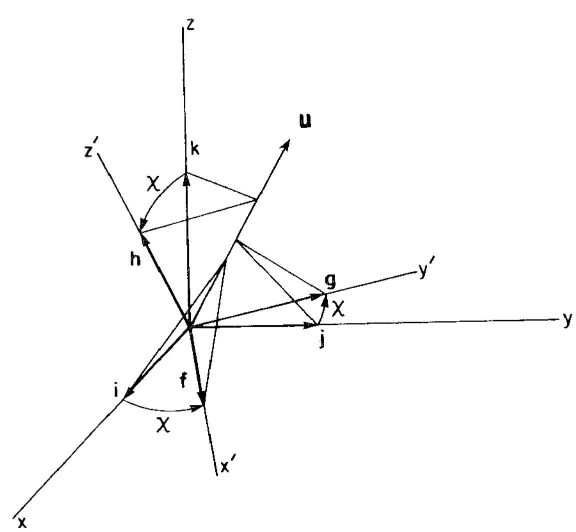

Figure 9.3.1 Euler rotation of reference frames.

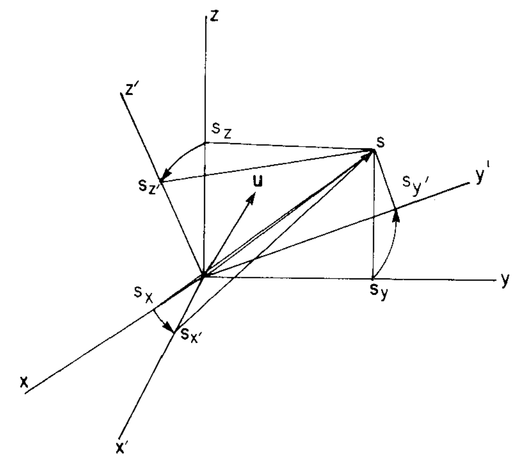

Figure 9.3.2 Vector **s** in x'-y'-z' and x-y-z frames.

each frame. The objective is to obtain relationships between components **s** and **s**′ ~~of the vector~~ in the x-y-z and x'-y'-z' frames, respectively. The direction cosine matrix **A** that provides the desired relation between the x'-y'-z' and x-y-z components of a vector is obtained if unit vectors **f**, **g**, and **h** along the x', y', and z' axes may be written in terms of χ and **u**; that is, Eq. 9.2.13 yields $\mathbf{A} = [\mathbf{f}, \mathbf{g}, \mathbf{h}]$.

Consider first the relation between **k** and **h**. Since **h** is obtained in the x-y-z frame by a rotation of **k** about **u**, its tip lies in the plane that is perpendicular to **u** and contains the tip of **k**, as shown in Fig. 9.3.3. The vector $\tilde{\mathbf{u}}\mathbf{k}$ lies in a parallel plane and has length equal to the radius of the circle that is formed by intersecting the plane perpendicular to **u** and the cone formed by rotating **k** about the **u** axis. The center N of the circle is located by the vector $(\mathbf{u}^T\mathbf{k})\mathbf{u}$.

A vector **a** is constructed to pass through the tip of **h** and to be perpendicular to the vector $\mathbf{k} - (\mathbf{u}^T\mathbf{k})\mathbf{u}$ from point N to the tip of **k**, as shown in Fig. 9.3.3. The vector $\mathbf{k} - (\mathbf{u}^T\mathbf{k})\mathbf{u}$ lies in the plane formed by **u** and **k**, so it is perpendicular to $\tilde{\mathbf{u}}\mathbf{k}$. Thus, vectors **a** and $\tilde{\mathbf{u}}\mathbf{k}$ are parallel. Furthermore, the lengths of vectors $\mathbf{h} - (\mathbf{u}^T\mathbf{k})\mathbf{u}$ and $\tilde{\mathbf{u}}\mathbf{k}$ are equal, since they are radii of the same circle. Thus, by simple trigonometry,

$$\mathbf{a} = \tilde{\mathbf{u}}\mathbf{k} \sin \chi$$

Similarly, the vector **b** in Fig. 9.3.3 is

$$\mathbf{b} = [\mathbf{k} - (\mathbf{u}^T\mathbf{k})\mathbf{u}] \cos \chi$$

Thus,

$$\begin{aligned}\mathbf{h} &= (\mathbf{u}^T\mathbf{k})\mathbf{u} + \mathbf{b} + \mathbf{a} \\ &= (\mathbf{u}^T\mathbf{k})\mathbf{u} + [\mathbf{k} - (\mathbf{u}^T\mathbf{k})\mathbf{u}] \cos \chi + \tilde{\mathbf{u}}\mathbf{k} \sin \chi\end{aligned}$$

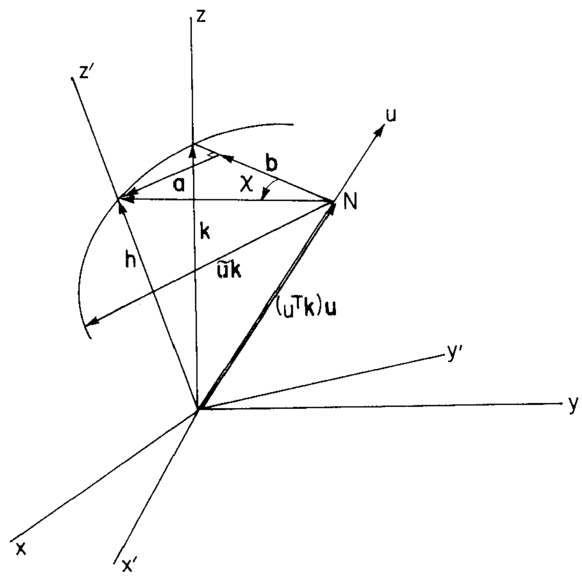

Figure 9.3.3 Relation between **k** and **h**.

A rearrangement of terms leads to

$$\mathbf{h} = \mathbf{k}\cos\chi + (\mathbf{u}^T\mathbf{k})\mathbf{u}(1-\cos\chi) + (\tilde{\mathbf{u}}\mathbf{k})\sin\chi \tag{9.3.1}$$

Euler Parameters By means of the trigonometric identities

$$1-\cos\chi = 2\sin^2\frac{\chi}{2}$$

$$\sin\chi = 2\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$

$$\cos\chi = 2\cos^2\frac{\chi}{2} - 1$$

and the set of four *Euler parameters*, defined as

$$e_0 \equiv \cos\frac{\chi}{2}$$

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \equiv \mathbf{u}\sin\frac{\chi}{2} \tag{9.3.2}$$

Eq. 9.3.1 can be written in the form

$$\mathbf{h} = \mathbf{k}\left(2\cos^2\frac{\chi}{2} - 1\right) + 2\mathbf{u}(\mathbf{u}^T\mathbf{k})\sin^2\frac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{k}\sin\frac{\chi}{2}\cos\frac{\chi}{2} \tag{9.3.3}$$

and manipulated to yield the desired result:

$$\mathbf{h} = [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{k} \tag{9.3.4}$$

Since the geometric relations between **f**-**i** and **g**-**j** are identical to that between **h**-**k**, the relation of Eq. 9.3.4 is valid for these pairs (Prob. 9.3.1); that is,

$$\begin{aligned} \mathbf{f} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{i} \\ \mathbf{g} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{j} \end{aligned} \tag{9.3.5}$$

From Eq. 9.2.13, using Eqs. 9.3.4 and 9.3.5 and the fact that $[\mathbf{i}, \mathbf{j}, \mathbf{k}] = \mathbf{I}$,

$$\begin{aligned} \mathbf{A} &= [\mathbf{f}, \mathbf{g}, \mathbf{h}] \\ &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})][\mathbf{i}, \mathbf{j}, \mathbf{k}] \\ &= (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}) \end{aligned} \tag{9.3.6}$$

More explicitly, the direction cosine matrix, written in terms of Euler parameters, is

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} \tag{9.3.7}$$

Example 9.3.1: Let the x'-y'-z' frame of Fig. 9.3.1 have its origin in common with that of the x-y-z reference frame and be rotated about a fixed vector $\mathbf{u} = (1/\sqrt{3})\ [1, 1, 1]^T$ by angle ϕ. The Euler parameters for orientation of the x'-y'-z' frame are thus $e_0 = \cos(\phi/2)$ and $\mathbf{e} = \mathbf{u}\sin(\phi/2)$. The direction cosine transformation matrix, from Eq. 9.3.7, is

$$\mathbf{A} = 2\begin{bmatrix} c^2 + \dfrac{s^2}{3} - \frac{1}{2} & \dfrac{s^2}{3} - \dfrac{sc}{\sqrt{3}} & \dfrac{s^2}{3} + \dfrac{sc}{\sqrt{3}} \\ \dfrac{s^2}{3} + \dfrac{sc}{\sqrt{3}} & c^2 + \dfrac{s^2}{3} - \frac{1}{2} & \dfrac{s^2}{3} - \dfrac{sc}{\sqrt{3}} \\ \dfrac{s^2}{3} - \dfrac{sc}{\sqrt{3}} & \dfrac{s^2}{3} + \dfrac{sc}{\sqrt{3}} & c^2 + \dfrac{s^2}{3} - \frac{1}{2} \end{bmatrix}$$

where $c \equiv \cos\phi/2$ and $s \equiv \sin\phi/2$. Geometrically, a vector $\mathbf{s}'$ that is fixed in and emanates from the origin of the x'-y'-z' frame sweeps out a cone in the x-y-z frame, with axis of revolution $\mathbf{u}$ (Prob. 9.3.2).

For convenience, denote the 4×1 column vector of Euler parameters as

$$\mathbf{p} = [e_0, \mathbf{e}^T]^T = [e_0, e_1, e_2, e_3]^T \tag{9.3.8}$$

The Euler parameters are not independent, since

$$e_0^2 + \mathbf{e}^T\mathbf{e} = \cos^2\frac{\chi}{2} + \mathbf{u}^T\mathbf{u}\sin^2\frac{\chi}{2} = 1$$

That is, they must satisfy the *Euler parameter normalization constraint*

$$\mathbf{p}^T\mathbf{p} = e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \tag{9.3.9}$$

It is possible to derive explicit formulas for the Euler parameters in terms of elements of the direction cosine matrix $\mathbf{A}$ from Eq. 9.3.7. Assume that nine direction cosines of a transformation matrix $\mathbf{A}$ are given. The *trace of* $\mathbf{A}$, denoted by $\mathrm{tr}\,\mathbf{A}$, is defined as

$$\mathrm{tr}\,\mathbf{A} = a_{11} + a_{22} + a_{33} \tag{9.3.10}$$

From Eq. 9.3.7,

$$\mathrm{tr}\,\mathbf{A} = 2(3e_0^2 + e_1^2 + e_2^2 + e_3^2) - 3 = 4e_0^2 - 1 \tag{9.3.11}$$

Thus,

$$e_0^2 = \frac{\mathrm{tr}\,\mathbf{A} + 1}{4} \tag{9.3.12}$$

Substitution of Eq. 9.3.12 into the diagonal elements of $\mathbf{A}$ in Eq. 9.3.7 results in

$$\begin{aligned} a_{11} &= 2(e_0^2 + e_1^2) - 1 \\ &= 2\left(\frac{\mathrm{tr}\,\mathbf{A} + 1}{4} + e_1^2\right) - 1 \end{aligned}$$

Thus,

$$e_1^2 = \frac{1 + 2a_{11} - \mathrm{tr}\,\mathbf{A}}{4} \tag{9.3.13}$$

Similarly,

$$e_2^2 = \frac{1 + 2a_{22} - \mathrm{tr}\,\mathbf{A}}{4} \tag{9.3.14}$$

$$e_3^2 = \frac{1 + 2a_{33} - \mathrm{tr}\,\mathbf{A}}{4} \tag{9.3.15}$$

It is interesting and computationally important to note that Eqs. 9.3.13 to 9.3.15 determine the magnitudes of the Euler parameters in terms of only the diagonal elements of the direction cosine matrix $\mathbf{A}$. To find the algebraic signs of the Euler parameters, off-diagonal terms of $\mathbf{A}$ must be used.

Equation 9.3.9 indicates that at least one Euler parameter must be nonzero. Consider first the case $e_0 \neq 0$. The sign of e_0 may be selected as positive or negative. From Eq. 9.3.7, subtracting symmetrically placed off-diagonal terms of $\mathbf{A}$ in Eq. 9.3.7 yields

$$\begin{aligned} a_{32} - a_{23} &= 4e_0 e_1 \\ a_{13} - a_{31} &= 4e_0 e_2 \\ a_{21} - a_{12} &= 4e_0 e_3 \end{aligned}$$

or

$$\begin{aligned} e_1 &= \frac{a_{32} - a_{23}}{4e_0} \\ e_2 &= \frac{a_{13} - a_{31}}{4e_0} \\ e_3 &= \frac{a_{21} - a_{12}}{4e_0} \end{aligned} \tag{9.3.16}$$

If e_0 calculated from Eq. 9.3.12 is nonzero and its sign is selected, then Eq. 9.3.16 can be used to determine e_1, e_2, and e_3. Suppose that, for the selected sign of e_0 and the computed values of e_1, e_2, and e_3, the angle of rotation and the axis of rotation are determined to be χ and $\mathbf{u}$, respectively. If the sign of e_0 is inverted, Eq. 9.3.16 shows that the signs of e_1, e_2, and e_3 are inverted. Changing the signs of all four parameters does not influence the transformation matrix, since variable elements in the matrix are quadratic in these variables (see Eq. 9.3.7).

Either sign may be selected for $e_0 \neq 0$ and still define the same physical orientation of the body. To see this, denote by $a > 0$ the magnitude of e_0 from Eq. 9.3.12 and denote $\cos(\chi^+/2) = e_0^+ = a$ and $\cos(\chi^-/2) = e_0^- = -a$, where $0 < \chi^+/2 \leq \pi/2$ and $\pi/2 \leq \chi^-/2 < \pi$. Since the cosine function is antisymmetric about $\pi/2$, that is, $\cos(\pi/2 + \theta) = -\cos(\pi/2 - \theta)$, and $\cos(\chi^+/2) = -\cos(\chi^-/2)$, there is a θ such that $\chi^+/2 = \pi/2 + \theta$ and $\chi^-/2 = \pi/2 - \theta$. Eliminating θ,

$$\frac{\chi^+}{2} + \frac{\chi^-}{2} = \pi$$

or

$$\chi^- = 2\pi - \chi^+$$

Thus, $\sin(\chi^-/2) = \sin(\chi^+/2)$. From Eq. 9.3.16, $\mathbf{e}^- = -\mathbf{e}^+$. The unit vector $\mathbf{u}$ in Eq. 9.3.2 defines the positive sense along the orientation axis for implementation of the rotation χ by the right-hand rule. Using the above and Eq. 9.3.2, $\mathbf{u}^- = -\mathbf{u}^+$, as shown in Fig. 9.3.4. Thus, if χ^- is selected, a rotation of $2\pi - \chi^+$ is implemented counterclockwise about the $\mathbf{u}^-$ axis, or equivalently clockwise about the $\mathbf{u}^+$ axis, as shown in Fig. 9.3.4, and precisely the same orientation is achieved.

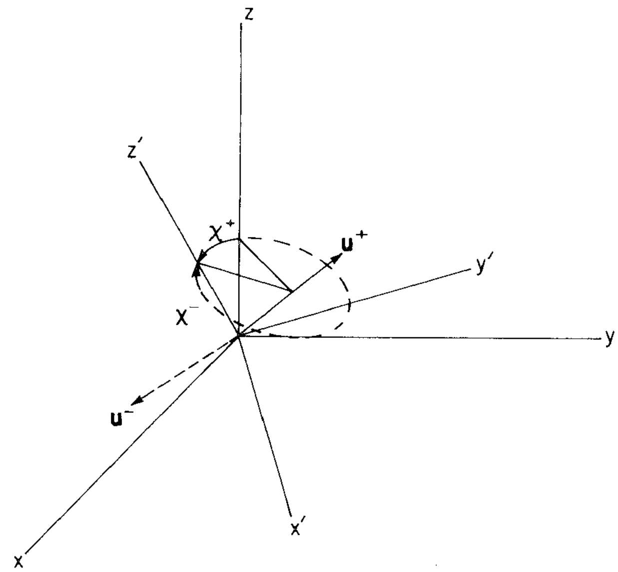

Figure 9.3.4 Alternative values of Euler parameters.

Consider next the case in which the value of e_0 calculated from Eq. 9.3.12 is zero. Then, Eq. 9.3.2 shows that $\chi = k\pi$, $k = \pm 1, \pm 3, \pm 5, \ldots$. Therefore, the sign of χ is immaterial. To find the algebraic signs of e_1, e_2, and e_3, symmetrically placed off-diagonal terms of the matrix **A** in Eq. 9.3.7 may be added to obtain

$$\begin{aligned} a_{21} + a_{12} &= 4e_1e_2 \\ a_{31} + a_{13} &= 4e_1e_3 \\ a_{32} + a_{23} &= 4e_2e_3 \end{aligned} \tag{9.3.17}$$

Since $e_0 = 0$, at least one of the three Euler parameters e_1, e_2, or e_3 from Eqs. 9.3.13 to 9.3.15 must be nonzero, and its sign may be selected as positive or negative. Then Eq. 9.3.17 can be used to determine the signs of the other two parameters.

Even though two distinct values of Euler parameters, $\mathbf{p}^+$ and $\mathbf{p}^-$ with $\mathbf{p}^- = -\mathbf{p}^+$, define the same orientation of the x'-y'-z' frame, the mapping from $\mathbf{p}$ to $\mathbf{A}(\mathbf{p})$ and hence $\mathbf{s} = \mathbf{A}(\mathbf{p})\mathbf{s}'$ is *locally one-to-one*; that is, if $|p_i^1 - p_i^2| < \delta$, $i = 1, 2, 3, 4,$ for a small δ, and if $\mathbf{p}^1 \neq \mathbf{p}^2$, then $\mathbf{A}(\mathbf{p}^1) \neq \mathbf{A}(\mathbf{p}^2)$. The two alternate values of $\mathbf{p}$ are never close together, because

$$(\mathbf{p}^+ - \mathbf{p}^-)^T(\mathbf{p}^+ - \mathbf{p}^-) = 4\mathbf{p}^{+T}\mathbf{p}^+ = 4$$

Thus, for neighboring orientations that are specified by $\mathbf{A}_1$ and $\mathbf{A}_2$, the associated Euler parameters $\mathbf{p}^1$ and $\mathbf{p}^2$ can be selected to be close to one another. This observation is important in kinematic and dynamic analysis, since the objective is to solve for $\mathbf{p}(t)$ to define the time history of the orientation of a body. Thus, continuity of $\mathbf{p}(t)$ may be used to solve for a unique value of $\mathbf{p}(t)$ once the sign of $\mathbf{p}(0)$ is selected.

Properties of Euler Parameters Important relations between Euler parameters, their time derivatives, and their variations and the associated transformation matrices, angular velocities, and virtual rotations are next derived. Derivation of some of the identities is given in the text. To avoid lengthy expansions, references are given to problems at the end of this chapter where calculations are outlined and the reader is invited to fill in the details.

Two 3×4 matrices $\mathbf{E}$ and $\mathbf{G}$ are first defined as

$$\mathbf{E} \equiv [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \tag{9.3.18}$$

$$\mathbf{G} \equiv [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix} \tag{9.3.19}$$

Observe that each row of $\mathbf{E}$ is orthogonal to $\mathbf{p}$; that is,

$$\begin{aligned} \mathbf{E}\mathbf{p} &= [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} \\ &= -e_0\mathbf{e} + \tilde{\mathbf{e}}\mathbf{e} + e_0\mathbf{e} = \mathbf{0} \end{aligned} \tag{9.3.20}$$

where Eq. 9.1.26 has been used. Similarly (Prob. 9.3.4),

$$\mathbf{G}\mathbf{p} = \mathbf{0} \tag{9.3.21}$$

A direct calculation shows that rows of $\mathbf{E}$ (Prob. 9.3.5) and rows of $\mathbf{G}$ (Prob. 9.3.6) are orthonormal; that is,

$$\mathbf{E}\mathbf{E}^T = \mathbf{G}\mathbf{G}^T = \mathbf{I} \tag{9.3.22}$$

However, $\mathbf{E}^T\mathbf{E}$ is a 4×4 matrix of the form

$$\mathbf{E}^T\mathbf{E} = \begin{bmatrix} 1 - e_0^2 & -e_0\mathbf{e}^T \\ -e_0\mathbf{e} & -\mathbf{e}\mathbf{e}^T + \mathbf{I}_3 \end{bmatrix} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^T \tag{9.3.23}$$

where the subscript is used to emphasize that $\mathbf{I}_4$ is the 4×4 identity matrix. Similarly,

$$\mathbf{G}^T\mathbf{G} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^T \tag{9.3.24}$$

The key relationship involving these matrices is found by evaluating the matrix product:

$$\begin{aligned}\mathbf{E}\mathbf{G}^T &= [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} -\mathbf{e}^T \\ \tilde{\mathbf{e}} + e_0\mathbf{I}\end{bmatrix} \\ &= \mathbf{e}\mathbf{e}^T + (\tilde{\mathbf{e}} + e_0\mathbf{I})(\tilde{\mathbf{e}} + e_0\mathbf{I}) \\ &= (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})\end{aligned} \tag{9.3.25}$$

Comparing Eq. 9.3.25 with the transformation matrix $\mathbf{A}$ of Eq. 9.3.6 reveals that

$$\mathbf{A} = \mathbf{E}\mathbf{G}^T \tag{9.3.26}$$

Thus the transformation matrix $\mathbf{A}$, with its quadratic terms in Euler parameters, is the product of two matrices whose terms are linear in Euler parameters.

Time Derivatives of Euler Parameters The first time derivative of Eq. 9.3.9 yields (Prob. 9.3.7)

$$\mathbf{p}^T\dot{\mathbf{p}} = \dot{\mathbf{p}}^T\mathbf{p} = 0 \tag{9.3.27}$$

Similarly, the first time derivatives of Eqs. 9.3.20 and 9.3.21 yield

$$\mathbf{E}\dot{\mathbf{p}} = -\dot{\mathbf{E}}\mathbf{p} \tag{9.3.28}$$

$$\mathbf{G}\dot{\mathbf{p}} = -\dot{\mathbf{G}}\mathbf{p} \tag{9.3.29}$$

Direct expansion, using Eqs. 9.3.18 and 9.3.19, shows (Prob. 9.3.8) that

$$\mathbf{E}\dot{\mathbf{G}}^T = \dot{\mathbf{E}}\mathbf{G}^T \tag{9.3.30}$$

Taking the time derivative of both sides of Eq. 9.3.26 and using Eq. 9.3.30,

$$\dot{\mathbf{A}} = \dot{\mathbf{E}}\mathbf{G}^T + \mathbf{E}\dot{\mathbf{G}}^T = 2\mathbf{E}\dot{\mathbf{G}}^T \tag{9.3.31}$$

The product $\mathbf{G}\dot{\mathbf{p}}$ can be expanded as

$$\mathbf{G}\dot{\mathbf{p}} = [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}_3]\begin{bmatrix}\dot{e}_0 \\ \dot{\mathbf{e}}\end{bmatrix} = -\dot{e}_0\mathbf{e} - \tilde{\mathbf{e}}\dot{\mathbf{e}} + e_0\dot{\mathbf{e}}$$

Transforming both sides of this equation to skew-symmetric matrices, using the $\tilde{}$ operation on both sides, yields

$$\begin{aligned}\widetilde{(\mathbf{G}\dot{\mathbf{p}})} &= -\dot{e}_0\tilde{\mathbf{e}} - \widetilde{(\tilde{\mathbf{e}}\dot{\mathbf{e}})} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \tilde{\dot{\mathbf{e}}}\tilde{\mathbf{e}} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \mathbf{e}\dot{\mathbf{e}}^T - \dot{\mathbf{e}}^T\mathbf{e}\mathbf{I} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \mathbf{e}\dot{\mathbf{e}}^T + e_0\dot{e}_0\mathbf{I} + e_0\tilde{\dot{\mathbf{e}}} \\ &= [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} -\dot{\mathbf{e}}^T \\ \tilde{\dot{\mathbf{e}}} + \dot{e}_0\mathbf{I}\end{bmatrix} \\ &= \mathbf{G}\dot{\mathbf{G}}^T\end{aligned} \tag{9.3.32}$$

where Eqs. 9.1.28 and 9.1.30 and the identity $\mathbf{p}^T\dot{\mathbf{p}} = e_0\dot{e}_0 + \mathbf{e}^T\dot{\mathbf{e}} = 0$ have been used.

Relationships between Euler Parameter Derivatives and Angular Velocity Relationships between the time derivatives of Euler parameters and angular velocity vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are needed in kinematic and dynamic analysis. Using Eqs. 9.3.26 and 9.3.31, Eq. 9.2.40 becomes

$$\tilde{\boldsymbol{\omega}}' = \mathbf{A}^T\dot{\mathbf{A}} = 2\mathbf{G}\mathbf{E}^T\mathbf{E}\dot{\mathbf{G}}^T$$

which, upon application of Eqs. 9.3.23 and 9.3.21, results in

$$\tilde{\boldsymbol{\omega}}' = 2\mathbf{G}\dot{\mathbf{G}}^T \tag{9.3.33}$$

Substituting Eq. 9.3.32 into Eq. 9.3.33 gives

$$\tilde{\boldsymbol{\omega}}' = 2\widetilde{(\mathbf{G}\dot{\mathbf{p}})}$$

Thus, the desired relationship between $\boldsymbol{\omega}'$ and $\dot{\mathbf{p}}$ is

$$\boldsymbol{\omega}' = 2\mathbf{G}\dot{\mathbf{p}} \tag{9.3.34}$$

Multiplying both sides of Eq. 9.3.34 on the left by $\mathbf{G}^T$ yields

$$\mathbf{G}^T\boldsymbol{\omega}' = 2\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$

which, upon application of Eqs. 9.3.24 and 9.3.27, results in the inverse transformation

$$\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\boldsymbol{\omega}' \tag{9.3.35}$$

Using Eqs. 9.3.26 and 9.3.34, angular velocity can be written in the x-y-z reference frame as

$$\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega}' = 2\mathbf{E}\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$

Application of Eqs. 9.3.24 and 9.3.27 yields

$$\boldsymbol{\omega} = 2\mathbf{E}\dot{\mathbf{p}} \tag{9.3.36}$$

The inverse transformation of Eq. 9.3.36 is (Prob. 9.3.9)

$$\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{E}^T\boldsymbol{\omega} \tag{9.3.37}$$

Relationships between Euler Parameter Variations and Virtual Rotations It was shown in Section 9.2 that the virtual displacements and rotations of a reference frame are related to variations in the generalized coordinates that are used to define position and orientation. The objective here is to develop relationships between variations in Euler parameters and virtual rotations of the x'-y'-z' frame whose orientation is defined by the Euler parameters. The principal tool in this development is the differential of calculus, which is a linear operator that satisfies the same rules of manipulation as the time derivative operator, to develop relationships between Euler parameter time derivatives and angular velocity.

To be more specific, taking the differential, or variation, of Eq. 9.3.9 yields

$$\mathbf{p}^T \delta\mathbf{p} = 0 \tag{9.3.38}$$

Since Euler parameter relations do not involve time explicitly, the differential operator satisfies the same rules as the time derivative operator, so it is not surprising that Eq. 9.3.38 is of exactly the same form as Eq. 9.3.27, with time derivative replaced by differential. Similarly, the total differential of Eq. 9.3.26 yields

$$\delta\mathbf{A} = \delta\mathbf{E}\mathbf{G}^T + \mathbf{E}\,\delta\mathbf{G}^T \tag{9.3.39}$$

which is analogous to the time derivative of the transformation matrix given in Eq. 9.3.31.

Replacing the time derivative operator by the differential operator, all Euler parameter relations that involve time derivatives can be replaced by analogous expressions in terms of differentials. Rather than repeating all these equations and identities, they will simply be used to obtain the desired relationships between variations in Euler parameters and virtual rotations.

Premultiplying both sides of Eq. 9.2.50 by $\mathbf{A}^T$,

$$\delta\tilde{\boldsymbol{\pi}}' = \mathbf{A}^T \delta\mathbf{A} \tag{9.3.40}$$

Substituting for $\delta\mathbf{A}$ from Eq. 9.3.39 and manipulating in exactly the same manner as in the derivation of Eq. 9.3.34 yields (Prob. 9.3.10)

$$\delta\boldsymbol{\pi}' = 2\mathbf{G}\,\delta\mathbf{p} \tag{9.3.41}$$

which provides the desired relationship between a variation in the Euler parameter vector and a virtual rotation of the x'-y'-z' frame.

Analogous to Eq. 9.3.36 (Prob. 9.3.11),

$$\delta\boldsymbol{\pi} = 2\mathbf{E}\,\delta\mathbf{p} \tag{9.3.42}$$

which relates a variation in Euler parameters to the virtual rotation vector in global coordinates.

Just as in Eq. 9.3.35, the inverse relation to Eq. 9.3.41 is (Prob. 9.3.12)

$$\delta\mathbf{p} = \tfrac{1}{2}\mathbf{G}^T \delta\boldsymbol{\pi}' \tag{9.3.43}$$

Finally, analogous to Eq. 9.3.37 (Prob. 9.3.13),

$$\delta\mathbf{p} = \tfrac{1}{2}\mathbf{E}^T \delta\boldsymbol{\pi} \tag{9.3.44}$$

These relationships provide transformations between virtual rotations and generalized coordinate variations that are written symbolically in Eq. 9.2.63. In the case of Euler parameters, the matrix $\mathbf{B}_i$ in Eq. 9.2.63 is simply $2\mathbf{G}_i$. The total differential and partial derivative calculations that are written symbolically in Eqs. 9.2.64 and 9.2.65 can now be written explicitly in terms of Euler parameters.

These results will be used extensively in formulating the Jacobian matrix of constraint equations for kinematic analysis.

Integrability of Angular Velocity and Virtual Rotation Virtual rotation $\delta\boldsymbol{\pi}'$, represented in the body-fixed frame, is related to differentials of Euler parameters in Eq. 9.3.41. This relation and the δ notation for virtual rotation raise the following question: Is $\delta\boldsymbol{\pi}'$ the differential of some function of Euler parameters? The answer is provided by investigating the right side of Eq. 9.3.41.

Consider the first component of Eq. 9.3.41, expanded using Eq. 9.3.19:

$$\delta\pi'_x = -2e_1\,\delta e_0 + 2e_0\,\delta e_1 + 2e_3\,\delta e_2 - 2e_2\,\delta e_3 \qquad \textbf{(9.3.45)}$$

According to a well-known theorem of calculus [25, 26], if functions $f_i(\mathbf{x})$, $i = 1, \ldots, n$, where $\mathbf{x} = [x_1, \ldots, x_n]^T$ are continuously differentiable, the differential form

$$\sum_i f_i(\mathbf{x})\,dx_i \qquad \textbf{(9.3.46)}$$

is *exact* (the differential of some function) if and only if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \qquad i, j = 1, \ldots, n \qquad \textbf{(9.3.47)}$$

Applying this test to the right side of Eq. 9.3.45, with $x_1 = e_0$, $x_2 = e_1$, $f_1 = -2e_1 = -2x_2$, and $f_2 = 2e_0 = 2x_1$,

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial e_1} = -2 \neq 2 = \frac{\partial f_2}{\partial e_0} = \frac{\partial f_2}{\partial x_1}$$

Thus, the differential on the right side of Eq. 9.3.45 is not exact. This shows that *virtual rotation is not integrable*; that is, virtual rotation is not the differential of any function of Euler parameters.

Note that the coefficient of $\dot{\mathbf{p}}$ on the right side of Eq. 9.3.34 is the same as the coefficient of $\delta\mathbf{p}$ on the right side of Eq. 9.3.41. Thus, *angular velocity is not integrable*; that is, it is not the time derivative of any vector. For this reason, angular velocity is often called a *quasi-coordinate* in the dynamics literature. Apart from motivating a fancy name, this fact has important consequences in integrating equations of motion that are written in terms of angular accelerations, which are time derivatives of angular velocities. This topic will be discussed in more detail in Chapter 11.

9.4 KINEMATIC CONSTRAINTS

Kinematic constraints on the absolute position and orientation of bodies in space and on the relative position and orientation of pairs of bodies that are connected by joints are derived in this section. Conditions for parallelism and orthogonality of pairs of vectors are defined to serve as building blocks for characterizing the kinematic constraints that are encountered in applications. Constraint equations that define a substantial library of joints between bodies in space are derived to provide the foundation for spatial kinematic analysis. In anticipation of the need for velocity and acceleration analysis, first and second derivatives of constraint functions are derived.

9.4.1 Joint Definition Frames

A typical body, denoted body i, is shown in Fig. 9.4.1. Its x_i'-y_i'-z_i' body-fixed reference frame is used to position and orient the body in space. A second x_i''-y_i''-z_i'' *joint definition frame* is attached to the body, with its origin at point P_i. To orient the x_i''-y_i''-z_i'' frame, unit vectors $\mathbf{f}_i$, $\mathbf{g}_i$, and $\mathbf{h}_i$ are defined along its coordinate axes. To define $\mathbf{h}_i$, a point Q_i is defined on the z_i'' axis, a unit distance from point P_i. Next, to define $\mathbf{f}_i$, a point R_i is defined on the x_i'' axis, a unit distance from point P_i. Finally, $\mathbf{g}_i = \tilde{\mathbf{h}}_i\mathbf{f}_i$.

The foregoing procedure defines the x_i''-y_i''-z_i'' frame in terms of *joint definition points*, P_i, Q_i, and R_i, hereafter taken as defining orthogonal unit vectors $\mathbf{f}_i$, $\mathbf{g}_i$, and $\mathbf{h}_i$. In terms of the unit vectors $\mathbf{f}_i'$, $\mathbf{g}_i'$, and $\mathbf{h}_i'$, represented in the x_i'-y_i'-z_i' frame, the transformation matrix from the x_i''-y_i''-z_i'' frame to the x_i'-y_i'-z_i' frame is obtained, as in Eq. 9.2.13, as

$$\mathbf{C}_i^P = [\mathbf{f}_i', \mathbf{g}_i', \mathbf{h}_i'] \tag{9.4.1}$$

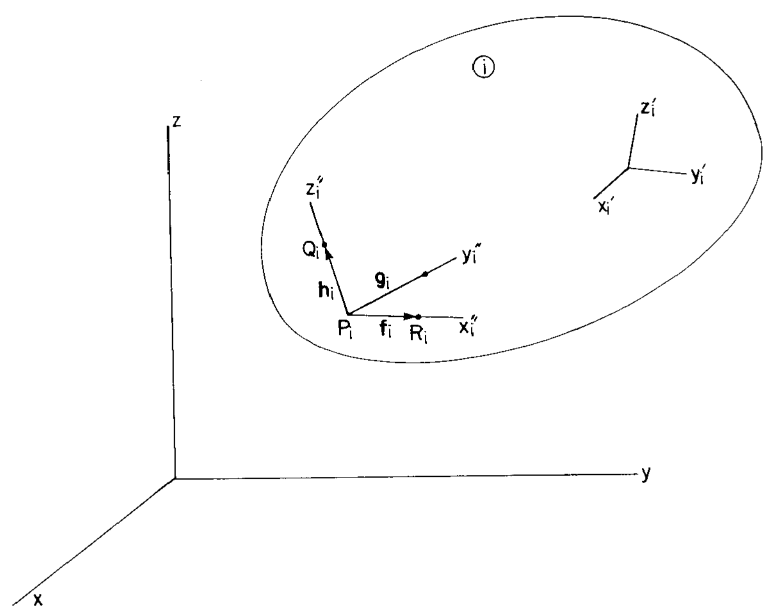

Figure 9.4.1 Construction of a joint definition frame.

Example 9.4.1: Consider the bar shown in Fig. 9.4.2, with a rotational joint at point P_2 that is pivoted about the x_2'' axis, which lies in the x_i'-y_i' plane and makes an angle of $\pi/4$ with the x_2' axis. Unit vectors along the z_2'' and x_2'' axes are, from Fig. 9.4.2,

$$\mathbf{h}_2' = [0, 0, 1]^T$$

$$\mathbf{f}_2' = \left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0\right]^T$$

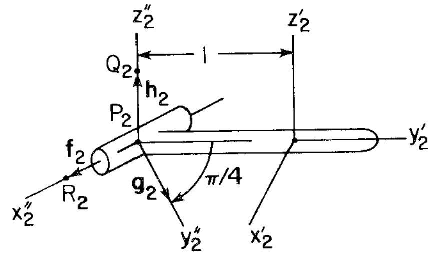

Figure 9.4.2 Oblique rotational axis.

Finally,

$$\mathbf{g}_2' = \tilde{\mathbf{h}}_2'\mathbf{f}_2' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

Thus, the transformation matrix $\mathbf{C}_2^P$ of Eq. 9.4.1 is

$$\mathbf{C}_2^P = [\mathbf{f}_2', \mathbf{g}_2', \mathbf{h}_2'] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9.4.2 Constraints on Vectors and Points

Consider the pair of rigid bodies, denoted bodies i and j, in Fig. 9.4.3. Reference points P_i and P_j and nonzero vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ are fixed in bodies i and j, respectively. Kinematic constraints between pairs of bodies are often characterized by conditions of orthogonality or parallelism of pairs of such vectors. The purpose here is to derive analytical conditions with which to define a library of kinematic connections.

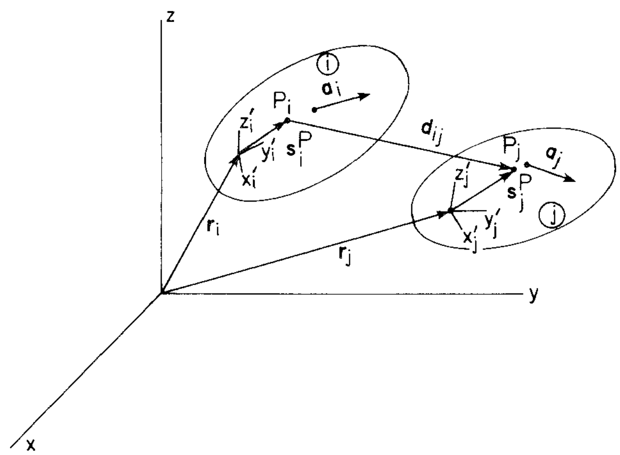

Figure 9.4.3 Vectors fixed in and between bodies.

Basic Constraints First, a necessary and sufficient condition that a pair of body-fixed nonzero vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ on bodies i and j, respectively, is orthogonal is that their scalar product be zero; that is,

$$\Phi^{d1}(\mathbf{a}_i, \mathbf{a}_j) \equiv \mathbf{a}_i^T\mathbf{a}_j = 0 \tag{9.4.2}$$

where the superscript notation $d1$ indicates the first form of dot or scalar product condition. Writing the vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ in terms of their respective body reference transformation matrices and body-fixed constant vectors, $\mathbf{a}_i = \mathbf{A}_i\mathbf{a}_i'$ and $\mathbf{a}_j = \mathbf{A}_j\mathbf{a}_j'$, Eq. 9.4.2 may be written in the form

$$\Phi^{d1}(\mathbf{a}_i, \mathbf{a}_j) = \mathbf{a}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{a}_j' = 0 \tag{9.4.3}$$

This condition is called the *dot-1 constraint* between vectors $\mathbf{a}_i$ and $\mathbf{a}_j$. Note that it is written in terms of a pair of body-fixed constant vectors and the transformation matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ that depend on the orientation generalized coordinates of bodies i and j, respectively. Thus, the dot-1 constraint restricts the relative orientation of a pair of bodies.

Example 9.4.2: The body in Example 9.4.1 is hinged relative to ground (body 1) about the y_1 axis, as shown in Fig. 9.4.4, to form an oblique pendulum. In order for the pendulum to be hinged on the y_1 axis, $\mathbf{f}_2$ must be perpendicular to $\mathbf{h}_1$ and $\mathbf{f}_1$. Hence, from Eq. 9.4.3,

$$\begin{aligned}\Phi^{d1}(\mathbf{f}_2, \mathbf{f}_1) &= \mathbf{f}_2'^T\mathbf{A}_2^T\mathbf{A}_1\mathbf{f}_1' \\ &= \left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0\right]\mathbf{A}_2^T[1, 0, 0]^T \\ &= \sqrt{2}\left(e_0^2 + e_1^2 - \tfrac{1}{2} - e_1e_2 + e_0e_3\right) = 0\end{aligned}$$

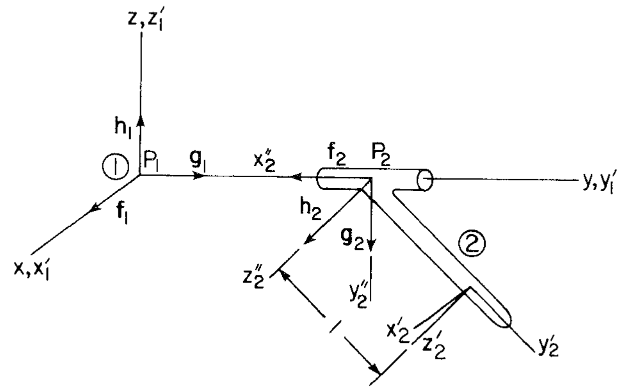

Figure 9.4.4 Oblique pendulum.

where $\mathbf{A}_1 = \mathbf{I}$ and $\mathbf{p} = [e_0, e_1, e_2, e_3]^T$ is the Euler parameter vector for body 2. Similarly,

$$\begin{aligned}\Phi^{d1}(\mathbf{f}_2, \mathbf{h}_1) &= \mathbf{f}_2'^T\mathbf{A}_2^T\mathbf{A}_1\mathbf{h}_1' \\ &= \left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0\right]\mathbf{A}_2^T[0, 0, 1]^T \\ &= \sqrt{2}(e_1e_3 + e_0e_2 - e_2e_3 - e_0e_1) = 0\end{aligned}$$

Since $\mathbf{f}_1$, $\mathbf{h}_1$, and $\mathbf{f}_2$ are unit vectors, these two equations are equivalent to the geometric condition that $\mathbf{f}_2$ and $\mathbf{g}_1$ be parallel.

The scalar product condition can also be used to prescribe orthogonality of a body-fixed vector $\mathbf{a}_i$ and a vector $\mathbf{d}_{ij}$ between bodies, shown in Fig. 9.4.3, provided $\mathbf{d}_{ij} \neq \mathbf{0}$. Analytically, this condition is defined using the scalar product as

$$\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij}) = \mathbf{a}_i^T\mathbf{d}_{ij} = 0 \tag{9.4.4}$$

where the superscript $d2$ indicates a second form of dot or scalar product condition. Writing the vector $\mathbf{d}_{ij}$ as

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P \tag{9.4.5}$$

Eq. 9.4.4 becomes

$$\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij}) = \mathbf{a}_i'^T\mathbf{A}_i^T(\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i) - \mathbf{a}_i'^T\mathbf{s}_i'^P = 0 \tag{9.4.6}$$

This condition is called the *dot-2 constraint* between vectors $\mathbf{a}_i$ and $\mathbf{d}_{ij}$.

Note that the dot-2 constraint is not symmetric as regards bodies i and j. To require that the vector $\mathbf{a}_j$ on body j be orthogonal to $\mathbf{d}_{ij}$, the dot-2 constraint of Eq. 9.4.6 can be used by simply interchanging indexes i and j. It is important to recall that the orthogonality condition of Eq. 9.4.6 breaks down if $\mathbf{d}_{ij} = \mathbf{0}$.

Example 9.4.3: Consider the pendulum in Example 9.4.2. To constrain P_2 to lie on the y_1' axis, two dot-2 constraints may be used to cause $\mathbf{d}_{12}$ to lie along the y_1' axis; equivalently, as long as $\mathbf{d}_{12} \neq \mathbf{0}$, $\mathbf{d}_{12}$ is perpendicular to $\mathbf{f}_1$ and $\mathbf{h}_1$. From Eq. 9.4.6,

$$\Phi^{d2}(\mathbf{f}_1, \mathbf{d}_{12}) = [1, 0, 0]\left\{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \mathbf{A}_2\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} - [1, 0, 0]\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= x_2 - 2e_1e_2 + 2e_0e_3 = 0$$

$$\Phi^{d2}(\mathbf{h}_1, \mathbf{d}_{12}) = [0, 0, 1]\left\{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \mathbf{A}_2\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right\}$$

$$= z_2 - 2e_2e_3 - 2e_0e_1 = 0$$

Providing $\mathbf{d}_{12} \neq \mathbf{0}$, these equations imply that $\mathbf{d}_{12}$ and $\mathbf{g}_1$ are parallel. Since they have point P_1 in common, they are collinear. Finally, since P_2 lies on the vector $\mathbf{d}_{12}$, it lies on the y_1' axis. If $\mathbf{d}_{12} = \mathbf{0}$, then points P_1 and P_2 coincide. Thus, even in this case, the equations imply that P_2 lies on the y_1' axis. Thus, the pair of dot-2 constraints is equivalent to the condition that P_2 lie on the y_1' axis.

The combined set of four constraint equations used in this example and in Example 9.4.2 has been shown to imply that P_2 lies on the y_1' axis and that $\mathbf{f}_2$ is parallel to the y axis. Since P_2 is on $\mathbf{f}_2$, $\mathbf{f}_2$ is collinear with the y_1' axis, which is the definition of the geometry of the joint. Thus, the four equations derived are equivalent to the geometry of the joint, even when $\mathbf{d}_{12} = \mathbf{0}$.

Consider next the pair of constraints derived in this example and the following pair of dot-2 constraints:

$$\Phi^{d2}(\mathbf{g}_2, \mathbf{d}_{12}) = 0$$
$$\Phi^{d2}(\mathbf{h}_2, \mathbf{d}_{12}) = 0$$

If $\mathbf{d}_{12} \neq \mathbf{0}$, these equations imply that $\mathbf{d}_{12}$ is parallel to $\mathbf{f}_2$. The pair of equations derived earlier in this example implies that P_2 lies on the y_1' axis and that $\mathbf{d}_{12}$ is orthogonal to the x_1'-y_1' plane. Hence, $\mathbf{f}_2$ is parallel to the y_1' axis, which implies the geometry of the joint.

Are the four dot-2 constraints defined in this example equivalent to the geometry of the joint? To see that the answer is no, note that when $\mathbf{d}_{12} = \mathbf{0}$ all four constraints are satisfied for any orientation of $\mathbf{f}_2$. If these four equations were used in kinematic analysis, all would be well as long as $\mathbf{d}_{12} \neq \mathbf{0}$. When P_2 coincides with P_1, however, the Jacobian of the constraints fails to have full row rank, and a singular matrix would occur in kinematic analysis. The more insidious problem is that, even if P_1 and P_2 are close but do not precisely coincide, the Jacobian becomes very ill-conditioned and numerical error occurs.

This example should adequately illustrate that constraint equations that are derived from the geometry of a joint may sometimes fail to imply the geometry of the joint. This situation cannot be tolerated in computer-aided kinematics or dynamics, since the mathematical model may fail to represent the geometry of the system.

It is often required that a pair of points on two bodies coincide. A necessary and sufficient condition for points P_i and P_j shown in Fig. 9.4.3 to coincide is that $\mathbf{d}_{ij} = \mathbf{0}$; that is,

$$\mathbf{\Phi}^s(P_i, P_j) = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P = \mathbf{0} \tag{9.4.7}$$

where the superscript designation s anticipates use of this equation in defining a *spherical joint* or *ball and socket joint.* Note that this vector equation consists of three scalar equations. It is called the *spherical constraint.* Note that Eq. 9.4.7 is symmetric with respect to body indexes.

Finally, it is often required that the distance between a pair of points on adjacent bodies be fixed by a physical connection that may be thought of as a link with a spherical joint on each end. A necessary and sufficient condition that the distance between points P_i and P_j in Fig. 9.4.3 be equal to $C \neq 0$ is simply

$$\Phi^{ss}(P_i, P_j, C) = \mathbf{d}_{ij}^T\mathbf{d}_{ij} - C^2 = 0 \tag{9.4.8}$$

which is called the *distance constraint* or *spherical–spherical constraint* and consists of a single scalar equation.

If $C = 0$ in Eq. 9.4.8, then this single equation implies the vector relation $\mathbf{d}_{ij} = \mathbf{0}$ (Prob. 9.4.3). While this might seem to be a practical way to reduce the number of constraint equations, it is not. To see why, denote by $\mathbf{q}$ the vector of generalized coordinates and observe that

$$\Phi_{\mathbf{q}}^{ss} = 2\mathbf{d}_{ij}^T(\mathbf{d}_{ij})_{\mathbf{q}} = \mathbf{0}$$

since $\mathbf{d}_{ij} = \mathbf{0}$ when $C = 0$. Thus, the constraint Jacobian does not have full row rank and cannot be effectively used in kinematic analysis. For this reason, use of the distance constraint is restricted to the case $C \neq 0$.

Example 9.4.4: Consider the compound pendulum shown in Fig. 9.4.5(a), where point P_2 on body 2 is constrained to point P_1 in ground with a distance

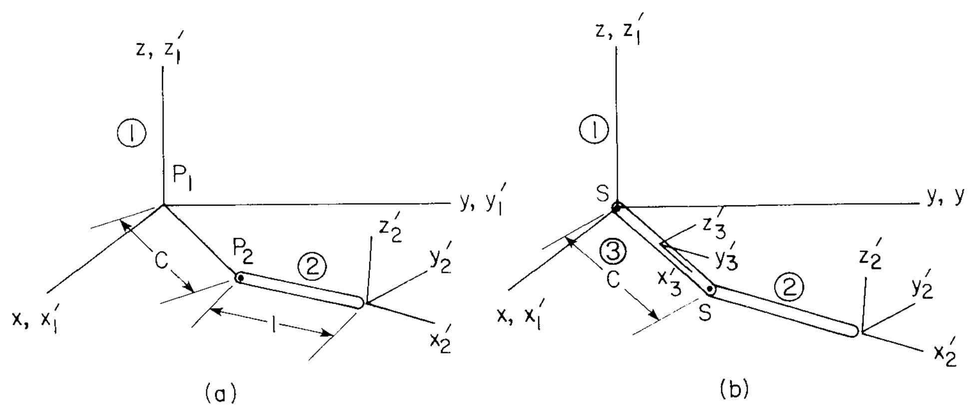

Figure 9.4.5 Compound pendulum.

constraint. From Eq. 9.4.5,

$$\begin{aligned}\mathbf{d}_{12} &= \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}_2'^P - \mathbf{r}_1 - \mathbf{A}_1\mathbf{s}_1'^P \\ &= [x_2, y_2, z_2]^T + \mathbf{A}_2[-1, 0, 0]^T \\ &= [x_2 - 2e_0^2 - 2e_1^2 + 1, y_2 - 2e_1e_2 - 2e_0e_3, z_2 - 2e_1e_3 + 2e_0e_2]^T\end{aligned}$$

Substituting $\mathbf{d}_{12}$ into Eq. 9.4.8,

$$\begin{aligned}\Phi^{ss}(P_1, P_2, C) = (x_2 - 2e_0^2 - 2e_1^2 + 1)^2 + (y_2 - 2e_1e_2 - 2e_0e_3)^2 \\ + (z_2 - 2e_1e_3 + 2e_0e_2)^2 - C^2 = 0\end{aligned}$$

A second model of the compound pendulum is shown in Fig. 9.4.5(b), where the distance constraint is replaced by a third body, with spherical joints at each end. This may look kinematically equivalent to the compound pendulum in Fig. 9.4.5(a), but it is not. Since the distance constraint is a single scalar equation, the compound pendulum in Fig. 9.4.5(a) has five degrees of freedom. On the other hand, the compound pendulum in Fig. 9.4.5(b) has six degrees of freedom, because the two spherical joints at the ends of body 3 yield six constraint equations. This extra degree of freedom is accounted for as the rotational degree of freedom of body 3 about its x_3' axis. Therefore, the two models are not equivalent.

The four basic constraint equations derived thus far form the foundation for defining a library of kinematic constraints between bodies. To see that these conditions may be efficiently used in defining other geometric relationships, two parallelism conditions are derived, using the dot-1 and dot-2 constraints. Consider the pair of bodies shown in Fig. 9.4.6, with joint definition frames located at points P_i and P_j on bodies i and j, respectively.

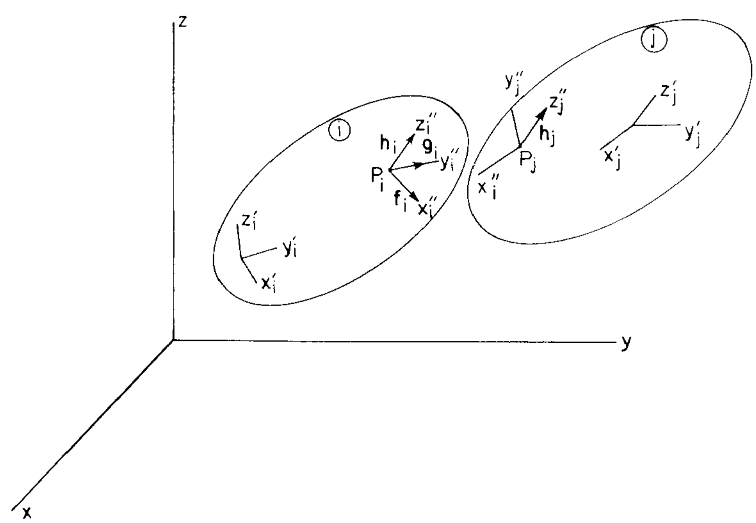

Figure 9.4.6 Parallel vectors in adjacent bodies.

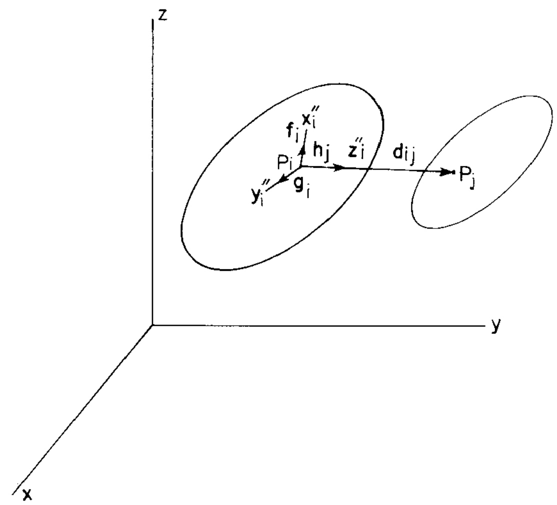

Figure 9.4.7 Parallel vectors on and between adjacent bodies.

Let the z_i'' and z_j'' axes be required to be parallel; that is, vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ are to be parallel. The vector $\mathbf{h}_j$ is parallel to $\mathbf{h}_i$ if and only if it is orthogonal to $\mathbf{f}_i$ and $\mathbf{g}_i$. Thus, the condition that $\mathbf{h}_i$ and $\mathbf{h}_j$ are parallel is simply

$$\mathbf{\Phi}^{p1}(\mathbf{h}_i, \mathbf{h}_j) = \begin{bmatrix} \Phi^{d1}(\mathbf{f}_i, \mathbf{h}_j) \\ \Phi^{d1}(\mathbf{g}_i, \mathbf{h}_j) \end{bmatrix} = \mathbf{0} \tag{9.4.9}$$

where the dot-1 constraint of Eq. 9.4.3 is employed and two scalar equations are obtained. This is called the *parallel-1 constraint.*

Finally, consider the condition that the vector $\mathbf{h}_i$ along the z_i'' axis on body i is to be parallel to the vector $\mathbf{d}_{ij}$, as shown in Fig. 9.4.7. Since $\mathbf{d}_{ij} \neq \mathbf{0}$ is parallel to $\mathbf{h}_i$ if and only if it is perpendicular to $\mathbf{f}_i$ and $\mathbf{g}_i$, the second parallelism condition is

$$\mathbf{\Phi}^{p2}(\mathbf{h}_i, \mathbf{d}_{ij}) = \begin{bmatrix} \Phi^{d2}(\mathbf{f}_i, \mathbf{d}_{ij}) \\ \Phi^{d2}(\mathbf{g}_i, \mathbf{d}_{ij}) \end{bmatrix} = \mathbf{0} \tag{9.4.10}$$

which is simply an application of the dot-2 constraint of Eq. 9.4.6. It is called the *parallel-2 constraint.* Note that the parallel-2 constraint breaks down if $\mathbf{d}_{ij} = \mathbf{0}$, since the dot-2 conditions break down in this case.

Example 9.4.5: Consider again the oblique pendulum in Example 9.4.2. To keep the pivot axis of the pendulum parallel to the y_1' axis, the parallel-1 constraint may be used to cause $\mathbf{f}_2$ and $\mathbf{g}_1$ in Fig. 9.4.4 to be parallel; that is,

$$\begin{aligned}\mathbf{\Phi}^{p1}(\mathbf{f}_2, \mathbf{g}_1) &= \begin{bmatrix} \Phi^{d1}(\mathbf{h}_2, \mathbf{g}_1) \\ \Phi^{d1}(\mathbf{g}_2, \mathbf{g}_1) \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{A}_2\mathbf{h}_2')^T\mathbf{g}_1 \\ (\mathbf{A}_2\mathbf{g}_2')^T\mathbf{g}_1 \end{bmatrix} \\ &= \begin{bmatrix} [0, 0, 1]\mathbf{A}_2^T[0, 1, 0]^T \\ \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]\mathbf{A}_2^T[0, 1, 0]^T \end{bmatrix} \\ &= \begin{bmatrix} 2(e_2e_3 - e_0e_1) \\ \sqrt{2}\,(e_1e_2 + e_0e_3 + e_0^2 + e_2^2 - 1/2) \end{bmatrix} = 0\end{aligned}$$

where the Euler parameters are for body 2. Constraining point P_2 to be on the y_1' axis, a complete set of constraint equations for the joint is obtained; that is, with

$$\begin{aligned}\mathbf{r}_2^P &= \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}_2'^P \\ &= \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \mathbf{A}_2\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_2 - 2e_1e_2 + 2e_0e_3 \\ y_2 - 2e_0^2 - 2e_2^2 + 1 \\ z_2 - 2e_2e_3 - 2e_0e_1 \end{bmatrix}\end{aligned}$$

the remaining two constraints are

$$\begin{aligned}x_2^P &= x_2 - 2e_1e_2 + 2e_0e_3 = 0 \\ z_2^P &= z_2 - 2e_2e_3 - 2e_0e_1 = 0\end{aligned}$$

Differentials and Derivatives of Basic Constraints In anticipation that differentials and derivatives of the four basic constraint equations with respect to generalized coordinates $(\mathbf{r}_i, \mathbf{p}_i)$ and $(\mathbf{r}_j, \mathbf{p}_j)$ of bodies i and j need to be calculated, it is instructive to use the vector and differential calculus methods of Section 9.2 and identities obtained in Section 9.3 to derive the needed differential and derivative expressions. The differential of the dot-1 constraint function of Eq. 9.4.2 was derived in Example 9.2.6 of Section 9.2. From Eq. 9.2.60,

$$\delta\Phi^{d1}(\mathbf{a}_i, \mathbf{a}_j) = -(\mathbf{a}_j'^T\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{a}}_i'\,\delta\boldsymbol{\pi}_i' + \mathbf{a}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{a}}_j'\,\delta\boldsymbol{\pi}_j') \qquad \textbf{(9.4.11)}$$

The differential of the dot-2 constraint function of Eq. 9.4.4 is

$$\delta\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij}) = \mathbf{a}_i^T\,\delta\mathbf{d}_{ij} + \mathbf{d}_{ij}^T\,\delta\mathbf{a}_i$$

Using $\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{s}_j^P - \mathbf{r}_i - \mathbf{s}_i^P$ and Eq. 9.2.53, this is (Prob. 9.4.4)

$$\begin{aligned}
\delta\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij}) &= \mathbf{a}_i'^T\mathbf{A}_i^T(\delta\mathbf{r}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\,\delta\boldsymbol{\pi}_i') \\
&\quad - \mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\mathbf{a}}_i'\,\delta\boldsymbol{\pi}_i' \\
&= \mathbf{a}_i'^T\mathbf{A}_i^T\,\delta\mathbf{r}_j - \mathbf{a}_i'^T\mathbf{A}_i^T\,\delta\mathbf{r}_i - \mathbf{a}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' \\
&\quad + (\mathbf{a}_i'^T\tilde{\mathbf{s}}_i'^P - \mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\mathbf{a}}_i')\,\delta\boldsymbol{\pi}_i'
\end{aligned} \tag{9.4.12}$$

The differential of the spherical constraint of Eq. 9.4.7, using Eq. 9.2.53, is

$$\delta\boldsymbol{\Phi}^s(P_i, P_j) = \delta\mathbf{r}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\,\delta\boldsymbol{\pi}_i' \tag{9.4.13}$$

Finally, the differential of the distance constraint of Eq. 9.4.8 is

$$\delta\Phi^{ss}(P_i, P_j, C) = 2\mathbf{d}_{ij}^T\,\delta\mathbf{d}_{ij}$$

Using the same expansion of $\delta\mathbf{d}_{ij}$ employed in deriving Eq. 9.4.12, this is

$$\delta\Phi^{ss}(P_i, P_j, C) = 2\mathbf{d}_{ij}^T(\delta\mathbf{r}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\,\delta\boldsymbol{\pi}_i') \tag{9.4.14}$$

Differentials of the parallel-1 and parallel-2 constraints of Eqs. 9.4.9 and 9.4.10 are obtained through direct application of Eqs. 9.4.11 and 9.4.12.

All constraint equations encountered can be written in the form

$$\boldsymbol{\Phi}(\mathbf{r}_i, \mathbf{A}_i, \mathbf{r}_j, \mathbf{A}_j) = \mathbf{0}$$

where i and j are numbers that identify the bodies connected. Taking the differential of both sides of each equation and manipulating has led to linear equations in virtual displacements and virtual rotations. Defining the coefficients of $\delta\mathbf{r}_i$ and $\delta\mathbf{r}_j$ as $\boldsymbol{\Phi}_{\mathbf{r}_i}$ and $\boldsymbol{\Phi}_{\mathbf{r}_j}$ and the coefficients of $\delta\boldsymbol{\pi}_i'$ and $\delta\boldsymbol{\pi}_j'$ as $\boldsymbol{\Phi}_{\boldsymbol{\pi}_i'}$ and $\boldsymbol{\Phi}_{\boldsymbol{\pi}_j'}$, respectively, the linearized constraint equations are

$$\delta\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathbf{r}_i}\,\delta\mathbf{r}_i + \boldsymbol{\Phi}_{\boldsymbol{\pi}_i'}\,\delta\boldsymbol{\pi}_i' + \boldsymbol{\Phi}_{\mathbf{r}_j}\,\delta\mathbf{r}_j + \boldsymbol{\Phi}_{\boldsymbol{\pi}_j'}\,\delta\boldsymbol{\pi}_j' = \mathbf{0}$$

The coefficients of $\delta\mathbf{r}_i$, $\delta\mathbf{r}_j$, $\delta\boldsymbol{\pi}_i'$, and $\delta\boldsymbol{\pi}_j'$ in the differentials of Eqs. 9.4.11 to 9.4.14 are tabulated in Table 9.4.1. The reader is cautioned that since $\delta\boldsymbol{\pi}'$ is not integrable, the Jacobian matrices $\boldsymbol{\Phi}_{\boldsymbol{\pi}'}$ in Table 9.4.1 should be interpreted only as coefficients of $\delta\boldsymbol{\pi}'$ in Eqs. 9.4.11 to 9.4.14, not as derivatives of $\boldsymbol{\Phi}$ with respect to a variable $\boldsymbol{\pi}'$, since $\boldsymbol{\pi}'$ does not exist.

TABLE 9.4.1 Partial Derivaties[a] of Basic Constraint Functions

Constraint function	$\boldsymbol{\Phi}_{\mathbf{r}_i}$	$\boldsymbol{\Phi}_{\mathbf{r}_j}$	$\boldsymbol{\Phi}_{\boldsymbol{\pi}_i'}$	$\boldsymbol{\Phi}_{\boldsymbol{\pi}_j'}$
$\Phi^{d1}(\mathbf{a}_i, \mathbf{a}_j)$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{a}_j'^T\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{a}}_i'$	$-\mathbf{a}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{a}}_j'$
$\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij})$	$-\mathbf{a}_i'^T\mathbf{A}_i^T$	$\mathbf{a}_i'^T\mathbf{A}_i^T$	$(\mathbf{a}_i'^T\tilde{\mathbf{s}}_i'^P - \mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\mathbf{a}}_i')$	$-\mathbf{a}_i'^T\mathbf{A}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'^P$
$\boldsymbol{\Phi}^s(P_i, P_j)$	$-\mathbf{I}$	$\mathbf{I}$	$\mathbf{A}_i\tilde{\mathbf{s}}_i'^P$	$-\mathbf{A}_j\tilde{\mathbf{s}}_j'^P$
$\Phi^{ss}(P_i, P_j, C)$	$-2\mathbf{d}_{ij}^T$	$2\mathbf{d}_{ij}^T$	$2\mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\mathbf{s}}_i'^P$	$-2\mathbf{d}_{ij}^T\mathbf{A}_j\tilde{\mathbf{s}}_j'^P$

[a] Technically, $\boldsymbol{\Phi}_{\boldsymbol{\pi}'}$ is not a partial derivative, but $\boldsymbol{\Phi}_{\mathbf{p}} = 2\boldsymbol{\Phi}_{\boldsymbol{\pi}'}\mathbf{G}$.

Using the relationship $\delta\boldsymbol{\pi}' = 2\mathbf{G}\,\delta\mathbf{p}$ of Eq. 9.3.41, a linearized constraint equation can be written in terms of variations of Euler parameters of the bodies involved; that is,

$$\delta\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathbf{r}_i}\,\delta\mathbf{r}_i + 2\boldsymbol{\Phi}_{\pi_i'}\mathbf{G}_i\,\delta\mathbf{p}_i + \boldsymbol{\Phi}_{\mathbf{r}_j} + 2\boldsymbol{\Phi}_{\pi_j'}\mathbf{G}_j\,\delta\mathbf{p}_j = \mathbf{0}$$

The coefficients of the differentials of $\delta\mathbf{r}_i$, $\delta\mathbf{r}_j$, $\delta\mathbf{p}_i$, and $\delta\mathbf{p}_j$ are thus the partial derivatives of the constraint functions with respect to these generalized coordinates. Partial derivatives with respect to $\mathbf{p}_i$ and $\mathbf{p}_j$ are thus obtained from Table 9.4.1 by multiplying $\Phi_{\pi_i'}$ and $\Phi_{\pi_j'}$ on the right by $2\mathbf{G}_i$ and $2\mathbf{G}_j$, respectively; that is,

$$\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{p}_i} \equiv \Phi_{\mathbf{p}_i} = 2\boldsymbol{\Phi}_{\pi_i'}\mathbf{G}_i$$

9.4.3 Absolute Constraints on a Body

Absolute constraints may be placed on the position of point P_i on body i, as shown in Fig. 9.4.8, and on the orientation of the body-fixed reference frame for body i. Six such constraint equations on individual generalized coordinates of body i may be written as

$$\begin{aligned}
\Phi^1 &= x_i^P - x_i^0 = 0\\
\Phi^2 &= y_i^P - y_i^0 = 0\\
\Phi^3 &= z_i^P - z_i^0 = 0\\
\Phi^4 &= e_{1i} - e_{1i}^0 = 0\\
\Phi^5 &= e_{2i} - e_{2i}^0 = 0\\
\Phi^6 &= e_{3i} - e_{3i}^0 = 0
\end{aligned} \tag{9.4.15}$$

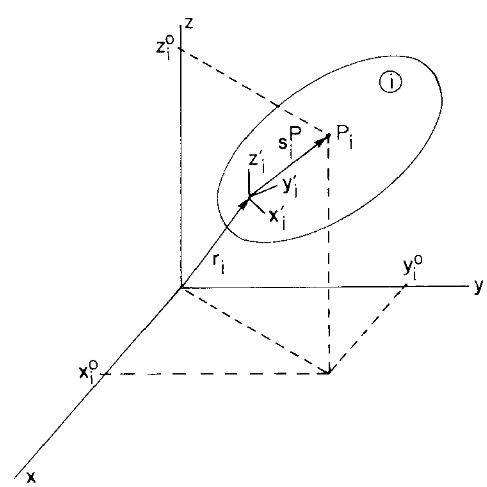

Figure 9.4.8 Absolute coordinate constraints.

Example 9.4.6: Absolute x and z constraints on points P_1 and P_2 on body 2 in Fig. 9.4.9 can be used to define a joint that permits body 2 to rotate about and slide along the global y axis (called a cylindrical joint). Writing

$$
\begin{aligned}
\mathbf{r}^{P_1} &= \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}_2'^{P_1} \\
&= \begin{bmatrix} x_2 + 2e_1e_2 - 2e_0e_3 \\ y_2 + 2e_0^2 + 2e_2^2 - 1 \\ z_2 + 2e_2e_3 + 2e_0e_1 \end{bmatrix} \\
\mathbf{r}^{P_2} &= \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}_2'^{P_2} \\
&= \begin{bmatrix} x_2 - 2e_1e_2 + 2e_0e_3 \\ y_2 - 2e_0^2 - 2e_2^2 + 1 \\ z_2 - 2e_2e_3 - 2e_0e_1 \end{bmatrix}
\end{aligned}
$$

where Euler parameters are for body 2, absolute x and z constraints on points are

$$
\begin{bmatrix} x_2 + 2e_1e_2 - 2e_0e_3 \\ z_2 + 2e_2e_3 + 2e_0e_1 \\ x_2 - 2e_1e_2 + 2e_0e_3 \\ z_2 - 2e_2e_3 - 2e_0e_1 \end{bmatrix} = \mathbf{0}
$$

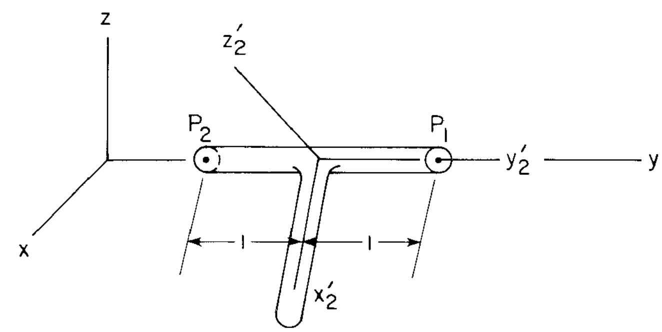

Figure 9.4.9 Cylindrical joint defined by absolute constraints on two points.

While any one of the first three constraints of Eq. 9.4.15, taken alone, is an *absolute coordinate constraint* that is physically meaningful, the last three conditions should be taken together to completely specify the orientation of body i, with the fourth Euler parameter determined by the normalization condition of Eq. 9.3.9. If all six of the constraints of Eq. 9.4.15 are employed, the position and orientation of body i relative to ground are fixed, yielding a *ground constraint*.

The vector to point P_i on body i in Fig. 9.4.8 is

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^P$$

The variation of this vector is, from Eq. 9.2.51,

$$\delta \mathbf{r}_i^P = \delta \mathbf{r}_i - \mathbf{A}_i \tilde{\mathbf{s}}_i'^P \, \delta \boldsymbol{\pi}_i'$$

Thus, the gradients of Φ^1, Φ^2, and Φ^3 in Eq. 9.4.15 are

$$\begin{bmatrix} \Phi^1_{\mathbf{r}_i} \\ \Phi^2_{\mathbf{r}_i} \\ \Phi^3_{\mathbf{r}_i} \end{bmatrix} = \mathbf{I}$$

$$\begin{bmatrix} \Phi^1_{\pi_i'} \\ \Phi^2_{\pi_i'} \\ \Phi^3_{\pi_i'} \end{bmatrix} = -\mathbf{A}_i \tilde{\mathbf{s}}_i'^P$$

From Eqs. 9.3.19 and 9.3.43,

$$\delta \mathbf{e}_i = \frac{\partial \mathbf{e}_i}{\partial \mathbf{p}_i} \delta \mathbf{p}_i = [\mathbf{0}, \mathbf{I}] \, \delta \mathbf{p}_i = \tfrac{1}{2}(\tilde{\mathbf{e}}_i + \mathbf{e}_{0_i}\mathbf{I}) \, \delta \boldsymbol{\pi}_i'$$

Thus, the gradients of Φ^4, Φ^5, and Φ^6 in Eq. 9.4.15 are

$$\begin{bmatrix} \Phi^4_{\mathbf{r}_i} \\ \Phi^5_{\mathbf{r}_i} \\ \Phi^6_{\mathbf{r}_i} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \Phi^4_{\pi_i'} \\ \Phi^5_{\pi_i'} \\ \Phi^6_{\pi_i'} \end{bmatrix} = \tfrac{1}{2}(\tilde{\mathbf{e}}_i + \mathbf{e}_{0_i}\mathbf{I})$$

An *absolute point constraint* on the position of point P_i on body i can be defined by the vector $\mathbf{r}_i^0$ in the x-y-z reference frame, as shown in Fig. 9.4.10.

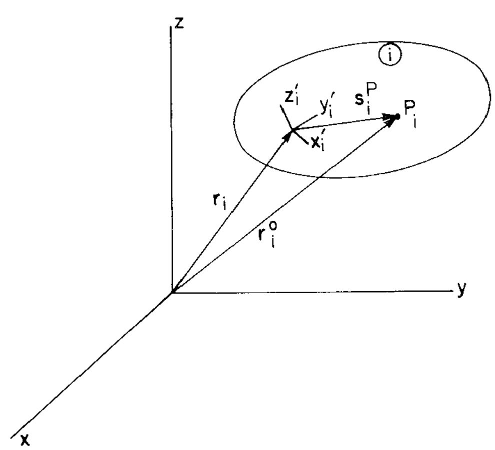

Figure 9.4.10 Absolute point constraint.

This vector condition is written in matrix form as

$$\mathbf{\Phi}^{P}(P_i) = \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^{P} - \mathbf{r}_i^0 = \mathbf{0} \tag{9.4.16}$$

While these three scalar constraint equations restrict the position of point P_i, they allow three rotational degrees of freedom of body i. From the expression for $\delta\mathbf{r}_i^P$ in Eq. 9.2.51,

$$\mathbf{\Phi}^{P}_{\mathbf{r}_i} = \mathbf{I}$$
$$\mathbf{\Phi}^{P}_{\pi_i'} = -\mathbf{A}_i\tilde{\mathbf{s}}_i'^{P}$$

9.4.4 Constraints between Pairs of Bodies

A variety of *spatial joints* between pairs of bodies is employed in the construction of mechanisms and machines. Constraint equations that define a library of such joints are derived in this subsection.

The distance between points P_i and P_j on bodies i and j can be fixed and equal to $C \neq 0$, as shown in Fig. 9.4.11, by direct application of the *distance constraint* of Eq. 9.4.8; that is,

$$\Phi^{ss}(P_i, P_j, C) = 0 \tag{9.4.17}$$

This scalar constraint equation permits five relative degrees of freedom between bodies i and j.

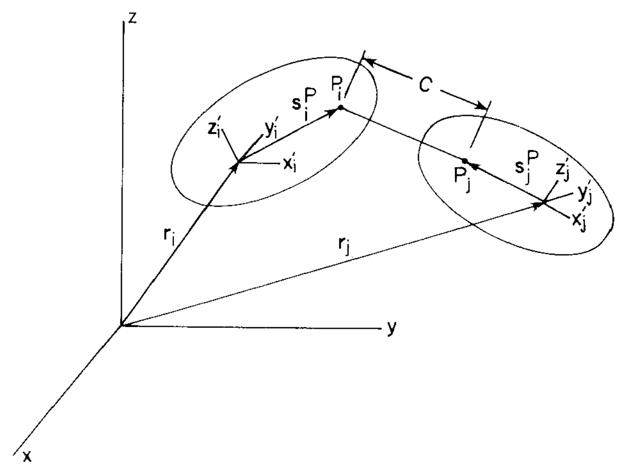

Figure 9.4.11 Distance constraint.

A *spherical joint* (or *ball and socket joint*) is defined by the condition that the center of the ball at point P_i on body i coincides with the center of the socket at P_j on body j, as shown in Fig. 9.4.12. This condition is simply the spherical constraint of Eq. 9.4.7; that is,

$$\mathbf{\Phi}^{s}(P_i, P_j) = \mathbf{0} \tag{9.4.18}$$

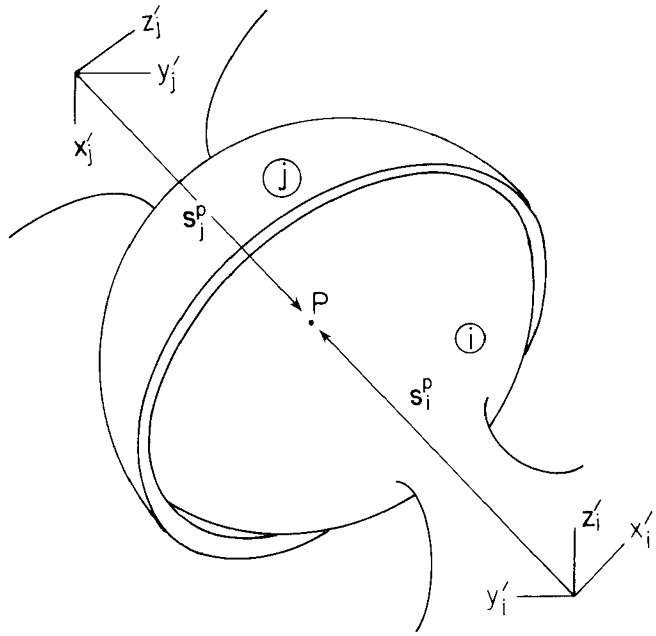

Figure 9.4.12 Spherical joint.

These three scalar constraint equations restrict only the relative position of points P_i and P_j. Three relative degrees of freedom remain.

The spherical constraint formulation does not involve a joint reference frame; so if the user does not wish to define specific frames, defaults that are parallel to the body-fixed reference frames are selected.

A *universal joint* between bodies i and j, shown in Fig. 9.4.13, is constructed with an intermediate body, or *cross,* that is pivoted in bodies i and j. The center of the cross of the universal joint is fixed in bodies i and j, defined by points P_i and P_j on the respective bodies. Points Q_i and Q_j on the arms of the cross between bodies i and j, respectively, are specified to determine the z'' axes of the joint definition frames, as shown in Fig. 9.4.13. The remaining axes of the joint definition frames are defined at the user's discretion.

The equations of constraint that characterise a universal joint are that points P_i and P_j coincide and that vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ are orthogonal. These conditions are specified by the constraint equations

$$\begin{aligned}\mathbf{\Phi}^{s}(P_i, P_j) &= \mathbf{0} \\ \Phi^{d1}(\mathbf{h}_i, \mathbf{h}_j) &= 0\end{aligned} \tag{9.4.19}$$

These four scalar constraint equations restrict the relative position of the bodies and rotation about the shafts of the cross. They allow two relative degrees of freedom between the bodies.

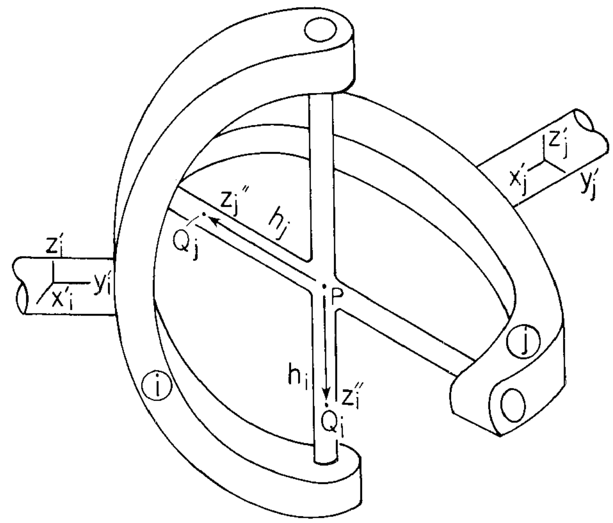

Figure 9.4.13 Universal joint.

Example 9.4.7: Consider the universal joint in Fig. 9.4.14. Let the driving angle of the first body be θ_1 and the output angle of the second body be θ_2, measured between the x-y plane and the y_2' axis. Both bodies are constrained to rotate in the x-y plane, as shown, so $\theta_2 = 0$ when $\theta_1 = 0$.

From Fig. 9.4.14,

$$\mathbf{h}_1' = [0, -1, 0]^T$$

$$\mathbf{h}_2' = [0, 0, -1]^T$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} \cos\phi & -\cos\theta_2 \sin\phi & \sin\theta_2 \sin\phi \\ \sin\phi & \cos\theta_2 \cos\phi & -\sin\theta_2 \cos\phi \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

From Eq. 9.4.3,

$$\Phi^{d1}(\mathbf{h}_1, \mathbf{h}_2) = \mathbf{h}_1'^T \mathbf{A}_1^T \mathbf{A}_2 \mathbf{h}_2'$$

$$= [0, -1, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\cos\theta_2 \sin\phi & \sin\theta_2 \sin\phi \\ \sin\phi & \cos\theta_2 \cos\phi & -\sin\theta_2 \cos\phi \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= -\cos\theta_1 \sin\theta_2 \cos\phi + \sin\theta_1 \cos\theta_2 = 0$$

Dividing by $\cos\theta_1 \cos\theta_2$,

$$\tan\theta_1 = \tan\theta_2 \cos\phi \tag{9.4.20}$$

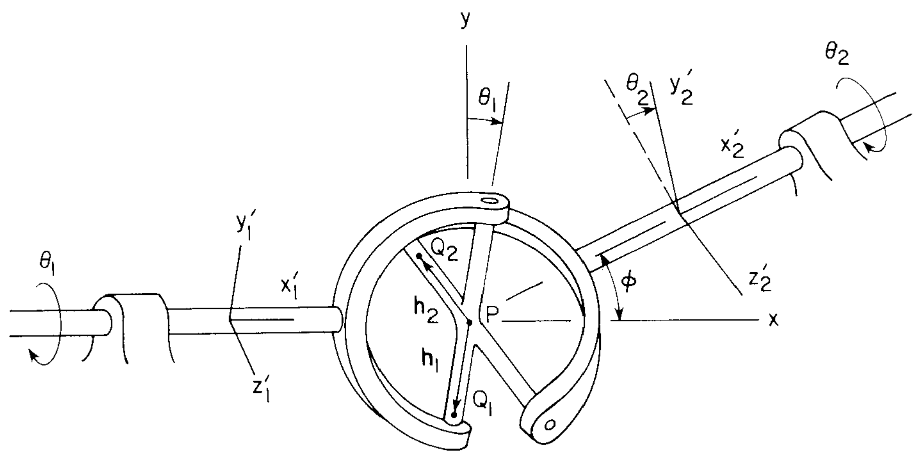

Figure 9.4.14 Universal joint example.

Hence,

$$\theta_2 = \text{Arctan}\,\frac{\tan\theta_1}{\cos\phi}$$

From Eq. 9.4.11, the variation of the dot-1 constraint is

$$-(\mathbf{h}_2'^T\mathbf{A}_2^T\mathbf{A}_1\tilde{\mathbf{h}}_1'\,\delta\boldsymbol{\pi}_1' + \mathbf{h}_1'^T\mathbf{A}_1^T\mathbf{A}_2\tilde{\mathbf{h}}_2'\,\delta\boldsymbol{\pi}_2') = 0$$

After substitution of $\tilde{\mathbf{h}}_i'$ and $\mathbf{A}_i$, $i = 1, 2$, with $\delta\boldsymbol{\pi}_1' = [\delta\theta_1, 0, 0]^T$ and $\delta\boldsymbol{\pi}_2' = [\delta\theta_2, 0, 0]^T$,

$$\begin{aligned}
&\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta_2\sin\phi & \cos\theta_2\cos\phi & \sin\theta_2 \\ \sin\theta_2\sin\phi & -\sin\theta_2\cos\phi & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\
&\quad\times \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \\
&\quad\times \begin{bmatrix} \cos\phi & -\cos\theta_2\sin\phi & \sin\theta_2\sin\phi \\ \sin\phi & \cos\theta_2\cos\phi & -\sin\theta_2\cos\phi \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta_2 \\ 0 \\ 0 \end{bmatrix} \\
&= -(\sin\theta_2\cos\phi\sin\theta_1 + \cos\theta_2\cos\theta_1)\,\delta\theta_1 \\
&\quad + (\cos\theta_1\cos\theta_2\cos\phi + \sin\theta_1\sin\theta_2)\,\delta\theta_2 = 0
\end{aligned} \tag{9.4.21}$$

Note that when $\phi = \pi/2$, hence $\cos\phi = 0$, for any $0 < \theta_2 < \pi/2$, Eq. 9.4.20 yields

$$\tan\theta_1 = 0$$

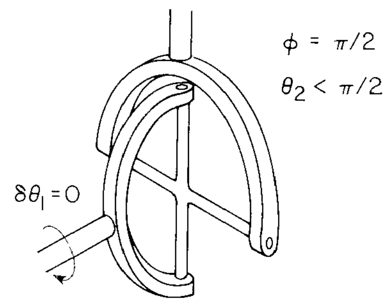

Figure 9.4.15 Singular behavior of universal joint.

which suggests that body 1 is locked. Similarly, from Eq. 9.4.21, with $\cos\phi = 0$,

$$\tan\theta_1\,\delta\theta_2 = \frac{2}{\tan\theta_2}\,\delta\theta_1$$

Since $\tan\theta_1 = 0$, this equation indicates that for any variation $\delta\theta_2$, $\delta\theta_1 = 0$. This singular behavior is interpreted as lock-up of the driving shaft, as illustrated in Fig. 9.4.15. Thus, the universal joint has a physically singular configuration, which must be avoided in applications.

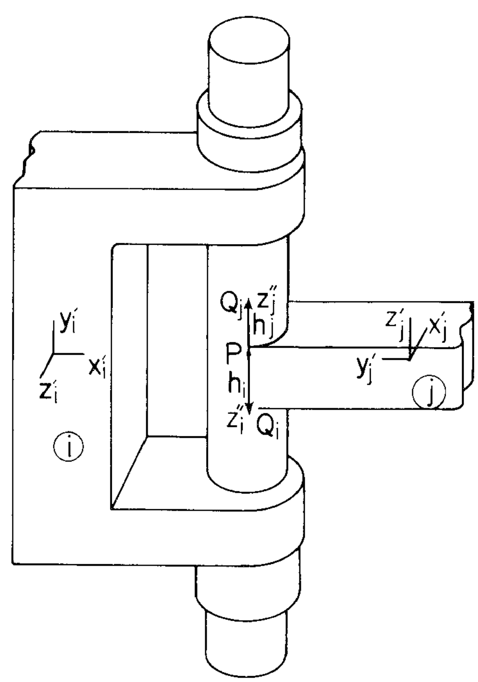

Figure 9.4.16 Revolute joint.

A *revolute joint* (or *rotational joint*) between bodies i and j is constructed with a bearing that allows relative rotation about a common axis, but precludes relative translation along this axis, as shown in Fig. 9.4.16. To define the revolute joint, the center of the joint is located on bodies i and j by points P_i and P_j. The axis of relative rotation is defined in bodies i and j by points Q_i and Q_j, and hence unit vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ along the respective z'' axes of the joint definition frames. The remaining joint definition frame axes are defined at the convenience of the user.

The analytical formulation of the revolute joint is that points P_i and P_j coincide and that body-fixed vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ are parallel, leading to the constraint equations

$$\begin{aligned}\mathbf{\Phi}^s(P_i, P_j) &= \mathbf{0}\\ \mathbf{\Phi}^{p1}(\mathbf{h}_i, \mathbf{h}_j) &= \mathbf{0}\end{aligned} \tag{9.4.22}$$

These five scalar constraint equations yield only one relative degree of freedom, rotation about the common axis of the bearing.

Example 9.4.8: Consider the revolute joint shown in Fig. 9.4.17. Since the orientation of body 1 does not change with respect to the x-y-z reference frame,

$$\mathbf{A}_1 = \mathbf{I}$$

From Fig. 9.4.17,

$$\begin{aligned}\mathbf{r}_1 = \mathbf{r}_2 &= \mathbf{0}\\ \mathbf{s}_1'^P = \mathbf{s}_2'^P &= \mathbf{0}\end{aligned}$$

Since body 2 rotates about the z axis, its orientation axis is

$$\mathbf{u}_2 = [0, 0, 1]^T$$

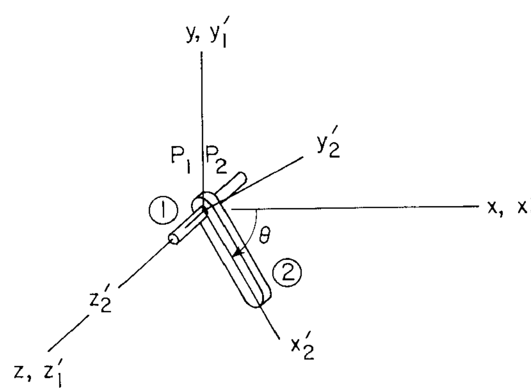

Figure 9.4.17 Revolute joint example.

Noting that rotation θ shown in Fig. 9.4.17 is clockwise, $\chi = -\theta$, and

$$e_0 = \cos\frac{\theta}{2}$$

$$\mathbf{e} = \left[0, 0, \ -\sin\frac{\theta}{2}\right]^T$$

Therefore,

$$\mathbf{A}_2 = 2\begin{bmatrix} e_0^2 - \frac{1}{2} & -e_0e_3 & 0 \\ e_0e_3 & e_0^2 - \frac{1}{2} & 0 \\ 0 & 0 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Eq. 9.4.7,

$$\mathbf{\Phi}^s = \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}_2'^P - \mathbf{r}_1 - \mathbf{A}_1\mathbf{s}_1'^P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \mathbf{I}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

for all θ. From Eq. 9.4.9,

$$\mathbf{\Phi}^{p1}(\mathbf{h}_1, \mathbf{h}_2) = \begin{bmatrix} \Phi^{d1}(\mathbf{f}_1, \mathbf{h}_2) \\ \Phi^{d1}(\mathbf{g}_1, \mathbf{h}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1'^T\mathbf{A}_1^T\mathbf{A}_2\mathbf{h}_2' \\ \mathbf{g}_1'^T\mathbf{A}_1^T\mathbf{A}_2\mathbf{h}_2' \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} [\cos\theta, \sin\theta, 0][0, 0, 1]^T \\ [-\sin\theta, \cos\theta, 0][0, 0, 1]^T \end{bmatrix} = \mathbf{0}$$

for all θ. Therefore, the revolute joint constraint equations of Eq. 9.4.22 are satisfied, for all θ.

A *cylindrical joint* is similar to a revolute joint in that it allows relative rotation about a common axis in two bodies. However, it does not preclude relative translation along this axis, as shown in Fig. 9.4.18. Joint definition points P_i and P_j are located on the common axis of rotation and additional points Q_i and Q_j on each body are defined along the axis of relative rotation, to establish the z''

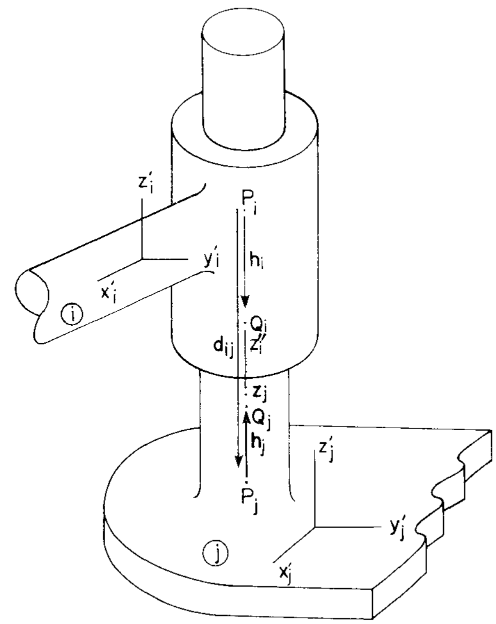

Figure 9.4.18 Cylindrical joint.

unit vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ on bodies i and j, respectively. The remaining axes of the joint definition frames are defined at the convenience of the user.

The analytical definition of the cylindrical joint is that vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ are collinear. Since these vectors and $\mathbf{d}_{ij}$ have points in common, collinearity is guaranteed by the conditions that $\mathbf{h}_i$ is parallel to both $\mathbf{h}_j$ and $\mathbf{d}_{ij}$, if $\mathbf{d}_{ij} \neq \mathbf{0}$; that is,

$$
\begin{aligned}
\mathbf{\Phi}^{p1}(\mathbf{h}_i, \mathbf{h}_j) &= \mathbf{0} \\
\mathbf{\Phi}^{p2}(\mathbf{h}_i, \mathbf{d}_{ij}) &= \mathbf{0}
\end{aligned}
\tag{9.4.23}
$$

Note that even when $\mathbf{d}_{ij} = \mathbf{0}$, P_i and P_j coincide and vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ have a point in common. Thus, the first parallelism condition of Eq. 9.4.23 implies collinearity, so Eq. 9.4.23 implies the geometry of the joint.

The cylindrical joint consists of four constraint equations; hence it allows two relative degrees of freedom between the bodies connected, relative rotation about the $\mathbf{h}_i = \mathbf{h}_j$ axis, and relative translation along this axis.

Example 9.4.9: Consider the cylindrical joint shown in Fig. 9.4.19. The only difference between this cylindrical joint and the revolute joint in Example 9.4.8 is

$$\mathbf{r}_2 = [0, 0, s]^T = \mathbf{d}_{12}$$

where s is a variable. It is shown in Example 9.4.8 that the parallel-2 constraint

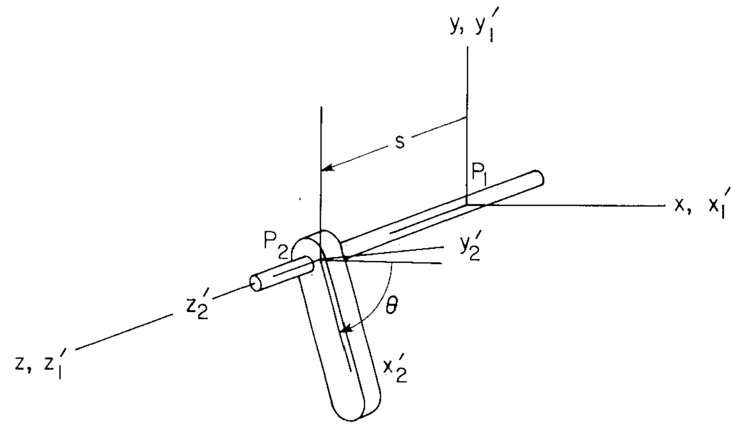

Figure 9.4.19 Cylindrical joint example.

of Eq. 9.4.23 is identically satisfied. From Eq. 9.4.10,

$$\mathbf{\Phi}^{p2}(\mathbf{h}_1, \mathbf{d}_{12}) = \begin{bmatrix} \Phi^{d2}(\mathbf{f}_1, \mathbf{d}_{12}) \\ \Phi^{d2}(\mathbf{g}_1, \mathbf{d}_{12}) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_1'^T \mathbf{A}_1^T \mathbf{d}_{12} \\ \mathbf{g}_1'^T \mathbf{A}_1^T \mathbf{d}_{12} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} [\cos\theta, \sin\theta, 0][0, 0, s]^T \\ [-\sin\theta, \cos\theta, 0][0, 0, s]^T \end{bmatrix} = \mathbf{0}$$

for all θ and s. Therefore, the cylindrical joint constraint of Eq. 9.4.23 is satisfied, for all θ and s.

The *translational joint* shown in Fig. 9.4.20 allows relative translation along a common axis between a pair of bodies, but precludes relative rotation about this axis. It is defined, as in the case of the cylindrical joint, by joint definition points P_i, Q_i, P_j, and Q_j along the axis of translation. The x'' axes of the joint definition frames on bodies i and j are selected so that they are perpendicular, defined by vectors $\mathbf{f}_i$ and $\mathbf{f}_j$, as shown in Fig. 9.4.20.

The analytical definition of a translational joint may be written by using the constraint equations of the cylindrical joint of Eq. 9.4.23 and adding the dot-1

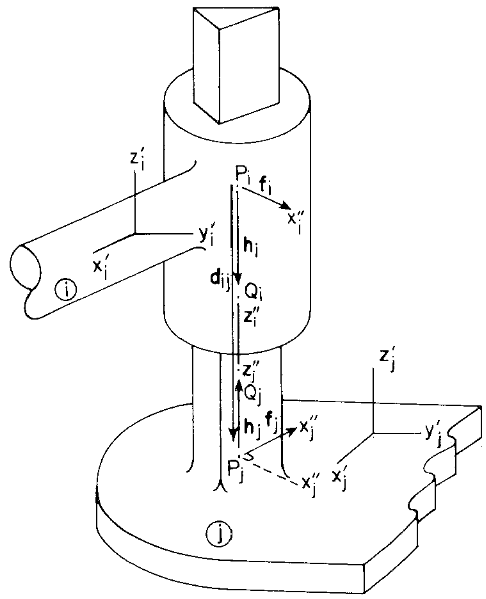

Figure 9.4.20 Translational joint.

condition that vectors $\mathbf{f}_i$ and $\mathbf{f}_j$ are orthogonal; that is,

$$
\begin{aligned}
\boldsymbol{\Phi}^{p1}(\mathbf{h}_i, \mathbf{h}_j) &= \mathbf{0} \\
\boldsymbol{\Phi}^{p2}(\mathbf{h}_i, \mathbf{d}_{ij}) &= \mathbf{0} \\
\Phi^{d1}(\mathbf{f}_i, \mathbf{f}_j) &= 0
\end{aligned}
\qquad \textbf{(9.4.24)}
$$

Since this joint comprises five constraint equations, only one relative translational degree of freedom exists between the bodies.

Example 9.4.10: Consider the translational joint shown in Fig. 9.4.21, where

$$
\begin{aligned}
\mathbf{A}_1 &= \mathbf{I} \\
\mathbf{A}_2 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{r}_1 &= 0 \\
\mathbf{r}_2 &= [0, 0, s]^T
\end{aligned}
$$

where s is the location of P_2 along the z axis. From Fig. 9.4.21,

$$\mathbf{s}_1'^P = \mathbf{s}_2'^P = \mathbf{0}$$

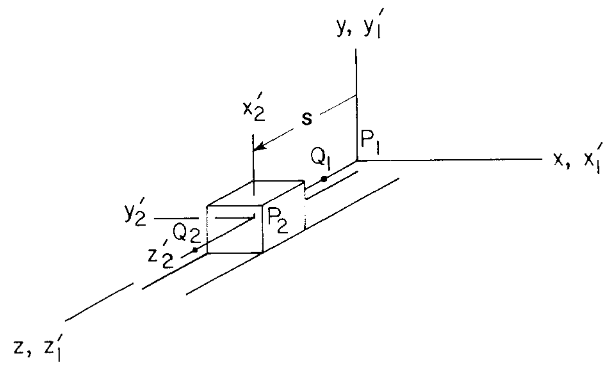

Figure 9.4.21 Translational joint example.

From Eq. 9.4.9,

$$\boldsymbol{\Phi}^{p1}(\mathbf{h}_1, \mathbf{h}_2) = \begin{bmatrix} \Phi^{d1}(\mathbf{f}_1, \mathbf{h}_2) \\ \Phi^{d1}(\mathbf{g}_1, \mathbf{h}_2) \end{bmatrix}$$

$$= \begin{bmatrix} [1, 0, 0]\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ [0, 1, 0]\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} [0, -1, 0][0, 0, 1]^T \\ [1, 0, 0][0, 0, 1]^T \end{bmatrix} = \mathbf{0}$$

From Eq. 9.4.10,

$$\boldsymbol{\Phi}^{p2}(\mathbf{h}_1, \mathbf{d}_{12}) = \begin{bmatrix} \Phi^{d2}(\mathbf{f}_1, \mathbf{d}_{12}) \\ \Phi^{d2}(\mathbf{g}_1, \mathbf{d}_{12}) \end{bmatrix}$$

$$= \begin{bmatrix} [1, 0, 0][0, 0, s]^T \\ [0, 1, 0][0, 0, s]^T \end{bmatrix} = \mathbf{0}$$

for all s. Finally, from Eq. 9.4.2,

$$\Phi^{d1}(\mathbf{f}_1, \mathbf{f}_2) = [1, 0, 0]\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= [0, -1, 0][1, 0, 0]^T = 0$$

Therefore, the translational joint constraint equations of Eq. 9.4.24 are satisfied for all s.

The *screw joint* shown in Fig. 9.4.22 is a cylindrical joint between bodies i and j, with the additional condition that relative translation along the common axis of rotation is specified by a *screw pitch* α times the relative angle of rotation between the bodies. The relative angle θ of rotation is defined as the angle between the body-fixed x_i'' and x_j'' axes, counterclockwise taken as positive, including the cumulative angle of rotation. In terms of vectors shown in Fig. 9.4.22, the advance condition for the screw is

$$\mathbf{h}_i^T\mathbf{d}_{ij} = \alpha(\theta + 2n\pi - \theta_0) \tag{9.4.25}$$

where θ_0 is the angle between the body-fixed x'' axes when $P_i = P_j$ and n is the cumulative number of revolutions; that is, $0 \leq (1/\alpha)\mathbf{h}_i^T\mathbf{d}_{ij} - 2n\pi + \theta_0 \leq 2\pi$.

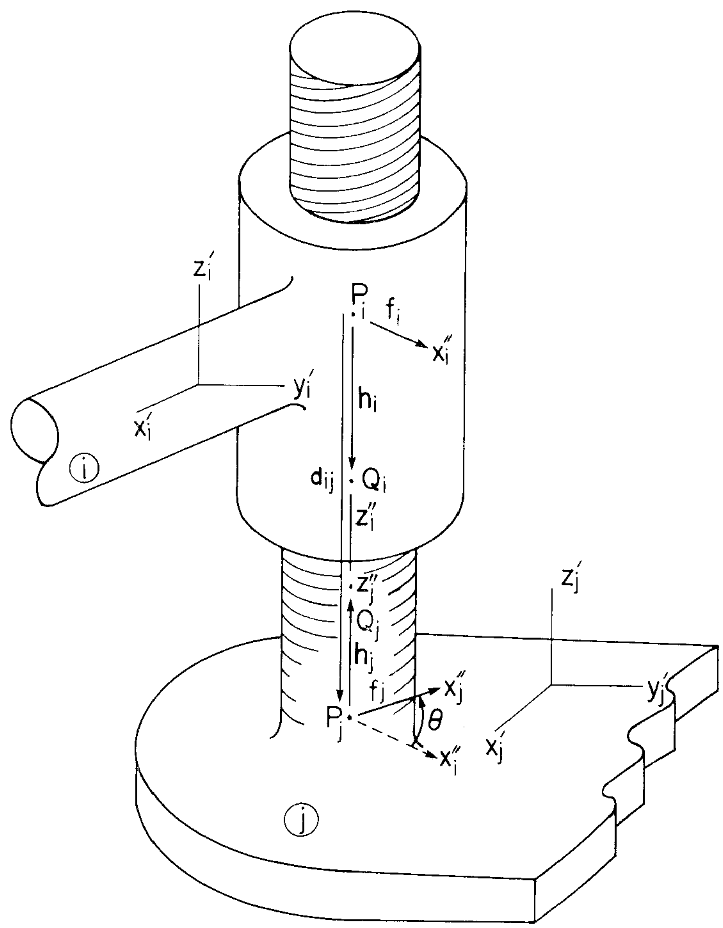

Figure 9.4.22 Screw joint.

The screw joint constraint equations are thus the cylindrical joint equations of Eq. 9.4.23 and

$$\Phi^{scr}(\mathbf{h}_i, \mathbf{d}_{ij}, \alpha, \theta_0) = \Phi^{d2}(\mathbf{h}_i, \mathbf{d}_{ij}) - \alpha(\theta + 2n\pi - \theta_0) = 0 \qquad \textbf{(9.4.26)}$$

The differential of Eq. 9.4.26 is calculated using the differential of the dot-2 constraint in Eq. 9.4.12 and the differential of θ in Eq. 9.2.61. Similarly, the partial derivatives of Φ^{scr} are obtained from the differentials of the dot-2 constraint in Table 9.4.1 and differentials of θ from Eq. 9.2.61, using $\delta\boldsymbol{\pi}' = 2\mathbf{G}\,\delta\mathbf{p}$ from Eq. 9.3.41, to obtain

$$\begin{aligned}
&\frac{\partial\theta}{\partial\mathbf{r}_i} = \frac{\partial\theta}{\partial\mathbf{r}_i} = \mathbf{0}\\
&\theta_{\boldsymbol{\pi}'_i} = -\mathbf{h}_i'^T\\
&\frac{\partial\theta}{\partial\mathbf{p}_i} = -2\mathbf{h}_i'^T\mathbf{G}_i\\
&\theta_{\boldsymbol{\pi}'_j} = \mathbf{h}_i'^T\mathbf{A}_i^T\mathbf{A}_j\\
&\frac{\partial\theta}{\partial\mathbf{p}_j} = 2\mathbf{h}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{G}_j
\end{aligned} \qquad \textbf{(9.4.27)}$$

Example 9.4.11: Consider the cylindrical joint in Example 9.4.9 and Eq. 9.4.25, that is,

$$\begin{aligned}
\mathbf{h}_1^T\mathbf{d}_{12} &= \mathbf{h}_1'^T\mathbf{A}_1^T\mathbf{d}_{12}\\
&= [0, 0, 1]\mathbf{I}[0, 0, s]^T\\
&= [0, 0, 1][0, 0, s]^T\\
&= s
\end{aligned}$$

Therefore, Eq. 9.4.25 is satisfied if

$$s = \alpha(\theta + 2n\pi - \theta_0)$$

9.4.5 Composite Constraints between Pairs of Bodies

Quite often in applications, a pair of bodies is connected by an intermediate body (or *coupler*) that serves only to define the kinematic constraints between bodies that are connected, called a *composite joint.* In such cases, it is convenient and computationally efficient to derive equivalent kinematic constraints between the pair of bodies connected, without introducing the coupler as a separate body, with its associated Cartesian generalized coordinates. Several such composite joints are derived in this section.

The *spherical–spherical composite joint* shown in Fig. 9.4.23 consists of bodies i and j and a coupler that contains spherical joints at each end, attached to

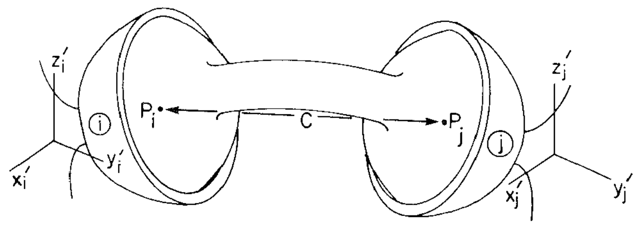

Figure 9.4.23 Spherical–spherical composite joint.

bodies i and j. The joint is defined by locating joint definition points P_i and P_j in bodies i and j, respectively, and by specifying the distance $C \neq 0$ between spherical joints on the coupler, as shown in Fig. 9.4.23. The orientation of joint definition frames is arbitrary, with defaults taken as parallel to the body-fixed reference frames.

The analytical definition of the spherical–spherical joint is simply that the distance between points P_i and P_j be equal to $C \neq 0$; that is,

$$\Phi^{ss}(P_i, P_j, C) = 0 \tag{9.4.28}$$

Since this joint is characterized by only one constraint equation, it allows five relative degrees of freedom between the bodies connected.

The *revolute–spherical composite joint* shown in Fig. 9.4.24 is comprised of a coupler with a spherical joint connected to body j and a revolute joint connected to body i. Joint definition points P_j and P_i define the centers of the respective joints on the pair of bodies. The axis of rotation in body i is defined by the z_i'' joint definition axis, and hence the unit vector $\mathbf{h}_i$. The remaining axes of the joint definition frames are arbitrary. It is required that the vector $\mathbf{d}_{ij} \neq \mathbf{0}$

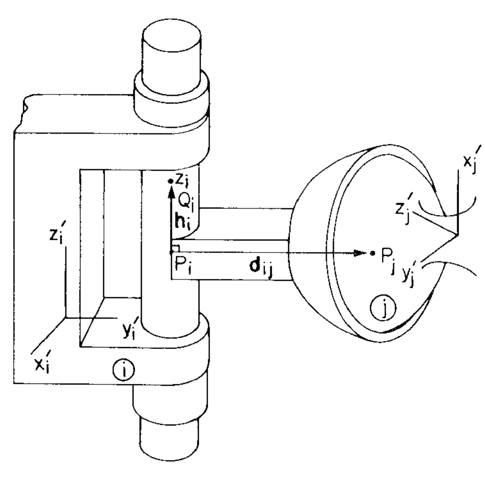

Figure 9.4.24 Revolute–spherical composite joint.

between points P_i and P_j be orthogonal to the axis of rotation of the revolute joint through P_i in the coupler.

The analytical definition of the revolute–spherical joint is that the distance between points P_i and P_j be equal to $C \neq 0$ and that vectors $\mathbf{h}_i$ and $\mathbf{d}_{ij}$ be orthogonal; that is,

$$\begin{aligned} \Phi^{ss}(P_i, P_j, C) &= 0 \\ \Phi^{d2}(\mathbf{h}_i, \mathbf{d}_{ij}) &= 0 \end{aligned} \tag{9.4.29}$$

Since two constraint equations characterize this joint, it allows four relative degrees of freedom between the bodies connected.

The *revolute–revolute composite joint with parallel axes* is defined by a coupler with parallel rotational joints that are at a fixed distance $C \neq 0$ apart, as shown in Fig. 9.4.25. Joint definition frames are fixed in bodies i and j, with their z'' axes along parallel axes of rotation, characterized by vectors $\mathbf{h}_i$ and $\mathbf{h}_j$. It is required that the vector $\mathbf{d}_{ij} \neq \mathbf{0}$ from P_i to P_j be orthogonal to each of the rotational axes. The remaining axes of the joint definition frames are arbitrary.

The analytical formulation of this composite joint is obtained by requiring that vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ be parallel, that $\mathbf{h}_i$ and $\mathbf{d}_{ij}$ be orthogonal, and that the distance between points P_i and P_j be $C \neq 0$; that is,

$$\begin{aligned} \mathbf{\Phi}^{p1}(\mathbf{h}_i, \mathbf{h}_j) &= \mathbf{0} \\ \Phi^{d2}(\mathbf{h}_i, \mathbf{d}_{ij}) &= 0 \\ \Phi^{ss}(P_i, P_j, C) &= 0 \end{aligned} \tag{9.4.30}$$

Since this constraint is characterized by four scalar equations, it allows two relative degrees of freedom between the bodies connected.

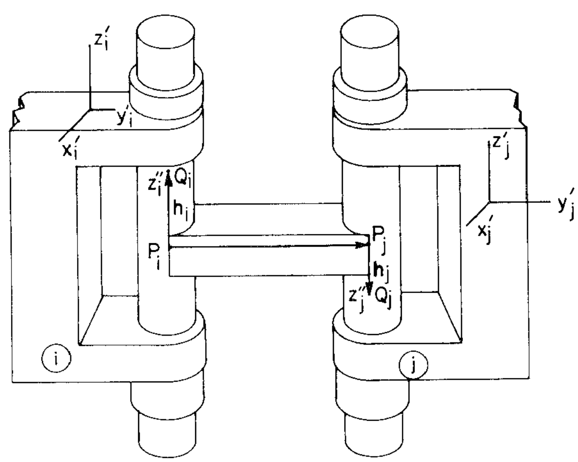

Figure 9.4.25 Revolute–revolute composite joint with parallel axes.

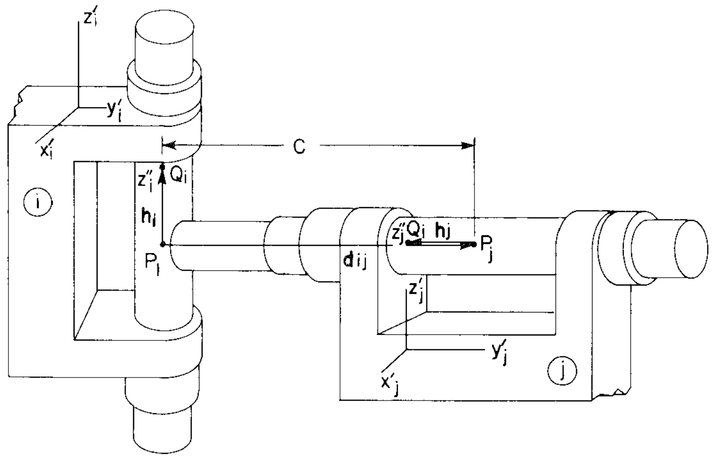

Figure 9.4.26 Revolute–revolute composite joint with orthogonal intersecting axes.

The *revolute–revolute composite joint with orthogonal intersecting axes* shown in Fig. 9.4.26 consists of a coupler with revolute joints connected to bodies i and j. In this case, the axes of the revolute joints, z_i'' and z_j'', intersect and are orthogonal. The z'' axes in bodies i and j are characterized by unit vectors $\mathbf{h}_i$ and $\mathbf{h}_j$, respectively. The remaining axes of the joint definition frames are arbitrary.

The analytical definition of this joint is obtained by requiring that $\mathbf{h}_i$ be perpendicular to $\mathbf{d}_{ij} \neq \mathbf{0}$, $\boldsymbol{h}_j$ be parallel to $\mathbf{d}_{ij}$, and the distance between points P_i and P_j be $C \neq 0$; that is,

$$\begin{aligned} \Phi^{d2}(\mathbf{h}_i, \mathbf{d}_{ij}) &= 0 \\ \boldsymbol{\Phi}^{p2}(\mathbf{h}_j, \mathbf{d}_{ji}) &= \mathbf{0} \\ \Phi^{ss}(P_i, P_j, C) &= 0 \end{aligned} \tag{9.4.31}$$

Since this composite joint is characterized by four scalar equations, it allows two relative degrees of freedom between the bodies connected.

The *revolute–cylindrical composite joint* shown in Fig. 9.4.27 consists of a coupler that is constrained to body i by a revolute joint about the $\mathbf{h}_i$ axis on body i and to body j through a cylindrical joint about the $\mathbf{h}_j$ axis. Vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ are required to be orthogonal. The geometrical conditions that define the revolute–cylindrical joint are that $\mathbf{h}_j$ be perpendicular to $\mathbf{h}_i$ and that $\mathbf{h}_j$ pass through point P_i.

The analytical definition of this joint is obtained by requiring that $\mathbf{h}_i$ and $\mathbf{h}_j$ be perpendicular and that, providing $\mathbf{d}_{ij} \neq \mathbf{0}$, $\mathbf{h}_j$ be parallel to $\mathbf{d}_{ij}$; that is,

$$\begin{aligned} \Phi^{d1}(\mathbf{h}_i, \mathbf{h}_j) &= 0 \\ \boldsymbol{\Phi}^{p2}(\mathbf{h}_j, \mathbf{d}_{ji}) &= \mathbf{0} \end{aligned} \tag{9.4.32}$$

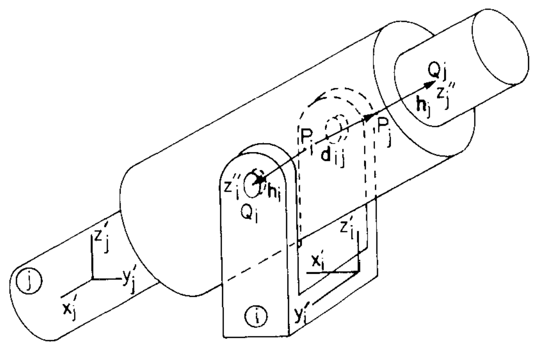

Figure 9.4.27 Revolute–cylindrical composite joint.

Note that if $\mathbf{d}_{ij} = \mathbf{0}$ then $P_i = P_j$ and the geometric conditions of the joint are satisfied. Thus, Eq. 9.4.32 is equivalent to the geometric definition of the joint. Since three scalar constraint equations comprise the definition of the revolute–cylindrical joint, there are three relative degrees of freedom between bodies i and j.

The *revolute–translational composite joint* shown in Fig. 9.4.28 is comprised of bodies i and j and a coupler between the bodies that is pivoted about $\mathbf{h}_i$ in body i. Body j can translate along vector $\mathbf{h}_j$, but cannot rotate about this axis relative to the coupler. The geometric conditions that define the revolute–translational composite joint include the conditions of the revolute–cylindrical joint, but in addition require that $\mathbf{f}_j$ and $\mathbf{h}_i$ remain parallel.

Analytical conditions that define the revolute–translational joint are that vectors $\mathbf{h}_i$ and $\mathbf{f}_j$ be parallel and that, if $\mathbf{d}_{ij} \neq \mathbf{0}$, it must be parallel to vector $\mathbf{h}_j$: that is,

$$\begin{aligned} \mathbf{\Phi}^{p1}(\mathbf{h}_i, \mathbf{f}_j) &= \mathbf{0} \\ \mathbf{\Phi}^{p2}(\mathbf{h}_j, \mathbf{d}_{ij}) &= \mathbf{0} \end{aligned} \qquad \textbf{(9.4.33)}$$

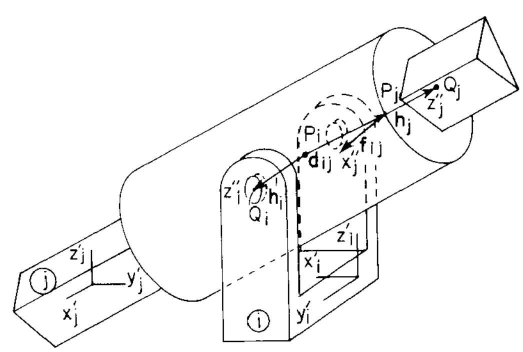

Figure 9.4.28 Revolute–translational composite joint.

In the case where $\mathbf{d}_{ij} = \mathbf{0}$, the second of Eqs. 9.4.33 reduces to the condition that $P_i = P_j$; hence $\mathbf{h}_j$ passes through point P_i. The four scalar constraint equations of Eq. 9.4.33 allow two relative degrees of freedom between bodies i and j.

Example 9.4.12: Consider the slider–crank mechanism shown in Fig. 9.4.29, where the slider (body 2) is to translate along the global x axis. Several kinematically equivalent models can be created using composite joints as couplers between bodies 1 and 2 and different combinations of absolute constraints on the slider. Each model has two bodies, suppressing ground, yielding 14 generalized coordinates.

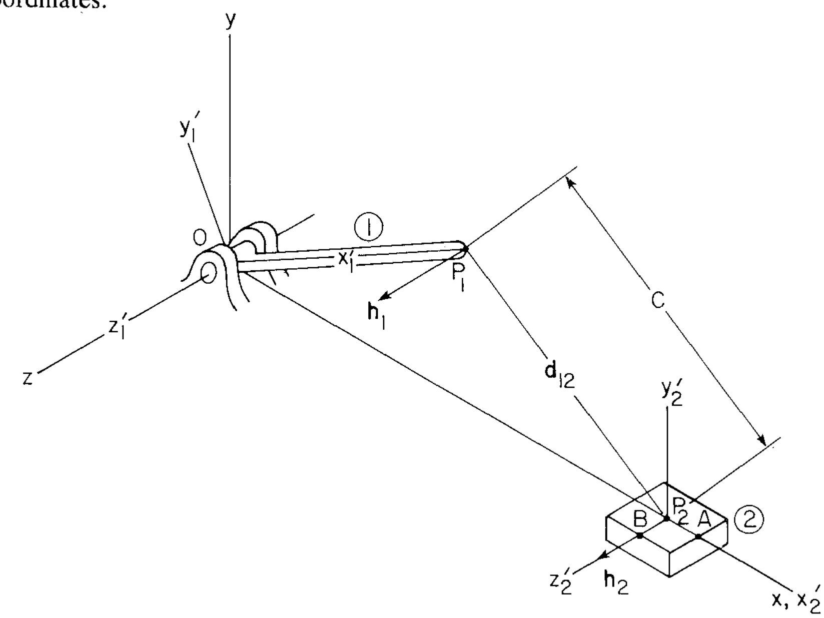

Figure 9.4.29 Slider–crank with composite joint.

(1) Model 1: Spherical-Spherical Joint	
Body 2: absolute constraints	
$y_2^P = 0$	1
$z_2^P = 0$	1
$e_1 = 0$	1
$e_2 = 0$	1
$e_3 = 0$	1
Spherical–spherical joint (coupler)	1
Revolute joint at O	5
Euler parameter normalization constraints	2
	13
DOF $= 14 - 13 = 1$.	

(2) Model 2: Revolute–Spherical Joint	
Body 2: absolute constraints	
$y_2^P = 0$	1
$e_1 = 0$	1
$e_2 = 0$	1
$e_3 = 0$	1
Revolute–spherical joint (coupler)	2
Revolute joint at O	5
Euler parameter normalization constraints	2
	13
DOF $= 14 - 13 = 1$.	

(3) Model 3: Revolute–Revolute Joint (Parallel Axes)	
Vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ shown in Fig. 9.4.29 define the revolute–revolute joint; that is, it is required that $\mathbf{h}_1$ and $\mathbf{h}_2$ be parallel.	
Body 2: absolute constraints	
$y_2^P = 0$	1
$y_2^A = 0$	1
Revolute–revolute joint (coupler)	4
Revolute joint at O	5
Euler parameter normalization constraints	2
	13
DOF $= 14 - 13 = 1$.	

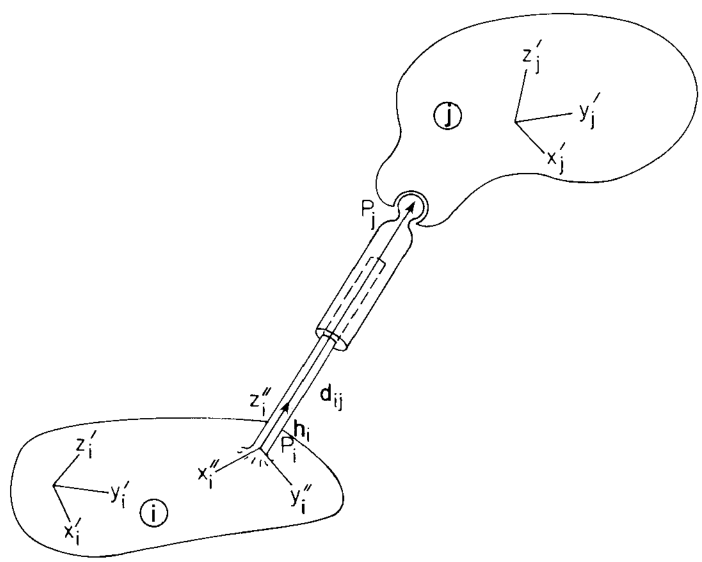

Figure 9.4.30 Strut composite joint.

The *strut composite joint* shown in Fig. 9.4.30 consists of a coupler that has a cylindrical joint about the fixed vector $\mathbf{h}_i$ in body i and a spherical joint at point P_j in body j. Point P_j on the strut lies on the axis of the coupler's cylindrical joint. The geometrical definition of this composite joint is that vector $\mathbf{h}_i$ passes through point P_j.

Providing $\mathbf{d}_{ij} \neq \mathbf{0}$, the geometry of the joint is guaranteed by the condition that $\mathbf{h}_i$ and $\mathbf{d}_{ij}$ be parallel; that is,

$$\boldsymbol{\Phi}^{p2}(\mathbf{h}_i, \mathbf{d}_{ij}) = \mathbf{0} \tag{9.4.34}$$

Note that when $\mathbf{d}_{ij} = \mathbf{0}$ points P_j and P_i coincide, so $\mathbf{h}_i$ passes through P_j. Thus, Eq. 9.4.34 defines the geometry of the strut, even when $\mathbf{d}_{ij} = \mathbf{0}$.

9.5 DRIVING CONSTRAINTS

Actuators may be employed to specify the time history of the position or orientation of one body relative to another or relative to ground in a mechanical system. Such time-dependent constraints are called *driving constraints*. A library of driving constraints that may be used for the kinematic analysis of mechanical systems is presented in this section.

In the case where constraints placed on the x, y, or z coordinates of point P_i on body i, as shown in Fig. 9.4.8, are dependent on time, the first three constraints of Eq. 9.4.15 may be written as time-dependent *absolute drivers*; that is,

$$\begin{aligned} \Phi^{1d} &= x_i^P - C_1(t) = 0 \\ \Phi^{2d} &= y_i^P - C_2(t) = 0 \\ \Phi^{3d} &= z_i^P - C_3(t) = 0 \end{aligned} \tag{9.5.1}$$

Each of these drivers restricts one degree of freedom of the motion of body i, relative to the x-y-z global reference frame.

In the case where the distance $C \neq 0$ defined in Fig. 9.4.11 and Eq. 9.4.17 is specified as a function of time (e.g., by a hydraulic or electrical actuator), a time-dependent *distance driver* is defined of the form

$$\Phi^{ssd} = \Phi^{ss}(P_i, P_j, 0) - (C(t))^2 \tag{9.5.2}$$

where $C(t) \neq 0$ is the distance between points P_i and P_j that are connected by the actuator. This scalar constraint equation eliminates one relative degree of freedom between bodies i and j.

If a pair of bodies is connected by a translational, cylindrical, screw, or strut joint, there is an axis of relative translation defined by the collinear body-fixed vector $\mathbf{h}_i$ and vector $\mathbf{d}_{ij}$, as shown in Fig. 9.5.1. The directed distance from point P_i to P_j is $\mathbf{h}_i^T\mathbf{d}_{ij}$, which is constrained to be equal to a specified function $C(t)$. In the notation of the dot-2 constraint function of Eq. 9.4.6, the *relative translational*

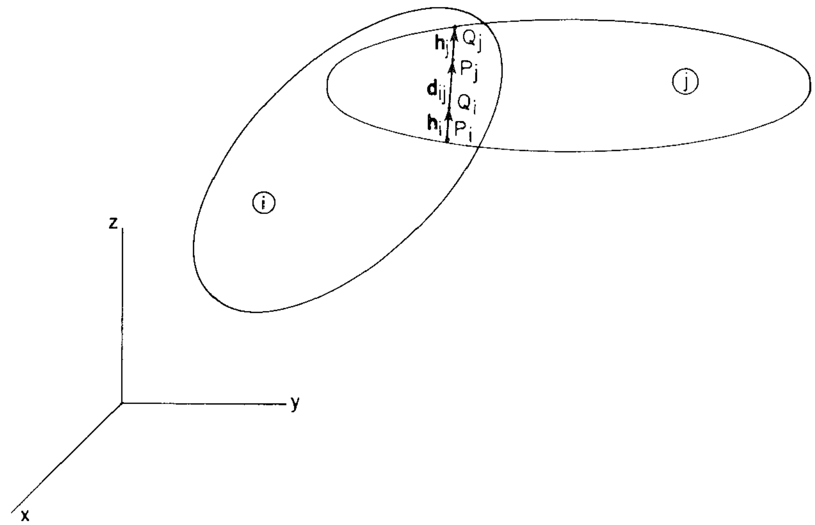

Figure 9.5.1 Relative translational driver.

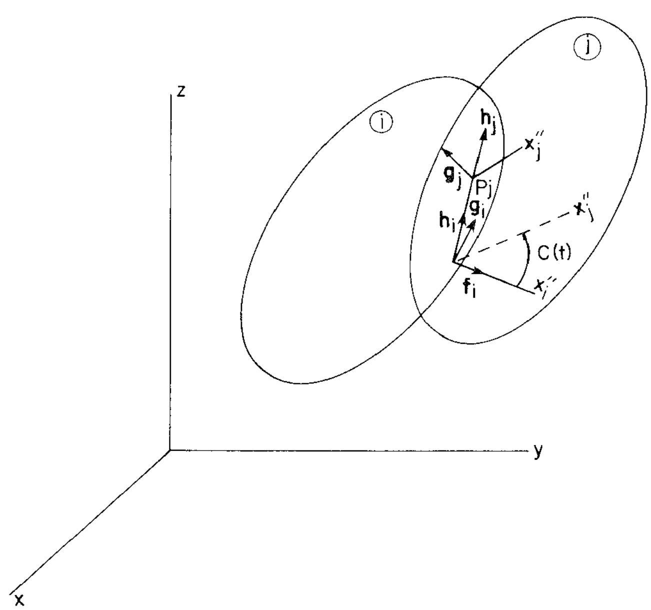

Figure 9.5.2 Relative rotational driver.

driver in translational, cylindrical, and strut screw joints is

$$\Phi^{td} = \Phi^{d2}(\mathbf{h}_i, \mathbf{d}_{ij}) - C(t) = 0 \tag{9.5.3}$$

Note that $C(t)$ can be zero or negative. This driving constraint is added to the kinematic constraint equations associated with a translational, cylindrical, screw, or strut joint to impose the driving condition.

The angle from the body-fixed x_i'' axis to the x_j'' axis, measured counterclockwise as positive, in a revolute, cylindrical, or screw joint is specified as $C(t)$, as shown in Fig. 9.5.2. The analytical definition of the *relative rotational driver* in revolute, cylindrical, and screw joints is obtained using the relative angle of rotation of Eq. 9.2.31; that is,

$$\Phi^{rotd} \equiv \theta + 2n\pi - C(t) = 0 \tag{9.5.4}$$

where n is the number of revolutions that have occurred; that is, $0 \leq C(t) - 2n\pi < 2\pi$.

Variations of each of the foregoing driving constraints are identical to the previously derived variations of corresponding kinematic constraints, since time is suppressed in calculating the variations with respect to position and orientation.

9.6 POSITION, VELOCITY, AND ACCELERATION ANALYSIS

A subtle distinction between the kinematic analysis of spatial and planar systems involves the use of angular velocity and angular acceleration variables in spatial analysis that are not time derivatives of generalized coordinates. Since position analysis requires direct computation with generalized coordinates, a different set of variables is employed for position and for velocity and acceleration analysis. The formulation of the constraint equations and their variations from Sections 9.4 and 9.5, together with the basic relations obtained in Sections 9.1 through 9.3, are used here to derive expressions that contribute to the equations for position, velocity, and acceleration of spatial systems.

9.6.1 Position Analysis

The generalized coordinate vector for body i in a system is

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{p}_i \end{bmatrix} \tag{9.6.1}$$

The composite set of generalized coordinates for the entire system is

$$\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_{nb}^T]^T \tag{9.6.2}$$

where nb is the number of bodies in the system.

In addition to the kinematic and driving constraints derived in Sections 9.4 and 9.5, the Euler parameter generalized coordinates of each body must satisfy the normalization constraint of Eq. 9.3.9:

$$\Phi_i^{\mathbf{p}} = \mathbf{p}_i^T\mathbf{p}_i - 1 = 0, \qquad i = 1, \ldots, nb \tag{9.6.3}$$

The vector of *Euler parameter normalization constraint* equations is thus

$$\mathbf{\Phi}^{\mathbf{p}} = [\Phi_1^{\mathbf{p}}, \ldots, \Phi_{nb}^{\mathbf{p}}]^T = \mathbf{0} \tag{9.6.4}$$

The combined system of kinematic, driving, and Euler parameter normalization constraint equations that determines the position and orientation of the system is

$$\mathbf{\Phi}(\mathbf{q}, t) \equiv \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \\ \mathbf{\Phi}^{\mathbf{p}}(\mathbf{q}) \end{bmatrix} = \mathbf{0} \tag{9.6.5}$$

It is presumed, for the purpose of kinematic analysis, that an adequate number of independent driving constraints has been specified so that Eq. 9.6.5 comprises $7nb$ equations in $7nb$ generalized coordinates.

To solve the nonlinear position equations in Eq. 9.6.5, the Jacobian matrix of the system must be calculated. First, the gradient of the kinematic constraint equations with respect to the generalized coordinates of body i is

$$\mathbf{\Phi}_{\mathbf{q}_i}^K = [\mathbf{\Phi}_{\mathbf{r}_i}^K, \mathbf{\Phi}_{\mathbf{p}_i}^K] = [\mathbf{\Phi}_{\mathbf{r}_i}^K, 2\mathbf{\Phi}_{\boldsymbol{\pi}_i'}^K\mathbf{G}_i] \tag{9.6.6}$$

where the matrices $\mathbf{\Phi}_{\mathbf{r}_i}^K$ and $\mathbf{\Phi}_{\boldsymbol{\pi}_i'}^k$ are derived for each of the kinematic constraints in Section 9.4. Similarly, the gradient of driving constraints with respect to the generalized coordinates of body i is

$$\mathbf{\Phi}_{\mathbf{q}_i}^D = [\mathbf{\Phi}_{\mathbf{r}_i}^D, \mathbf{\Phi}_{\mathbf{p}_i}^D] = [\mathbf{\Phi}_{\mathbf{r}_i}^D, 2\mathbf{\Phi}_{\boldsymbol{\pi}_i'}^D\mathbf{G}_i] \tag{9.6.7}$$

where the matrices $\mathbf{\Phi}_{\mathbf{r}_i}^D$ and $\mathbf{\Phi}_{\boldsymbol{\pi}_i'}^D$ are calculated in Section 9.5 for each of the driving constraints. Finally, a direct calculation of the ith Euler parameter normalization constraint of Eq. 9.6.3 yields

$$\Phi_{i_{\mathbf{q}_i}}^{\mathbf{p}} = [\mathbf{0}, 2\mathbf{p}_i^T] \tag{9.6.8}$$

For purposes of velocity and acceleration analysis, the variation of the Euler parameter normalization constraint is obtained, using the differential of Eqs. 9.6.3 and 9.3.43, as

$$\delta\Phi_i^{\mathbf{p}} = 2\mathbf{p}_i^T\,\delta\mathbf{p}_i = \mathbf{p}_i^T\mathbf{G}_i^T\,\delta\boldsymbol{\pi}_i' = 0$$

for arbitrary $\delta\boldsymbol{\pi}_i'$. Thus, using Eq. 9.3.21,

$$\Phi_{i_{\boldsymbol{\pi}_i'}}^{\mathbf{p}} = \mathbf{p}_i^T\mathbf{G}_i^T = \mathbf{0} \tag{9.6.9}$$

The derivatives of kinematic, driving, and Euler parameter normalization constraints of Eq. 9.6.6 to 9.6.8 may be combined to form the *Jacobian* of the constraint equation of Eq. 9.6.5 as

$$\mathbf{\Phi}_{\mathbf{q}} = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}}^{K} \\ \mathbf{\Phi}_{\mathbf{q}}^{D} \\ \mathbf{\Phi}_{\mathbf{q}}^{\mathbf{p}} \end{bmatrix} \tag{9.6.10}$$

If the kinematic, driving, and Euler parameter normalization constraints are independent and if DOF drivers have been specified, the Jacobian is nonsingular. Thus, provided that the system can be assembled at a nominal position, the implicit function theorem guarantees the existence of a unique solution for the position and orientation of the system in a neighborhood of the assembled configuration.

Even more pronounced than for planar kinematics in Chapter 3, the kinematic constraint equations for spatial systems are highly nonlinear and complicated, yielding little potential for an explicit solution. Therefore, an iterative technique is adopted for the solution of Eq. 9.6.5. The *Newton-Raphson method* is most commonly used. It is based on iterative computations, according to the algorithm of Section 4.5; that is,

$$\begin{aligned} \mathbf{\Phi}_{\mathbf{q}}\, \Delta\mathbf{q}^{(j)} &= -\mathbf{\Phi}(\mathbf{q}^{(j)}, t) \\ \mathbf{q}^{(j+1)} &= \mathbf{q}^{(j)} + \Delta\mathbf{q}^{(j)} \end{aligned} \tag{9.6.11}$$

where $\mathbf{q}^{(0)}$ is the initial estimate of the assembled configuration and improved estimates are obtained by solving the sequence of equations in Eq. 9.6.11, until convergence is obtained. Numerical methods for solving these equations are presented in Chapter 4.

9.6.2 Velocity Analysis

Note, from Eq. 9.6.9, that the velocity equation in $\boldsymbol{\omega}'$ variables associated with the Euler parameter normalization constraint is identically satisfied and can, therefore, be ignored. Since the Euler parameter normalization velocity equation in $\boldsymbol{\omega}'$ is identically satisfied, differentiation of this equation yields an identically satisfied acceleration equation in $\dot{\boldsymbol{\omega}}'$. Therefore, for purposes of velocity and acceleration analysis, in terms of angular velocities and angular accelerations, the Euler parameter normalization constraint is not required.

Taking time derivatives of the kinematic and driving constraint equations yields the *velocity equations*

$$\sum_{i=1}^{nb} \left\{ \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}_i}^{K} \\ \mathbf{\Phi}_{\mathbf{r}_i}^{D} \end{bmatrix} \dot{\mathbf{r}}_i + \begin{bmatrix} \mathbf{\Phi}_{\boldsymbol{\pi}_i'}^{K} \\ \mathbf{\Phi}_{\boldsymbol{\pi}_i'}^{D} \end{bmatrix} \boldsymbol{\omega}_i' \right\} = \begin{bmatrix} -\mathbf{\Phi}_t^{K} \\ -\mathbf{\Phi}_t^{D} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{v}^{K} \\ \mathbf{v}^{D} \end{bmatrix} \tag{9.6.12}$$

Since time does not arise explicitly in the kinematic constraint equations,

$$-\mathbf{\Phi}_t^{K} = \mathbf{0} \equiv \mathbf{v}^{K} \tag{9.6.13}$$

Time does, however, arise explicitly in the driving constraints, requiring that $\mathbf{v}^{D}$

be calculated for each of the driving constraints presented in Section 9.5. From Eq. 9.5.1,

$$\begin{bmatrix} \nu^{1d} \\ \nu^{2d} \\ \nu^{3d} \end{bmatrix} = -\begin{bmatrix} \Phi_t^{1d} \\ \Phi_t^{2d} \\ \Phi_t^{3d} \end{bmatrix} = \begin{bmatrix} \dot{C}_1(t) \\ \dot{C}_2(t) \\ \dot{C}_3(t) \end{bmatrix} \tag{9.6.14}$$

From Eq. 9.5.2,

$$\nu^{ssd} = -\Phi_t^{ssd} = 2C(t)\dot{C}(t) \tag{9.6.15}$$

From Eq. 9.5.3,

$$\nu^{td} = -\Phi_t^{td} = \dot{C}(t) \tag{9.6.16}$$

Finally, from Eq. 9.5.4,

$$\nu^{rotd} = -\Phi_t^{rotd} = \dot{C}(t) \tag{9.6.17}$$

These vectors and Eq. 9.6.13 form the right side of the velocity equation in Eq. 9.6.12, for $\dot{\mathbf{r}}_i$ and $\boldsymbol{\omega}_i'$. If $\dot{\mathbf{p}}_i$ is desired, it can be obtained from Eq. 9.3.35.

It is interesting to note that the coefficient matrix in the velocity equations of Eq. 9.6.12 is different from the Jacobian matrix in position analysis. Therefore, when velocity analysis is carried out using angular velocities, the velocity coefficient matrix must be employed and care must be taken that both the Jacobian for position analysis and the coefficient matrix in velocity analysis be nonsingular.

9.6.3 Acceleration Analysis

Kinematic acceleration equations are obtained by differentiating the velocity equations of Eq. 9.6.12. Since the terms arising in Eq. 9.6.12 involve the rotation matrix $\mathbf{A}$ for each body in the system, it is helpful to recall the identity of Eq. 9.2.40:

$$\dot{\mathbf{A}} = \mathbf{A}\tilde{\boldsymbol{\omega}}' \tag{9.6.18}$$

Note that explicit time dependence in constraint equations arises only in the driving constraints of Section 9.5 and that, in each of these constraints, the functions involved are sums of functions that depend on only generalized coordinates or time. Thus, $\boldsymbol{\Phi}_{\mathbf{q}t} = \mathbf{0}$ and differentiation of Eq. 9.6.12 yields the *acceleration equation*

$$\sum_{i=1}^{nb} \left\{ \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{r}_i}^K \\ \boldsymbol{\Phi}_{\mathbf{r}_i}^D \end{bmatrix} \ddot{\mathbf{r}}_i + \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{\pi}_i'}^K \\ \boldsymbol{\Phi}_{\boldsymbol{\pi}_i'}^D \end{bmatrix} \dot{\boldsymbol{\omega}}_i' \right\} = -\begin{bmatrix} \boldsymbol{\Phi}_{tt}^K \\ \boldsymbol{\Phi}_{tt}^D \end{bmatrix} - \sum_{i=1}^{nb} \left\{ \begin{bmatrix} \dot{\boldsymbol{\Phi}}_{\mathbf{r}_i}^K \\ \dot{\boldsymbol{\Phi}}_{\mathbf{r}_i}^D \end{bmatrix} \dot{\mathbf{r}}_i + \begin{bmatrix} \dot{\boldsymbol{\Phi}}_{\boldsymbol{\pi}_i'}^K \\ \dot{\boldsymbol{\Phi}}_{\boldsymbol{\pi}_i'}^D \end{bmatrix} \boldsymbol{\omega}_i' \right\} \equiv \begin{bmatrix} \gamma^K \\ \gamma^D \end{bmatrix} \tag{9.6.19}$$

To evaluate the terms in Eq. 9.6.19, the matrices presented in Table 9.4.1 and in related constraint variation identities derived in Sections 9.4 and 9.5 may be differentiated to evaluate terms on the right of Eq. 9.6.19. For the dot-1 constraint, differentiation of terms arising in Table 9.4.1, using Eq. 9.6.18, yields (Prob. 9.6.1)

$$
\begin{aligned}
\gamma^{d1} &= \mathbf{a}_j'^T[\mathbf{A}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i' - \tilde{\boldsymbol{\omega}}_j'\mathbf{A}_j^T\mathbf{A}_i]\tilde{\mathbf{a}}_i'\boldsymbol{\omega}_i' \\
&\quad + \mathbf{a}_i'^T[\mathbf{A}_i^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}_j' - \tilde{\boldsymbol{\omega}}_i'\mathbf{A}_i^T\mathbf{A}_j]\tilde{\mathbf{a}}_j'\boldsymbol{\omega}_j' \\
&= -\mathbf{a}_j'^T[\mathbf{A}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_j'\tilde{\boldsymbol{\omega}}_j'\mathbf{A}_j^T\mathbf{A}_i]\mathbf{a}_i \\
&\quad + 2\boldsymbol{\omega}_j'^T\tilde{\mathbf{a}}_j'\mathbf{A}_j^T\mathbf{A}_i\tilde{\mathbf{a}}_i'\boldsymbol{\omega}_i'
\end{aligned}
\tag{9.6.20}
$$

Similarly, for the dot-2 constraint (Prob. 9.6.2),

$$
\begin{aligned}
\gamma^{d2} &= -\mathbf{a}_i'^T\tilde{\boldsymbol{\omega}}_i'\mathbf{A}_i^T(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \\
&\quad + [(\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j'^P - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P)^T\mathbf{A}_i\tilde{\mathbf{a}}_i' + \mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{a}}_i']\boldsymbol{\omega}_i' \\
&\quad + \mathbf{a}_i'^T[\mathbf{A}_i^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}_j' - \tilde{\boldsymbol{\omega}}_i'\mathbf{A}_i^T\mathbf{A}_j]\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j' \\
&= 2\boldsymbol{\omega}_i'^T\tilde{\mathbf{a}}_i'\mathbf{A}_i^T(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + 2\mathbf{s}_j'^{PT}\tilde{\boldsymbol{\omega}}_j'\mathbf{A}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{a}_i' \\
&\quad -\mathbf{s}_i'^{PT}\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{a}_i' - \mathbf{s}_j'^{PT}\tilde{\boldsymbol{\omega}}_j'\tilde{\boldsymbol{\omega}}_j'\mathbf{A}_j^T\mathbf{A}_i\mathbf{a}_i' - \mathbf{d}_{ij}^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{a}_i'
\end{aligned}
\tag{9.6.21}
$$

For the spherical constraint (Prob. 9.6.3),

$$
\begin{aligned}
\gamma^{s} &= -\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i' + \mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j \\
&= \mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P - \mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j'^P
\end{aligned}
\tag{9.6.22}
$$

Finally, for the spherical–spherical constraint (Prob. 9.6.4),

$$
\begin{aligned}
\gamma^{ss} &= 2[\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j'^P - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P]^T \\
&\quad \times [(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) - \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i' + \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j'] \\
&\quad - 2\mathbf{d}_{ij}^T[\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i - \mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j'] \\
&= -2(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)^T(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i) + 2\mathbf{s}_j'^{PT}\tilde{\boldsymbol{\omega}}_j'\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j'^P \\
&\quad + 2\mathbf{s}_i'^{PT}\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P - 4\mathbf{s}_j'^{PT}\tilde{\boldsymbol{\omega}}_j'\mathbf{A}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P \\
&\quad + 4(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)(\mathbf{A}_j\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j' - \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i') \\
&\quad -2\mathbf{d}_{ij}^T[\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i' - \mathbf{A}_j\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j']
\end{aligned}
\tag{9.6.23}
$$

Using differentials of the absolute constraints of Eq. 9.4.15, their velocity equations may be differentiated to obtain (Prob. 9.6.5)

$$
\begin{aligned}
\begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{bmatrix} &= \mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_i'^{\mathrm{P}}\boldsymbol{\omega}_i' \\
&= -\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P
\end{aligned}
\tag{9.6.24}
$$

$$
\begin{bmatrix} \gamma^4 \\ \gamma^5 \\ \gamma^6 \end{bmatrix} = -\tfrac{1}{2}(\tilde{\dot{\mathbf{e}}}_i + \dot{e}_{0_i}\mathbf{I})\boldsymbol{\omega}_i'
\tag{9.6.25}
$$

For the point constraint of Eq. 9.4.16 (Prob. 9.6.6),

$$\begin{aligned}\boldsymbol{\gamma}^P &= \mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i' \\ &= -\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i'^P\end{aligned} \tag{9.6.26}$$

Finally, for the angle θ in the screw joint of Eq. 9.4.27 (Prob. 9.6.7),

$$\gamma^\theta = -\mathbf{h}_i'^T(\mathbf{A}_i^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}_j' - \tilde{\boldsymbol{\omega}}_i'\mathbf{A}_i^T\mathbf{A}_j)\boldsymbol{\omega}_j' \tag{9.6.27}$$

Thus, for the screw joint of Eq. 9.4.26, Eqs. 9.6.21 and 9.6.27 yield

$$\gamma^{scr} = \gamma^{d2} - \alpha\gamma^\theta \tag{9.6.28}$$

For the driving constraints of Eq. 9.5.1, which depend explicitly on time, the foregoing derivative results and direct differentiation of time-dependent terms yield

$$\begin{bmatrix} \gamma^{1d} \\ \gamma^{2d} \\ \gamma^{3d} \end{bmatrix} = \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{bmatrix} + \begin{bmatrix} \ddot{C}_1(t) \\ \ddot{C}_2(t) \\ \ddot{C}_3(t) \end{bmatrix} \tag{9.6.29}$$

For the spherical–spherical driving constraint of Eq. 9.5.2,

$$\gamma^{ssd} = \gamma^{ss} + 2C(t)\ddot{C}(t) + 2\dot{C}(t)\dot{C}(t) \tag{9.6.30}$$

For the translational driving constraint of Eq. 9.5.3,

$$\gamma^{td} = \gamma^{d2} + \ddot{C}(t) \tag{9.6.31}$$

Finally, for the rotational driving constraint of Eq. 9.5.4,

$$\gamma^{\text{rotd}} = \gamma^\theta + \ddot{C}(t) \tag{9.6.32}$$

The right sides of acceleration equations presented in Eqs. 9.6.20 to 9.6.32 may now be substituted into Eq. 9.6.19 to obtain a set of equations to determine accelerations $\ddot{\mathbf{r}}_i$ and $\dot{\boldsymbol{\omega}}_j'$. Note that the coefficient matrix in these acceleration equations is identical to the coefficient matrix in the velocity equations of Eq. 9.6.12, yielding efficiency in calculating the solution of the acceleration equations.

PROBLEMS

Section 9.1

9.1.1. Use the Cartesian component forms of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ to verify that Eq. 9.1.6 is valid.

9.1.2. Carry out the expansion indicated to verify that Eq. 9.1.11 is valid.

9.1.3. Carry out the expansion indicated to verify that Eq. 9.1.16 is valid.

9.1.4. Use the component representation of physical vectors and the associated properties of vector addition (Eq. 9.1.5) to verify that Eq. 9.1.19 is valid.

9.1.5. Expand both sides of Eq. 9.1.25 and verify that it is valid.

9.1.6. Verify Eq. 9.1.28 by expanding both sides and show that they are equal.

9.1.7. Verify Eqs. 9.1.29 and 9.1.30 by expanding both sides and show that they are equal.

9.1.8. Given points P with $\mathbf{r}^P = [1, 1, 1]^T$, Q with $\mathbf{r}^Q = [2, 1, 0]^T$, and R with $\mathbf{r}^R = [2, 2, 0]^T$, construct unit vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ that define an x'-y'-z' Cartesian reference frame with origin at point P, using the method of Example 9.1.6.

9.1.9. Use the definition of matrix addition and properties of differentiation to verify that Eq. 9.1.34 is valid.

9.1.10. Verify that Eqs. 9.1.35 to 9.1.37 are valid.

Section 9.2

9.2.1. Find the direction cosine representations of unit vectors in the same direction as the vectors $\mathbf{a} = [1, 1, 1]^T$ and $\mathbf{b} = [1, 1, 0]^T$.

9.2.2. If the vector $\mathbf{a}$ in Prob. 9.2.1 is along the x' axis of a Cartesian x'-y'-z' reference frame and if $\mathbf{b}$ is in the first quadrant of its x'-y' plane, find the unit coordinate vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$. *Hint:* Since $\mathbf{a}$ and $\mathbf{b}$ lie in the x'-y' plane, $\tilde{\mathbf{a}}\mathbf{b}$ must be along the z' axis. Form the transformation matrix from the x'-y'-z' frame to the x-y-z frame.

9.2.3. Expand the product $\mathbf{A}^T\mathbf{A}$ using Eq. 9.2.13 to show that Eq. 9.2.14 is correct.

9.2.4. If $(\mathbf{B} - \mathbf{C})\mathbf{a} = \mathbf{0}$ for arbitrary $\mathbf{a}$, show that $\mathbf{B} = \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ matrices and $\mathbf{a}$ is in R^n.

9.2.5. If $\mathbf{A}_i$ and $\mathbf{A}_j$ are orthogonal matrices, show that $\mathbf{A}_{ij}$ of Eq. 9.2.25 is orthogonal.

9.2.6. Verify that Eq. 9.2.31 uniquely defines the angle θ, $0 \leq \theta < 2\pi$, in Fig. 9.2.10.

9.2.7. Show that Eq. 9.2.38 is correct.

9.2.8. Show that Eqs. 9.2.42 and 9.2.43 are correct.

9.2.9. Use Eqs. 9.2.41 to 9.2.43 to show that

$$\begin{aligned}\ddot{\mathbf{r}}^P &= \ddot{\mathbf{r}} - \tilde{\mathbf{s}}^P\dot{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}^P\boldsymbol{\omega} \\ &= \ddot{\mathbf{r}} - \mathbf{A}\tilde{\mathbf{s}}'^P\dot{\boldsymbol{\omega}}' - \mathbf{A}\boldsymbol{\omega}'\tilde{\mathbf{s}}'^P\boldsymbol{\omega}'\end{aligned}$$

9.2.10. Write the vector $\mathbf{r}^P$ for the point $\mathbf{s}'^P$ of Example 9.2.4, with $\mathbf{r} = \mathbf{0}$, explicitly as a function of ϕ. Differentiate the result to obtain $\dot{\mathbf{r}}^P$ and $\ddot{\mathbf{r}}^P$ and compare with the results presented in Examples 9.2.4 and 9.2.5.

9.2.11 If matrices $\mathbf{A}_{m\times p}$ and $\mathbf{B}_{p\times n}$ depend on a variable $\mathbf{q}$, then

$$\delta\mathbf{A} = \left[\frac{\partial a_{ij}}{\partial \mathbf{q}}\delta\mathbf{q}\right] \quad \text{and} \quad \delta\mathbf{B} = \left[\frac{\partial b_{ij}}{\partial \mathbf{q}}\delta\mathbf{q}\right]$$

are linear in $\delta\mathbf{q}$. Use this result to show that

$$\delta(\mathbf{AB}) = (\delta\mathbf{A})\mathbf{B} + \mathbf{A}(\delta\mathbf{B})$$

In particular, show that this result and $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ lead to Eq. 9.2.44.

9.2.12. Use Eq. 9.2.46 to show that the virtual rotation in Example 9.2.2 is $\delta\boldsymbol{\pi} = \delta\boldsymbol{\pi}' = [0, 0, \delta\phi]^T$. *Hint:* Follow the calculation carried out in Example 9.2.4.

9.2.13 Use the results of Prob. 9.2.11 and the fact that $\mathbf{s}'^P$ is constant to obtain Eq. 9.2.48.

Section 9.3

9.3.1. Repeat the derivation of Eq. 9.3.4 for the **f-i** and **g-j** relations to show that Eq. 9.3.5 is correct.

9.3.2. Show that as ϕ is incremented by 120° in Example 9.3.1, beginning from $\phi = 0$, that the vector $\mathbf{s}' = [1, 0, 0]^T$ becomes $\mathbf{s}(0) = \mathbf{s}'$, $\mathbf{s}(120°) = [0, 0, 1]^T$, $\mathbf{s}(240°) = [0, 0, 1]^T$, and $\mathbf{s}(360°) = \mathbf{s}'$. Interpret these results physically in terms of the cone that is swept out by $\mathbf{s}$ as ϕ varies.

9.3.3. Use the relations of Eqs. 9.3.11 to 9.3.15 with **A** of Eq. 9.2.17 to recover the geometry of the rotation of Example 9.2.2.

9.3.4. Show that each row of **G** is orthogonal to **p** (Eq. 9.3.20).

9.3.5. Show that the rows of **E** are orthogonal unit vectors (Eq. 9.3.22) (a) using the expanded form of matrix **E** in Eq. 9.3.18, and (b) using the compact form of matrix **E** in Eq. 9.3.18 and evaluating the matrix product

$$\mathbf{E}\mathbf{E}^T = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} -\mathbf{e}^T \\ -\tilde{\mathbf{e}} + e_0 I \end{bmatrix}$$

9.3.6. Repeat Prob. 9.3.5 for matrix **G**.

9.3.7. Take the time derivative of the identity $\mathbf{p}^T\mathbf{p} = 1$ to show that $\mathbf{p}^T\dot{\mathbf{p}} = 0$ and $\dot{\mathbf{p}}^T\mathbf{p} = 0$.

9.3.8. Use the definitions of Eqs. 9.3.18 and 9.3.19 to expand the products $\dot{\mathbf{E}}\mathbf{G}^T$ and $\dot{\mathbf{E}}\mathbf{G}^T$ and show that they are equal.

9.3.9. Show that the inverse transformation of Eq. 9.3.37 is valid.

9.3.10. Carry out the detailed derivation of Eq. 9.3.41, following the procedure outlined in the text.

9.3.11. Carry out the detailed derivation of Eq. 9.3.42, following the procedure outlined in the text.

9.3.12. Carry out the detailed derivation of Eq. 9.3.43, following the procedure outlined in the text.

9.3.13. Carry out the detailed derivation of Eq. 9.3.44, following the procedure outlined in the text.

Section 9.4

9.4.1 For Fig. P9.4.1, write three constraint equations that require that the bottom plane (x_2'-y_2') of body 2 slides on the x-y plane (x_1'-y_1'). *Hint:* The origin of the body 2 reference frame must be in the x-y plane, and z_2' is parallel to the z axis if and only if the x_2' and y_2' axes are perpendicular to the z axis.

9.4.2. Body 2 is constrained to translate along the y axis, as shown in Fig. P9.4.2. Show that the constraint equations of Problem 9.4.1 plus $x_2 = 0$ and $\Phi^{d2}(\mathbf{d}_{12}, \mathbf{f}_2) = 0$ define the constraint, provided $\mathbf{d}_{12} \neq \mathbf{0}$. Show an orientation of body 2 that satisfies these constraint equations when $\mathbf{d}_{12} = \mathbf{0}$, but violates the geometry of the constraint.

9.4.3. Show that if $C = 0$ in Eq. 9.4.8 then $\mathbf{d}_{ij} = \mathbf{0}$.

9.4.4. Carry out manipulations to show that Eq. 9.4.12 is valid.

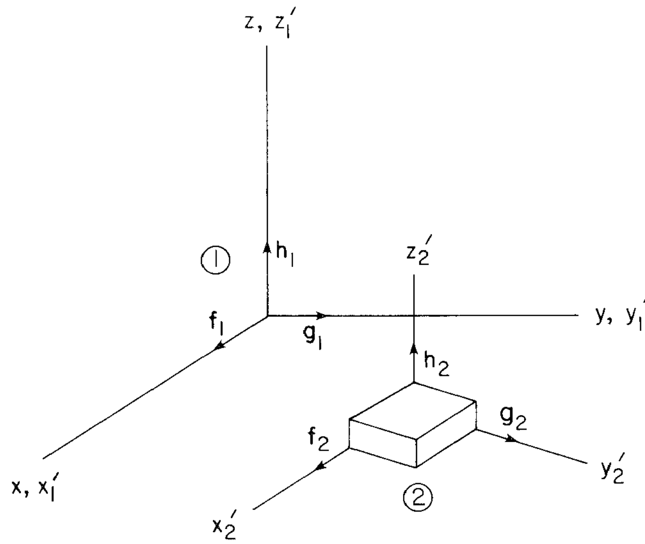

Figure P9.4.1

Section 9.6

9.6.1. Verify that Eq. 9.6.20 is valid.
9.6.2. Verify that Eq. 9.6.21 is valid.
9.6.3. Verify that Eq. 9.6.22 is valid.
9.6.4. Verify that Eq. 9.6.23 is valid.

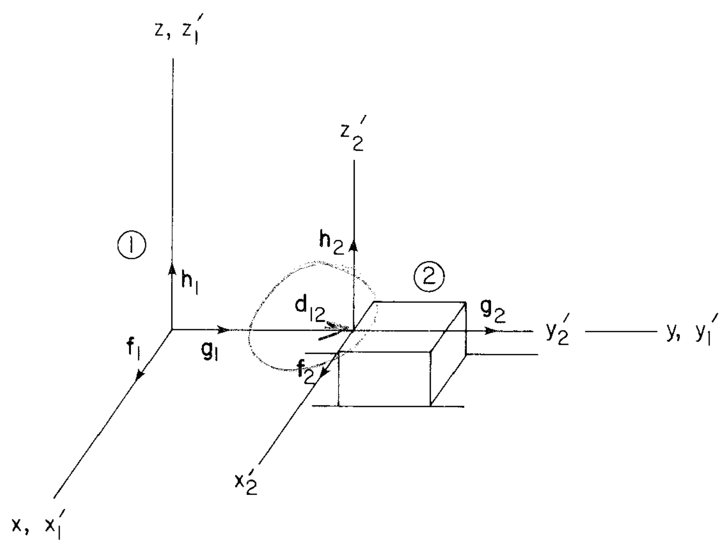

Figure P9.4.2

9.6.5. Verify that Eqs. 9.6.24 and 9.6.25 are valid.

9.6.6. Verify that Eq. 9.6.26 is valid.

9.6.7. Verify that Eq. 9.6.27 is valid.

SUMMARY OF KEY FORMULAS

Algebraic Vectors

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \qquad \textbf{(9.1.21)}$$

$$\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}, \qquad \tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a}, \qquad \tilde{\mathbf{a}}\mathbf{a} = \mathbf{0} \qquad \textbf{(9.1.23, 25, 26)}$$

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I} \qquad \textbf{(9.1.28)}$$

$$\widetilde{(\tilde{\mathbf{a}}\mathbf{b})} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}} \qquad \textbf{(9.1.29, 30)}$$

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} + \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{b}}\tilde{\mathbf{a}} + \mathbf{b}\mathbf{a}^T, \qquad \widetilde{(\mathbf{a}+\mathbf{b})} = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \qquad \textbf{(9.1.31, 32)}$$

Transformation of Coordinates

$$\mathbf{s} = \mathbf{A}\mathbf{s}', \quad \mathbf{s}' = \mathbf{A}^T\mathbf{s}, \quad \mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I} \qquad \textbf{(9.2.10, 14, 15)}$$

$$\mathbf{A} = [\mathbf{f}, \mathbf{g}, \mathbf{h}], \quad \mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \qquad \textbf{(9.2.13, 16)}$$

$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T, \quad \tilde{\mathbf{s}}' = \mathbf{A}^T\tilde{\mathbf{s}}\mathbf{A}, \quad \mathbf{A}_{ij} = \mathbf{A}_i^T\mathbf{A}_j \qquad \textbf{(9.2.21, 22, 25)}$$

Velocity, Acceleration, and Angular Velocity

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{s}^P = \dot{\mathbf{r}} + \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P \qquad \textbf{(9.2.33, 37, 38)}$$

$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\boldsymbol{\omega}}', \quad \tilde{\boldsymbol{\omega}}' = \mathbf{A}^T\dot{\mathbf{A}} \qquad \textbf{(9.2.39, 40)}$$

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{A}}\mathbf{s}'^P \qquad \textbf{(9.2.41)}$$

$$\ddot{\mathbf{A}} = \tilde{\dot{\boldsymbol{\omega}}}\mathbf{A} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\dot{\boldsymbol{\omega}}}' + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}' \qquad \textbf{(9.2.42, 43)}$$

Virtual Displacements and Rotations

$$\delta\mathbf{A} = \delta\tilde{\boldsymbol{\pi}}\mathbf{A} = \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}', \qquad \delta\tilde{\boldsymbol{\pi}}' = \mathbf{A}^T\,\delta\mathbf{A} \qquad \textbf{(9.2.47, 50)}$$

$$\delta\mathbf{r}^P = \delta\mathbf{r} + \delta\mathbf{A}\mathbf{s}'^P = \delta\mathbf{r} + \delta\tilde{\boldsymbol{\pi}}\mathbf{s}^P = \delta\mathbf{r} + \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}'\mathbf{s}'^P \qquad \textbf{(9.2.48, 49, 51)}$$

$$\delta\mathbf{s} = \delta\mathbf{A}\mathbf{s}' = \mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}'\mathbf{s}' = -\mathbf{A}\tilde{\mathbf{s}}'\,\delta\boldsymbol{\pi}' \qquad \textbf{(9.2.52, 53)}$$

$$\delta(\mathbf{g}^T\mathbf{h}) = \mathbf{h}^T\,\delta\mathbf{g} + \mathbf{g}^T\,\delta\mathbf{h}, \qquad \delta(\tilde{\mathbf{g}}\mathbf{h}) = \tilde{\mathbf{g}}\,\delta\mathbf{h} - \tilde{\mathbf{h}}\,\delta\mathbf{g} \qquad \textbf{(9.2.55, 56)}$$

Euler Parameter Definitions

$$e_0 = \cos(\chi/2), \qquad \mathbf{e} = \mathbf{u}\sin(\chi/2) \qquad \textbf{(9.3.2)}$$

$$\mathbf{A} = (2e_0^2 - 1)\mathbf{I} + (\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}) \qquad \textbf{(9.3.6)}$$

$$\mathbf{p} = [e_0, \mathbf{e}^T]^T = [e_0, e_1, e_2, e_3]^T, \quad \mathbf{p}^T\mathbf{p} = 1 \tag{9.3.8, 9}$$

Euler Parameter Relations

$$\mathbf{E} = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]_{3\times4}, \qquad \mathbf{G} = [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}]_{3\times4} \tag{9.3.18, 19}$$

$$\mathbf{E}\mathbf{E}^T = \mathbf{G}\mathbf{G}^T = \mathbf{I}, \qquad \mathbf{E}^T\mathbf{E} = \mathbf{G}^T\mathbf{G} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^T \tag{9.3.22, 23, 24}$$

$$\mathbf{A} = \mathbf{E}\mathbf{G}^T \tag{9.3.26}$$

$$\boldsymbol{\omega}' = 2\mathbf{G}\dot{\mathbf{p}}, \qquad \dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\boldsymbol{\omega}' \tag{9.3.34, 35}$$

$$\boldsymbol{\omega} = 2\mathbf{E}\dot{\mathbf{p}}, \qquad \dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{E}^T\boldsymbol{\omega} \tag{9.3.36, 37}$$

$$\delta\boldsymbol{\pi}' = 2\mathbf{G}\,\delta\mathbf{p}, \qquad \delta\mathbf{p} = \tfrac{1}{2}\mathbf{G}^T\,\delta\boldsymbol{\pi}' \tag{9.3.41, 43}$$

$$\delta\boldsymbol{\pi} = 2\mathbf{E}\,\delta\mathbf{p}, \qquad \delta\mathbf{p} = \tfrac{1}{2}\mathbf{E}^T\,\delta\boldsymbol{\pi} \tag{9.3.42, 44}$$