CHAPTER ELEVEN

Dynamics of Spatial Systems

Many of the basic ideas used in Chapter 6 for planar dynamics are applicable for spatial dynamics. The main difference is that angular velocity effects in spatial dynamics are more intricate. They lead to nonlinear terms in the equations of motion of even a single body. A self-contained derivation of the equations of motion of spatial rigid body systems is presented here, including a definition of the inertia matrix and development of its basic properties. The numerical aspects of spatial dynamic analysis are presented in this chapter only as regards differences from corresponding methods for the dynamics of planar systems in Chapter 7.

Greater emphasis is placed in this chapter on the formulation of spatial equations of motion for multibody systems and less on manipulation with analytical examples than was the case in the study of planar system dynamics in Chapter 6. The reason for this distinction is the algebraic complexity of representing the orientation of bodies in space and the more extensively nonlinear character of spatial equations of motion than was the case in the study of planar system dynamics. In spite of the analytical complexity of spatial dynamic equations, the mathematical theory and numerical methods developed and illustrated for planar systems in Part I are directly applicable and provide a firm foundation for computer implementation of solution methods. It is suggested that the reader concentrate on the similarity in form of the equations of spatial multibody dynamics and the simpler and more intuitively appealing equations of planar dynamics. Similarity in the form of the equations is exploited in creating reliable computer implementations.

From the viewpoint of the user of established computer software, such as the DADS program [27], essentially all the algebraic and numerical complexity is hidden in the implementation of computer intensive calculations. This happy circumstance is an illustration of a situation in which equations of spatial dynamics that are virtually intractable from the point of view of the human analyst are practically implementable on modern high-speed digital computers, whose ever-increasing power makes spatial kinematic and dynamic analysis practical. It is the responsibility of the engineer to fully understand the basic formulation, however, in order to avoid analytical and numerical difficulties. As is shown in this chapter, if physically meaningful models are generated and care is taken to define the data that characterize spatial systems, reliable numerical results may be obtained and understood.

11.1 EQUATIONS OF MOTION OF A RIGID BODY

Equations of Motion from Newton's Laws Consider the rigid body shown in Fig. 11.1.1, which is located in space by a vector $\mathbf{r}$ and a set of generalized coordinates that defines the orientation of the x'-y'-z' body-fixed frame, relative to an inertial x-y-z reference frame. A differential mass $dm(P)$ at a typical point P is located on the body vector $\mathbf{s}^P$; that is, $\mathbf{s}'^P$ is constant. As a model of a rigid body, let a distance constraint act between each differential element in the body.

Forces that act on a differential element of mass at point P include the external force $\mathbf{F}(P)$ per unit of mass at point P and the internal force $\mathbf{f}(P, R)$ per unit of masses at points P and R, as shown in Fig. 11.1.1. Internal forces in this model of a rigid body are due only to gravitational interaction and distance constraints, so they are collinear (see the historical footnote on this subject in Section 6.1).

Newton's equation of motion for differential mass $dm(P)$ is

$$\ddot{\mathbf{r}}^P\, dm(P) - \mathbf{F}(P)\, dm(P) - \int_m \mathbf{f}(P, R)\, dm(R)\, dm(P) = \mathbf{0} \qquad \textbf{(11.1.1)}$$

where integration of the internal force $\mathbf{f}(P, R)$ is taken over the entire body.

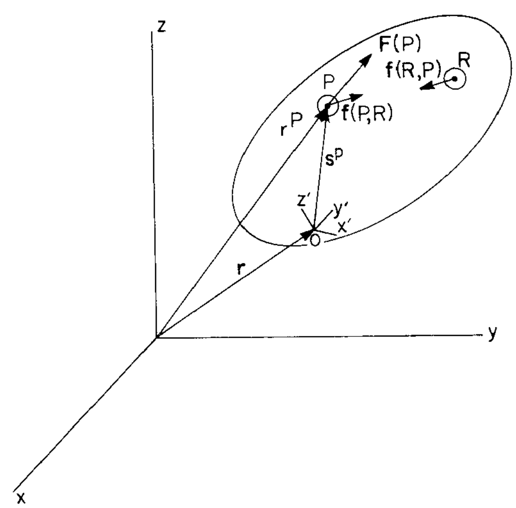


Figure 11.1.1 Forces acting on a rigid body in space.

Let $\delta\mathbf{r}^P$ denote a *virtual displacement* of point P; that is, an infinitesimal variation in the location of point P that is consistent with the allowed motion of point P. Premultiplying both sides of Eq. 11.1.1 by $\delta\mathbf{r}^{PT}$ and integrating over the total mass of the body yields

$$\int_m \delta\mathbf{r}^{PT}\ddot{\mathbf{r}}^P\, dm(p) - \int_m \delta\mathbf{r}^{PT}\mathbf{F}(P)\, dm(P) - \int_m\int_m \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\, dm(R)\, dm(P) = 0 \tag{11.1.2}$$

Manipulation of the double integral that appears in Eq. 11.1.2 shows that (Prob. 11.1.1)

$$\begin{aligned}
&\int_m\int_m \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\, dm(R)\, dm(P)\\
&\quad=\frac{1}{2}\int_m\int_m \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\, dm(R)\, dm(P)\\
&\quad\quad+\frac{1}{2}\int_m\int_m \delta\mathbf{r}^{RT}\mathbf{f}(R, P)\, dm(P)\, dm(R)\\
&\quad=\frac{1}{2}\int_m\int_m (\delta\mathbf{r}^P - \delta\mathbf{r}^R)^T\mathbf{f}(P, R)\, dm(P)\, dm(R)
\end{aligned} \tag{11.1.3}$$

where the first equality simply rewrites the expression and interchanges dummy variables P and R of integration in the second integral. The second equality follows from interchange of the order of integration and uses Newton's third law of action and reaction; that is, $\mathbf{f}(P, R)\, dm(P)\, dm(R) = -\mathbf{f}(R, P)\, dm(R)\, dm(P)$.

For a rigid body, with distance constraints between all points,

$$(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = C$$

Taking the differential of both sides yields

$$(\delta\mathbf{r}^P - \delta\mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = 0 \tag{11.1.4}$$

Since the internal force $\mathbf{f}(P, R)$ in the present model of a rigid body acts between points P and R; that is,

$$\mathbf{f}(P, R) = k(\mathbf{r}^P - \mathbf{r}^R) \tag{11.1.5}$$

where k is a constant, from Eq. 11.1.4, the right side of Eq. 11.1.3 is zero.

Collinearity of $\mathbf{f}(P, R)$ and $\mathbf{f}(R, P)$ (i.e., Eq. 11.1.5) is not implied by Newton's third law for general force fields. Examples of noncollinearity of $\mathbf{f}(P, R)$ and $\mathbf{f}(R, P)$ include charged particles in an electric field, dipoles in a magnetic field, and particles in a nonuniform gravitational field. In the presence of such effects, the integral on the right of Eq. 11.1.3 may not be zero and such forces must be accounted for as externally applied forces. Note that the right side of Eq. 11.1.3 is the virtual work of the internal forces in a rigid body. Attention in this text is focused on rigid bodies, for which the internal forces are workless.

Substituting Eqs. 11.1.4 and 11.1.5 into Eq. 11.1.3,

$$\int_m \int_m \delta\mathbf{r}^{PT}\mathbf{f}(P, R)\, dm(P)\, dm(R) = 0 \tag{11.1.6}$$

Using this result, Eq. 11.1.2 simplifies to

$$\int_m \delta\mathbf{r}^{PT}\ddot{\mathbf{r}}^P\, dm(P) - \int_m \delta\mathbf{r}^{PT}\mathbf{F}(P)\, dm(P) = 0 \tag{11.1.7}$$

which must hold for all virtual displacements $\delta\mathbf{r}^P$ that are consistent with constraints on the body.

To take full advantage of Eq. 11.1.7, the virtual displacement of point P may be written in terms of a virtual displacement of the origin of the x'-y'-z' frame and a virtual rotation of the body. From Eq. 9.2.49, a virtual displacement of point P is

$$\delta\mathbf{r}^P = \delta\mathbf{r} - \mathbf{A}\tilde{\mathbf{s}}'^P\, \delta\boldsymbol{\pi}' \tag{11.1.8}$$

Similarly, from Eqs. 9.2.41 and 9.2.43, the acceleration of point P may be written as

$$\begin{aligned}\ddot{\mathbf{r}}^P &= \ddot{\mathbf{r}} + \ddot{\mathbf{A}}\mathbf{s}'^P \\ &= \ddot{\mathbf{r}} + \mathbf{A}\tilde{\dot{\boldsymbol{\omega}}}'\mathbf{s}'^P + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P\end{aligned} \tag{11.1.9}$$

Substituting Eqs. 11.1.8 and 11.1.9 into the variational equation of Eq. 11.1.7 yields

$$\begin{aligned}&\int_m (\delta\mathbf{r}^T + \delta\boldsymbol{\pi}'^T\tilde{\mathbf{s}}'^P\mathbf{A}^T)(\ddot{\mathbf{r}} + \mathbf{A}\tilde{\dot{\boldsymbol{\omega}}}'\mathbf{s}'^P + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P)\, dm(P) \\ &\qquad - \int_m (\delta\mathbf{r}^T + \delta\boldsymbol{\pi}'^T\tilde{\mathbf{s}}'^P\mathbf{A}^T)\mathbf{F}(P)\, dm(P) = 0\end{aligned} \tag{11.1.10}$$

for all $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$ that are consistent with constraints that act on the body. Expanding the integrals of Eq. 11.1.10 yields (Prob. 11.1.2)

$$\begin{aligned}&\delta\mathbf{r}^T\ddot{\mathbf{r}}\int_m dm(P) + \delta\mathbf{r}^T(\mathbf{A}\tilde{\dot{\boldsymbol{\omega}}}' + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}')\int_m \mathbf{s}'^P\, dm(P) \\ &\qquad + \delta\boldsymbol{\pi}'^T\int_m \mathbf{s}'^P\, dm(P)\mathbf{A}^T\ddot{\mathbf{r}} + \delta\boldsymbol{\pi}'^T\int_m \tilde{\mathbf{s}}'^P\tilde{\dot{\boldsymbol{\omega}}}'\mathbf{s}'^P\, dm(P) \\ &\qquad + \delta\boldsymbol{\pi}'^T\int_m \tilde{\mathbf{s}}'^P\,\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P\, dm(P) - \delta\mathbf{r}^T\int_m \mathbf{F}(P)\, dm(P) \\ &\qquad - \delta\boldsymbol{\pi}'^T\int_m \tilde{\mathbf{s}}'^P\mathbf{F}'(P)\, dm(P) = 0\end{aligned} \tag{11.1.11}$$

for all $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$ that are consistent with constraints.

Equations of Motion with Centroidal Body-Fixed Reference Frame To simplify the form of Eq. 11.1.11, a body-fixed x'-y'-z' reference frame is selected with its origin at the *center of mass* (or *centroid*) of the body. It is thus called a *centroidal body reference frame.* The total mass is

$$m \equiv \int_m dm(P) \tag{11.1.12}$$

and, by definition of the centroid,

$$\int_m \mathbf{s}'^P \, dm(P) = \mathbf{0} \tag{11.1.13}$$

The total external force acting on the body is simply

$$\mathbf{F} \equiv \int_m \mathbf{F}(P) \, dm(P) \tag{11.1.14}$$

and the *moment* (or *torque*) of the external forces with respect to the origin of the x'-y'-z' frame is

$$\mathbf{n}' \equiv \int_m \tilde{\mathbf{s}}'^P \mathbf{F}'(P) \, dm(P) \tag{11.1.15}$$

The fourth integral in Eq. 11.1.11 may be written as

$$\begin{aligned} \int_m \tilde{\mathbf{s}}'^P \tilde{\dot{\boldsymbol{\omega}}}' \mathbf{s}'^P \, dm(P) &= -\left(\int_m \tilde{\mathbf{s}}'^P \tilde{\mathbf{s}}'^P \, dm(P) \right) \dot{\boldsymbol{\omega}}' \\ &\equiv \mathbf{J}' \dot{\boldsymbol{\omega}}' \end{aligned} \tag{11.1.16}$$

where the constant *inertia matrix* (or *inertia tensor*) $\mathbf{J}'$ with respect to the x'-y'-z' centroidal frame is defined as

$$\begin{aligned} \mathbf{J}' &\equiv -\int_m \tilde{\mathbf{s}}'^P \tilde{\mathbf{s}}'^P \, dm(P) \\ &= \int_m \begin{bmatrix} (y'^P)^2 + (z'^P)^2 & -x'^P y'^P & -x'^P z'^P \\ -x'^P y'^P & (x'^P)^2 + (z'^P)^2 & -y'^P z'^P \\ -x'^P z'^P & -y'^P z'^P & (x'^P)^2 + (y'^P)^2 \end{bmatrix} dm(P) \end{aligned} \tag{11.1.17}$$

where each term in the matrix on the right is integrated separately. The diagonal elements in the matrix $\mathbf{J}'$ are called *moments of inertia* and the off-diagonal terms are called *products of inertia.* Since symmetrically placed off-diagonal elements are equal, $\mathbf{J}'$ is symmetric; that is, $\mathbf{J}' = \mathbf{J}'^T$.

The integrand of the fifth integral in Eq. 11.1.11 may be expanded, using Eqs. 9.1.25 and 9.1.31, as

$$\begin{aligned} \tilde{\mathbf{s}}'^P \tilde{\boldsymbol{\omega}}' \tilde{\boldsymbol{\omega}}' \mathbf{s}'^P &= -\tilde{\mathbf{s}}'^P \tilde{\boldsymbol{\omega}}' \tilde{\mathbf{s}}'^P \boldsymbol{\omega}' \\ &= -\tilde{\boldsymbol{\omega}}' \tilde{\mathbf{s}}'^P \tilde{\mathbf{s}}'^P \boldsymbol{\omega}' - \boldsymbol{\omega}' \mathbf{s}'^{PT} \tilde{\mathbf{s}}'^P \boldsymbol{\omega}' - \mathbf{s}'^P \boldsymbol{\omega}'^T \tilde{\boldsymbol{\omega}}' \mathbf{s}'^P \\ &= -\tilde{\boldsymbol{\omega}}' \tilde{\mathbf{s}}'^P \tilde{\mathbf{s}}'^P \boldsymbol{\omega}' \end{aligned}$$

Integrating both sides over the mass of the body yields

$$\int_m \tilde{\mathbf{s}}'^{PT}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P\,dm(P) = \tilde{\boldsymbol{\omega}}'\Big(-\int_m \tilde{\mathbf{s}}'^P\tilde{\mathbf{s}}'^P\,dm(P)\Big)\boldsymbol{\omega}' = \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \quad \mathbf{(11.1.18)}$$

Substituting Eqs. 11.1.12 to 11.1.18 into Eq. 11.1.11 yields the *variational Newton–Euler equations of motion* for a rigid body with a centroidal body-fixed reference frame,

$$\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}'] = 0 \quad \mathbf{(11.1.19)}$$

which must hold for all virtual displacements $\delta\mathbf{r}$ and virtual rotations $\delta\boldsymbol{\pi}'$ of the centroidal x'-y'-z' frame that are consistent with constraints that act on the body.

Example 11.1.1: A body shown in Fig. 11.1.2 is constrained so that point P on its z' axis moves on the surface of a unit sphere. This constraint is defined by

$$\Phi = (\mathbf{r} + \mathbf{A}\mathbf{h}')^T(\mathbf{r} + \mathbf{A}\mathbf{h}') - 1 = 0$$

where $\mathbf{h}'$ is the z' axis unit vector, $\mathbf{h}' = [0, 0, 1]^T$. Taking the variation of both sides of this equation yields

$$\begin{aligned}
\delta\Phi &= 2(\mathbf{r} + \mathbf{A}\mathbf{h}')^T(\delta\mathbf{r} + \delta\mathbf{A}\mathbf{h}') \\
&= 2(\mathbf{r} + \mathbf{A}\mathbf{h}')^T\,\delta\mathbf{r} + 2(\mathbf{r} + \mathbf{A}\mathbf{h}')^T\mathbf{A}\,\delta\tilde{\boldsymbol{\pi}}'\mathbf{h}' \\
&= 2(\mathbf{r} + \mathbf{A}\mathbf{h}')^T\,\delta\mathbf{r} - (2\mathbf{r}^T\mathbf{A}\tilde{\mathbf{h}}' + 2\mathbf{h}'^T\tilde{\mathbf{h}}')\,\delta\boldsymbol{\pi}' \\
&= 2(\mathbf{r} + \mathbf{A}\mathbf{h}')^T\,\delta\mathbf{r} - (2\mathbf{r}^T\mathbf{A}\tilde{\mathbf{h}}')\,\delta\boldsymbol{\pi}' = 0
\end{aligned}$$

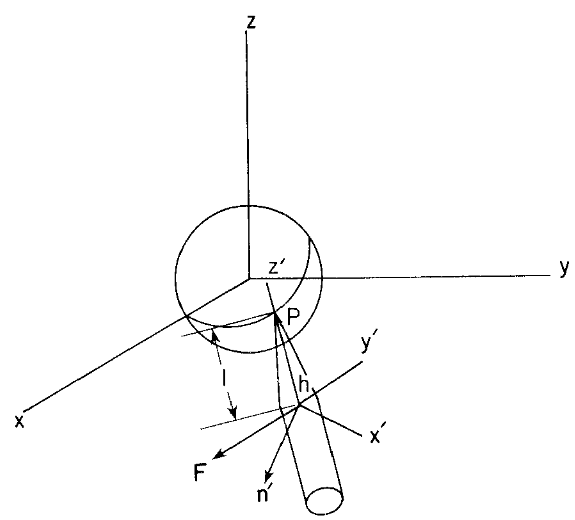


Figure 11.1.2 Pendulum with point on unit sphere.

Since Eq. 11.1.19 must hold for all $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$ that satisfy the above equation, the Lagrange multiplier theorem guarantees the existence of a multiplier λ such that

$$\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F} + 2\lambda(\mathbf{r} + \mathbf{A}\mathbf{h}')] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}' + 2\lambda\tilde{\mathbf{h}}'\mathbf{A}^T\mathbf{r}] = 0$$

which must hold for all $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$. Thus, the equations of motion are

$$m\ddot{\mathbf{r}} + 2(\mathbf{r} + \mathbf{A}\mathbf{h}')\lambda = \mathbf{F}$$

$$\mathbf{J}'\dot{\boldsymbol{\omega}}' + 2\tilde{\mathbf{h}}'\mathbf{A}^T\mathbf{r}\lambda = \mathbf{n}' - \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}'$$

and the constraint equation.

If no constraints act on a body, (i.e., it is free to move in space), then $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$ are arbitrary and their coefficients in Eq. 11.1.19 must be zero. This yields the *Newton–Euler equations of motion* for an unconstrained body:

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} \\ \mathbf{J}'\dot{\boldsymbol{\omega}}' &= \mathbf{n}' - \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{aligned} \tag{11.1.20}$$

The Newton–Euler equations of Eq. 11.1.20 may be thought of as differential equations in $\mathbf{r}$ and $\boldsymbol{\omega}'$. In some cases, they may be integrated for these variables. Integration for orientation of the body is possible, however, only after a set of orientation generalized coordinates, such as Euler parameters, is introduced. This is required, since $\boldsymbol{\omega}'$ cannot be integrated directly (Prob. 11.1.3).

Equations of Motion for a Body with One Point Fixed in Space Consider the special case of a body, such as the top shown in Fig. 11.1.3, with a point O'' fixed in the inertial reference frame. Since point O'' will not generally be the center of mass of the body, to avoid confusion, define an x''-y''-z'' body-fixed reference frame with origin at point O''. Since point O'' is

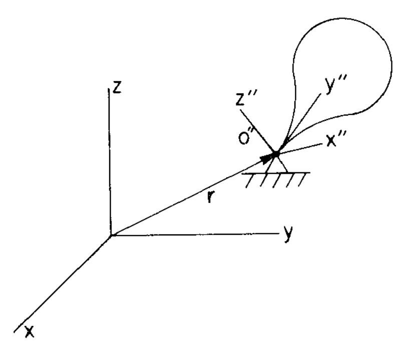


Figure 11.1.3 Body with O'' fixed in space.

fixed in space, $\mathbf{r}$ is constant, its velocity and acceleration are zero, and kinematically admissible virtual displacements of point O'' are zero; that is,

$$\begin{aligned} \mathbf{r} &= \mathbf{c} \\ \dot{\mathbf{r}} = \ddot{\mathbf{r}} &= \mathbf{0} \\ \delta\mathbf{r} &= \mathbf{0} \end{aligned} \tag{11.1.21}$$

The general variational form of the equations of motion in a noncentroidal reference frame of Eq. 11.1.11 can be written in terms of the x''-y''-z'' frame, using the relations of Eq. 11.1.21. Using Eqs. 11.1.15 to 11.1.18, all of which are valid for the noncentroidal x''-y''-z'' reference frame, Eq. 11.1.11 reduces to

$$\delta\boldsymbol{\pi}''^T[\mathbf{J}''\dot{\boldsymbol{\omega}}'' + \tilde{\boldsymbol{\omega}}''\mathbf{J}''\boldsymbol{\omega}'' - \mathbf{n}''] = 0 \tag{11.1.22}$$

for all virtual rotations $\delta\boldsymbol{\pi}''$ that are consistent with any additional constraints that may be imposed on motion of the body.

If no orientation constraints are imposed on the body; for example, if the body of Fig. 11.1.3 is free to move as a pendulum or a spinning top with a spherical joint at point O'', then $\delta\boldsymbol{\pi}''$ of Eq. 11.1.22 is arbitrary and its coefficient must be zero, leading to the *Euler equations of motion*:

$$\mathbf{J}''\dot{\boldsymbol{\omega}}'' = \mathbf{n}'' - \tilde{\boldsymbol{\omega}}''\mathbf{J}''\boldsymbol{\omega}'' \tag{11.1.23}$$

Since these equations are nonlinear in angular velocities that appear on the right side, closed-form solution is generally not possible.

Example 11.1.2: As a special case of a body with a point fixed in space, consider a pendulum that is constrained to rotate about the global y axis, as shown in Fig. 11.1.4. The body-fixed x''-y''-z'' frame is selected so that the y'' axis coincides with the global y axis and is the axis of rotation. By construction of the

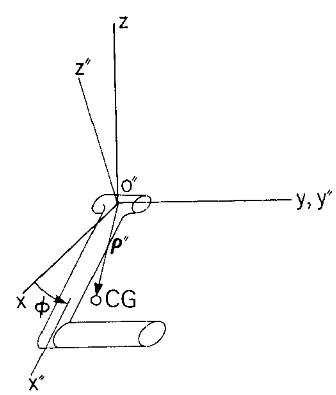


Figure 11.1.4 Pendulum rotating about y axis.

body-fixed reference frame and definition of the angle ϕ of rotation defined in Fig. 11.1.4, the transformation from the x''-y''-z'' frame to the inertial x-y-z frame is

$$\mathbf{s} \equiv \mathbf{A}\mathbf{s}'' = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \mathbf{s}''$$

A direct calculation indicates that angular velocity, acceleration, and virtual rotation will be about the y'' axis and are related to the angle ϕ by

$$\begin{aligned} \boldsymbol{\omega}'' &= [0, \dot{\phi}, 0]^T \\ \dot{\boldsymbol{\omega}}'' &= [0, \ddot{\phi}, 0]^T \\ \delta\boldsymbol{\pi}'' &= [0, \delta\phi, 0]^T \end{aligned} \tag{11.1.24}$$

Substituting these relations into Eq. 11.1.22 yields the variational form of the equations of motion of the pendulum as

$$[0, \delta\phi, 0]\left\{\begin{bmatrix} J_{x''y''}\ddot{\phi} \\ J_{y''y''}\ddot{\phi} \\ J_{z''y''}\ddot{\phi} \end{bmatrix} + \begin{bmatrix} J_{z''y''}\dot{\phi}^2 \\ 0 \\ -J_{x''y''}\dot{\phi}^2 \end{bmatrix} - \begin{bmatrix} n_{x''} \\ n_{y''} \\ n_{z''} \end{bmatrix}\right\} = 0 \tag{11.1.25}$$

The gravitational force acts at the center of mass and is $\mathbf{F} = -mg\mathbf{k}$, in the x-y-z frame. In the x''-y''-z'' frame, this is

$$\mathbf{F}'' = \mathbf{A}^T\mathbf{F} = mg\begin{bmatrix} \sin\phi \\ 0 \\ -\cos\phi \end{bmatrix}$$

The torque $\mathbf{n}''$ is thus, from Eq. 11.1.15,

$$\mathbf{n}'' = \tilde{\boldsymbol{\rho}}''\mathbf{F}'' = mg\begin{bmatrix} -\rho_{y''}\cos\phi \\ \rho_{z''}\sin\phi + \rho_{x''}\cos\phi \\ -\rho_{y''}\sin\phi \end{bmatrix}$$

Since $\delta\phi$ in Eq. 11.1.25 is arbitrary, its coefficient must be zero, yielding the differential equation of motion of the pendulum as

$$J_{y''y''}\ddot{\phi} = n_{y''} = mg(\rho_{z''}\sin\phi + \rho_{x''}\cos\phi) \tag{11.1.26}$$

Note that only the moment of inertia about the y'' axis of the body appears in the equation of motion.

To determine reaction torques that act at the rotational joint located at point O'' in Fig. 11.1.4, the general form of reaction torque may be written as

$$\mathbf{T}'' = [T_{x''}, 0, T_{z''}]^T \tag{11.1.27}$$

The y'' component of the reaction torque is zero, since the rotational joint permits free rotation of the body about that axis. Adding the reaction torque of Eq. 11.1.27 to Eq. 1.1.22 yields a variational equation of motion that accounts for all torques acting on the body. Since motion of the body, with point O'' fixed in space, under the action of the reaction torques is equivalent to imposing the rotational joint constraint, the virtual rotation $\delta\boldsymbol{\pi}''$, with $\mathbf{T}''$ included, is arbitrary, and the coefficient of $\delta\boldsymbol{\pi}''$ in Eq. 11.1.22 must be zero, leading to Eq. 11.1.26 and

$$\begin{aligned} T_{x''} &= J_{z''y''}\dot{\phi}^2 + J_{x''y''}\ddot{\phi} - n_{x''} \\ T_{z''} &= -J_{x''y''}\dot{\phi}^2 + J_{z''y''}\ddot{\phi} - n_{z''} \end{aligned} \tag{11.1.28}$$

Thus, if Eq. 11.1.26 is solved for ϕ, $\dot{\phi}$, and $\ddot{\phi}$, the reaction torques due to asymmetry of the pendulum are determined by Eq. 11.1.28. This result shows that the products of inertia have a substantial influence on the forces that act in a system, if asymmetry is present.

11.2 PROPERTIES OF CENTROID AND MOMENTS AND PRODUCTS OF INERTIA

Centroid To find the centroid of a body, let a noncentroidal body-fixed x''-y''-z'' reference frame with origin at O'' be given. Let the centroid be designated as point O', and let the x'-y'-z' reference frame have its origin at O', as shown in Fig. 11.2.1.

The vector $\boldsymbol{\rho}''$ in the x''-y''-z'' frame that locates the centroid O' may be determined by writing $\mathbf{s}'^P = \mathbf{C}(\mathbf{s}''^P - \boldsymbol{\rho}'')$, where $\mathbf{C}$ is the constant orthogonal transformation matrix from the x''-y''-z'' frame to the x'-y'-z' frame; that is,

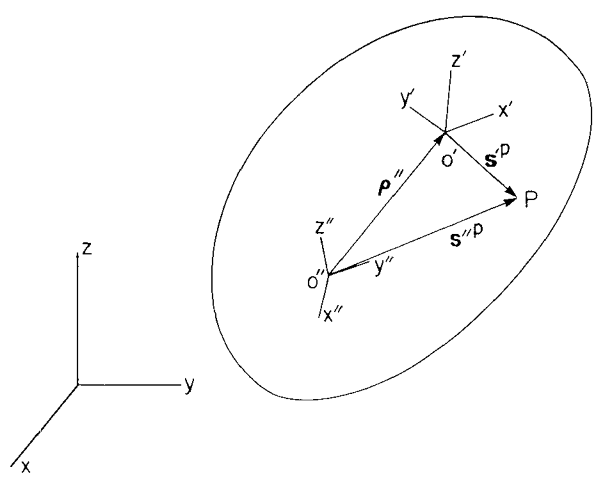


Figure 11.2.1 Location of centroid.

$\mathbf{s}' = \mathbf{C}\mathbf{s}''$. Using Eq. 11.1.13,

$$\begin{aligned}\mathbf{0} &= \int_m \mathbf{s}'^P \, dm(P) = \int_m \mathbf{C}(\mathbf{s}''^P - \boldsymbol{\rho}'') \, dm(P) \\ &= \mathbf{C} \int_m \mathbf{s}''^P \, dm(P) - \mathbf{C}\boldsymbol{\rho}'' \int_m dm(P) \\ &= \mathbf{C} \int_m \mathbf{s}''^P \, dm(P) - m \, \mathbf{C}\boldsymbol{\rho}''\end{aligned}$$

Premultiplying both sides of this equation by $\mathbf{C}^T = \mathbf{C}^{-1}$,

$$\boldsymbol{\rho}'' = \frac{1}{m} \int_m \mathbf{s}''^P \, dm(P) \tag{11.2.1}$$

Example 11.2.1: Consider the homogeneous body shown in Fig. 11.2.2. By Eq. 11.2.1, its centroid in the x''-y''-z'' frame is

$$\begin{aligned}\boldsymbol{\rho}'' &= \frac{1}{m} \int_0^{10} \int_0^{10} \int_{9-0.2y''}^{11+0.2y''} \gamma [x'', y'', z'']^T \, dz'' \, dy'' \, dx'' \\ &= \frac{\gamma}{m} \int_0^{10} \int_0^{10} [x''(2 + 0.4y''), y''(2 + 0.4y''), 20 + 4y'']^T \, dy'' \, dx'' \\ &= \frac{\gamma}{m} \int_0^{10} \left[40x'', \frac{700}{3}, 400\right]^T dx'' \\ &= \frac{\gamma}{m} \left[2000, \frac{7000}{3}, 4000\right]^T\end{aligned}$$

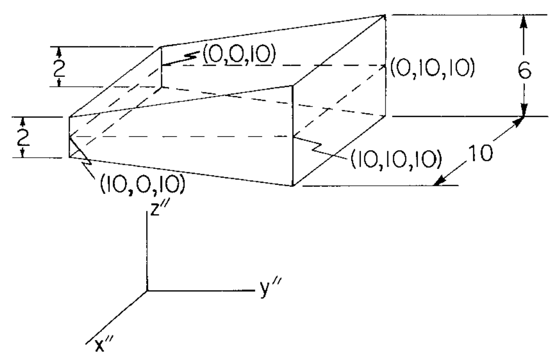


Figure 11.2.2 Homogeneous body.

where γ is mass density of the material. The mass of the body is $m = \gamma 400$. Thus, the centroid is located by

$$\boldsymbol{\rho}'' = [5, \tfrac{35}{6}, 10]^T$$

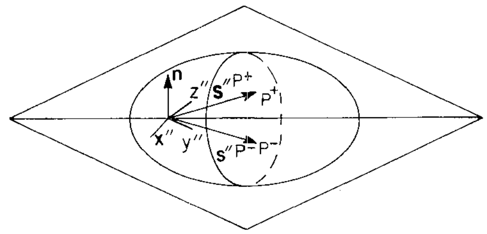


Figure 11.2.3 Body with plane of symmetry.

Consider the body shown in Fig. 11.2.3, for which the location of points and mass distribution are symmetric about a plane with normal $\mathbf{n}$. Select a body x''-y''-z'' frame so that its origin is on the plane of symmetry. Thus, for points P^+ and P^- and associated differential masses that are symmetrically placed with respect to the *plane of symmetry*,

$$\begin{aligned} \mathbf{n}^T\mathbf{s}''^{P^-} &= -\mathbf{n}^T\mathbf{s}''^{P^+} \\ dm\,(P^-) &= dm\,(P^+) \end{aligned} \tag{11.2.2}$$

Taking the scalar product of both sides of Eq. 11.2.1 with $\mathbf{n}$ and using Eq. 11.2.2 to change variables of integration,

$$\begin{aligned} \mathbf{n}^T\boldsymbol{\rho}'' &= \frac{1}{m}\int_m \mathbf{n}^T\mathbf{s}''^P\,dm(P) \\ &= \frac{1}{m}\int_{m^+} \mathbf{n}^T\mathbf{s}''^{P^+}\,dm(P^+) + \frac{1}{m}\int_{m^-} \mathbf{n}^T\mathbf{s}''^{P^-}\,dm(P^-) \\ &= \frac{1}{m}\int_{m^+} \mathbf{n}^T\mathbf{s}''^{P^+}\,dm(P^+) + \frac{1}{m}\int_{m^+} \mathbf{n}^T(-\mathbf{s}''^{P^+})\,dm(P^+) \\ &= 0 \end{aligned} \tag{11.2.3}$$

Equation 11.2.3 shows that if a body has a plane of symmetry, then the centroid lies on that plane. This agrees with the result of Example 11.2.1, in which the planes $x''=5$ and $z''=10$ are planes of symmetry. If a body has an *axis of symmetry*; that is, every plane that contains the axis of symmetry is a plane of symmetry, then the centroid lies on the axis of symmetry. Note that if a body is geometrically symmetric, but is asymmetrically nonhomogeneous, these helpful geometric results are no longer valid.

Transformation of Inertia Matrices The inertia matrix with respect to the x''-y''-z'' frame in Fig. 11.2.1 is defined as

$$\mathbf{J}'' = -\int_m \tilde{\mathbf{s}}''^P\tilde{\mathbf{s}}''^P\,dm(P) \tag{11.2.4}$$

To obtain a relationship between $\mathbf{J}''$ and the inertia matrix $\mathbf{J}'$ with respect to the x'-y'-z' centroidal reference frame, substitute $\mathbf{s}''^P = \boldsymbol{\rho}'' + \mathbf{C}^T\mathbf{s}'^P$ into Eq. 11.2.4 to obtain

$$\mathbf{J}'' = -\int_m (\tilde{\boldsymbol{\rho}}'' + \widetilde{\mathbf{C}^T\mathbf{s}'^P})(\tilde{\boldsymbol{\rho}}'' + \widetilde{\mathbf{C}^T\mathbf{s}'^P})\, dm(P)$$

Using Eq. 9.2.21 and expanding terms, this becomes

$$\mathbf{J}'' = -m\tilde{\boldsymbol{\rho}}''\tilde{\boldsymbol{\rho}}'' - \tilde{\boldsymbol{\rho}}''\mathbf{C}^T \int_m \tilde{\mathbf{s}}'^P\, dm(P)\mathbf{C} - \mathbf{C}^T \int_m \tilde{\mathbf{s}}'^P\, dm(P)\mathbf{C}\tilde{\boldsymbol{\rho}}'' - \mathbf{C}^T \int_m \tilde{\mathbf{s}}'^P\tilde{\mathbf{s}}'^P\, dm(P)\mathbf{C} \tag{11.2.5}$$

Since the x'-y'-z' frame is centroidal,

$$\int_m \tilde{\mathbf{s}}'^P\, dm(P) = \widetilde{\left(\int_m \mathbf{s}'^P\, dm(P)\right)} = \mathbf{0}$$

Using this result and Eq. 11.1.17, Eq. 11.2.5 becomes

$$\mathbf{J}'' = \mathbf{C}^T\mathbf{J}'\mathbf{C} - m\tilde{\boldsymbol{\rho}}''\tilde{\boldsymbol{\rho}}'' \tag{11.2.6}$$

where $\mathbf{C}$ is the orthogonal transformation matrix from the x''-y''-z'' frame to the centroidal x'-y'-z' frame; that is, $\mathbf{s}' = \mathbf{C}\mathbf{s}''$. Using Eq. 9.1.28, Eq. 11.2.6 may be written in the more conventional form [4, 7, 9, 35]

$$\mathbf{J}'' = \mathbf{C}^T\mathbf{J}'\mathbf{C} + m(\boldsymbol{\rho}''^T\boldsymbol{\rho}''\mathbf{I} - \boldsymbol{\rho}''\boldsymbol{\rho}''^T) \tag{11.2.7}$$

As a special case, if $\mathbf{C} = \mathbf{I}$ (i.e., if the x'-y'-z' and x''-y''-z'' frames are parallel) then moments of inertia on the diagonal of Eq. 11.2.7 are related by

$$\begin{aligned} J_{x''x''} &= J_{x'x'} + m(\rho_{y''}^2 + \rho_{z''}^2) \\ J_{y''y''} &= J_{y'y'} + m(\rho_{x''}^2 + \rho_{z''}^2) \\ J_{z''z''} &= J_{z'z'} + m(\rho_{x''}^2 + \rho_{y''}^2) \end{aligned} \tag{11.2.8}$$

Thus, moments of inertia with respect to a noncentroidal x''-y''-z'' frame are those with respect to the parallel centroidal x'-y'-z' frame plus the mass of the body times the square of the distances between the respective parallel axes. This is the so-called *parallel axis theorem*. Products of inertia for parallel axes are obtained from off-diagonal terms in Eq. 11.2.7 as

$$\begin{aligned} J_{x''y''} &= J_{x'y'} - m\rho_{x''}\rho_{y''} \\ J_{x''z''} &= J_{x'z'} - m\rho_{x''}\rho_{z''} \\ J_{y''z''} &= J_{y'z'} - m\rho_{y''}\rho_{z''} \end{aligned} \tag{11.2.9}$$

Consider a special case in which one of the planes of an x''-y''-z'' frame is a plane of symmetry (e.g., the x''-y'' plane). Then, using the symmetry argument employed in deriving Eq. 11.2.3 (Prob. 11.2.2),

$$J_{z''x''} = J_{z''y''} = 0 \tag{11.2.10}$$

Similarly, if the x''-z'' plane is a plane of symmetry,

$$J_{x''y''} = J_{z''y''} = 0 \tag{11.2.11}$$

and if the y''-z'' plane is a plane of symmetry,

$$J_{x''y''} = J_{x''z''} = 0 \tag{11.2.12}$$

If any two coordinate planes of the x''-y''-z'' frame are planes of symmetry (i.e., have both geometric and mass distribution symmetry), then, from Eqs. 11.2.10 to 11.2.12, all products of inertia are zero (Prob. 11.2.3). This is a very helpful property, since bodies in mechanical systems often have two planes of symmetry. A common case of pairs of planes of symmetry is homogeneous bodies of revolution about one of the axes of a reference frame.

Example 11.2.2: The moments and products of inertia of the body of Fig. 11.2.2 with respect to a centroidal x'-y'-z' frame that is parallel to the x''-y''-z'' frame can be calculated using the parallel axis theorem. Using the results of Example 11.2.1 in Eq. 11.2.8,

$$J_{x'x'} = J_{x''x''} - 400\gamma\left[\left(\frac{35}{6}\right)^2 + 100\right]$$

$$J_{y'y'} = J_{y''y''} - 400\gamma[25 + 100]$$

$$J_{z'z'} = J_{z''z''} - 400\gamma\left[25 + \left(\frac{35}{6}\right)^2\right]$$

which may be evaluated using the results of Prob. 11.2.3. Since the x'-y' and y'-z' planes are planes of symmetry,

$$J_{x'y'} = J_{x'z'} = J_{y'z'} = 0$$

Principal Axes Even if a body is not symmetric, it is possible to find a centroidal reference frame with respect to which the products of inertia are all zero. Let the x''-y''-z'' and x'-y'-z' frames have their origins at the centroid of a body. Then, $\boldsymbol{\rho}'' = \mathbf{0}$. Presume $\mathbf{J}''$ is known, in general, as a full symmetric matrix. Multiplying Eq. 11.2.7 on the left by $\mathbf{C}$ and multiplying it on the right by $\mathbf{C}^T$,

$$\mathbf{J}' = \mathbf{C}\mathbf{J}''\mathbf{C}^T \tag{11.2.13}$$

The task is now to find an orthogonal transformation matrix $\mathbf{C}$ from the x''-y''-z'' frame to the x'-y'-z' frame for which $\mathbf{J}'$ is diagonal.

From the definition of $\mathbf{J}''$ in Eq. 11.2.4, for any vector $\mathbf{a}$ in R^3,

$$\begin{aligned}\mathbf{a}^T\mathbf{J}''\mathbf{a} &= -\int_m \mathbf{a}^T\tilde{\mathbf{s}}''\tilde{\mathbf{s}}''\mathbf{a}\,dm \\ &= \int_m \mathbf{a}^T\tilde{\mathbf{s}}''^T\tilde{\mathbf{s}}''\mathbf{a}\,dm \\ &= \int_m (\tilde{\mathbf{s}}''\mathbf{a})^T(\tilde{\mathbf{s}}''\mathbf{a})\,dm \geq 0 \end{aligned} \tag{11.2.14}$$

Thus, $\mathbf{J}''$ is *positive semidefinite.* In fact, providing that mass density is nowhere zero in the body, $\mathbf{a}^T\mathbf{J}''\mathbf{a} = \mathbf{0}$ implies that $\tilde{\mathbf{s}}''\mathbf{a} = \mathbf{0} = -\tilde{\mathbf{a}}\mathbf{s}''$, for all $\mathbf{s}''$ in the body. If there are three linearly independent vectors $\mathbf{s}_i''$, $i = 1, 2, 3$, to points in the body, then

$$\tilde{\mathbf{a}}[\mathbf{s}_1'', \mathbf{s}_2'', \mathbf{s}_3''] = \mathbf{0}$$

and since $[\mathbf{s}_1'', \mathbf{s}_2'', \mathbf{s}_3'']$ is nonsingular, $\tilde{\mathbf{a}} = \mathbf{0}$ and $\mathbf{a} = \mathbf{0}$. Thus, in this case, $\mathbf{J}''$ is *positive definite.* This is the case for any homogeneous body whose volume is not zero.

The 3×3 positive semidefinite matrix $\mathbf{J}''$ has three orthonormal *eigenvectors,* denoted $\mathbf{f}''$, $\mathbf{g}''$, and $\mathbf{h}''$ in the x''-y''-z'' frame, with corresponding nonnegative eigenvalues [22], ζ_i, $i = 1, 2, 3$, ordered such that $\zeta_1 \geq \zeta_2 \geq \zeta_3 \geq 0$, where

$$\begin{aligned}\mathbf{J}''\mathbf{f}'' &= \zeta_1\mathbf{f}'' \\ \mathbf{J}''\mathbf{g}'' &= \zeta_2\mathbf{g}'' \\ \mathbf{J}''\mathbf{h}'' &= \zeta_3\mathbf{h}''\end{aligned} \tag{11.2.15}$$

Since $-\mathbf{g}''$ and $-\mathbf{h}''$ are also eigenvectors, their signs can be arranged so that $\mathbf{f}''$, $\mathbf{g}''$, and $\mathbf{h}''$ are unit coordinate vectors for a right-hand x'-y'-z' Cartesian frame. The transformation matrix $\mathbf{C}^T$ from the x'-y'-z' frame to the x''-y''-z'' frame is, by Eq. 9.2.13,

$$\mathbf{C}^T = [\mathbf{f}'', \mathbf{g}'', \mathbf{h}''] \tag{11.2.16}$$

Substituting $\mathbf{C}^T$ from Eq. 11.2.16 into Eq. 11.2.13,

$$\begin{aligned}\mathbf{J}' &= \mathbf{C}\mathbf{J}''\mathbf{C}^T \\ &= [\mathbf{f}'', \mathbf{g}'', \mathbf{h}'']^T\mathbf{J}''[\mathbf{f}'', \mathbf{g}'', \mathbf{h}''] \\ &= [\mathbf{f}'', \mathbf{g}'', \mathbf{h}'']^T[\mathbf{J}''\mathbf{f}'', \mathbf{J}''\mathbf{g}'', \mathbf{J}''\mathbf{h}''] \\ &= [\mathbf{f}'', \mathbf{g}'', \mathbf{h}'']^T[\zeta_1\mathbf{f}'', \zeta_2\mathbf{g}'', \zeta_3\mathbf{h}''] \\ &= [\mathbf{f}'', \mathbf{g}'', \mathbf{h}'']^T[\mathbf{f}'', \mathbf{g}'', \mathbf{h}'']\begin{bmatrix}\zeta_1 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3\end{bmatrix} \\ &= \begin{bmatrix}\zeta_1 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3\end{bmatrix}\end{aligned} \tag{11.2.17}$$

Thus, in the x'-y'-z' frame, $J_{x'x'} = \zeta_1$, $J_{y'y'} = \zeta_2$, $J_{z'z'} = \zeta_3$, and $J_{x'y'} = J_{x'y'} = J_{y'z'} = 0$. The axes of such a centroidal frame are called *principal axes* and the associated moments of inertia are called *principal moments of inertia.* By construction, the principal products of inertia are zero.

Example 11.2.3: The triangular plate shown in Fig. 11.2.4 has negligible thickness, so the mass is considered concentrated in the x-y plane, with mass density γ per unit area. Evaluating the integrals in Eq. 11.1.17 in the x''-y''-z'' frame,

$$\begin{aligned} J''_x &= \gamma \int_0^1 \int_0^{1-x} y^2\, dy\, dx \\ &= \gamma \int_0^1 \frac{(1-x)^3}{3}\, dx \\ &= \frac{\gamma}{12} \end{aligned}$$

$$J''_{yy} = J''_{xx} = \frac{\gamma}{12}$$

$$\begin{aligned} J''_{zz} &= \gamma \int_0^1 \int_0^{1-x} (x^2 + y^2)\, dy\, dx \\ &= \gamma \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx \\ &= \frac{\gamma}{6} \end{aligned}$$

$$\begin{aligned} J''_{xy} &= -\gamma \int_0^1 \int_0^{1-x} xy\, dy\, dx \\ &= -\gamma \int_0^1 \frac{x(1-x)^2}{2}\, dx \\ &= -\frac{\gamma}{24} \end{aligned}$$

$$J''_{xz} = J''_{yz} = 0$$

Thus, the inertia matrix in the x''-y''-z'' frame is

$$\mathbf{J}'' = \frac{\gamma}{24} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The centroid is located by Eq. 11.2.1, with $m = \gamma/2$:

$$\begin{aligned} \boldsymbol{\rho}'' &= \frac{2}{\gamma} \int_0^1 \int_0^{1-x} [x, y, 0]^T \gamma\, dy\, dx \\ &= [\tfrac{1}{3}, \tfrac{1}{3}, 0]^T \end{aligned}$$

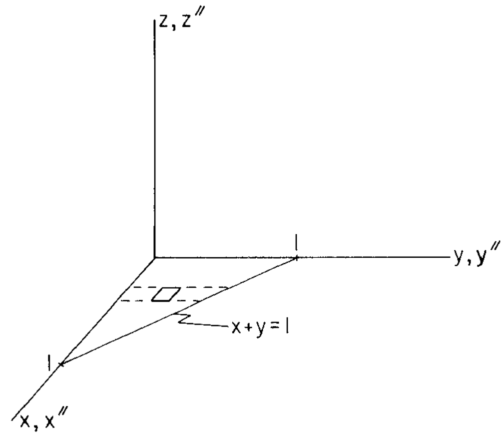


Figure 11.2.4 Triangular plate in x-y plane.

For a centroidal x'-y'-z' frame parallel to the x''-y''-z'' noncentroidal frame shown in Fig. 11.2.5, the parallel axis theorem applies and, from Eqs. 11.2.8 and 11.2.9,

$$\mathbf{J}' = \mathbf{J}'' - m\begin{bmatrix} (\rho_{y''}^2 + \rho_{z''}^2) & -\rho_{x''}\rho_{y''} & -\rho_{x''}\rho_{z''} \\ -\rho_{x''}\rho_{y''} & (\rho_{x''}^2 + \rho_{z''}^2) & -\rho_{y''}\rho_{z''} \\ -\rho_{x''}\rho_{z''} & -\rho_{y''}\rho_{z''} & (\rho_{x''}^2 + \rho_{y''}^2) \end{bmatrix}$$

$$= \frac{\gamma}{72}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The eigenvalues of $\mathbf{J}'$ are obtained by solving the *characteristic equation* [22]:

$$|\mathbf{J}' - \zeta\mathbf{I}| = \begin{bmatrix} \frac{\gamma}{36} - \zeta & \frac{\gamma}{72} & 0 \\ \frac{\gamma}{72} & \frac{\gamma}{36} - \zeta & 0 \\ 0 & 0 & \frac{\gamma}{18} - \zeta \end{bmatrix}$$

$$= \left(\frac{\gamma}{36} - \zeta\right)^2\left(\frac{\gamma}{18} - \zeta\right) - \left(\frac{\gamma}{18} - \zeta\right)\left(\frac{\gamma}{72}\right)^2$$

$$= \left[\left(\frac{\gamma}{36} - \zeta\right)^2 - \left(\frac{\gamma}{72}\right)^2\right]\left(\frac{\gamma}{18} - \zeta\right) = 0$$

whose solutions are

$$\zeta_1 = \frac{\gamma}{18}, \qquad \zeta_2 = \frac{\gamma}{24}, \qquad \zeta_3 = \frac{\gamma}{72}$$

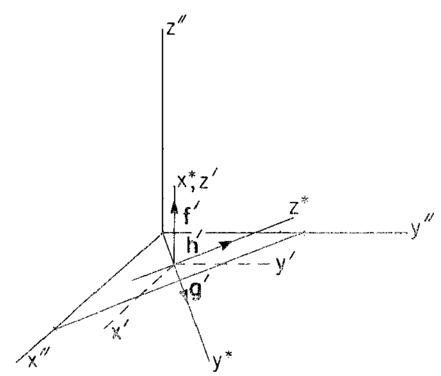


Figure 11.2.5 Principal axes for triangular plate.

Substituting these values of ζ into Eq. 11.2.15 yields the eigenvectors

$$\mathbf{f}' = [0, 0, 1]^T$$

$$\mathbf{g}' = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T$$

$$\mathbf{h}' = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T$$

Principal axes are shown in Fig. 11.2.5, in which the inertia matrix is

$$\mathbf{J}^* = \begin{bmatrix} \frac{\gamma}{18} & 0 & 0 \\ 0 & \frac{\gamma}{24} & 0 \\ 0 & 0 & \frac{\gamma}{72} \end{bmatrix}$$

Inertia Properties of Complex Bodies Components of machines are often made up of combinations of subcomponents that have standard shapes (e.g., circles, disks, cylinders, spheres, and rectangular solids). A typical example of such a component is shown in Fig. 11.2.6, in which each subcomponent and void are of some standard shape, typical of those resulting from standard manufacturing processes. The objective of this subsection is to develop expressions for the inertia properties of the composite body using easily calculated properties of individual subcomponents.

Let an x''-y''-z'' frame be fixed in the composite body in a convenient location and orientation (e.g., as shown in Fig. 11.2.6). Using this frame, the centroid of the composite body may be obtained using the definition of Eq. 11.2.1, employing the property that an integral over the entire mass may be

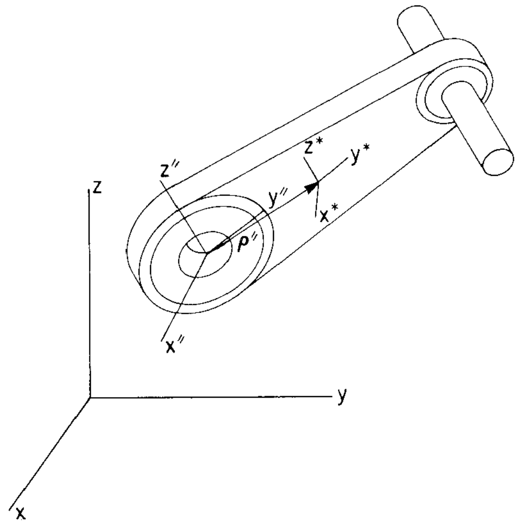


Figure 11.2.6 Composite body made up of subcomponents.

written as the sum of integrals over subcomponents m_i, to obtain

$$\begin{aligned}\boldsymbol{\rho}'' &= \frac{1}{m}\sum_{i=1}^{k}\int_{m_i}\mathbf{s}''^P\,dm(P)\\ &= \frac{1}{m}\sum_{i=1}^{k} m_i\boldsymbol{\rho}_i'' \end{aligned} \tag{11.2.18}$$

where $m = \sum_{i=1}^{k} m_i$. To use this result, the centroid $\boldsymbol{\rho}_i''$ is first located in the x''-y''-z'' frame, using mass and centroid location information from Table 11.2.1 or from direct numerical calculation. Equation 11.2.18 is then used to locate the centroid of the composite body in the x''-y''-z'' frame.

The inertia matrix $\mathbf{J}^*$ with respect to the x^*-y^*-z^* centroidal reference frame of the composite body (e.g., as shown in Fig. 11.2.6) is now to be calculated. The notation x^*-y^*-z^* is selected here to avoid confusion with the centroidal x_i'-y_i'-y_i' frame of subcomponent i. The defining equation of Eq. 11.1.17 may be written and its integral evaluated as a sum of integrals over subcomponents that make up the composite body to obtain

$$\begin{aligned}\mathbf{J}^* &= -\int_m \tilde{\mathbf{s}}^{*P}\tilde{\mathbf{s}}^{*P}\,dm(P)\\ &= \sum_{i=1}^{k} -\int_{m_i}\tilde{\mathbf{s}}^{*P}\tilde{\mathbf{s}}^{*P}\,dm(P)\\ &= \sum_{i=1}^{k}\mathbf{J}_i^* \end{aligned} \tag{11.2.19}$$

To use Eq. 11.2.19, the inertia matrix $\mathbf{J}_i^*$ of subcomponent i in the x^*-y^*-z^* frame must be calculated. To do this, a convenient x_i'-y_i'-z_i' centroidal frame is defined for subcomponent i in the composite body. Denote by $\mathbf{J}_i'$ the inertia matrix of subcomponent i, with respect to its selected centroidal x'-y'-z' frame and by $\boldsymbol{\rho}_i'$ the vector that locates the centroid of the subcomponent in the x^*-y^*-z^* frame. Since the x^*-y^*-z^* frame is not centroidal for the individual

TABLE 11.2.1 Inertia Properties of Some Homogeneous Bodies

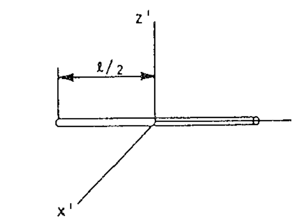
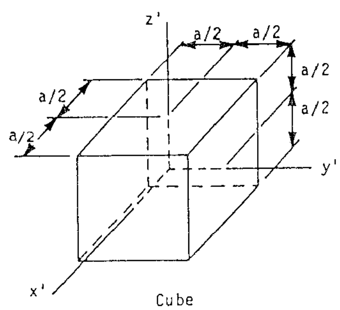
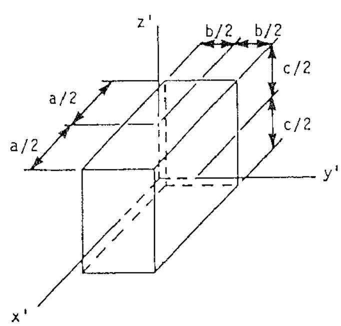
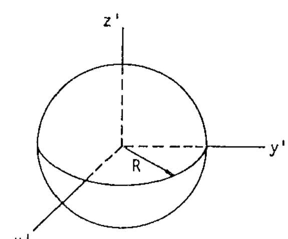
Body	Mass and Moment of Inertia (γ = mass density) (A = cross-sectional area)
 Thin Rod	$m = \gamma \ell A$ $J_{x'x'} = J_{z'z'} = \frac{m}{12}\ell^2$ $J_{y'y'} = 0$
 Cube	$m = \gamma a^3$ $J_{x'x'} = J_{y'y'} = J_{z'z'} = \frac{1}{6}ma^2$
 Rectangular Prism	$m = \gamma abc$ $J_{x'x'} = \frac{1}{12}m(b^2 + c^2)$ $J_{y'y'} = \frac{1}{12}m(a^2 + c^2)$ $J_{z'z} = \frac{1}{12}m(a^2 + b^2)$

TABLE 11.2.1 Continued

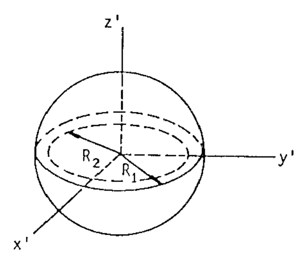
Body	Mass and Moment of Inertia (γ = mass density) (A = cross-sectional area)



Sphere

$$m = \tfrac{4}{3}\pi\gamma R^2$$

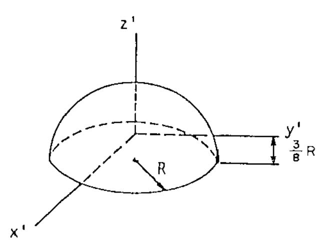
$$J_{x'x'} = J_{y'y'} = J_{z'z'} = \tfrac{2}{5}mR^2$$



Hollow Sphere

$$m = \tfrac{4}{3}\pi\gamma(R_1^2 - R_2^2)$$

$$J_{x'x'} = J_{y'y'} = J_{z'z'} = \tfrac{2}{5}\, m \frac{R_1^5 - R_2^5}{R_1^3 R_2^3}$$

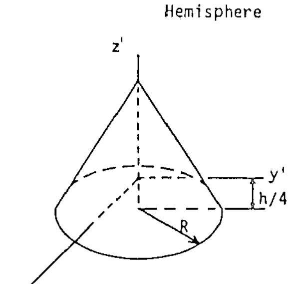


Hemisphere

$$m = \tfrac{2}{3}\pi\gamma R^3$$

$$J_{x'x'} = J_{y'y'} = \tfrac{83}{320}mR^2$$

$$J_{z'z'} = \tfrac{2}{5}mR^2$$



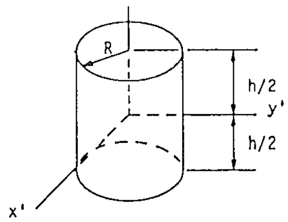
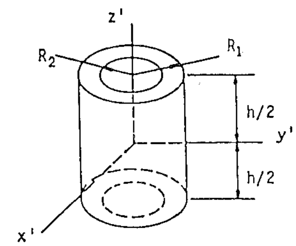
Right Circular Cone

$$m = \tfrac{1}{3}\pi\gamma R^2 h$$

$$J_{x'x'} = J_{y'y'} = \tfrac{3}{80}m(4R^2 + h^2)$$

$$J_{z'z'} = \tfrac{3}{10}mR^2$$

TABLE 11.2.1 Continued

Body	Mass and Moment of Inertia (γ = mass density) (A = cross-sectional area)
 Right Circular Cylinder	$m = \pi\gamma R^2 h$ $J_{x'x'} = J_{y'y'} = \frac{1}{12}m(3R^2 + h^2)$ $J_{z'z'} = \frac{1}{2}mR^2$
 Hollow Right Circular Cylinder	$m = \pi\gamma h(R_1^2 - R_1^2)$ $J_{x'x'} = J_{y'y'} = \frac{1}{2}m(3R_1^2 + 3R_2^2 + h^2)$ $J_{z'z'} = \frac{1}{2}m(R_1^2 + R_2^2)$

subcomponents, the inertia transformation of Eq. 11.2.7 must be used to calculate the inertia matrix of subcomponent i with respect to the composite body centroidal x^*-y^*-z^* frame. Denoting by $\mathbf{C}_i'$ the transformation matrix from the x^*-y^*-z^* frame to the x_i'-y_i'-z_i' frame, Eq. 11.2.7 yields

$$\mathbf{J}_i^* = \mathbf{C}_i'^T\mathbf{J}_i'\mathbf{C}_i' + m_i(\boldsymbol{\rho}_i'^T\boldsymbol{\rho}_i'\mathbf{I} - \boldsymbol{\rho}_i'\boldsymbol{\rho}_i'^T) \qquad \textbf{(11.2.20)}$$

This result may be substituted into Eq. 11.2.19 to obtain the desired inertia matrix for the composite body. The inertia properties of a few homogeneous bodies for use in such calculations are provided in Table 11.2.1.

If there are voids in a composite body, Eq. 11.2.19 may be used by assigning $-\mathbf{J}_i^*$ as the inertia matrix of the void, where $\mathbf{J}_i^*$ is the inertia matrix of a body that would occupy the void, with the same material density as the body from which it is removed.

Example 11.2.4: The body of Fig. 11.2.7 is made up of a cylinder with hemispherical ends, all made of the same homogeneous material of density γ. Since all three coordinate planes are planes of symmetry, the centroid is at the origin of the x-y-z frame and all products of inertia with respect to this frame are

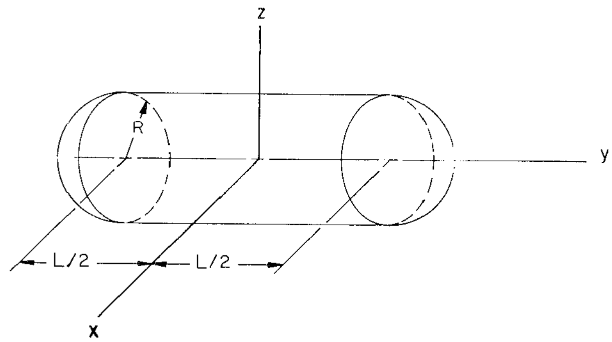


Figure 11.2.7 Cylinder with spherical ends.

zero. Using Eqs. 11.2.19 and 11.2.20, with $\mathbf{C}_i' = \mathbf{I}$, and Table 11.2.1, the moments of inertia of the composite body are

$$J_{xx} = J_{zz} = \frac{\pi\gamma R^2 L}{12}(3R^2 + L^2) + 2\left[\frac{83\pi\gamma R^5}{480} + \left(\frac{2\pi\gamma R^3}{3}\right)\left(\frac{4L + 3R}{8}\right)^2\right]$$

$$J_{yy} = \frac{\pi\gamma R^4 L}{2} + 2\left[\frac{4\pi\gamma R^5}{15}\right]$$

Example 11.2.5: The body of Fig. 11.2.8 is formed by a rectangular bar with a hole of radius $R = c/2$ about the x axis. Since all three coordinate planes are planes of symmetry, the centroid is at the origin of the x-y-z frame and the products of inertia with respect to this frame are zero.

From Eq. 11.2.20, with $\mathbf{J}_1$ as the inertia matrix of the bar without a hole

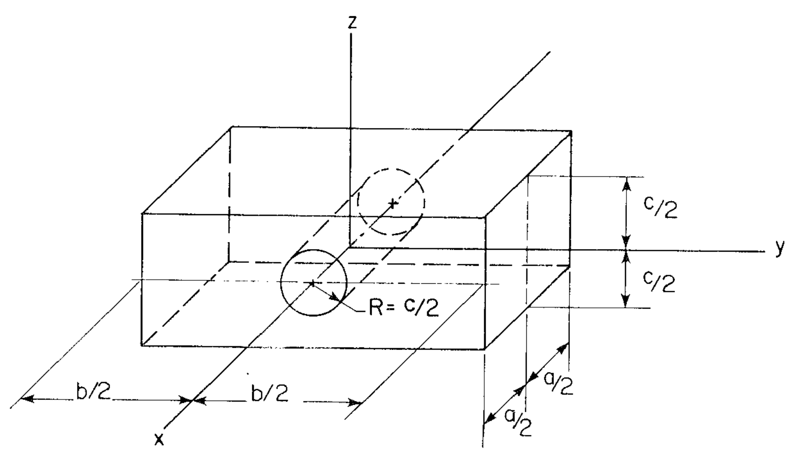


Figure 11.2.8 Rectangular solid with hole.

and $\mathbf{J}_2$ as the inertia matrix of a cylinder that would occupy the hole,

$$\mathbf{J} = \mathbf{J}_1 - \mathbf{J}_2$$

Using Table 11.2.1,

$$J_{xx} = \frac{\gamma abc}{12}(b^2 + c^2) - \frac{\pi\gamma c^4 a}{32}$$

$$J_{yy} = \frac{\gamma abc}{12}(a^2 + c^2) - \frac{\pi\gamma c^2 a}{48}\left(\frac{3c^2}{4} + a^2\right)$$

$$J_{zz} = \frac{\gamma abc}{12}(a^2 + b^2) - \frac{\pi\gamma c^2 a}{48}\left(\frac{3c^2}{4} + a^2\right)$$

The relatively elementary examples studied in this section show that the transformation relations for inertial properties and the theory of calculating centroids and inertia properties of complex bodies can be used systematically. The reader may, however, despair over the extent of such computations for complex composite bodies, such as automobile bodies, aircraft landing gear assemblies, vending machine components, and robot arms. Indeed, manual application of the methods used in this section would lead to extensive algebraic manipulations that are extremely time consuming and prone to error when carried out analytically by the engineer.

The availability of large-scale computer codes for system dynamic simulation creates new opportunities for large-scale applications, but requires detailed inertia data that have not normally been calculated in engineering practice. This dilemma may be alleviated by emerging computer-aided design and computer-aided engineering systems that contain extensive geometric modeling software that describes the geometry of complex mechanical assemblies. The methods developed and illustrated in this section form the foundation for computer implementation with geometric modelers to determine the locations of centroids and moments and products of inertia with respect to the reference frames specified by the engineer. The technical challenge in implementing such methods is computer generation of the geometric definition of complex bodies, which is the domain of solid modeling methods that are emerging in the form of practical computational tools in modern computer-aided design and computer-aided engineering systems.

Once geometric and material property information is defined and entered into a computer data base, implementation of the analytical methods presented in this section for calculating the locations of centroids and moments and products of inertia is a relatively simple matter. The viewpoint taken for the remainder of this text is that such modern computer-aided engineering tools are available to the engineer for the computation of the inertial properties of machine components, or that the engineer is willing to manually carry out the tedious computations illustrated in this section to obtain the inertial properties of bodies in mechanical systems.

11.3 EQUATIONS OF MOTION OF CONSTRAINED SPATIAL SYSTEMS

Consider now nb bodies that form a constrained multibody system. The composite set of generalized coordinates for the system is

$$\begin{aligned} \mathbf{r} &= [\mathbf{r}_1^T, \mathbf{r}_2^T, \ldots, \mathbf{r}_{nb}^T]^T \\ \mathbf{p} &= [\mathbf{p}_1^T, \mathbf{p}_2^T, \ldots, \mathbf{p}_{nb}^T]^T \end{aligned} \tag{11.3.1}$$

The kinematic and driving constraints that act on the system are derived in Chapter 9 in the form

$$\mathbf{\Phi}(\mathbf{r}, \mathbf{p}, t) = \mathbf{0} \tag{11.3.2}$$

In addition, the Euler parameter normalization constraints

$$\mathbf{\Phi}^{\mathbf{p}} \equiv \begin{bmatrix} \mathbf{p}_1^T\mathbf{p}_1 - 1 \\ \vdots \\ \mathbf{p}_{nb}^T\mathbf{p}_{nb} - 1 \end{bmatrix} = \mathbf{0} \tag{11.3.3}$$

must hold.

Constrained equations of motion are derived in this section using the results of Section 11.1. The equations of motion are first written in terms of angular accelerations and then in terms of second derivatives of Euler parameters.

11.3.1 Newton–Euler Form of Constrained Equations of Motion

To implement the variational Newton–Euler equations, Eq. 11.1.19 is evaluated for each body in the system and the resulting equations are added to obtain a Newton–Euler variational equation of motion for the system. To simplify notation, define

$$\begin{aligned} \delta\mathbf{r} &= [\delta\mathbf{r}_1^T, \delta\mathbf{r}_2^T, \ldots, \delta\mathbf{r}_{nb}^T]^T \\ \mathbf{M} &\equiv \mathrm{diag}(m_1\mathbf{I}_3, m_2\mathbf{I}_3, \ldots, m_{nb}\mathbf{I}_3) \\ \delta\boldsymbol{\pi}' &= [\delta\boldsymbol{\pi}_1'^T, \delta\boldsymbol{\pi}_2'^T, \ldots, \delta\boldsymbol{\pi}_{nb}'^T]^T \\ \mathbf{F} &\equiv [\mathbf{F}_1^T, \mathbf{F}_2^T, \ldots, \mathbf{F}_{nb}^T]^T \\ \mathbf{J}' &\equiv \mathrm{diag}(\mathbf{J}_1', \mathbf{J}_2', \ldots, \mathbf{J}_{nb}') \\ \boldsymbol{\omega}' &\equiv [\boldsymbol{\omega}_1'^T, \boldsymbol{\omega}_2'^T, \ldots, \boldsymbol{\omega}_{nb}'^T]^T \\ \mathbf{n}' &\equiv [\mathbf{n}_1'^T, \mathbf{n}_2'^T, \ldots, \mathbf{n}_{nb}'^T]^T \\ \tilde{\boldsymbol{\omega}}' &\equiv \mathrm{diag}(\tilde{\boldsymbol{\omega}}_1', \tilde{\boldsymbol{\omega}}_2', \ldots, \tilde{\boldsymbol{\omega}}_{nb}') \end{aligned} \tag{11.3.4}$$

Using this notation, the sum of Eqs. 11.1.19 over all bodies in the system may be written as (Prob. 11.3.1)

$$\delta\mathbf{r}^T[\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}'] = 0$$

which must hold for all virtual displacements $\delta\mathbf{r}$ and virtual rotations $\delta\boldsymbol{\pi}'$ that are consistent with constraints. Forces and torques that act on the system may be partitioned into *applied forces and torques* $\mathbf{F}^A$ and $\mathbf{n}'^A$ and *constraint forces and torques* $\mathbf{F}^C$ and $\mathbf{n}'^C$, respectively. For all constraints treated in this text, forces of constraint do no work as long as virtual displacements and rotations are consistent with constraints; that is,

$$\delta\mathbf{r}^T\mathbf{F}^C + \delta\boldsymbol{\pi}'^T\mathbf{n}'^C = 0$$

Thus, with $\mathbf{F} = \mathbf{F}^A + \mathbf{F}^C$, $\mathbf{n}' = \mathbf{n}'^A + \mathbf{n}'^C$, and this relation, the *variational equation of motion for a constrained system* is

$$\delta\mathbf{r}^T[\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}^A] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}'^A] = 0 \tag{11.3.5}$$

which must hold for all kinematically admissible virtual displacements and rotations.

Virtual displacements $\delta\mathbf{r}$ and virtual rotations $\delta\boldsymbol{\pi}'$ are kinematically admissible for constraints of Eq. 11.3.2 if

$$\boldsymbol{\Phi}_{\mathbf{r}}\,\delta\mathbf{r} + \boldsymbol{\Phi}_{\pi'}\,\delta\boldsymbol{\pi}' = \mathbf{0} \tag{11.3.6}$$

where $\boldsymbol{\Phi}_{\mathbf{r}}$ and $\boldsymbol{\Phi}_{\pi'}$ are assembled using the results of Chapter 9. Euler parameter normalization constraints are excluded, since they are automatically satisfied by $\delta\boldsymbol{\pi}'$. Since Eq. 11.3.5 must hold for all $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$ that satisfy Eq. 11.3.6, by the Lagrange multiplier theorem of Section 6.3, there exists a Lagrange multiplier vector $\boldsymbol{\lambda}$ such that

$$\delta\mathbf{r}^T[\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}^A + \boldsymbol{\Phi}_{\mathbf{r}}^T\boldsymbol{\lambda}] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}'^A + \boldsymbol{\Phi}_{\pi'}^T\boldsymbol{\lambda}] = 0 \tag{11.3.7}$$

for arbitrary $\delta\mathbf{r}$ and $\delta\boldsymbol{\pi}'$. The coefficients of these arbitrary variations must be zero, yielding the *constrained Newton–Euler equations* of motion

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \boldsymbol{\Phi}_{\mathbf{r}}^T\boldsymbol{\lambda} &= \mathbf{F}^A \\ \mathbf{J}'\dot{\boldsymbol{\omega}}' + \boldsymbol{\Phi}_{\pi'}^T\boldsymbol{\lambda} &= \mathbf{n}'^A - \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{aligned} \tag{11.3.8}$$

To complete the equations of motion, acceleration equations associated with the kinematic constraints of Eq. 11.3.2 must be obtained. The *velocity equation* obtained in Section 9.6.2, by taking the time derivative of Eq. 11.3.2, is

$$\boldsymbol{\Phi}_{\mathbf{r}}\dot{\mathbf{r}} + \boldsymbol{\Phi}_{\pi'}\boldsymbol{\omega}' = -\boldsymbol{\Phi}_t \equiv \mathbf{v} \tag{11.3.9}$$

The time derivative of this equation, calculated in Section 9.6.3, yields the *acceleration equation*

$$\boldsymbol{\Phi}_{\mathbf{r}}\ddot{\mathbf{r}} + \boldsymbol{\Phi}_{\pi'}\dot{\boldsymbol{\omega}}' = \boldsymbol{\gamma} \tag{11.3.10}$$

where the vector $\boldsymbol{\gamma}$ is defined in Chapter 9 for each constraint equation.

Combining Eqs. 11.3.8 and 11.3.10, the *system acceleration equations* are

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{\Phi}_{\mathbf{r}}^T \\ \mathbf{0} & \mathbf{J}' & \mathbf{\Phi}_{\pi'}^T \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_{\pi'} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^A \\ \mathbf{n}'^A - \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \\ \boldsymbol{\gamma} \end{bmatrix} \tag{11.3.11}$$

These equations of motion, taken with the kinematic constraint equations of Eq. 11.3.2 and the velocity equations of Eq. 11.3.9, yield a mixed system of *differential–algebraic equations of motion* for the system.

Technically, Eq. 11.3.11 is a system of mixed first-order differential–algebraic equations for velocity variables $\dot{\mathbf{r}}$ and $\boldsymbol{\omega}'$ and the algebraic variables $\boldsymbol{\lambda}$. It is not a second-order differential–algebraic system, since angular velocity $\boldsymbol{\omega}'$ is not integrable. Equation 11.3.11 must be augmented by first-order kinematic velocity equations in terms of Euler parameter time derivatives and the kinematic constraint equations of Eq. 11.3.2. The details of formulating such a reduced first-order system of differential–algebraic equations of motion are discussed in more detail in Section 11.6.

Just as shown in Section 6.3 for planar systems, if $\delta\dot{\mathbf{r}}^T\mathbf{M}\,\delta\dot{\mathbf{r}} + \delta\boldsymbol{\omega}'^T\mathbf{J}'\,\delta\boldsymbol{\omega}' > 0$ for all nonzero $\delta\dot{\mathbf{r}}$ and $\delta\boldsymbol{\omega}'$ such that $\mathbf{\Phi}_{\mathbf{r}}\,\delta\dot{\mathbf{r}} + \mathbf{\Phi}_{\pi'}\,\delta\boldsymbol{\omega}' = \mathbf{0}$, and if $[\mathbf{\Phi}_{\mathbf{r}}, \mathbf{\Phi}_{\pi'}]$ has full row rank, then the coefficient matrix in Eq. 11.3.11 is nonsingular (Prob. 11.3.2).

Initial conditions on position and orientation and on velocity must be provided to define the dynamics of a system. Since the orientation of a body is specified by Euler parameters, initial conditions on the position and orientation must be specified in the form

$$\mathbf{\Phi}^I(\mathbf{r}, \mathbf{p}, t_0) = \mathbf{0} \tag{11.3.12}$$

where $\mathbf{r}$ and $\mathbf{p}$ must satisfy Eq. 11.3.2 at t_0 and the Euler parameter normalization constraints of Eq. 11.3.3. Equations 11.3.2, 11.3.3, and 11.3.12 should uniquely determine $\mathbf{r}(t_0)$ and $\mathbf{p}(t_0)$.

If angular velocity is used in formulating the equations of motion, the velocity initial conditions should be given in the form

$$\mathbf{B}_{\mathbf{r}}^I\dot{\mathbf{r}} + \mathbf{B}_{\omega'}^I\boldsymbol{\omega}' = \mathbf{v}^I \tag{11.3.13}$$

at $t = t_0$, where $\mathbf{B}_{\mathbf{r}}^I$, $\mathbf{B}_{\omega'}^I$, and $\mathbf{v}^I$ may depend on $\mathbf{r}$ and $\mathbf{p}$ at t_0. Furthermore, Eqs. 11.3.9 and 11.3.13 should uniquely determine $\dot{\mathbf{r}}(t_0)$ and $\boldsymbol{\omega}'(t_0)$.

11.3.2 Euler Parameter Form of Constrained Equations of Motion

The equations of motion can also be written in terms of the Euler parameters, with Lagrange multipliers to account for both the kinematic and Euler parameter normalization constraints. To write the velocity equation of Eq. 11.3.9 in terms of the Euler parameter derivatives, Eq. 9.3.34 may be used to obtain

$$\mathbf{\Phi}_{\mathbf{r}}\dot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{p}}\dot{\mathbf{p}} = -\mathbf{\Phi}_t \equiv \boldsymbol{\nu} \tag{11.3.14}$$

where

$$\mathbf{\Phi}_{\mathbf{p}_i} = 2\mathbf{\Phi}_{\boldsymbol{\pi}_i'}\mathbf{G}_i \tag{11.3.15}$$

To write the constraint acceleration equation of Eq. 11.3.10 in terms of Euler parameters, a relation between $\dot{\boldsymbol{\omega}}'$ and $\ddot{\mathbf{p}}$ is needed. Differentiating Eq. 9.3.34,

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{G}\ddot{\mathbf{p}} + 2\dot{\mathbf{G}}\dot{\mathbf{p}}$$

From Eq. 9.3.19,

$$\begin{aligned}\dot{\mathbf{G}}\dot{\mathbf{p}} &= [-\dot{\mathbf{e}}, \dot{\tilde{\mathbf{e}}} + \dot{e}_0\mathbf{I}]\begin{bmatrix}\dot{e}_0 \\ \dot{\mathbf{e}}\end{bmatrix} \\ &= -\dot{\mathbf{e}}\dot{e}_0 - \dot{\tilde{\mathbf{e}}}\dot{\mathbf{e}} + \dot{e}_0\dot{\mathbf{e}} \\ &= \mathbf{0}\end{aligned} \tag{11.3.16}$$

Thus,

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{G}\ddot{\mathbf{p}} \tag{11.3.17}$$

To obtain the inverse relation to Eq. 11.3.17, differentiate Eq. 9.3.35 to obtain

$$\ddot{\mathbf{p}} = \tfrac{1}{2}(\dot{\mathbf{G}}^T\boldsymbol{\omega}' + \mathbf{G}^T\dot{\boldsymbol{\omega}}') \tag{11.3.18}$$

Using Eq. 9.3.34,

$$\dot{\mathbf{G}}^T\boldsymbol{\omega}' = 2\dot{\mathbf{G}}^T\mathbf{G}\dot{\mathbf{p}} \tag{11.3.19}$$

Differentiating both sides of Eq. 9.3.24,

$$\dot{\mathbf{G}}^T\mathbf{G} = -[\mathbf{G}^T\dot{\mathbf{G}} + \dot{\mathbf{p}}\mathbf{p}^T + \mathbf{p}\dot{\mathbf{p}}^T]$$

Substituting this result into Eq. 11.3.19 and using Eqs. 9.3.27 and 11.3.16,

$$\dot{\mathbf{G}}^T\boldsymbol{\omega}' = -2\dot{\mathbf{p}}^T\dot{\mathbf{p}}\mathbf{p} \tag{11.3.20}$$

From Eqs. 9.3.35 and 9.3.22,

$$\dot{\mathbf{p}}^T\dot{\mathbf{p}} = \tfrac{1}{4}\boldsymbol{\omega}'^T\mathbf{G}\mathbf{G}^T\boldsymbol{\omega}' = \tfrac{1}{4}\boldsymbol{\omega}'^T\boldsymbol{\omega}'$$

Substituting this result and Eq. 11.3.20 into Eq. 11.3.18 yields the desired result:

$$\ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\dot{\boldsymbol{\omega}}' - \tfrac{1}{4}\boldsymbol{\omega}'^T\boldsymbol{\omega}'\mathbf{p} \tag{11.3.21}$$

Note the quadratic term in $\boldsymbol{\omega}'$ in this relation, which is more complicated than Eq. 9.3.35 for $\dot{\mathbf{p}}$ in terms of $\boldsymbol{\omega}'$.

Substituting Eq. 11.3.17 into Eq. 11.3.10 and using Eq. 11.3.15, the kinematic acceleration equations are obtained, in terms of Euler parameters, as

$$\mathbf{\Phi}_{\mathbf{r}}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{p}}\ddot{\mathbf{p}} = \boldsymbol{\gamma} \tag{11.3.22}$$

where $\boldsymbol{\gamma}$ is defined in Section 9.6.3.

Differentiating the Euler parameter constraint equation of Eq. 11.3.3 yields the Euler parameter velocity equation

$$\mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}}\dot{\mathbf{p}} = \mathbf{0} \tag{11.3.23}$$

where

$$\mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}} = 2\begin{bmatrix} \mathbf{p}_1^T & \mathbf{0} & \mathbf{0} \dots \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2^T & \mathbf{0} \dots \mathbf{0} \\ \vdots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \dots \mathbf{p}_{nb}^T \end{bmatrix}$$

Differentiating Eq. 11.3.23 yields the Euler parameter acceleration equation

$$\mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}}\ddot{\mathbf{p}} = -2\begin{bmatrix} \dot{\mathbf{p}}_1^T\dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{p}}_{nb}^T\dot{\mathbf{p}}_{nb} \end{bmatrix} \equiv \boldsymbol{\gamma}^{\mathbf{p}} \tag{11.3.24}$$

Similarly, the total differential of Eq. 11.3.2 may be calculated to obtain

$$\mathbf{\Phi}_{\mathbf{r}}\,\delta\mathbf{r} + \mathbf{\Phi}_{\mathbf{p}}\,\delta\mathbf{p} = \mathbf{0} \tag{11.3.25}$$

which defines kinematically admissible virtual displacements of the system. Likewise, variations in Euler parameters must satisfy

$$\mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}}\,\delta\mathbf{p} = \mathbf{0} \tag{11.3.26}$$

which is obtained by taking the differential of Eq. 11.3.3.

Substituting for $\delta\boldsymbol{\pi}'$ from Eq. 9.3.41, for $\boldsymbol{\omega}'$ from Eq. 9.3.34, for $\dot{\boldsymbol{\omega}}'$ from Eq. 11.3.17, and using Eqs. 9.3.24, 9.3.29, 9.3.32, Eq. 11.3.5 may be rewritten as

$$\delta\mathbf{r}^T[\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}^A] + \delta\mathbf{p}^T[4\mathbf{G}^T\mathbf{J}'\mathbf{G}\ddot{\mathbf{p}} - 8\dot{\mathbf{G}}^T\mathbf{J}'\dot{\mathbf{G}}\mathbf{p} - 2\mathbf{G}^T\mathbf{n}'^A] = 0 \tag{11.3.27}$$

where

$$\mathbf{G} = \mathrm{diag}(\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_{nb})$$

Equation 11.3.27 must hold for all $\delta\mathbf{p}$ and $\delta\mathbf{r}$ that satisfy Eqs. 11.3.25 and 11.3.26.

Applying the Lagrange multiplier theorem to append the virtual displacement and Euler parameter variation constraints of Eqs. 11.3.25 and 11.3.26, there exist associated Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^{\mathbf{p}}$ such that

$$\delta\mathbf{r}^T[\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}^A + \mathbf{\Phi}_{\mathbf{r}}^T\boldsymbol{\lambda}] + \delta\mathbf{p}^T[4\mathbf{G}^T\mathbf{J}'\mathbf{G}\ddot{\mathbf{p}} - 8\dot{\mathbf{G}}^T\mathbf{J}'\dot{\mathbf{G}}\mathbf{p} - 2\mathbf{G}^T\mathbf{n}'^A + \mathbf{\Phi}_{\mathbf{p}}^T\boldsymbol{\lambda} + \mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}T}\boldsymbol{\lambda}^{\mathbf{p}}] = 0 \tag{11.3.28}$$

which must hold for arbitrary $\delta\mathbf{r}$ and $\delta\mathbf{p}$. Since coefficients of these arbitrary variations must be zero, and appending the acceleration equations of Eqs. 11.3.22 and 11.3.24, the *Euler parameter system acceleration equation* is obtained as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{\Phi}_{\mathbf{r}}^T & \mathbf{0} \\ \mathbf{0} & 4\mathbf{G}^T\mathbf{J}'\mathbf{G} & \mathbf{\Phi}_{\mathbf{p}}^T & \mathbf{\Phi}_{\mathbf{p}}^{PT} \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_{\mathbf{p}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{P}} \\ \boldsymbol{\lambda} \\ \boldsymbol{\lambda}^{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^A \\ 2\mathbf{G}^T\mathbf{n}'^A + 8\dot{\mathbf{G}}^T\mathbf{J}'\dot{\mathbf{G}}\mathbf{p} \\ \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathbf{p}} \end{bmatrix} \tag{11.3.29}$$

This system of equations, taken with the kinematic and Euler parameter normalization constraints of Eqs. 11.3.2 and 11.3.3 and the associated velocity equations of Eqs. 11.3.14 and 11.3.23, comprise the mixed system of *differential–algebraic equations of motion* of the system in terms of Euler parameters. A direct extension of the argument outlined in Prob. 11.3.2 may be used to show that the coefficient matrix in Eq. 11.3.29 is nonsingular (Prob. 11.3.3).

Initial conditions must be given on $\mathbf{r}$ and $\mathbf{p}$ at t_0, as in Eq. 11.3.12. For velocities, however, initial conditions should be given on $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$. Since initial conditions are most naturally given on angular velocities, rather than on $\dot{\mathbf{p}}$, it is desirable to retain initial conditions in the form of Eq. 11.3.13. This can be done by substituting for $\boldsymbol{\omega}'$ from Eq. 9.3.34 into Eq. 11.3.13 to obtain

$$\mathbf{B}_{\mathbf{r}}^{I}\dot{\mathbf{r}} + 2\mathbf{B}_{\omega'}\mathbf{G}\dot{\mathbf{p}} = \mathbf{v}^{I} \tag{11.3.30}$$

which, with Eqs. 11.3.14 and 11.3.23, determine $\dot{\mathbf{r}}(t_0)$ and $\dot{\mathbf{p}}(t_0)$.

Given a complete set of initial conditions on $\mathbf{r}$, $\mathbf{p}$, $\dot{\mathbf{r}}$, and $\dot{\mathbf{p}}$ that satisfy Eqs. 11.3.2, 11.3.3, 11.3.14, and 11.3.23, at t_0, Eq. 11.3.29, and Eqs. 11.3.2, 11.3.3, 11.3.14, and 11.3.23 for $t > t_0$ constitute a mixed system of second-order differential–algebraic equations of motion for a constrained mechanical system. This contrasts with the first-order character of Eq. 11.3.11, as noted earlier. The expanded system of mixed differential–algebraic equations in $\mathbf{r}$ and $\mathbf{p}$ may thus be integrated, using any of the techniques developed in Chapter 7 for solving differential–algebraic equations of motion for planar systems. Only increased dimensionality and the detailed form of the equations of motion change.

A note of some importance regarding the difference in forms of Eqs. 11.3.11 and 11.3.29 is in order. The upper-left terms in the coefficient matrix on the left of Eq. 11.3.11 are constant matrices $\mathbf{M}$ and $\mathbf{J}'$, whereas the generalized coordinate dependent matrix $\mathbf{G}^T\mathbf{J}'\mathbf{G}$ appears on the left of Eq. 11.3.29. Even though both coefficient matrices in Eqs. 11.3.11 and 11.3.29 are nonsingular for meaningful physical systems, the appearance of generalized coordinate dependent terms in the matrix of Eq. 11.3.29 may influence the performance of matrix factorization algorithms and numerical integration methods. These considerations are discussed in more detail in Section 11.6.

11.4 INTERNAL FORCES

As in Chapter 6 for planar systems, the internal forces associated with springs, dampers, and actuators can be accounted for using the principle of virtual work. The virtual work approach is even more attractive for spatial systems than for planar systems, since relationships for virtual rotations are available.

11.4.1 Translational Spring–Damper–Actuator

Consider first the *translational spring–damper–actuator* (or TSDA) shown in Fig. 11.4.1, which connects points P_i and P_j on bodies i and j, respectively. The vector

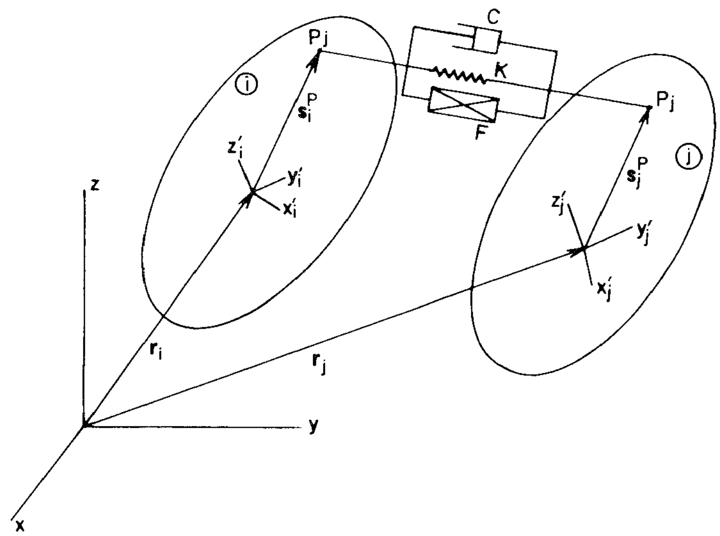


Figure 11.4.1 Translational spring–damper–actuator.

from P_i to P_j is

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P \tag{11.4.1}$$

Thus, the length ℓ of the spring–damper–actuator is given by

$$\ell^2 = \mathbf{d}_{ij}^T\mathbf{d}_{ij} \tag{11.4.2}$$

Differentiating with respect to time yields

$$2\ell\dot{\ell} = 2\mathbf{d}_{ij}^T\dot{\mathbf{d}}_{ij} \tag{11.4.3}$$

which may be solved for the time rate of change of length as

$$\begin{aligned}\dot{\ell} &= \left(\frac{\mathbf{d}_{ij}}{\ell}\right)^T(\dot{\mathbf{r}}_j + \dot{\mathbf{A}}_j\mathbf{s}_j'^P - \dot{\mathbf{r}}_i - \dot{\mathbf{A}}_i\mathbf{s}_i'^P) \\ &= \left(\frac{\mathbf{d}_{ij}}{\ell}\right)^T(\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\boldsymbol{\omega}_j' - \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\boldsymbol{\omega}_i')\end{aligned} \tag{11.4.4}$$

where Eq. 9.2.40 has been used to write the time derivatives of transformation matrices in terms of angular velocities.

Note that if ℓ approaches zero an indeterminate fraction occurs in the first term of Eq. 11.4.4. In this case, L'Hospital's rule [25, 26] may be used to obtain

$$\lim_{\ell \to 0} \frac{\mathbf{d}_{ij}}{\ell} = \dot{\mathbf{d}}_{ij}/\dot{\ell}|_{\ell=0} \tag{11.4.5}$$

which is used only when the length of the spring–damper–actuator approaches zero.

The magnitude of the force that acts in the spring–damper–actuator, with tension taken as positive, is

$$f = k(\ell - \ell_0) + c\dot{\ell} + F(\ell, \dot{\ell}) \tag{11.4.6}$$

where k is the spring coefficient, c is the damping coefficient, and $F(\ell, \dot{\ell})$ is a general actuator force. The virtual work of this force is simply

$$\delta W = -f\,\delta\ell \tag{11.4.7}$$

where the variation $\delta\ell$ in length is obtained by taking the differential of Eq. 11.4.2 and dividing by ℓ, as in the derivation of Eq. 11.4.4, to obtain

$$\delta\ell = \left(\frac{\mathbf{d}_{ij}}{\ell}\right)^T (\delta\mathbf{r}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\,\delta\boldsymbol{\pi}_i') \tag{11.4.8}$$

where Eq. 9.2.50 has been used to expand the differential of the transformation matrix. Substituting this result into Eq. 11.4.7 yields

$$\delta W = -\frac{f}{\ell}\mathbf{d}_{ij}^T(\delta\mathbf{r}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j'^P\,\delta\boldsymbol{\pi}_j' - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i'^P\,\delta\boldsymbol{\pi}_i') \tag{11.4.9}$$

The coefficients of virtual displacements and virtual rotations in Eq. 11.4.9 yield the angular orientation form of the TSDA generalized force as

$$\begin{aligned}
\mathbf{Q}_i &= f/\ell\begin{bmatrix}\mathbf{d}_{ij}\\ \tilde{\mathbf{s}}_i'^P\mathbf{A}_i^T\mathbf{d}_{ij}\end{bmatrix}\\
\mathbf{Q}_j &= -f/\ell\begin{bmatrix}\mathbf{d}_{ij}\\ \tilde{\mathbf{s}}_j'^P\mathbf{A}_j^T\mathbf{d}_{ij}\end{bmatrix}
\end{aligned} \tag{11.4.10}$$

In the case where the generalized forces associated with Euler parameter orientation coordinates are desired, the virtual rotations in Eq. 11.4.9 may be written in terms of the variations in Euler parameters, using Eq. 9.3.41. Making this substitution and identifying the coefficients of virtual displacements and variations in Euler parameters, the Euler parameter form of the TSDA generalized forces is obtained as

$$\begin{aligned}
\mathbf{Q}_i &= f/\ell\begin{bmatrix}\mathbf{d}_{ij}\\ 2\mathbf{G}_i^T\tilde{\mathbf{s}}_i'^P\mathbf{A}_i^T\mathbf{d}_{ij}\end{bmatrix}\\
\mathbf{Q}_j &= -f/\ell\begin{bmatrix}\mathbf{d}_{ij}\\ 2\mathbf{G}_j^T\tilde{\mathbf{s}}_j'^P\mathbf{A}_j^T\mathbf{d}_{ij}\end{bmatrix}
\end{aligned} \tag{11.4.11}$$

The effect of the TSDA force on the equations of motion is accounted for by inserting the angular orientation generalized forces of Eq. 11.4.10 into the right side of Eq. 11.3.11 to obtain the angular acceleration equations of motion.

Similarly, the Euler parameter generalized forces of Eq. 11.4.11 may be substituted into the right side of Eq. 11.3.29 to obtain the Euler parameter acceleration equations.

11.4.2 Rotational Spring–Damper–Actuator

A *rotational spring–damper–actuator* (or RSDA) can act about the common axis of a revolute, cylindrical, or screw joint between bodies i and j, as shown in Fig. 11.4.2. The cumulative relative angle of rotation is $\theta + 2n\pi$, where n is the number of revolutions from the free angle of the spring and θ is measured from the joint definition x_i'' axis in body i to the joint definition x_j'' axis in body j. The angle θ is determined by Eq. 9.2.31. Its differential $\delta\theta$ and time derivative $\dot{\theta}$ are determined by Eqs. 9.2.61 and 9.2.62.

The magnitude of the torque exerted by the RSDA, with a counterclockwise torque exerted on the x_j''-y_j''-z_j'' frame relative to the x_i''-y_i''-z_i'' frame as positive, is

$$n = k_\theta(\theta + 2n_{\text{rev}}\pi) + c_\theta\dot{\theta} + N(\theta + 2n_{\text{rev}}\pi,\ \dot{\theta}) \tag{11.4.12}$$

where k_θ is the spring coefficient, spring torque is zero when x_i'' and x_j'' initially coincide and $n_{\text{rev}} = 0$, c_θ is the damping coefficient, and $N(\theta + 2n\pi,\ \dot{\theta})$ is a general actuator torque. Since n is positive counterclockwise and $\delta\theta$ is positive counterclockwise, the virtual work of this torque is

$$\delta W = -n\,\delta\theta \tag{11.4.13}$$

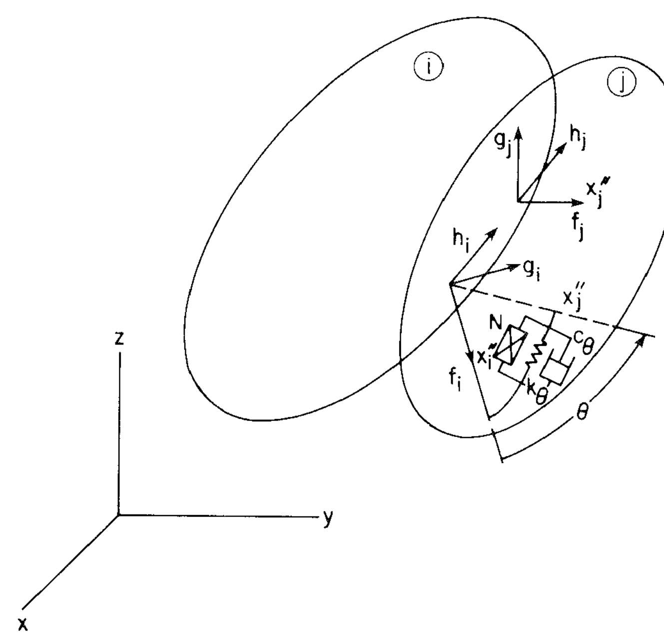


Figure 11.4.2 Rotational spring–damper–actuator.

Equation 9.2.62 may be used to write $\dot{\theta}$ in terms of angular velocities; that is,

$$\dot{\theta} = -\mathbf{h}_i'^T(\mathbf{A}_i^T\mathbf{A}_j\boldsymbol{\omega}_j' - \boldsymbol{\omega}_i') \tag{11.4.14}$$

Similarly, substituting from Eq. 9.2.61 into Eq. 11.4.13,

$$\delta W = -n\mathbf{h}_i'^T(\mathbf{A}_i^T\mathbf{A}_j\,\delta\boldsymbol{\pi}_j' - \delta\boldsymbol{\pi}_i') \tag{11.4.15}$$

Thus, the generalized forces due to this element on bodies i and j are

$$\begin{aligned} \mathbf{Q}_i &= \begin{bmatrix} \mathbf{0} \\ n\mathbf{h}_i' \end{bmatrix} \\ \mathbf{Q}_j &= -\begin{bmatrix} \mathbf{0} \\ n\mathbf{A}_j^T\mathbf{A}_i\mathbf{h}_i' \end{bmatrix} \end{aligned} \tag{11.4.16}$$

If generalized forces associated with Euler parameter generalized coordinates are desired, then $\mathbf{Q}_i$ and $\mathbf{Q}_j$ in Eq. 11.4.15 must be multiplied on the left by $2\mathbf{G}_i$ and $2\mathbf{G}_j$, respectively.

11.5 INVERSE DYNAMICS, EQUILIBRIUM ANALYSIS, AND REACTION FORCES IN JOINTS

The inverse dynamics of kinematically driven spatial systems follow exactly the same approach as for planar systems presented in Section 6.4. The kinematics problem is first solved for position, velocity, and acceleration using the formulation of Chapter 9. The equations of motion are then solved algebraically for Lagrange multipliers associated with the constraints. Finally, using the results presented later in this section, the reaction forces and torques associated with both the kinematic and driving constraints are calculated. While the details of the terms in the equations of constraint and equations of spatial motion differ, the matrix form of these equations is identical to that encountered for planar systems in Section 6.4, so precisely the same analysis sequence and matrix subroutines may be used for spatial inverse dynamic analysis.

Equilibrium analysis for spatial systems follows exactly the same pattern as in the case of planar systems in Section 6.5. While detailed expressions in the equations of equilibrium are different in the case of spatial systems, precisely the same matrix form is encountered and the same basic principles may be employed as in Section 6.5. Either dynamic settling to an equilibrium position, direct solution of the equilibrium equations, or minimum total potential energy for conservative systems may be used to determine equilibrium positions. The numerical considerations in spatial system equilibrium analysis are identical to those in planar systems presented in Section 6.5.

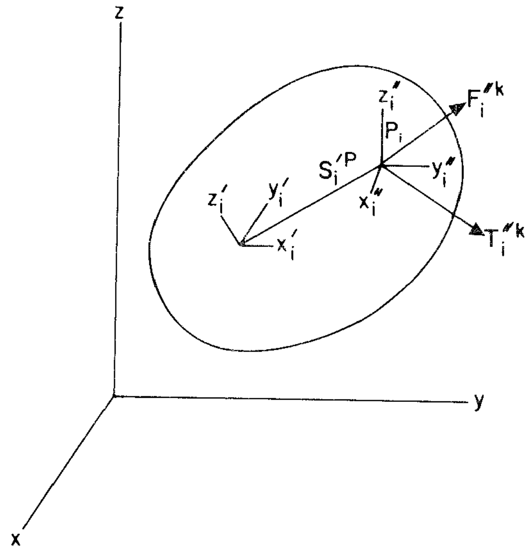


Figure 11.5.1 Reaction force and torque on body i at point P.

Consider a typical joint k, with joint definition point P (see Fig. 11.5.1), constraint equations $\mathbf{\Phi}^k = \mathbf{0}$, and associated Lagrange multipliers $\boldsymbol{\lambda}^k$. In terms of the x_i''-y_i''-z_i'' joint definition frame, joint reaction forces are $\mathbf{F}_i''^k$ and reaction torques are $\mathbf{T}_i''^k$, as shown in Fig. 11.5.1. Equating the virtual work of these reaction forces and torques to the negative of the variational terms in the equations of motion associated with constraint k, as in Eqs. 6.6.1 and 6.6.2,

$$-(\delta \mathbf{r}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^{kT} \boldsymbol{\lambda}^k + \delta \boldsymbol{\pi}_i'^T \mathbf{\Phi}_{\boldsymbol{\pi}_i'}^{kT} \boldsymbol{\lambda}^k) = \delta \mathbf{r}_i''^{PT} \mathbf{F}_i''^k + \delta \boldsymbol{\pi}_i''^T \mathbf{T}_i''^k \tag{11.5.1}$$

which must hold for arbitrary virtual displacements.

To take advantage of Eq. 11.5.1, all virtual displacements must be represented in the joint definition frame. From Eq. 9.2.51,

$$\delta \mathbf{r}_i^P = \delta \mathbf{r}_i - \mathbf{A}_i \tilde{\mathbf{s}}_i'^P \, \delta \boldsymbol{\pi}_i' \tag{11.5.2}$$

In terms of virtual displacement and rotation, represented in joint definition frame coordinates,

$$\begin{aligned} \delta \mathbf{r}_i^P &= \mathbf{A}_i \mathbf{C}_i \, \delta \mathbf{r}_i''^P \\ \delta \boldsymbol{\pi}_i' &= \mathbf{C}_i \, \delta \boldsymbol{\pi}_i'' \end{aligned} \tag{11.5.3}$$

where $\mathbf{C}_i$ is the direction cosine transformation matrix from the x_i''-y_i''-z_i'' joint definition frame to the x_i'-y_i'-z_i' centroidal reference frame. From Eqs. 11.5.2 and 11.5.3,

$$\delta \mathbf{r}_i = \mathbf{A}_i \mathbf{C}_i \, \delta \mathbf{r}_i''^P + \mathbf{A}_i \tilde{\mathbf{s}}_i'^P \mathbf{C}_i \, \delta \boldsymbol{\pi}_i'' \tag{11.5.4}$$

Substituting the virtual rotation and displacement from Eqs. 11.5.3 and 11.5.4 into Eq. 11.5.1,

$$\delta\mathbf{r}_i^{\prime\prime PT}(\mathbf{F}_i^{\prime\prime k} + \mathbf{C}_i^T\mathbf{A}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT}\boldsymbol{\lambda}^k) + \delta\boldsymbol{\pi}_i^{\prime\prime T}(\mathbf{T}_i^{\prime\prime k} + \mathbf{C}_i^T\mathbf{\Phi}_{\boldsymbol{\pi}_i}^{kT}\boldsymbol{\lambda}^k + \mathbf{C}_i^T\tilde{\mathbf{s}}_i^{\prime PT}\mathbf{A}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT}\boldsymbol{\lambda}^k) = 0 \quad \textbf{(11.5.5)}$$

which must hold for arbitrary $\delta\mathbf{r}_i^{\prime\prime P}$ and $\delta\boldsymbol{\pi}_i^{\prime\prime}$. Therefore,

$$\begin{aligned}\mathbf{F}_i^{\prime\prime k} &= -\mathbf{C}_i^T\mathbf{A}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT}\boldsymbol{\lambda}^k \\ \mathbf{T}_i^{\prime\prime k} &= -\mathbf{C}_i^T(\mathbf{\Phi}_{\boldsymbol{\pi}_i}^{kT} - \tilde{\mathbf{s}}_i^{\prime P}\mathbf{A}_i^T\mathbf{\Phi}_{\mathbf{r}_i}^{kT})\boldsymbol{\lambda}^k\end{aligned} \quad \textbf{(11.5.6)}$$

which are the desired expressions for joint reaction forces and torques on body i at joint k in the joint reference frame.

11.6 NUMERICAL CONSIDERATIONS IN SOLVING SPATIAL DIFFERENTIAL–ALGEBRAIC EQUATIONS OF MOTION

If the Euler parameter form of the equations of motion in Eq. 11.3.29 is employed, with the kinematic constraints of Eq. 11.3.2 and Euler parameter normalization constraints of Eq. 11.3.3, then any of the methods presented in Chapter 7 for integrating mixed differential–algebraic equations of motion may be directly applied. This approach has been used successfully and continues to serve as a reliable method for dynamic analysis.

If, on the other hand, the Newton–Euler form of the acceleration equations of Eq. 11.3.11, written in terms of angular accelerations, is employed, some modification of the methods presented in Chapter 7 is required. One alternative is to solve Eq. 11.3.11 for $\ddot{\mathbf{r}}$ and $\dot{\boldsymbol{\omega}}'$ and then to apply Eq. 11.3.21 to calculate $\ddot{\mathbf{p}}$ and integrate for Euler parameter generalized coordinates, just as would have been done if Eq. 11.3.29 had been used to determine $\ddot{\mathbf{p}}$. Even this slight modification in the basic algorithm is attractive, since the coefficient submatrix in the (2, 2) position in Eq. 11.3.11 is constant, whereas the corresponding matrix in Eq. 11.3.29 depends on generalized coordinates and must be reevaluated each time the acceleration equations are evaluated.

Both of the foregoing alternatives work with second time derivatives of generalized coordinates. Both, therefore, permit direct implementation of all four of the numerical integration algorithms of Section 7.3. Since the derivatives of kinematic and Euler parameter constraint equations with respect to all generalized coordinates are readily computed, independent generalized coordinates can be identified and the generalized coordinate partitioning or hybrid algorithm of Section 7.3 can be directly implemented. At the time of this writing, the hybrid algorithm is implemented with Euler parameter second time derivatives in the DADS software system [27] for the dynamic analysis of constrained mechanical systems. This algorithm is both theoretically sound and has been demonstrated to be computationally reliable for broad classes of applications.

Improved methods for the numerical solution of mixed differential–algebraic systems, however, will likely be developed and implemented in the near future.

Another alternative for formulating mixed differential–algebraic equations for constrained mechanical systems is to define an intermediate variable $\mathbf{s} \equiv \dot{\mathbf{r}}$ and to form a first-order system of differential–algebraic equations, using Eqs. 9.3.35 and 11.3.11, as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{\Phi}_{\mathbf{r}}^T \\ \mathbf{0} & \mathbf{J}' & \mathbf{\Phi}_{\pi'}^T \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_{\pi'} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{s}} \\ \dot{\boldsymbol{\omega}}' \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{n}' - \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \\ \boldsymbol{\gamma} \end{bmatrix}$$

$$\dot{\mathbf{r}} = \mathbf{s} \qquad (11.5.7)$$

$$\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T(\mathbf{p})\boldsymbol{\omega}'$$

In addition, the kinematic and Euler parameter normalization constraints of Eqs. 11.3.2 and 11.3.3 and the constraint velocity equations of Eq. 11.3.9 must hold; that is,

$$\mathbf{\Phi}(\mathbf{r}, \mathbf{p}, t) = \mathbf{0}$$

$$\mathbf{\Phi}^{\mathbf{p}}(\mathbf{p}) = \mathbf{0} \qquad (11.5.8)$$

$$\mathbf{\Phi}_{\mathbf{r}}\dot{\mathbf{r}} + \mathbf{\Phi}_{\pi'}\boldsymbol{\omega}' = \boldsymbol{\nu}$$

The first-order system of differential–algebraic equations of Eqs. 11.5.7 and 11.5.8, taken with initial conditions that satisfy Eq. 11.5.8 at the initial time, determines the motion of the constrained mechanical system. It is shown in Reference 48 that this system may be numerically solved using the direct numerical integration method of Section 7.3.

PROBLEMS

Section 11.1

11.1.1. Carry out the manipulations in detail to derive Eq. 11.1.3.

11.1.2. Carry out the manipulations in detail to derive Eq. 11.1.11.

11.1.3. Show that $\boldsymbol{\omega}'$ is not a total differential, because it cannot be integrated. (*Hint*: Use Eq. 9.3.34 to express $\boldsymbol{\omega}'$ in terms of Euler parameter differentials and show that each component of $\boldsymbol{\omega}'$ is not an exact differential.)

11.1.4. Let the pendulum of Example 11.1.2 be permitted to rotate about and slide along the y axis of an inertial reference frame, as shown in Fig. P11.1.4. Write the equations of motion in terms of ϕ and s.

Section 11.2

11.2.1. Calculate the moments of inertia of the body of Example 11.2.1 (Fig. 11.2.2) with respect to the x''-y''-z'' frame. Use these results, $\boldsymbol{\rho}''$ from Example 11.2.1, and the

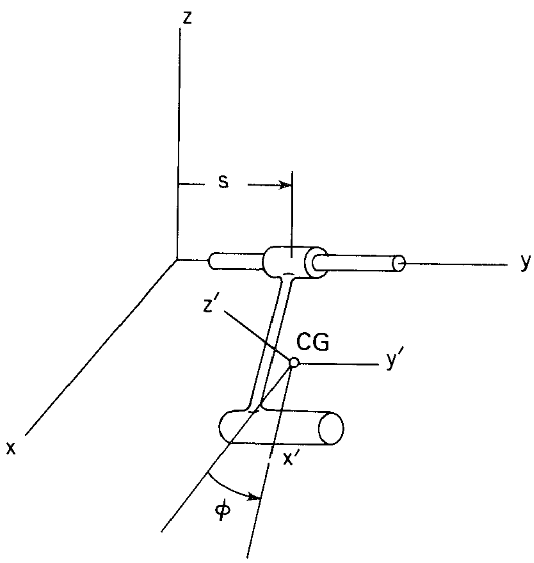


Figure P11.1.4

parallel axis transformation of Example 11.2.2 to evaluate the moments of inertia of the body with respect to the centroidal x'-y'-z' frame.

11.2.2. Verify that Eqs. 11.2.10 to 11.2.12 are valid under the stated conditions.

11.2.3. Show that, if any pair of coordinate planes are planes of symmetry for a body, then all products of inertia with respect to this body-fixed reference frame are zero.

Section 11.3

11.3.1. Expand Eq. 11.3.5, using the definitions of Eq. 11.3.4, and show that it is just

$$\sum_{i=1}^{nb} \{\delta\mathbf{r}_i^T[m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i^A] + \delta\boldsymbol{\pi}_i'^T[\mathbf{J}_i'\dot{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_i'\mathbf{J}_i'\boldsymbol{\omega}_i' - \mathbf{n}_i'^A]\} = 0$$

11.3.2. Use the argument presented in Section 6.3 to show that the coefficient matrix in Eq. 11.3.11 is nonsingular if kinetic energy is positive for all nonzero virtual velocities that satisfy the homogeneous velocity equations of Eq. 11.3.9 and if the constraint Jacobian has full row rank.

11.3.3. Repeat the argument of Prob. 11.3.2 to show that, under the same hypotheses, the coefficient matrix in Eq. 11.3.29 is nonsingular.

SUMMARY OF KEY FORMULAS

Newton–Euler Equations of Motion of a Body

$$\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\boldsymbol{\pi}'^T[\mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' - \mathbf{n}'] = 0 \qquad \textbf{(11.1.19)}$$

$$\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\mathbf{p}^T[4\mathbf{G}^T\mathbf{J}'\mathbf{G}\ddot{\mathbf{p}} - 8\dot{\mathbf{G}}^T\mathbf{J}'\dot{\mathbf{G}}\mathbf{p} - 2\mathbf{G}^T\mathbf{n}'] = 0 \qquad \textbf{(11.3.27)}$$

Inertia Properties

$$\int_m \mathbf{s}'^P \, dm(P) = \mathbf{0}, \qquad \boldsymbol{\rho}'' = \frac{1}{m}\int_m \mathbf{s}''^P \, dm(P) \qquad \textbf{(11.1.13, 11.2.1)}$$

$$\mathbf{J}' = -\int_m \tilde{\mathbf{s}}'^P \tilde{\mathbf{s}}'^P \, dm(P), \qquad \mathbf{J}'' = -\int_m \tilde{\mathbf{s}}''^P \tilde{\mathbf{s}}''^P \, dm(P) \qquad \textbf{(11.1.17, 11.2.4)}$$

$$\mathbf{J}'' = \mathbf{C}^T \mathbf{J}' \mathbf{C} + m(\boldsymbol{\rho}''^T \boldsymbol{\rho}'' \mathbf{I} - \boldsymbol{\rho}'' \boldsymbol{\rho}''^T), \qquad \mathbf{s}' = \mathbf{C}\mathbf{s}'' \qquad \textbf{(11.2.7)}$$

$$\boldsymbol{\rho}'' = \frac{1}{m}\sum_{i=1}^{k} m_i \boldsymbol{\rho}_i'', \qquad \mathbf{J}' = \sum_{i=1}^{k} \mathbf{J}_i' \qquad \textbf{(11.2.18, 19)}$$

Lagrange Multiplier Constrained Equations of Motion

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \boldsymbol{\Phi}_{\mathbf{r}}^T \\ \mathbf{0} & \mathbf{J}' & \boldsymbol{\Phi}_{\boldsymbol{\pi}'}^T \\ \boldsymbol{\Phi}_{\mathbf{r}} & \boldsymbol{\Phi}_{\boldsymbol{\pi}'} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^A \\ \mathbf{n}'^A - \tilde{\boldsymbol{\omega}}' \mathbf{J}' \boldsymbol{\omega}' \\ \boldsymbol{\gamma} \end{bmatrix} \qquad \textbf{(11.3.11)}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \boldsymbol{\Phi}_{\mathbf{r}}^T & \mathbf{0} \\ \mathbf{0} & 4\mathbf{G}^T \mathbf{J}' \mathbf{G} & \boldsymbol{\Phi}_{\mathbf{p}}^T & \mathbf{0} \\ \boldsymbol{\Phi}_{\mathbf{r}} & \boldsymbol{\Phi}_{\mathbf{p}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_{\mathbf{p}}^{\mathbf{p}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \boldsymbol{\lambda} \\ \boldsymbol{\lambda}^{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^A \\ 2\mathbf{G}^T \mathbf{n}'^A + 8\dot{\mathbf{G}}^T \mathbf{J}' \dot{\mathbf{G}} \mathbf{p} \\ \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathbf{p}} \end{bmatrix} \qquad \textbf{(11.3.29)}$$

Joint Reaction Forces

$$\mathbf{F}_i''^k = -\mathbf{C}_i^T \mathbf{A}_i^T \boldsymbol{\Phi}_{\mathbf{r}_i}^{kT} \boldsymbol{\lambda}^k, \qquad \mathbf{T}_i''^k = -\mathbf{C}_i^T (\boldsymbol{\Phi}_{\boldsymbol{\pi}_i}^{kT} + \tilde{\mathbf{s}}_i'^{PT} \mathbf{A}_i^T \boldsymbol{\Phi}_{\mathbf{r}_i}^{kT}) \boldsymbol{\lambda}^k \qquad \textbf{(11.5.6)}$$