CHAPTER SIX

Dynamics of Planar Systems

The equations of motion of a rigid body and constrained systems of rigid bodies are derived beginning with Newton's laws for a particle and a model of rigid bodies. The *virtual work* approach is introduced to develop a self-contained derivation of both variational and differential equation formulations. The variational, or virtual work, formulation is best suited for multibody dynamics. The Lagrange multiplier form of constrained equations of motion is derived, and properties of the resulting differential–algebraic equations of motion that are important for numerical solution are developed. Special forms of the equations of motion that are suitable for inverse dynamic and equilibrium analysis are developed. Finally, reaction forces that act on bodies due to joints are derived from Lagrange multipliers.

6.1 EQUATIONS OF MOTION OF A PLANAR RIGID BODY

Consider the rigid body shown in Fig. 6.1.1, located in the x-y plane by the vector $\mathbf{r}$ and angle of rotation ϕ. A differential mass $dm(P)$ at point P is located on the body by the vector $\mathbf{s}^P$. Forces acting on this differential element of mass include external forces $\mathbf{F}(P)$ per unit of mass at point P and internal forces $\mathbf{f}(P, R)$ per units of mass located at points P and R, as shown in Fig. 6.1.1. As a *model of a rigid body,* let a distance constraint (a massless bar with revolute joints at both ends) act between each pair of differential elements (thought of as particles) in the body. With this model, physical internal forces $\mathbf{f}(P, R)\,dm(R)\,dm(P)$ and $\mathbf{f}(R, P)\,dm(P)\,dm(R)$ of interaction on $dm(P)$ and $dm(R)$ act along the massless links between points P and R, due to the distance constraint between points P and R, so they are equal in magnitude, opposite in direction, and colinear.*

* Colinearity of $\mathbf{f}(P, R)$ and $\mathbf{f}(R, P)$ in the model used here for a rigid body is a function of the model. While $\mathbf{f}(P, R)\,dm(R)\,dm(P) = -\mathbf{f}(R, P)\,dm(P)\,dm(R)$ follows from Newton's law of action–reaction, in general Newton said nothing about colinearity. This matter has been studied by many scholars and is elegantly summarized by Truesdell's essay "Whence the Law of Moment of Momentum?" in Reference 32. The rigid-body model used here is adequate to represent internal

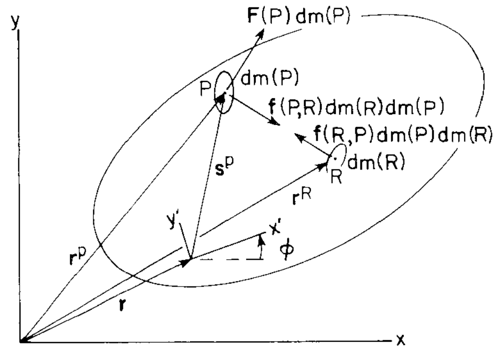

Figure 6.1.1 Forces acting on a rigid body.

6.1.1 Variational Equations of Motion from Newton's Equations

Newton's equations of motion [7, 9] for differential mass $dm(P)$ are

$$\ddot{\mathbf{r}}^P\, dm(P) = \mathbf{F}(P)\, dm(P) + \left[\int_m \mathbf{f}(P, R)\, dm(R)\right] dm(P) \tag{6.1.1}$$

where integration of the internal forces acting on $dm(P)$ is taken over the total mass of the body. Equation 6.1.1 is difficult to use, since it explicitly involves the internal forces that act within the body. Furthermore, it is written for every differential element of mass, yielding a ridiculously large number of equations of motion. The reason for this difficulty is that Eq. 6.1.1 fails to take advantage of the kinematic characteristics of the rigid body, which relate motion of all differential elements in the body. A unifying concept that can be used to resolve this dilemma and yield broadly applicable tools for mechanical system dynamics [35] is the variational or *virtual work* approach.

Let $\delta\mathbf{r}^P$ denote an arbitrary *virtual displacement* of point P; that is, a small variation in the location of point P that is permitted to occur with time held fixed. The variation of a vector may be thought of as the partial differential of the vector with only physical coordinates varied and time held fixed. The δ operator is just the differential operator of calculus with time held fixed. Thus, it is

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gravitational attraction and the effects of stress distributions (as a limit of an elastic body) with symmetric stress tensors. The model used here requires that, if forces of interaction occur in a body due to electric or magnetic field effects, they must be accounted for as external forces. This model of a rigid body is adequate for commonly used engineering materials and machine dynamics applications. For more philosophical considerations, the reader may wish to consult Reference 32 and references cited therein.

possible to relate the virtual displacement $\delta\mathbf{r}^P$ of point P to variations $\delta\mathbf{q}$ in generalized coordinates $\mathbf{q}$ of a body, once the relation $\mathbf{r}^P = \mathbf{r}^P(\mathbf{q})$ is established. This calculation is temporarily postponed to permit development of equations of dynamics in which it is of value.

Premultiplying both sides of Eq. 6.1.1 by $\delta\mathbf{r}^{PT}$ and integrating over the total mass m of the body yields

$$\int_m \delta\mathbf{r}^{PT}\ddot{\mathbf{r}}^P\,dm(P) = \int_m \delta\mathbf{r}^{PT}\mathbf{F}(P)\,dm(P) + \iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P) \tag{6.1.2}$$

which must hold for arbitrary $\delta\mathbf{r}^P$.

Manipulation of the double integral appearing on the right of Eq. 6.1.2 yields

$$\begin{aligned}\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P) &= \frac{1}{2}\iint_{mm} [\delta\mathbf{r}^{PT}\mathbf{f}(P, R) + \delta\mathbf{r}^{PT}\mathbf{f}(P, R)]\,dm(R)\,dm(P)\\ &= \frac{1}{2}\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P) + \frac{1}{2}\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P)\end{aligned}$$

Since P and R are dummy variables of integration, they may be renamed without affecting the value of the integral; that is,

$$\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P) = \iint_{mm} \delta\mathbf{r}^{RT}\mathbf{f}(R, P)\,dm(P)\,dm(R)$$

Furthermore, the order of carrying out the integration can be reversed without affecting the value of the integral [26], and $\mathbf{f}(P, R)\,dm(R)\,dm(P) = -\mathbf{f}(R, P)\,dm(P)\,dm(R)$ can be used to obtain

$$\begin{aligned}&\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P)\\ &\qquad = \frac{1}{2}\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\,dm(R)\,dm(P) - \frac{1}{2}\iint_{mm} \delta\mathbf{r}^{RT}\mathbf{f}(P, R)\,dm(R)\,dm(P)\end{aligned}$$

Combining the integrals on the right and using the fact from differential calculus [25, 26, 35] that $\delta\mathbf{r}^P - \delta\mathbf{r}^R = \delta(\mathbf{r}^P - \mathbf{r}^R)$,

$$\iint_{mm} \delta\mathbf{r}^{PT}\,\mathbf{f}(P, R)\,dm(R)\,dm(P) = \frac{1}{2}\iint_{mm} \delta(\mathbf{r}^P - \mathbf{r}^R)^T\mathbf{f}(P, R)\,dm(R)\,dm(P)$$

Recall that the definition of a rigid body and the model used here require that the distance between points P and R be constant; that is,

$$(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = c$$

Taking the differential of both sides, using the rules of differential calculus (Prob. 6.1.1), yields

$$\delta(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = 0$$

Thus, $\delta(\mathbf{r}^P - \mathbf{r}^R)$ is orthogonal to $\mathbf{r}^P - \mathbf{r}^R$. Since $\mathbf{f}(P, R)$ acts along the line between P and R in the model used here, it is also orthogonal to $\delta(\mathbf{r}^P - \mathbf{r}^R)$, so $\delta(\mathbf{r}^P - \mathbf{r}^R)\mathbf{f}(P, R) = 0$. Thus,

$$\iint_{mm} \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\, dm(R)\, dm(P) = 0 \tag{6.1.3}$$

Using the result of Eq. 6.1.3, Eq. 6.1.2 simplifies to

$$\int_m \delta\mathbf{r}^{PT}\ddot{\mathbf{r}}^P\, dm(P) = \int_m \delta\mathbf{r}^{PT}\mathbf{F}(P)\, dm(P) \tag{6.1.4}$$

which must hold for all $\delta\mathbf{r}^P$ that are consistent with constraints on $\mathbf{r}^P$ that define rigid-body motion. It is important to note that while Eq. 6.1.2 holds for arbitrary $\delta\mathbf{r}^P$, Eq. 6.1.3 does not. The price for eliminating the second term on the right of Eq. 6.1.2 is that $\delta\mathbf{r}^P$ must be consistent with the definition of rigid-body motion. This result is sometimes called *D'Alembert's principle* [4, 35] or the *principle of virtual work*. To take full advantage of Eq. 6.1.4, the virtual displacement $\delta\mathbf{r}^P$ of point P must be written in terms of variations in generalized coordinates of the body.

The vector $\mathbf{r}^P$ that locates point P may be written, using Eq. 2.4.8, as

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \tag{6.1.5}$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{6.1.6}$$

and $\mathbf{r}$ and ϕ are generalized coordinates that locate the body in the plane. An expression for the virtual displacement of point P, in terms of variations in the generalized coordinates, may be obtained by taking the differential of Eq. 6.1.5 (Prob. 6.1.2):

$$\delta\mathbf{r}^P = \delta\mathbf{r} + \delta\phi\mathbf{B}\mathbf{s}'^P \tag{6.1.7}$$

where, as in Eq. 2.6.3,

$$\mathbf{B} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \tag{6.1.8}$$

To write the acceleration term on the left of Eq. 6.1.4 in terms of time derivatives of the generalized coordinates, Eq. 6.1.5 may be differentiated with respect to time to obtain, as in Eq. 2.6.4,

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P \tag{6.1.9}$$

Differentiating this result with respect to time yields, as in Eq. 2.6.9,

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{B}\mathbf{s}'^P - \dot{\phi}^2\mathbf{A}\mathbf{s}'^P \tag{6.1.10}$$

Equations 6.1.7 and 6.1.10 may be used to expand Eq. 6.1.4, obtaining the general *variational form of the equations of planar motion* (Prob. 6.1.3):

$$\begin{aligned}\delta\mathbf{r}^T\ddot{\mathbf{r}}\int_m dm(P) &+ [\delta\mathbf{r}^T(\ddot{\phi}\mathbf{B} - \dot{\phi}^2\mathbf{A}) + \delta\phi\ddot{\mathbf{r}}^T\mathbf{B}]\int_m \mathbf{s}'^P\, dm(P) \\ &+ \delta\phi\int_m \mathbf{s}'^{PT}\mathbf{B}^T[\ddot{\phi}\mathbf{B} - \dot{\phi}^2\mathbf{A}]\mathbf{s}'^P\, dm(P) \\ &= \delta\mathbf{r}^T\int_m \mathbf{F}(P)\, dm(P) + \delta\phi\int_m \mathbf{s}'^{PT}\mathbf{B}^T\mathbf{F}(P)\, dm(P)\end{aligned} \tag{6.1.11}$$

Even though Eq. 6.1.4 must hold for all $\delta\mathbf{r}^P$ that are consistent with constraints that define rigid-body motion, Eq. 6.1.11 holds for arbitrary $\delta\mathbf{r}$ and $\delta\phi$. This is true since $\delta\mathbf{r}^P$ given by Eq. 6.1.7 is consistent with Eq. 6.1.5, which defines rigid-body motion, for arbitrary $\delta\mathbf{r}$ and $\delta\phi$.

6.1.2 Variational Equations of Motion with Centroidal Coordinates

The general form of the equations of motion of a planar body of Eq. 6.1.11 can be substantially simplified if special features of the body-fixed x'-y' frame are exploited. The first integral on the left of Eq. 6.1.11 is simply the total mass of the body; that is,

$$m = \int_m dm(P) \tag{6.1.12}$$

regardless of the location of the x'-y' frame. If the origin of the x'-y' frame is located at the *center of mass* or *centroid* of the body, then, by definition of the centroid,

$$\int_m \mathbf{s}'^P\, dm(P) = \mathbf{0} \tag{6.1.13}$$

and the second integral on the left of Eq. 6.1.11 vanishes.

Direct matrix multiplications verify that

$$\begin{aligned}\mathbf{B}^T\mathbf{B} &= \mathbf{I} \\ \mathbf{B}^T\mathbf{A} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\end{aligned} \tag{6.1.14}$$

Thus, the first term in the third integral on the left of Eq. 6.1.11, after $\ddot{\phi}$ is moved outside the integral, becomes simply the *polar moment of inertia* of the body with respect to the origin of the body-fixed x'-y' frame; that is,

$$J' \equiv \int_m \mathbf{s}'^{PT}\mathbf{s}'^{P}\,dm(P) \tag{6.1.15}$$

A direct expansion of the second term in the integrand of the third integral on the left of Eq. 6.1.11 shows that (Prob. 6.1.4)

$$\mathbf{s}'^{PT}\mathbf{B}^T\mathbf{A}\mathbf{s}'^{P} = 0$$

To interpret terms on the right of Eq. 6.1.11, note first that the resultant force acting on the body, including applied forces and forces of contact with other bodies, is

$$\mathbf{F} = \int_m \mathbf{F}(P)\,dm(P) \tag{6.1.16}$$

The second integral on the right of Eq. 6.1.11 may be expanded to see that it defines the *torque,* or *moment* of the forces acting on the body, about an axis perpendicular to the x-y plane, counterclockwise taken as positive; that is, writing $\mathbf{F}$ in terms of body-fixed components and using Eq. 6.1.14, the torque acting on the body is

$$\begin{aligned} n &= \int_m \mathbf{s}'^{PT}\mathbf{B}^T\mathbf{F}(P)\,dm(P) \\ &= \int_m \mathbf{s}'^{PT}\mathbf{B}^T\mathbf{A}\mathbf{F}'\,dm(P) \\ &= \int_m [-s^P_{y'}F_{x'}(P) + s^P_{x'}F_{y'}(P)]\,dm(P) \end{aligned} \tag{6.1.17}$$

Substituting Eqs. 6.1.12 through 6.1.17 in Eq. 6.1.11,

$$\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\phi[J'\ddot{\phi} - n] = 0 \tag{6.1.18}$$

for arbitrary virtual displacements $\delta\mathbf{r}$ and $\delta\phi$. Equation 6.1.18 is called the *variational equation of motion* of a rigid body with a *centroidal body-fixed reference frame* in the plane. It may be thought of as an extension of the *principle of virtual work,* in that the first term is the virtual work of the unbalanced force $m\ddot{\mathbf{r}} - \mathbf{F}$ and the second term is the virtual work of the unbalanced torque $J'\ddot{\phi} - n$. Recall that Eq. 6.1.18 is valid only if the origin of the body-fixed x'-y' frame is at the centroid of the body.

6.1.3 Differential Equations of Motion

If all forces acting on a body have been accounted for, since $\delta\mathbf{r}$ and $\delta\phi$ are arbitrary in Eq. 6.1.18, their coefficients must be zero and the conventional differential equations for motion of a rigid body in the plane are obtained as

$$m\ddot{\mathbf{r}} = \mathbf{F}$$
$$J'\ddot{\phi} = n \tag{6.1.19}$$

To see that this is true, first set $\delta\mathbf{r} = 0$ and $\delta\phi = 0.001$ in Eq. 6.1.18. This shows that the last of Eqs. 6.1.19 holds. A similar argument can be given with $\delta\phi = 0$ and each component of $\delta\mathbf{r}$, in turn, equal to zero to obtain the first of Eqs. 6.1.19.

It is important to remember that Eqs. 6.1.19 are valid only if the vector $\mathbf{F}$ of forces and the torque n acting on the body represent all force effects, that is, both applied and constraint forces.

Example 6.1.1: The tractor shown in Fig. 6.1.2 has no suspension and can be modeled as one body. It has mass m and polar moment of inertia J'. A driving force T_r is generated at the rear wheels, and ground reaction forces F_f and F_r are applied at the front and rear wheels, respectively.

A body fixed x_1'-y_1' frame is located at the center of mass 0_1 of the tractor. Tire force due to penetration into the soil is modeled as

$$F = f(d) = \begin{cases} kd, & d \geqslant 0 \\ 0, & d < 0 \end{cases}$$

where k is a spring constant and d is tire deflection.

For a small pitch angle ϕ_1, tire deflections d_f and d_r for the front and rear wheels are, respectively,

$$d_f = h_0 - (y_1 + \ell_f\phi_1)$$
$$d_r = h_0 - (y_1 - \ell_r\phi_1)$$

where h_0 is the height of the center of mass when the tractor is level and there is

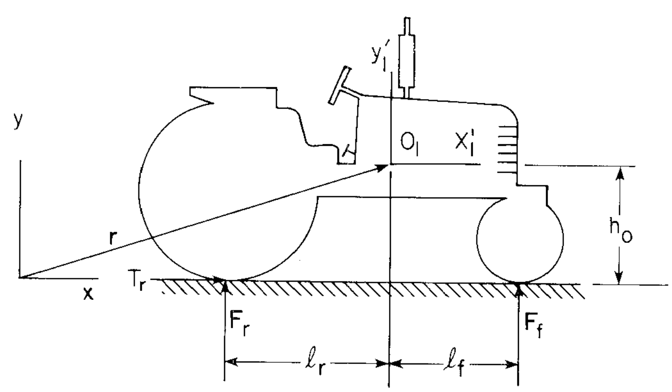


Figure 6.1.2 Plane motion of a tractor.

no tire deflection. Hence, the supporting forces are

$$F_f = k(h_0 - (y_1 + \ell_f \phi_1))$$
$$F_r = k(h_0 - (y_1 - \ell_r \phi_1))$$

Since there are no kinematic constraints acting on the tractor and all applied forces have been accounted for, the equations of motion of Eq. 6.1.19 are

$$m\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} = \begin{bmatrix} T_r \\ k(h_0 - y_1 - \ell_f \phi_1) + k(h_0 - y_1 + \ell_r \phi_1) - mg \end{bmatrix}$$

$$J\ddot{\phi}_1 = \ell_f k(h_0 - y_1 - \ell_f \phi_1) - \ell_r k(h_0 - y_1 + \ell_r \phi_1) + T_r h_0$$

Note that these are linear differential equations. The small pitch angle approximation used in deriving tire deflection leads to the linear form. Most often, the equations of machine dynamics are nonlinear.

6.1.4 Properties of the Centroid and Polar Moment of Inertia

Consider the planar body shown in Fig. 6.1.3, with a noncentroidal x''-y'' body-fixed reference frame. Dimensions and mass distribution in the body are known, and the location of the centroid in the x''-y'' frame is to be found.

Let vector $\boldsymbol{\rho}''$ in the x''-y'' frame locate the centroid of the body and define an x'-y' frame with its origin at the centroid (a body-fixed *centroidal reference frame*) and with axes parallel to the x''-y'' frame. To determine $\boldsymbol{\rho}''$, the equation

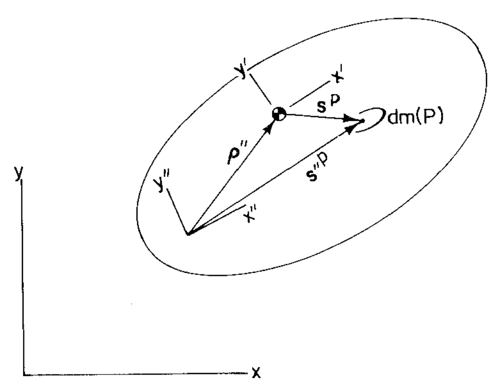


Figure 6.1.3 Location of a centroid.

that defines the centroid, Eq. 6.1.13, is

$$\begin{aligned}\mathbf{0} &= \int_m \mathbf{s}'^P \, dm(P) \\ &= \int_m (\mathbf{s}''^P - \boldsymbol{\rho}'') \, dm(P) \\ &= \int_m \mathbf{s}''^P \, dm(P) - \boldsymbol{\rho}'' \int_m dm(P) \\ &= \int_m \mathbf{s}''^P dm(P) - m\boldsymbol{\rho}''\end{aligned}$$

since $\boldsymbol{\rho}''$ does not depend on P. Thus, the vector $\boldsymbol{\rho}''$ that locates the centroid in the x''-y'' frame is

$$\boldsymbol{\rho}'' = \frac{1}{m} \int_m \mathbf{s}''^P \, dm(P) \tag{6.1.20}$$

Let the polar moment of inertia J'' with respect to the x''-y'' frame be calculated as

$$J'' = \int_m \mathbf{s}''^{PT} \mathbf{s}''^P \, dm(P) \tag{6.1.21}$$

The goal now is to use this information and the location $\boldsymbol{\rho}''$ of the centroid in the x''-y'' frame to calculate the polar moment of inertia with respect to the x'-y' frame.

Substituting $\mathbf{s}''^P = \boldsymbol{\rho}'' + \mathbf{s}'^P$ from Fig. 6.1.3 into Eq. 6.1.21 and manipulating,

$$\begin{aligned}J'' &= \int_m (\boldsymbol{\rho}'' + \mathbf{s}'^P)^T (\boldsymbol{\rho}'' + \mathbf{s}'^P) \, dm(P) \\ &= \boldsymbol{\rho}''^T \boldsymbol{\rho}'' \int_m dm(P) + 2\boldsymbol{\rho}''^T \int_m \mathbf{s}'^P \, dm(P) + \int_m \mathbf{s}'^{PT} \mathbf{s}'^P \, dm(P)\end{aligned}$$

Since the x'-y' frame has its origin at the centroid, the next to last term on the right is zero. Furthermore, the last term is J'. Thus,

$$J'' = J' + m \, |\boldsymbol{\rho}''|^2 \tag{6.1.22}$$

This relation is called the *parallel axis theorem* [4, 7, 9] for polar moment of inertia.

Masses, polar moments of inertia, and centroid locations for a few commonly encountered homogeneous plane bodies are given in Table 6.1.1.

Consider a body that has an axis of both physical and mass distribution symmetry that passes through the origin of the x''-y'' frame, with unit normal

TABLE 6.1.1 Mass and Polar Moment of Inertia of Planar Bodies

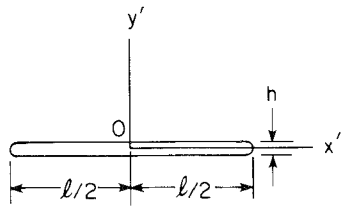
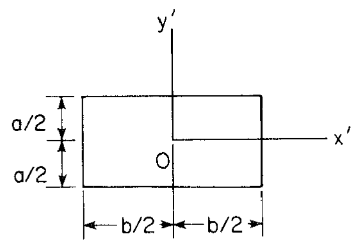
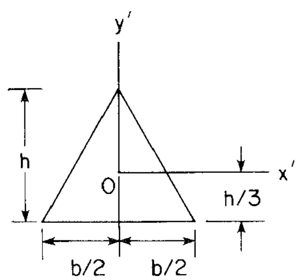
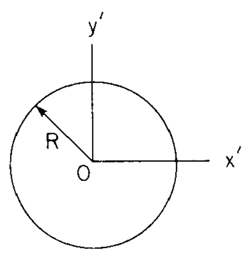
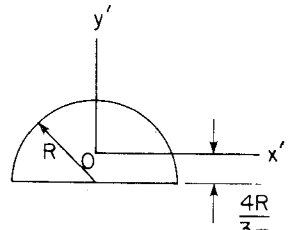
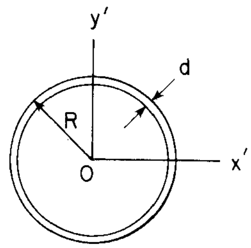
Body	Mass and polar moment of inertia (γ = mass per unit area)
 Thin Rod	$m = \gamma \ell h, \quad h = \text{height}$ $J' = \frac{m}{12}\ell^2$
 Rectangle	$m = \gamma ab$ $J' = \frac{1}{12}m(a^2 + b^2)$
 Isosceles Triangle	$m = \frac{1}{2}\gamma bh$ $J' = m\left(\frac{b^2}{24} + \frac{h^2}{18}\right)$
 Circle	$m = \gamma \pi R^2$ $J' = \frac{1}{2}mR^2$

TABLE 6.1.1 (continued)

Body	Mass and polar moment of inertia (γ = mass per unit area)
 Semicircle	$m=\frac{1}{2}\gamma\pi R^2$ $J'=mR^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)$
 Thin Ring	$m=2\pi R\gamma d,\qquad d=\text{width}$ $J'=mR^2$

vector $\mathbf{k}$, as shown by the dashed line in Fig. 6.1.4. Then, for any point P there is a point P^R that is symmetrically placed across the line of symmetry. Thus,

$$\mathbf{k}^T\mathbf{s}''^P\,dm(P)=-\mathbf{k}^T\mathbf{s}''^{P^R}\,dm(P^R)$$

Premultiplying both sides of Eq. 6.1.20 by $\mathbf{k}^T$,

$$\begin{aligned}\mathbf{k}^T\boldsymbol{\rho}''&=\frac{1}{m}\mathbf{k}^T\int_m \mathbf{s}''^P\,dm(P)\\&=\frac{1}{m}\int_m \mathbf{k}^T\mathbf{s}''^P\,dm(P)\end{aligned}$$

Denoting by m^+ the mass on the side of the line of symmetry toward which $\mathbf{k}$ points and by m^- the mass on the other side, $m^+=m^-=m/2$ and

$$\begin{aligned}\mathbf{k}^T\boldsymbol{\rho}''&=\frac{1}{m}\left(\int_{m^+}\mathbf{k}^T\mathbf{s}''^P\,dm(P)+\int_{m^-}\mathbf{k}^T\mathbf{s}''^{P^R}\,dm(P^R)\right)\\&=\frac{1}{m}\left(\int_{m^+}\mathbf{k}^T\mathbf{s}''^P\,dm(P)-\int_{m^+}\mathbf{k}^T\mathbf{s}''^P\,dm(P)\right)\\&=0\end{aligned}$$

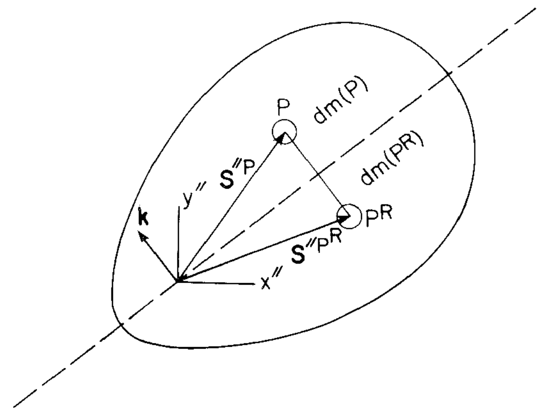


Figure 6.1.4 Body with axis of symmetry.

Thus, the centroid of the body lies on each axis of symmetry of the body. In particular, if a body has two distinct axes of symmetry, the centroid must lie at their intersection. Note that the standard shapes of Table 6.1.1 have one or more axes of symmetry, on which their centroids lie.

Note that even if a body has a geometric axis of symmetry, if its mass is not symmetrically distributed, the centroid will not be on the geometric axis of symmetry.

6.1.5 Inertial Properties of Composite Bodies

Components of machines are often made up of combinations of subcomponents that have standard shapes, such as rods, circles, rings, rectangles, and triangles. A typical example of such a *component* (or *composite body*) is shown in Fig. 6.1.5, in which all subcomponents and voids have some standard shape typical of those resulting from common manufacturing processes. The objective of this subsection

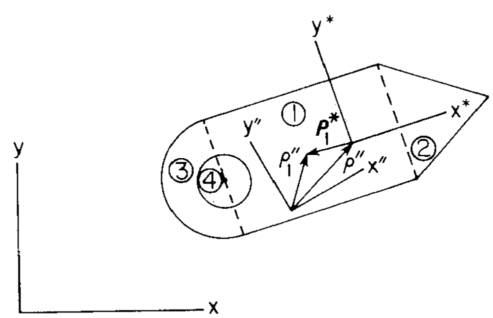


Figure 6.1.5 Body made up of subcomponents.

is to develop expressions for inertia properties of composite bodies using easily calculated properties of individual subcomponents.

Let an x''-y'' frame be fixed to the composite body in a convenient location (e.g., as shown in Fig. 6.1.5). Using this frame, the centroid of the complex body may be obtained by applying the definition of Eq. 6.1.20, employing the property that an integral over the entire mass may be written as the sum of integrals over masses m_i of subcomponents to obtain

$$\begin{aligned}\boldsymbol{\rho}'' &= \frac{1}{m}\sum_{i=1}^{k}\int_{m_i}\mathbf{s}''^P\,dm_i(P)\\ &= \frac{1}{m}\sum_{i=1}^{k} m_i\boldsymbol{\rho}_i''\end{aligned}\tag{6.1.23}$$

where $m = \sum_{i=1}^{k} m_i$. To use this result, the centroids $\boldsymbol{\rho}_i''$ are first located in the x''-y'' frame using mass and centroid location information such as that found in Table 6.1.1 or from direct numerical calculation. Equation 6.1.23 is then used to locate the centroid of the composite body in the x''-y'' frame.

Denote by x^*-y^* the centroidal frame of the composite body, as shown in Fig. 6.1.5, to avoid confusion with centroidal x_i'-y_i' frames on each subcomponent. To calculate the polar moment of inertia J^* with respect to the x^*-y^* composite body centroidal frame, the defining equation of Eq. 6.1.15 may be written and its integral evaluated as a sum of integrals over subcomponents m_i that make up the composite body to obtain

$$\begin{aligned}J^* &= \int_m \mathbf{s}^{*PT}\mathbf{s}^{*P}\,dm(P)\\ &= \sum_{i=1}^{k}\left[\int_{m_i}\mathbf{s}_i^{*PT}\mathbf{s}_i^{*P}\,dm_i(P)\right]\\ &= \sum_{i=1}^{k} J_i^*\end{aligned}\tag{6.1.24}$$

For each subcomponent i, the polar moment of inertia J_i' with respect to its centroidal x_i'-y_i' frame is often known. Using Eq. 6.1.22, the polar moment of inertia J_i^* with respect to the centroidal x^*-y^* frame of the composite body, which is noncentroidal for each of the individual subcomponents, is

$$J_i^* = J_i' + m_i\,|\boldsymbol{\rho}_i^*|^2 \tag{6.1.25}$$

where $\boldsymbol{\rho}_i^*$ is the vector that locates the centroid of the subcomponent in the x^*-y^* frame, as shown in Fig. 6.1.5.

If there are voids in a composite body, Eq. 6.1.24 may be used by including material that would have been in the void in a subcomponent with the void removed. The void is then treated as a subcomponent, and $-J_i^*$ is assigned as the polar moment of inertia of the void, where J_i^* is the polar moment of inertia of a body that would occupy the void with the same material density as the subcomponent from which it is removed.

Example 6.1.2: The pendulum shown in Fig. 6.1.6 consists of a slender rod and a disk with a hole in it. The rod has a mass of 2.8 kg and the disk has a density of 8000 kg/m^3 and a thickness of 0.01 m. Find the centroid and polar moment of inertia of the pendulum.

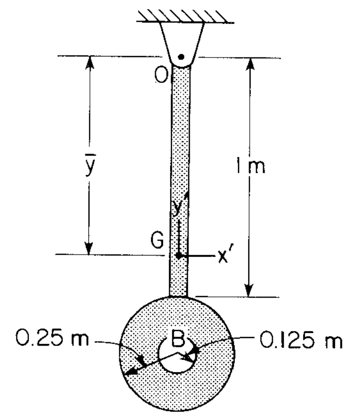


Figure 6.1.6 Composite pendulum.

Let m_d denote the mass of a disk that does not have the hole, and let m_h be the mass of the disk that would be removed to create the hole. Then

$$m_d = 8000[\pi(0.25)^2(0.01)] = 15.71 \text{ kg}$$

$$m_h = 8000[\pi(0.125)^2(0.01)] = 3.93 \text{ kg}$$

Since the y axis is an axis of symmetry of the composite body, the centroid lies on it. Let the distance between the centroid of the composite pendulum and point O be $\bar{y}$. From Eq. 6.1.23,

$$\bar{y} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_r y_r + (m_d - m_h) y_d}{m_r + (m_d - m_h)}$$

$$= \frac{2.80(0.5) + 15.71(1.25) - (3.93)(1.25)}{2.80 + 15.71 - 3.93} = 1.11 \text{ m}$$

where m_r, m_d, and m_h are masses of the rod, disk, and material that would have been in the hole, and y_r, y_d, and y_h are the distances from point O to the centroids of the rod, disk, and hole, respectively. From Table 6.1.1, the polar moment of inertia of the rod, with respect to its centroid, is

$$J_r = \frac{m_r \ell^2}{12} = \frac{2.8(1)^2}{12} = 0.233 \text{ kg} \cdot \text{m}^2$$

From Eq. 6.1.25, the polar moment of inertia of the rod, with respect to the centroid, point G of the assembly, is

$$\begin{aligned} J'_r &= 0.233 + m_r\rho^2 \\ &= 0.233 + 2.8(1.11 - 0.5)^2 = 1.275 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

where $\rho = |\boldsymbol{\rho}|$.

Similarly, from Table 6.1.1 and Eq. 6.1.25, the polar moment of inertia of the disk, with respect to point G, is

$$\begin{aligned} J'_d &= \tfrac{1}{2}m_d R^2 + m_d\rho^2 \\ &= 0.5(15.71)(0.25)^2 + (15.71)(1.25 - 1.11)^2 \\ &= 0.799 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

Similarly, for the hole,

$$\begin{aligned} J'_h &= \tfrac{1}{2}m_h R^2 + m_h\rho^2 \\ &= 0.5(3.93)(0.125)^2 + (3.93)(1.25 - 1.11)^2 \\ &= 0.108 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

The polar moment of inertia of the pendulum about point G is, therefore,

$$\begin{aligned} J' &= J'_r + J'_d - J'_h \\ &= 1.275 + 0.799 - 0.108 \\ &= 1.966 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

6.2 VIRTUAL WORK AND GENERALIZED FORCE

The variational equation of motion of a rigid body in the plane derived in Section 6.1 is used to assemble the variational equation of motion of a constrained multibody system in Section 6.3. Concepts of virtual work and generalized force are introduced in this section as a foundation for the multibody variational method.

The *virtual work* of a force $\mathbf{F}$ that acts at the origin of the x'-y' frame and a torque n that acts on the body in the plane may be written in terms of its virtual displacement $\delta\mathbf{r}$ and virtual rotation $\delta\phi$ as

$$\begin{aligned} \delta W &= \delta\mathbf{r}^T\mathbf{F} + \delta\phi n \\ &\equiv \delta\mathbf{q}^T\mathbf{Q} \end{aligned} \qquad \textbf{(6.2.1)}$$

where

$$\begin{aligned} \mathbf{q} &= [\mathbf{r}^T, \phi]^T \\ \mathbf{Q} &= [\mathbf{F}^T, n]^T \end{aligned} \qquad \textbf{(6.2.2)}$$

The basic idea in defining *generalized force* $\mathbf{Q}$ associated with generalized

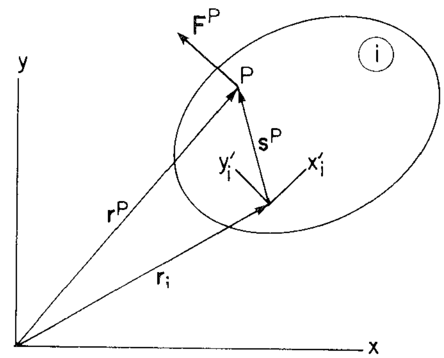


Figure 6.2.1 Force at point P on body i.

coordinate $\mathbf{q}$ is to first write the virtual work of a set of forces and moments that act on a body in terms of products of physical virtual displacements and rotations and physical forces and torques. Next, virtual displacements and rotations are written in terms of variations $\delta\mathbf{q}$ in generalized coordinates. Finally, coefficients of all variations of generalized coordinates that are used in the formulation are collected. These coefficients are defined to be generalized forces associated with the corresponding generalized coordinates. Note that virtual work is the key quantity that is preserved in defining generalized force.

Consider a force $\mathbf{F}^P$ that acts at point P on body i, as shown in Fig. 6.2.1. Since

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i\mathbf{s}'^P$$

and

$$\delta\mathbf{r}_i^P = \delta\mathbf{r}_i + \delta\phi_i\mathbf{B}_i\mathbf{s}'^P$$

the virtual work of force $\mathbf{F}^P$ is

$$\begin{aligned}\delta W &= \delta\mathbf{r}_i^{PT}\mathbf{F}^P \\ &= \delta\mathbf{r}_i^T\mathbf{F}^P + \delta\phi_i\mathbf{s}'^{PT}\mathbf{B}_i^T\mathbf{F}^P\end{aligned}$$

Thus, for a general force $\mathbf{F}^P$ acting at point P on body i, the corresponding generalized force is

$$\mathbf{Q} = \begin{bmatrix} \mathbf{F}^P \\ \mathbf{s}'^{PT}\mathbf{B}_i^T\mathbf{F}^P \end{bmatrix} \qquad \textbf{(6.2.3)}$$

Note that the third component of generalized force $\mathbf{Q}$ in Eq. 6.2.3 is just the moment of $\mathbf{F}^P$ about the origin of the x_i'-y_i' frame, as in Eq. 6.1.17.

If force $\mathbf{F}^P$ were fixed in body i (e.g., in the case of a rocket thrustor attached to a spacecraft), then $\mathbf{F}^P = \mathbf{A} \cdot \mathbf{F}'^P$, and Eq. 6.2.3 becomes

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A}_i\mathbf{F}'^P \\ \mathbf{s}'^{PT}\mathbf{B}_i^T\mathbf{A}_i\mathbf{F}'^P \end{bmatrix}$$

Since $\mathbf{B}_i = \mathbf{A}_i\mathbf{R}$, $\mathbf{s}'^{PT}\mathbf{B}_i^T\mathbf{A}_i = \mathbf{s}'^{PT}\mathbf{R}^T = (\mathbf{R}\mathbf{s}'^P)^T$. Thus, for a body-fixed force $\mathbf{F}'^P$,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A}_i \\ (\mathbf{R}\mathbf{s}'^P)^T \end{bmatrix} \mathbf{F}'^P \tag{6.2.4}$$

Using the notation of Eqs. 6.2.2, 6.2.3, or 6.2.4, $\delta\mathbf{q} = [\delta\mathbf{r}^T, \delta\phi]^T$, and

$$\mathbf{M} = \text{diag}(m, m, J')$$

the variational equation of motion of Eq. 6.1.18 may be written in the form

$$\delta\mathbf{q}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = 0 \tag{6.2.5}$$

A common force element that is encountered in mechanical system dynamics is a compliant connection between points P_i and P_j on bodies i and j, respectively, as shown in Fig. 6.2.2. A compliant element in applications may consist of a spring, a damper, or a force actuator such as a hydraulic cylinder that exerts a force along the vector between points P_i and P_j. Such an element is called a *translational spring–damper–actuator.*

Since this force element exerts equal and opposite forces on bodies i and j, rather than calculating generalized forces on each body separately, as in the case of external applied forces, it is instructive to make direct use of the virtual work definition of generalized force. Consider first the vector $\mathbf{d}_{ij}$ from point P_i to point P_j:

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}'_j - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}'_i \tag{6.2.6}$$

The length of this element is calculated from

$$\ell^2 = \mathbf{d}_{ij}^T\mathbf{d}_{ij} \tag{6.2.7}$$

Note that ℓ may be positive or negative, depending on the relative orientation at assembly of the system. Since mechanical force elements are usually designed with care so that their length can never be zero, they should be modeled to

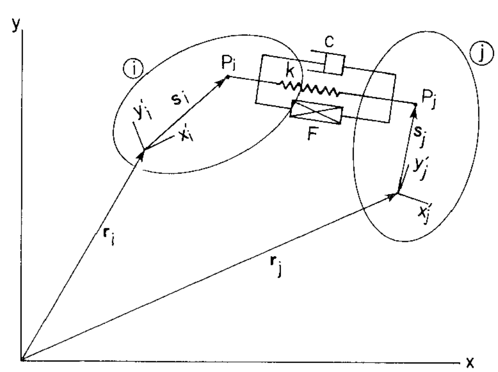


Figure 6.2.2 Translational spring–damper–actuator.

preclude the pathological case $\ell = 0$. The reader is encouraged to adopt the convention $\ell > 0$.

Differentiating Eq. 6.2.7,

$$2\ell\dot{\ell} = 2\mathbf{d}_{ij}^T\dot{\mathbf{d}}_{ij}$$

As long as ℓ is not equal to zero, dividing both sides by ℓ and differentiating Eq. 6.2.6 yields

$$\dot{\ell} = \left(\frac{\mathbf{d}_{ij}}{\ell}\right)^T(\dot{\mathbf{r}}_j + \mathbf{B}_j\mathbf{s}_i'\dot{\phi}_j - \dot{\mathbf{r}}_i - \mathbf{B}_i\mathbf{s}_i'\dot{\phi}_i) \qquad \textbf{(6.2.8)}$$

In case ℓ approaches zero, the limit of the right side of Eq. 6.2.8 may be taken as ℓ approaches zero, using L'Hospital's rule [26], to obtain

$$\lim_{\ell\to 0}\frac{\mathbf{d}_{ij}}{\ell} = \lim_{\ell\to 0}\frac{\dot{\mathbf{d}}_{ij}}{\dot{\ell}} = \left.\frac{\dot{\mathbf{d}}_{ij}}{\dot{\ell}}\right|_{\ell=0}$$

providing $\dot{\ell} \neq 0$ when $\ell = 0$. Substituting from this result into Eq. 6.2.8 and solving for $(\dot{\ell})^2$

$$(\dot{\ell})^2 = \dot{\mathbf{d}}_{ij}'^T\mathbf{d}_{ij} \qquad \textbf{(6.2.9)}$$

when $\ell = 0$. The sign of $\dot{\ell}$ is selected as the value it had just prior to the occurrence of $\ell = 0$.

Defining tension in the compliant element as positive; that is, a force that tends to draw the bodies together is positive, the force acting in the element is written as

$$f = k(\ell - \ell_0) + c\dot{\ell} + F(\ell, \dot{\ell}, t) \qquad \textbf{(6.2.10)}$$

where k is the spring coefficient, ℓ_0 is the free length of the spring, c is the damping coefficient, and F is a general actuator force that may depend on ℓ, $\dot{\ell}$, or time t.

Making direct use of the definition of virtual work, the virtual work of the force f is simply the product of the force and the variation $\delta\ell$ in the length of the element. Since a positive $\delta\ell$ tends to separate the bodies and the sign convention for f is that tension is positive, tending to pull the bodies together, for positive f and $\delta\ell$, the work done on the bodies is negative. Thus, the virtual work is

$$\delta W = -f\delta\ell \qquad \textbf{(6.2.11)}$$

Taking the differential of both sides of Eq. 6.2.7 and manipulating, as in the derivation of the velocity relation of Eq. 6.2.8,

$$\delta\ell = \left(\frac{\mathbf{d}_{ij}}{\ell}\right)^T(\delta\mathbf{r}_j + \mathbf{B}_j\mathbf{s}_j'\,\delta\phi_j - \delta\mathbf{r}_i - \mathbf{B}_i\mathbf{s}_i'\,\delta\phi_i) \qquad \textbf{(6.2.12)}$$

where $\mathbf{B}_i$ is defined in Eq. 6.1.8. Substituting this result into Eq. 6.2.11, the virtual work of the spring–damper–actuator force is

$$\delta W = -\frac{f}{\ell}\mathbf{d}_{ij}^T(\delta\mathbf{r}_j + \mathbf{B}_j\mathbf{s}_j'\,\delta\phi_j - \delta\mathbf{r}_i - \mathbf{B}_i\mathbf{s}_i'\,\delta\phi_i) \tag{6.2.13}$$

By definition, the generalized forces acting on bodies i and j are simply the coefficients of variations in generalized coordinates that define the position and orientation of each body; that is,

$$\begin{aligned} \mathbf{Q}_i &= \frac{f}{\ell}\begin{bmatrix} \mathbf{d}_{ij} \\ \mathbf{d}_{ij}^T\mathbf{B}_i\mathbf{s}_i' \end{bmatrix} \\ \mathbf{Q}_j &= -\frac{f}{\ell}\begin{bmatrix} \mathbf{d}_{ij} \\ \mathbf{d}_{ij}^T\mathbf{B}_j\mathbf{s}_j' \end{bmatrix} \end{aligned} \tag{6.2.14}$$

Note that as ℓ approaches zero in Eq. 6.2.14 an indeterminate fraction occurs, which may be resolved by applying L'Hospital's rule as ℓ approaches zero to obtain

$$\left.\begin{aligned} \mathbf{Q}_i &= \frac{f}{\dot{\ell}}\begin{bmatrix} \dot{\mathbf{d}}_{ij} \\ \dot{\mathbf{d}}_{ij}^T\mathbf{B}_i\mathbf{s}_i' \end{bmatrix} \\ \mathbf{Q}_j &= -\frac{f}{\dot{\ell}}\begin{bmatrix} \dot{\mathbf{d}}_{ij} \\ \dot{\mathbf{d}}_{ij}^T\mathbf{B}_j\mathbf{s}_j' \end{bmatrix} \end{aligned}\right\}, \qquad \ell = 0 \tag{6.2.15}$$

if $\dot{\ell} \neq 0$, where the fact that $\mathbf{d}_{ij} = \mathbf{0}$ when $\ell = 0$ has been used.

Consider next the pair of bodies shown in Fig. 6.2.3, with a revolute joint defined at point P and a torsional compliant element acting between the

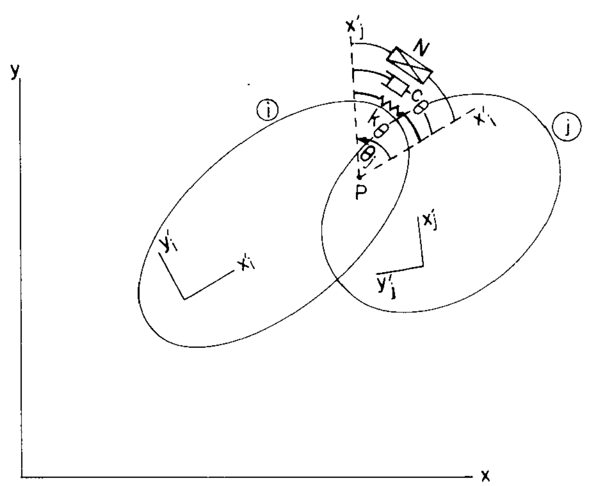


Figure 6.2.3 Rotational spring–damper–actuator.

body-fixed x_i' and x_j' axes. This compliant element exerts torques of equal magnitude but opposite orientation on bodies i and j; hence it is called a *rotational spring–damper–actuator*. A positive torque is one that acts counterclockwise on body i and clockwise on body j.

The angle θ_{ij} from the x_i' axis to the x_j' axis, counterclockwise taken as positive, is

$$\theta_{ij} = \phi_j - \phi_i \tag{6.2.16}$$

Taking the time derivative of this relation yields

$$\dot{\theta}_{ij} = \dot{\phi}_j - \dot{\phi}_i \tag{6.2.17}$$

The torque that acts between the bodies is

$$n = k_\theta(\theta_{ij} - \theta_0) + c_\theta \dot{\theta}_{ij} + N(\theta_{ij}, \dot{\theta}_{ij}, t) \tag{6.2.18}$$

where k_θ is a torsional spring coefficient, θ_0 is the free angle of the spring, c_θ is a torsional damping coefficient, and N is a general actuator torque that may depend on θ_{ij}, $\dot{\theta}_{ij}$, and time t. The virtual work of the torque n is simply the negative of its product with the relative virtual rotation $\delta\theta_{ij}$; that is,

$$\delta W = -n\,\delta\theta_{ij} \tag{6.2.19}$$

since when $\theta_{ij} > 0$, a positive torque draws the axes together, but a positive $\delta\theta_{ij}$ separates them. From Eq. 6.2.16, the relative virtual rotation is just

$$\delta\theta_{ij} = \delta\phi_j - \delta\phi_i \tag{6.2.20}$$

Substituting this result into Eq. 6.2.19, the virtual work is

$$\delta W = -n(\delta\phi_j - \delta\phi_i) \tag{6.2.21}$$

By definition, the generalized forces acting on bodies i and j are simply the coefficients of their respective generalized coordinate variations; that is,

$$\begin{aligned} \mathbf{Q}_i &= \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \\ \mathbf{Q}_j &= -\begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \end{aligned} \tag{6.2.22}$$

6.3 EQUATIONS OF MOTION OF CONSTRAINED PLANAR SYSTEMS

Consider now mechanical systems that are made up of a collection of rigid bodies in the plane, with kinematic constraints between bodies. The variational and differential equations of motion of Eqs. 6.1.18 and 6.1.19 are valid for each body in the system, provided that all forces that act on each body are accounted for, including constraint reaction forces. In this section, the variational formulation of the equations of dynamics is extended to include the effect of kinematic constraints between bodies. Lagrange multipliers are introduced to account for the effect of kinematic constraints.

6.3.1 Variational Equations of Motion for Planar Systems

The variational equations of motion for each body i in a planar multibody system, given by Eq. 6.2.5, may be summed to obtain the *system variational equation of motion,*

$$\sum_{i=1}^{nb} \delta \mathbf{q}_i^T [\mathbf{M}_i \ddot{\mathbf{q}}_i - \mathbf{Q}_i] = 0 \qquad \textbf{(6.3.1)}$$

for arbitrary $\delta \mathbf{q}_i$, $i = 1, \ldots, nb$, provided all forces that act are included in $\mathbf{Q}$. To treat a multibody system, it is convenient to define a composite state variable vector, a composite mass matrix, and a composite vector of generalized forces as

$$\begin{aligned} \mathbf{q} &= [\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_{nb}^T]^T \\ \mathbf{M} &= \mathrm{diag}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_{nb}) \\ \mathbf{Q} &= [\mathbf{Q}_1^T, \mathbf{Q}_2^T, \ldots, \mathbf{Q}_{nb}^T]^T \end{aligned} \qquad \textbf{(6.3.2)}$$

Using this notation, the variational equation of Eq. 6.3.1 may be written in the more compact form:

$$\delta \mathbf{q}^T [\mathbf{M} \ddot{\mathbf{q}} - \mathbf{Q}] = 0$$

Equations 6.2.5 for each body or Eq. 6.3.1 for the entire system are difficult to apply, since the generalized forces must include both *constraint forces* and *applied forces* (defined here to be all forces acting on or between bodies in the system except forces of constraint). Thus, forces due to gravity and spring–damper–actuators are treated as applied forces. To make systematic distinction between applied and constraint forces, let $\mathbf{F}_i^A$ and n_i^A denote applied force and torque on body i and $\mathbf{F}_i^C$ and n_i^C denote constraint force and torque on body i. Similarly, $\mathbf{Q}_i^A$ and $\mathbf{Q}_i^C$ denote generalized applied and constraint forces on body i. Finally, consistent with the composite vector notation of Eq. 6.3.2, $\mathbf{Q}^A$ and $\mathbf{Q}^C$ denote composite vectors of generalized applied and constraint forces acting on the system. In each case,

$$\begin{aligned} \mathbf{F}_i &= \mathbf{F}_i^A + \mathbf{F}_i^C, \qquad n_i = n_i^A + n_i^C \\ \mathbf{Q}_i &= \mathbf{Q}_i^A + \mathbf{Q}_i^C \\ \mathbf{Q} &= \mathbf{Q}^A + \mathbf{Q}^C \end{aligned}$$

Using the preceding notation, Eq. 6.3.1 may be written as

$$\sum_{i=1}^{nb} \delta\mathbf{q}_i^T[\mathbf{M}_i\ddot{\mathbf{q}}_i - \mathbf{Q}_i^A] - \sum_{i=1}^{nb} \delta\mathbf{q}_i^T\mathbf{Q}_i^C = 0$$

or, using the notation of Eq. 6.3.2, Eq. 6.3.1 is

$$\delta\mathbf{q}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A] - \delta\mathbf{q}^T\mathbf{Q}^C = 0$$

These equations must hold for arbitrary $\delta\mathbf{q}_i$ and $\delta\mathbf{q}$, respectively.

Note that, whereas $\delta\mathbf{q}_i^T\mathbf{Q}_i^C$ is the virtual work of constraint forces that act on only body i,

$$\sum_{i=1}^{nb} \delta\mathbf{q}_i^T\mathbf{q}_i^C = \delta\mathbf{q}^T\mathbf{Q}^C$$

is the total virtual work of constraint forces that act on all bodies of the system. By Newton's law of action and reaction, if there is no friction in kinematic joints, constraint forces act perpendicular to contacting surfaces and are equal in magnitude. Thus, if attention is restricted to *virtual displacements that are consistent with the constraints* that act on the system, then the virtual work of all constraint forces is zero; that is,

$$\sum_{i=1}^{nb} \delta\mathbf{q}_i^T\mathbf{Q}_i^C = \delta\mathbf{q}^T\mathbf{Q}^C = 0$$

It is important to note that this is true even though $\delta\mathbf{q}_i^T\mathbf{Q}_i^C \neq 0$ for an individual body, even for virtual displacements that are consistent with constraints. It is only after the variational equations for individual bodies of Eq. 6.2.5 are summed to obtain the system variational equation of Eq. 6.3.1 that the effect of constraint forces can be eliminated.

To summarize, the system variational equation of motion of Eq. 6.3.1 may be written in either of the following forms of *constrained variational equations of motion*:

$$\begin{aligned}\sum_{i=1}^{nb} \delta\mathbf{q}_i^T[\mathbf{M}_i\ddot{\mathbf{q}}_i - \mathbf{Q}_i^A] &= 0\\ \delta\mathbf{q}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A] &= 0\end{aligned} \tag{6.3.3}$$

for all virtual displacements $\delta\mathbf{q}$ that are *consistent with constraints* that act on the system.

The combined set of kinematic and driving constraints is written in the form

$$\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0} \tag{6.3.4}$$

where the specific form of constraint equations is presented in Chapter 3. Since generalized coordinate variations (or virtual displacements) $\delta\mathbf{q}$ are considered to occur with time held fixed, the condition for a *kinematically admissible virtual displacement* $\delta\mathbf{q}$ is obtained by taking the differential of Eq. 6.3.4 with time held fixed; that is,

$$\mathbf{\Phi_q}\,\delta\mathbf{q} = \mathbf{0} \tag{6.3.5}$$

where the Jacobian is evaluated at a state $\mathbf{q}$ that satisfies Eq. 6.3.4. Thus, the *constrained variational equations of motion* are that Eq. 6.3.3 hold for all virtual displacements $\delta\mathbf{q}$ that satisfy Eq. 6.3.5.

Example 6.3.1: Consider the simple pendulum shown in Fig. 6.3.1. The generalized coordinate vector is $\mathbf{q} = [x_1, y_1, \phi_1]^T$. The virtual work of the gravitational force (applied force), from Eq. 6.2.1, is

$$\begin{aligned}\delta W &= [\delta x_1, \delta y_1]^T \begin{bmatrix} 0 \\ -mg \end{bmatrix} \\ &= [\delta x_1, \delta y_1, \delta\phi_1]^T \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}\end{aligned}$$

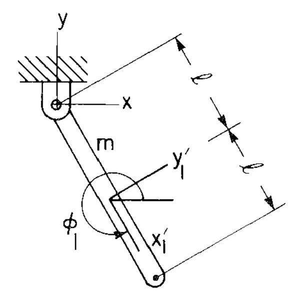


Figure 6.3.1 Simple pendulum.

where m is mass of the pendulum. Therefore, the generalized applied force $\mathbf{Q}^A$ is

$$\mathbf{Q}^A = [0, -mg, 0]^T$$

From Eq. 6.3.3,

$$\delta\mathbf{q}^T[\mathbf{M\ddot{q}} - \mathbf{Q}^A] = [\delta x_1, \delta y_1, \delta\phi_1] \begin{bmatrix} m\ddot{x}_1 \\ m\ddot{y}_1 + mg \\ (m\ell^2/3)\ddot{\phi}_1 \end{bmatrix} = 0 \tag{6.3.6}$$

where the polar moment of inertia of the pendulum is $m\ell^2/3$. The constraint equation is

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} x_1 - \ell\cos\phi_1 \\ y_1 - \ell\sin\phi_1 \end{bmatrix} = \mathbf{0} \tag{6.3.7}$$

The condition for kinematically admissible virtual displacements of Eq. 6.3.5 is

$$\begin{bmatrix} 1 & 0 & \ell\sin\phi_1 \\ 0 & 1 & -\ell\cos\phi_1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta y_1 \\ \delta\phi_1 \end{bmatrix} = \mathbf{0} \tag{6.3.8}$$

Hence, the variational equation of motion of the pendulum of Eq. 6.3.6 must hold for all virtual displacements $\delta\mathbf{q}$ that satisfy Eq. 6.3.8.

Example 6.3.2: Consider the two-body slider–crank model shown in Fig. 6.3.2. The virtual work of gravitational forces, from Eq. 6.2.1, is

$$\delta W = \delta\mathbf{r}_1^T\mathbf{F}_1 + \delta\mathbf{r}_2^T\mathbf{F}_2 = \begin{bmatrix}\delta x_1\\ \delta y_1\\ \delta\phi_1\\ \delta x_2\\ \delta y_2\\ \delta\phi_2\end{bmatrix}^T \begin{bmatrix}0\\ -m_1g\\ 0\\ 0\\ -m_2g\\ 0\end{bmatrix}$$

Hence the generalized applied force vector is

$$\mathbf{Q}^A = [0,\ -m_1g,\ 0,\ 0,\ -m_2g,\ 0]^T$$

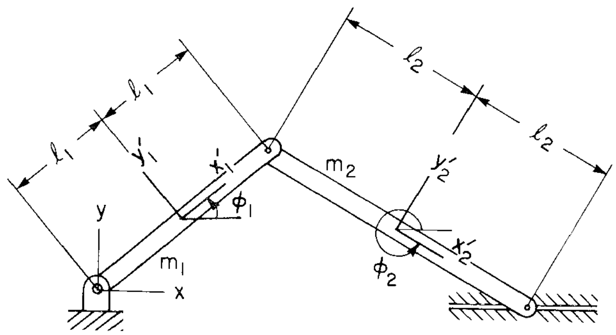


Figure 6.3.2 Two-body slider–crank.

From Eq. 6.3.3,

$$\delta\mathbf{q}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A] = \begin{bmatrix}\delta x_1\\ \delta y_1\\ \delta\phi_1\\ \delta x_2\\ \delta y_2\\ \delta\phi_2\end{bmatrix}^T \begin{bmatrix} m_1\ddot{x}_1\\ m_1\ddot{y}_1 + m_1g\\ \left(\frac{m_1\ell_1^2}{3}\right)\ddot{\phi}_1\\ m_2\ddot{x}_2\\ m_2\ddot{y}_2 + m_2g\\ \left(\frac{m_2\ell_2^2}{3}\right)\ddot{\phi}_2 \end{bmatrix} = 0 \tag{6.3.9}$$

The constraint equation is

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} x_1 - \ell_1\cos\phi_1\\ y_1 - \ell_1\sin\phi_1\\ x_2 - 2\ell_1\cos\phi_1 - \ell_2\cos\phi_2\\ y_2 - 2\ell_1\sin\phi_1 - \ell_2\sin\phi_2\\ 2\ell_1\sin\phi_1 + 2\ell_2\sin\phi_2 \end{bmatrix} = \mathbf{0} \tag{6.3.10}$$

The condition for kinematically admissible virtual displacements of Eq. 6.3.5 is

$$\begin{bmatrix} 1 & 0 & \ell_1 \sin\phi_1 & 0 & 0 & 0 \\ 0 & 1 & -\ell_1\cos\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 2\ell_1\sin\phi_1 & 1 & 0 & \ell_2\sin\phi_2 \\ 0 & 0 & -2\ell_1\cos\phi_1 & 0 & 1 & -\ell_2\cos\phi_2 \\ 0 & 0 & 2\ell_1\cos\phi_1 & 0 & 0 & 2\ell_2\cos\phi_2 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta y_1 \\ \delta\phi_1 \\ \delta x_2 \\ \delta y_2 \\ \delta\phi_2 \end{bmatrix} = \mathbf{0} \qquad \textbf{(6.3.11)}$$

The equation of motion of Eq. 6.3.9 thus holds for all virtual displacements $\delta\mathbf{q}$ that satisfy Eq. 6.3.11.

6.3.2 Lagrange Multipliers

A classical method in mechanics is to introduce Lagrange multipliers to reduce the variational equation of Eq. 6.3.3 to a mixed system of differential–algebraic equations. The conventional method of introducing Lagrange multipliers in this context may be found in References 7 and 9. A more mathematically precise introduction of Lagrange multipliers is employed here, using a theorem of optimization theory [28, 33].

Lagrange Multiplier Theorem: Let $\mathbf{b}$ be an n vector of constants, $\mathbf{x}$ be an n vector of variables, and $\mathbf{A}$ be an $m \times n$ constant matrix. If

$$\mathbf{b}^T\mathbf{x} = 0 \qquad \textbf{(6.3.12)}$$

holds for all $\mathbf{x}$ that satisfy

$$\mathbf{A}\mathbf{x} = \mathbf{0} \qquad \textbf{(6.3.13)}$$

then there exists an m vector $\boldsymbol{\lambda}$ of *Lagrange multipliers* such that

$$\mathbf{b}^T\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} = 0 \qquad \textbf{(6.3.14)}$$

for arbitrary $\mathbf{x}$.

Example 6.3.3: Consider $\mathbf{b} = [1, 3, 2]^T$, $\mathbf{x} = [x_1, x_2, x_3]^T$, and $\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. Note that

$$\begin{aligned} \mathbf{b}^T\mathbf{x} &= [1, 3, 2][x_1, x_2, x_3]^T \\ &= [-1, 0, 1][x_1, x_2, x_3]^T + [2, 3, 1][x_1, x_2, x_3]^T \end{aligned}$$

So $\mathbf{b}^T\mathbf{x} = 0$ for all $\mathbf{x}$ such that

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

Then, by the Lagrange multiplier theorem, there exists $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T$ that satisfies

$$[1, 3, 2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [\lambda_1, \lambda_2]\begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 + 3x_2 + 2x_3 + \lambda_1(-x_1 + x_3) + \lambda_2(2x_1 + 3x_2 + x_3)$$

$$= (1 - \lambda_1 + 2\lambda_2)x_1 + (3 + 3\lambda_2)x_2 + (2 + \lambda_1 + \lambda_2)x_3 = 0$$

Since this equation holds for all $\mathbf{x}$, it is equivalent to a system of three equations; that is,

$$1 - \lambda_1 + 2\lambda_2 = 0$$

$$3 + 3\lambda_2 = 0$$

$$2 + \lambda_1 + \lambda_2 = 0$$

The solution is

$$\boldsymbol{\lambda} = [-1, -1]^T$$

It is important to note that the Lagrange multiplier theorem does *not* say that $\mathbf{b}^T\mathbf{x} = 0$ for all $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ for *any* $\mathbf{b}$ and $\mathbf{A}$. In fact, there must be a close relation between $\mathbf{A}$ and $\mathbf{b}$ in order for this to be true (Prob. 6.3.2).

Equation 6.3.3 corresponds to Eq. 6.3.12, and the kinematic admissibility condition of Eq. 6.3.5 corresponds to Eq. 6.3.13. Since Eq. 6.3.3 must hold for all $\delta\mathbf{q}$ that satisfy Eq. 6.3.5, the Lagrange multiplier theorem guarantees the existence of a Lagrange multiplier vector $\boldsymbol{\lambda}$ such that

$$[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A]^T\,\delta\mathbf{q} + \boldsymbol{\lambda}^T\boldsymbol{\Phi}_\mathbf{q}\,\delta\mathbf{q} = [\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_\mathbf{q}^T\boldsymbol{\lambda} - \mathbf{Q}^A]^T\,\delta\mathbf{q} = 0 \qquad \textbf{(6.3.15)}$$

for arbitrary $\delta\mathbf{q}$. Therefore, the coefficient of $\delta\mathbf{q}$ in Eq. 6.3.15 must be $\mathbf{0}$, yielding the *Lagrange multiplier form of the equations of motion*:

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_\mathbf{q}^T\boldsymbol{\lambda} = \mathbf{Q}^A \qquad \textbf{(6.3.16)}$$

In addition to these equations of motion, recall the velocity and acceleration equations of Eqs. 3.6.10 and 3.6.11:

$$\begin{aligned} \boldsymbol{\Phi}_\mathbf{q}\dot{\mathbf{q}} &= -\boldsymbol{\Phi}_t \equiv \boldsymbol{\nu} \\ \boldsymbol{\Phi}_\mathbf{q}\ddot{\mathbf{q}} &= -(\boldsymbol{\Phi}_\mathbf{q}\dot{\mathbf{q}})_\mathbf{q}\dot{\mathbf{q}} - 2\boldsymbol{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \boldsymbol{\Phi}_{tt} \equiv \boldsymbol{\gamma} \end{aligned} \qquad \textbf{(6.3.17)}$$

These equations, together with Eq. 6.3.16, comprise the complete set of *constrained equations of motion* for the system.

6.3.3 Mixed Differential–Algebraic Equations of Motion

Equation 6.3.16 and the second of Eqs. 6.3.17 may be written in matrix form as

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_\mathbf{q}^T \\ \boldsymbol{\Phi}_\mathbf{q} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^A \\ \boldsymbol{\gamma} \end{bmatrix} \qquad \textbf{(6.3.18)}$$

This is a *mixed system of differential–algebraic equations* (DAE), since no derivatives of the Lagrange multiplier $\boldsymbol{\lambda}$ appear. Furthermore, Eq. 6.3.4 and the first of Eqs. 6.3.17 must be satisfied.

Example 6.3.4: Consider the simple pendulum of Example 6.3.1. By successive differentiation of Eq. 6.3.7 with respect to time, the velocity and acceleration equations are

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \begin{bmatrix} 1 & 0 & \ell \sin\phi_1 \\ 0 & 1 & -\ell\cos\phi_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\phi}_1 \end{bmatrix} = \mathbf{0} \equiv \mathbf{v}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -\begin{bmatrix} \ell\cos\phi_1\dot{\phi}_1^2 \\ \ell\sin\phi_1\dot{\phi}_1^2 \end{bmatrix} \equiv \boldsymbol{\gamma} \tag{6.3.19}$$

From Eqs. 6.3.6, 6.3.8, and 6.3.19, the equations of motion are

$$\begin{bmatrix} m & 0 & 0 & 1 & 0 \\ 0 & m & 0 & 0 & 1 \\ 0 & 0 & m\ell^2/3 & \ell\sin\phi_1 & -\ell\cos\phi_1 \\ 1 & 0 & \ell\sin\phi_1 & 0 & 0 \\ 0 & 1 & -\ell\cos\phi & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \\ -\ell\cos\phi_1\dot{\phi}_1^2 \\ -\ell\sin\phi_1\dot{\phi}_1^2 \end{bmatrix} \tag{6.3.20}$$

Example 6.3.5: In Example 6.3.2, the variational equation of motion and kinematic constraint equations of the two-body slider–crank are written as Eqs. 6.3.9 and 6.3.10. The acceleration equation is

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -\begin{bmatrix} \ell_1\cos\phi_1\dot{\phi}_1^2 \\ \ell_1\sin\phi_1\dot{\phi}_1^2 \\ 2\ell_1\cos\phi_1\dot{\phi}_1^2 + \ell_2\cos\phi_2\dot{\phi}_2^2 \\ 2\ell_1\sin\phi_1\dot{\phi}_1^2 + \ell_2\sin\phi_2\dot{\phi}_2^2 \\ -2\ell_1\sin\phi_1\dot{\phi}_1^2 - 2\ell_2\sin\phi_2\dot{\phi}_2^2 \end{bmatrix} \equiv \boldsymbol{\gamma}$$

Applying the Lagrange multiplier theorem, the equations of motion are as given on page 226.

It is important to know when Eq. 6.3.18 has a unique solution for accelerations and Lagrange multipliers. A practical condition for the coefficient matrix in Eq. 6.3.18 to be nonsingular is the following theorem:

Constrained Dynamic Existence Theorem: Let $\boldsymbol{\Phi}_{\mathbf{q}}$ have full row rank and let the kinetic energy of the system be positive for any nonzero virtual velocity that is consistent with the constraints; that is,

$$\delta\dot{\mathbf{q}}^T\mathbf{M}\,\delta\dot{\mathbf{q}} > 0 \tag{6.3.21}$$

$$
\begin{bmatrix}
m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & m_1\ell_1^2/3 & 0 & 0 & 0 & \ell_1 \sin\phi_1 & -\ell_1\cos\phi_1 & 2\ell_1 \sin\phi_1 & -2\ell_1\cos\phi_1 & 2\ell_1\cos\phi_1 \\
0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & m_2\ell_2^2/3 & 0 & 0 & \ell_2\sin\phi_2 & -2\ell_2\cos\phi_2 & 2\ell_2\cos\phi_2 \\
1 & 0 & \ell_1\sin\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\ell_1\cos\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\ell_1\sin\phi_1 & 1 & 0 & \ell_2\sin\phi_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\ell_1\cos\phi_1 & 0 & 1 & -\ell_2\cos\phi_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\ell_1\cos\phi_1 & 0 & 0 & 2\ell_2\cos\phi_2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5
\end{bmatrix}
=
\begin{bmatrix}
0 \\ -m_1 g \\ 0 \\ 0 \\ -m_2 g \\ 0 \\ -\ell_1\cos\phi_1\dot{\phi}_1^2 \\ -\ell_1\sin\phi_1\dot{\phi}_1^2 \\ -2\ell_1\cos\phi_1\dot{\phi}_1^2 - \ell_2\cos\phi_2\dot{\phi}_2^2 \\ -2\ell_1\sin\phi_1\dot{\phi}_1^2 - \ell_2\sin\phi_2\dot{\phi}_2^2 \\ 2\ell_1\sin\phi_1\dot{\phi}_1^2 + 2\ell_2\sin\phi_2\dot{\phi}_2^2
\end{bmatrix}
$$

for all $\delta\dot{\mathbf{q}} \neq \mathbf{0}$ that satisfy

$$\mathbf{\Phi_q}\, \delta\dot{\mathbf{q}} = \mathbf{0} \tag{6.3.22}$$

Then the coefficient matrix in Eq. 6.3.18 is nonsingular and $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$ are uniquely determined.

If the mass matrix is positive definite [22], which is normally the case, and if the constraints are independent, then the preceding conditions are satisfied and Eq. 6.3.18 has a unique solution. To prove this result, use the fact that the coefficient matrix in Eq. 6.3.18 is nonsingular if [22] the only solution of the homogeneous equation

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi_q^T} \\ \mathbf{\Phi_q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mathbf{0}$$

is $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$. To show this, the preceding matrix equation can be written as two separate equations:

$$\mathbf{My} + \mathbf{\Phi_q^T z} = \mathbf{0} \tag{6.3.23}$$

$$\mathbf{\Phi_q y} = \mathbf{0} \tag{6.3.24}$$

Multiplying Eq. 6.3.23 on the left by $\mathbf{y}^T$,

$$\mathbf{y}^T\mathbf{My} + \mathbf{y}^T\mathbf{\Phi_q^T z} = \mathbf{y}^T\mathbf{My} = 0$$

where the fact that $\mathbf{y}$ satisfies Eq. 6.3.24 has been used. Therefore, $\mathbf{y} = \mathbf{0}$. Thus, Eq. 6.3.23 reduces to

$$\mathbf{\Phi_q^T z} = \mathbf{0}$$

But the left side of this equation is just a linear combination of columns of the matrix $\mathbf{\Phi_q^T}$, with the components of $\mathbf{z}$ as coefficients in the expansion. Since $\mathbf{\Phi_q}$ has full row rank, its transpose is made up of linearly independent columns. Hence, the preceding equation requires that $\mathbf{z} = \mathbf{0}$. This completes the proof.

6.3.4 Initial Conditions

A set of additional conditions that, together with the kinematic constraints, determines initial position and velocity must be defined to initiate motion of the system. Recall from Chapter 3 that the system constraint equations involve both kinematic and driving constraints of the form

$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \end{bmatrix} = \mathbf{0} \tag{6.3.25}$$

which must hold for all time.

A system of initial conditions on position and orientation may be used to define the position of the system at the initial time t_0. *Initial position conditions* are defined in the form

$$\mathbf{\Phi}^I(\mathbf{q}(t_0), t_0) = \mathbf{0} \tag{6.3.26}$$

where it is required that the number of equations in Eq. 6.3.26 be equal to the number of system degrees of freedom. Furthermore, Eq. 6.3.25 at t_0 and Eq. 6.3.26 must be independent to uniquely determine the *initial position* $\mathbf{q}(t_0)$.

Initial velocity conditions may be given in the form

$$\mathbf{B}^I\dot{\mathbf{q}}(t_0) = \mathbf{v}^I \tag{6.3.27}$$

which must contain the same number of equations as the number of system degrees of freedom. Furthermore, it is required that the first of Eqs. 6.3.17 at t_0 and Eq. 6.3.27 uniquely determine the *initial velocity* $\dot{\mathbf{q}}(t_0)$. Thus, a complete set of initial conditions that are consistent with constraints is obtained.

Example 6.3.6: Consider again the simple pendulum of Example 6.3.1. Let the initial orientation of the pendulum be $\phi_1 = 3\pi/2$. Hence, Eq. 6.3.26 is written, with $t_0 = 0$, as

$$\mathbf{\Phi}^I(\mathbf{q}(0), 0) = \phi_1(0) - \frac{3\pi}{2} = 0 \tag{6.3.28}$$

Equation 6.3.7 at $t = 0$ and Eq. 6.3.28 determine the initial position $\mathbf{q}(0)$, since they are independent. Let the initial angular velocity of the simple pendulum be

$$\mathbf{B}^I\dot{\mathbf{q}}(0) = \dot{\phi}_1(0) - 2\pi = 0 \tag{6.3.29}$$

The velocity equation can be obtained by differentiating Eq. 6.3.7 with respect to time, yielding

$$\mathbf{\Phi}_\mathbf{q}\dot{\mathbf{q}}(0) = \begin{bmatrix} 1 & 0 & \ell \sin \phi_1(0) \\ 0 & 1 & -\ell \cos \phi_1(0) \end{bmatrix} \begin{bmatrix} \dot{x}_1(0) \\ \dot{y}_1(0) \\ \dot{\phi}_1(0) \end{bmatrix} = \mathbf{0} \tag{6.3.30}$$

Equations 6.3.29 and 6.3.30 uniquely determine the initial velocity $\dot{\mathbf{q}}(0)$.

While it is not necessary that initial position and velocity conditions be given on the same variables, as is done in Example 6.3.6, this is often the most natural situation in applications. The essential considerations are (1) that the number of initial conditions on both position and velocity be equal to the number of degrees of freedom of the system after the constraints of Eq. 6.3.25 are specified and (2) that Eqs. 6.3.25 and 6.3.26 uniquely determine $\mathbf{q}(t_0)$ and the velocity equation associated with Eqs. 6.3.25 and 6.3.27 uniquely determine $\dot{\mathbf{q}}(t_0)$. These conditions will be met if the following matrices are nonsingular at t_0:

$$\begin{bmatrix} \mathbf{\Phi}_\mathbf{q} \\ \mathbf{\Phi}^I_\mathbf{q} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{\Phi}_\mathbf{q} \\ \mathbf{B}^I \end{bmatrix}$$

These considerations are identical to the selection of a complete set of drivers in kinematic analysis (see Section 3.5).

6.4 INVERSE DYNAMICS OF KINEMATICALLY DRIVEN SYSTEMS

The equations of motion of a constrained dynamic system derived in Section 6.3, specifically Eqs. 6.3.18, are valid regardless of the number of degrees of freedom of the system. If driving constraints equal in number to the number of kinematic degrees of freedom of the system are appended to the kinematic constraints, then the system is kinematically determined and the position, velocity, and acceleration equations of Chapter 3 completely determine the motion of the system. In this special case, the constraint Jacobian of Eq. 6.3.4 is square and nonsingular; that is,

$$|\mathbf{\Phi_q}(\mathbf{q}, t)| \neq 0 \tag{6.4.1}$$

This very special situation yields simplified results that are applicable for the analysis of kinematically driven systems.

Expanding the equations of motion of Eq. 6.3.18,

$$\mathbf{M\ddot{q}} + \mathbf{\Phi_q^T\lambda} = \mathbf{Q}^A \tag{6.4.2}$$

$$\mathbf{\Phi_q\ddot{q}} = \boldsymbol{\gamma} \tag{6.4.3}$$

Since the Jacobian is nonsingular, Eq. 6.4.3 may be solved to obtain

$$\mathbf{\ddot{q}} = \mathbf{\Phi_q^{-1}}\boldsymbol{\gamma} \tag{6.4.4}$$

In reality, the algebraic equations of Eq. 6.4.3 are solved numerically for the acceleration $\mathbf{\ddot{q}}$, but the notation of Eq. 6.4.4 emphasizes this direct relationship.

Once the acceleration vector is known, Eq. 6.4.2 may be solved for $\boldsymbol{\lambda}$ as

$$\boldsymbol{\lambda} = \mathbf{\Phi_q^{-1T}}[\mathbf{Q}^A - \mathbf{M\ddot{q}}] \tag{6.4.5}$$

since the transpose of a nonsingular matrix is nonsingular. In reality, Eq. 6.4.2 is solved numerically for $\boldsymbol{\lambda}$ after $\mathbf{\ddot{q}}$ is found from Eq. 6.4.3.

As is shown in Section 6.6, the Lagrange multiplier $\boldsymbol{\lambda}$ uniquely determines the constraint forces and torques that act in the system. This fact motivates the use of the terminology *inverse dynamics* for systems that are kinematically determined. For such systems, the engineer may specify desired driving constraint relations and solve Eq. 6.4.3 for $\mathbf{\ddot{q}}$. Equation 6.4.2 is then solved for the Lagrange multipliers. The reaction force relations of Section 6.6 are then used to determine forces that would be required to impose the driving constraints that have been specified. Such analysis is valuable in assessing the availability of actuators that can provide forces and torques that are required to achieve the desired motion.

Example 6.4.1: Consider the simple pendulum of Examples 6.3.1, 6.3.4, and 6.3.6 with the kinematic driver $\Phi^D = \phi_1 - 3\pi/2 - 2\pi t_0$. From Eq. 6.3.20, Eqs. 6.4.2 and 6.4.3 are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m\ell^2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell \sin \phi_1 & -\ell \cos \phi_1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} \tag{6.4.6}$$

$$\begin{bmatrix} 1 & 0 & \ell \sin \phi_1 \\ 0 & 1 & -\ell \cos \phi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \end{bmatrix} = - \begin{bmatrix} \ell \cos \phi_1 \dot{\phi}_1^2 \\ \ell \sin \phi_1 \dot{\phi}_1^2 \\ 0 \end{bmatrix} \tag{6.4.7}$$

The solution of Eq. 6.4.7 is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \end{bmatrix} = \begin{bmatrix} -\ell \cos \phi_1 \dot{\phi}_1^2 \\ -\ell \sin \phi_1 \dot{\phi}_1^2 \\ 0 \end{bmatrix} \tag{6.4.8}$$

Substituting Eq. 6.4.8 into Eq. 6.4.6, the Lagrange multipliers are determined as

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} m\ell \cos \phi_1 \dot{\phi}_1^2 \\ -mg + m\ell \sin \phi_1 \dot{\phi}_1^2 \\ -mg\ell \cos \phi_1 \end{bmatrix}$$

Even though the reaction force can be calculated using the general method presented in Section 6.6, notice that λ_1 is the x component of the centripetal force on the pendulum, λ_2 is the y component of the centripetal force minus the weight of the pendulum, and λ_3 is the moment due to the weight.

6.5 EQUILIBRIUM CONDITIONS

A system is said to be in *equilibrium* if it remains stationary under the action of applied forces; that is, if

$$\ddot{\mathbf{q}} = \dot{\mathbf{q}} = \mathbf{0} \tag{6.5.1}$$

If there is adequate damping in a system, the system equations of motion may be formulated and integrated until all motion damps out and an equilibrium state is obtained.

An alternative approach is to substitute the equilibrium definition of Eq. 6.5.1 into the equations of motion, in this case Eq. 6.4.2, to obtain

$$\mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{Q}^A \tag{6.5.2}$$

These equations constitute the system *equilibrium equations*. The state $\mathbf{q}$ and Lagrange multiplier $\boldsymbol{\lambda}$ are determined by Eq. 6.5.2 and the constraint equations of

Eq. 6.3.4. While this approach is computationally feasible, it suffers from the requirement that a good estimate of the equilibrium position is needed as a starting point for iterative computation. Furthermore, the equilibrium equations of Eq. 6.5.2 are valid for both stable and unstable states of equilibrium. Thus, an algorithm that is based on solving the equilibrium equations alone can converge to either a stable or an unstable state of equilibrium, depending on which is nearest to the initial estimate.

To avoid the difficulty associated with unstable equilibrium configurations, for *conservative mechanical systems* [7, 9], the *principle of minimum total potential energy* [34] may be employed. It states that a system is in a state of stable equilibrium if the total potential energy takes on a strict relative minimum at that position.

The total potential energy of a system is defined as

$$TPE = SE - W(F) \tag{6.5.3}$$

where SE is the strain energy of compliant components and $-W(F)$ is the potential energy of all forces that act on the system. In the case of linear translational and rotational springs, the strain energy is

$$SE = \tfrac{1}{2}k(\ell - \ell_0)^2 \quad \text{and} \quad \tfrac{1}{2}k_\theta(\theta - \theta_0)^2 \tag{6.5.4}$$

respectively. For a constant force $\mathbf{F}^P$ that acts at a point P and a torque n on the body, the potential energy is

$$-W(F) = -(x^P F_x^P + y^P F_y^P + n\phi) \tag{6.5.5}$$

For conservative systems, the condition that defines a state $\mathbf{q}_0$ of stable equilibrium is that the total potential energy be minimized; that is,

$$TPE(\mathbf{q}_0) \leq TPE(\mathbf{q}) \tag{6.5.6}$$

for all states $\mathbf{q}$ near the state of stable equilibrium; that is, for some small positive ε, for all $\mathbf{q}$ such that

$$(\mathbf{q} - \mathbf{q}_0)^T(\mathbf{q} - \mathbf{q}_0) \leq \varepsilon \tag{6.5.7}$$

A valuable fact that is exploited in numerical minimization techniques that determine $\mathbf{q}_0$ to satisfy Eq. 6.5.6, consistent with the constraints, is that the generalized applied force $\mathbf{Q}^A$ is the negative of the gradient of the total potential energy [7, 9]; that is

$$\frac{\partial TPE}{\partial \mathbf{q}} = -\mathbf{Q}^{A^T} \tag{6.5.8}$$

Thus, if it is known that forces acting on a system are conservative, the generalized forces provide the gradient of the total potential energy, which is needed by numerical minimization algorithms. A numerical method of implementing this idea is presented in Chapter 7.

Example 6.5.1: Consider a simple pendulum with a torsional spring attached, as shown in Fig. 6.5.1. Let m and 2ℓ be the mass and length of the pendulum, respectively. The total potential energy is

$$TPE = \tfrac{1}{2}k_\theta(\phi_1 - \phi_0)^2 + mgy_1 \qquad \textbf{(6.5.9)}$$

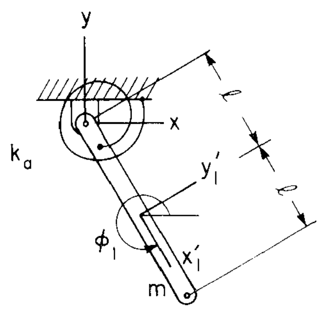


Figure 6.5.1 Simple pendulum with a torsional spring.

where k_θ is the torsional spring constant and ϕ_0 is the angle at which the spring is not deflected. The constraint equation is

$$\mathbf{\Phi} = \begin{bmatrix} x_1 - \ell \cos \phi_1 \\ y_1 - \ell \sin \phi_1 \end{bmatrix} = \mathbf{0} \qquad \textbf{(6.5.10)}$$

Substituting the second equation of Eq. 6.5.10 into Eq. 6.5.9,

$$TPE = \tfrac{1}{2}k_\theta(\phi_1 - \phi_0)^2 + mg\ell \sin \phi_1$$

By the minimum potential energy theorem, the pendulum is in equilibrium when

$$\frac{\partial TPE}{\partial \phi_1} = k_\theta(\phi_1 - \phi_0) + mg\ell \cos \phi_1 = 0 \qquad \textbf{(6.5.11)}$$

This is a nonlinear equation, which cannot be easily solved analytically. Equation 6.5.11 can be written as

$$-k_\theta(\phi_1 - \phi_0) = mg\ell \cos \phi_1$$

which shows that the pendulum is in equilibrium when the spring torque is equal to the moment due to gravity. With the generalized coordinate $\mathbf{q} = [x_1, y_1, \phi_1]^T$, the gradient of total potential energy with respect to $\mathbf{q}$ of Eq. 6.5.8 gives the generalized force:

$$\frac{\partial TPE}{\partial \mathbf{q}} = [0, mg, k_\theta(\phi_1 - \phi_0)] = -\mathbf{Q}^T$$

Example 6.5.2: Consider the double pendulum shown in Fig. 6.5.2. At the tip of the second body, a constant force F is applied horizontally. Let m_1 and m_2 be masses of the first and second bars, respectively, and the lengths be as shown. The total potential energy of Eq. 6.5.3 may be written in terms of ϕ_1 and ϕ_2,

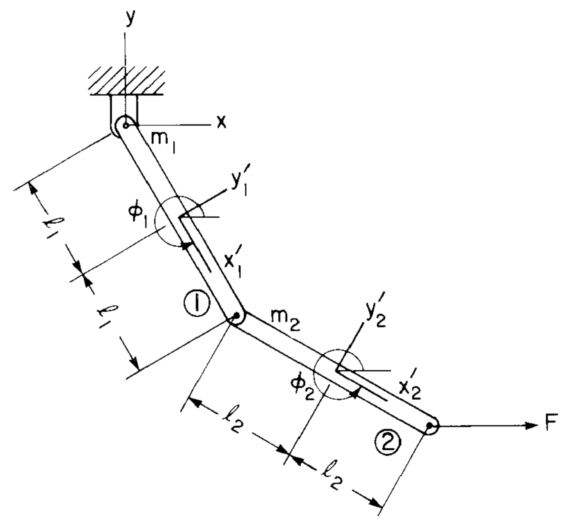


Figure 6.5.2 Double pendulum.

which can be regarded as independent, as

$$TPE = m_1 g\ell_1 \sin\phi_1 + m_2 g(2\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2) - F(2\ell_1 \cos\phi_1 + 2\ell_2 \cos\phi_2)$$

By the principle of minimum total potential energy, the system is in equilibrium when

$$\frac{\partial TPE}{\partial\phi_1} = \frac{\partial TPE}{\partial\phi_2} = 0$$

$$\frac{\partial TPE}{\partial\phi_1} = m_1 g\ell_1 \cos\phi_1 + 2m_2 g\ell_1 \cos\phi_1 + 2F\ell_1 \sin\phi_1 = 0$$

$$\frac{\partial TPE}{\partial\phi_2} = m_2 g\ell_2 \cos\phi_2 + 2F\ell_2 \sin\phi_2 = 0$$

Solving these equations,

$$\phi_1 = \text{Arctan}\left[-\frac{(m_1 + 2m_2)g}{2F} \right]$$

$$\phi_2 = \text{Arctan}\left[-\frac{m_2 g}{2F} \right]$$

6.6 CONSTRAINT REACTION FORCES

Consider a kinematic or driving constraint between bodies i and j of a multibody system, numbered as constraint k. Denote by $\mathbf{\Phi}^k = \mathbf{0}$ its kinematic constraint

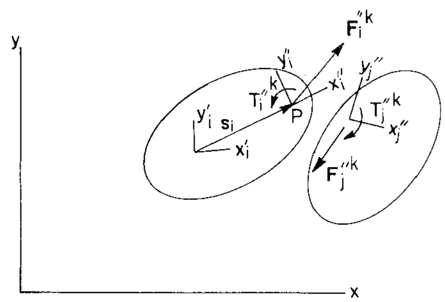


Figure 6.6.1 Constraint reaction force.

equation. The variational equation of motion associated with body i may be written, from Eq. 6.3.15, by setting $\delta\mathbf{q}_j = \mathbf{0}$ for $j \neq i$ to obtain

$$\delta\mathbf{q}_i^T\mathbf{M}_i\ddot{\mathbf{q}}_i + \sum_{\ell \neq k} \delta\mathbf{q}_i^T\mathbf{\Phi}_{\mathbf{q}_i}^{\ell T}\boldsymbol{\lambda}^\ell + \delta\mathbf{q}_i^T\mathbf{\Phi}_{\mathbf{q}_i}^{kT}\boldsymbol{\lambda}^k = \delta\mathbf{q}_i^T\mathbf{Q}_i^A \tag{6.6.1}$$

where $\boldsymbol{\lambda}^\ell$ is the subvector of $\boldsymbol{\lambda}$ corresponding to constraint ℓ and constraint k that acts between bodies i and j has been singled out for special attention. As a consequence of Eq. 6.3.15, Eq. 6.6.1 holds for arbitrary $\delta\mathbf{q}_i$.

If constraint k were broken and replaced by reaction forces in the joint that occur during dynamics in the system, as shown schematically in Fig. 6.6.1, then the resulting motion will be identical to that predicted by Eq. 6.6.1. If the equations of motion with reaction forces due to constraint k included were formulated and constraint k were deleted, the last term on the left of Eq. 6.6.1 would disappear and be replaced by the virtual work of the constraint reaction force and torque; that is,

$$-(\delta\mathbf{r}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT}\boldsymbol{\lambda}^k + \delta\phi_i\mathbf{\Phi}_{\phi_i}^{kT}\boldsymbol{\lambda}^k) = \delta\mathbf{r}_i''^{PT}\mathbf{F}_i''^k + \delta\phi_i T_i''^k \tag{6.6.2}$$

where the expression on the left of Eq. 6.6.1 has been expanded in terms of physical generalized coordinates. Equation 6.6.2 must hold for arbitrary $\delta\mathbf{r}_i$ and $\delta\phi_i$. For a more detailed derivation of Eq. 6.6.2, which shows that the multipliers associated with each joint are unaffected by replacing a joint by its reaction forces, see Reference 35.

The objective is to use Eq. 6.6.2 to develop relations for joint reaction forces $\mathbf{F}_i''^k$ and torques $T_i''^k$ in terms of the Lagrange multipliers. First, recall that

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^P \tag{6.6.3}$$

whose differential yields

$$\delta\mathbf{r}_i^P = \delta\mathbf{r}_i + \mathbf{B}_i\mathbf{s}_i'^P\,\delta\phi_i \tag{6.6.4}$$

Since virtual displacement is a vector quantity and the constant transformation matrix $\mathbf{C}_i$ transforms vectors in the x_i''-y_i'' frame to the x_i'-y_i' frame,

$$\delta\mathbf{r}_i^P = \mathbf{A}_i\mathbf{C}_i\,\delta\mathbf{r}_i''^P \tag{6.6.5}$$

Substituting from Eq. 6.6.5 into Eq. 6.6.4 yields

$$\delta\mathbf{r}_i = \mathbf{A}_i\mathbf{C}_i\,\delta\mathbf{r}_i''^P - \mathbf{B}_i\mathbf{s}_i'^P\,\delta\phi_i \tag{6.6.6}$$

Substituting this result into Eq. 6.6.2,

$$-\delta\mathbf{r}_i''^{PT}\mathbf{C}_i^T\mathbf{A}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT}\boldsymbol{\lambda}^k - \delta\phi_i(\mathbf{\Phi}_{\phi_i}^{kT} - \mathbf{s}_i'^{PT}\mathbf{B}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT})\boldsymbol{\lambda}^k = \delta\mathbf{r}_i''^{PT}\mathbf{F}_i''^k + \delta\phi_i T_i''^k \tag{6.6.7}$$

Since virtual displacements $\delta\mathbf{r}_i''^P$ and rotations $\delta\phi_i$ are arbitrary, their coefficients on both sides of Eq. 6.6.7 must be equal, yielding the desired results:

$$\mathbf{F}_i''^k = -\mathbf{C}_i^T\mathbf{A}_i^T\mathbf{\Phi}_{r_i}^{kT}\boldsymbol{\lambda}^k \tag{6.6.8}$$

$$T_i''^k = (\mathbf{s}_i'^{PT}\mathbf{B}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT} - \mathbf{\Phi}_{\phi_i}^{kT})\boldsymbol{\lambda}^k \tag{6.6.9}$$

Note that the product of the last two terms on the right of Eq. 6.6.8 must be the negative of the joint reaction force in the global x-y frame. As in Eq. 6.6.8, the first term on the right of Eq. 6.6.9 is the negative of the moment of the joint reaction force about the origin of the x'-y' frame; that is, it is associated with transfer of the force from the origin of the body-fixed x'-y' frame to the origin P of the x''-y'' frame, as shown in Fig. 6.6.1.

Equations 6.6.8 and 6.6.9 may be readily programmed to provide joint reaction forces and torques at a joint in the body-fixed joint x''-y'' reference frame.

Example 6.6.1: Consider the kinematically driven simple pendulum of Example 6.4.1, shown in Fig. 6.6.2. The Lagrange multiplier vector is obtained in Example 6.4.1 as

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}^r \\ \boldsymbol{\lambda}^d \end{bmatrix} = \begin{bmatrix} m\ell\cos\phi_1\dot{\phi}_1^2 \\ -mg + m\ell\sin\phi_1\dot{\phi}_1^2 \\ -mg\ell\cos\phi_1 \end{bmatrix}$$

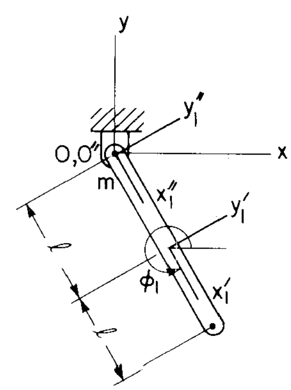


Figure 6.6.2 Reaction force on a simple pendulum.

Since the x_1'-y_1' and x_1''-y_1'' frames are parallel, $\mathbf{C}$ is an identity matrix. From Eq. 6.6.8, the joint reaction force at point O, in the x_1''-y_1'' frame, is

$$\mathbf{F}'' = -\begin{bmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} m\ell\cos\phi_1\dot{\phi}_1^2 \\ -mg + m\ell\sin\phi_1\dot{\phi}_1^2 \\ -mg\ell\cos\phi_1 \end{bmatrix}$$

$$= \begin{bmatrix} -m\ell\dot{\phi}_1^2 + mg\sin\phi_1 \\ mg\cos\phi_1 \end{bmatrix} \tag{6.6.10}$$

The reaction torque from Eq. 6.6.9, in this case the driving torque required to create the specified motion of the pendulum, is

$$T = \left\{[-\ell, 0]\begin{bmatrix} -\sin\phi_1 & \cos\phi_1 \\ -\cos\phi_1 & -\sin\phi_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - [\ell\sin\phi_1, -\ell\cos\phi_1, 1]\right\}\boldsymbol{\lambda}$$

$$= [0, 0, -1]\begin{bmatrix} m\cos\phi_1\dot{\phi}_1^2 \\ -mg + m\sin\phi_1\dot{\phi}_1^2 \\ -mg\ell\cos\phi_1 \end{bmatrix}$$

$$= mg\ell\cos\phi_1 \tag{6.6.11}$$

The force of Eq. 6.6.10 is the reaction force in the revolute joint at point O in the x_1''-y_1'' frame. The torque of Eq. 6.6.11 is necessary to generate the motion specified by the driving constraint $\phi_1 = 3\pi/2 + 2\pi t$.

Since the driver implies that $\ddot{\phi}_1 = 0$, the summation of moments about point O must be zero; that is,

$$T - mg\ell\cos\phi_1 = 0$$

This direct application of the equations of motion for the body confirms that the result of Eq. 6.6.11 is correct. Similarly, using the fact that the acceleration of the center of mass is $\ell\dot{\phi}_1^2$ in the negative x_1' direction, equating mass times this acceleration to the sum of the gravitational and reaction forces, Eq. 6.6.10 is verified.

It is instructive to expand Eqs. 6.6.8 and 6.6.9 for a few of the standard joints of Chapter 3. Consider first an absolute x constraint of Eq. 3.2.3. Using the Jacobian of Eq. 3.2.5, the reaction force and torque on body i, from Eqs. 6.6.8 and 6.6.9, are

$$\mathbf{F}_i''^{ax(i)} = -\mathbf{C}_i^T\mathbf{A}_i^T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda^{ax(i)}$$

$$T_i''^{ax(i)} = [-x_i'^P\sin\phi_i - y_i'^P\cos\phi_i - (-x_i'^P\sin\phi_i - y_i'^P\cos\phi_i)]\lambda^{ax(i)} = 0$$

As expected, the constraint supports no reaction torque at point P. Transforming $\mathbf{F}_i''^{ax(i)}$ to the global frame, by multiplying by $\mathbf{C}_i$ and then $\mathbf{A}_i$ yields

$$\mathbf{F}_i^{ax(i)} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda^{ax(i)}$$

Thus, there is no reaction force in the global y direction and the global x component of reaction force is just the Lagrange multiplier $\lambda^{ax(i)}$.

For a revolute joint of Eq. 3.3.10, using the Jacobian of Eq. 3.3.12 in Eqs. 6.6.8 and 6.6.9,

$$\mathbf{F}_i''^{r(i,j)} = -\mathbf{C}_i^T\mathbf{A}_i^T(\mathbf{I})\boldsymbol{\lambda}^{r(i,j)} = -\mathbf{C}_i^T\mathbf{A}_i^T\boldsymbol{\lambda}^{r(i,j)}$$

$$T_i''^{k(i,j)} = [\mathbf{s}_i'^{PT}\mathbf{B}_i^T(\mathbf{I}) - (\mathbf{s}_i'^{PT}\mathbf{B}_i^T)]\boldsymbol{\lambda}^{r(i,j)} = 0$$

As expected, the revolute joint supports no reaction torque. Premultiplying the reaction force by $\mathbf{C}_i$ and then $\mathbf{A}_i$ yields the reaction force in the global x-y frame as

$$\mathbf{F}_i^{r(i,j)} = -\boldsymbol{\lambda}^{r(i,j)}$$

Thus, for the revolute joint, the negative of the Lagrange multiplier is the reaction force on body i, in the global x-y frame.

Finally, for the translational joint of Eq. 3.3.13, using the Jacobian of Eq. 3.3.14 in Eqs. 6.6.8 and 6.6.9,

$$\begin{aligned}
\mathbf{F}_i''^{t(i,j)} &= -\mathbf{C}_i^T\mathbf{A}_i^T[-\mathbf{B}_i\mathbf{v}_i', \mathbf{0}]\boldsymbol{\lambda}^{t(i,j)} \\
&= \mathbf{C}_i^T\mathbf{A}_i^T\mathbf{R}\mathbf{A}_i\mathbf{v}_i'\lambda_1^{t(i,j)} \\
&= \lambda_1^{t(i,j)}\mathbf{C}_i^T\mathbf{A}_i^T\mathbf{R}\mathbf{v}_i \\
T_i''^{t(i,j)} &= \{\mathbf{s}_i'^{PT}\mathbf{B}_i^T[-\mathbf{B}_i\mathbf{v}_i', \mathbf{0}] \\
&\quad + [(\mathbf{r}_j - \mathbf{r}_i)^T\mathbf{A}_i\mathbf{v}_i' + \mathbf{s}_j'^{PT}\mathbf{A}_{ij}^T\mathbf{v}_i', \mathbf{v}_j'^T\mathbf{A}_{ij}^T\mathbf{v}_i']\}\boldsymbol{\lambda}^{t(i,j)} \\
&= [(\mathbf{r}_j - \mathbf{r}_i)^T\mathbf{A}_i + \mathbf{s}_j'^{PT}\mathbf{A}_{ij}^T - \mathbf{s}_i'^{PT}, \mathbf{v}_j'^T\mathbf{A}_{ij}^T]\mathbf{v}_i'\boldsymbol{\lambda}^{t(i,j)}
\end{aligned}$$

Note that the reaction force $\mathbf{F}_i^{t(i,j)}$ is perpendicular to the vector $\mathbf{v}_i$ along the translational joint, as should be expected, and that the reaction torque is not generally zero.

PROBLEMS

Section 6.1

6.1.1. A virtual displacement $\delta\mathbf{r}^P$ is just the differential of the vector $\mathbf{r}^P$. Since the differential of calculus obeys rules of differentiation,

$$\begin{aligned}
\delta(\mathbf{r}^{P^T}\mathbf{r}^R) &= \delta(x^P x^R + y^P y^R) \\
&= \delta x^P x^R + x^P\,\delta x^R + \delta y^P y^R + y^P\,\delta y^R \\
&= [\delta x^P, \delta y^P]\begin{bmatrix} x^R \\ y^R \end{bmatrix} + [x^P, y^P]\begin{bmatrix} \delta x^R \\ \delta y^R \end{bmatrix} \\
&\equiv \delta\mathbf{r}^{P^T}\mathbf{r}^R + \mathbf{r}^{P^T}\,\delta\mathbf{r}^R
\end{aligned}$$

Use this result to show that if $(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = \text{constant}$ then

$$[\delta(\mathbf{r}^P - \mathbf{r}^R)]^T(\mathbf{r}^P - \mathbf{r}^P) = 0$$

6.1.2. With x, y, and ϕ as generalized coordinates, Eq. 6.1.5 may be written as

$$\mathbf{r}^P = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s_{x'}^P \\ s_{y'}^P \end{bmatrix}$$

where $s_{x'}^P$ and $s_{y'}^P$ are constants. Use the chain rule of differential calculus to expand

$$\delta\mathbf{r}^P = \frac{\partial \mathbf{r}^P}{\partial x}\delta x + \frac{\partial \mathbf{r}^P}{\partial y}\delta y + \frac{\partial \mathbf{r}^P}{\partial \phi}\delta\phi$$

and verify that Eq. 6.1.7 is correct.

6.1.3. Verify that Eq. 6.1.11 is correct.

6.1.4. Use the second of Eqs. 6.1.14 to verify that for any $\mathbf{s}'^P$, $\mathbf{s}'^{PT}\mathbf{B}^T\mathbf{A}\mathbf{s}'^P = 0$.

6.1.5. Find the centroid and the polar moment of inertia with respect to the centroid of the machine element shown in Fig. P6.1.5, whose density is 8000 kg/m^3 and thickness is 0.01 m.

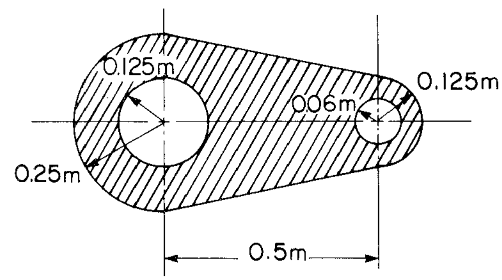


Figure P6.1.5

6.1.6. Find the centroid and the polar moment of inertia with respect to the centroid of the thin metal object shown in Fig. P6.1.6, whose mass per unit area is 80 kg/m^2.

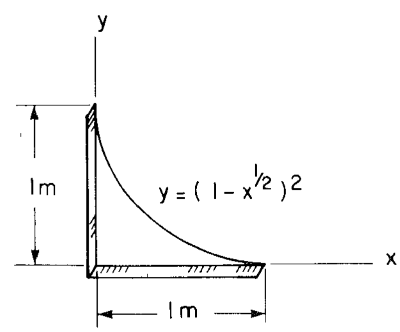


Figure P6.1.6

Section 6.2

6.2.1. In place of the rigid coupler in the revolute–translational composite joint of Fig. 3.3.8, the translational actuator shown in Fig. P6.2.1 is inserted, where the actuator

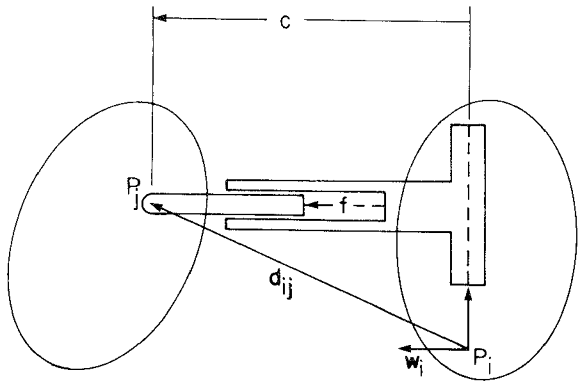


Figure P6.2.1

force is given as a function of the variable distance c as $f = g(c)$. Derive generalized forces $\mathbf{Q}_i$ and $\mathbf{Q}_j$ on bodies i and j, respectively, due to this force element (*Hint*: Use the fact that $c = \mathbf{w}_i^T\mathbf{d}_{ij}$ and use Eqs. 3.3.16, 3.3.17, and 3.3.18 to form $\delta W = f\,\delta c$.)

6.2.2. Derive generalized forces on bodies i and j that are connected by the translational joint of Fig. 3.3.5 or 3.5.9, with a spring-damper–actuator attached between points P_i and P_j. [*Hint*: Use the force expression of Eq. 6.2.10 and the fact that $\ell = C(t)$ in Eq. 3.5.12.]

Section 6.3

6.3.1. Write the variational equations of motion of Eqs. 6.3.3 and 6.3.5 for the double pendulum shown in Fig. P6.3.1, where the masses of the uniform bars are m_1 and m_2.

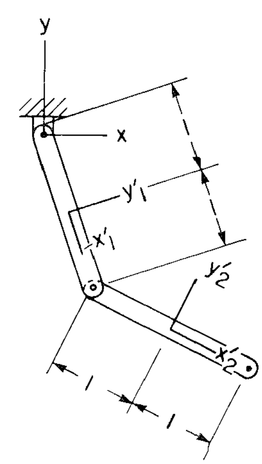


Figure P6.3.1

6.3.2. Show that, if $\mathbf{b}_1 = [1, 3, 1]^T$ and $\mathbf{A}$ is as defined in Example 6.3.3, then there is an $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$, but $\mathbf{b}_1^T\mathbf{x} \neq 0$. Show further that the conclusion of the Lagrange multiplier theorem fails; that is, there is no $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T$ such that $\mathbf{b}_1^T\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{Ax} = 0$ for all $\mathbf{x}$.

6.3.3. Write the mixed differential–algebraic equations of motion of Eq. 6.3.18 for the double pendulum of Prob. 6.3.1.

6.3.4. Define initial conditions for the double pendulum of Prob. 6.3.1, with both bars vertical and with $\dot{\phi}_1 = \dot{\phi}_2 = 1$ rad/s at $t = 0$.

Section 6.4

6.4.1. Consider the mechanism of Prob. 6.3.1. Let the dimensions be in m and the mass of the bar be 20 kg. The system is driven as in Prob. 3.6.1. Write the equations of motion of Eqs. 6.4.2 and 6.4.3 and find the Lagrange multipliers. Give a physical interpretation of the Lagrange multipliers.

6.4.2. Consider the mechanism of Prob. 3.6.2. For the purpose of dynamic analysis, the x_2'-y_2' frame is moved to the centroid of the pendulum, as shown in Fig. P6.4.2. Use the masses of the block and the pendulum and the lengths given in the figure. The drivers are defined as

$$x_1 - at - bt^2 = 0$$

$$\phi_2 - \frac{3\pi}{2} - \omega t = 0$$

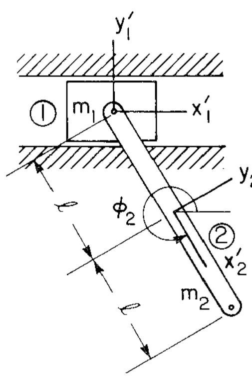


Figure P6.4.2

Write the equations of motion of Eqs. 6.4.2 and 6.4.3 and find the Lagrange multipliers.

6.4.3. Consider the mechanism of Example 3.6.2. Figure 3.6.2 is redrawn here for the purpose of dynamic analysis (Fig. P6.4.3). Use the same driver as in Example 3.6.2 and write the equations of motion of Eqs. 6.4.2 and 6.4.3 and find the Lagrange multipliers.

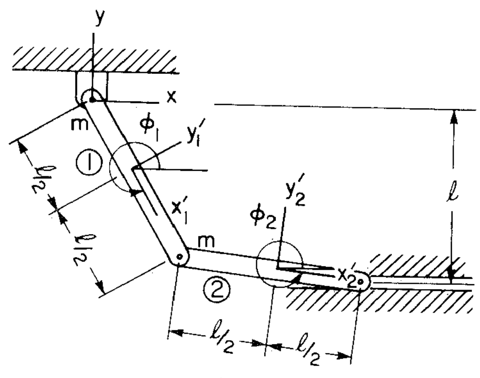


Figure P6.4.3

Section 6.5

6.5.1. Find the equilibrium position of the block shown in Fig. P6.5.1 supported by a linear spring. Use the minimum potential energy theorem and confirm the result by using the equilibrium condition $\sum F = 0$. Let m and k be the mass of the block and spring constant, respectively, and let $\ell_0 = 0$ be the free length of the spring.

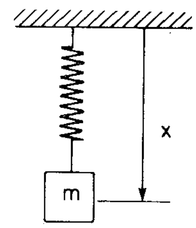


Figure P6.5.1

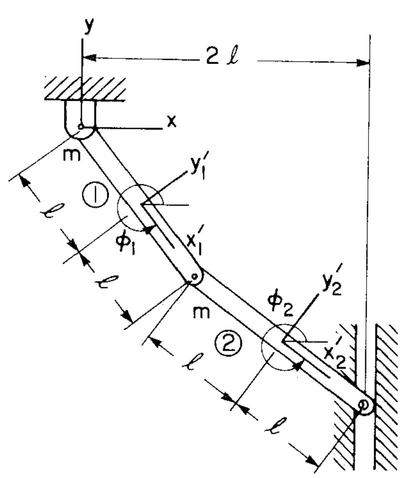


Figure P6.5.2

6.5.2. Consider the double pendulum of Example 6.5.2. The tip of the second bar is constrained by the vertical absolute constraint shown in Fig. P6.5.2. Let $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$. Write the total potential energy and constraint equation in terms of ϕ_1 and ϕ_2. Confirm that $\phi_1 = 5.0233$ rad and $\phi_2 = 5.4795$ rad is the equilibrium configuration.

Section 6.6

6.6.1. Choose reasonable x''-y'' frames and continue Prob. 6.4.1 to find the constraint reaction forces.

6.6.2. Repeat Prob. 6.6.1 for Prob. 6.4.2.

6.6.3. Repeat Prob. 6.6.1 for Prob. 6.4.3.

6.6.4. Derive expressions for constraint reaction forces and torques, in terms of Lagrange multipliers, for the following joints:
(a) Relative distance constraint of Eq. 3.3.7.
(b) Revolute–translational constraint of Eq. 3.3.17.
(c) Relative angle driver of Eq. 3.5.7.
Interpret the physical significance of the Lagrange multiplier.